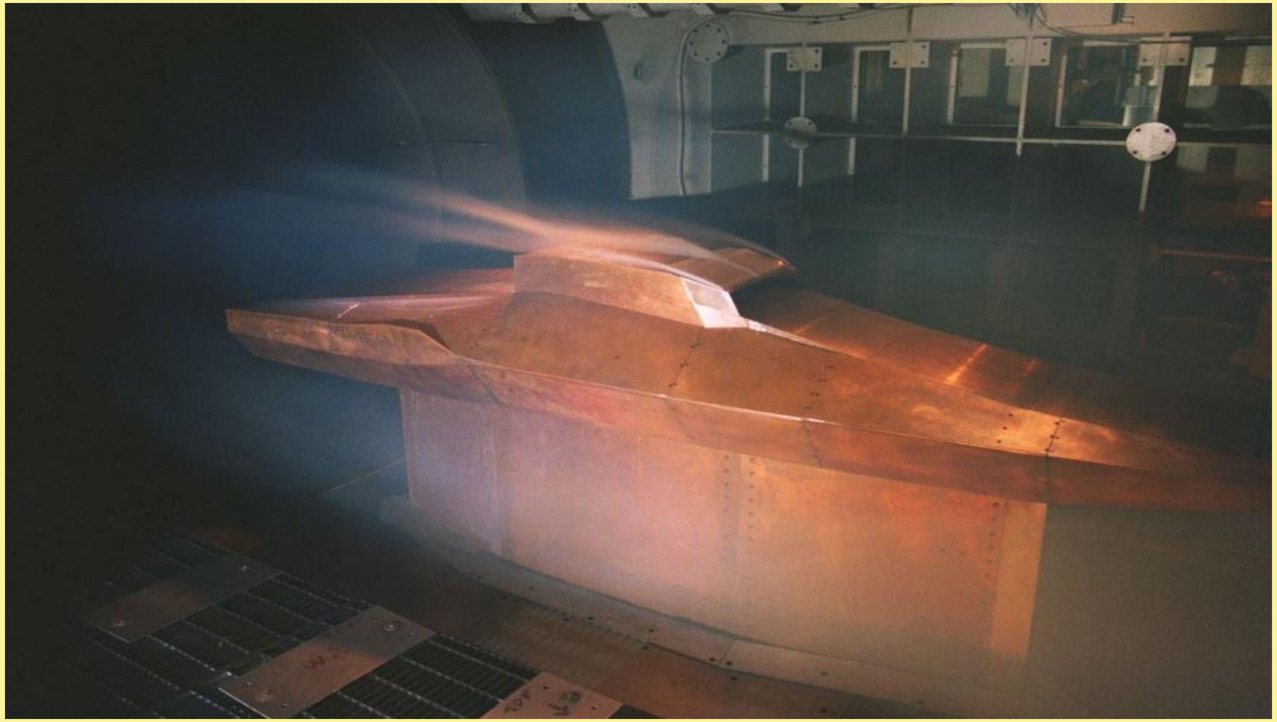
Dimensional Analysis



Hydraulics

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Introduction

- Many flow problems can only be investigated experimentally
- Few problems in fluids can be solved by analysis alone
- One must know how to plan experiments
- Correlate other experiments to a specific problem.

Introduction

- Usually, the goal is to make the experiment widely applicable
- Similitude is used to make experiments more applicable
- Laboratory flows are studied under carefully controlled conditions

Example Problem

Pipe-Flow Example: Pressure Drop per Unit Length

The pressure drop per unit length that develops along a pipe as the result of friction can not be explained analytically without the use of experimental data

First, we determine the important variables in the flow related to pressure drop:

$$\Delta\rho_l = f(D, \rho, \mu, V)$$

D is the diameter of the pipe, ρ is the density of the fluid, μ is the viscosity of the fluid, and V is the flow velocity.

