$$M_2(k_1,k_2) = \sum_{i,j} x[i,j]\, x[i-k_1, j-k_2]$$

$i' = i-k_1$, $j' = j-k_2$

$$= \sum_{i',j'} x[i'+k_1, j'+k_2]\, x[i',j']$$

$$= M_2[-k_1; -k_2]$$

$$M_3[k_1,k_2,l_1,l_2] = \sum_{i,j} x[i,j]\, x[i-k_1, j-k_2]\, x[i-l_1, j-l_2]$$

$\left.\begin{array}{l} i' = i-k_1 \\ j' = j-k_2 \end{array}\right\} = M_3[-k_1; -k_2; l_1-k_1; l_2-k_2]$ ②

$= M_3[l_1; l_2; k_1; k_2]$ ①

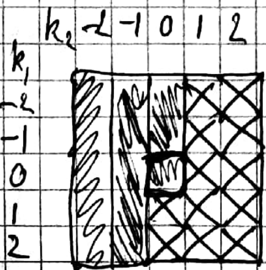
can also get from ① & ② & ①.

$\left.\begin{array}{l} i' = i-l_1 \\ j' = j-l_2 \end{array}\right\} = M_3[k_1-l_1; k_2-l_2; -l_1; -l_2]$

$\left.\begin{array}{l} i' = i-k_1-l_1 \\ j' = \cancel{j-k_2-l_2} \end{array}\right\} \cancel{= M_3[}$

There are also symmetry rules due to circular indexing mod W/N/L...

②nd also



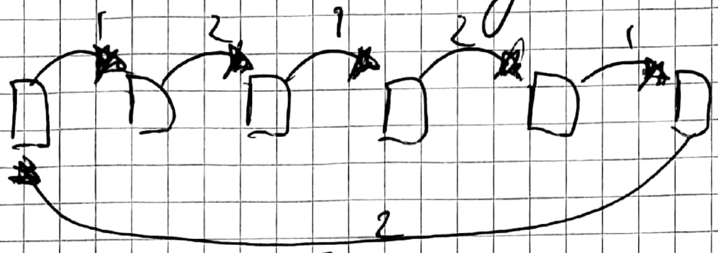
: take

: exclude

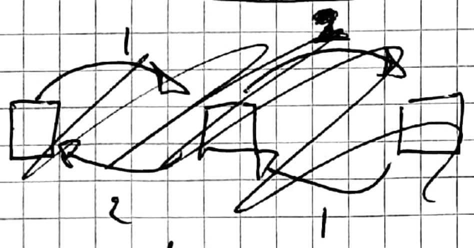
$$W\left(\frac{W-1}{2}\right) + \frac{W-1}{2} + 1 = (W+1)\frac{W-1}{2} + 1$$

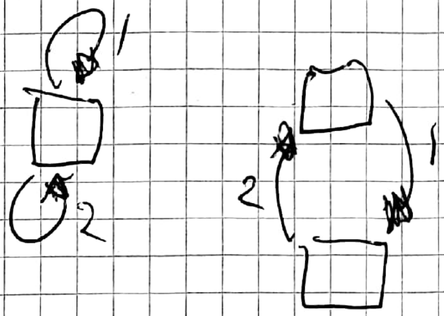
$$= \frac{W^2+1}{2}.$$

Rules 1 & 2 are enough:



: Typical

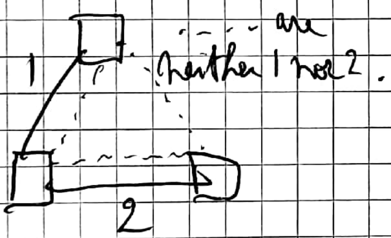


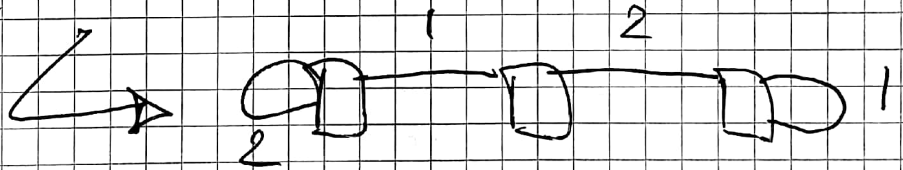


Hmmm.....
not a cycle but has 3 elements.

Well, 1 is reversible: $1^{-1} = 1$.
Same for 2: $2^{-1} = 2$.

So, edges are undirected.





this is fine.

I haven't seen any 4's or 5's.

6 is max because $r_1 \circ r_2 \circ r_1 \circ r_2 \circ r_1 \circ r_2 = id$ (to check)

If any nodes are "out of bounds" we just ignore them

↳ indices outside $\left(-\frac{W-1}{2} ; +\frac{W-1}{2}\right)$.

Big MRA speeding it up.

