Let us start with a 2R Manipulator robot, one of the most common robots.

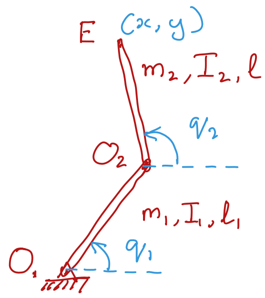
2R = 2 Revolute joints

Two most common type of joints

Revolute (R) & Prismatic (P)

2R Manipulator

Also called planar elbow manipulator



E - end effector

(x, y) - end effector position

(q_1, q_2) - joint angles

Note q_2 convention.

Assume origin at O_1.

Let us assume that motors are connected to both joints O_1 and O_2 and we have the ability to control either the torques τ_1 and τ_2 applied at these joints or control the angles q_1 and q_2.

Angles are sometimes θ_1 & θ_2, or p_1 & p_2 in the book.

We will study later how (hardware, algorithm, and software) we can control τ_1 & τ_2 or q_1 & q_2.

Let us consider 3 tasks

Task 1 (T1) - Given arbitrary trajectory of end effector (given x, y function of time), make the robot follow this trajectory.

Task 2 (T2) - Given a location of a wall, make the robot touch the wall and apply a constant force on the wall.

Task 3 (T3) - Make the robot behave like a virtual spring is connected from E to a given point (x_0, y_0).

Now,

$$x = l_1 \cos q_1 + l_2 \cos q_2$$

&

$$y = l_1 \sin q_1 + l_2 \sin q_2$$

or using simplified notation

$$\left.\begin{aligned} x &= l_1 c q_1 + l_2 c q_2 \\ y &= l_1 s q_1 + l_2 s q_2 \end{aligned}\right\} - 1$$

Differentiating 1, we get

$$\dot{x} = -l_1 s q_1 \cdot \dot{q}_1 - l_2 s q_2 \cdot \dot{q}_2$$

$$\dot{y} = l_1 c q_1 \cdot \dot{q}_1 + l_2 c q_2 \cdot \dot{q}_2$$

$\Rightarrow$

End-effector velocity

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & -l_2 s q_2 \\ l_1 c q_1 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad - 2$$

We will also need the reverse relationships. Given x & y, we need to be able to solve for q_1 & q_2 using 1.

Option 1 - Solve numerically

Option 2 - Derive closed-form expression

- Hard in general
- Multiple solutions

Cosine rule + switch to acute angle

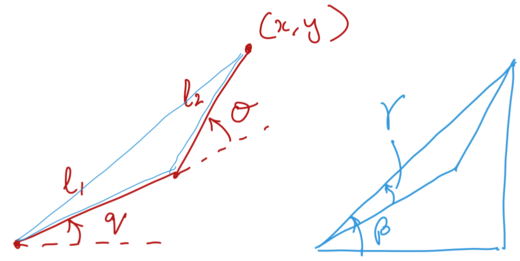
$$\theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2 l_1 l_2}\right)$$

$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos}\right)$$

$\beta - \gamma$

$$q_2 = q_1 + \theta$$

(3)

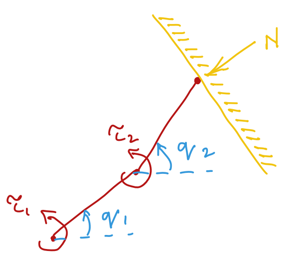


Control both motors in position control mode to achieve above q_1 & q_2 at each time step.

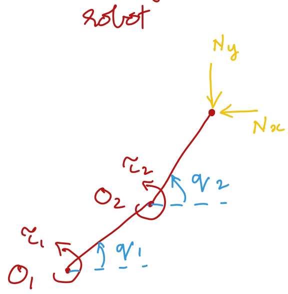
First level answer to T1

We will later start using the notation x_d and y_d (and q_{1d} and q_{2d}) here for desired values (they are not necessarily actual values).

Task T2



FBD of entire robot



Forces applied by manipulator

F_y

F_x

Neglect gravity for the moment.