Example Problem

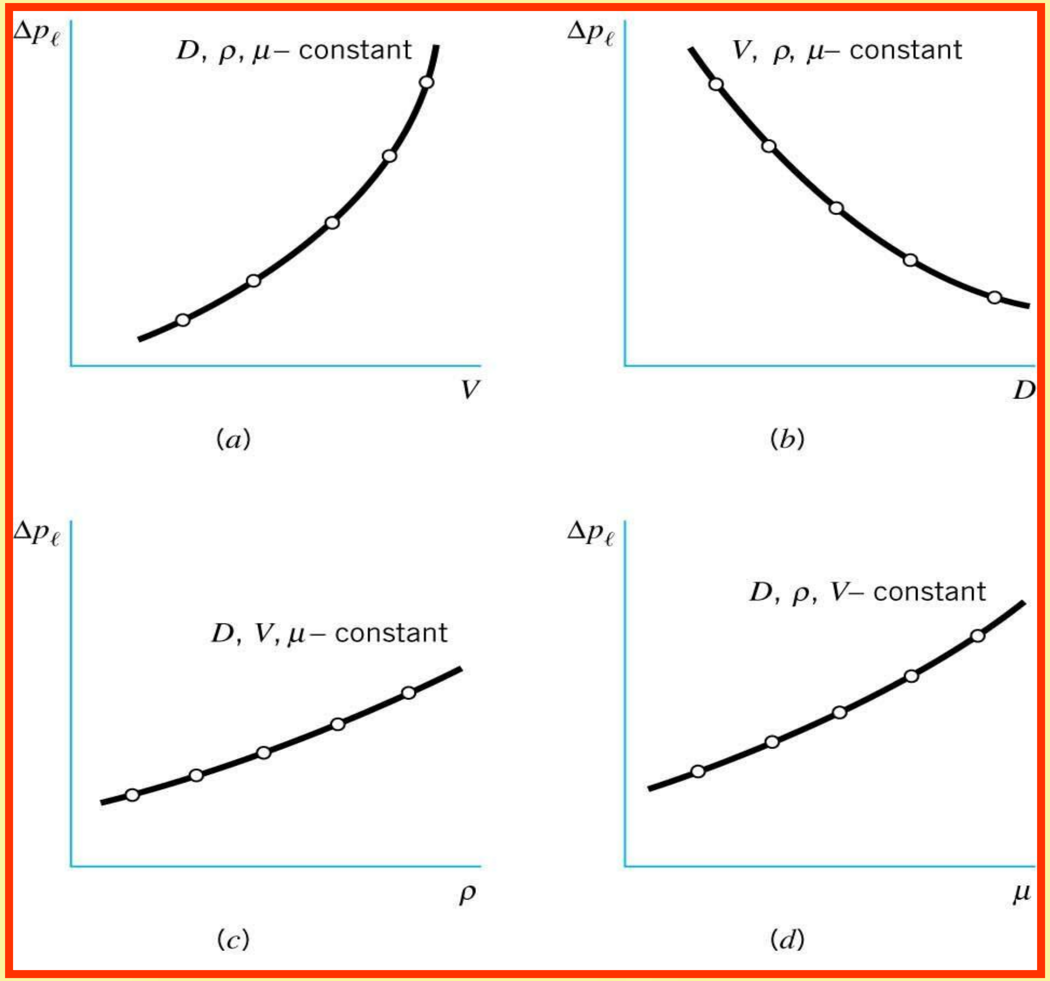
Pipe-Flow Example: Pressure Drop per Unit Length

$$\Delta\rho_l = f(D, \rho, \mu, V)$$

So, how do we approach this problem?

Logically, it seems that we could vary one variable at a time holding the other constants

Example Problem



So, now we have done five experiments for each plot with the other variables held constant (20 total experiments).

What have we gained?

Our analysis is very narrow and specific, not widely applicable.

Now what if do 10 points for each variable, and let the other three variable vary for 10 values.

Example Problem

Total combinations 10x10x10x10:

10,000 experiments!

