J. Appl. Prob. 13, 377–379 (1976)
Printed in Israel

THE MEAN NUMBER OF MAXIMA ABOVE HIGH LEVELS IN GAUSSIAN RANDOM FIELDS

A. M. HASOFER, *University of New South Wales*

Abstract

An asymptotic formula for the mean number of maxima above a level of an n-dimensional stationary Gaussian field has been given by Nosko without proof. In this note a short general proof of this formula is given.

RANDOM FIELDS; MAXIMA ABOVE A LEVEL; GAUSSIAN PROCESSES

Introduction

Belyayev [1] attributes to Nosko [2] the following asymptotic result for $M_\xi(S)$, the mean number of maxima above a level ξ of an n-dimensional stationary Gaussian field $X(t)$, $t = t_1, \cdots, t_n$, with mean zero, in the Lebesgue measurable set S of measure $w(S)$:

$$(1)\quad M_\xi(S)/w(S) = (2\pi)^{-\frac{1}{2}(n+1)}\mu^{-n+\frac{1}{2}}|A|^{\frac{1}{2}}\xi^{n-1}\exp(-\xi^2/2\mu)[1+O(1/\xi)], \cdots$$

where $|A|$ is the determinant of the covariance matrix of the partial derivatives $\partial X/\partial t_i$ and μ is the variance of X.

However, a perusal of Nosko's paper reveals that the result is stated without proof. A thorough literature search carried out by the author and some colleagues failed to produce a published proof of the general result. Only a partial proof is given in Nosko [3], applicable only to a two-dimensional isotropic Gaussian field, and the method used cannot be generalized to more than two dimensions.

In this note a short proof of (1) for the general case is given.

The exact mean number of maxima above a level

The following exact result is given by Belyayev [1]. Let $X(t)$, $(t = t_1, \cdots, t_n)$ be a separable, stationary Gaussian field with mean zero, twice differentiable in quadratic mean. Let $X_i(t) = \partial X(t)/\partial t_i$, $X_{ij}(t) = \partial^2 X(t)/\partial t_i \partial t_j$.

Let the following conditions be satisfied:

$$\max_{i,j} E\,|X_{ij}(t+h) - X_{ij}(t)|^2 \leqq \frac{C}{|\log|h||^{1+\varepsilon}}$$

for $C > 0$, $\varepsilon > 0$; $i, j = 1, \cdots, n$.

Received 6 May 1975. The research embodied in this paper was carried out while the author was on study leave in the Department of Civil Engineering, Massachusetts Institute of Technology.

We introduce the following notation. Let $\Delta = (z_{ij})$ be a symmetric $n \times n$ matrix. Then $z = \mathrm{Vec}(z_{ij})$ will denote the column vector of length $n(n+1)/2$ obtained from the symmetric matrix (z_{ij}) by placing the successive columns on and above the main diagonal under one another. Let $y = (y_i)$ be a vector of length n. Let $\varphi(x, y, z)$ be the joint density function of $X(t)$, $X_i(t)$, $X_{ij}(t)$. Let S be a Lebesgue measurable set in R^n, and $M_\xi(S)$ the mean number of maxima of $X(t)$ above ξ in S. Let $w(S)$ be the Lebesgue measure of S. Let $|\Delta|$ represent the determinant of Δ. Then

$$m_\xi = M_\xi(S)/w(S) = (-1)^n \int_L |\Delta| \varphi(x, \mathbf{0}, z)\, dx\, dz, \cdots \tag{2}$$

where L is the set of points (x, z) for which Δ is negative definite, and $x > \xi$.

Proof of the asymptotic formula

The basic idea of the proof is simply that as $\xi \to \infty$, the z-set for which Δ is negative definite extends to the whole z-space.

To prove this we analyse more closely the function φ. First we carry out a number of simplifying transformations.

(1) It is easy to see that there always exists an orthogonal transformation of the field coordinates t which will make the derivatives X_i uncorrelated.

