Ampere's law (Exercise)*

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1 Worked out exercises

1.1

Problem 1: Two wires each carrying current I are perpendicular to xy plane. The current in one of them is into the plane denoted by a cross sign and the current in the other wire is out of the plane denoted by a filled circle. If the linear distance between the positions of two wires is “2a”, then find the net magnetic field at a distance ”b” on the perpendicular bisector of the line joining the positions of two wires.

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Magnetic field at perpendicular bisector

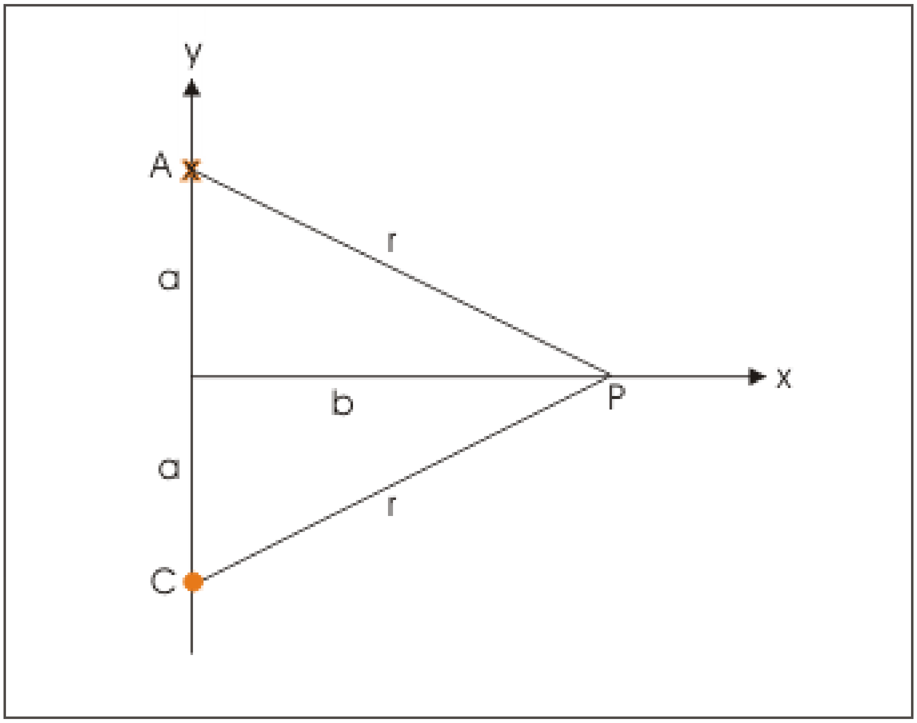

Figure 1: Magnetic field at perpendicular bisector

Solution : The magnitudes of magnetic fields due to wires at A and B are equal. Applying Ampere's law, the magnetic field due to each wire is :

$$B = \frac{\mu_0 I}{2\pi r}$$

The magnetic fields are directed tangential to the circle drawn containing point "P" with centers "A" and "B" as shown in the figure. Each magnetic field makes an angle say "θ" with the bisector. The components in y-direction cancel out, whereas x-components add up. Clearly, the net magnetic field is directed in negative x – direction. The magnitude of net magnetic field is :

Magnetic field at perpendicular bisector

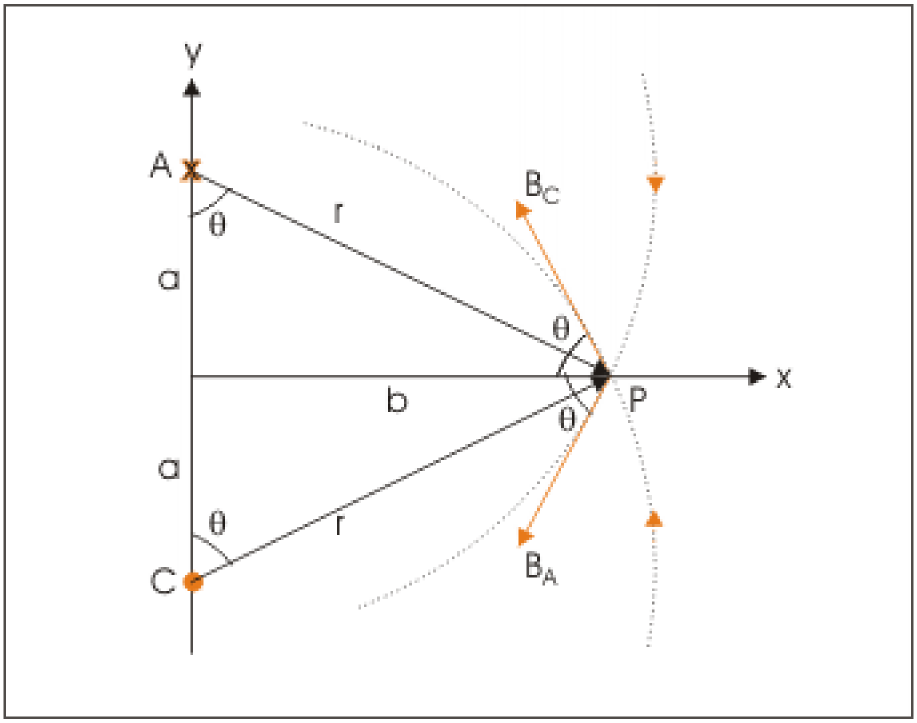

Figure 2: Magnetic field at perpendicular bisector

$$\Rightarrow B = 2X\frac{\mu_0 I\cos\theta}{2\pi r} = \frac{\mu_0 I\cos\theta}{\pi r}$$

Now,

$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{(a^2+b^2)}}$$

and

$$r = \sqrt{(a^2+b^2)}$$

Putting these expressions in the equation for the magnetic field at "P", we have :

$$\Rightarrow B = \frac{\mu_0 I\cos\theta}{\pi r} = \frac{\mu_0 I a}{\pi\sqrt{(a^2+b^2)}\sqrt{(a^2+b^2)}}$$

$$\Rightarrow B = \frac{\mu_0 I a}{\pi\left(a^2+b^2\right)}$$

1.2

Problem 2: Five straight wires, carrying current I, are perpendicular to the plane of drawing. Four of them are situated at the corners and fifth wire is situated at the center of a square of side "a". Two of the wires at the corners are flowing into the plane whereas the remaining three are flowing out of the plane. Find the net magnetic field at the center of square.

