Notes on Poisson Model for Big MRA

William Leeb

1 Setup

We consider the following observation model. X = (X[0], ..., X[L-1]) is a random vector of length L, drawn from some fixed distribution. For fixed n, we observe a random vector Y of length n + L, generated as follows. Points are chosen in $\{1, ..., n\}$ according to a Poisson process with parameter γn . For each point i that is chosen from 1 to n, a random vector X from the distribution is then placed in the large vector, with element 0 at location i, with overlapping vectors being added together.

If M_i denotes the number of hits at location $i, 1 \leq i \leq n$, then by definition of the Poission process M_i 's are iid and $M_i \sim \operatorname{Poisson}(\gamma)$. Conditional on the value of $M = (M_1, \ldots, M_n)$, if we let $X_1^i, \ldots, X_{M_i}^i$ denote the random vectors with position 0 located at i, then $X_{k_1}^i$ and $X_{k_2}^i$ are independent for $k_1 \neq k_2$.

With this notation, if $Y \in \mathbb{R}^{n+L}$ is the observed vector, we can write each entry as:

$$Y[i] = \sum_{j=0}^{L-1} \sum_{k=1}^{M_{i-j}} X_k^{i-j}[j].$$
 (1)

We will denote by \mathcal{M}_l the moments of X:

$$\mathcal{M}_1[i] = \mathbb{E}X[i], \quad 0 \le i \le L - 1,\tag{2}$$

$$\mathcal{M}_2[i,j] = \mathbb{E}X[i|X[j], \quad 0 \le i, j \le L - 1, \tag{3}$$

and

$$\mathcal{M}_3[i, j, k] = \mathbb{E}X[i]X[j]X[k], \quad 0 \le i, j, k \le L - 1.$$
 (4)

We will also denote by \mathcal{L}_l the autocorrelations of X:

$$\mathcal{L}_1 = \sum_{i=0}^{L-1} \mathcal{M}_1[i],\tag{5}$$

$$\mathcal{L}_2(\Delta) = \sum_{i=0}^{L-1} \mathcal{M}_2[i, i + \Delta], \tag{6}$$

and

$$\mathcal{L}_3(\Delta_1, \Delta_2) = \sum_{i=0}^{L-1} \mathcal{M}_3[i, i + \Delta_1, i + \Delta_2]. \tag{7}$$

Note that in the strongly-separated model, the first three observed moments are, respectively, \mathcal{L}_1 , $\mathcal{L}_2(\Delta)$, and $\mathcal{L}_3(\Delta_1, \Delta_2)$.

In this notation, we will show that the first moment of the data is $\gamma \mathcal{L}_1$, the second moment vector is $(\gamma \mathcal{L}_1)^2 + \gamma \mathcal{L}_2(\Delta)$, and the third moment matrix is $(\gamma \mathcal{L}_1)^3 + \gamma \mathcal{L}_1 \cdot (\gamma \mathcal{L}_2(\Delta_1) + \gamma \mathcal{L}_2(\Delta_2) + \gamma \mathcal{L}_2(\Delta_2 - \Delta_1)) + \gamma \mathcal{L}_3(\Delta_1, \Delta_2)$. In particular, from the first three moments of the Poisson process model, once can recover the first three moments from the strongly-separated model, with the Poisson rate γ playing the role of the "occupancy factor". So if recovery is possible for the strongly-separated model, it is also possible for the Poisson process model.

2 The first moment of Y

To compute the first moment of Y, we will first condition on $M = (M_1, \ldots, M_n)$, and then average over M. We have:

$$\mathbb{E}[Y[i]|M] = \sum_{j=0}^{L-1} \sum_{k=1}^{M_{i-j}} \mathbb{E}X_k^{i-j}[j] = \sum_{j=0}^{L-1} \sum_{k=1}^{M_{i-j}} \mathcal{M}_1[j] = M_{i-j} \sum_{j=0}^{L-1} \mathcal{M}_1[j].$$
 (8)

Now taking expectations over M we see:

$$\mathbb{E}Y[i] = \gamma \sum_{i=0}^{L-1} \mathcal{M}_1[j] = \gamma \mathcal{L}_1. \tag{9}$$

