Bachelorthesis

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[5]

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2.1 Minkowski space ✓

First we need to know what a metric is. The mathematical definition is as follows[see differentialgeometrie.pdf p.85] A metric $d : X \times X \to \mathbb{R}$ is a function, that satisfies the conditions:

(i) $d(x, y) = d(y, x)$;

(ii) $d(x, y) \geq 0$, with equality if and only if $x = y$;

(iii) $d(x, y) + d(x, z) \geq d(x, z)$

for any $x, y, z \in X$.

But in physics, we express the metric as an invariant line element:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{2.1}$$

so for getting $d(x, y)$ one have to take the square root of $\mathrm{d}s^2$ and integrate. $g_{\mu\nu}(x)$ is the *metric tensor*.

For the of ordinary Minkowski space the metric tensor is now $g_{\mu\nu}(x) = \eta_{\mu\nu}$ so that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{2.2}$$

Where $\eta_{\mu\nu}$ is of the form

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{2.3}$$

2.2 De Sitter space ✓

De Sitter space is a good approximation of the geometry of our universe today and in the past while inflation[1]. It is a solution of the Einstein's equations with positive energy and is a submanifold of the Minkowski space.

[1]inflation: a well proofed theory of exponential expansion of space and mass in the early universe

In order to understand the form of the De Sitter metric, let's start with the Riemann tensor[2] for a maximally symmetric n-dimensional manifold:

$$R_{\mu\nu\lambda\xi} = \kappa(g_{\mu\lambda}g_{\nu\xi} - g_{\mu\xi}g_{\nu\lambda}) \tag{2.4}$$

Here κ is a measure of the curvature.[3]

The already known Minkowski space is a maximally symmetric spacetime with a vanishing curvature $\kappa = 0$. If the curvature is positive, which means $\kappa > 0$, then it is called De Sitter space. For building its metric, we first take the five-dimensional Minkowski space:

$$\mathrm{d}s_5^2 = -\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 \tag{2.5}$$

And a corresponding hyperboloid:

$$-u^2 + x^2 + y^2 + z^2 + w^2 = 1 \tag{2.6}$$

We use new coordinates τ, χ, θ and ϕ:

$$\begin{aligned} u &= \sinh(\tau) \\ w &= \cosh(\tau)\cos\chi \\ x &= \cosh(\tau)\sin\chi\cos\theta \\ y &= \cosh(\tau)\sin\chi\sin\theta\cos\phi \\ z &= \cosh(\tau)\sin\chi\sin\theta\sin\phi \end{aligned} \tag{2.7}$$

So in the end the metric on the hyperboloid looks like:

$$ds^2 = -\mathrm{d}\tau^2 + \cosh^2(\tau)\,\mathrm{d}\Omega_3^2 \tag{2.8}$$

with $\mathrm{d}\Omega_3^2 = \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$ which is the metric on a three-sphere $\mathbb{S}^3$.[4]

Just for completion, a maximally symmetric spacetime with negative curvature $\kappa < 0$ is called Anti-De Sitter space (further reading: [1]).

2.3 Schwarzschild geometry ✓

The Schwarzschild geometry is a source-free solution of Einstein's equation with spherical symmetry. The latter means, the solution is invariant under rotations. At large distances it approaches the ordinary Minkowski space. The spacetime metric of the Schwarzschild geometry looks like this:

[see chapter21 in [2]]

$$\mathrm{d}s^2 = -\frac{r-2GM}{r}\,\mathrm{d}t^2 + \frac{r}{r-2GM}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{2.9}$$

[2]It should be known, that in general relativity we are in a four-dimensional Riemann manifold, where the Riemann tensor gives us the curvature.

[3]It is normalized and more specifically, it's the Ricci curvature, but this and more information about κ is not needed for understanding the De Sitter space.

[4]A sphere is a n-dimensional manifold in Euclidean $(n+1)$ - dimensional space.

The G is Newton's gravitational constant, M is a mass parameter, which comes from idealization if one is looking at the black hole from a distance $r \gg 2GM$. The term in the brackets is often shorten by $\mathrm{d}\Omega_2^2$ which is the metric on the two-sphere $\mathbb{S}^2$.

The most interesting radii are $r = 0$ and $r = 2GM$. At $r = 0$ we have a singularity, i.e. the sphere $\mathbb{S}^2$ goes to zero size and the Schwarzschild metric diverges. This can be described by the Riemann tensor $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ which encodes the tidal effects[5] on free-falling objects.

$r_s \equiv 2GM$ is called the *Schwarzschild radius*. At this radius, the metric has a singularity too, but that is just because of our choice of coordinates. Here the signs of $\mathrm{d}r^2$ and $\mathrm{d}t^2$ switch, so the coordinate r becomes timelike, and the coordinate t becomes spacelike. That causes that everything under r_s will inevitably fall into the singularity. So nothing, even massless particles like photons, can move forward in ordinary time. This means for an observer in $r > r_s$, everything in $r < r_s$ is invisible and inversely.

That is, why r_s is often called the *event horizon* or just *horizon*. In addition, the closer someone is to r_s while sending a signal, the lower will be its energy when it reaches $r \gg 2GM$. This phenomenon is called *gravitational redshift*.

3 Basic ways of visualising the black hole problems on paper

3.1 The Kruskal extension ✓(still questions)

From here on, we will use $r_s = 2GM = 1$.[6] Instead of using the coordinates (t, r, Ω) like in the previous section, we now introduce the Kruskal-Szekeres coordinates, because they are a better choice for near-horizon physics.

First we parametrize the radial null geodesics in the Schwarzschild geometry as

$$t = \pm r_* + C, \tag{3.1}$$

where C is some constant of motion and r_* is a new radial coordinate defined as

$$r_* \equiv r + \log(r - 1). \tag{3.2}$$

also called the *tortoise coordinate*[7], because now we have an infinite coordinate range that fits in a finite geodesic distance.

The Kruskal-Szekeres coordinates are then defined as

$$U \equiv -e^{\frac{r_* - t}{2}} \tag{3.3}$$

$$V \equiv e^{\frac{r_* + t}{2}}. \tag{3.4}$$

[5]tidal effects: The nearer an object is to a black hole, the more deformed it becomes because of the gravitational force. Some also call this spaghettification. In the end, it will be destroyed before it reaches the singularity, except in Planck-scale physics.

[6]This means, the Schwarzschild geometry looks like $\mathrm{d}s^2 = -\frac{r-1}{r}\mathrm{d}t^2 + \frac{r}{r-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$

[7]The name *"tortoise"* has its origin in the paradox of Achilles and the tortoise.

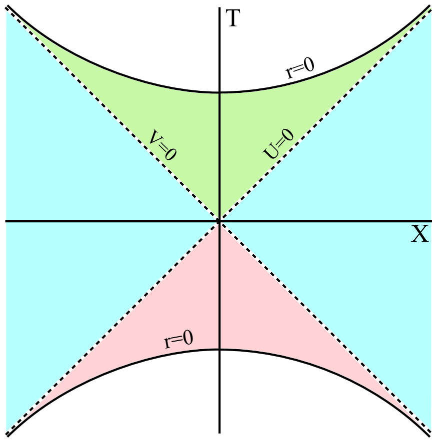

Figure 1. This is the XT plane of the **Kruskal extension**. The horizons are the dashed lines. The blue wedges are the exterior, the green wedge is the future interior and the past interior is in red. Nothing can escape the green wedge into the blue wedges, because there is no radial null geodesic which would connect the wedges.

Their lines of constant U and V are radial null geodesics and these coordinates have the property, that

$$UV = (1 - r)e^r. \tag{3.5}$$

This means we have a singularity at $UV = 1$ and the horizon is at $U = 0$ or $V = 0$. The metric looks now like

$$\mathrm{d}s^2 = -\frac{2}{r}e^{-r}\left(\mathrm{d}U\mathrm{d}V + \mathrm{d}V\mathrm{d}U\right) + r^2\mathrm{d}\Omega_2^2 \tag{3.6}$$

Because this metric still has an off-diagonal tensor, we define another set of coordinates

$$\begin{aligned} U &= T - X \\ V &= T + X \end{aligned} \tag{3.7}$$

Now the metric looks as follows:

$$\mathrm{d}s^2 = \frac{4}{r}e^{-r}\left(-\mathrm{d}T^2 + \mathrm{d}X^2\right) + r^2\mathrm{d}\Omega_2^2 \tag{3.8}$$

Note that there is now no singularity at $r = 1$.

