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Cumulants: A Powerful Tool in Signal Processing

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The impulse response of a linear, time-invariant system is related in a simple closed-form solution to the output cumulants, when the input is assumed to be non-Gaussian and independent. This expression permits the use of one-dimensional processing of the output cumulants for identification of non-minimum-phase systems, and opens new directions in other signal processing applications.

I. PROBLEM STATEMENT

We address the problem of identifying the impulse response of a finite-dimensional, Linear, Time-Invariant (LTI) system, when output (perhaps noisy) observations are provided. The input is unknown, but is assumed to be stationary, non-Gaussian, independent, and identically distributed (i.i.d.) When the input is Gaussian and/or output autocorrelation samples are used, one may obtain only the spectrally equivalent minimum phase (MP) part of the system. The underlying reason is that second-order output statistics are unaffected by all-pass factors, and as such the autocorrelation is a "phase-blind" sequence. To recover a general non-minimum-phase (NMP) model, we need phase sensitive higher order output statistics. The statistics that we propose are the *cumulants* (in frequency domain known as *polyspectra,*) whose relationship with LTI systems, [1], is described by

$$c_k^y(m_1, \cdots, m_{k-1}) = \gamma_k^x \sum_{i=0}^{\infty} h(i)\, h(i+m_1) \cdots h(i+m_{k-1}) \qquad (1)$$

or, in the frequency domain by

$$S_k^y(\omega_1, \cdots, \omega_{k-1}) = \gamma_k^x H(\omega_1) \cdots H(\omega_{k-1})\, H\left(-\sum_{i=1}^{k-1} \omega_i\right) \qquad (2)$$

where $c_k^y(\cdot)$ denotes the output kth-order cumulant, γ_k^x the input kth-order cumulant, $S_k^y(\cdot)$ the output kth-order spectrum, and $h(i)$ [$H(\omega)$] stands for the impulse response (transfer function) of the underlying NMP model. Output autocorrelation and spectrum, can be viewed as special cases of (1) and (2), respectively, when $k = 2$. The input non-Gaussianity is necessary ($\gamma_k^x \neq 0$ for some $k > 2$,) but the i.i.d. assumption can be relaxed by a *kth-order whiteness* assumption, [10].

Based on (2), Lii and Rosenblatt [2] have shown that using higher-order periodogram techniques, the amplitude and phase of $H(\omega)$ can be estimated from output data only (up to a scale and time-delay ambiguity.) The high-variance and low-resolution characteristics of the *Fourier-type* methods [2]–[4] suggested the use of cumulants in conjunction with parametric (MA, AR, ARMA) models [7]–[10]. The *novelty* of this letter (Section II) is to provide a very simple *closed-form solution* of the impulse response (IR) samples in terms of the output cumulants, useful in both the parametric and nonparametric approaches. The potential of this method in various signal processing tasks is analyzed in Section III. In Section IV, we discuss properties of the proposed solution, and comment on its computational aspects.

II. MAIN RESULT

A. FIR Case (MA Models)

Considering the third-order output cumulant [c.f. (1) with $k = 3$] we obtain

$$c_3^y(m_1, m_2) \equiv E\{y(k)\, y(k+m_1)\, y(k+m_2)\}$$
$$= \gamma_3^x \sum_{i=0}^{p} h(i)\, h(i+m_1)\, h(i+m_2) \qquad (3)$$

where $\gamma_3^x \equiv E\{x^3(k)\} \neq 0$, and p denotes the order of the MA model. If we substitute $m_1 = p$, $m_2 = k$, and $m_1 = m_2 = -p$ in (3), and assume that $h(0) = 1$, it is easy to show that $c_3^y(p, k) = \gamma_3^x h(p)\, h(k)$, and $c_3^y(-p, -p) = \gamma_3^x h(p)$. Hence

$$h(k) = \frac{c_3^y(p, k)}{c_3^y(-p, -p)}, \quad \text{for } k = 0, 1, \cdots, p. \qquad (4)$$

Equation (4) states that the IR $\{h(k)\}$ of an LTI system is identical (within a scale factor) to the third-order output cumulant sequence $\{c_3^y(p, k)\}$, and consequently the true NMP system can be recovered using output cumulants only. This can also be verified using a graphical interpretation of (3) (see also Fig. 1). Therefore, from an identification viewpoint, $\{c_3^y(p, k)\}$ plays the role of the cross-correlation sequence $\{r_{xy}(k)\}$, because

$$\sigma_x^2\, h(k) = r_{xy}(k) \equiv E\{x(i)\, y(i+k)\}.$$

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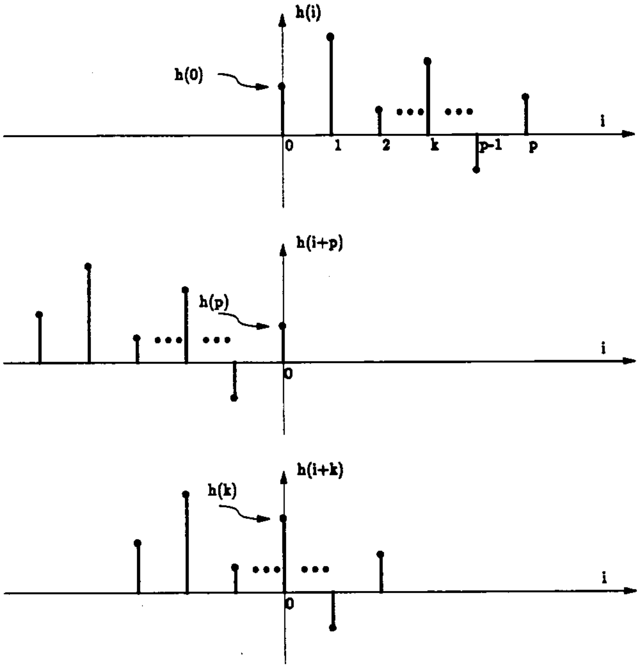