3 The second moment of Y

Again, we will condition on M first, and then take the expectation over M. Fix $i_1 \neq i_2$, and let $\Delta = i_2 - i_1$. Then:

$$Y_{i_1}Y_{i_2} = \sum_{j_1=0}^{L-1} \sum_{j_2=0}^{L-1} \sum_{k_1=1}^{M_{i_1-j_1}} \sum_{k_2=1}^{M_{i_2-j_2}} X_{k_1}^{i_1-j_1}[j_1] X_{k_2}^{i_2-j_2}[j_2].$$
 (10)

We break up the double sum over j_1 and j_2 into two terms: one where $j_2 \neq j_1 + \Delta$, and one where $j_2 = j_1 + \Delta$ or equivalently $i_1 - j_1 = i_2 - j_2$. In the first case, all the terms are independent, and so the expectation factors. In the second case, when $k_1 \neq k_2$ we have independence, but otherwise not. This

gives (all expectations are conditional on M):

$$\mathbb{E}Y_{i_{1}}Y_{i_{2}} = \sum_{j_{1}=0}^{L-1} \sum_{j_{2}=0}^{L-1} \sum_{k_{1}=1}^{M_{i_{1}-j_{1}}} \sum_{k_{2}=1}^{M_{i_{2}-j_{2}}} \mathbb{E}X_{k_{1}}^{i_{1}-j_{1}}[j_{1}]X_{k_{2}}^{i_{2}-j_{2}}[j_{2}]$$

$$= \sum_{j_{1}-j_{2} \neq \Delta} \sum_{k_{1}} \sum_{k_{2}} \mathbb{E}X_{k_{1}}^{i_{1}-j_{1}}[j_{1}]X_{k_{2}}^{i_{2}-j_{2}}[j_{2}]$$

$$+ \sum_{j_{1}=0}^{L-1} \sum_{k_{1} \neq k_{2}} \mathbb{E}X_{k_{1}}^{i_{1}-j_{1}}[j_{1}]X_{k_{2}}^{i_{1}-j_{1}}[j_{1} + \Delta]$$

$$+ \sum_{j_{1}=0}^{L-1} \sum_{k_{1}=1}^{M_{i_{1}-j_{1}}} \mathbb{E}X_{k_{1}}^{i_{1}-j_{1}}[j_{1}]X_{k_{1}}^{i_{1}-j_{1}}[j_{1} + \Delta]$$

$$= \sum_{j_{1}-j_{2} \neq \Delta} M_{i_{1}-j_{1}} M_{i_{2}-j_{2}} \mathcal{M}_{1}[j_{1}] \mathcal{M}_{1}[j_{2}]$$

$$+ \sum_{j_{1}=0}^{L-1} M_{i_{1}-j_{1}} (M_{i_{1}-j_{1}} - 1) \mathcal{M}_{1}[j_{1}] \mathcal{M}_{1}[j_{1} + \Delta]$$

$$+ \sum_{j_{1}=0}^{L-1} M_{i_{1}-j_{1}} \mathcal{M}_{2}[j_{1}, j_{1} + \Delta]. \tag{11}$$

Now take expectations over the Poisson random variables, using this fact:

Lemma 3.1. If $M \sim Poisson(\gamma)$, then

$$\mathbb{E}\binom{M}{k} = \frac{\gamma^k}{k!}.\tag{12}$$

We get (now the expectation is over M and X):

$$\mathbb{E}Y_{i_{1}}Y_{i_{2}} = \sum_{j_{1}-j_{2}\neq\Delta} \mathbb{E}M_{i_{1}-j_{1}}M_{i_{2}-j_{2}}\mathcal{M}_{1}[j_{1}]\mathcal{M}_{1}[j_{2}]$$

$$+ \sum_{j_{1}=0}^{L-1} \mathbb{E}M_{i_{1}-j_{1}}(M_{i_{1}-j_{1}}-1)\mathcal{M}_{1}[j_{1}]\mathcal{M}_{1}[j_{1}+\Delta]$$

$$+ \sum_{j_{1}=0}^{L-1} \mathbb{E}M_{i_{1}-j_{1}}\mathcal{M}_{2}[j_{1},j_{1}+\Delta]$$