Five straight wires, carrying current I

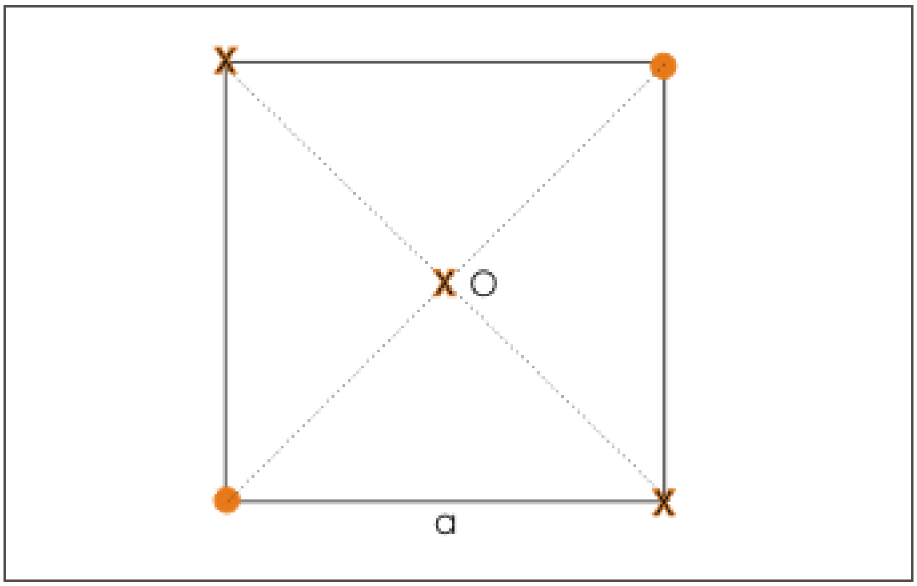

Figure 3: Five straight wires, carrying current I

Solution : According to Ampere's law[1], the magnetic field due to a straight wire carrying current "I" at a perpendicular distance "r" is given as :

$$B = \frac{\mu_0 I}{2\pi R}$$

The wires at the corners carry equal currents and the center "O" is equidistant from these wires. Thus, magnetic fields due to these four wires have equal magnitude. In order to find the directions of magnetic fields, we draw circles containing point of observation "O". The direction of magnetic field is tangential to the circle. Applying Right hand thumb rule for straight wire, we determine the orientation of magnetic field as shown in the figure. Clearly, the net magnetic field due to these four wires at the center is zero.

[1] "Ampere's law": Section Basis of Ampere law <http://cnx.org/content/m31895/latest/#section-1>

