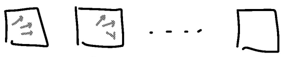
Video sequences

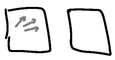
* Sequence of images:



t_0 t_1 $t_k = t_0 + k\,\Delta t$

eg: $\Delta t = \frac{1}{24}$ [s]

* Difference from stereo



large motion

* Types of motion:
  - Single (static camera and moving object or static object and moving camera)
  - Multiple motions (multiple objects and camera moving)

Motion analysis

* Problems:

1) Correspondence (easier in video) finding corresponding pts between the frames - easier bcz not much mvmt b/w frames
2) Reconstruction (easier in stereo) reconstruct the corrdinates of th epts in the world and motion vectors...diff in video bcz the rays thru will be allmost II becz less dist
3) Motion segmentation distinguish between the motion of one object and the other

Motion of rigid objects

3D motion vectors (world) A vector in the moving object helps to know the motion of the camera

A change of pixels in frames

2D projected motion (camera)

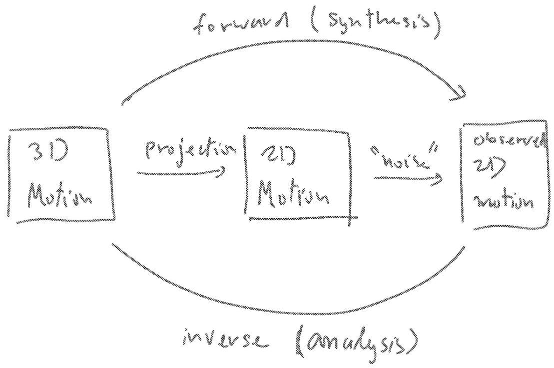
3D not ncessarily reflected properly to projected, so the observed may vary from truth

2D observed motion (image)

⇑

optical flow

Motion of rigid objects



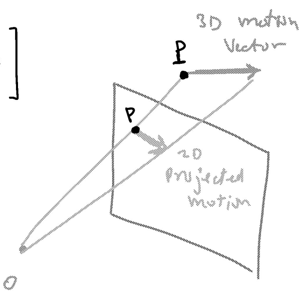
view image

Basic equations of projected motion

$$\underline{P} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad p = \begin{bmatrix} x \\ y \\ f \end{bmatrix}$$

$$p = f \, \frac{\underline{P}}{Z}$$

Z coordinate of $\underline{P}$



$p, \underline{P}, Z$ all depend on t

Basic equations of projected motion

* Motion in image:

$$\frac{dp}{dt} = \frac{d}{dt}\left(f \frac{\underline{P}}{Z}\right) = f\left(\frac{1}{Z}\frac{d\underline{P}}{dt} - \frac{1}{Z^2}\frac{dZ}{dt}\underline{P}\right)$$

$$\Rightarrow \quad v = f\left(\frac{1}{Z} V - \frac{1}{Z^2} V_Z \, \underline{P}\right)$$

$$\boxed{v = \frac{f}{Z^2}\left(Z V - V_Z \, \underline{P}\right)}$$

projected 2D motion

3D motion

Basic equations of projected motion

* In component form:

$$\begin{cases} v_x = \frac{f}{Z^2}\left(Z V_x - V_z X\right) \\ v_y = \frac{f}{Z^2}\left(Z V_y - V_z Y\right) \\ v_z = \frac{f}{Z^2}\left(Z V_z - V_z Z\right) = 0 \end{cases}$$

projected motion vectors do not have z component

Basic equations of projected motion

* Moving to image coordinates from world coords.

$$p = \frac{f P}{Z} \Rightarrow P = \frac{p}{f} Z$$

$$v = \frac{f}{Z^2}\left(Z V - V_z P\right)$$

$$= \frac{f}{Z^{\cancel{2}}}\left(\cancel{Z} V - V_z \frac{p}{f} \cancel{Z}\right)$$

$$\boxed{v = \frac{f}{Z} V - \frac{p}{Z} V_z}$$

projected motion equation

Decomposing projected motion

Plugging V_x, V_y, V_z in projected motion equations:

$$\boxed{\begin{cases} v_x = v_x^{(T)} + v_x^{(\omega)} \\ v_y = v_y^{(T)} + v_y^{(\omega)} \\ v_z = 0 \end{cases}}$$

Projected motion

$$\boxed{\begin{cases} v_x^{(T)} = \frac{T_z x - T_x f}{Z} \\ v_y^{(T)} = \frac{T_z y - T_y f}{Z} \end{cases}}$$

translational component

$$\boxed{\begin{cases} v_x^{(\omega)} = -\omega_y f + \omega_z y + \frac{\omega_x x y}{f} - \frac{\omega_y x^2}{f} \\ v_y^{(\omega)} = \omega_x f - \omega_z x - \frac{\omega_y x y}{f} + \frac{\omega_x y^2}{f} \end{cases}}$$

rotational component

Pure translational motion

$$\omega = 0 \Rightarrow \begin{cases} v_x = v_x^{(T)} = \dfrac{\tau_z x - \tau_x f}{z} \\ v_y = v_y^{(T)} = \dfrac{\tau_z y - \tau_y f}{z} \end{cases}$$