(2) We change the scale of the t_i as well as the scale of X to make the variances of X and the X_i equal to unity.

By considering the spectral representation of X, it is then trivial to reach the following conclusions

(1) X and X_i are uncorrelated for all i;
(2) X and X_{ij} are uncorrelated for $i \neq j$;
(3) X_i and X_j are uncorrelated for $i \neq j$;
(4) X_i and X_{jk} are uncorrelated for all i, j, k.

Thus we can write

$$\varphi(x, y, z) = \varphi_1(x)\varphi_2(y)\varphi_3(z \mid x),$$

and we note that

$$\varphi_2(\mathbf{0}) = (2\pi)^{-\frac{1}{2}n} = (-1)^n \alpha, \text{ say.}$$

Thus

$$m_\xi = \alpha \int_\xi^\infty \varphi_1(x) dx \int_P |\Delta| \varphi_3(z \mid x) dz,$$

where P is the set of z for which Δ is negative definite.

On account of (2) above, the only effect of conditioning the variables X_{ij} on

$X = x$ will be to shift the mean of the X_{ij} by an amount $-x$, and to change the covariance matrix in a way independent of x. If we make the change of variables $z_{ij} = v_{ij} - x\delta_{ij}$, where δ_{ij} is the Kronecker delta, we find

$$m_\xi = \alpha \int_\xi^\infty \varphi_1(x)dx \int_{U(x)} | V - xI | \varphi_4(v)dv,$$

where $V = (v_{ij})$, $v = \mathrm{Vec}(v_{ij})$, $\varphi_4(v)$ is a multivariate normal distribution independent of x, and $U(x)$ is the set of points v for which the matrix $V - xI$ is negative definite.

We now claim that by making x sufficiently large, we can make $U(x)$ contain the sphere $\sigma(R)$ centred at the origin with arbitrary radius R. In fact, since V is symmetric, we can diagonalize it by means of an orthogonal matrix P. Thus

$$P'(V - xI)P = P'VP - xI = \mathrm{diag}\,(w_1 - x, \cdots, w_n - x),$$

say.

Thus Δ will be negative definite provided $x > \max_i w_i$. But $\max_i w_i \leqq \| V \|$, where $\| \cdot \|$ denotes any matrix norm. Thus, in particular

$$\max_i w_i \leqq \max_j \sum_i | v_{ij} | \leqq n \max_{i,j} | v_{ij} | \leqq n | v |,$$

and $U(x)$ will contain the sphere $\sigma(R)$ provided $x > nR$.

Next we note that $| V - xI |$ is the characteristic polynomial of the matrix V, so that the coefficient of x^n is $(-1)^n$. It is now easy to estimate that the relative error made in replacing $U(x)$ by the whole space of v and the determinant $| \Delta |$ by $(-1)^n x^n$ is $O(1/\xi)$. Thus

$$m_\xi = \alpha \int_\xi^\infty (-1)^n x^n \varphi_1(x)dx\,[1 + O(1/\xi)].$$

And, since $\int_\xi^\infty x^n \varphi_1(x)dx = \xi^{n-1}\varphi_1(\xi)[1 + O(1/\xi)]$, we obtain the asymptotic formula

$$m_\xi = (2\pi)^{-\frac{1}{2}(n+1)} \xi^{n-1} \exp(-\xi^2/2)\,[1 + O(1/\xi)].$$

Finally, reverting to the original coordinates of t and X, we obtain Nosko's formula (1).

References

[1] BELYAYEV, YU. K. (1972) Point processes and first passage problems. *Proc. 6th Berkeley Symp. Math. Statist. Prob.* **2,** 1–17.

[2] NOSKO, V. P. (1969) The characteristics of excursions of Gaussian homogeneous fields above a high level. *Proceedings of the U.S.S.R.–Japan Symposium on Probability, Novosibirsk.*

[3] NOSKO, V. P. (1970) 'Shines' of Gaussian random fields. *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **25,** 4, 18–22.