Notes on Poisson Model for Big MRA

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1 Setup

We consider the following observation model. $X = (X[0], \dots, X[L-1])$ is a random vector of length L, drawn from some fixed distribution. For fixed n, we observe a random vector Y of length $n + L$, generated as follows. Points are chosen in $\{1, \dots, n\}$ according to a Poisson process with parameter γn. For each point i that is chosen from 1 to n, a random vector X from the distribution is then placed in the large vector, with element 0 at location i, with overlapping vectors being added together.

If M_i denotes the number of hits at location i, $1 \le i \le n$, then by definition of the Poission process M_i's are iid and $M_i \sim \text{Poisson}(\gamma)$. Conditional on the value of $M = (M_1, \dots, M_n)$, if we let $X_1^i, \dots, X_{M_i}^i$ denote the random vectors with position 0 located at i, then $X_{k_1}^i$ and $X_{k_2}^i$ are independent for $k_1 \neq k_2$.

With this notation, if $Y \in \mathbb{R}^{n+L}$ is the observed vector, we can write each entry as:

$$Y[i] = \sum_{j=0}^{L-1} \sum_{k=1}^{M_{i-j}} X_k^{i-j}[j]. \tag{1}$$

We will denote by $\mathcal{M}_l$ the moments of X:

$$\mathcal{M}_1[i] = \mathbb{E}X[i], \quad 0 \le i \le L-1, \tag{2}$$

$$\mathcal{M}_2[i,j] = \mathbb{E}X[i]X[j], \quad 0 \le i,j \le L-1, \tag{3}$$

and

$$\mathcal{M}_3[i,j,k] = \mathbb{E}X[i]X[j]X[k], \quad 0 \le i,j,k \le L-1. \tag{4}$$

We will also denote by $\mathcal{L}_l$ the autocorrelations of X:

$$\mathcal{L}_1 = \sum_{i=0}^{L-1} \mathcal{M}_1[i], \tag{5}$$

$$\mathcal{L}_2(\Delta) = \sum_{i=0}^{L-1} \mathcal{M}_2[i, i+\Delta], \tag{6}$$

and

$$\mathcal{L}_3(\Delta_1, \Delta_2) = \sum_{i=0}^{L-1} \mathcal{M}_3[i, i+\Delta_1, i+\Delta_2]. \tag{7}$$

Note that in the strongly-separated model, the first three observed moments are, respectively, $\mathcal{L}_1$, $\mathcal{L}_2(\Delta)$, and $\mathcal{L}_3(\Delta_1, \Delta_2)$.

In this notation, we will show that the first moment of the data is $\gamma\mathcal{L}_1$, the second moment vector is $(\gamma\mathcal{L}_1)^2 + \gamma\mathcal{L}_2(\Delta)$, and the third moment matrix is $(\gamma\mathcal{L}_1)^3 + \gamma\mathcal{L}_1 \cdot (\gamma\mathcal{L}_2(\Delta_1) + \gamma\mathcal{L}_2(\Delta_2) + \gamma\mathcal{L}_2(\Delta_2 - \Delta_1)) + \gamma\mathcal{L}_3(\Delta_1, \Delta_2)$. In particular, from the first three moments of the Poisson process model, once can recover the first three moments from the strongly-separated model, with the Poisson rate γ playing the role of the "occupancy factor". So if recovery is possible for the strongly-separated model, it is also possible for the Poisson process model.

2 The first moment of Y

To compute the first moment of Y, we will first condition on $M = (M_1, \dots, M_n)$, and then average over M. We have:

$$\mathbb{E}[Y[i]|M] = \sum_{j=0}^{L-1} \sum_{k=1}^{M_{i-j}} \mathbb{E}X_k^{i-j}[j] = \sum_{j=0}^{L-1} \sum_{k=1}^{M_{i-j}} \mathcal{M}_1[j] = M_{i-j} \sum_{j=0}^{L-1} \mathcal{M}_1[j]. \tag{8}$$