Directions of magnetic fields

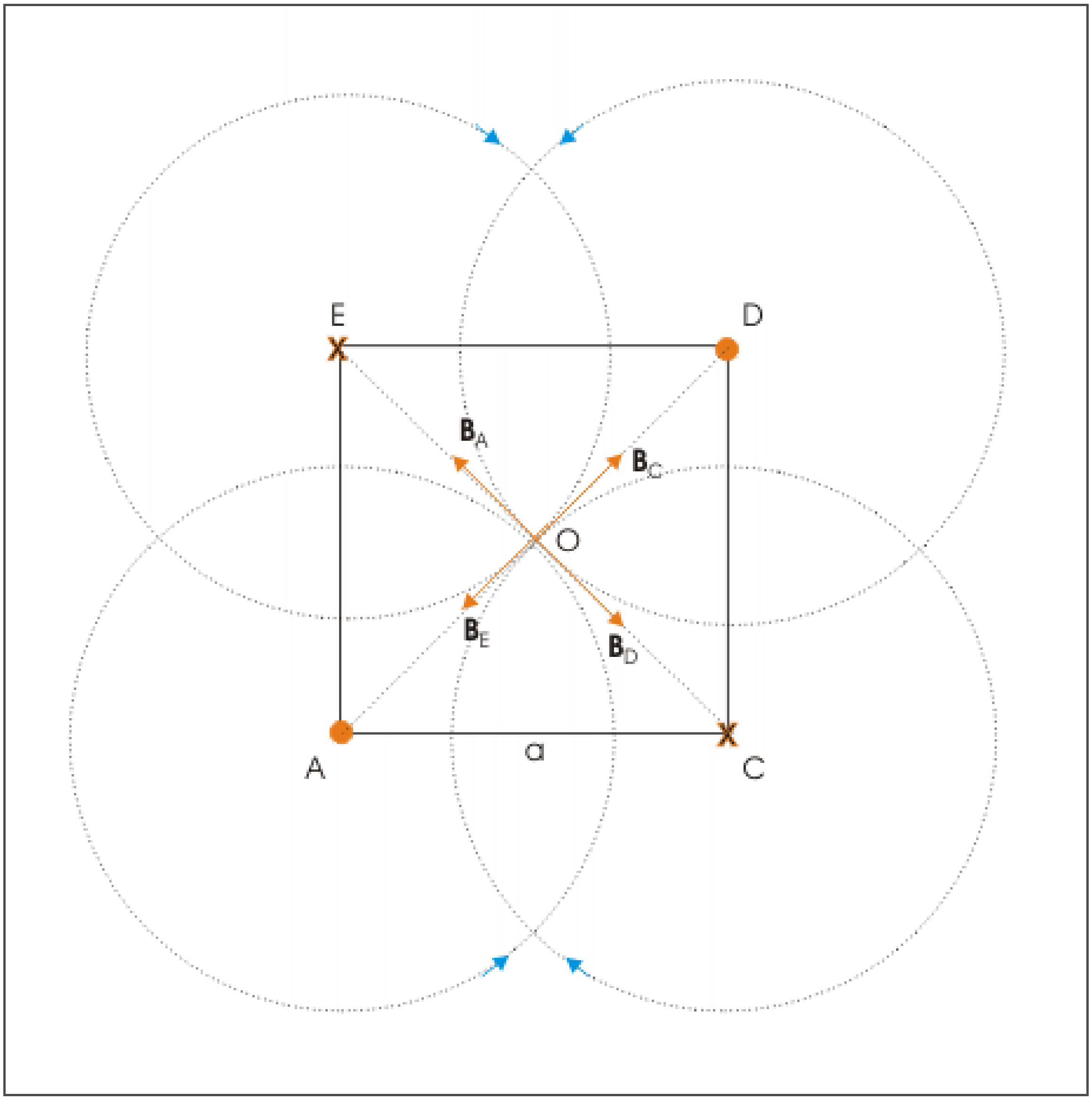

Figure 4: Directions of magnetic fields

Now, magnetic field at a point on the wire itself is zero. Thus, magnetic fields due to all the five wires at the center "O" is zero.

It is interesting to note that if straight wires with currents are arranged differently, for example, two currents out of the plane at A and C respectively and the other two currents into the plane at D and E respectively are arranged, then magnetic fields do not cancel and there is net non-zero magnetic field at "O" due to currents in four wires.

1.3

Problem 3: There are five long wires perpendicular to the plane of drawing, each carrying current I as shown by filled circles (out of plane) and crosses (into the plane) in the figure below. Determine closed line integrals $\oint \mathbf{B}.d\mathbf{l}$ for each of the four contours in the direction of integration shown.

Currents and Ampere's loops

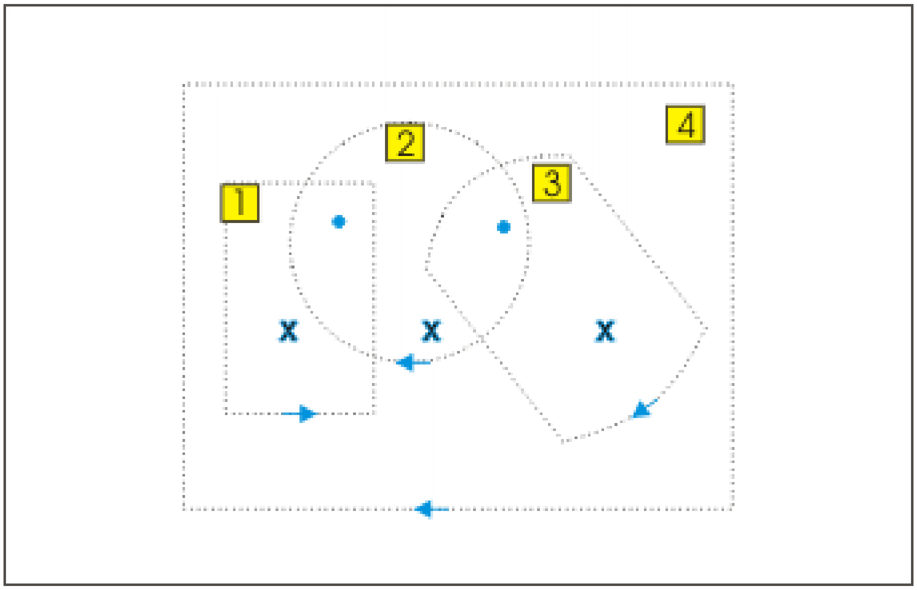

Figure 5: Currents and Ampere's loops

Solution :
According to Ampere's law[2], the closed line integral is related to enclosed current as :

$$\oint \mathbf{B}.\mathbf{l} = \mu_0 I$$

For loop 1, the integration direction is anticlockwise. The current out of the page is positive and current into the page is negative. There are one in and one out current here. The net current is zero. Hence,

$$\Rightarrow \oint \mathbf{B}.\mathbf{l} = 0$$

For loop 2, the integration direction is clockwise. The current out of the page is negative and current into the page is positive. There are one in and two out current here. The net current is one out current i.e. "-I". Hence,

$$\oint \mathbf{B}.\mathbf{l} = -\mu_0 I$$

For loop 3, the integration direction is clockwise. The current out of the page is negative and current into the page is positive. There are one in and one out current here. The net current is zero. Hence,

[2]"Ampere's law": Section Statement of Ampere law <http://cnx.org/content/m31895/latest/#section-2>

$$\oint \mathbf{B}.d\mathbf{l} = 0$$

For loop 4, the integration direction is clockwise. The current out of the page is negative and current into the page is positive. There are three in and two out current here. The net current is one in current i.e. "I". Hence,

$$\oint \mathbf{B}.\mathbf{l} = \mu_0 I$$

1.4

Problem 4: The magnetic field in a region is given by relation :

$$\mathbf{B} = 5\hat{\mathbf{l}}$$

Closed line integral of magnetic field

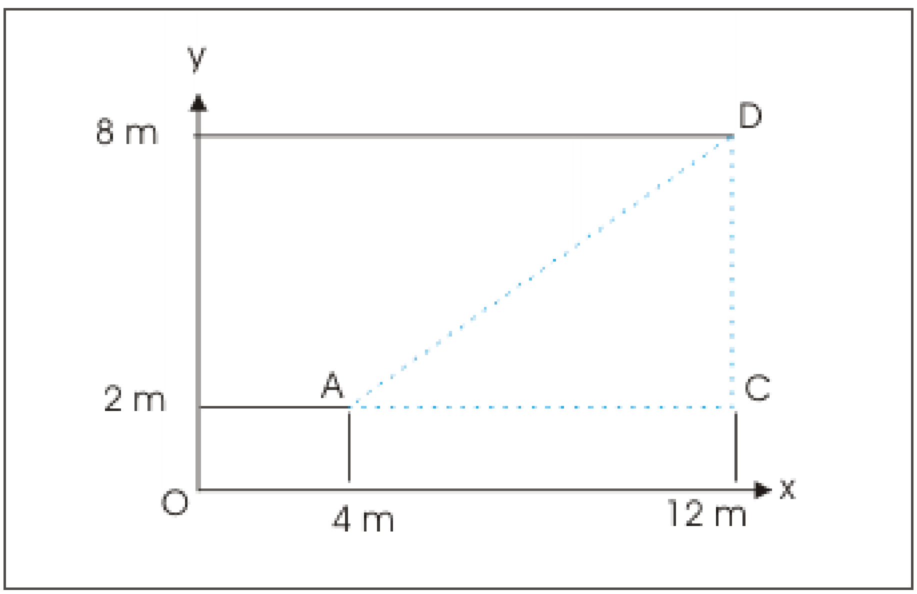

Figure 6: Closed line integral of magnetic field

where **i** is the unit vector in x direction. Determine closed line integral of magnetic field along the triangle ACD.

