CS 512 - 02 [FALL 24]

ASSIGNMENT - 0

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PART - A

(1) $3P + 2q = 3\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

$$= \begin{bmatrix} 6 \\ 3 \\ 22 \end{bmatrix}$$

(2) $\hat{P} = \frac{P}{\|P\|}$; $\|p\| = \sqrt{2^2 + (-1)^2 + 4^2}$

$$\|p\| = \sqrt{21}$$

$$\hat{P} = \frac{1}{\sqrt{21}}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{21} \\ -1/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix}$$

③ $\|p\| = \sqrt{21}$

Unit vector along y-axis, $e_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\|e_y\| = \sqrt{0^2+1^2+0^2} = 1$

dot product, $x \cdot y = \|x\|\|y\| \cos\theta$

$\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{21} \cdot 1 \cdot \cos\theta$

$\sqrt{21} \cdot \cos\theta = 2\times0 + (-1)(1) + 4\times0) = -1$

$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{21}}\right)$

④ $p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$

from previous, Y axis $\Rightarrow \cos\theta_y = -1/\sqrt{21}$

$\cos\theta = \frac{P_x/P_y/P_z}{\|p\|}$ X axis $\Rightarrow \cos\theta_x = 2/\sqrt{21}$

(normalization by coordi length) z axis $\Rightarrow \cos\theta_z = 4/\sqrt{21}$

Cos of angle that p-vector makes with x, y, z axis.

⑤ $p \cdot q \Rightarrow \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} \Rightarrow -3 + 20 \Rightarrow 17$

$\cos\theta \cdot \|p\|\|q\| = 17$

$\cos\theta = \frac{17}{\sqrt{21}\times\sqrt{34}}$; $\theta = \cos^{-1}\left(\frac{17}{\sqrt{21}\cdot\sqrt{34}}\right)$

$\theta = \cos^{-1}\left(\frac{17}{\sqrt{714}}\right)$

(6) $p \cdot q = q \cdot p = 17$ (from (5))

(7)

$\cos\theta = \dfrac{17}{\sqrt{714}}$ $\qquad p \cdot q = \|p\| \|q\| \cos\theta$

$q \cdot q = \sqrt{21} \cdot \sqrt{34} \cdot \dfrac{17}{\sqrt{714}} = 17$

(8)

Scalar projection of q on $\hat{p}$ = $\dfrac{q \cdot \hat{p}}{\|\hat{p}\|}$ ← 1

$= \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{21} \\ -1/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix}$

$= \dfrac{-3}{\sqrt{21}} + \dfrac{20}{\sqrt{21}} \Rightarrow \dfrac{17}{\sqrt{21}}$

(9) Let u be the vector $\perp r$ to p.

angle b/w $p + r \Rightarrow 90°$

$\theta = \cos^{-1}\left(\dfrac{p \cdot u}{\|p\| \|u\|}\right)$

$\cos\theta = \dfrac{p \cdot u}{\|p\| \|u\|}$

$\cos 90° = \dfrac{p \cdot u}{\|p\| \|u\|} = 0$

$\Rightarrow p \cdot u = 0$

Let $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $p = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

$p \cdot u = 2x - y + 4z = 0$

Assume :- $x = 0$; $y = 1$

$-1 + 4z = 0 \Rightarrow z = 1/4 \Rightarrow u = \begin{bmatrix} 0 \\ 1 \\ 1/4 \end{bmatrix}$

Assume $x = 1$; $y = 2$

$2 - 2 + 4z = 0 \Rightarrow z = 0 \Rightarrow u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

⑩ $p \times q$ + $q \times p$

$$p \times q = \begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix} = \vec{i}(-5-12) - \vec{j}(10-0) + \vec{k}(6-0)$$

$$= -17\vec{i} - 10\vec{j} + 6\vec{k}$$

$\vec{i}, \vec{j}, \vec{k}$ are unit vectors ⇒ ~~$p \times q = [-17\ 10\ 6]$~~

$$p \times q = \begin{bmatrix} -17 \\ -10 \\ 6 \end{bmatrix}$$

$$q \times p = \begin{vmatrix} i & j & k \\ 0 & 3 & 5 \\ 2 & -1 & 4 \end{vmatrix} = \vec{i}(12+5) - \vec{j}(0-10) + \vec{k}(0-6)$$

$$= \vec{i}(17) + \vec{j}(10) - 6\vec{k}$$

$$q \times p = \begin{bmatrix} 17 \\ 10 \\ -6 \end{bmatrix} \qquad p \times q = -(q \times p)$$