Now taking expectations over M we see:

$$\mathbb{E}Y[i] = \gamma \sum_{j=0}^{L-1} \mathcal{M}_1[j] = \gamma\mathcal{L}_1. \tag{9}$$

3 The second moment of Y

Again, we will condition on M first, and then take the expectation over M. Fix $i_1 \neq i_2$, and let $\Delta = i_2 - i_1$. Then:

$$Y_{i_1} Y_{i_2} = \sum_{j_1=0}^{L-1} \sum_{j_2=0}^{L-1} \sum_{k_1=1}^{M_{i_1-j_1}} \sum_{k_2=1}^{M_{i_2-j_2}} X_{k_1}^{i_1-j_1}[j_1] X_{k_2}^{i_2-j_2}[j_2]. \tag{10}$$

We break up the double sum over j_1 and j_2 into two terms: one where $j_2 \neq j_1 + \Delta$, and one where $j_2 = j_1 + \Delta$ or equivalently $i_1 - j_1 = i_2 - j_2$. In the first case, all the terms are independent, and so the expectation factors. In the second case, when $k_1 \neq k_2$ we have independence, but otherwise not. This gives (all expectations are conditional on M):

$$\begin{aligned}
\mathbb{E}Y_{i_1}Y_{i_2} &= \sum_{j_1=0}^{L-1}\sum_{j_2=0}^{L-1}\sum_{k_1=1}^{M_{i_1-j_1}}\sum_{k_2=1}^{M_{i_2-j_2}} \mathbb{E}X_{k_1}^{i_1-j_1}[j_1]X_{k_2}^{i_2-j_2}[j_2] \\
&= \sum_{j_1-j_2\neq\Delta}\sum_{k_1}\sum_{k_2}\mathbb{E}X_{k_1}^{i_1-j_1}[j_1]X_{k_2}^{i_2-j_2}[j_2] \\
&\quad + \sum_{j_1=0}^{L-1}\sum_{k_1\neq k_2}\mathbb{E}X_{k_1}^{i_1-j_1}[j_1]X_{k_2}^{i_1-j_1}[j_1+\Delta] \\
&\quad + \sum_{j_1=0}^{L-1}\sum_{k_1=1}^{M_{i_1-j_1}}\mathbb{E}X_{k_1}^{i_1-j_1}[j_1]X_{k_1}^{i_1-j_1}[j_1+\Delta] \\
&= \sum_{j_1-j_2\neq\Delta} M_{i_1-j_1}M_{i_2-j_2}\mathcal{M}_1[j_1]\mathcal{M}_1[j_2] \\
&\quad + \sum_{j_1=0}^{L-1} M_{i_1-j_1}(M_{i_1-j_1}-1)\mathcal{M}_1[j_1]\mathcal{M}_1[j_1+\Delta] \\
&\quad + \sum_{j_1=0}^{L-1} M_{i_1-j_1}\mathcal{M}_2[j_1,j_1+\Delta]. \qquad (11)
\end{aligned}$$

Now take expectations over the Poisson random variables, using this fact:

Lemma 3.1. *If $M \sim Poisson(\gamma)$, then*

$$\mathbb{E}\binom{M}{k} = \frac{\gamma^k}{k!}. \qquad (12)$$

We get (now the expectation is over M and X):

$$
\begin{aligned}
\mathbb{E}Y_{i_1}Y_{i_2} =& \sum_{j_1-j_2\neq\Delta} \mathbb{E}M_{i_1-j_1}M_{i_2-j_2}\mathcal{M}_1[j_1]\mathcal{M}_1[j_2] \\
&+ \sum_{j_1=0}^{L-1} \mathbb{E}M_{i_1-j_1}(M_{i_1-j_1}-1)\mathcal{M}_1[j_1]\mathcal{M}_1[j_1+\Delta] \\
&+ \sum_{j_1=0}^{L-1} \mathbb{E}M_{i_1-j_1}\mathcal{M}_2[j_1,j_1+\Delta] \\
=& \sum_{j_1-j_2\neq\Delta} \gamma^2\mathcal{M}_1[j_1]\mathcal{M}_1[j_2] + \sum_{j_1=0}^{L-1} \gamma^2\mathcal{M}_1[j_1]\mathcal{M}_1[j_1+\Delta] \\
&+ \sum_{j_1=0}^{L-1} \gamma\mathcal{M}_2[j_1,j_1+\Delta] \\
=& \left(\gamma\sum_{j=0}^{L-1}\mathcal{M}_1[j]\right)^2 + \gamma\sum_{j=0}^{L-1}\mathcal{M}_2[j,j+\Delta] \\
=& (\gamma\mathcal{L}_1)^2 + \gamma\mathcal{L}_2(\Delta).
\end{aligned} \tag{13}
$$