Solution :

For line segment AC, magnetic field and length element are in the same direction. Applying Ampere's law :

$$\int_{AC} \mathbf{B}.\mathbf{l} = \int_{AC} 5x\cos 0^\circ = \int_{AC} 5x$$

$$\Rightarrow \int_{AC} \mathbf{B}.\mathbf{l} = 5\int_{AC} x = 5X8 = 40 \quad Tm$$

We see that magnetic field is perpendicular to line segment CD. Therefore, magnetic line integral for this segment is equal to zero.

For the line segment DA, the length is $\sqrt{(8^2 + 6^2)} = 10$ m.

Closed line integral of magnetic field

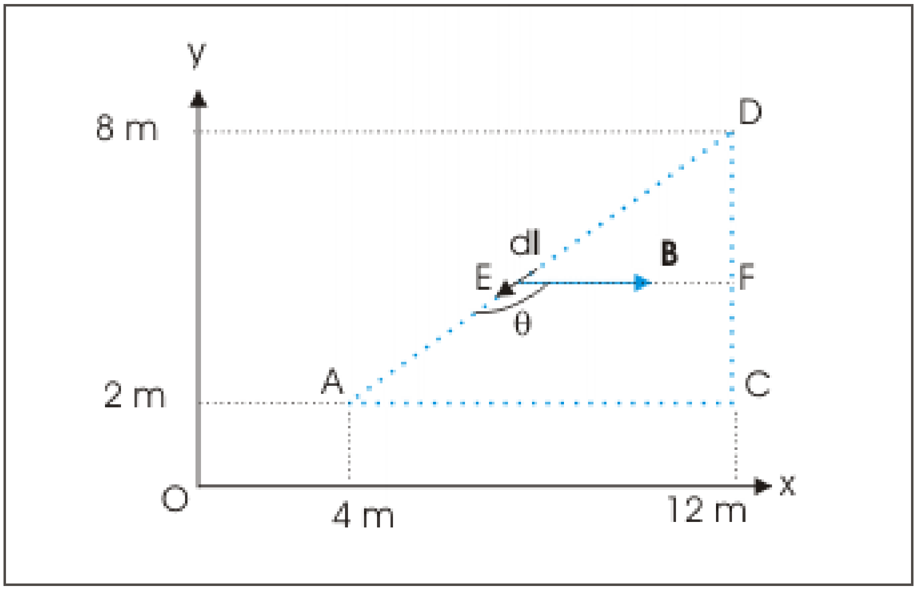

Figure 7: Closed line integral of magnetic field

$$\int_{DA} \mathbf{B}.\mathbf{l} = \int_{DA} 5x\cos\angle AEF = 5\int_{DA} x\cos\left(\pi - \angle DEF\right) = -5\int_{DA} x\cos\angle DEF$$

$$\Rightarrow \int_{DA} \mathbf{B}.\mathbf{l} = -5\int_{DA} x\cos\angle DAC = -5\int_{DA} \frac{AC}{AD} x = -5\int_{DA} \frac{8}{10} x$$

$$\Rightarrow \int_{DA} \mathbf{B}.\mathbf{l} == -4\int_{DA} x = -4XDA = -40 \quad Tm$$

Adding two values, the value of closed line integral is zero.

$$\oint \mathbf{B}.\mathbf{l} = 0$$

Thus, we see that current though the region is zero even though there exists magnetic field in the region.

1.5

Problem 5: Straight wires are mounted tightly over a long hollow cylinder of radius "R" such that they are parallel to the axis of cylinder. The perpendicular cross-section of the arrangement is shown in the figure below. If there are N such wires each carrying a current I, then determine magnetic field inside and outside the cylinder.