⑪ From question 9,

$\perp r$ to p → $2x - y + 4z = 0$ — ①

To find $u \perp r$ to q:

$$q \cdot u = 0$$

$$q = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} ; \quad u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$3y + 5z = 0 \quad \text{—②}$$

~~$2x - y + 3y + 4z = 0$~~

$$① = ② \Rightarrow 2x - y + 4z = 3y + 5z$$

$$2x - 4y - z = 0$$

Assume: $x = 0; y = 1 \Rightarrow z = -4 \Rightarrow u = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$

Assume: $x = 1; y = 2 \Rightarrow z = -6 \Rightarrow u = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$

Cross product of 2 vectors is $\perp$ to both.

$\Rightarrow p \times q = \begin{bmatrix} -17 \\ -10 \\ 6 \end{bmatrix}$

(15). A set of vectors $\{x_1, \ldots, x_n\} \subset \mathbb{R}^m$ is linearly independent if no vectors can be represented as linear combination of remaining vectors.

Linearly dependent $\rightarrow$ $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$

$x_i \rightarrow$ vectors ; $\alpha_i \rightarrow$ scalars.

$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i = 0$

$\alpha_1 p + \alpha_2 q + \alpha_3 r = 0$;

$$\begin{bmatrix} 2\alpha_1 \\ -\alpha_1 \\ 4\alpha_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3\alpha_2 \\ 5\alpha_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ -2\alpha_3 \\ 2\alpha_3 \end{bmatrix} = 0$$

$2\alpha_1 + \alpha_3 = 0$ — (1)

$-\alpha_1 + 3\alpha_2 - 2\alpha_3 = 0$ — (2)

$4\alpha_1 + 5\alpha_2 + 2\alpha_3 = 0$ — (3)

(2) $\times 5 \rightarrow \cancel{-5\alpha_1} + 15\alpha_2 - 10\alpha_3 = 0$

(3) $\times 3 \Rightarrow +12\alpha_1 + 15\alpha_2 + 6\alpha_3 = 0$

$\quad (-) \quad (-) \quad (-)$

$-17\alpha_1 \quad -16\alpha_3 = 0$

from (1) $\alpha_3 = -2\alpha_1$

$-17\alpha_1 - 16(-2\alpha_1) = 0$

$-17\alpha_1 + 32\alpha_1 = 0 \Rightarrow \alpha_1 = 0$

$\Rightarrow \alpha_3 = 0$

from (2) $\Rightarrow 0 + 3\alpha_2 - 0 = 0$

$\Rightarrow \alpha_2 = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow p, q, r$ are linearly independent.

(13) $p^T q + pq^T$

$$p^T q = \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} = 0 - 3 + 20 = 17$$

dot / inner product

$$pq^T = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 5 \end{bmatrix}$$

outer product

$$= \begin{bmatrix} 2\times 0 & 2\times 3 & 2\times 5 \\ -1\times 0 & -1\times 3 & -1\times 5 \\ 4\times 0 & 4\times 3 & 4\times 5 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 10 \\ 0 & -3 & -5 \\ 0 & 12 & 20 \end{bmatrix}$$

PART B

(1) $X + 2Y$

$$X + 2Y = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} + 2 \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -1 & 4 \\ +5 & 3 & -2 \\ 5 & 6 & 0 \end{bmatrix}$$

(2) XY

$$XY = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8+3+0 & -2+0+0 & 4-3+0 \\ -4+9+4 & +1+0+8 & -2-9+4 \\ 12+6-2 & -3+0-4 & 6-6-2 \end{bmatrix} = \begin{bmatrix} 11 & -2 & 1 \\ 9 & 9 & -7 \\ 16 & -7 & -2 \end{bmatrix}$$

$$YX = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+1+6 & 4-3+4 & 0-4-4 \\ 6+0-9 & 3+0-6 & 0+0+6 \\ 2-2+3 & 1+6+2 & 0+8-2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 5 & -8 \\ -3 & -3 & 6 \\ 3 & 9 & 6 \end{bmatrix}$$

③ $(XY)^T$ and $Y^T X^T$

$$(XY)^T \Rightarrow \begin{bmatrix} 11 & 9 & 16 \\ -2 & 9 & -7 \\ 1 & -7 & -2 \end{bmatrix}$$

$$Y^T X^T = \begin{bmatrix} 4 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+3+0 & -4+9+4 & 12+6-2 \\ -2+0+0 & 1+0+8 & -3+0-4 \\ 4-3+0 & -2-9+4 & 6-6-2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 16 \\ -2 & 9 & -7 \\ 1 & -7 & -2 \end{bmatrix}$$