This metric defines a geometry over the full XT plane, which can be seen in Figure 1. The **right blue wedge** is, in the old Schwarzschild coordinates, former $r > 1$, $-\infty < t < \infty$. If one wants to continue to $r < 1$, there is a branch cut defined in (3.3), which allows us to either go to the regions with $X^2 - T^2 < 0$ and $T > 0$ which is the **green wedge**, or the **red wedge** with $T < 0$ and also $X^2 - T^2 < 0$. At last we have the left blue wedge in which we also can have $r \gg 1$. Both blue wedges are asymptotically Minkowski regions.

wie sieht dieser branch cut genau aus?

The singularity $r = 0$, which you find at the top and the bottom of Figure 1, is the hyperboloid $X^2 - T^2 = -1$. It has two connected components, one at each boundary of the green and the red regions. These two regions are also called the future and the past interiors, while the other two are called the original/new exteriors (right/left blue wedges).

All regions together can interpret the full Schwarzschild metric as a wormhole connecting two nearly flat universes, both acting at $r \gg 1$ as if there were a point source of mass M. Signals can not travel through the wormhole, but two observers coming from opposite sides could meet in the middle and compare notes.

how the hell does he know that?

3.2 Penrose diagrams

We want to know the causal structure of spacetime by asking the question which points can receive signals from which other points. For throwing out some irrelevant information, we will know introduce the so called *conformal compactification*.

> We have two spacetimes with metrics, which are related as $g'_{\mu\nu}(x) = e^{2\omega(x)}g_{\mu\nu}(x)$ with a smooth real function $\omega(x)$. Or in other words, they differ only by multiplication with a positive scalar function on spacetime. The important thing is, those metrics have the same null geodesics. This goes not without saying, because only timelike/spacelike curves in one metric will be timelike/spacelike curves in the other, but not geodesics. Two metrics with this kind of relation are called *conformally equivalent*. This gives us a way to represent the asymptotic behaviour of spacetimes at large distances.

Now, *conformal compactification* is including infinity as a boundary of spacetime in a manifold by taking a function $\omega(x)$ that diverges while we approach infinity, so that infinity is brought into a finite distance.

3.2.1 of Minkowski space ✓

We will now show this on an example, the ordinary flat Minkowski space, whose metric in spherical coordinates looks like

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2 \tag{3.9}$$

Because the interesting things are happening in the asymptotical behavior at $r \to \infty$ and $|t| \to \infty$, we parametrize them with the aid of $\arctan(x)$, so that the boundary is pulled in a finite distance.

$$\begin{aligned} T + R &\equiv \arctan(t + r) \\ T - R &\equiv \arctan(t - r) \end{aligned} \tag{3.10}$$

And now the metrics looks as follows:

$$\mathrm{d}s^2 = \frac{1}{\cos^2(T+R)\cos^2(T-R)}\left[-\mathrm{d}T^2 + \mathrm{d}R^2 + \left(\frac{\sin(2R)}{2}\right)^2 \mathrm{d}\Omega_2^2\right]. \tag{3.11}$$

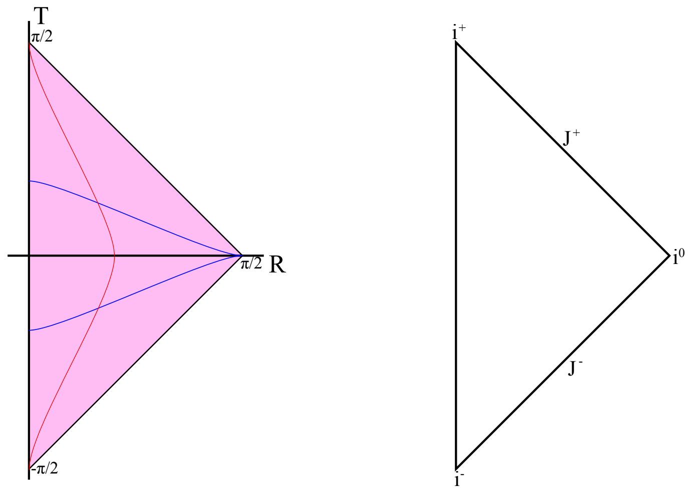

Figure 2. On the left you see the full **Minkowski space** in pink in the RT plane. We removed the prefactor of (3.11), because it would diverge at the boundary. On the right we formalize this in to a **Penrose diagram**.

This seems to be quite complicated, but if you have a look at **Figure 2**, you will see, why we were doing this. The new ranges of our coordinates are $|T \pm R| < \pi/2, R \geq 0$, and the spacetime was compactified by including the boundary at $|T \pm R| = \pi/2$.

The new boundary which are illustrated in **Figure 2** on the right side, is divided into five parts:

i^+ :	future timelike infinity	J^+ :	future null infinity
i^- :	past timelike infinity	J^- :	past null infinity
i^0 :	spatial infinity		

So timelike curves come from i^- and go to i^+, same for null curves with $J^{\mp}$, the spatial geodesics are ending at i^0. Massless particles are entering/leaving at $i^{\mp}$ and massive particles at $J^{\mp}$. The scattering matrix[8] maps the states on $J^- \cup i^-$ to the states on $J^+ \cup i^+$.

From this diagram, we learn that the Minkowski space does *not* have *event horizons*.

But what about the other two dimensions, the angulars i.e. the $\mathbb{S}^2$. In the Penrose diagram of the Minkowski space it is suppressed at each point. This is allowed because of the spherical symmetry of the metric (3.11), so nearly no information about the spacetime is lost by this simplification.

In a Penrose diagram, there are only two ways having a boundary:

1. The $\mathbb{S}^2$ has infinite size. For example in the Minkowski case at $|T \pm R| = \pi/2$
2. The $\mathbb{S}^2$ collapses to zero. In the Minkowski case this happens at $R = 0$

Within the diagram, the radius of $\mathbb{S}^2$ is also changing as we move around.

[8] also called S-matrix

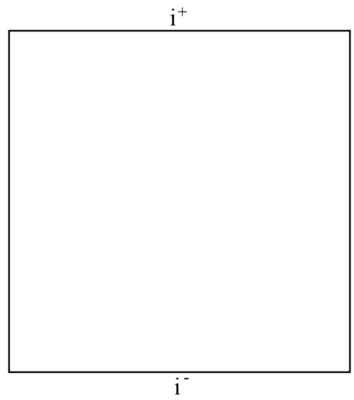

Figure 3. This is a Penrose diagram of the De Sitter space. As you can see, $i^{\pm}$ are spacelike surfaces instead of points like in **Figure 2**.

3.2.2 of De Sitter space ✓

As we already know from (2.8), the metric of the De Sitter space looks like

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \cosh^2\tau \mathrm{d}\Omega_3^2 \tag{3.12}$$

As we can see in **Figure 3**, the De Sitter space has nether an infinite spatial boundary i^0, nor any light-like infinity $J^{\mp}$. That means, it is impossible to find a formulation of De Sitter space in quantum theory or an S-matrix since existing well-defined theories of quantum gravity needs either of these.

But it has *event horizons*: on the left and right boundaries in **Figure 3** the $\mathbb{S}^3$ shrinks to zero size, while it goes to infinite size at $i^{\pm}$. This means, observers, who are moving on timelike geodesics at vertical straight lines in the diagram **Figure 3** could be unable to communicate.

3.2.3 of Schwarzschild geometry ✓

For the Penrose diagram of the Schwarzschild geometry, we take the Kruskal Szekeres coordinates of (3.8). That lightens the compactification we need for the Penrose diagram a lot, because the difference between (T, X, Ω) coordinates and the Minkowski space coordinates (t, r, Ω) is just their range.[9] So the transformation is as it was for Minkowski space:

$$\begin{aligned} T' + X' &\equiv \arctan(T + X) \\ T' - X' &\equiv \arctan(T - X) \end{aligned} \tag{3.13}$$

So before we had the diamond[10] $|R \pm T| < \pi/2$ and threw out the $R = 0$ region. If this is not clear, then have a look again at **Figure 2** for the first one. If you could add $R < 0$,

[9] Minkowski: $-\infty < t < \infty, r \geq 0$; Kruskal: $X^2 - T^2 > -1$

[10] Here the diamond is just the geometrical shape which looks like a diamond also called rhombus.