$$= \sum_{j_{1}-j_{2}\neq\Delta} \gamma^{2}\mathcal{M}_{1}[j_{1}]\mathcal{M}_{1}[j_{2}] + \sum_{j_{1}=0}^{L-1} \gamma^{2}\mathcal{M}_{1}[j_{1}]\mathcal{M}_{1}[j_{1}+\Delta]$$

$$+ \sum_{j_{1}=0}^{L-1} \gamma\mathcal{M}_{2}[j_{1},j_{1}+\Delta]$$

$$= \left(\gamma \sum_{j=0}^{L-1} \mathcal{M}_{1}[j]\right)^{2} + \gamma \sum_{j=0}^{L-1} \mathcal{M}_{2}[j,j+\Delta]$$

$$= (\gamma \mathcal{L}_{1})^{2} + \gamma \mathcal{L}_{2}(\Delta). \tag{13}$$

But the first term in the sum is just the square of the first moment of Y; so from the first two moments we can recover $\gamma \mathcal{L}_2(\Delta)$, which is just the expected power spectrum of the random vector X, i.e. the usual second moment we have been working with.

4 The third moment of Y

For three distinct i_1 , i_2 and i_3 , we let $\Delta_1 = i_2 - i_1$ and $\Delta_2 = i_3 - i_1$. We have:

$$Y_{i_{1}}Y_{i_{2}}Y_{i_{3}} = \sum_{j_{1}=0}^{L-1} \sum_{j_{2}=0}^{L-1} \sum_{j_{3}=0}^{L-1} \sum_{k_{1}=1}^{L-1} \sum_{k_{2}=1}^{M_{i_{1}-j_{1}}} \sum_{k_{3}=1}^{M_{i_{3}-j_{3}}} X_{k_{1}}^{i_{1}-j_{1}}[j_{1}]X_{k_{2}}^{i_{2}-j_{2}}[j_{2}]X_{k_{3}}^{i_{3}-j_{3}}[j_{3}].$$
 (14)

We will break up the outer three sums into disjoint sums with the following ranges of indices:

1.
$$j_2 = j_1 + \Delta_1$$
 and $j_3 = j_2 + \Delta_2 - \Delta_1$.

2.
$$j_2 = j_1 + \Delta_1$$
 and $j_3 \neq j_2 + \Delta_2 - \Delta_1$.

3.
$$j_2 \neq j_1 + \Delta_1$$
 and $j_3 = j_1 + \Delta_2$.

4.
$$j_2 \neq j_1 + \Delta_1$$
 and $j_3 \neq j_1 + \Delta_2$ and $j_3 = j_2 + \Delta_2 - \Delta_1$.

5.
$$j_2 \neq j_1 + \Delta_1$$
 and $j_3 \neq j_1 + \Delta_2$ and $j_3 \neq j_2 + \Delta_2 - \Delta_1$.

For Case 1, we have $\ell \equiv i_1 - j_1 = i_2 - j_2 = i_3 - j_3$. We further break up the sum:

$$\sum_{j=0}^{L-1} \sum_{k_{1}=1}^{M_{\ell}} \sum_{k_{2}=1}^{M_{\ell}} \sum_{k_{3}=1}^{M_{\ell}} X_{k_{1}}^{\ell}[j] X_{k_{2}}^{\ell}[j+\Delta_{1}] X_{k_{3}}^{\ell}[j+\Delta_{2}]$$

$$= \sum_{j=0}^{L-1} \sum_{k_{i} \text{ distinct}} X_{k_{1}}^{\ell}[j] X_{k_{2}}^{\ell}[j+\Delta_{1}] X_{k_{3}}^{\ell}[j+\Delta_{2}]$$

$$+ \sum_{j=0}^{L-1} \sum_{k_{1}=k_{2}\neq k_{3}} X_{k_{1}}^{\ell}[j] X_{k_{2}}^{\ell}[j+\Delta_{1}] X_{k_{3}}^{\ell}[j+\Delta_{2}]$$

$$+ \sum_{j=0}^{L-1} \sum_{k_{1}=k_{3}\neq k_{2}} X_{k_{1}}^{\ell}[j] X_{k_{2}}^{\ell}[j+\Delta_{1}] X_{k_{3}}^{\ell}[j+\Delta_{2}]$$

$$+ \sum_{j=0}^{L-1} \sum_{k_{2}=k_{3}\neq k_{1}} X_{k_{1}}^{\ell}[j] X_{k_{2}}^{\ell}[j+\Delta_{1}] X_{k_{3}}^{\ell}[j+\Delta_{2}]$$