(4) $|X| = \begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{vmatrix}$

$= 2(-6-8) - 1(2-12) + 0$

$= -28 + 10 \Rightarrow -18 \Rightarrow |X| \neq 0 \Rightarrow$ $|Z| \neq 0$ and the column vectors are not linearly dependent

$|Z| = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{vmatrix} = 2(8-5) - 0 - (1-12)$

$= 6 + 11 \Rightarrow 17$

(5) If a matrix, X has the row vectors forming an orthogonal set, then $XX^T = I \Rightarrow$ X would be a orthogonal matrix.

Let us find if either XX^T, YY^T or ZZ^T equals I

$XX^T = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 0 & 4 & -2 \end{bmatrix}$

$= \begin{bmatrix} 4+1+0 & -2+3+0 & 6+2+0 \\ -2+3+0 & +1+9+16 & -3+6-8 \\ 6+2+0 & -3+6-8 & 9+4+4 \end{bmatrix}$

$= \begin{bmatrix} 5 & 1 & 8 \\ 1 & 26 & -5 \\ 8 & -5 & 17 \end{bmatrix} \neq I$

$YY^T = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 16+1+4 & 12+0-6 & 4-2+2 \\ 12+0-6 & 9+0+9 & 3+0-3 \\ 4-2+2 & 3+0-3 & 1+4+1 \end{bmatrix} = \begin{bmatrix} 21 & 6 & 4 \\ 6 & 18 & 0 \\ 4 & 0 & 6 \end{bmatrix}$

$YY^T \neq I$

$$ZZ^T = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \\ -1 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+0-5 & 6+0-2 \\ 2+0-5 & 1+16+25 & 3+4+10 \\ 6+0-2 & 3+4+10 & 9+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 & 4 \\ -3 & 42 & 17 \\ 4 & 17 & 14 \end{bmatrix} \neq I$$

~~XXI≠~~ $\Rightarrow$ X, Y, Z None of are Orthogonal & do not have row vectors forming orthogonal set

(6) Inverse Formula; $A^{-1} = \frac{1}{|A|} \text{adj}(A)$

$(\text{adj}(A))_{ij} = (-1)^{i+j} |A_{ji}|$

$$X^{-1} = \frac{1}{|X|} \text{adj}(X)$$

from question (a) $|X| = -18$

$$X^{-1} = \frac{1}{-18} \begin{bmatrix} -6-8 & -(2+12) & (-2-9) \\ -(2+0) & +(-4-0) & -(4-3) \\ +(4-0) & -(8-0) & +(6+1) \end{bmatrix}^T$$

$$= -\frac{1}{18} \begin{bmatrix} -14 & +10 & -11 \\ +2 & -4 & -1 \\ 4 & -8 & 7 \end{bmatrix}^T$$

$$= \begin{bmatrix} 14/18 & -2/18 & -4/18 \\ -10/18 & 4/18 & 8/18 \\ 11/18 & 1/18 & -7/18 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} 7/9 & -1/9 & -2/9 \\ -5/9 & 2/9 & 4/9 \\ 11/18 & 1/18 & -7/18 \end{bmatrix}$$

Find $X^{-1}X^{-T}$

$$Y^{-1} = \frac{1}{|Y|}\operatorname{adj}(Y) \; ; \; |Y| = 4(0+6)+1(3+3)+2(6-0)$$

$$= 24 + 6 + 12 = 42$$

$$= \frac{1}{42}\begin{bmatrix} +(0+6) & -(3+3) & +(6-0) \\ -(-1-4) & +(4-2) & -(8+1) \\ +(3-0) & -(-12-6) & +(0+3) \end{bmatrix}$$

$$= \frac{1}{42}\begin{bmatrix} 6 & -6 & 6 \\ 5 & 2 & -9 \\ 3 & 18 & 3 \end{bmatrix}^T = \frac{1}{42}\begin{bmatrix} 6 & 5 & 3 \\ -6 & 2 & 18 \\ 6 & -9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/7 & 5/42 & 1/14 \\ -1/7 & 1/21 & 3/7 \\ 1/7 & -3/14 & 1/14 \end{bmatrix}$$

