3D Reconstruction

Given $\{p_i\}_{i=1}^{m} \leftrightarrow \{p_i'\}_{i=1}^{m}$ find $\{\underline{P}_i\}_{i=1}^{m}$

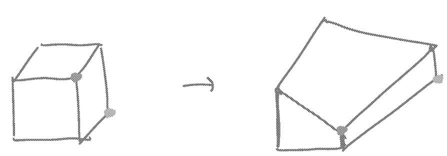
2D 2D 3D

Cases:

1) know intrinsic + extrinsic $\Rightarrow$ Absolute reconstruction
2) know intrinsic $\Rightarrow$ Euclidean reconstruction (up to scale)
3) None are known $\Rightarrow$ Reconstruction up to unknown 3D projective map

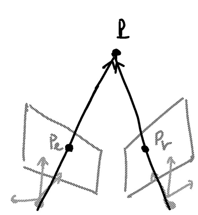
3D Reconstruction

Reconstruction up to unknown 3D projective map



Absolute reconstruction

* Triangulation algorithm:
  - Send ray from O_ℓ through p_ℓ and ray from O_r through p_r
  - Reconstruct $\underline{P}$ at rays intersection.



* step 1: move to camera coords:

$$\begin{cases} \bar{p}_\ell = K_\ell^* p_\ell \\ \bar{p}_r = K_r^* p_r \end{cases} \Rightarrow \begin{cases} p_\ell = K_\ell^{*-1} \bar{p}_\ell \\ p_r = K_r^{*-1} \bar{p}_r \end{cases}$$

image camera

Absolute reconstruction

* Step 2: Find intersection of rays:

scalars

left ray: $a\,p_\ell$

right ray: $b\,p_r^*$

intersection: $a\,p_\ell = b\,p_r^*$

- The ray $b\,p_r^*$ is right ray in left coordinates

$$p_r^* = M_{left \leftarrow right}\, P_r = R\,p_r + T$$

Absolute reconstruction

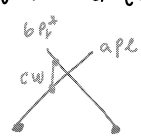
* To find intersection solve:

$$a\,p_\ell = b\,p_r^*$$

$$\Leftrightarrow \quad a\,p_\ell = b\,R\,p_r + T \quad \rightarrow \text{ solve for } a, b$$

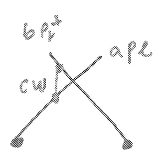
* problem:

due to inaccuracies rays do not intersect (no solution)



Absolute reconstruction

$$\begin{cases} a\,p_\ell + c\,w = b\,p_r^* & \leftarrow R\,p_r + T \\ w = p_\ell \times p_r^* = p_\ell \times R\,p_r \end{cases}$$



unknowns

$$\underline{a}\,p_\ell + \underline{c}\,(p_\ell \times R\,p_r) - \underline{b}\,R\,p_r = T$$

$$\begin{bmatrix} | & | & | \\ p_\ell & p_\ell \times R\,p_r & -R\,p_r \\ | & | & | \end{bmatrix} \begin{bmatrix} a \\ c \\ b \end{bmatrix} = T \Rightarrow \text{solve for } a, b, c$$

Absolute reconstruction

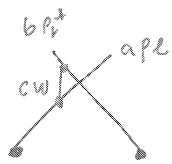
- Summary:

1) Given $\bar{p}_l, \bar{p}_r, R, T, k_l^*, k_r^*$ solve for a, b, c
   $\Rightarrow a_0, b_0, c_0$

2) Compute P:

$$P = \frac{1}{2}(a_0 p_l + b_0 R p_r + T)$$

$$P = a_0 p_l + \frac{1}{2} c_0 w$$



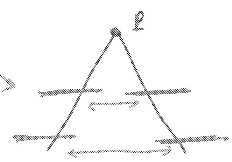
3) Convert P to world coordinates (from left coords.)

$$\underline{P}^{(w)} = R_l P + T_l$$

Euclidean reconstruction

- Have intrinsic parameters but do not know extrinsic
- Need correspondence of 8 points in two views
  $\Rightarrow$ weak calibration (Get F)
- Cannot recover baseline
  $\Rightarrow$ Reconstruction up to unknown scale

Same disparity with different baseline
$\Rightarrow$ different Z for same disparity



Euclidean Reconstruction

Algorithm:

1) use 8-points alg. to recover F
2) Compute E
3) Normalize E
4) Recover T up to unknown sign
5) Recover R up to unknown sign
6) Resolve ambiguity in T and R
7) Reconstruct up to unknown scale