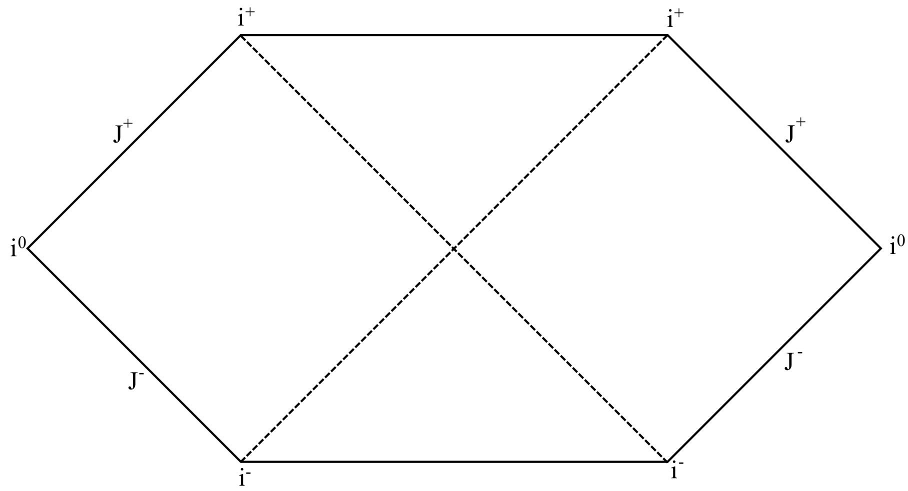

Figure 4. This is the **Penrose diagram** for the **Schwarzschild geometry**. The horizons are marked in dashed lines. As you can see, there are boundaries like the ones of the Minkowski space on both sides. In this diagram, the spacetime boundaries are marked explicitly with dashed lines.

then you would have a diamond instead of the triangle. But instead we now have the coordinate range of $|X' \pm T'| < \pi/2$ while throwing out $|T'| > \pi/4$. The final Kruskal diagram can be seen in **Figure** 4. You can imagine two additional triangles on top and at the bottom of it, which would be the regions we cut out.

3.3 The apparent horizon N

Now, how does that lead us to black holes? First of all, real astrophysical black holes are a result of gravitational collapse, like at the end of a star live. But considering massive particles would mean, that we have to include all interactions between them. So for making it convenient, we instead imagine a black hole arise from a spherically symmetric infalling shell of photons. That also leads to the fact, that there is no obstacle while the black hole is formed, like the Schwarzschild radius.

But now the horizon extends into the Minkowski space and we have to make a difference between the *actual horizon* and the *apparent horizon*. When you are passing the actual horizon, you will not notice it immediately, even though your fate has already been sealed. This leads to some kind of acausal nature of horizons: Their locations depend on events that have not yet happened.

For defining the apparent horizon, which will help us, to avoid this acasual nature, we first notice, that the Schwarzschild horizon can be detected locally in time, for any sphere of constant r with $r < 1$.[11] Any null geodesics of these spheres which starts out orthogonal, converges towards other null geodesics of this kind. And if we have a compact 2-dim surface, it is called a *closed trapped surface*.

[11] Here, the Schwarzschild radius is set to $2GM = 1$

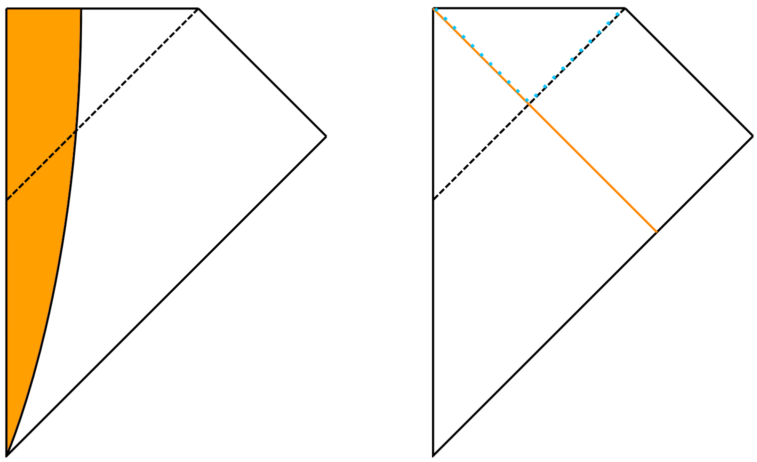

Figure 5. In both diagrams, the upper boundary is the singularity while the left boundary is the origin of the polar coordinates. The other two boundaries are asymptotic to the ones of the Minkowski space. On the left side we have a collapsing cloud of massive particles shown in orange, which forms the black hole. On the right side we have a black hole forming out of an infalling shell of photons, where we have the Schwarzschild geometry above the orange line, and a piece of Minkowski space below it. The event horizon is illustrated as a dashed line, the apparent horizon is shown with blue dots.

An apparent horizon is now a surface which is a boundary of a connected set of closed trapped surfaces. For to describe the real black hole in one diagram, we take a mixture of Minkowski space and Schwarzschild solution into a Penrose diagram as shown in **Figure 5.** The apparent horizon is illustrated with blue dots. And as you can see, it does form itself in the same moment when the photon shell crosses the event horizon.

4 What is Quantum field theory? ✓

Here the Hilbert space is like an infinite tensor product over all points in space, while we have finite degrees of freedom at each point. For example, we take a scalar field $\phi(x)$ where there is only a one degree of freedom at each spatial point.

Then the free scalar field of mass m's Hamiltionian looks like

$$H = \frac{1}{2}\int \mathrm{d}^3x \left(\pi(x)^2 + \vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x) + m^2\phi(x)^2\right). \tag{4.1}$$

The function $\pi(x)$ is the canonical conjugated momentum to ϕ which can be put out to tender as $\frac{\partial\mathscr{L}}{\partial\dot{\phi}(x)}$. For beginners, the ϕ would be something like the spacial coordinate x in theoretical mechanics and π is the companion piece to p, the momentum. Together they obey

$$\begin{aligned}[\phi(x),\pi(y)] &= i\delta^3(x-y)\\ [\phi(x),\phi(y)] &= 0\\ [\pi(x),\pi(y)] &= 0.\end{aligned} \tag{4.2}$$

This is consistent to the fact that for each field at a each point, we have an individual tensor factor. Note that x and y are four dimensional coordinates living in Minkowski space.

The Hamiltonian results out of the Lorentz-invariant action[12]:

$$S = -\frac{1}{2}\int \mathrm{d}^4x \left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right) \tag{4.3}$$

Where in general action is defined as [3]:

$$S = S[q] = \int_{t_1}^{t_2} \mathrm{d}t\mathscr{L}(q,\dot{q},t) \tag{4.4}$$

In quantum mechanics, we are often interested in describing the ground state wavefunction $|\Omega\rangle$. If we do this for the free massive scalar field, we find that [7]

$$\langle\phi|\Omega\rangle \propto \exp\left[-\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}^3y\phi(x)\phi(y)K(x,y)\right]. \tag{4.5}$$

with

$$K(x,y) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\sqrt{\vec{k}^2+m^2}. \tag{4.6}$$

$K(x,y)$ is a propagator and normally includes at least four coordinates, two for the beginning state and two for the final state. If we want to include time-ordering, we would use the Green's function which would be a Heaviside step function multiplied with this propagator: $G(\mathbf{r}_2,t_2;\mathbf{r}_1,t_1) = \Theta(t_2-t_1)\cdot K(\mathbf{r}_2,t_2;\mathbf{r}_1,t_1)$.[8]

But we now would like to generalize theories with interactions, so we study the vacuum expectation values of products of Heisenberg picture fields which look like $\phi(t,x) \equiv e^{iHt}\phi(x)e^{-iHt}$. In our case of the free massive scalar field we have this solution of equation of motion:

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^2}\frac{1}{\sqrt{2\omega_k}}\left[a_{\vec{k}}e^{i(\vec{k}\cdot\vec{x}-\omega t)} + a^\dagger_{\vec{k}}e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\right] \tag{4.7}$$

where we defined $\omega_k \equiv \sqrt{\vec{k}^2+m^2}$. The $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ should be known out of the basics of

[12]For germanspeakers: Because it is often confused because of television, action in physics means *Wirkung*.

quantum mechanics as the creation and annihilation operators[13], and they obey

$$
\begin{aligned}
[a_{\vec{k}}, a^\dagger_{\vec{k}'}] &= (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \\
[a_{\vec{k}}, a_{\vec{k}'}] &= 0 \\
[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] &= 0 \\
[H, a_{\vec{k}}] &= -\omega_k a_{\vec{k}}.
\end{aligned}
$$

Now we are searching for a more abstract form of (4.7) where a_n and $a_n^\dagger$ have the standard algebra, in the following way:

$$\phi = \sum_n \left(f_n a_n + f_n^* a_n^\dagger \right) \tag{4.8}$$

Here we use a basis of solutions $f_n(x)$ of

$$\left(\partial_\mu \partial^\mu - m^2 \right) f(t, \vec{x}) = 0 \tag{4.9}$$

that have a time dependence of the form $e^{-i\omega t}$ with $\omega > 0$. We also define the same f_n in a way, that they are orthonormal in the Klein-Gordon[14] norm, so that

$$(f_1, f_2)_{KG} \equiv i \int d^3x \left(f_1^* \dot{f}_2 - \dot{f}_1^* f_2 \right). \tag{4.10}$$

Now we can have a look at the expectation values (perhaps you remember, that this is the interesting part), which are called *correlation functions*. With just a one-point function, it has to vanish

$$\langle \Omega | \phi(t, \vec{x}) | \Omega \rangle = 0. \tag{4.11}$$

because of the translation invariance in vacuum and how a_n and $a_n^\dagger$ act on the vacuum. In other words, it is forbidden, that one particle just arise from the vacuum but never vanishes oder that one particle had always existed end suddenly disappears. This is, why Quantum field theory is called a multiparticle theory.