Jan 16 2018 ♡♡

$X : L \times L$

indexing is 1-based

$M_1 = m\langle X, \mathbb{1}_{L\times L}\rangle$

$$M_2[k_1,k_2] = \underbrace{B_2[k_1,k_2]}_{\text{bias}} + m\boxed{\langle X, ZP^*(C_{k_1k_2}(ZP(X)))\rangle}$$

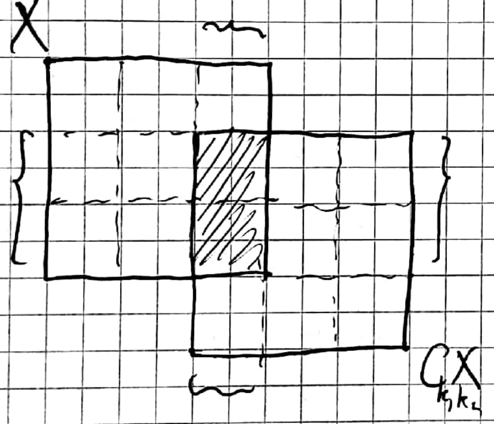
~~A_{k_1,k_2}~~ has a bunch of 0's in it.

$$\sum_{i=1}^{L}\sum_{j=1}^{L} X[i,j]\, X[i\pm k_1, j\pm k_2]$$

±? ; also, if out of bounds 0.

i: $k_1 \geq 0$: $(1+k_1):L$

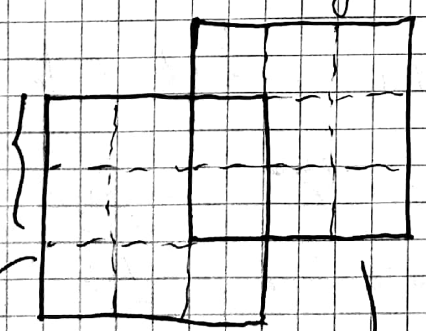
j: $k_2 \geq 0$: $(1+k_2):L$



i: $k_1 \geq 0$: $1 : L-k_1$

j: $k_2 \geq 0$: $1 : L-k_2$ ✓

What if k_1, k_2 not ≥ 0?



$k_1 = -1$

$k_2 = 2$

it's like changing roles of i and $i+k_1$.

Do change of variable: $i' = i+k_1$;

then, $i = i' - k_1 = i' + \underbrace{(-k_1)}_{\geq 0}$.

$1+k_1 : L \to 1 : L+k_1$

$2:3 \to 1:2$

$1 : L-k_1 \;//\; 1:2$

$\to 1-k_1 : L \;//\; 2:3$

We actually do end up computing all possible "shifts + innerproduct" in FFT? Conv! That's specific to Π_2 though; for Π_3 we will subsample, and triple products, so....

filter2(X, x, 'full')

= conv2(X, rot90(X, 2))

← built-in.

This is most likely a winning strategy for Π_2 since we consider all possible shifts.

Not sure about derivatives though.

conv2 and rot90 are both linear, so, should be fine if we have the adjoints.

$$\Pi_3[k_1 k_2 l_1 l_2] = B_3[k_1 k_2 l_1 l_2] + m \sum_{ij} X[i,j]X[i+k_1, j+k_2]X[i+l_1, j+l_2]$$

if fix (k_1, k_2), can reduce to $\langle X \cdot X_{k_1 k_2}, X_{l_1 l_2} \rangle$.

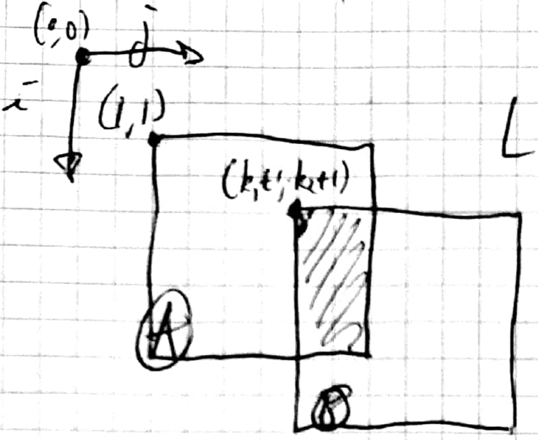
$$\sum_{i=1}^{L} \sum_{j=1}^{L} x[i,j]x[i+k_1, j+k_2]$$

$$= \sum_{i'=1+k_1}^{L+k_1} \sum_{j'=1+k_2}^{L+k_2} x[i'-k_1, j'-k_2]x[i', j'] \text{ conf.}$$

$i' = i+k_1$

$j' = j+k_2$

How about reason in terms of <u>constraints and coordinates</u>?



$L=3$

Might generalize better.