Analyze two cases:

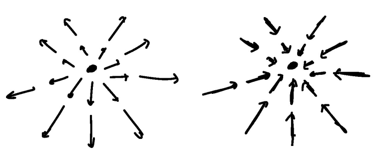
1) $\tau_z \neq 0$ (translation in z) — airplane landing front view

2) $\tau_z = 0$ (no translation in z) — car traveling side view

Pure translational motion

Case 1: $\tau_z \neq 0$ (and pure translation) — radial motion field

$$\begin{cases} v_x = \dfrac{\tau_z x - \tau_x f}{z} = \dfrac{\tau_z}{z}\left(x - \dfrac{\tau_x}{\tau_z} f\right) \equiv \underline{\dfrac{\tau_z}{z}(x - x_0)} \\ v_y = \dfrac{\tau_z y - \tau_y f}{z} = \dfrac{\tau_z}{z}\left(y - \dfrac{\tau_y}{\tau_z} f\right) \equiv \underline{\dfrac{\tau_z}{z}(y - y_0)} \end{cases}$$



$\tau_z > 0$ $\qquad$ $\tau_z < 0$

Instantaneous epipole

$$(x_0, y_0) \equiv \left(\frac{\tau_x}{\tau_z} f, \frac{\tau_y}{\tau_z} f\right)$$

In homogeneous coords:

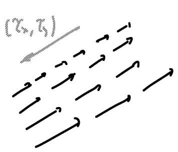
$$(\tau_x f, \tau_y f, \tau_z)$$

Pure translational motion

Case 2: $\tau_z = 0$ (and pure translation)

$$\begin{cases} v_x = \dfrac{\tau_z x - \tau_x f}{z} = \underline{-\dfrac{\tau_x}{z} f} \\ v_y = \dfrac{\tau_z y - \tau_y f}{z} = \underline{-\dfrac{\tau_y}{z} f} \end{cases}$$

linear motion field



Instantaneous epipole in homogeneous coordinates is a point at infinity

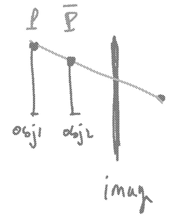
$$(\tau_x f, \tau_y f, \underset{0}{\tau_z}) \Rightarrow \text{direction}$$

Motion parallax

apparent motion of two instantaneously coincident points

Point P projected motion:

$$\begin{cases} v_x = v_x^{(T)} + v_x^{(\omega)} \\ v_y = v_y^{(T)} + v_y^{(\omega)} \end{cases}$$



Point $\bar{P}$ projected motion:

$$\begin{cases} \bar{v}_x = \bar{v}_x^{(T)} + \bar{v}_x^{(\omega)} \\ \bar{v}_y = \bar{v}_y^{(T)} + \bar{v}_y^{(\omega)} \end{cases}$$

Relative motion field

$$\left.\begin{aligned} \Delta v_x &= v_x - \bar{v}_x = v_x^{(T)} - \bar{v}_x^{(T)} \\ \Delta v_y &= v_y - \bar{v}_y = v_y^{(T)} - \bar{v}_y^{(T)} \end{aligned}\right\}$$

The depth z does not appear in the rotational component $\Rightarrow v^{(\omega)} = \bar{v}^{(\omega)}$

$$\Delta v_x = (x-x_0)\frac{\tau_z}{z} - (x-x_0)\frac{\tau_z}{\bar{z}} = (x-x_0)\tau_z\left(\frac{1}{z} - \frac{1}{\bar{z}}\right)$$

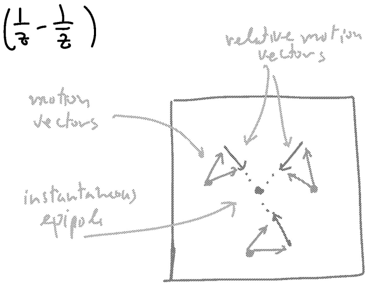
$$\Delta v_y = (y-y_0)\frac{\tau_z}{z} - (y-y_0)\frac{\tau_z}{\bar{z}} = (y-y_0)\tau_z\left(\frac{1}{z} - \frac{1}{\bar{z}}\right)$$

- The relative motion field is always radial and defines an instantaneous epipole.