For a two-point function with equal times it looks like this:

$$\langle \Omega | \phi(0, \vec{x}) \phi(0, \vec{y}) | \Omega \rangle = \frac{1}{4\pi^2} \frac{m}{|\vec{x} - \vec{y}|} K_1(m|\vec{x} - \vec{y}|). \tag{4.12}$$

This function scales with

- $\frac{1}{|\vec{x}-\vec{y}|}$ for $|\vec{x} - \vec{y}| \ll m^{-1}$
- $e^{-m|\vec{x}-\vec{y}|}$ for $|\vec{x} - \vec{y}| \gg m^{-1}$

[13] They are defined: $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} + \frac{i}{m\omega} \hat{p} \right)$ and $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} - \frac{i}{m\omega} \hat{p} \right)$ with the feature that if you let them operate with a wave function $|n\rangle$ they act like this: $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. And $a^\dagger$ is the adjoint of a which means: $\langle \psi | a | \phi \rangle^* = \langle \phi | a^\dagger | \psi \rangle$ just for reminding.

[14] Perhaps you already noticed, that (4.9) is the Klein-Gordon Equation which is nothing else than the relativistic Schrödinger Equation. It is just valid for spin 0 particles.

but is "gapless" for $m = 0$, because the m^{-1}, which is also called *correlation length*, is infinite, so exicted states' energies can be arbitrarily close to the ground state energy. A massless scalar field is also invariant under the *conformal group*. This means here we can scale the spacetime like $x'^{\mu} = \lambda x^{\mu}$, so it's scale-invariant. The quantum field theory with this larger symmetry group is called *conformal field theory* or *CFT*.

$K_1(m|\vec{x} - \vec{y}|)$ is some hyperbolic Besselfunction which is in general a solution of differential equation like the Klein-Gordon equation in hyperbolic coordinates. It contains an exponential function, which is, why we have the second point in the listing above. But I will not do further explanations about this function because it is not necessary here.

We need to mention that (4.12) is a correlation function which is not only just valid in free scalar field theory but also is divergent in short distance power-law.

Now that we contemplate the two-point function without the time, we add the time difference to our calculations. But we need to order the function in time so we use the time-ordering operator T, so that the two-point function now is:

$$\langle \Omega | T\phi(t, \vec{x})\phi(t', \vec{y}) | \Omega \rangle = \frac{1}{4\pi^2} \frac{m}{\sqrt{|\vec{x} - \vec{y}|^2 - (t - t')^2 + i\epsilon}} K_1 \left(m\sqrt{|\vec{x} - \vec{y}|^2 - (t - t')^2 + i\epsilon} \right) \tag{4.13}$$

This is also called a *transition amplitude*. T takes care that time increases while going left. The factor ϵ should go to zero and reminds us, that there could be other ways of complex integration because of the square root not leading to the timelike-seperated points.

I think going left in a feynman diagram, but i am not sure

5 What is Entanglement and what is it telling us about the spacetime? ✓

In relativistic QFT, the ground state has correlations between field operators at spatially separated points. Here we can use *entanglement* as an explanation.

But at first, let's start from the beginning:
We have ρ which is a quantum state on Hilbert space $\mathcal{H}$ and is called *density matrix*. Quantum states are illustrated in operators, here: ρ is a non-negative hermitian operator of trace 1. If it can be written in the form[15]

$$\rho = |\psi\rangle\langle\psi|, \tag{5.1}$$

the quantum state is called *pure*. If a state is not pure, it is *mixed*.

While doing an experiment, we will a measure an outcome i, which is always related to a projection operator Π_i with a probability of measuring i, that looks like:

$$P(i) = \mathrm{tr}(\rho\Pi_i). \tag{5.2}$$

[15] Here $|\psi\rangle$ is some element of $\mathcal{H}$ with norm 1.

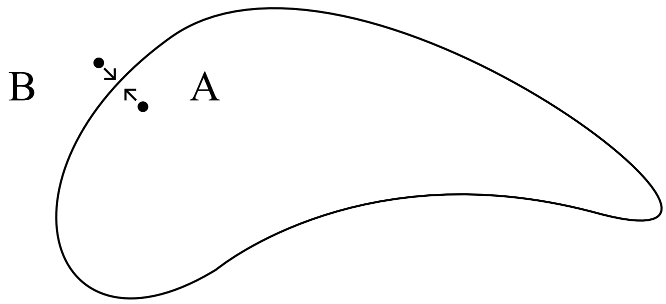

Figure 6. This is a decomposed Hilbert space into regions A and B by their tensor product of their fields. The to dots symbolise the two-point functions which are diverging while aproaching each other (the arrows). The boundary between these two regions is called the *entangling surface*.

For finding out, whether a given state ρ is pure or mixed, we define a function S for convenience:

$$S(\rho) \equiv -\mathrm{tr}(\rho \log \rho) \tag{5.3}$$

And $S(\rho)$ is called the *Von Neumann Entropy* or *information entropy.* Its proberties are:

- for any unitary operator U: $S(U^\dagger \rho U) = S(\rho)$
- $S(\rho) \geq 0$, with equality if and only if ρ is pure.
- for d is the dimension of $\mathcal{H}$: $S(\rho) \leq \log d$, with equality if and only if ρ is maximally mixed.
- The entropy of the average over a set of states is at least equal to the average of all their individual entropies. This is also called *concavity* and is defined as:

$$S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i), \tag{5.4}$$

  while λ_i is any set of non-negative numbers with $\sum_i \lambda_i = 1$.

Now, let's have a look at an entangled state written in the two-qubit state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \tag{5.5}$$

Here, the full state is pure, but the reduced state on either qubit ($|00\rangle$ or $|11\rangle$) is mixed.

Just a short explanation what 'qubits' are:
It is the spin 1/2-particle system with a standard basis $\{|0\rangle, |1\rangle\}$. So the particles can just have spin 'up' or 'down'. A ket in the form $|00\rangle$ for example, the first particle has

spin up in one region, the second spin up in an other region.

In **Figure 6** the Hilbert space was decomposed into a tensor product of the degrees of freedom in a region A and the its complement B. From the end of chapter 4 we know that the correlation functions are divergent at short distances which means that, if the two points at the boundary between A and B, the correlation becomes infinite. But what does that mean in respect to entanglement? Well, in the ground state there must be an infinite amount of entanglement between neighboring regions.

Another way of illustrating entanglement in QFT is the Reeh-Schlieder theorem [16]. This says, that while acting on the vacuum $|\Omega\rangle$ with operators of any region A, a set of states which is dense in full Hilbert space can be produced. This means if a local operator acts on the vacuum for example in your bedroom, we can create the moon. This is possible because of the highly entangled nature of the vacuum.

Soll ich das wirklich reinschreiben?

6 Rindler decomposition

Now that we know what entanglement is, we would also like to know, how much who is entangled with whom. In **Figure 7** you can see a method which will help us, to reach that goal, the Rindler decomposition of Minkowski space.

Therefore we split the Hilbert space into a factor $\mathcal{H}_L$ that acts on the fields $x < 0$ and $\mathcal{H}_R$ for $x > 0$. And each factor has its own basis of states with which we can decompose the vacuum.

We now introduce the *Lorentz boost*[17] *operator* K_x, which mixes x and t but does not act on the y or z direction. This operator exist in any relativistic QFT and looks in the free massive theory like this:

$$K_x = \frac{1}{2}\int \mathrm{d}^3x \left[x(\dot{\phi}^2 + \vec{\nabla}\phi \cdot \vec{\nabla}\phi + m^2\phi^2) + t\dot{\phi}\partial_x\phi\right]. \tag{6.1}$$

It is not explicitly time-dependent, because if you plug it in Heisenberg's equation of motion the time-dependence of the fields cancels the explicit time dependence. In all four regions i.e. wedges of **Figure 7** the action of K_x is well defined.