$\boxed{\text{indexing 1-based}}$

///: (A) $1 \leq i \leq L$

$1 \leq j \leq L$

(B) $k_1+1 \leq i \leq k_1+L$

$k_2+1 \leq j \leq k_2+L$

$\max(1, 1+k_1) \leq i \leq \min(L, L+k_1)$

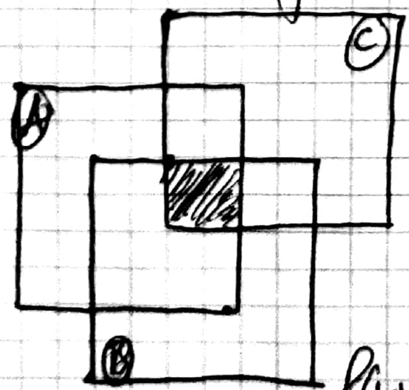
$\max(1, 1+k_2) \leq j \leq \min(L, L+k_2)$

"in" region.

Then need to locate "in" region in each of (A) and (B);

as is — subtract $(k_1; k_2)$: valid by def.

Should work same for 3 images:



So, for given k_1, k_2, ℓ_1, ℓ_2

there are 3 operators which extract a window of X: T_1, T_2, T_3

$$f(X) = \Pi_3[k_1 k_2 \ell_1 \ell_2] = B_3[k_{12}\, \ell_{12}]$$
$$+ m\, \mathbb{1}^T\left(T_1(X) \odot T_2(X) \odot T_3(X)\right)\mathbb{1}$$

$$Df(X)[\dot{X}] = m\, \mathbb{1}^T\left(T_1(\dot{X}) \odot T_2 \odot T_3 + T_1 \odot \dot{T}_2 \odot T_3 + T_1 \odot T_2 \odot \dot{T}_3\right)\mathbb{1}$$
$$= m \left\langle T_1^{*}(T_2 \odot T_3) + T_2^{*}(T_1 \odot T_3) + T_3^{*}(T_1 \odot T_2) ; \dot{X} \right\rangle$$

So, want the adjoint of

$$T_i(X) = X_{I,J} \quad \text{(extraction)}$$

$$= \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix} X \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$Y = T_i^*(A) =$ place values of A in X where T_i would extract them, and rest as 0.

```
X = zeros(L, L);
X(I, J) = A;
```

$$f(X) = m \, \mathbb{1}^T \left(T_1(X) \odot T_2(X) \right) \mathbb{1}$$

$$Df(X)[\dot{X}] = m \left\langle \underline{T_1^*(T_2(X)) + T_2^*(T_1(X))}, \dot{X} \right\rangle$$

Chat w/ Suvrit @ MIT Jan 18

- determinantal pt processes, MCMC sampling, learning (even w/ cardinality fixed)
  - $\to$ nice on pos. def. matrices
- GMM: condition # of Hess to explain good behavior of his reformulation
- Stable polynomials, strongly Rayleigh measures
  Kadison - Singer $\to$ Wibisono (now in Wisconsin Madison)
- Mike Jordan's phd student Andre has thesis on ODE view of accelerat and didn't see a way for M.

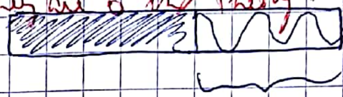
BigPRA

$\boxed{\text{Jan 31 2018}}$ clean signal

L-1 L: signal L-1: separation

Zero-padded
- Correlations with shift $-(L-1)$ —— $(L-1)$ are non-zero.

everything else is 0. Negative is symmetric, so: $0 - (L-1)$ are relevant.
not really: another signal could occur. In any case: we'd rather compute them. We can state they are 0 in theory.

- when we optimize with variable $y =$  : $L + (L-1) = 2L - 1$

We're hoping this will be 0, but not the case yet.

So, really, we want 0-padded auto-correlations of y to match

data from $0..L-1$,

0 from $L..W-1$ (to ∞ actually, but trivial for $W+$.)

Zero-padded ~~auto~~ correlations correspond to the first entries of circulant auto correlations (accessible through FFT) if we pad with $W-1$ zeros at least.
(ok to do more, but wasteful.)

This shouldn't be enough anyway given phase retrieval ill-posedness in 1D:

$x \in \mathbb{R}^L$ $\quad Zp(x) = \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^{L+(L-1) = 2L-1}$

$u = F^{-1}\left(|F(Zp(x))|^2\right)$: contains L distinct real numbers: one would think it would be enough.

$$f(x) = \frac{1}{2}\| F(u) - \overbrace{F(Zp(x)) \odot \overline{F(Zp(x))}}^{= R(x)} \|_2^2$$

$Df(x)[\dot{x}] = \langle R(x), \odot F(Zp(x)) \cdot$ ----

$\nabla f(x) = 2 \cdot Zp^*(F^*(R(x) \odot F(Zp(x))))$

See Matlab quick test — second order — zero-padded m: indeed not enough.

$g_d : \mathbb{R}^d \to \mathbb{R}^{d}$: gives the zero-padded auto-correlations:

$$g_d(y)[k] = \langle y_{k:d-1}, y_{0:d-k-1} \rangle \qquad k = 0..d-1$$

zero-based

Can be implemented with FFT, but that's not the point here.