Relative motion field

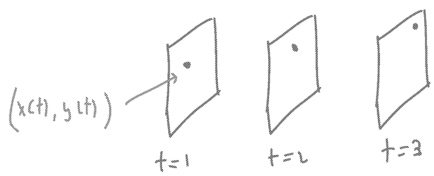
$$\Delta v_x = (x-x_0)\tau_z\left(\frac{1}{z} - \frac{1}{\bar{z}}\right)$$

$$\Delta v_y = (y-y_0)\tau_z\left(\frac{1}{z} - \frac{1}{\bar{z}}\right)$$



Optical flow estimation

- optical flow = motion observed in images
- Assume image brightness per object point is constant.



$$I(x(t), y(t), t) = C$$

Brightness constancy (Horn 1974) is the tendency for objects to maintain their perceived brightness under varying illumination conditions

not considering lighting changes

The linearized form of the incremental update to the SSD error (9.28) is often called the optical flow constraint or brightness constancy constraint equation (Horn and Schunck 1981)

Optical flow estimation

$$I(x(t), y(t), t) = C$$

$$\boxed{\frac{d}{dt}\Big(I(x(t), y(t), t)\Big) = 0}$$

OFCE
optical flow constraint equation

$$\frac{\partial I}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial I}{\partial t} = 0$$

$$\boxed{\nabla I \cdot V = -I_t}$$

$$\begin{cases} V = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right) \\ \nabla I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}\right) \\ I_t = \frac{\partial I}{\partial t} \end{cases}$$

Optical flow estimation

$$\boxed{\nabla I \cdot V = -I_t}$$

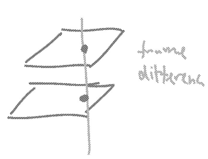
intensity gradient

optical flow

time derivative (frame difference)

$$V = ?$$

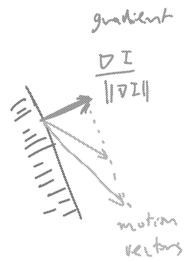
$$\boxed{\frac{\nabla I}{\|\nabla I\|} \cdot V = -\frac{I_t}{\|\nabla I\|}}$$



The aperture problem

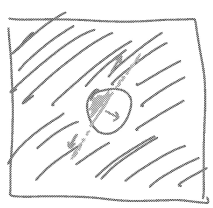
OFCE: $$\frac{\nabla I}{\|\nabla I\|} \cdot v = \frac{-I_t}{\|\nabla I\|}$$

projection of motion vector onto gradient



• Problem:
Different motion vectors create the same projection.

The aperture problem



• It is only possible to extract the component of the motion vector parallel to the image gradient

$$V_\perp = \frac{-I_t}{\|\nabla I\|}$$

Second order optical flow estimation

Assume constant spatial gradient (in addition to constant intensity):

$$\begin{cases} I\left(x(t), y(t), t\right) = C_1 \\ \nabla I\left(x(t), y(t), t\right) = C_2 \end{cases}$$

$$\frac{d}{dt}\left(\nabla I\left(x(t), y(t), t\right)\right) = 0$$

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial}{\partial x} I\left(x(t), y(t), t\right) \\ \frac{\partial}{\partial y} I\left(x(t), y(t), t\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second order optical flow estimation

$$\frac{d}{dt}\begin{bmatrix}\frac{\partial}{\partial x} I(x(t), y(t), t) \\ \frac{\partial}{\partial y} I(x(t), y(t), t)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} I_x \\ I_y\end{bmatrix} = \begin{bmatrix} I_{xx}x_t + I_{xy}y_t + I_{xt} \\ I_{yx}x_t + I_{yy}y_t + I_{yt}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

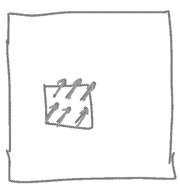
$$\begin{bmatrix} I_{xx} & I_{xy}\\ I_{yx} & I_{yy}\end{bmatrix} V = \begin{bmatrix}-I_{xt}\\-I_{yt}\end{bmatrix}$$

Solve for V
(noisy estimate)

Block based optical flow estimation

• Assume the motion in a small patch is constant.

$$E(V) = \sum_{\substack{(x,y)\in\\ \text{patch}}} \left(\nabla I(x,y)^T \mathbf{V} + I_t(x,y)\right)^2$$



$$V^* = \underset{V}{\arg\min}\, E(V)$$

$$\nabla E(V) = 0$$

Block based optical flow estimation

Lucas-Kanade

$$E(V) = \sum \left(I_x \cdot x_t + I_y \cdot y_t + I_t\right)^2$$

$$\frac{\partial E}{\partial x_t} = 0 = 2\sum\left(I_x x_t + I_y y_t + I_t\right) I_x$$

$$\frac{\partial E}{\partial y_t} = 0 = 2\sum\left(I_x x_t + I_y y_t + I_t\right) I_y$$

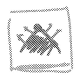
$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$$

Block based optical flow estimation

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$$

structure tensor (known)   optical flow (unknown)   (known)

- solve for optical flow vector in patch (x_t, y_t)
- solution exist when the structure tensor matrix is non singular (at corners).