$$+ \sum_{j=0}^{L-1} \sum_{k_{1}=k_{2}=k_{3}} X_{k_{1}}^{\ell}[j] X_{k_{2}}^{\ell}[j+\Delta_{1}] X_{k_{3}}^{\ell}[j+\Delta_{2}].$$

$$+ \sum_{j=0}^{L-1} \sum_{k_{1}=k_{2}=k_{3}} X_{k_{1}}^{\ell}[j] X_{k_{2}}^{\ell}[j+\Delta_{1}] X_{k_{3}}^{\ell}[j+\Delta_{2}].$$

$$(15)$$

For term (a), the expectation conditional on M is:

$$\sum_{j=0}^{L-1} M_{\ell}(M_{\ell} - 1)(M_{\ell} - 2)\mathcal{M}[j]\mathcal{M}[j + \Delta_1]\mathcal{M}[j + \Delta_2].$$
 (16)

Using Lemma 3.1, the unconditional expectation of (a) is then:

$$\gamma^{3} \sum_{j=0}^{L-1} \mathcal{M}_{1}[j] \mathcal{M}_{1}[j+\Delta_{1}] \mathcal{M}_{1}[j+\Delta_{2}]. \tag{17}$$

For term (b), the expectation conditional on M is:

$$\sum_{j=0}^{L-1} M_{\ell}(M_{\ell} - 1) \mathcal{M}_2[j, j + \Delta_1] \mathcal{M}_1[j + \Delta_2]$$
(18)

and then again using Lemma 3.1 we get the expected value:

$$\gamma^2 \sum_{j=0}^{L-1} \mathcal{M}_2[j, j + \Delta_1] \mathcal{M}_1[j + \Delta_2]. \tag{19}$$

Similarly, the expected values of terms (c) and (d) are:

$$\gamma^2 \sum_{j=0}^{L-1} \mathcal{M}_2[j, j + \Delta_2] \mathcal{M}_1[j + \Delta_1]. \tag{20}$$

and

$$\gamma^2 \sum_{j=0}^{L-1} \mathcal{M}_2[j + \Delta_1, j + \Delta_2] \mathcal{M}_1[j]. \tag{21}$$

Finally, the expected value of term (e) is easily shown to be:

$$\gamma \sum_{j=0}^{L-1} \mathcal{M}_3[j, j + \Delta_1, j + \Delta_2]. \tag{22}$$

This concludes the computation for Case 1.

Moving onto Case 2, we have $\ell_1 \equiv i_1 - j_1 = i_2 - j_2$, and also define $\ell_2 \equiv i_3 - j_3$. By definition, $\ell_1 \neq \ell_2$. The sum is:

$$\sum_{j_{1}=0}^{L-1} \sum_{j_{3}\neq j_{1}+\Delta_{2}} \sum_{1\leq k_{1},k_{2}\leq M_{\ell_{1}}} \sum_{k_{3}=1}^{M_{\ell_{2}}} X_{k_{1}}^{\ell_{1}}[j_{1}] X_{k_{2}}^{\ell_{1}}[j_{1}+\Delta_{1}] X_{k_{3}}^{\ell_{2}}[j_{3}]$$

$$= \sum_{j_{1}=0}^{L-1} \sum_{j_{3}\neq j_{1}+\Delta_{2}} \sum_{k_{3}=1}^{M_{\ell_{2}}} \left\{ \sum_{1\leq k_{1}\neq k_{2}\leq M_{\ell_{1}}} X_{k_{1}}^{\ell_{1}}[j_{1}] X_{k_{2}}^{\ell_{1}}[j_{1}+\Delta_{1}] X_{k_{3}}^{\ell_{2}}[j_{3}] + \sum_{k_{1}=1}^{M_{\ell_{1}}} X_{k_{1}}^{\ell_{1}}[j_{1}] X_{k_{1}}^{\ell_{1}}[j_{1}+\Delta_{1}] X_{k_{3}}^{\ell_{2}}[j_{3}] \right\}. \tag{23}$$