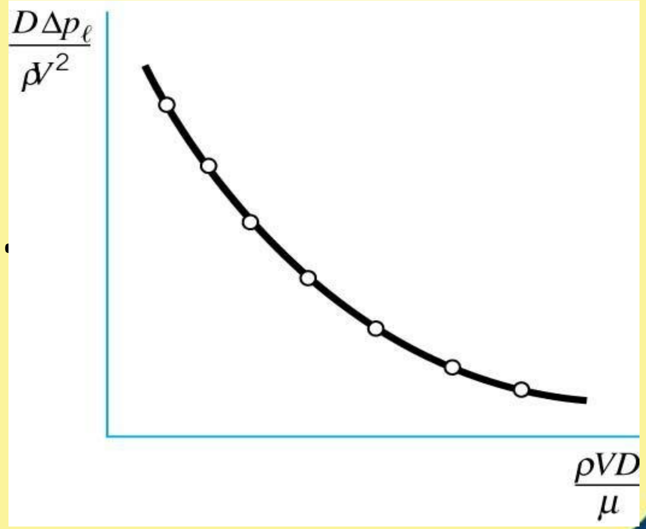
More applicable, but very expensive, At $50/experiment =$500,000

Dimensional Analysis

Fortunately, there is a simpler approach: Dimensionless Groups

The original list of variables can be collected into two dimensionless groups.

$$\frac{D\Delta p_l}{\rho V^2} = \Phi\left(\frac{\rho V D}{\mu}\right)$$



Now instead of working with 5 variables, there are only two.

The experiments would consist of varying the independent variable and determining the dependent variable which is related to the pressure drop.

Now, the curve is universal for any smooth walled, laminar pipe flow.

Dimensional Analysis

Dimensions are Mass (M), Length (L), Time (T), Force (F or MLT^{-2})

Then, we check our dimensionless groups

$$V \doteq LT^{-1}$$

$$\mu \doteq FL^{-2}T$$

$$\Delta p_l \doteq FL^{-3}$$

$$D \doteq L$$

$$\rho \doteq FL^{-4}T^2$$

Substituting, we see no dimensions on our two variables:

$$\frac{D\Delta p_l}{\rho V^2} \doteq \frac{L(F/L^3)}{(FL^{-4}T^2)(LT^{-1})^2} \doteq F^0L^0T^0$$

$$\frac{\rho VD}{\mu} \doteq \frac{(FL^{-4}T^2)(LT^{-1})(L)}{(FL^{-2}T)} \doteq F^0L^0T^0$$

*** Not only have we reduced the number of variables from five to two, but the dimensionless plot is independent of the system of units used.**

So, how do we know what groups of dimensionless variables to form?

Buckingham Pi Theorem

Buckingham Pi Theorem is a systematic way of forming dimensionless groups:

If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k-r independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables.

The dimensionless products are referred to as "pi terms".

Requires that equation have dimensional homogeneity:

$$\boldsymbol{u_1 = f(u_2, u_3, \dots, u_k)}$$

Dimensions on the left side = dimension on the right side

Then if pi terms are formed, they are dimensionless products one each side.

$$\Pi_1 = \Phi(\Pi_2, \Pi_3, \dots, \Pi_{k-r})$$

*The required number of pi terms is fewer than the original number of variables by r, where r is the minimum number of reference dimensions needed to describe the original set of variables (M, L, T, or F).

Buckingham Pi Theorem

Systematic Approach: Example Pipe Flow

Step 1. List all the variables that are involved in the problem:

$$\Delta\rho_l = f(D, \rho, \mu, V)$$

Step 2. Express each of the variables in terms of basic dimensions:

Step 3. Determine the require number of pi terms:

The basic dimensions are F,L,T or M,L,T, noting F = MLT^{-2}, 3 total

$V \doteq LT^{-1}$

$\mu \doteq FL^{-2}T$

$\Delta p_l \doteq FL^{-3}$

$D \doteq L$

$\rho \doteq FL^{-4}T^2$

Then the number of pi terms are the number of variables, 5 minus the number of basic dimensions, 3. So there should be two pi terms for this case.

Buckingham Pi Theorem

Step 4. Select a number of repeating variables, where the number required is equal to the number of reference dimensions.

We choose three independent variables as the repeating variables—there can be more than one set of repeating variables.

Repeating variables: D, V, and ρ

We note the these three variables by themselves are dimensionally independent; you can not form a dimensionless group with them alone.