But the first term in the sum is just the square of the first moment of Y; so from the first two moments we can recover $\gamma\mathcal{L}_2(\Delta)$, which is just the expected power spectrum of the random vector X, i.e. the usual second moment we have been working with.

4 The third moment of Y

For three distinct i_1, i_2 and i_3, we let $\Delta_1 = i_2 - i_1$ and $\Delta_2 = i_3 - i_1$. We have:

$$
\begin{aligned}
&Y_{i_1}Y_{i_2}Y_{i_3} \\
&= \sum_{j_1=0}^{L-1}\sum_{j_2=0}^{L-1}\sum_{j_3=0}^{L-1}\sum_{k_1=1}^{M_{i_1-j_1}}\sum_{k_2=1}^{M_{i_2-j_2}}\sum_{k_3=1}^{M_{i_3-j_3}} X_{k_1}^{i_1-j_1}[j_1]X_{k_2}^{i_2-j_2}[j_2]X_{k_3}^{i_3-j_3}[j_3].
\end{aligned} \tag{14}
$$

We will break up the outer three sums into disjoint sums with the following ranges of indices:

1. $j_2 = j_1 + \Delta_1$ and $j_3 = j_2 + \Delta_2 - \Delta_1$.
2. $j_2 = j_1 + \Delta_1$ and $j_3 \neq j_2 + \Delta_2 - \Delta_1$.
3. $j_2 \neq j_1 + \Delta_1$ and $j_3 = j_1 + \Delta_2$.
4. $j_2 \neq j_1 + \Delta_1$ and $j_3 \neq j_1 + \Delta_2$ and $j_3 = j_2 + \Delta_2 - \Delta_1$.
5. $j_2 \neq j_1 + \Delta_1$ and $j_3 \neq j_1 + \Delta_2$ and $j_3 \neq j_2 + \Delta_2 - \Delta_1$.

For Case 1, we have $\ell \equiv i_1 - j_1 = i_2 - j_2 = i_3 - j_3$. We further break up the sum:

$$\begin{aligned}
&\sum_{j=0}^{L-1} \sum_{k_1=1}^{M_\ell} \sum_{k_2=1}^{M_\ell} \sum_{k_3=1}^{M_\ell} X_{k_1}^{\ell}[j] X_{k_2}^{\ell}[j+\Delta_1] X_{k_3}^{\ell}[j+\Delta_2] \\
&= \underbrace{\sum_{j=0}^{L-1} \sum_{k_i \text{distinct}} X_{k_1}^{\ell}[j] X_{k_2}^{\ell}[j+\Delta_1] X_{k_3}^{\ell}[j+\Delta_2]}_{\text{(a)}} \\
&\quad + \underbrace{\sum_{j=0}^{L-1} \sum_{k_1=k_2\neq k_3} X_{k_1}^{\ell}[j] X_{k_2}^{\ell}[j+\Delta_1] X_{k_3}^{\ell}[j+\Delta_2]}_{\text{(b)}} \\
&\quad + \underbrace{\sum_{j=0}^{L-1} \sum_{k_1=k_3\neq k_2} X_{k_1}^{\ell}[j] X_{k_2}^{\ell}[j+\Delta_1] X_{k_3}^{\ell}[j+\Delta_2]}_{\text{(c)}} \\
&\quad + \underbrace{\sum_{j=0}^{L-1} \sum_{k_2=k_3\neq k_1} X_{k_1}^{\ell}[j] X_{k_2}^{\ell}[j+\Delta_1] X_{k_3}^{\ell}[j+\Delta_2]}_{\text{(d)}} \\
&\quad + \underbrace{\sum_{j=0}^{L-1} \sum_{k_1=k_2=k_3} X_{k_1}^{\ell}[j] X_{k_2}^{\ell}[j+\Delta_1] X_{k_3}^{\ell}[j+\Delta_2]}_{\text{(e)}}.
\end{aligned} \tag{15}$$