Magnetic field due to current in tightly packed straight wires

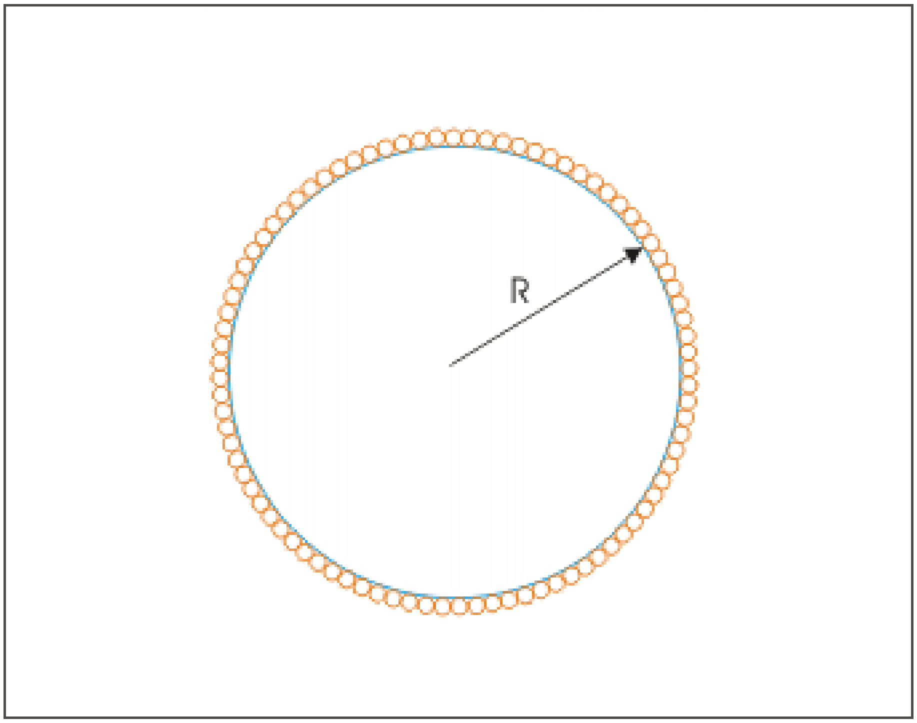

Figure 8: Magnetic field due to current in tightly packed straight wires

Solution : We draw an Ampere loop of radius "r" for applying Ampere's law[3] at a point inside the cylinder. But there is no current inside. Hence,

[3]"Ampere's law": Section Statement of Ampere law <http://cnx.org/content/m31895/latest/#section-2>

Magnetic field due to current in tightly packed straight wires

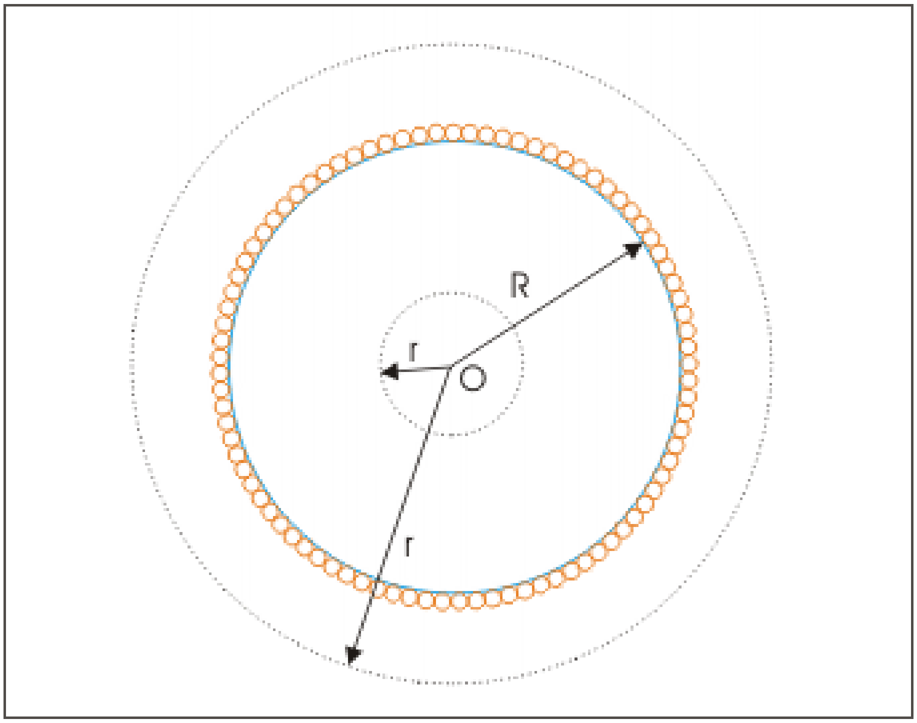

Figure 9: Magnetic field due to current in tightly packed straight wires

$$\oint \mathbf{B}.\mathbf{l} = 0$$

Here integration of dl along the loop is equal to perimeter of loop i.e. 2πr. Hence, B = 0. For determining magnetic field at an outside point, we draw an Ampere loop of radius "r". Here, the total current is NI. Hence,

$$\oint \mathbf{B}.\mathbf{l} = \mu_0 NI$$

$$\Rightarrow 2\pi rB = \mu_0 NI$$

$$\Rightarrow B = \frac{\mu_0 NI}{2\pi r}$$

1.6

Problem 6: A cylindrical conductor of radius R carries current I distributed uniformly across the cross-section. Draw the curve showing variation of magnetic field as we move away from the axis of conductor in perpendicular direction.

Solution : Let the perpendicular direction to the axis be x-axis. The magnetic field at a point inside the conductor is given by :

$$B = \frac{\mu_0 I x}{2\pi R^2}$$

Clearly, magnetic field increases linearly as move away from the axis towards the edge of conductor and attains the maximum at the surface, when x=R and magnetic field is given as:

$$B = \frac{\mu_0 I R}{2\pi R^2} = \frac{\mu_0 I}{2\pi R}$$

The magnetic field at a point outside the conductor is :

$$B = \frac{\mu_0 I}{2\pi x}$$

The magnetic field is inversely proportional to the linear distance "x". The required plot of magnetic field .vs. x is as shown in the figure below :