Find $Y^{-1}Y^{-T}$

$$X^{-1} X^{-T} = 1 \begin{bmatrix} 7/9 & -1/9 & -2/9 \\ -5/9 & 2/9 & 4/9 \\ 11/18 & 1/18 & -7/18 \end{bmatrix} \begin{bmatrix} 7/9 & -5/9 & 11/18 \\ -1/9 & 2/9 & 1/18 \\ -2/9 & 4/9 & -7/18 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{54}{81} & \frac{-45}{81} & \frac{90}{162} \\ \frac{-45}{81} & \frac{45}{81} & \frac{-81}{162} \\ \frac{90}{162} & \frac{-81}{162} & \frac{171}{18^2} \end{bmatrix} \neq I$$

$$y^{-1} \cdot y^{-T} = \begin{bmatrix} \frac{5}{126} & \frac{1}{63} & \frac{2}{49} \\ \frac{1}{63} & \frac{46}{441} & 0 \\ 2/49 & 0 & 1/14 \end{bmatrix} \neq I$$

$\Rightarrow$ X^{-1} & Y^{-1} are not orthogonal matrix & do not have row vectors forming orthogonal set.

(7) $z^{-1} = \frac{1}{|z|} \, Adj(z)$

from (4) $|z| = 17$

$$= \frac{1}{17} \begin{bmatrix} +(8-5) & -(2-15) & +(1-12) \\ -(0+1) & +(4+3) & -(2+0) \\ +(0+4) & -(10+1) & +(8-0) \end{bmatrix}^T$$

$$= \frac{1}{17} \begin{bmatrix} 3 & 13 & -11 \\ -1 & 7 & -2 \\ 4 & -11 & 8 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3/17 & -1/17 & 4/17 \\ 13/17 & 7/17 & -11/17 \\ -11/17 & -2/17 & 8/17 \end{bmatrix}$$

$$|z^{-1}| = \frac{3}{\cancel{\quad}}\left[\frac{(56-22)}{17^3} + \frac{(104-121)}{17^3} + \frac{4(-26+77)}{17^3}\right]$$

$$= \frac{289}{17^3}$$

$$= 0.05 \neq 0$$

$\det(z^{-1}) \neq 0 \Rightarrow z^{-1}$ is linearly independent

(8) product - XS

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2+4+0 \\ 1+12+0 \\ -3+8+0 \end{bmatrix} = \begin{bmatrix} +2 \\ 13 \\ 5 \end{bmatrix}$$

(4) From X, let the row vectors of X be

$x_1 = [2\ 1\ 0]$; $x_2 = [-1\ 3\ 4]$; $x_3 = [3\ 2\ -2]$

Normalized $\hat{S} = \dfrac{S}{\|S\|}$

$\|S\| = \sqrt{(-1)^2 + 4^2 + 0} = \sqrt{17}$

Scalar proj of x_1 on $S \Rightarrow x_1 \cdot \hat{S} = x_1 \cdot \dfrac{S}{\|S\|}$

$= \dfrac{1}{\sqrt{17}} [2\ 1\ 0] \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \dfrac{1}{\sqrt{17}} \times [-2 + 4 + 0]$

$= +2/\sqrt{17}$

$x_2 \Rightarrow x_2 \cdot \hat{S} \Rightarrow x_2 \cdot \dfrac{S}{\|S\|} = \dfrac{1}{\sqrt{17}} [-1\ 3\ 4] \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

$= \dfrac{1}{\sqrt{17}} (1 + 12 + 0) = 13/\sqrt{17}$

$x_3 \rightarrow \dfrac{1}{\sqrt{17}} [3\ 2\ -2] \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \dfrac{1}{\sqrt{17}} (-3 + 8 + 0)$

$= 5/\sqrt{17}$

(10) Vector projection of $\vec{u}$ on $\vec{v} \Rightarrow$ finding a component of $\vec{u}$ parallel to $\vec{v}$.

$$\text{Proj}_v u = \left(\frac{u \cdot v}{\|v\|^2} \right) v$$

Normalized $\vec{s}, \hat{s} = \dfrac{s}{\|s\|} = \dfrac{1}{\sqrt{17}}\begin{bmatrix}-1\\4\\0\end{bmatrix}$

$x_1 = \begin{bmatrix}2 & 1 & 0\end{bmatrix}$

Vector $\text{proj}_{\hat{s}}\, x_1 = \left(\dfrac{x_1 \cdot \hat{s}}{\|\hat{s}\|^2}\right)\hat{s}$

$\|\hat{s}\| = 1$ (as it normalized already)

$\text{Proj}_{\hat{s}}\, x_1 = \left(\dfrac{1}{\sqrt{17}}\begin{bmatrix}2 & 1 & 0\end{bmatrix}\begin{bmatrix}-1\\4\\0\end{bmatrix}\right)\dfrac{1}{\sqrt{17}}\begin{bmatrix}-1\\4\\0\end{bmatrix}$