Nachrechnen (siehe QED übung)

So in the right blue wedge this operator is evolving forward in time, which is indicated by the red line with the arrow pointing in the increasing t direction. On the contrary K_x is evolving backwards in time in the left blue wedge, where the arrow on the red line is pointing at decreasing t direction. The upper and lower wedge are again the future and the past region where the action of K_x is spacelike. This figure looks very alike the Kruskal extension in **Figure 1** but be careful and don't mix them up.

[16] If you are interested in the detail [7]

[17] A Lorentz boost is a rotation-free Lorentz transformation, which is a Galilei-transformation in relativistic.[2] p.7

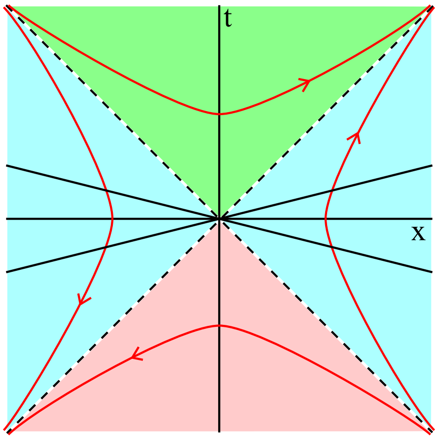

Figure 7. The **Rindler decomposition** of **Minkowski space**. The blue wedges are the Rindler wedges, the red one is the past wedge and the green one is the future wedge. The straight lines in black are slices of the Rindler time. The *red lines* are the action of the *boost operator* K_x.

6.1 Eigenstate and Euklidean path integral in general ✓

We now introduce the Euclidean path integral because we want to find the eigenstates of the left and right Rindler wedges.

First of all we have a ground state $|\Omega\rangle$ of H and a vacuum state $|0\rangle$, so we can write for a very long time T

$$\begin{aligned} e^{-iHT}|0\rangle &= \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle \\ &= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle \end{aligned}$$

Now we enclose and get the ground state

$$|\Omega\rangle = \lim_{T\to\infty} \frac{e^{iE_0T}}{\langle\Omega|0\rangle} e^{-iHT}|0\rangle$$

We can define E_0 with $H_0|0\rangle = |0\rangle$ so

$$|\Omega\rangle = \frac{1}{\langle\Omega|0\rangle} \lim_{T\to\infty} e^{-iHT}|0\rangle$$

(see p.86 in [6]) But this equation does still tell us nothing about the entanglement. So let's continue:
Let a time-independent field ϕ act on this ground state:

$$\langle\phi|\Omega\rangle = \frac{1}{\langle\Omega|0\rangle} \lim_{T\to\infty} \langle\phi|e^{-iHT}|0\rangle$$

Now use the Euclidean path integral formalism, rotate t about 90° into the complex plane: $t \to -it_E$ and choose the early boundary condition $\phi = 0$, so that

$$\langle \phi | \Omega \rangle \propto \int_{\hat{\phi}(t_E=-\infty)=0}^{\hat{\phi}(t_E=0)=\phi} D\hat{\phi} e^{-I_E} \tag{6.2}$$

with the Euclidean action for a free massive scalar field

$$I_E[\hat{\phi}] = \frac{1}{2} \int \mathrm{d}^3x \mathrm{d}t_E \left[(\partial_{t_E}\hat{\phi})^2 + (\vec{\nabla}\hat{\phi})^2 + m^2\hat{\phi}^2 \right] \tag{6.3}$$

Note that $\hat{\phi}$ compared to ϕ is time-dependent.

6.2 The ground states of the Rindler wedges ✓(bis auf Fragen)

But instead of integrating from $t_E = -\infty$ to 0, we choose the range of π like in **Figure** 8. In addition our Hilbert space is separated in to $\mathcal{H}_L$ and $\mathcal{H}_R$ with the fields ϕ_L and ϕ_R so we write

$$\langle \phi_L \phi_R | \Omega \rangle \propto \langle \phi_R | e^{-\pi K_R} \Theta | \phi_L \rangle_L \tag{6.4}$$

Here the K_R is the operator K_x but in the right Rindler wedge while the operator Θ which is antiunitary[18],also called CPT and exists in all quantum field theories. It acts on a scalar field in the Heisenberg picture like $\Theta^\dagger \phi(t,x,y,z)\Theta = \phi^\dagger(-t,-x,y,z)$ which gives a map between the to Hilbert spaces. This is important in (6.4) because

warum e hoch -pi, was ist das i in der summe, was ist omega i und v.a. warum steht manchmal ein L oder ein R außerhalb der Kets

1. $\langle \phi_L \phi_R | \Omega \rangle$ can now be described just with a matrix in $\mathcal{H}_R$
2. the ϕ_L is playing the role of a final state, the ϕ_R the role of the initial state in **Figure** 8.

Now we can evaluate (8):

$$\begin{aligned} \langle \phi | \Omega \rangle &\propto \sum_i e^{-\pi\omega_i} \langle i | \Theta | \phi_L \rangle \langle \phi_R | i \rangle_R \\ &\propto \sum_i e^{-\pi\omega_i} \langle \phi_L | i^* \rangle_L \langle \phi_R | i \rangle_R \end{aligned} \tag{6.5}$$

We inserted a complete set of K_R eigenstates, used that Θ is antiunitary and defined: $|i^*\rangle_L = \Theta^\dagger |i\rangle_R$, so now the ground state is:

$$|\Omega\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\pi\omega_i} |i^*\rangle_L |i\rangle_R \tag{6.6}$$

In this connection Z is the partition function, a constant which is given through the normalization condition[19].

[18] unitary would mean for a linear operator A: $A^\dagger A = \mathbb{1}$ so $\langle Ax|Ay\rangle = \langle x|y\rangle$, antiunitary would be a antilinear operator A ($\langle x|A^\dagger\rangle = \langle y|Ax\rangle$) which also fulfills: $\langle Ax|Ay\rangle = \langle y|x\rangle$.

[19] The sum over all probabilities is equal to one. see p.11 [9]

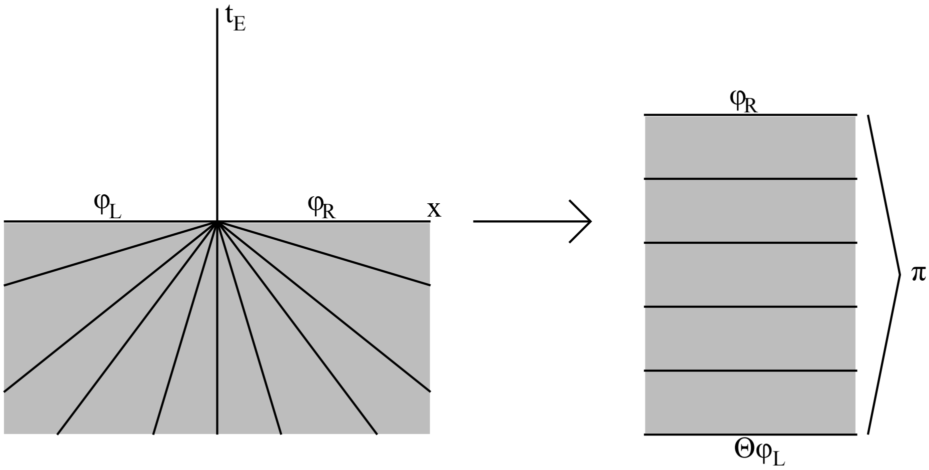

Figure 8. This is the Euclidean path integral representation changing (6.2) into a calculable integral, after we choose to integrate over an angle π. Note that φ_R and φ_L are the ϕs in the text.

Or we compute the density matrix of the right Rindler wedge (because see above: we just need the matrix of $\mathcal{H}_R$)

$$\rho_R = \frac{1}{Z}\sum_i e^{-2\pi\omega_i} \left|i\right\rangle_R \left\langle i\right| \tag{6.7}$$

which is the thermal density matrix with temperature $T = \frac{1}{2\pi}$.