Buckingham Pi Theorem

Step 5. Form a pi term by multiplying one of the non repeating variables by the product of repeating variables each raised to an exponent that will make the combination dimensionless. The first group chosen usually includes the dependent variable.

$$\Pi_1 = \Delta p_l D^a V^b \rho^c$$

Product should be dimensionless:

So, we need to solve for the exponent values.

$$\left(FL^{-3}\right)(L)^a\left(LT^{-1}\right)^b\left(FL^{-4}T^2\right)^c \doteq F^0L^0T^0$$

Buckingham Pi Theorem

Step 5 (continued).

$$1 + c = 0 \quad (for\ F)$$

$$-3 + a + b - 4c = 0 \quad (for\ L)$$

$$-b + 2c = 0 \quad (for\ T)$$

Solving the set of algebraic equations, we obtain: a = 1, b = -2, c = -1:

$$\Pi_1 = \frac{\Delta p_l D}{\rho V^2}$$

μ is a remaining non repeating variable, so we can form another group:

$$\Pi_2 = \mu D^a V^b \rho^c$$

Buckingham Pi Theorem

$\left(FL^{-2}T\right)(L)^a\left(LT^{-1}\right)^b\left(FL^{-4}T^2\right)^c \doteq F^0L^0T^0$ Solving, a = -1, b = -1, and c = -1

$$\Longrightarrow \quad \Pi_2 = \frac{\mu}{DV\rho}$$

Step 6. Repeat Step 5. for each of the remaining repeating variables.

We could have chosen D, V and μ as another repeating group (later).

Step 7. Check all the resulting pi terms to make sure they are dimensionless.

$$\Pi_1 = \frac{\Delta p_l D}{\rho V^2} \doteq \frac{\left(FL^{-3}\right)(L)}{\left(FL^{-4}T^2\right)\left(LT^{-1}\right)^2} \doteq F^0L^0T^0$$

$$\Pi_2 = \frac{\mu}{DV_\rho} = \frac{\left(FL^{-2}T\right)}{(L)\left(FL^{-4}T^2\right)\left(LT^{-1}\right)} \doteq F^0L^0T^0$$

Step 8. Express the final form as relationship among the pi terms and think about what it means.

$$\Pi_1 = \Phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

For our case,

$$\frac{\Delta p_l D}{\rho V^2} = \phi'\left(\frac{\mu}{DV\rho}\right) \quad \text{or} \quad \frac{D\Delta p_l}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Reynolds Number

Pressure drop depends on the Reynolds Number.

Choosing Variables: Buckingham Pi

One of the most important aspects of dimensional analysis is choosing the variables important to the flow, however, this can also prove difficult.

We do not want to choose so many variables that the problem becomes cumbersome.

Often we use engineering simplifications, to obtain first order results sacrificing some accuracy, but making the study more tangible.

Most variables fall in to the categories of geometry, material property, and external effects:

Choosing Variables: Buckingham Pi

External Effects: Denotes a variable that produces a change in the system, pressures, velocity, or gravity.

Material Properties: Bind the relationship between external effects and the fluid response. Viscosity, and density of the fluid.

Geometry: lengths and angles, usually very important and obvious variables.

Choosing Variables: Buckingham Pi

We must choose the variables such that they are independent:

$$\boldsymbol{f(p, q, r, \dots, u, v, w, ..) = 0} \qquad \boldsymbol{q = f_1(u, v, w, ..)}$$

Then, u, v, and w are not necessary in f if they only enter the problem through q, or q is not necessary in f.

Class Question

The time period T of water surface waves is known to depend on the wave length λ, depth of flow D, density of the fluid ρ, acceleration due to gravity g and surface tension σ. Obtain the dimensionless form of the functional relationship

Solution:

$$T = fn(\lambda , D, \rho, g, \sigma)$$

List of the dimensions of each variable as follows

T	λ	D	ρ	g	σ
$[T]$	$[L]$	$[L]$	$[ML^{-3}]$	$[LT^{-2}]$	$[MT^{-2}]$

There are a total of six variables, n=6

Number of primary dimensions, m=3

Hence there are (6-3)=3 dimensionless Pi terms

Select λ, g, and ρ as repeating variables.