$= \dfrac{1}{17}\begin{bmatrix}-2+4+0\end{bmatrix}\begin{bmatrix}-1\\4\\0\end{bmatrix}$

$= \dfrac{2}{17}\begin{bmatrix}-1\\4\\0\end{bmatrix} = \begin{bmatrix}-2/17\\8/17\\0\end{bmatrix}$

$x_2 = \begin{bmatrix}-1 & 3 & 4\end{bmatrix}$

$\text{Proj}_{\hat{s}}\, x_2 = \left[\begin{bmatrix}-1 & 3 & 4\end{bmatrix}\dfrac{1}{\sqrt{17}}\begin{bmatrix}-1\\4\\0\end{bmatrix}\right]\dfrac{1}{\sqrt{17}}\begin{bmatrix}-1\\4\\0\end{bmatrix}$

$= \dfrac{1}{17}(1+12+0)\begin{pmatrix}-1\\4\\0\end{pmatrix}$

$= \begin{bmatrix}-13/17\\52/17\\0\end{bmatrix}$

$x_3 = [3 \quad 2 \quad -2]$

$$\text{Proj}_{\hat{s}}\, x_3 = \left(\frac{1}{17}[3 \; 2 \; -2]\begin{bmatrix}-1\\4\\0\end{bmatrix}\right)\begin{bmatrix}-1\\4\\0\end{bmatrix}$$

$$= \left(\frac{1}{17}(-3+8+0)\right)\begin{bmatrix}-1\\4\\0\end{bmatrix}$$

$$= \frac{5}{17}\begin{bmatrix}-1\\4\\0\end{bmatrix} = \begin{bmatrix}-5/17\\20/17\\0\end{bmatrix}$$

(11) let y be the linear comb of columns of X with coefficients being elements of s.

$$y = Xs = \begin{bmatrix}2 & 1 & 0\\-1 & 3 & 4\\3 & 2 & -2\end{bmatrix}\begin{bmatrix}-1\\4\\0\end{bmatrix}$$

$$= (-1)\begin{bmatrix}2\\-1\\3\end{bmatrix} + 4\begin{bmatrix}1\\3\\2\end{bmatrix} + 0\begin{bmatrix}0\\4\\-2\end{bmatrix}$$

$$y = \begin{bmatrix}-2\\1\\-3\end{bmatrix} + \begin{bmatrix}4\\12\\8\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\13\\5\end{bmatrix}$$

= general product method for (8) question.

(12) $Yt = s \Rightarrow t = Y^{-1}s$ from (6), taking Y^{-1}

$$= \begin{bmatrix} 1/7 & 5/42 & 1/14 \\ -1/7 & 1/21 & 3/7 \\ 1/7 & -3/14 & 1/14 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 + \frac{20}{42} + 0 \\ +1/7 + 4/21 + 0 \\ -1/7 - 12/14 + 0 \end{bmatrix} = \begin{bmatrix} 14/42 \\ 7/21 \\ -14/14 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

(13) $Zt = s \Rightarrow t = Z^{-1}s$ from (7), taking Z^{-1}

$$= \begin{bmatrix} 3/17 & -1/17 & 4/17 \\ 13/17 & 7/17 & -11/17 \\ -11/17 & -2/17 & 8/17 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3-4+0}{17} \\ (-13+28+0)/17 \\ (11-8+0)/17 \end{bmatrix} = \begin{bmatrix} -7/17 \\ 15/17 \\ 3/17 \end{bmatrix}$$

Reason multiply by Z^{-1} on both sides: $Z.Z^{-1}.t = Z^{-1}s$

$\Rightarrow t = Z^{-1}s \quad (ZZ^{-1} = I)$

Part C   Eigenvalue - Eigenvector pair $\Rightarrow (\lambda I - A)x = 0$

$\Rightarrow |(\lambda I - A)| = 0$

① $M = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$. Find $\lambda I - A$

$\lambda I - \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} = \Rightarrow \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda - 3 & -2 \\ 1 & \lambda - 4 \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda - 3 & -2 \\ 1 & \lambda - 4 \end{vmatrix} = 0$

$(\lambda - 3)(\lambda - 4) - (-2)(1) = 0$

$(\lambda - 3)(\lambda - 4) + 2 = 0$

$\lambda^2 - 7\lambda + 12 + 2 = 0 \Rightarrow \lambda^2 - 7\lambda + 14 = 0$ ①

Solve quadratic eq. ①

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{7 \pm \sqrt{49 - 56}}{2} = \dfrac{7 \pm \sqrt{-7}}{2}$