Variation of magnetic field

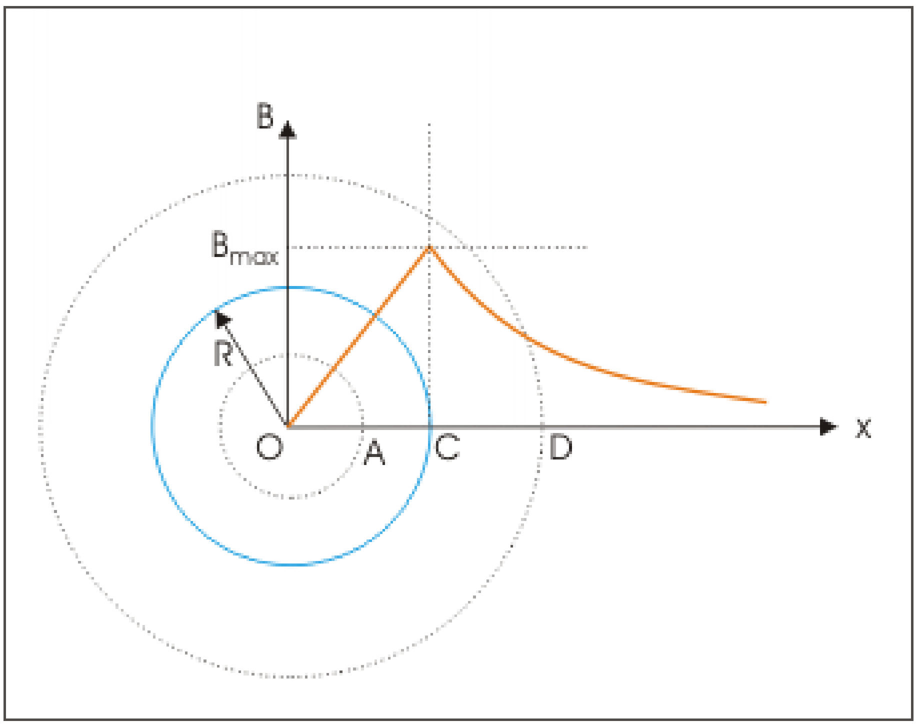

Figure 10: Variation of magnetic field

1.7

Problem 7: A long annular cylindrical conductor of radii "a" and "b" carries current I. The perpendicular cross section of annular cylinder is shown in the figure below. If the current distribution in the annular region is uniform, determine magnetic field at a point in the annular region at a radial distance "r" from the axis.

Magnetic field due to current in annular cylindrical conductor

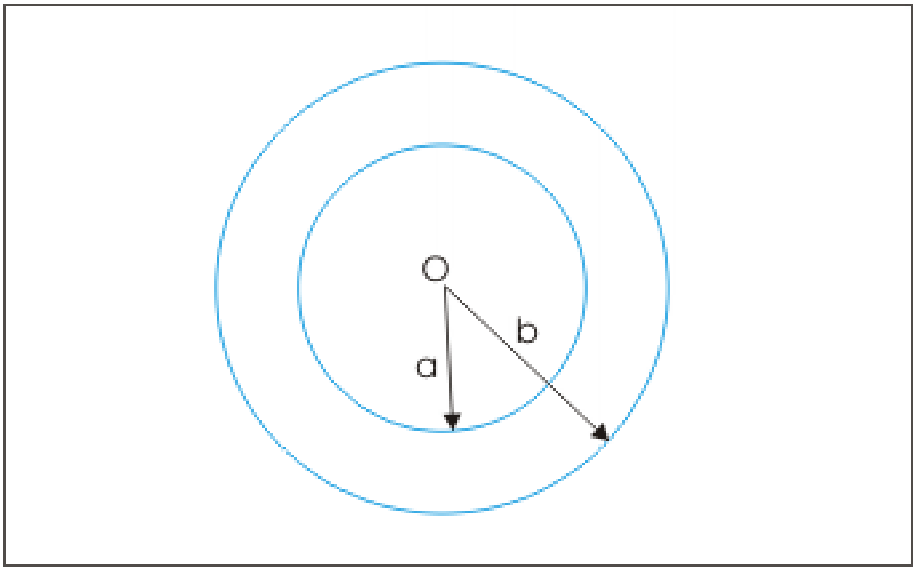

Figure 11: Magnetic field due to current in annular cylindrical conductor

Solution : According to Ampere's law[4],

$$\oint \mathbf{B}.\mathbf{l} = \mu_0 I$$

In order to evaluate this equation, we need to know the current in the annular region from r=a to r=r. For this we need the value of current density. Here, total current is given. Dividing this by the total area of the region gives us the current density,

$$J = \frac{I}{\pi \left(b^2 - a^2\right)}$$

The net current through the Ampere loop of radius "r" falling in the annular region is given by multiplying current density with the annular area between r=a and r=r. Applying Ampere's law for a loop of radius r,

[4]"Biot - Savart Law": Section Experimental verification of Biot-Savart's law <http://cnx.org/content/m31057/latest/#section-2>

Magnetic field due to current in annular cylindrical conductor

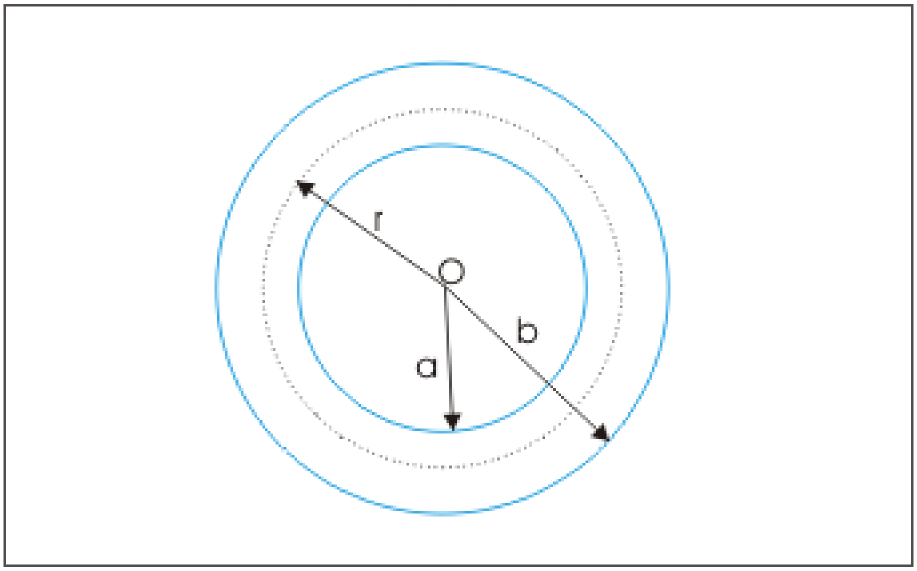

Figure 12: Magnetic field due to current in annular cylindrical conductor

$$\oint \mathbf{B.l} = \mu_0 \pi \left(r^2 - a^2\right) J = \frac{\mu_0 \pi \left(r^2 - a^2\right) I}{\pi \left(b^2 - a^2\right)}$$

$$\Rightarrow 2\pi r B = \frac{\mu_0 I \left(r^2 - a^2\right)}{\left(b^2 - a^2\right)}$$

$$\Rightarrow B = \frac{\mu_0 I \left(r^2 - a^2\right)}{2\pi r \left(b^2 - a^2\right)}; \quad a < r < b$$

1.8

Problem 8: A long annular cylindrical conductor of radii "a" and "b" carries a current. If the current distribution in the annular region is given as J = kr, where k is a constant, then determine magnetic field at a point in the annular region at a radial distance "r" from the axis.