$\cancel{g_d\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right)}$ $\quad g_{L+(L-1)}\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = g\left(\begin{bmatrix} g_L(x) \\ 0 \end{bmatrix}\right)$, $x \in \mathbb{R}^L$, $0 \in \mathbb{R}^{L-1}$; $g_L(x) \to L$, $0 \to L-1$; $= 2L-1$

this (L-1) could be anything: what is special about L-1?

$g_{2L-1} \circ zp = zp \circ g_L$, with appropriate sizes.

zp: adds L-1 zeros

So, if $y \in \mathbb{R}^{2L-1}$ is our optimization variable,

and $M_2 = g_L(x)$ are the measured moments,

we want $\quad g_{\underbrace{2L-1}_{L+K}}(y) \sim \underbrace{zp}_{\text{add K zeros}}(M_2)$, to favor $y \sim zp(x)$;

and not merely $\quad [g_{2L-1}(y)]_{0:L-1} \sim M_2$

Separation (min distance between to starting pts of x in observation): $W \geq 2L-1$.

2L-1 is necessary so that when we shift the micro (get auto-correlations) we can get all zero-padded auto-corr of x w/o interference (one occurrence touching another one.)

requires shifts from 0..L-1: need the extra L-1 spacing. But don't need more.

This separation of L-1 allows us to compute an estimate of $g_L(x)$, which is all we need and also all we can get. THE SIZE OF THE OPTIM VARIABLE IS SEPARATE!

Scanned by CamScanner

$F : \mathbb{R}^d \to \mathbb{C}^d$

$F(x)$

This won't work: you need the FT's of zero-padded signals.

0-based indexing

$$\hat{x}[k] = F(x)[k] = \sum_{l=0}^{d-1} x[l] \, e^{-2\pi i \frac{kl}{d}}$$

$$x[l] = \frac{1}{d} \sum_{k=0}^{d-1} \hat{x}[k] \, e^{2\pi i \frac{kl}{d}}$$

let $x \in \mathbb{R}^d$, with zero-based indexing and out-of-bounds indexing gives 0 (zero-padding):

$0 \leq l \leq d-1$, $0 \leq l-s \leq d-1$ $\to$ $\max(0,s) \leq l \leq \min(d-1, d-1+s)$

$$\Gamma_2^{x}[s] = \sum_{l=-\infty}^{\infty} x[l] \, x[l-s]$$

(change of var $l-s \mapsto l$ shows $\pm s$ is irrelevant.)

no conjugate for $\mathbb{R}$, but it would make sense..

$$= \frac{1}{d^2} \sum_{l} \sum_{k_1=0}^{d-1} \sum_{k_2=0}^{d-1} \hat{x}[k_1] \hat{x}[k_2] \exp\left(\frac{2\pi i}{d}[k_1 l + k_2(l-s)]\right)$$

$$= \frac{1}{d^2} \sum_{k_1=0}^{d-1} \sum_{k_2=0}^{d-1} \hat{x}[k_1] \hat{x}[k_2] \, e^{-\frac{2\pi i}{d} k_2 s} \cdot \sum_{l} e^{2\pi i \frac{(k_1+k_2) l}{d}}$$

$\to$ have to limit range of l to $[\max(0,s), d-1+\min(0,s)]$ for the Fourier formula to be valid.

Otherwise, need to do

$$x[l] = \begin{cases} \text{F.T.} & \text{if } l \in 0..d-1 \\ 0 & \text{otherwise} \end{cases}$$

if $k_1 + k_2 = 0$,

$$\sum_{l=\dots}^{\dots} e^{2\pi i \frac{(k_1+k_2) l}{d}}$$

$= d-1+\min(0,s) - \max(0,s) + 1$

$= d - |s|$. Even if $k_1+k_2 \neq 0$, get non trivial thing.

So, what you really want is:

$x \in \mathbb{R}^L$, zero based indexing, 0 out of bounds

$$\hat{x}_d[k] = F_d(x)[k] = \sum_{l=0}^{d-1} x[l]\, e^{-2\pi i \frac{kl}{d}}$$: the only change is that $d \neq L$ in general.

k: $0..d-1$; l will be accessed out of bounds

FT of x zero padded to length d (or truncated)

$$x[l] = \frac{1}{d} \sum_{k=0}^{d-1} \hat{x}_d[k]\, e^{+2\pi i \frac{kl}{d}}$$

l: $0..d-1$, with $L..d-1$ being 0 by construction.

$$M_2(x)[s] = \sum_{l=-\infty}^{\infty} x[l]\, x[l-s]$$ ← this formula is independent of d. well defined as is!

need $0 \leq l \leq d-1$

$0 \leq l-s \leq d-1$ for a chosen d: still free to choose it!

$$= \sum_{l=\max(0,s)}^{\min(0,s)+d-1} x[l]\, x[l-s]$$

$$= \frac{1}{d^2} \sum_{\substack{k_1=0\\k_2=0}}^{d-1} \hat{x}_d[k_1]\, \hat{x}_d[k_2] \boxed{\sum_{l=\cdots}^{\cdots} e^{-2\pi i \frac{k_1 l + k_2(l-s)}{d}}}$$

$$= \left(e^{2\pi i \frac{k_2 s}{d}}\right) \cdot \sum_{l=\max(0,s)}^{\min(0,s)+d-1} e^{-2\pi i \frac{l(k_1+k_2)}{d}}$$ annoying.