I term: $\pi_1 = T\lambda^a g^b \rho^c$

$$M^0L^0T^0 = (T)(L)^a(LT^{-2})^b(ML^{-3})^c$$

$1 - 2b = 0$ $\qquad \therefore b = 1/2$

$c = 0$

$a + b - 3c = 0$ $\qquad \therefore a = -1/2$

$$\therefore \pi_1 = \frac{T\sqrt{g}}{\sqrt{\lambda}}$$

II term: $\pi_2 = D\lambda^a g^b \rho^c$

By inspection it is easy to see that $\pi_2 = D/\lambda$

III term: $$\pi_3 = \sigma\lambda^a g^b \rho^c$$

$$M^0L^0T^0 = (MT^{-2})(L)^a(LT^{-2})^b(ML^{-3})^c$$

$1 + c = 0$ $\qquad \therefore c = -1$

$-2 - 2b = 0$ $\qquad \therefore b = -1$

$a + b - 3c = 0$ $\qquad \therefore a = -2$

$$\therefore \pi_3 = \frac{\sigma}{\lambda^2 g\rho}$$

Hence $$\frac{T\sqrt{g}}{\sqrt{\lambda}} = fn\left[\frac{D}{\lambda}, \frac{\sigma}{\lambda^2 g\rho}\right]$$

Choosing Variables: Buckingham Pi

1. Clearly define the problem. What is the main variable of interest (the dependent variable)?
2. Consider the basic laws that govern the phenomenon. Even a crude theory that describes the essential aspects of the system may be helpful.
3. Start the variable selection process by grouping the variables into three broad classes: geometry, material properties, and external effects.
4. Consider other variables that may not fall into one of the above categories. For example, time will be an important variable if any of the variables are time dependent.

Choosing Variables: Buckingham Pi

5. Be sure to include all quantities that enter the problem even though some of them may be held constant (e.g., the acceleration of gravity, g). For a dimensional analysis it is the dimensions of the quantities that are important—not specific values!

6. Make sure that all variables are independent. Look for relationships among subsets of the variables.

Uniqueness of Pi term

Now, back to our example of pressure drop, but choose a different repeating group (D, V, μ).

If we evaluate, we find

$$\frac{\Delta p_l D^2}{V_\mu}$$

The other pi term remains the same.

$$\frac{\Delta p_l D^2}{V_\mu} = \phi_1\left(\frac{\rho V D}{\mu}\right)$$

Uniqueness of Pi term

But, we note that the L.H.S, is simply what we had before multiplied by the Reynolds Number.

$$\left(\frac{\Delta p_l D}{\rho V^2}\right)\left(\frac{\rho V D}{\mu}\right) = \frac{\Delta p_l D^2}{V_\mu}$$

There is not a unique set of pi terms, but rather a set number of pi terms. In this case there are always two.

Uniqueness of Pi term

If we take three pi terms, we can form another by multiplying

$$\Pi_1 = \phi(\Pi_2, \Pi_3)$$

$$\Pi_2' = \Pi_2{}^a \Pi_3{}^b \quad \Longrightarrow \quad \Pi_1 = \phi(\Pi_2', \Pi_3)$$

or

$$\Pi_1 = \phi(\Pi_2, \Pi_2')$$

*Often the set of pi terms chosen is based on previous flow analysis.

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V l}{\mu}$	Reynolds number, Re	$\frac{Inertia\ force}{Viscous\ force}$	Generally of importance in all types of fluid dynamics Problems
$\frac{V}{\sqrt{gl}}$	Froude number, Fr	$\frac{Inertia\ force}{gravitational\ force}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{pressure\ force}{inertia\ force}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_\upsilon}$	Cauchy number, Ca	$\frac{Inertia\ Force}{Compressibility\ force}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number, Ma	$\frac{Inertia}{Compressibility\ force}$	Flows in which the compressibility of the fluid is important
$\frac{\omega l}{V}$	Strouhal number, St	$\frac{Inertia\ (local)\ force}{Inertia\ (convective) force}$	Unsteady flow with a characteristic frequency of Oscillation
$\frac{\rho V^2 l}{\sigma}$	Weber number, We	$\frac{Inertia}{Surface\ tension\ force}$	Problems in which surface tension is important