For term (a), the expectation conditional on M is:

$$\sum_{j=0}^{L-1} M_\ell(M_\ell-1)(M_\ell-2)\mathcal{M}[j]\mathcal{M}[j+\Delta_1]\mathcal{M}[j+\Delta_2]. \tag{16}$$

Using Lemma 3.1, the unconditional expectation of (a) is then:

$$\gamma^3 \sum_{j=0}^{L-1} \mathcal{M}_1[j]\mathcal{M}_1[j+\Delta_1]\mathcal{M}_1[j+\Delta_2]. \tag{17}$$

For term (b), the expectation conditional on M is:

$$\sum_{j=0}^{L-1} M_\ell(M_\ell-1)\mathcal{M}_2[j, j+\Delta_1]\mathcal{M}_1[j+\Delta_2] \tag{18}$$

and then again using Lemma 3.1 we get the expected value:

$$\gamma^2 \sum_{j=0}^{L-1} \mathcal{M}_2[j, j+\Delta_1]\mathcal{M}_1[j+\Delta_2]. \tag{19}$$

Similarly, the expected values of terms (c) and (d) are:

$$\gamma^2 \sum_{j=0}^{L-1} \mathcal{M}_2[j, j+\Delta_2]\mathcal{M}_1[j+\Delta_1]. \tag{20}$$

and

$$\gamma^2 \sum_{j=0}^{L-1} \mathcal{M}_2[j+\Delta_1, j+\Delta_2]\mathcal{M}_1[j]. \tag{21}$$

Finally, the expected value of term (e) is easily shown to be:

$$\gamma \sum_{j=0}^{L-1} \mathcal{M}_3[j, j+\Delta_1, j+\Delta_2]. \tag{22}$$

This concludes the computation for Case 1.

Moving onto Case 2, we have $\ell_1 \equiv i_1 - j_1 = i_2 - j_2$, and also define $\ell_2 \equiv i_3 - j_3$. By definition, $\ell_1 \neq \ell_2$. The sum is:

$$\begin{aligned}
&\sum_{j_1=0}^{L-1} \sum_{j_3 \neq j_1+\Delta_2} \sum_{1 \leq k_1, k_2 \leq M_{\ell_1}} \sum_{k_3=1}^{M_{\ell_2}} X_{k_1}^{\ell_1}[j_1] X_{k_2}^{\ell_1}[j_1+\Delta_1] X_{k_3}^{\ell_2}[j_3] \\
&= \sum_{j_1=0}^{L-1} \sum_{j_3 \neq j_1+\Delta_2} \sum_{k_3=1}^{M_{\ell_2}} \Bigg\{ \sum_{1 \leq k_1 \neq k_2 \leq M_{\ell_1}} X_{k_1}^{\ell_1}[j_1] X_{k_2}^{\ell_1}[j_1+\Delta_1] X_{k_3}^{\ell_2}[j_3] \\
&\quad + \sum_{k_1=1}^{M_{\ell_1}} X_{k_1}^{\ell_1}[j_1] X_{k_1}^{\ell_1}[j_1+\Delta_1] X_{k_3}^{\ell_2}[j_3] \Bigg\}.
\end{aligned} \tag{23}$$