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6.3 In free massive scalar theory ✓

Let's search solutions for the free massive scalar theory in Rindler decomposition to have an example. First, we introduce hyperboloidal coordinates for the right and left wedge in **Figure 7** omitting a length scale ℓ because of simplicity (see next subsection 6.4).

$$\begin{aligned} x &= e^{\xi_R}\cosh\tau_R = -e^{-\xi_L}\cosh\tau_L \\ t &= e^{\xi_R}\sinh\tau_R = e^{-\xi_L}\sinh\tau_L \end{aligned} \tag{6.8}$$

The coordinate ranges are: $-\infty < \xi_{L,R} < \infty$, $-\infty < \tau_{L,R} < \infty$. And the translation of τ_R forwards and τ_L backwards in time is the evolution of the boost operator K_x, while the $\xi_{L,R}$ coordinates form hyperboloidal orbits of K_x. They only cover the blue wedges of 7 but not the future or past Rindler wedges. There are so called Rindler horizons at $\xi_R = -\infty$ and $\xi_L = \infty$ but they are again just there because of choice of coordinates and not actual event horizons. We get the corresponding metric by plugging in (6.8) into $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}\vec{y}^2$:

$$\mathrm{d}s^2 = e^{2\xi_R}(-\mathrm{d}\tau_R^2 + \mathrm{d}\xi_R^2) + \mathrm{d}\vec{y}^2 = e^{-2\xi_L}(-\mathrm{d}\tau_L^2 + \mathrm{d}\xi_L^2) + \mathrm{d}\vec{y}^2 \tag{6.9}$$

The solutions of the massive wave equation should now be of the form:

$$
\begin{aligned}
f_{R\omega k} &= e^{-i\omega\tau_R} e^{i\vec{k}\vec{y}} \psi_{Rk\omega}(\xi_R) \\
f_{L\omega k} &= e^{-i\omega\tau_L} e^{i\vec{k}\vec{y}} \psi_{Lk\omega}(\xi_L)
\end{aligned}
\tag{6.10}
$$

Note that $\omega > 0$ and the ψs obey the equations

$$
\begin{aligned}
\left[-\partial_{\xi_R}^2 + (m^2 + \vec{k}^2)e^{2\xi_R} - \omega^2\right] \psi_{Rk\omega} &= 0 \\
\left[-\partial_{\xi_L}^2 + (m^2 + \vec{k}^2)e^{-2\xi_L} - \omega^2\right] \psi_{Lk\omega} &= 0
\end{aligned}
\tag{6.11}
$$

These are in fact just the Schrödinger equations of non-relativistic particles in an exponential potential. We could write down the solution in terms of Bessel functions, but for us it is enough to have a look at the normalizable solutions, in other words, the Klein-Gordon norm from (4.10). They oscillate at negative ξ_R, accordingly positive ξ_L, and decay exponentially at positive ξ_R, accordingly negative ξ_L.

Those modes in (6.10) are having certain boost energies, namely ω in the right wedge and $-\omega$ in the left wedge. The field which we formulate in terms of them looks as follows:

$$
\phi = \sum_{\omega,k} \left(f_{R\omega k} a_{R\omega k} + f_{L\omega k} a_{L\omega k} + f_{R\omega k}^* a_{R\omega k}^\dagger + f_{L\omega k}^* a_{L\omega k}^\dagger \right)
\tag{6.12}
$$

Here $a_{L,R\omega k}^\dagger$ is again a creation operator which creates states on the Rindler vacuum state $|0\rangle$ with the boost energies we discussed above. And of course $a_{L,R\omega k}$ annihilates those created states.

Now it is possible to rewrite the ground state of (6.6) into a product state over all modes:

$$
|\Omega\rangle = \bigotimes_{\omega,k} \left[\sqrt{1 - e^{-2\pi\omega}} \sum_n e^{-\pi\omega n} |n\rangle_{L\omega(-k)} |n\rangle_{R\omega k} \right]
\tag{6.13}
$$

The number of particles on top of the Rindler vacuum is considered by n in each mode. Note, that there is a sign flip of k, which comes from the CPT configuration earlier. In other words, the operator Θ sends $\tau_R \to -\tau_L$, $\xi_R \to -\xi_L$, and $\vec{y}$ to itself. If we apply this to the right-hand modes $f_{R\omega k}$ it sends positive frequency modes to negative frequency modes. This means we have to take the complex conjugate to get the coefficient of the annihilation operator, and this flips the sign of k.

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6.4 The Unruh temperature ✓

Imagine a quantum field in its vacuum state, means there are no particles present from an inertial observer point of view. In this case there is no temerature measurable. But if we now have an accelerating observer in this field, he or she would experience a radiation which leads to a temperature, the so called Unruh temperature:

$$
T_{\text{Unruh}} = \frac{\hbar a}{2\pi k_B c}
\tag{6.14}
$$

And this phenomenon, which has still not been experimental detected (because for 1 Kelvin one needs $10^{20}\frac{\mathrm{m}}{\mathrm{s}^2}$ acceleration [4]) is called the Unruh effect. As you can see, if we set $\hbar = k_B = c = 1$ then (6.14) looks like the temperature at the end of chapter 6.2 except for the acceleration a.

In the choice of Rindler coordingates $\xi_{R,L}$ and $\tau_{R,L}$ in (6.8), there was a suppressed length scale ℓ. This ℓ is also the inverse of the acceleration of an observer at $\xi_R = 0$ with τ_R as his or hers proper time. If we would have included this length scale earlier, the temperature would have looked like $T = \frac{1}{2\pi\ell}$. This means in the end, that an observer with $a = \frac{1}{\ell}$ will be exposed to the Unruh temperature in (6.14).

6.5 Entanglement in the Rindler decomposition ✓

What happens, if we want to cross the $x = 0$ surface? In order to find that out, we put the system in a mixed state with

$$\rho = \rho_L \otimes \rho_R \tag{6.15}$$

instead of having a ground state $|\Omega\rangle$. Here ρ_L and ρ_R are the thermal density matrices, which we get if we trace out the respectively other one in the vacuum $|\Omega\rangle$. If the fields are completely discontinuous like in (6.15) , the gradient term of the Hamiltonian will diverge at $x = 0$. If you are an observer in the left or right Rindlers wedge it seems, that you just have vacuum state, but the energy is infinite.

Its typical field fluctuation is given by $\frac{1}{\epsilon}$ where ϵ is a short-distance length cutoff. So it is valid that

was ist damit genau gemeint?

$$\partial_x \phi|_{x=0} \propto \frac{1}{\epsilon^2}. \tag{6.16}$$

Which means, that the gradient term in the Hamiltonian contributes

$$\mathrm{d}x \int \mathrm{d}^2 y (\partial_x \phi)^2 \propto \epsilon \int \mathrm{d}^2 y \frac{1}{\epsilon^4} = \frac{A}{\epsilon^3} \tag{6.17}$$

The smaller ϵ is the bigger becomes the energy and ϵ is even to the third power.

This is called a *firewall*: A huge concentration of energy at $x = 0$, that annihilates anybody who tries to jump through the Rindler horizon into the future wedge.

For example the product states $|00\rangle$ and $|11\rangle$ of the states $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ which shall both have smooth horizons, should have them too, because of linearity of quantum mechanics. But as we just saw, no product state possibly can have a smooth horizon in QFT. So, for going smoothly through the Rindlers horizon we need not only any entanglement but it must have the *right entanglement* too.

7 Contemplation of a fixed black hole

7.1 Approximation from Schwarzschild to Rindler ✓

Upto now we still considered the quantum field theory and classical black holes apart from each other. Combining them would require a theory of quantum gravity on which i will not go deeper in this thesis. But we can start with a much simpler problem that would be the quantum field theory in a fixed black hole backround. Physically this means, that we're sending G to zero and the mass of the black hole M to infinity so that the Schwarzschild radius $r_s = 2GM$ ist fixed. This can be assured by sending $\frac{M}{m_p}$ to infinity, where $m_p = \frac{1}{\sqrt{8\pi G}}$ is the Planck mass.

This approximation is justified, which we can see by the example of a black hole with the mass of our sun:

$$\frac{m_p}{M_{solar}} \approx 2.4 \cdot 10^{-39} \tag{7.1}$$

In this type of scale, the Schwarzschild radius is in order of kilometers (sun: $r_s = 3\text{km}$) where we could imagine doing experiments.