Fig. 1. $\Sigma_{i=0}^{p} h(i)\, h(i+p)\, h(i+k)$ is always a scalar multiple of $h(k)$.

where σ_x^2 denotes the input variance. Notice though, that for the $r_{xy}(k)$ estimation we need both input and output (I/O) data, whereas for the $c_3^y(p, k)$ estimation output data are sufficient.

The order p, if not known, has to be determined via cumulant statistics as in [9]. Moreover, combining (4) and (3) with $m_1 = m_2 = 0$, the input cumulant can be expressed as

$$\gamma_3^x = \frac{c_3^y(0, 0)}{\sum_{k=0}^{p} [c_3^y(p, k)/c_3^y(-p, -p)]^3}. \tag{5}$$

It is interesting that expressions similar to (4) and (5) can be derived for output cumulants of any order. This is particularly useful when $\gamma_3^x = 0$ (e.g., when the input distribution is symmetric,) but $\gamma_k^x \neq 0$ for some $k > 3$. As an example, substituting $k = 4$ in (3), we may obtain

$$h(k) = \frac{c_4^y(p, p, k)}{c_4^y(-p, -p, -p)}, \quad \text{for } k = 0, 1, \cdots, p. \tag{6}$$

B. IIR Case (AR, ARMA Models)

Assuming that our model is stable, and that the input process is purely random (in the Wold's sense) there will exist a finite constant M such that $c_3^y(m_1, m_2) \approx 0$ for every $m_1 > M$. In this case, the results of the previous subsection apply, and the IR is expressed in a *nonparametric* fashion as

$$h(k) \approx \frac{c_3^y(M, k)}{c_3^y(-M, -M)}. \tag{7}$$

Alternatively, if a *parametric* AR or ARMA model is adapted, one may use the equivalence between the $\{h(k)\}$ and $\{c_3^y(M, k)\}$ sequences, to obtain the ARMA model (in an I/O or state-space approach) that "best" fits the $\{c_3^y(M, k)\}$ sequence. Notice, that with output data only, we always have a scale ambiguity.

In both approaches, the Hankel matrix of the $\{c_3^y(p, k)\}$ sequence is formed, and the Singular Value Decomposition (SVD) is employed to yield both the ARMA parameters and the order of the underlying model, as in [10]. SVD distinguishes itself from other solutions because it is robust with respect to noise, finite precision errors, incorrect model order, and imperfect modeling. Basically, with the key equation (7), we have transformed the stochastic realization problem to a deterministic realization, or model fitting problem. Consequently, any classical deterministic realization technique, such as the Ho-Kalman algorithm, or the SVD approach of [6] can be employed to obtain the NMP ARMA model which can be shown to be *stable* along the lines of [10].

III. Areas of Application

Keeping only the significant singular values of the Hankel matrix formed by the $\{c_3^y(p, k)\}$ sequence, one may obtain a *reduced NMP ARMA model* that approximates the original model in the spectral norm. Moreover, for *phase reconstruction* purposes we omit the $c_3^y(-p, -p)$ constant in (3), and after taking the Fourier transform of $h(k)$ we obtain

$$\begin{aligned}\phi(\omega) &= \arg\,[H(\omega)] \\ &= \arg\left[\sum_{k=0}^{p} c_3^y(p, k)\, e^{-j\omega k}\right]\end{aligned} \tag{8}$$

which is a *closed-form solution* of the system's phase in terms of the output cumulants. Furthermore, once the ARMA model has been obtained we may use it for deconvolution (see also [2],) harmonic retrieval, and spectral estimation. For the latter, using the $\{c_3^y(p, k)\}$ samples as constraints one may obtain a unique representative in the family of extrapolating spectra described in [5].

IV. Conclusions–Discussion

The *main contribution* of this letter is a very simple, time-domain solution of the stochastic realization problem that requires *one-dimensional processing*. As opposed to the methods of [2]–[4], [7], and [8] we proved that one-dimensional versions of the output cumulants are sufficient for NMP system identification. The output cumulants are computed through sample averaging, e.g.,

$$\hat{c}_3^y(p, k) = \frac{1}{N} \sum_{i=0}^{N} y(i)\, y(i + p)\, y(i + k). \tag{9}$$

The estimator in (9), and consequently the IR estimator, can be shown to be *consistent* along the lines of [2] and [10]. Additionally, as noted in [3], when symmetrically distributed noise is added to the output, the cumulants remain unaffected. Therefore, the only error introduced comes from (9) when N is finite. Although the SVD results in smooth estimates, further reduction of the error variance can be achieved if we use cumulants of different orders and average over the corresponding estimates of $h(k)$. Finally, accurate estimates can be obtained using optical implementation of the triple correlation as described in [3].

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