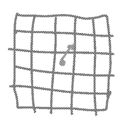


Weighted block based optical flow estimation

- Give preference to satisfying OFCE at the center of the patch.

$$E(x_t, y_t) = \sum w(x,y) \left(I_x x_t + I_y y_t + I_t \right)^2$$

$$w(x,y) = \frac{1}{\| (x,y) - (x_c, y_c) \| + 1}$$



or

$$w(x,y) = \exp\left(- \| (x,y) - (x_c, y_c) \|^2 / \sigma^2 \right)$$

Weighted block based optical flow estimation

$$E(x_t, y_t) = \sum w(x,y) \left(I_x x_t + I_y y_t + I_t \right)^2$$

$$\nabla E = 0 \Rightarrow \begin{cases} \sum w(x,y) \left(I_x x_t + I_y y_t + I_t \right) I_x = 0 \\ \sum w(x,y) \left(I_x x_t + I_y y_t + I_t \right) I_y = 0 \end{cases}$$

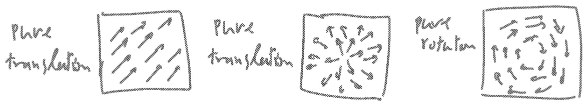
$$\begin{bmatrix} \sum w I_x^2 & \sum w I_x I_y \\ \sum w I_x I_y & \sum w I_y^2 \end{bmatrix} v = \begin{bmatrix} -\sum w I_x I_t \\ -\sum w I_y I_t \end{bmatrix}$$

Affine optical flow estimation

- Assume the motion in a local neighbourhood is described by an affine map (instead of constant)
- Model: optical flow vector at location (x,y)

$$v(x,y) \equiv \begin{bmatrix} u \\ v \end{bmatrix} \equiv \begin{bmatrix} a_1 \\ a_4 \end{bmatrix} + \begin{bmatrix} a_2 & a_3 \\ a_5 & a_6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Constant motion is a special case of the affine case when $a_2 = a_3 = a_5 = a_6$



Affine optical flow estimation

$$v(x,y;a) = \begin{bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{bmatrix}$$

6 unknowns instead of 2 in constant model

$$E(a) = \sum \left(I_x x_t + I_y y_t + I_t \right)^2$$

$$= \sum \left(I_x v^{(x)} + I_y v^{(y)} + I_t \right)^2$$

$$= \sum \left(I_x (a_1 + a_2 x + a_3 y) + I_y (a_4 + a_5 x + a_6 y) + I_t \right)^2$$

$$a^* = \underset{a}{\operatorname{argmin}}\, E(a) \implies \nabla E(a) = 0$$

Affine optical flow estimation

$$E(a) = \sum \left(I_x (a_1 + a_2 x + a_3 y) + I_y (a_4 + a_5 x + a_6 y) + I_t \right)^2$$

$$\nabla E(a) = \left[\frac{\partial E}{\partial a_1}, \frac{\partial E}{\partial a_2}, \dots \frac{\partial E}{\partial a_6} \right] = [0, \dots, 0]$$

$$\frac{\partial E}{\partial a_1} = \sum 2 \left(I_x (a_1 + a_2 x + a_3 y) + I_y (a_4 + a_5 x + a_6 y) + I_t \right) I_x$$

$$\frac{\partial E}{\partial a_2} = \sum 2 \left(I_x (a_1 + a_2 x + a_3 y) + I_y (a_4 + a_5 x + a_6 y) + I_t \right) I_x x$$

$$\vdots$$

$$\frac{\partial E}{\partial a_6} = \sum 2 \left(I_x (a_1 + a_2 x + a_3 y) + I_y (a_4 + a_5 x + a_6 y) + I_t \right) I_y y$$

Affine optical flow estimation

$$\underbrace{\begin{bmatrix} \sum I_x^2 & \sum I_x^2 x & \sum I_x^2 y & \sum I_x I_y & \sum I_x I_y x & \sum I_x I_y y \\ \sum I_x^2 x & \sum I_x^2 x^2 & \sum_x xy & \sum I_x I_y x & \sum I_x I_y x^2 & \sum I_x I_y xy \\ \vdots & & & & & \vdots \\ \sum I_x I_y y & \sum I_y xy & \sum I_x I_y y^2 & \sum I_y^2 y & \sum I_y^2 xy & \sum I_y^2 y^2 \end{bmatrix}}_{\text{known}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}}_{\text{unknown}} = \underbrace{\begin{bmatrix} -\sum I_x I_t \\ -\sum I_x I_t x \\ \vdots \\ -\sum I_y I_t y \end{bmatrix}}_{\text{known}}$$

Solve for $a_1, \ldots, a_6$