Try differently: take FT of $M_2(x)$:

$$F_d(M_2(x))[t] = \sum_{s=0}^{d-1} \left[\sum_{l=-\infty}^{\infty} x[l]\, x[l-s]\right] e^{\overset{?}{-}+2\pi i \frac{st}{d}}$$

→ well defined for any s, and is 0 for $s \geq d$.

$$= \sum_{l=-\infty}^{\infty} x[l] \left[\sum_{s=0}^{d-1} x[l-s]\, e^{\overset{?}{-}+2\pi i \frac{st}{d}}\right] \quad s' = l-s$$

$$= \sum_{l=-\infty}^{\infty} x[l] \sum_{s'=l}^{l-(d-1)} x[s']\, e^{\overset{?}{-}+2\pi i \frac{(l-s')t}{d}}$$

$$F_d(\Pi_2(x))[t] = \sum_{l=-\infty}^{\infty} x[l]\, e^{2\pi i \frac{lt}{d}} \underbrace{\sum_{s'=l-(d-1)}^{l} x[s']\, e^{-2\pi i \frac{s't}{d}}}$$

signs in exponents??

ok to replace w/ $l = 0 .. d-1$ owing to $x[l]$ being 0 otherwise. If so, s' can start at 0.

Could even go from $0 .. L-1$, but unclear that would help.

need to make this independent of l.

I want it to be equal to $|F_d(x)[t]|^2$, but ranges are weird.

$$= \sum_{l=0}^{d-1} x[l]\, e^{2\pi i \frac{lt}{d}} \boxed{\sum_{s'=0}^{l} x[s']\, e^{-2\pi i \frac{s't}{d}}}$$

$$= \sum_{s'=0}^{d-1} x[s']\, e^{-2\pi i \frac{s't}{d}} - \sum_{s'=l+1}^{d-1} x[s']\, e^{-2\pi i \frac{s't}{d}}$$

$$= \hat{x}_d[t] - \boxed{\sum_{s'=l+1}^{L-1} x[s']\, e^{-2\pi i \frac{s't}{d}}}$$

can we get rid of this?

only necessary for $l+1 \le L-1$, as otherwise the sum is trivially 0: $l \le L-2$.

Well, for $t=0$ we're already in trouble.

What is happening? Recall that we only care for $t \in 0 .. L-1$

and $d = 2L-1$ or more.

Might be easier to rely on cyclic result first,
then argue that zeropadding makes it equiv. to cyclic!
(since we only care about first half of results.)

START HERE

let $x \in \mathbb{R}^L$ be my signal; out-of-bounds is 0.

$x_d \in \mathbb{R}^d$ is zero padded to length $d \geq L$.
and indexing is now cyclic for x_d.

$\hat{x}_d = F(x_d)$: FT of x_d. : fft(x, d) in Matlab

~~By the~~ standard formula of autocorrelation with cyclic indexing:

$$\hat{x}_d[k] = \sum_{l=0}^{d-1} \underbrace{x_d[l]}_{= x[l] \text{ over that range and with our conventions.}} e^{-2\pi i \frac{kl}{d}}$$

$$P(x_d)[k] = |\hat{x}_d[k]|^2 = \hat{x}_d[k] \cdot \overline{\hat{x}_d[k]}$$

$$= \sum_{l_1=0}^{d-1} \sum_{l_2=0}^{d-1} x_d[l_1] \overline{x_d[l_2]} \, e^{2\pi i \frac{k(l_2 - l_1)}{d}}$$

Take FT^{-1} of $P(x_d)$ to get auto-correlation of x_d:

$$F^{-1}(P(x_d))[s] = \frac{1}{d} \sum_{k=0}^{d-1} P(x_d)[k] \, e^{2\pi i \frac{ks}{d}}$$

$$= \frac{1}{d} \sum_{l_1=0}^{d-1} \sum_{l_2=0}^{d-1} x_d[l_1] \overline{x_d[l_2]} \underbrace{\boxed{\sum_{k=0}^{d-1} e^{2\pi i \frac{l_2 - l_1 + s}{d} k}}}_{= d \cdot \delta_{k = l_1 - l_2 \text{ mod } d} = d \cdot \delta_{l_2 = l_1 - s \text{ mod } d}}$$

this step uses cyclic indexing because the δ is mod d

$$= \sum_{l=0}^{d-1} x_d[l] \overline{x_d[\underbrace{l - s}_{\text{depends on } s \text{ now, so also mod.}}]}$$

for this range, non-zero if $l \in 0..L-1$

$\cancel{F^{-1}(P_{x_d}[s]))}$

$$F^{-1}(P(x_d))[s] = \sum_{l=0}^{L-1} x[l]\,\overline{x_d[l-s]}$$

$$x_d(l-s) = x_d(l-s \bmod d)$$

$$= \begin{cases} x[l-s \bmod d] & \text{if } \overbrace{(l-s \bmod d)}^{0..d-1} \leq L-1, \\ 0 & \text{otherwise} \end{cases}$$