Euclidean Reconstruction

step 1: use 8 points algorithm to recover F

step 2: compute E:

$$F \equiv K_r^{*^{-T}} E K_\ell^{*^{-1}}$$

$$\Rightarrow E = K_r^{*^{T}} F K_\ell^{*}$$

Euclidean Reconstruction

Step 3: Normalize E:

$\Rightarrow$ Find $\hat{E}$ with baseline $= 1$

$$E^T E = (R^T [T]_\times)^T R^T [T]_\times$$

$$= [T]_\times^T R R^T [T]_\times = [T]_\times^T [T]_\times$$

$$= \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

Euclidean Reconstruction

$$E^T E = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} T_y^2 + T_z^2 & -T_x T_y & -T_x T_z \\ \sim & T_z^2 + T_x^2 & \sim \\ \sim & \sim & T_x^2 + T_y^2 \end{bmatrix}$$

$$tr[E^T E] = T_y^2 + T_z^2 + T_z^2 + T_x^2 + T_x^2 + T_y^2 = 2(T_x^2 + T_y^2 + T_z^2) = 2\|T\|^2$$

$$\hat{E} \equiv \frac{2}{tr(E^T E)} E \Rightarrow$$ $\hat{E}$ has baseline of 1 $\left(tr[\hat{E}^T \hat{E}] = 2\|T\|^2 = 1\right)$

Euclidean Reconstruction

* Step 4: Recover $\hat{T}$ $(\|\hat{T}\| = 1)$

$$\hat{E}^T\hat{E} = \begin{bmatrix} \hat{T}_y^2+\hat{T}_z^2 & -\hat{T}_x\hat{T}_y & -T_xT_z \\ \sim & \hat{T}_z^2+\hat{T}_x^2 & \sim \\ \sim & \sim & \hat{T}_x^2+\hat{T}_y^2 \end{bmatrix} \equiv \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$\begin{cases} \hat{T}_y^2+\hat{T}_z^2 = \alpha_{11} \Rightarrow 1-\hat{T}_x^2 = \alpha_{11} \Rightarrow \hat{T}_x = \pm\sqrt{1-\alpha_{11}} \\ -\hat{T}_x\hat{T}_y = \alpha_{12} \Rightarrow \hat{T}_y = -\dfrac{\alpha_{12}}{\hat{T}_x} \\ -\hat{T}_x\hat{T}_z = \alpha_{13} \Rightarrow \hat{T}_z = \dfrac{-\alpha_{13}}{\hat{T}_x} \end{cases}$$

we have $\hat{T}$ up to unknown sign

Euclidean Reconstruction

* Step 5: Recover R

$$\hat{E} = R^T[\hat{T}]_x \equiv \begin{bmatrix} -\hat{E}_1^T- \\ -\hat{E}_2^T- \\ -\hat{E}_3^T- \end{bmatrix}$$

$$\hat{E}^T = [\hat{T}]_x^T R = -[\hat{T}]_x R \equiv -[\hat{T}]_x [r_1\ r_2\ r_3]$$

$$[\hat{E}_1\ \hat{E}_2\ \hat{E}_3] = -[\hat{T}]_x [r_1\ r_2\ r_3]$$ ← $\hat{T}$ is rank deficient and so cannot multiply by $[\hat{T}_x]^{-1}$ to find R

$$\hat{E}_i = -[\hat{T}]_x r_i = -\hat{T}\times r_i = r_i \times \hat{T}$$

Euclidean Reconstruction

$$\hat{E}_i = r_i \times \hat{T}$$

Define $w_i = \hat{E}_i \times \hat{T}$

$$r_i = -w_i + w_j \times w_k$$ ← not shown here

$$\begin{cases} r_1 = -w_1 + w_2 \times w_3 \\ r_2 = -w_2 + w_3 \times w_1 \\ r_3 = -w_3 + w_1 \times w_2 \end{cases} \qquad R \equiv [r_1\ r_2\ r_3]$$

Euclidean Reconstruction

* step 6: resolve sign ambiguity

  $\hat{T}$ is known up to sign

  $\Rightarrow$ w_i is known up to sign

  $\Rightarrow$ R is known up to sign

- possible signs: $T, R \rightarrow (++, +-, -+, --)$
- Reconstruct (using triangulation) each option
  choose the solution where all z are positive

* step 7: triangulate using $R, \hat{T} \Rightarrow \{P_i\}_{i=1}^{m}$