Dimensionless Group

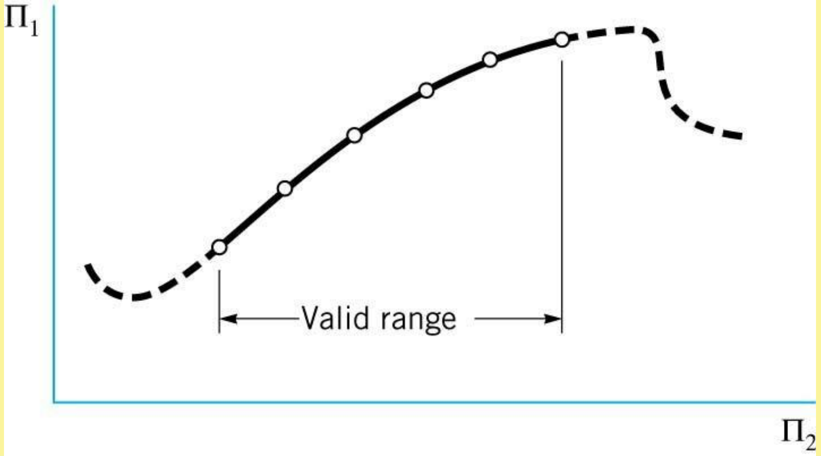
If only one pi variable exists in a fluid phenomenon, the functional relationship must be a constant.

$$\Pi_1 = C$$

The constant must be determine from experiment.

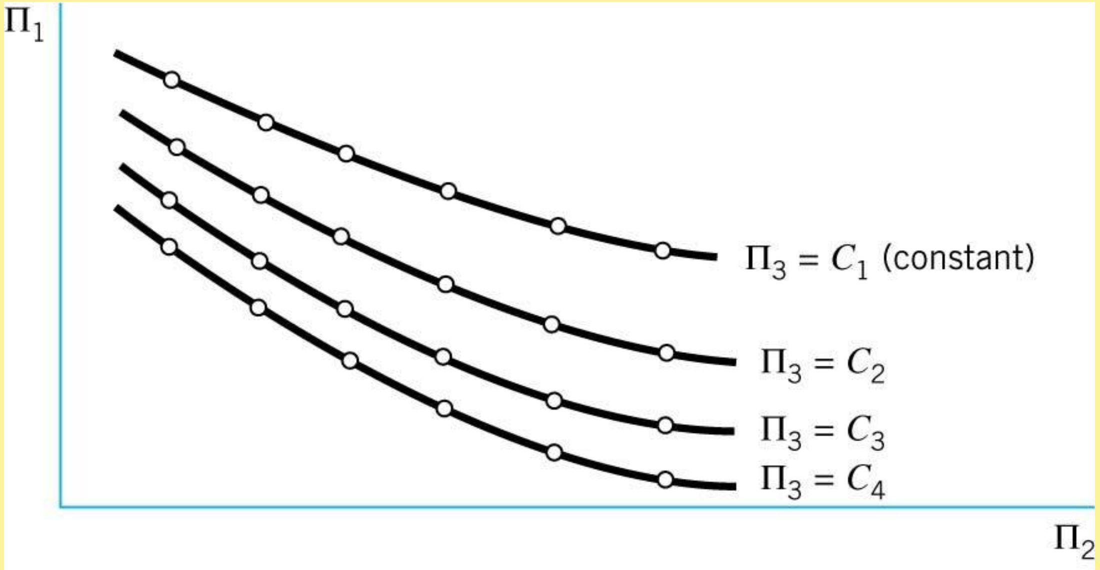
If we have two pi terms, we must be careful not to over extend the range of applicability, but the relationship can be presented pretty easily graphically:

$$\Pi_1 = \phi(\Pi_2)$$



Dimensionless Group

If we have three pi groups, we can represent the data as a series of curves, however, as the number of pi terms increase the problem becomes less tractable, and we may resort to modeling specific characteristics.



Similitude

"Model":

Often we want to use models to predict real flow phenomenon.

We obtain similarity between a model and a prototype by equating pi terms.

In these terms we must have geometric, kinematic, and dynamic similarity.

Geometric similarity: A model and a prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear scale ratio.