$l = 0 .. L-1$

$s = 0 .. L-1$: because for x it's not interesting after that.

$l-s = -(L-1) ... (L-1)$; given this range:

$$l-s \bmod d = \begin{cases} l-s & \text{if } l-s \geq 0, \\ l-s+d & \text{if } l-s < 0, \end{cases}$$

provided $d \geq L$.

Furthermore, if $d \geq 2L-1$ (*), then

- if $l-s \geq 0$:

$$x_d[l-s] = x_d[l-s \bmod d] = x_d[l-s] = x[l-s];$$

- if $l-s < 0$:

$$x_d[l-s] = x_d[l-s \bmod d] = x_d[\overbrace{\underbrace{l-s+d}_{\geq -(L-1)+2L-1 \,(*) \,= L}}^{\leq d-1}] = 0 = x[l-s].$$

Thus, in all cases, w/ the "0 out of bounds" convention,

$$\boxed{F^{-1}(P(x_d))[\Delta]} = \sum_{\ell=0}^{L-1} x[\ell]\overline{x[\ell-\Delta]},$$

$$= \boxed{\sum_{\ell=\Delta}^{L-1} x[\ell]\overline{x[\ell-\Delta]}}$$

for $d \geq 2L-1$ and $\Delta = 0..L-1$. □

In Matlab, $F(x_d) = \mathrm{fft}(x, d)$ ← zero pads.

Triple correlations

$$B(x_d)[k_1,k_2] = \hat{x}_d[k_1]\overline{\hat{x}_d[k_2]}\hat{x}_d[k_2-k_1]$$

$$= \sum_{\ell_1=0}^{d-1}\sum_{\ell_2=0}^{d-1}\sum_{\ell_3=0}^{d-1} x_d[\ell_1]\overline{x_d[\ell_2]}x_d[\ell_3]\, e^{-2\pi i \frac{k_1\ell_1 - k_2\ell_2 + (k_2-k_1)\ell_3}{d}}$$

Take a 2D inverse FT: F^{-1} along k_1 and k_2:

$$F^{-1}(B(x_d))[\Delta_1,\Delta_2] = \frac{1}{d^2}\sum_{k_1=0}^{d-1}\sum_{k_2=0}^{d-1} B(x_d)[k_1,k_2]\, e^{2\pi i \frac{k_1\Delta_1 + k_2\Delta_2}{d}}$$

$$= \frac{1}{d^2}\sum_{\ell_1\ell_2\ell_3} x_d[\ell_1]\overline{x_d[\ell_2]}x_d[\ell_3] \boxed{\sum_{k_1,k_2} e^{2\pi i \frac{k_1(\Delta_1-\ell_1+\ell_3)+k_2(\Delta_2+\ell_2-\ell_3)}{d}}}$$

$$\hookrightarrow = d\cdot\delta_{\Delta_1 = \ell_1-\ell_3 \bmod d} \cdot d\cdot\delta_{\Delta_2=\ell_3-\ell_2 \bmod d}$$

$\ell_3 = \ell_1 - \Delta_1 \ldots$ $\qquad$ $\ell_3 = \ell_2+\Delta_2 \ldots$

$\ell_1 = \Delta_1 + \ell_3 \ldots$ $\qquad$ $\ell_2 = \ell_3 - \Delta_2 \ldots$

$$= \sum_{\ell_3=0}^{d-1}\sum_{\ell_2=0}^{d-1}\sum_{\substack{\ell_1=0 \\ \ell_1=\ell_3+\Delta_1 \bmod d \\ \ell_2=\ell_3-\Delta_2 \bmod d}}^{d-1} x_d[\ell_1]\overline{x_d[\ell_2]}x_d[\ell_3]$$

using cyclic indexing of x_d

$$= \sum_{\ell=0}^{d-1} x_d[\ell+\Delta_1]\overline{x_d[\ell-\Delta_2]}x_d[\ell]$$: now need to reduce this to a formula with x, not x_d.