$\lambda = \dfrac{7 \pm \sqrt{7}\sqrt{-1}}{2} = \dfrac{7 \pm i\sqrt{7}}{2}$

$\boxed{\lambda_1 = \dfrac{7 + i\sqrt{7}}{2} \ ; \ \lambda_2 = \dfrac{7 - i\sqrt{7}}{2}}$

Eigen vectors:

λ_1

$\lambda I - A = \begin{bmatrix} \dfrac{7 + i\sqrt{7}}{2} - 3 & -2 \\ 1 & \dfrac{7 + i\sqrt{7}}{2} - 4 \end{bmatrix}$

$= \begin{bmatrix} \dfrac{1 + i\sqrt{7}}{2} & -2 \\ 1 & \dfrac{-1 + i\sqrt{7}}{2} \end{bmatrix}$

from, $(\lambda I - A)x = 0$

$$\begin{bmatrix} \frac{1+i\sqrt{7}}{2} & -2 \\ 1 & \frac{-1+i\sqrt{7}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x\left(\frac{1+i\sqrt{7}}{2}\right) - 2y \\ x + y\left(\frac{-1+i\sqrt{7}}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y\left(\frac{-1+i\sqrt{7}}{2}\right) = 0$$

$$x = -y\left(\frac{-1+i\sqrt{7}}{2}\right)$$

Assume $y = 1$

$$x = \frac{1-i\sqrt{7}}{2}$$

$$\boxed{\Rightarrow \text{for } \lambda_1,\ x_1 = \begin{bmatrix} (1-i\sqrt{7})/2 \\ 1 \end{bmatrix}}$$

λ_2

$$\lambda I - A = \begin{bmatrix} \frac{7-i\sqrt{7}}{2} - 3 & -2 \\ 1 & \frac{7-i\sqrt{7}}{2} - 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-i\sqrt{7}}{2} & -2 \\ 1 & -\frac{(1+i\sqrt{7})}{2} \end{bmatrix}$$

$(\lambda I - A)x = 0$

$$\begin{bmatrix} (1-i\sqrt{7})/2 & -2 \\ 1 & -\frac{(1+i\sqrt{7})}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{x(1-i\sqrt{7})}{2} - 2y \\ x - \frac{y(1+i\sqrt{7})}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \cancel{\frac{x(1-i\sqrt{7})}{2} = 2y}$$

$$x = \frac{y(1+i\sqrt{7})}{2}$$

Assume $y = 1 \Rightarrow x = \frac{(1+i\sqrt{7})}{2}$

for λ_2,
$$\boxed{x_2 = \begin{bmatrix} (1+i\sqrt{7})/2 \\ 1 \end{bmatrix}}$$

(2) dot product of eigenvectors of M

$$x_1 \cdot x_2 = \begin{bmatrix} (1-i\sqrt{7})/2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} (1+i\sqrt{7})/2 \\ 1 \end{bmatrix}$$

$$i^2 = -1$$

$$= \frac{1^2 - 4i^2}{4} + 1 \Rightarrow \frac{+8+4}{4} = \frac{+12}{4} = \cancel{4/2}$$

$$\boxed{= 3}$$

(3) dot product of Eigen vectors of N

$$N = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}$$

Eigenvalue - Eigenvector pair $\Rightarrow (\lambda I - N)x = 0$

$$\Rightarrow |(\lambda I - N)| = 0$$

$$\lambda I - N = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} \lambda-5 & 3 \\ 3 & \lambda-6 \end{bmatrix}$$

$$|\lambda I - N| = \begin{vmatrix} \lambda-5 & 3 \\ 3 & \lambda-6 \end{vmatrix} = (\lambda-5)(\lambda-6) - 9$$

$$\Rightarrow \lambda^2 - 11\lambda + 30 - 9 = 0$$

$$\lambda^2 - 11\lambda + 21 = 0$$

Solve

$$\lambda = \frac{-(-11) \pm \sqrt{121 - 84}}{2} = \frac{11 \pm \sqrt{37}}{2}$$

$$\lambda_1 = \frac{11 + \sqrt{37}}{2} \; ; \; \lambda_2 = \frac{11 - \sqrt{37}}{2}$$

