Formalism for autocorrelation derivations

(T,N)B

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Let $x_{(1)}, \ldots, x_{(|s|)}$ denote the (independent) realizations of the random signal x in the observation y, starting at (deterministic) positions $s_{(1)}, \ldots, s_{(|s|)}$. Let I_{ij} be the indicator variable for whether position i is in the support of occurrence j, that is, it is one if i is in $\{s_{(j)}, \ldots, s_{(j)} + L - 1\}$, and zero otherwise. Then,

$$y[i] = \sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] + \varepsilon[i].$$

This gives a simple expression for the first autocorrelation of y:

$$a_y^1 = \mathbb{E}_y \left\{ \frac{1}{N} \sum_{i=0}^{N-1} y[i] \right\} \tag{1}$$

$$= \frac{1}{N} \mathbb{E}_{x_{(1)}, \ldots, x_{(|s|)}, \varepsilon} \left\{ \sum_{i=0}^{N-1} \sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] + \varepsilon[i] \right\}. \tag{2}$$

Now switch the sums over i and j, and observe that I_{ij} is zero unless $i = s_{(j)} + t$ for t in the range $0, \ldots, L-1$. Hence,

$$a_y^1 = \frac{1}{N} \sum_{j=1}^{|s|} \mathbb{E}_{x_{(j)}} \left\{ \sum_{t=0}^{L-1} x_{(j)}[t] \right\} + \frac{1}{N} \mathbb{E}_{\varepsilon} \left\{ \sum_{i=0}^{N-1} \varepsilon[i] \right\}. \tag{3}$$

Since the noise has zero mean and $x_{(1)}, \ldots, x_{(|s|)}$ are independent and all distributed as x, we further find:

$$a_y^1 = \frac{|s|}{N} L a_x^1 = \gamma a_x^1. \tag{4}$$

To address the second-order moments, we resort to the separation conditions. Indeed, consider this expression:

$$a_y^2[\ell] = \mathbb{E}_y \left\{ \frac{1}{N} \sum_{i=0}^{N-1} y[i]y[i+\ell] \right\} \tag{5}$$

$$= \frac{1}{N}\mathbb{E}_{x_{(1)},\ldots,x_{(|s|)},\varepsilon} \left\{ \left(\sum_{j=1}^{|s|} I_{ij} x_{(j)}[i - s_{(j)}] + \varepsilon[i] \right) \left(\sum_{j'=1}^{|s|} I_{i+\ell,j'} x_{(j')}[i + \ell - s_{(j')}] + \varepsilon[i+\ell] \right) \right\}. \tag{6}$$