Solution : This question is similar to earlier question with one difference that areal current density is not uniform. We see here that the current distribution in the annular region is given as J=kr. Clearly, current density increases as we move from inner edge to the outer edge of the annular cylinder. The current in the small strip đr is :

$$I = 2\pi r r J = 2\pi r r k r = 2\pi k r^2 r$$

Magnetic field due to current in annular cylindrical conductor

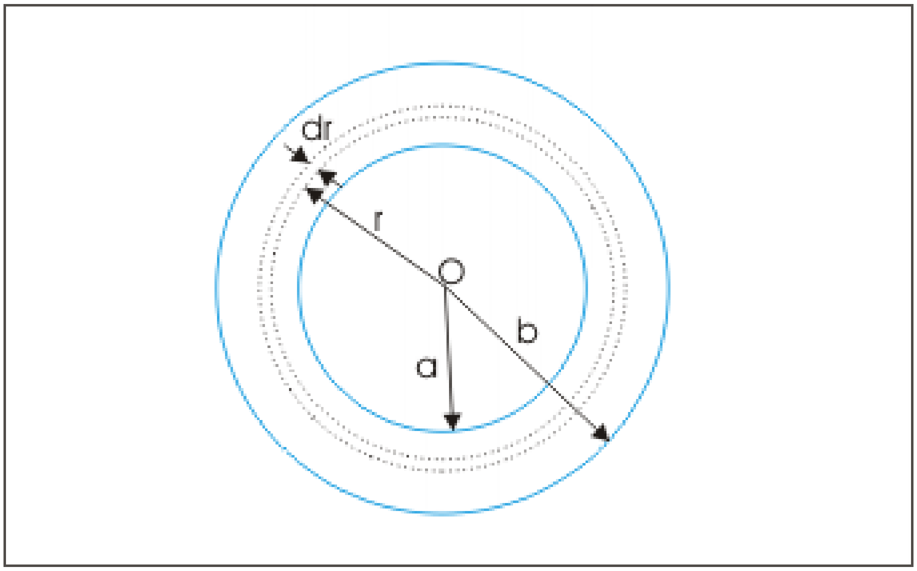

Figure 13: Magnetic field due to current in annular cylindrical conductor

Applying Ampere's law for a loop of radius r and considering that current is distributed from r=a to r=r,

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 \oint dI = \mu_0 \int_a^r 2\pi k r^2 dr = 2\pi\mu_0 k \int_a^r r^2 dr$$

$$\Rightarrow 2\pi r B = 2\pi\mu_0 k \left[\frac{r^3}{3}\right]_a^r = \frac{2\pi\mu_0 k \left(r^3 - a^3\right)}{3}$$

$$\Rightarrow B = \frac{\mu_0 k \left(r^3 - a^3\right)}{3r}; \quad a < r < b$$

1.9

Problem 9: A long solenoid having 1000 turns per meter carries a current of 1 A. A long straight conductor of radius 0.5 cm and carrying a current of 10π A is placed coaxially along the axis of solenoid. Compare magnetic fields due to two currents at that point. Also determine magnetic field at a point on the surface of straight conductor.

Solution : The magnetic field due to solenoid is uniform inside the solenoid and is given as :

$$B_S = \mu_0 n I = 4\pi X 10^{-7} X 1000 X 1 = 4\pi X 10^{-4} \quad T$$

The magnetic field due to straight conductor on its surface is :

$$B_C = \frac{\mu_0 I}{2\pi R} = \frac{4\pi X 10^{-7} X 10\pi}{2\pi X 0.5 X 10^{-2}} = 4\pi X 10^{-4} \quad T$$

Magnetic field due solenoid and straight conductor

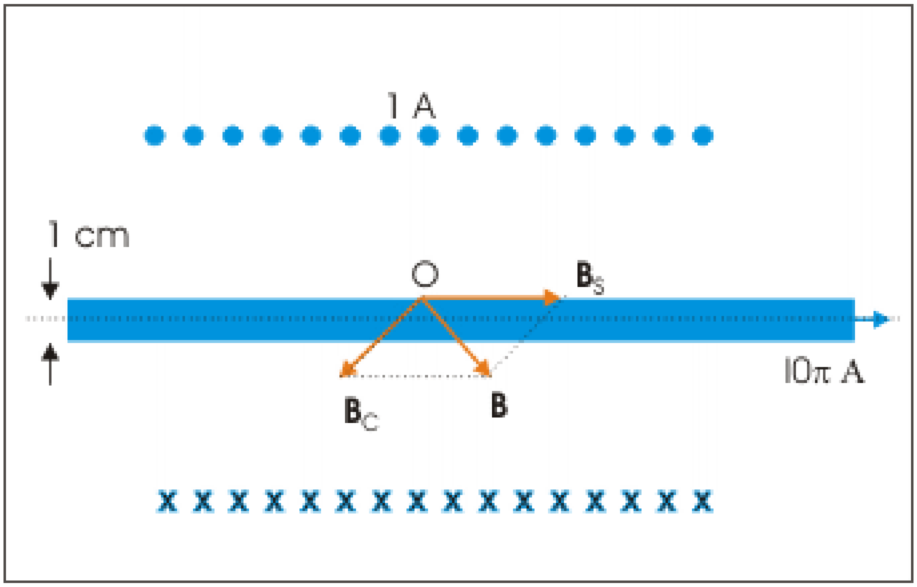

Figure 14: Magnetic field due solenoid and straight conductor

The magnetic field due to straight conductor is tangential to the circumference and hence is perpendicular to magnetic field due to solenoid. The resultant magnetic field is, therefore,

$$B = \sqrt{\left(2X16\pi^2 X10^{-8}\right)} = \sqrt{2}X4\pi X10^{-4} = 1.778X10^{-3} \quad T$$

Both solenoid and straight conductor produces equal magnetic field at the surface of conductor. It is interesting to observe that a straight conductor requires a current of magnitude which is 10π i.e. 31.4 times the current in solenoid. This illustrates the effectiveness of solenoid over a straight conductor in setting up a magnetic field with respect to straight conductor. For this reason, a solenoid is generally used as a magnet in application situations.