$(\lambda I - A)x = 0$

for λ_1 eigen vector x_1

$$\begin{bmatrix} \frac{11+\sqrt{37}}{2} - 5 & 3 \\ 3 & \frac{11+\sqrt{37}}{2} - 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{11+\sqrt{37}-10}{2} x_1 + 3y = 0$$

$$\frac{1+\sqrt{37}}{\cancel{2}} x_1 + 6y = 0$$

$$x_1 = \frac{-6y}{1+\sqrt{37}}$$

for $y = 1$; $\quad x_1 = \frac{-6}{1+\sqrt{37}}$

$$\Rightarrow x_1 = \begin{bmatrix} -6/1+\sqrt{37} \\ 1 \end{bmatrix}$$

for λ_2 eigen vector x_2

$$\begin{bmatrix} \frac{11-\sqrt{37}}{2} - 5 & 3 \\ 3 & \frac{11-\sqrt{37}}{2} - 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-\sqrt{37}}{2} & 3 \\ 3 & \frac{-1+\sqrt{37}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1-\sqrt{37}}{2}x + 3y = 0$$

$$x = \frac{-6y}{1-\sqrt{37}}$$

for $y=1$, $x = \frac{-6}{1-\sqrt{37}}$

~~vector~~ $x_2 = \begin{bmatrix} -6/1-\sqrt{37} \\ 1 \end{bmatrix}$

dot product, $x_1 \cdot x_2 = \begin{bmatrix} \frac{-6}{1+\sqrt{37}} \\ 1 \end{bmatrix} \begin{bmatrix} \frac{-6}{1-\sqrt{37}} \\ 1 \end{bmatrix}$

$$= \frac{36}{1^2-37} - 1 \quad = \frac{36}{-36} + 1$$

$$= -1+1 = 0$$

(4) The dot product eigenvectors is zero. ⇒ The eigenvectors are orthogonal to each other.

This is because the matrix N is symmetric

⇒ $N = N^T$

(5) $Pt = 0$ Check if P is invertible

$P^{-1} = \frac{1}{0}$

$|P| = 16 - 16 = 0$

$\Rightarrow$ P is non invertible.

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2t_1 + 4t_2 = 0$

$t_1 = -2t_2$

$$t = \begin{bmatrix} -2t_2 \\ t_2 \end{bmatrix}$$

$Pt = 0 \Rightarrow$ when $t = \begin{bmatrix} -2t_2 \\ t_2 \end{bmatrix}$

trivial solution, $t_2 = 0$

$$\Rightarrow \begin{bmatrix} -2(0) \\ 0 \end{bmatrix} \Rightarrow t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(6) Non trivial Solution,

$$t = \begin{bmatrix} -2t_2 \\ t_2 \end{bmatrix}$$

if $t_2 = 1 \Rightarrow t = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$t_2 = 2 \Rightarrow t = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$t_2 = -1 \Rightarrow t = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

M

(7) $|M| = 12 - (2) = 14$

$$M^{-1} = \frac{1}{14} \begin{array}{cc} a_{11} & a_{12} \\ \multicolumn{2}{c}{\begin{bmatrix} +4 & +1 \\ -2 & +3 \end{bmatrix}^{T}} \\ a_{21} & a_{22} \end{array}$$

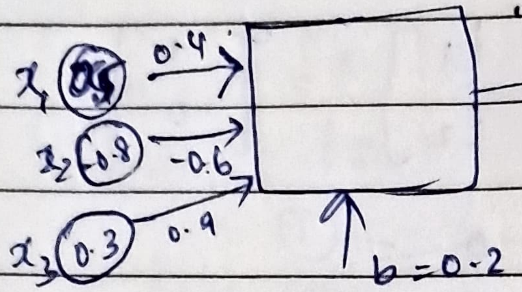
$$= \frac{1}{14} \begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ 1/14 & 3/14 \end{bmatrix}$$

$|M| \neq 0$ and M^{-1} exists.

$\Rightarrow Mt = 0$ implies $t = 0$ (trivial solution)

Part D

(1)



$$y = w_1 x_1 + w_2 x_2 + w_3 x_3 + b$$

$$= (0.5)(0.4) + (-0.8)(-0.6) + (0.3)(0.9) + 0.2$$

$$= 0.2 + 0.48 + 0.27 + 0.2$$

$$y = 1.15$$

(2)

$$\sigma(y) = \frac{1}{1+e^{-y}} = \frac{1}{1+e^{-1.15}}$$

$$= \frac{1}{1+0.316} = 0.759$$

$$\sigma(y) \simeq 0.76$$

(3) Rectified Linear Unit, $RELU(x) = max(0, x)$

$$RELU(y) = max(0, 1.15)$$

$$= 1.15$$

(4)

$$y = W \cdot X + b$$

$$= \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4+0.6 \\ 0.2+1.4 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$$

$$\text{ReLU}(y) = \begin{bmatrix} \max(0, 1.1) \\ \max(0, 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$$