Taking expectations conditional on M, we then get:

$$\begin{aligned}
&\sum_{j_1=0}^{L-1} \sum_{j_3 \neq j_1+\Delta_2} \Bigg(M_{\ell_1}(M_{\ell_1}-1) M_{\ell_2} \mathcal{M}_1[j_1]\mathcal{M}_1[j_1+\Delta_1]\mathcal{M}_1[j_3] \\
&\quad + M_{\ell_1} M_{\ell_2} \mathcal{M}_2[j_1, j_1+\Delta_1]\mathcal{M}_1[j_3] \Bigg).
\end{aligned} \tag{24}$$

Taking expectations over M and using Lemma 3.1 then gives:

$$\gamma^3 \sum_{j_1=0}^{L-1} \sum_{j_3 \neq j_1+\Delta_2} \mathcal{M}_1[j_1]\mathcal{M}_1[j_1+\Delta_1]\mathcal{M}_1[j_3] \tag{25}$$

$$+ \gamma^2 \sum_{j_1=0}^{L-1} \sum_{j_3 \neq j_1+\Delta_2} \mathcal{M}_2[j_1, j_1+\Delta_1]\mathcal{M}_1[j_3]. \tag{26}$$

Similarly, Cases 3 and 4 give the expressions:

$$\gamma^3 \sum_{j_1=0}^{L-1} \sum_{j_2 \neq j_1+\Delta_1} \mathcal{M}_1[j_1]\mathcal{M}_1[j_1+\Delta_2]\mathcal{M}_1[j_2] \tag{27}$$

$$+\gamma^2 \sum_{j_1=0}^{L-1} \sum_{j_2 \neq j_1+\Delta_1} \mathcal{M}_2[j_1, j_1+\Delta_2]\mathcal{M}_1[j_2] \tag{28}$$

and

$$\gamma^3 \sum_{j_2=0}^{L-1} \sum_{j_1 \neq j_2} \mathcal{M}_1[j_1]\mathcal{M}_1[j_2+\Delta_1]\mathcal{M}_1[j_2+\Delta_2] \tag{29}$$

$$+\gamma^2 \sum_{j_2=0}^{L-1} \sum_{j_1 \neq j_2} \mathcal{M}_2[j_2+\Delta_1, j_2+\Delta_2]\mathcal{M}_1[j_1]. \tag{30}$$

Finally, in Case 5 we have $i_1 - j_1$, $i_2 - j_2$, and $i_3 - j_3$ are all pairwise distinct. Consequently, the X variables are always independent, and the expectation conditional on M (letting $\ell_q = i_q - j_q$, $q = 1, 2, 3$),

$$\sum_{j_1, j_2, j_3} M_{\ell_1} M_{\ell_2} M_{\ell_3} \mathcal{M}_1[j_1]\mathcal{M}_1[j_2]\mathcal{M}_1[j_3]; \tag{31}$$

since the M_{ℓ_q}'s are pairwise independent, $q = 1, 2, 3$, the expectation over M then yields:

$$\gamma^3 \sum_{j_1, j_2, j_3} \mathcal{M}_1[j_1]\mathcal{M}_1[j_2]\mathcal{M}_1[j_3]. \tag{32}$$

Now we add all the terms from Cases 1 to 5. Expressions (17), (25), (27), (29), and (32) sum to the expression:

$$(\gamma \mathcal{L}_1)^3. \tag{33}$$

Note that this is obtained directly from the first moment. Expressions (19), (20), (21), (26),(28), and (30) sum to the expression:

$$\gamma \mathcal{L}_1 \cdot (\gamma \mathcal{L}_2(\Delta_1) + \gamma \mathcal{L}_2(\Delta_2) + \gamma \mathcal{L}_2(\Delta_2 - \Delta_1)). \tag{34}$$

Again, note that this is obtained directly from the first two moments. Finally, expression (22) is simply:

$$\gamma \mathcal{L}_3(\Delta_1, \Delta_2) \tag{35}$$

which is the usual third-order auto-correlation.

5 Signal plus noise

The expected values of the non-zero (for second moment) and off-diagonal (for third moment) terms are the same as without noise, as is true for the strongly-separated case. The same proof of almost sure convergence from my notes for the strongly-separated case also goes through verbatim.