Formalism for autocorrelation derivations

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Let $x_{(1)}, \ldots, x_{(|s|)}$ denote the (independent) realizations of the random signal x in the observation y, starting at (deterministic) positions $s_{(1)}, \ldots, s_{(|s|)}$. Let I_{ij} be the indicator variable for whether position i is in the support of occurrence j, that is, it is one if i is in $\{s_{(j)}, \ldots, s_{(j)} + L - 1\}$, and zero otherwise. Then,

$$y[i] = \sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] + \varepsilon[i]. \tag{1}$$

This gives a simple expression for the first autocorrelation of y. Indeed,

$$a_y^1 = \mathbb{E}_y \left\{ \frac{1}{N} \sum_{i=0}^{N-1} y[i] \right\} \tag{2}$$

$$= \frac{1}{N} \mathbb{E}_{x_{(1)}, \ldots, x_{(|s|)}, \varepsilon} \left\{ \sum_{i=0}^{N-1} \sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] + \varepsilon[i] \right\}. \tag{3}$$

Now switch the sums over i and j, and observe that I_{ij} is zero unless $i = s_{(j)} + t$ for t in the range $0, \ldots, L - 1$. Hence,

$$a_y^1 = \frac{1}{N} \sum_{j=1}^{|s|} \mathbb{E}_{x_{(j)}} \left\{ \sum_{t=0}^{L-1} x_{(j)}[t] \right\} + \frac{1}{N} \mathbb{E}_{\varepsilon} \left\{ \sum_{i=0}^{N-1} \varepsilon[i] \right\}. \tag{4}$$

Since the noise has zero mean and $x_{(1)}, \ldots, x_{(|s|)}$ are independent and all distributed as x, we further find:

$$a_y^1 = \frac{|s|L}{N} a_x^1 = \gamma a_x^1. \tag{5}$$

To address the second-order moments, we resort to the separation conditions. Indeed, consider this expression:

$$
\begin{aligned}
N \cdot a_y^2[\ell] &= \mathbb{E}_y \left\{ \sum_{i=0}^{N-1} y[i]y[i+\ell] \right\} \\
&= \sum_{i=0}^{N-1} \mathbb{E}_{x_{(1)},\ldots,x_{(|s|)},\varepsilon} \Bigg\{ \left(\sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] + \varepsilon[i] \right) \cdot \\
&\qquad \left(\sum_{j'=1}^{|s|} I_{i+\ell,j'} x_{(j')}[i + \ell - s_{(j')}] + \varepsilon[i+\ell] \right) \Bigg\} \\
&= \sum_{i=0}^{N-1} \mathbb{E}_{x_{(1)},\ldots,x_{(|s|)},\varepsilon} \Bigg\{ \sum_{j=1}^{|s|} \sum_{j'=1}^{|s|} I_{ij} I_{i+\ell,j'} x_{(j)}[i - s_{(j)}] x_{(j')}[i + \ell - s_{(j')}] \\
&\qquad + \sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] \varepsilon[i+\ell] \\
&\qquad + \sum_{j'=1}^{|s|} I_{i+\ell,j'} x_{(j')}[i + \ell - s_{(j')}] \varepsilon[i] \\
&\qquad + \varepsilon[i]\varepsilon[i+\ell] \Bigg\}.
\end{aligned}
$$

The cross-terms vanish in expectation since ε is zero mean and independent from the signal occurrences. The last term vanishes in expectation unless $\ell = 0$ since distinct entries of ε are independent. For $\ell = 0$, $\mathbb{E}\{\varepsilon[i]^2\} = \sigma^2$. Finally, using the separation property, observe that if $I_{ij} I_{i+\ell,j'}$ is nonzero, then it is equal to one, $j = j'$ and $i = s_{(j)} + t$ for some t in $0, \ldots, L - \ell - 1$. Then, switch the order of summations to get

$$
N \cdot a_y^2[\ell] = \sum_{j=1}^{|s|} \mathbb{E}_{x_{(j)}} \left\{ \sum_{t=0}^{L-\ell-1} x_{(j)}[t] x_{(j)}[t+\ell] \right\} + N\sigma^2 \delta[\ell], \tag{6}
$$

where $\delta[0] = 1$ and $\delta[\ell \neq 0] = 0$. Since each $x_{(j)}$ is distributed as x, they all have the same autocorrelations as x and we finally get

$$
a_y^2[\ell] = \gamma a_x^2[\ell] + \sigma^2 \delta[\ell]. \tag{7}
$$

We now turn to the third-order autocorrelations. These involve the sum

$$
\sum_{i=0}^{N-1} y[i]y[i+\ell_1]y[i+\ell_2]. \tag{8}
$$

Using (1), we find that this quantity can be expressed as a sum of eight terms:

1. $\sum_{i=0}^{N-1} \sum_{j,j',j''=1}^{|s|} I_{ij} I_{i+\ell_1,j'} I_{i+\ell_2,j''} x_{(j)}[i - s_{(j)}] x_{(j')}[i + \ell_1 - s_{(j')}] x_{(j'')}[i + \ell_2 - s_{(j'')}]$
2. $\sum_{i=0}^{N-1} \sum_{j,j'=1}^{|s|} I_{ij} I_{i+\ell_1,j'} x_{(j)}[i - s_{(j)}] x_{(j')}[i + \ell_1 - s_{(j')}] \varepsilon[i + \ell_2]$
3. $\sum_{i=0}^{N-1} \sum_{j,j''=1}^{|s|} I_{ij} I_{i+\ell_2,j''} x_{(j)}[i - s_{(j)}] \varepsilon[i + \ell_1] x_{(j'')}[i + \ell_2 - s_{(j'')}]$
4. $\sum_{i=0}^{N-1} \sum_{j',j''=1}^{|s|} I_{i+\ell_1,j'} I_{i+\ell_2,j''} \varepsilon[i] x_{(j')}[i + \ell_1 - s_{(j')}] x_{(j'')}[i + \ell_2 - s_{(j'')}]$
5. $\sum_{i=0}^{N-1} \sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] \varepsilon[i + \ell_1] \varepsilon[i + \ell_2]$
6. $\sum_{i=0}^{N-1} \sum_{j'=1}^{|s|} I_{i+\ell_1,j'} \varepsilon[i] x_{(j')}[i + \ell_1 - s_{(j')}] \varepsilon[i + \ell_2]$
7. $\sum_{i=0}^{N-1} \sum_{j''=1}^{|s|} I_{i+\ell_2,j''} \varepsilon[i] \varepsilon[i + \ell_1] x_{(j'')}[i + \ell_2 - s_{(j'')}]$
8. $\sum_{i=0}^{N-1} \varepsilon[i] \varepsilon[i + \ell_1] \varepsilon[i + \ell_2]$

Terms 2–4 and 8 vanish in expectation since odd moments of centered Gaussian variables are zero. For the first term, we use the fact that the separation condition implies

$$I_{ij} I_{i+\ell_1,j'} I_{i+\ell_2,j''} = 1 \iff \\ j = j' = j'' \text{ and } i = s_{(j)} + t \text{ with } t \in \{0, \dots L - \max(\ell_1, \ell_2) - 1\}. \tag{9}$$

(Otherwise, the product of indicators is zero.) This allows to reduce the summations over j, j', j'' to a single sum over j. Then, witching the order of summation with i, we get that the first term is equal to

$$\sum_{j=1}^{|s|} \sum_{t=0}^{L-\max(\ell_1,\ell_2)-1} x_{(j)}[t] x_{(j)}[t + \ell_1] x_{(j)}[t + \ell_2]. \tag{10}$$

In expectation over the realizations $x_{(j)}$, using again that they are i.i.d. with the same distribution as x, this first term yields $|s| L a_x^3[\ell_1, \ell_2]$. Now consider the fifth term. Taking expectation against ε yields

$$\sum_{i=0}^{N-1} \sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] \sigma^2 \delta[\ell_1 - \ell_2]. \tag{11}$$

Switch the order of summation over i and j again to get

$$\sigma^2 \delta[\ell_1 - \ell_2] \sum_{j=1}^{|s|} \sum_{t=0}^{L-1} x_{(j)}[t]. \tag{12}$$

Now taking expectation against the signal occurrences yields $|s| L \sigma^2 a_x^1 \delta[\ell_1 - \ell_2]$. A similar reasoning for terms 6 and 7 yields this final formula for the third-order autocorrelations of y:

$$a_y^3[\ell_1, \ell_2] = \gamma a_x^3[\ell_1, \ell_2] + \gamma \sigma^2 a_x^1 \left(\delta[\ell_1] + \delta[\ell_2] + \delta[\ell_1 - \ell_2] \right). \tag{13}$$