Scanned by CamScanner

since $x_d[l] \neq 0$ only for $l = 0..L-1$ in $0..d-1$:

$$F^{-1}(B(x_d))[\Delta_1;\Delta_2] = \sum_{l=0}^{L-1} x[l]\,\overline{x_d[l-\Delta_2]}\,x_d[l+\Delta_1]$$

if $\Delta_1, \Delta_2 = 0 .. L-1$ (which is all we need),

(indeed: one is fixed, one shifts left, one shifts right. for the moving parts to intersect at least the fixed part, can do no more than $L-1$ move; even that will give many 0's.)

⚠ There are many structural 0's in the triple correlation: need to avoid them when we sample.

$l = 0..d-1$; $\Delta_1 = 0..L-1$; $\Delta_2 = 0..L-1$

$l - \Delta_2 = -(L-1)..(L-1)$

$l + \Delta_1 = 0..2L-2$

→ this one is easy:

$x_d[l+\Delta_1] = 0$ unless $0 \leq l+\Delta_1 \leq L-1$,

using that $l+\Delta_1 \in 0..2L-2 \leq d-1$:

we only access the first period of x_d if $d \geq 2L-1$

Then, restrict $l \leq L-1-\Delta_1$, and $x_d[l+\Delta_1] = x[l+\Delta_1]$.

Also, if $l - \Delta_2 \geq 0$: $x_d[l-\Delta_2] = x[l-\Delta_2]$,

if $l-\Delta_2 < 0$, $x_d[l-\Delta_2] = x_d[\underbrace{d+l-\Delta_2}_{L \leq \cdot \leq d-1 \text{ if } d \geq 2L-1}] = 0 = x[l-\Delta_2]$.

⇒ restrict $l \geq \Delta_2$

$$\boxed{F^{-1}(B(x_d))[\Delta_1;\Delta_2] = \sum_{\ell=\Delta_2}^{L-1-\Delta_1} x[\ell]\,x[\ell-\Delta_2]\,x[\ell+\Delta_1]}$$

renaming is more natural. Can't rename.

with $d \geq 2L-1$, $\Delta_1;\Delta_2 \in 0..L-1$.

this is 0 if the sum is empty: $\Delta_2 > L-1-\Delta_1 \equiv \Delta_1+\Delta_2 > L-1$.

$\Rightarrow$ Only consider entries with
$$\boxed{\begin{array}{l}\Delta_1 : 0..L-1\\ \Delta_2 : 0..L-1\\ \Delta_1+\Delta_2 \leq L-1\end{array}}$$

Then also factor in real symmetries if easy to do:
I don't see any, except for the obvious repetitions fact M_3 is real if x is real.

See Matlab code moments_via_Fourier_1D.m

$$\Rightarrow M_2 = \left[F_{1D}^{-1}(P(x_d))\right]_{0:L-1}$$

$$M_3 = \left[F_{2D}^{-1}(B(x_d))\right]_{\substack{0:L-1:\Delta_1,\\ 0:L-1:\Delta_2\\ \Delta_1+\Delta_2 \leq L-1}}$$

This and $M_1 = \sum_{\ell=0}^{L-1} x[\ell] = F(x_d)[0]$;

Ideally, find $x \in \mathbb{R}^L$ that matches those moments, but empirically it's better to search for x_d directly, ignoring the 0's at first?

What does this really mean? invariances? Ambiguities?

In general, I think that if you are interested in autocorrelations of any order with zero-padded shifts separated by up to $K-1$ entries (not, for the Power Spectrum, the first K entries), need to zero-pad w/ $K-1$ additional 0's, in order to be able to compute those correlations via FFT's.

in general, the condition on the shifts is that any pair of the copies of the signals has to overlap:

$|0-\Delta_1| \leq L-1$

$|0+\Delta_2| \leq L-1$

$|\Delta_1+\Delta_2| \leq L-1$

This should also generalize well to 2D etc: separable along dimensions.

BigMRA 1D

Feb 2, 2018

perhaps $L' = L+1$ is enough relaxation? 2L-1 seems arbitrary.

$$f(x) = \frac{1}{2m}\left[w_1 |R_1|^2 + w_2 \|R_2\|_2^2 + w_3 \|R_3\|_F^2 \right]$$

Take $x \in \mathbb{R}^{L'}$, with $L \le L'$

awkward: seems like formulas only work with $L' = L$?? no? unsure.

Let $\hat{x} = F_{L'+L'-1}(x)$: x zero-padded to $L'+L'-1$ if necessary, then DFT.

because we need $L'-1$ zeros to access zero-padded autocorrelations of x with shift differences up to $L'-1$:

$$R_1 = m\hat{x}[0] - M_1$$

$$R_2 = m\left[F^{-1}(|\hat{x}|^2)\right]_{0:L'-1} + \text{bias} - M_2$$
$\to \mathbb{R}^{L'}$ (1)

0..L-1 from data, rest is 0.