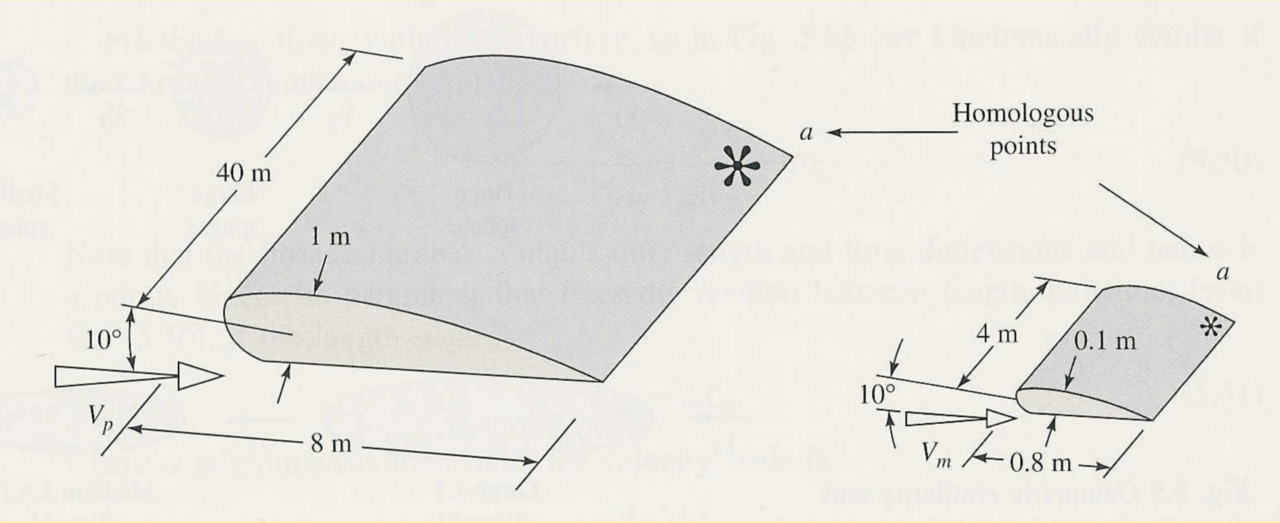
$$\frac{l_1}{l_2} = \frac{l_{1m}}{l_{2m}} \Longrightarrow \frac{l_{1m}}{l_1} = \frac{l_{2m}}{l_2}$$

All angles are preserved.
All flow directions are the same.
Orientations must be the same.

*Things that must be considered that are over-looked: roughness, scale of fasteners protruding.

Similitude

Geometric Similarity: Scale 1/10th



Similitude

Kinematic Similarity: Same length scale ratio and same time-scale ratio. The motion of the system is kinematically similar if homologous particles lie at homologous locations at homologous times.

This requires equivalence of dimensionless groups: **Reynolds Number, Froude Number, Mach numbers, etc.**

For a flow in which Froude Number and Reynolds Number is important:

Length scale:

Froude Number similarity: $$\frac{V_m}{\sqrt{g_m l_m}} = \frac{V}{\sqrt{gl}} \Rightarrow \frac{V_m}{V} = \sqrt{\frac{l_m}{l}} = \sqrt{\lambda_l}$$

Similitude

Reynolds Number similarity:

$$\frac{\rho_m V_m l_m}{\mu_m} = \frac{\rho V l}{\mu} \Rightarrow \frac{V_m}{V} = \frac{\mu_m}{\mu}\frac{\rho}{\rho_m}\frac{l}{l_m}$$

Then, $$\frac{\mu_m/\rho_m}{\mu/\rho} = \frac{\upsilon_m}{\upsilon} = (\lambda_l)^{3/2}$$ **Might relax condition.**

<u>Time scale:</u>

$$\frac{t_m}{t} = \frac{l_m/v_m}{l/v} = \sqrt{\lambda_l}$$

Similitude

Dynamic Similarity: The same length scale, time-scale, and force scale is required.

First, satisfy geometric, and kinematic similarity. Dynamic similarity then exists if the force and pressure coefficient are the same.

In order to ensure that the force and pressure coefficients are the same:

For compressible flow: Re, Mach, and specific heat ratio must be matched.

For incompressible flow with no free surface: Re matching only.

For incompressible flow with a free surface: Re, Froude, and possibly Weber number (surface tension effects), and cavitation number must be matched.