(5)

i/p $x = \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$; $W_0 = [0.5 \quad -0.3]$

$b_0 = [0.1]$

o/p $y = Wx + b$

$$= [0.5 \quad -0.3] \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix} + [0.1]$$

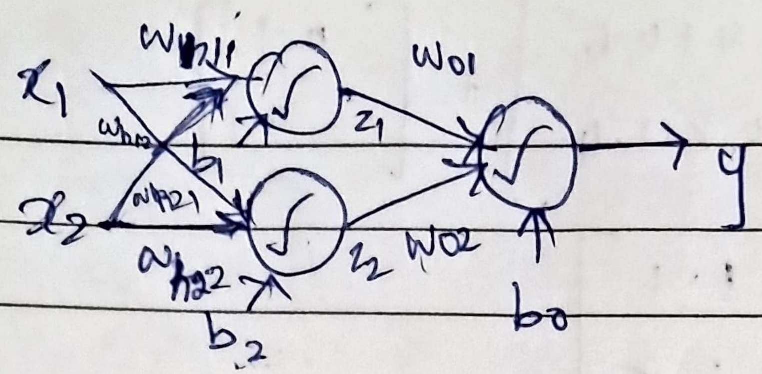
$$= (0.5)(1.1) + (-0.3)(1.5) + 0.1$$

$$= 0.55 - 0.45 + 0.1$$

$$y = 0.2$$

$$\sigma(y) = \frac{1}{1+e^{-0.2}} = \frac{1}{1+0.818} = \frac{1}{1.81} = 0.55$$

6



By forward pass, $z_i = g(W_h x + b_h)$
where g is the activation fn & $i = 1, 2$ represents neuron in hidden layer

let h be be the activation fn for o/p layer

$$y = h(W_o z + b_o)$$

Backward pass - $\frac{\partial L}{\partial x_1}$ calculation.

$$\frac{\partial L}{\partial x_1} = \sum_{i=1}^{2} \frac{\partial L}{\partial y} \cdot \frac{\partial y}{\partial z_i} \cdot \frac{\partial z_i}{\partial x_1}$$

Part E Gradient

① $f(x) = 2x^2 - 1$; $g(x) = 3x^2 + 4$; $h(x,y) = x^2 + y^2 + xy$

$$f'(x) = \frac{d f(x)}{dx} = 2\frac{d}{dx}(x^2) - \frac{d}{dx}(1)$$

$$= 2 \times 2x - 0$$

$$= 4x$$

$$f''(x) = \frac{d f'(x)}{dx} = \frac{d}{dx}(4x) = 4$$

② $$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + xy) = 2x + 0 + y(1)$$

$$= 2x + y$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial x}(x^2 + y^2 + xy) = 2y + x$$

③ Gradient of $h(x,y)$ is the collection of all its partial derivatives

$$\nabla h(x,y) = \begin{bmatrix} \partial h / \partial x \\ \partial h / \partial y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 2y + x \end{bmatrix}$$

④ $$\frac{d}{dx} f(g(x))$$

without

$$g(x) = 3x^2 + 4$$

$$f(x) = 2x^2 - 1$$

$$f(g(x)) = 2(3x^2 + 4)^2 - 1$$

$$= 2(9x^4 + 16 + 24x^2) - 1$$

$$= 18x^4 + 32 + 48x^2 - 1$$

$$= 18x^4 + 48x^2 + 31$$

$$\frac{d}{dx} f(g(x)) = 72x^3 + 96x$$

with

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx} f(g(x)) \cdot \frac{d}{dx} g(x)$$

$$= f'(g(x)) \cdot g'(x)$$

$$= f'(3x^2+4) \cdot \frac{d}{dx}(3x^2+4)$$

$$= f'(3x^2+4) \cdot 6x$$

$$\cancel{= \frac{d}{dx}\left[2(3x^2+4)-1\right] \cdot 6x}$$

$$\cancel{= \frac{d}{dx}(6x^2+7) \cdot 6x}$$

from ①, $f'(x) = 4x$ $\cancel{12x}$

$$\Rightarrow \frac{d}{dx} f(g(x)) = 4(3x^2+4) \cdot 6x$$

$$= (12x^2+16)\, 6x$$

$$= \underline{\underline{72x^3 + 96x}}$$