0..L-1 matches data

L..L'-1 should be 0.

$$R_3 = m\left[F_{2D}^{-1}(\hat{x}\hat{x}^* \circ C(\hat{x}))\right]_{\substack{0:L'-1\\0:L'-1\\ \Delta_1,\Delta_2 \le L'-1}} + \text{bias} - M_3$$
$\to \mathbb{R}^{L' \times L'}$, with

$M_3[\Delta_1, \Delta_2] = 0$ if $\Delta_1 + \Delta_2 \ge L$.

still correct.

$\hat{x} = F(Z(x))$, $Z: \mathbb{R}^{L'} \to \mathbb{R}^{2L'-1}$: zero pads

$F: \mathbb{R}^{2L'-1} \to \mathbb{C}^{2L'-1}$: 1D DFT in $2L'-1$.

$$g_1(x) = \frac{w_1}{2m} \left| m\hat{x}[0] - M_1 \right|^2$$

$$= \frac{w_1}{2m} \left| m \mathbb{1}_{L'}^T x - M_1 \right|^2$$

$$\boxed{\nabla g_1(x)} = \frac{w_1}{\cancel{m}} \cancel{m} \cdot (m\mathbb{1}_{L'}^T x - M_1)\mathbb{1}_{L'} = \boxed{w_1 R_1 \cdot \mathbb{1}_{L'}}$$

KINDA TEMPTING to optimize directly with $\hat{x}$, but then it's hard to impose that $\hat{x} \in \mathbb{C}^{2L'-1}$ comes from DFT of zeropadded $x \in \mathbb{R}^{L'}$: don't.

$$g_2(x) = \frac{w_2}{2m} \| m\, S\left(F^{-1}\left(F(Z(x)) \odot \overline{F(Z(x))}\right)\right) + \underbrace{N\sigma^2 e_0}_{\text{bins}} - M_2 \|_2^2$$

S: subsample entries $0 \ldots L'-1$ out of $2L'-1$ entries.

$e_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^L$

$$Dg_2(x)[\dot{x}] = \frac{w_2}{\cancel{m}} \left\langle R_2 \, ; \, \cancel{m}\, S\left(F^{-1}\left(F(Z(\dot{x})) \odot \overline{F(Z(x))} + F(Z(x)) \odot \overline{F(Z(\dot{x}))}\right)\right) \right\rangle$$

$$= w_2 \left\langle 2\mathrm{Re}\left\{F^{-*}(S^*(R_2))\right\} \odot F(Z(x)),\ F(Z(\dot{x})) \right\rangle$$

$$\boxed{\nabla g_2(x) = 2 w_2 Z^*\left[\underbrace{F^*\left[\mathrm{Re}\left\{F^{-*}(S^*(R_2))\right\} \odot \underbrace{F(Z(x))}_{\hat{x}}\right]}_{\text{see if simplifies}}\right]}$$

$S^* = \tilde{Z}$, but careful that we mean here to zero pad from L' to $2L'-1$, whereas Z is ~~also~~ zero pad from L' to $2L'-1$: same now.

$Z^* = \tilde{S}$: extract entries $0 \ldots L'-1$ from $2L'-1$.

$F^{-*} = \frac{1}{2L'-1} F$, $\frac{1}{2L'-1} F^* = F^{-1}$ (checked w/ dftmtx)

~~$\nabla g_2(x) = 2 w_2$~~ simplify notation.

$$\boxed{\nabla g_2(x) = 2 w_2 \tilde{S}\left[F^{-1}\left[\mathrm{Re}\left\{F(\tilde{Z}(R_2))\right\} \odot \hat{x}\right]\right]}$$

$$g_3(x) = \frac{w_3}{2m} \left\| m\, S\left[F_{2D}^{-1}\left(\hat{x}\hat{x}^* \odot C(\hat{x})\right)\right] + B_3 - M_3 \right\|_F^2$$

S: take $L' \times L'$ upper left and zero out lower right triangle. → only the upper left $L \times L$ triangle is non-zero.

B_3: bias (constant matrix)

$$Dg_3(x)[\dot{x}] = \frac{w_3}{\cancel{m}} \left\langle R_3 \, , \, \cancel{m}\, S\left[F_{2D}^{-1}\left(\dot{\hat{x}}\hat{x}^* \odot C(\hat{x}) + \hat{x}\dot{\hat{x}}^* \odot C(\hat{x}) + \hat{x}\hat{x}^* \odot C(\dot{\hat{x}})\right)\right] \right\rangle$$

$$\cancel{\nabla g_3(x)} = w_3 \left\langle 2H\left[F_{2D}^{-*}(S^*(R_3)) \odot \overline{C(\hat{x})}\right]\hat{x} + C^*\left(F_{2D}^{-*}(S^*(R_3)) \odot \overline{\hat{x}\hat{x}^*}\right), \dot{\hat{x}} \right\rangle$$

$H(A) = \frac{A + A^*}{2}$

S^*: zero pad to $(2L'-1) \times (2L'-1)$

$\dot{\hat{x}} = F_{1D}(Z(\dot{x}))$: still need to take $Z^*(F_{1D}^*(\ldots))$