1.10

Problem 10: A long cylindrical conductor of radii “a” is coaxially placed inside an annular cylindrical conductor of radii “b” and “c”. The perpendicular cross section of the coaxial annular cylinders is shown in the figure below. If currents in two conductors are I each but in opposite direction, then find magnetic field at a point (i) inside the inner conductor (ii) region between two cylinders (iii) inside annular cylinder and (iv) outside the annular cylinder. Assume current density to be uniform in both cylinders.

Magnetic field due to current in coaxial cylindrical conductors

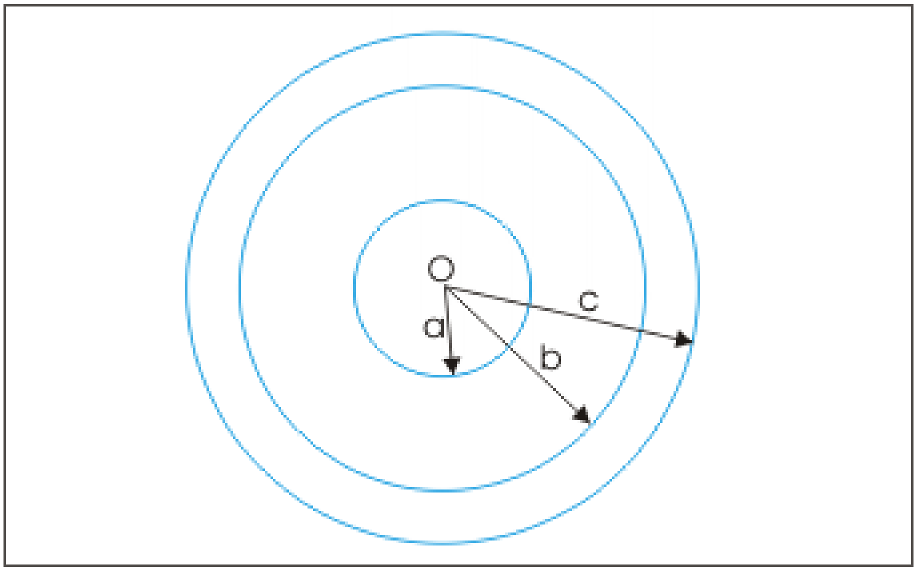

Figure 15: Magnetic field due to current in coaxial cylindrical conductors

Solution : We note that current densities in two cylinders are uniform. To find magnetic field at a point inside the inner cylinder, we first determine its current density.

$$J_i = \frac{I}{\pi a^2}$$

Note that current outside the Ampere loop in the inner cylinder and current in the outer conductor do not contribute towards enclosed current. Applying Ampere's law for a loop of radius r inside the inner cylinder,

$$\oint \mathbf{B}.\mathbf{l} = \mu_0 \pi r^2 J = \frac{\mu_0 \pi r^2 I}{\pi a^2} = \frac{\mu_0 I r^2}{a^2}$$

$$\Rightarrow 2\pi r B = \frac{\mu_0 I r^2}{a^2}$$

$$\Rightarrow B = \frac{\mu_0 I r^2}{2\pi r a^2}; \quad r < a$$

To find magnetic field at a point between inner and outer cylinders, we apply Ampere's law for a loop of radius r between the region (a<r<b). Note that outer conductor does not contribute towards enclosed current. Applying Ampere's law[5] for a loop of radius r between inner and outer cylinders,

$$\oint \mathbf{B}.\mathbf{l} = \mu_0 I$$

$$\Rightarrow 2\pi r B = \mu_0 I$$

[5]"Ampere's law": Section Statement of Ampere law <http://cnx.org/content/m31895/latest/#section-2>

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

To find magnetic field at a point inside the outer cylinder, we apply Ampere's law for a loop of radius r between the region (b<r<c). Note that current in the inner conductor and annular region defined by b<r<c contribute towards enclosed current. In order to find the enclosed current in the outer cylinder, we first determine its current density.

$$J_o = \frac{I}{\pi\left(c^2 - b^2\right)}$$

Further the current in inner and outer cylinders are opposite in direction. We observe here that current density of inner cylinder is greater as current I is divided by smaller area. Thus, we shall deduct the current through the annular region of outer cylinder from the current in inner cylinder. Applying Ampere's law for a loop of radius r inside the outer cylinder,

$$\oint \mathbf{B}.\mathbf{l} = \mu_0 I - \mu_0 X \frac{\pi\left(r^2 - b^2\right) XI}{\pi\left(c^2 - b^2\right)} = \mu_0 \left[I - \frac{\left(r^2 - b^2\right) XI}{\left(c^2 - b^2\right)}\right]$$

$$\Rightarrow 2\pi r B = \frac{\mu_0 I\left(c^2 - b^2 - r^2 + b^2\right)}{\left(c^2 - b^2\right)} = \frac{\mu_0 I\left(c^2 - r^2\right)}{\left(c^2 - b^2\right)}$$

$$\Rightarrow B = \frac{\mu_0 I\left(c^2 - r^2\right)}{2\pi r\left(c^2 - b^2\right)}$$

To find magnetic field at a point outside the outer cylinder, we apply Ampere's law for a loop of radius r (r>c). The net current the loop is zero. Hence,

$$\oint \mathbf{B}.\mathbf{l} = 0$$

$$\Rightarrow 2\pi r B = 0$$

$$\Rightarrow B = 0$$