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From the Training of the USA IMO Team

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Preface

This book contains 104 of the best problems used in the training and testing of the U.S. International Mathematical Olympiad (IMO) team. It is not a collection of very difficult, and impenetrable questions. Rather, the book gradually builds students' number-theoretic skills and techniques. The first chapter provides a comprehensive introduction to number theory and its mathematical structures. This chapter can serve as a textbook for a short course in number theory. This work aims to broaden students' view of mathematics and better prepare them for possible participation in various mathematical competitions. It provides in-depth enrichment in important areas of number theory by reorganizing and enhancing students' problem-solving tactics and strategies. The book further stimulates students' interest for the future study of mathematics.

In the United States of America, the selection process leading to participation in the International Mathematical Olympiad (IMO) consists of a series of national contests called the American Mathematics Contest 10 (AMC 10), the American Mathematics Contest 12 (AMC 12), the American Invitational Mathematics Examination (AIME), and the United States of America Mathematical Olympiad (USAMO). Participation in the AIME and the USAMO is by invitation only, based on performance in the preceding exams of the sequence. The Mathematical Olympiad Summer Program (MOSP) is a four-week intensive training program for approximately fifty very promising students who have risen to the top in the American Mathematics Competitions. The six students representing the United States of America in the IMO are selected on the basis of their USAMO scores and further testing that takes place during MOSP. Throughout MOSP, full days of classes and extensive problem sets give students thorough preparation in several important areas of mathematics. These topics include combinatorial arguments and identities, generating functions, graph theory, recursive relations, sums and products, probability, number theory, polynomials, functional equations, complex numbers in geometry, algorithmic proofs, combinatorial and advanced geometry, functional equations, and classical inequalities.

Olympiad-style exams consist of several challenging essay problems. Correct solutions often require deep analysis and careful argument. Olympiad questions can seem impenetrable to the novice, yet most can be solved with elementary high school mathematics techniques, when cleverly applied.

Here is some advice for students who attempt the problems that follow.

- Take your time! Very few contestants can solve all the given problems.
- Try to make connections between problems. An important theme of this work is that all important techniques and ideas featured in the book appear more than once!
- Olympiad problems don't "crack" immediately. Be patient. Try different approaches. Experiment with simple cases. In some cases, working backward from the desired result is helpful.
- Even if you can solve a problem, do read the solutions. They may contain some ideas that did not occur in your solutions, and they may discuss strategic and tactical approaches that can be used elsewhere. The solutions are also models of elegant presentation that you should emulate, but they often obscure the tortuous process of investigation, false starts, inspiration, and attention to detail that led to them. When you read the solutions, try to reconstruct the thinking that went into them. Ask yourself, "What were the key ideas? How can I apply these ideas further?"
- Go back to the original problem later, and see whether you can solve it in a different way. Many of the problems have multiple solutions, but not all are outlined here.
- Meaningful problem solving takes practice. Don't get discouraged if you have trouble at first. For additional practice, use the books on the reading list.

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Many problems are either inspired by or adapted from mathematical contests in different countries and from the following journals:

- *The American Mathematical Monthly*, United States of America
- *Crux*, Canada
- *High School Mathematics*, China
- *Mathematics Magazine*, United States of America
- *Revista Matematică Timişoara*, Romania

We did our best to cite all the original sources of the problems in the solution section. We express our deepest appreciation to the original proposers of the problems.

Abbreviations and Notation

Abbreviations

AHSME	American High School Mathematics Examination
AIME	American Invitational Mathematics Examination
AMC10	American Mathematics Contest 10
AMC12	American Mathematics Contest 12, which replaces AHSME
APMC	Austrian–Polish Mathematics Competition
ARML	American Regional Mathematics League
Balkan	Balkan Mathematical Olympiad
Baltic	Baltic Way Mathematical Team Contest
HMMT	Harvard–MIT Math Tournament
IMO	International Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
MOSP	Mathematical Olympiad Summer Program
Putnam	The William Lowell Putnam Mathematical Competition
St. Petersburg	St. Petersburg (Leningrad) Mathematical Olympiad

Notation for Numerical Sets and Fields

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the set of integers modulo n
$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set of nonnegative integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers
$\mathbb{Q}^0$	the set of nonnegative rational numbers
$\mathbb{Q}^n$	the set of n-tuples of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}^0$	the set of nonnegative real numbers
$\mathbb{R}^n$	the set of n-tuples of real numbers
$\mathbb{C}$	the set of complex numbers
$[x^n](p(x))$	the coefficient of the term x^n in the polynomial $p(x)$

Notation for Sets, Logic, and Number Theory

$\|A\|$	the number of elements in the set A
$A \subset B$	A is a proper subset of B
$A \subseteq B$	A is a subset of B
$A \setminus B$	A without B (set difference)
$A \cap B$	the intersection of sets A and B
$A \cup B$	the union of sets A and B
$a \in A$	the element a belongs to the set A
$n \mid m$	n divides m
$\gcd(m, n)$	the greatest common divisor of m, n
$\operatorname{lcm}(m, n)$	the least common multiple of m, n
$\pi(n)$	the number of primes $\leq n$
$\tau(n)$	number of divisors of n
$\sigma(n)$	sum of positive divisors of n
$a \equiv b \pmod{m}$	a and b are congruent modulo m
φ	Euler's totient function
$\operatorname{ord}_m(a)$	order of a modulo m
μ	Möbius function
$\overline{a_k a_{k-1} \ldots a_0}_{(b)}$	base-b representation
$S(n)$	the sum of digits of n
$(f_1, f_2, \ldots, f_m)$	factorial base expansion
$\lfloor x \rfloor$	floor of x
$\lceil x \rceil$	celling of x
$\{x\}$	fractional part of x
e_p	Legendre's function
$p^k \\| n$	p^k fully divides n
f_n	Fermat number
M_n	Mersenne number