Now we must descide, which geometry we use. Either the two-sided Schwarzschild geometry or the one-sided collapse geometry. The disadvantage of the latter is, that it is only Schwarzschild after the infalling matter has gone in. So for avoiding the infalling shell problem, we use the former. The Schwarzschild geometry is approximatly the region of Minkowski space which is near the Rindler horizon in the Rindler decomposition, if $r \approx 1$ and the angular arrangement is not quite too big. We now sketch this for the right exterior ($r > 1$) by using the tortoise coordinate from (3.2):

$$r_* = r + log(r-1)$$

The Schwarzschild metric then is

$$\mathrm{d}s^2 = \frac{r-1}{r}\left(-\mathrm{d}t^2 + \mathrm{d}r_*^2\right) + r^2\mathrm{d}\Omega_2^2 \tag{7.2}$$

Now let's say $y_1 = \theta, y_2 = \varphi$ where the two angulars are orthogonal coordinates on the sphere. Near the horizon $r \approx 1$ the metric then looks like

$$\mathrm{d}s^2 \approx e^{r_*-1}\left(-\mathrm{d}t^2 + \mathrm{d}r_*^2\right) + \mathrm{d}\vec{y}^2 \tag{7.3}$$

This quite resembles the right Rindler wedge metric. It even becomes equal, if we insert

$$\begin{aligned} r_* &= 2\xi_R + 1 - \log 4 \\ t &= 2\tau_R \end{aligned} \tag{7.4}$$

which leads to

$$\Rightarrow \qquad \mathrm{d}t^2 = 4\mathrm{d}\tau_R^2 \qquad \mathrm{d}r_*^2 = 4\mathrm{d}\xi_R^2 \qquad e^{r_*-1} = e^{2\xi_R+1-1-\log 4} \qquad \mathrm{d}\vec{y}^2 = \mathrm{d}\vec{y}^2$$

and finaly like in (6.9)

$$\mathrm{d}s^2 = e^{2\xi_R}(-\mathrm{d}\tau_R^2 + \mathrm{d}\xi_R^2) + \mathrm{d}\vec{y}^2$$

How this looks like in Penrose diagrams if we do the same thing for the other three wedges too, can be seen in **Figure 9**. The outcome of this is, that for every initial state

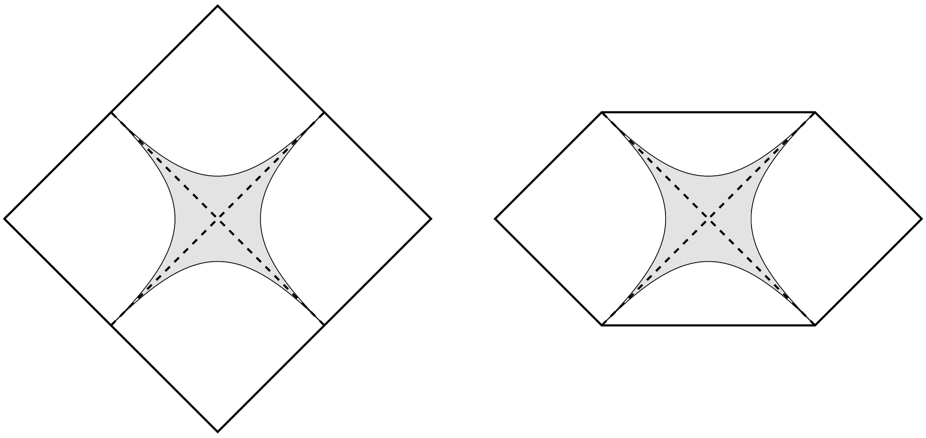

Figure 9. At the left, one can see the Rindler Penrose, at the right there is the Schwarzschild Penrose. The two grey regions in the middle are the regions near the horizon. As you can see, they approximate each other well. Note that a $\mathbb{R}^2$ is suppressed at each point instead of $\mathbb{S}^2$ which is why the left Penrose doesn't look like the Minkowski Penrose in **Figure 2**.

in a Schwarzschild geometry the left and right exteriors must be thermally entangled, assumed that the Schwarzschild looks like Minkowski space near them. The relation (7.4) between Schwarzschild and Rindler time leads to

$$T_{\text{Hawking}} = \frac{T_{\text{Unruh}}}{2} = \frac{1}{4\pi} \tag{7.5}$$

This alliance between temperature and time can be made, because the time one is exposed to the radiation is proportional to the temperature.

With all constants the Hawking temperature is

$$T_{\text{Hawking}} = \frac{1}{4\pi r_s} = \frac{\hbar c^3}{8\pi k_B G M} \tag{7.6}$$

So, the bigger the temperature is, the bigger is the size of the black hole. For a solarmass one for example, the temperature is about $6 \cdot 10^{-8}\,\mathrm{K}$.

7.2 Schwarzschild modes N

We now have a look at free fields in the Schwarzschild geometry. For the beginning we need to find a set of modes f that solve the free scalar equation of motion:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = m^2\phi \tag{7.7}$$

which is the *Klein-Gordons equation for curved space* and where $g_{\mu\nu}$ is the Schwarzschild metric and g is its determinant. Its solutions are also modes in (4.8), were we can study its properties in an appropriate quantum state such as the Hartle-Hawking state[20]

We now focus on the right exterior of the Schwarzschild geometry, where we use the coordinates (t, r, Ω). The solutions are having the form

$$f_{\omega lm} = \frac{1}{r} Y_{lm}(\Omega) e^{-i\omega t} \psi_{\omega l}(r) \tag{7.8}$$

Let's put these into the equation (7.7) above, use the tortoise coordinates from (3.2) and plug in

$$\begin{aligned} g^{\mu\nu} &= diag\left(\frac{1}{\frac{1}{r} - 1}, 1 - \frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^2 \sin\theta} \right) \\ \Rightarrow \sqrt{-g} &= r^2 \sin\theta \end{aligned}$$

to reform it into a Schrödinger equation:

$$-\frac{d^2}{dr_*^2} \Psi_{\omega l} + V(r)\Psi_{\omega l} = \omega^2 \Psi_{\omega l} \tag{7.9}$$

with the effective Potential of

$$V(r) = \frac{r-1}{r^3} \left(m^2 r^2 + l(l+1) + \frac{1}{r} \right) \tag{7.10}$$

Let's have a look at the mass m: For simlicity, we consider the Compton wavelength $\frac{1}{m}$ can be[21]

(i) $\frac{1}{m} \ll r_s$ which is the *massive case* and

(ii) $\frac{1}{m} \gg r_s$ which is the *massless case.*

For to explain, why case (ii) is more interesting for us, we first need to have a look at case (i).

Here the potential goes to m^2 for $r \gg 1$ which means that massive modes will only propagate till near infinity if $\omega \geq m$. Because we assumed, that $m \gg 1$, any modes with an energy ω of order of the Schwarzschild radius *will stay near the horizon.*

But we found out earlier, that the temperature of black holes is of order $\frac{1}{r_s}$. This means, that $\omega \approx 1$ would be the most interesting energy, but if the mass is already much bigger than one, we cannot examine the propagation to infinity because ω would also be much bigger than one.

So from now on we remain within the case of $m^2 = 0$, the massless case. Here the asymptotic behavior of the potential is

$$V \approx \begin{cases} \frac{l(l+1)}{r_*^2} & r_* \to \infty \\ (l^2 + l + 1)e^{r_* - 1} & r_* \to -\infty \end{cases} \tag{7.11}$$

[20] The Hartle-Hawking state is the pure state for the Schwarzschild geometry, where the two exteriors of the Rindler space are thermally entangled as seen in the chapter 7.1 above.

[21] The *Schwarzschild radius* r_s *is still 1*, but I sometimes write r_s to make some things better understandable.

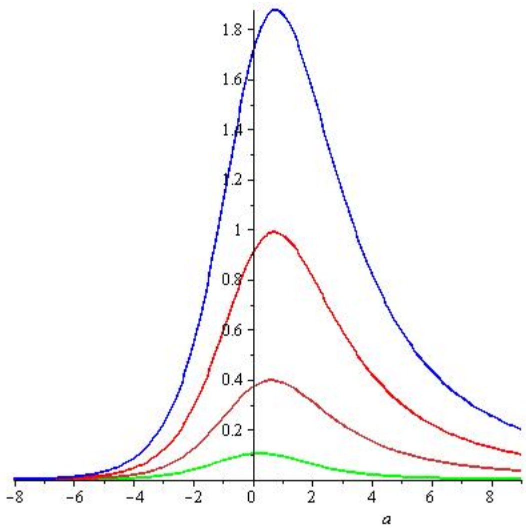

Figure 10. These are the selfmade plots of $V(r_*)$ for $l = \{0, 1, 2, 3\}$.

If you wonder, why r_* can go to infinity, then send r in the tortoise coordinate (3.2) to 1 (which would be the Schwarzschild radius). You will get minus infinity. So this just means, that r approaches r_s.

So the first approachment in (7.11) leads to the vanishment of the potential at spatial infinity polynomially in r_*, but near the horizon it vanishes exponentially. The barrier between these two regions of (7.11) can be seen in **Figure 10**. For $\ell \gg 1$ we will find the peak at $r = \frac{3}{2}$ (which means in the figure at approx. $r_* = 0.8$) with a height of order ℓ^2.