Taking expectations conditional on M, we then get:

$$\sum_{j_{1}=0}^{L-1} \sum_{j_{3}\neq j_{1}+\Delta_{2}} \left(M_{\ell_{1}}(M_{\ell_{1}}-1)M_{\ell_{2}}\mathcal{M}_{1}[j_{1}]\mathcal{M}_{1}[j_{1}+\Delta_{1}]\mathcal{M}_{1}[j_{3}] + M_{\ell_{1}}M_{\ell_{2}}\mathcal{M}_{2}[j_{1},j_{1}+\Delta_{1}]\mathcal{M}_{1}[j_{3}] \right).$$
(24)

Taking expectations over M and using Lemma 3.1 then gives:

$$\gamma^{3} \sum_{j_{1}=0}^{L-1} \sum_{j_{3} \neq j_{1} + \Delta_{2}} \mathcal{M}_{1}[j_{1}] \mathcal{M}_{1}[j_{1} + \Delta_{1}] \mathcal{M}_{1}[j_{3}]$$
 (25)

$$+ \gamma^2 \sum_{j_1=0}^{L-1} \sum_{j_3 \neq j_1 + \Delta_2} \mathcal{M}_2[j_1, j_1 + \Delta_1] \mathcal{M}_1[j_3].$$
 (26)

Similarly, Cases 3 and 4 give the expressions:

$$\gamma^{3} \sum_{j_{1}=0}^{L-1} \sum_{j_{2} \neq j_{1} + \Delta_{1}} \mathcal{M}_{1}[j_{1}] \mathcal{M}_{1}[j_{1} + \Delta_{2}] \mathcal{M}_{1}[j_{2}]$$
 (27)

$$+ \gamma^2 \sum_{j_1=0}^{L-1} \sum_{j_2 \neq j_1 + \Delta_1} \mathcal{M}_2[j_1, j_1 + \Delta_2] \mathcal{M}_1[j_2]$$
 (28)

and

$$\gamma^{3} \sum_{j_{2}=0}^{L-1} \sum_{j_{1} \neq j_{2}} \mathcal{M}_{1}[j_{1}] \mathcal{M}_{1}[j_{2} + \Delta_{1}] \mathcal{M}_{1}[j_{2} + \Delta_{2}]$$
 (29)

$$+ \gamma^2 \sum_{j_2=0}^{L-1} \sum_{j_1 \neq j_2} \mathcal{M}_2[j_2 + \Delta_1, j_2 + \Delta_2] \mathcal{M}_1[j_1].$$
 (30)

Finally, in Case 5 we have i_1-j_1 , i_2-j_2 , and i_3-j_3 are all pairwise distinct. Consequently, the X variables are always independent, and the expectation conditional on M (letting $\ell_q=i_q-j_q,\ q=1,2,3$),

$$\sum_{j_1, j_2, j_3} M_{\ell_1} M_{\ell_2} M_{\ell_3} \mathcal{M}_1[j_1] \mathcal{M}_1[j_2] \mathcal{M}_1[j_3]; \tag{31}$$

since the M_{ℓ_q} 's are pairwise independent, q=1,2,3, the expectation over M then yields:

$$\gamma^3 \sum_{j_1, j_2, j_3} \mathcal{M}_1[j_1] \mathcal{M}_1[j_2] \mathcal{M}_1[j_3]. \tag{32}$$

Now we add all the terms from Cases 1 to 5. Expressions (17), (25), (27), (29), and (32) sum to the expression:

$$(\gamma \mathcal{L}_1)^3. \tag{33}$$

Note that this is obtained directly from the first moment. Expressions (19), (20), (21), (26),(28), and (30) sum to the expression:

$$\gamma \mathcal{L}_1 \cdot (\gamma \mathcal{L}_2(\Delta_1) + \gamma \mathcal{L}_2(\Delta_2) + \gamma \mathcal{L}_2(\Delta_2 - \Delta_1)). \tag{34}$$

Again, note that this is obtained directly from the first two moments. Finally, expression (22) is simply:

$$\gamma \mathcal{L}_3(\Delta_1, \Delta_2) \tag{35}$$

which is the usual third-order auto-correlation

5 Signal plus noise

The expected values of the non-zero (for second moment) and off-diagonal (for third moment) terms are the same as without noise, as is true for the strongly-separated case. The same proof of almost sure convergence from my notes for the strongly-separated case also goes through verbatim.