Model Scales

Fluid flow models are usually designed for one most dominant force and occasionally for two

If dominant force is gravity then Froude number must be same in model and prototype

If dominant force is viscous force then Reynolds number must be same in model and prototype

Class Problem

For Froude model law, find the ratios of velocity, discharge, force, work and power in terms of length scale

Solution: **In Froude model law the model and prototype Froude numbers are same. Hence**

$$(Fr)_m = \frac{V_m}{\sqrt{gL_m}} = (Fr)_p = \frac{V_p}{\sqrt{gL_p}}$$

The gravity g is same for both model and prototype. Hence if length ration $L_m/L_p=L_r$ and $\rho_m/\ \rho_p=\ \rho_r$

Velocity ratio V_r

$$V_r = \frac{V_m}{V_p} = \sqrt{\frac{L_m}{L_p}} = \sqrt{L_r}$$

Class Problem

Solution contd.

Discharge ratio Q_r = (Velocity*area)$_r$ $$Q_r = V_r L_r^2 = L_r^{5/2}$$

Force ratio F_r = $(\rho L^2 V^2)_r$ $$F_r = (\rho_r L_r^3)$$

Work ratio E_r = Energy ratio= Force*distance $$E_r = (\rho_r L_r^4)$$

Power = Force * Velocity $$P_r = (\rho_r L_r^3) * (L_r^{1/2}) = \rho_r L_r^{7/2}$$

Distorted Models

The idea behind similitude is that we simply equate all the Pi terms

In reality it is always not possible to satisfy all the known requirements so as to be able to equate all the Pi terms

Example; Study of open channel or free surface flows

Froude number similarity gives

$$\frac{V_m}{\sqrt{g_m l_m}} = \frac{V_p}{\sqrt{g_p l_p}}$$

Under same gravitational field

$$\frac{V_m}{V_p} = \sqrt{\frac{l_m}{l_p}} = \sqrt{\lambda_r}$$

Distorted Models

Reynolds number similarity gives
$$\frac{\rho_m V_m l_m}{\mu_m} = \frac{\rho_p V_p l_p}{\mu_p}$$

Velocity scale is
$$\frac{V_m}{V_p} = \frac{\mu_m}{\mu_p}\frac{\rho_p}{\rho_m}\frac{l_p}{l_m}$$

Since velocity scale must be equal to $\sqrt{\lambda_r}$ due to Froude number similarity

Ratio of kinematic viscosity
$$\frac{\frac{\mu_m}{\rho_m}}{\frac{\mu_p}{\rho_p}} = \frac{\nu_m}{\nu_p} = (\lambda_r)^{1.5}$$

Very difficult to find such liquids that satisfy the above relation

Distorted Models

To overcome difficulties like before the modeling is done using distorted scales; Distorted models

In open channel the vertical dimension is used to simulate Froude's law while other

two dimension are scaled to suit available space

In distorted models

Horizontal scale (length and width) L_r Vertical scale (depth) h_r

Cross sectional area ratio= area in model/area in prototype $\dfrac{(By)_m}{(By)_p} = L_r h_r$

Distorted Models

Froude number ratio=1 → $\frac{F_m}{F_p} = 1 = \frac{\frac{V_m}{\sqrt{gy_m}}}{\frac{V_p}{\sqrt{gy_p}}}$ → $\frac{(\frac{V_m}{V_p})^2}{\frac{y_m}{y_p}} = \frac{V_r^2}{h_r} = 1$

Thus velocity ratio V_r $V_r = \sqrt{h_r}$

Discharge ratio Q_r = Area ratio*velocity ratio $= L_r h_r^{1.5}$

Slope ratio S_r $= \frac{S_m}{S_p} = \frac{h_r}{L_r}$

Time ratio T_r $= \frac{L_r}{V_r} = \frac{L_r}{\sqrt{h_r}}$

For distorted models find Manning's ratio n_r

Class Problem

In a tidal model, the horizontal scale ratio is 1/500. The vertical scale is 1/50. What model period would correspond to a prototype period of 12 hours .

Solution: Horizontal Scale = L_r= 1/500 Vertical Scale = h_r= 1/50

Time ratio T_r

$$T_r = \frac{L_r}{\sqrt{h_r}} = \frac{1/500}{\sqrt{1/50}} = 0.01414$$

$$T_r = \frac{T_m}{T_p} \quad \rightarrow \quad T_m = T_p T_r = (12*60*60)*0.01414 = 610s$$