For modes with less energy then the height of the barrier, we have two regions outside the black hole to look at:

1.) $r \gg \frac{3}{2}$: Here the geometry is weakly curved, so modes propagate like in Minkowski space.

2.) $1 < r < \frac{3}{2}$: This near-horizon region can be called "thermal atmosphere" or "the zone". The first name was given because the modes, which propagate mostly near the horizon, must deal with a Boltzmann distribution with temperature $T_{\text{Hawking}} \approx 1$.

If we one approches this problem on the base of Schrödinger equation (7.9), it can be viewed as a scattering problem, see **Figure 11**. The modes can come from two directions and leave in two directions namely: coming from a white hole horizon at $r_* \to -\infty$ or from J_- at $r_* \to \infty$, which is the past null infinity if you remember, and leaving into the black hole horizon at $r_* \to -\infty$ or through J_+ at $r_* \to \infty$, which would be the future null infinity.

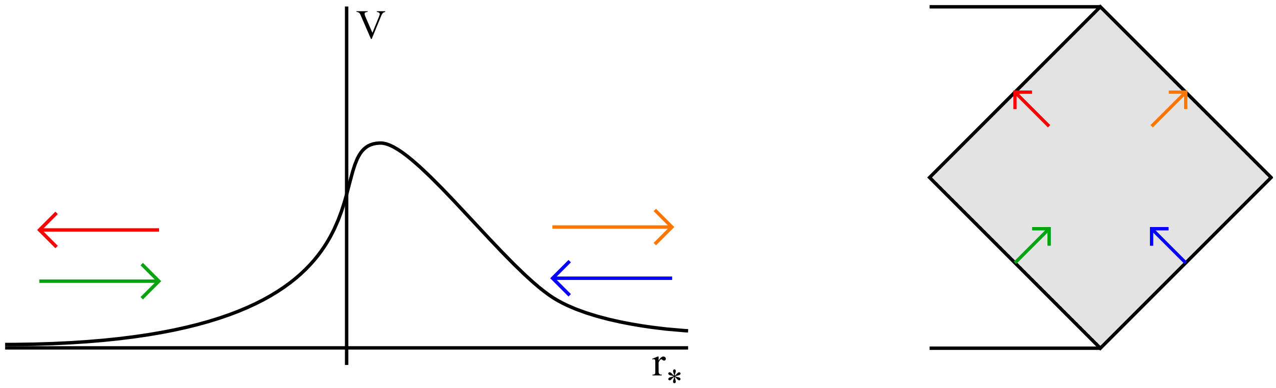

Figure 11. Bla

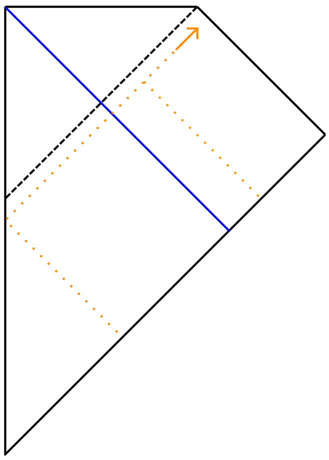

Figure 12. Bla

8 Information problem

8.1 Black hole radiation according to Hawking N

8.2 Entropy and thermodynamics ✓

If a black hole has a temperature and an energy, it must also have an entropy. So let's remember the inner energy in statistical mechanics[22] and derive it to:

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{1}{T}\bigg|_{V,N=const.} \tag{8.1}$$

[22] $\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N$

For the black hole we have $T = \frac{1}{8\pi GM}$ and $M = E$. If we assume that $S(E=0) = 0$, we can write:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}M} &= 8\pi GM \\
&\Leftrightarrow \int_0^S \mathrm{d}S' = \int_0^M 8\pi M' dM' \\
&\Leftrightarrow S = 4\pi GM^2 && \Big| \, r_s = 2GM \\
&\Leftrightarrow S = \frac{r_s^2 \pi}{G} && \Big| \, A = 4\pi r_s^2 \text{ and } l_p = \sqrt{8\pi G} \\
\Leftrightarrow \; S &= \frac{A}{4G} = 2\pi \frac{A}{l_p} \qquad (8.2)
\end{aligned}$$

For a black hole of the mass of our sun, this entropy would be 10^{78} which is enormous! If we take the sun like it is, the entropy would "just" be 10^{60}.

Historical the entropy of a black hole was discovered before its temperature. With help of classical general relativity, we can see that the area of an event horizon of a black hole never decreases which looks quite like the second law of thermodynamics. Together with certain formal definition of the entropy, where it is proportional to the horizon area and a temperature indirect proportional to the Schwarzschild radius, the first law of thermodynamics with $\mathrm{d}M = T\mathrm{d}S$ is satisfied, too.

Jacob Bekenstein was holding out that this entropy should be that kind of statistical entropy of a black hole, that counts the number of ways it could have formed itself. In a thought experiement he was throwing some systems with own entropy into a black hole and discovered that the interior entropy was growing faster, than the exterior entropy was sinking because of the systems loss. So this means, that the entropy must be given by some constant proportional to the horizon area in Planck units. Bekenstein called this the *Genereralized Second Law*.

As Hawking published his paper about the temperature of a black hole, Bekensteins theory strongly reinforced. This is why the entropy of a black hole is often called **Bekenstein-Hawking entropy**.

In *string theory* this idea of an entropy counting microstates is strong supported. For example in many situations where we count the states of a long vibrating string we can see how big the entropy of a black hole is. In some supersymmetric cases it is even possible to compute the $\frac{1}{4}$ in equation (8.2).

8.3 Evaporation N

8.4 What happens to the information while evaporation? N

Steven Hawking said in his paper, it is inconsistent with quantum mechanics, that the black hole's entropy counts the number of ways it could have been formed which most people would think in the first place. [2]

The idea behind these thoughts is that the outgoing radiation of a black hole is completely independent of details of the initial state of photons. In explicit we make a diagonal density matrix

$$\rho \propto \bigotimes_{\omega,l,m} \left(\sum_n |n\rangle \langle n|_{\omega,l,m} P_{abs}(\omega,l) e^{-\beta\omega n} \right) \tag{8.3}$$

which leads to the emission rate

das gehört weiter nach oben

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\omega \mathrm{d}\omega}{2\pi} \frac{P_{abs}(\omega,l)}{e^{\beta\omega}-1} \tag{8.4}$$

where $\beta = \frac{1}{T_{Hawking}}$, and $P_{abs}(\omega,l)$ is the absorption probability of a "blue" mode.

This should remind you of the Rindler result which means that this reduced density matrix for the right or left Rindler wedge is just a thermal density matrix. But back then, we did not calculate in the gravity, so there will be catastrophic consequences once we turn on the gravity again.

Now, if a black hole was orginally formed in some pure state $|\psi\rangle$, its outside radiation field becomes more and more mixed as we move forward in time. Because we are normally only looking at the late radiation outside of the black hole, this doesn't seem problematic.

While the black hole evaporates and becomes smaller, its entanglement entropy is always increasing as seen in (8.3). The problem is, its size decreases until it is Planckian[23] and in those kinds of systems our common physics can't help us any more.

What happens to the entropy now? One of two things must happen:

(1) The evaporation stops at Planck size. This rest is also called "remnant" and its entanglement entropy must be enormously big, bigger than that of a black hole with a comparable mass.

(2) The black hole finishes the evaporation till there is nothing left. The law of energy conservation prohibits that the last boost of photons contains enough entanglement entropy for reproducing the initial state. But if information can not get lost, we would have to violate the quantum mechanics here. So in the end we would have a mixed state with an entropy comparable to the one of the initial horizons entropy.

The option number (1) is in fact possible, but it would mean that there are objects with an infinit amount of states below any finite energy. Also if a black hole can form out of photons and gravitons why should it not be possible for it to disapear entirely back into photons and gravitons.

Option (2) in contrast seems to be the better choice, but it also means that black holes can destroy information. So one must admitt that gravity and quantum mechanics are inconstistent, they have no theory in common.

Let's have a look at an other option, which is nearly similar to option number (2):

[23] The planck scale begins at the planck length $l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1,6 \cdot 10^{-35}$m, while a proton is about the size of $8,4 \cdot 10^{-16}$m

(3) While evaporating, the information is hidden in some entanglement between the Hawking photons blasted by the black hole (or the rest of it). In the end we have a pure state of the radiation field instead of a mixed state like in (2). This is just possible if we don't look at too many photons at once, because in complicated states any small subsystem looks thermal, so it can justify (8.3).

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