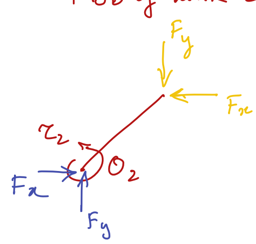
Static equilibrium

$\Rightarrow$ $\Sigma M_{O_2} = 0$ & $\Sigma M_{O_1} = 0$ using FBD of each link respectively

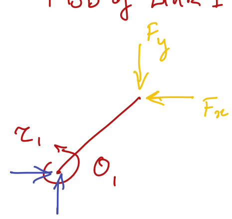
$\Rightarrow$

$$\boxed{\begin{aligned} F_y l_2 c q_2 - F_x l_2 s q_2 &= \tau_2 \\ F_y l_1 c q_1 - F_x l_1 s q_1 &= \tau_1 \end{aligned}} \quad - (4)$$

FBD of Link 2



FBD of Link 1



Note: These FBDs and (4) are valid for any F_x, F_y

(3) along with (4) answers T2.

For T3 and next-level answer to T1, need to understand the dynamics.

Lagrange's Equations : Lagrangian $\mathcal{L} = K - V$

K - Kinetic energy, V - potential energy.

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i'} \quad (5)$$

Q_i' s are generalized forces derived using principle of virtual work.

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2}_{\text{pure rotation of } L1} + \underbrace{\frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\dot{q}_2^2}_{\text{rotation of } L_2 \text{ about c.G.}} + \underbrace{\frac{1}{2} m_2 v_{c_2}^2}_{\text{translation of } L_2}$$

$$v_{c_2}^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2 \dot{q}_2}{2}\right)^2 + 2 l_1 \dot{q}_1 \frac{l_2}{2} \dot{q}_2 \cos(q_2 - q_1)$$

$$V = m_1 g \frac{l_1}{2} s q_1 + m_2 g \left(l_1 s q_1 + \frac{l_2}{2} s q_2\right)$$

$$\boxed{\begin{aligned}&\frac{1}{3}m_1 l_1^2\ddot{q}_1 + m_2 l_1^2\ddot{q}_1 + m_2\frac{l_1 l_2}{2}\ddot{q}_2\cos(q_2-q_1) - m_2\frac{l_1 l_2}{2}\dot{q}_2(\dot{q}_2-\dot{q}_1)\sin(q_2-q_1) + m_1 g\frac{l_1}{2}cq_1 + m_2 g l_1 cq_1 = \tau_1 \quad (3)\\ &\frac{1}{8}m_2 l_2^2\ddot{q}_2 + m_2\frac{l_2^2}{4}\ddot{q}_2 + m_2\frac{l_1 l_2}{2}\ddot{q}_1\cos(q_2-q_1) - m_2\frac{l_1 l_2}{2}\dot{q}_1(\dot{q}_2-\dot{q}_1)\sin(q_2-q_1) + m_2 g\frac{l_2}{2}sq_2 = \tau_2 \quad (6)\end{aligned}}$$

Save this, will need several times later.

Next, we note that (4) is valid for any end-effector F_x & F_y (not just wall reaction).

$F_x = kx$ — more generally $F_x = k_x(x-x_0)$ | From (1) $F_x = k(l_1 cq_1 + l_2 cq_2)$

$F_y = ky$ — $F_y = k_y(y-y_0)$ | $F_y = k(l_1 sq_1 + l_2 sq_2)$

From (4)

$$\boxed{\begin{aligned}&k(l_1 sq_1 + l_2 sq_2)l_2 cq_2 - k(l_1 cq_1 + l_2 cq_2)l_2 sq_2 = \tau_{2_s}\\ &\&\\ &k(l_1 sq_1 + l_2 sq_2)l_1 cq_1 - k(l_1 cq_1 + l_2 cq_2)l_1 sq_1 = \tau_{1_s}\end{aligned}}$$

This is what we want.

(7)

Set motor torques to be $\tau_1 + \tau_{1s}$, and $\tau_2 + \tau_{2s}$, respectively! Answer to T3

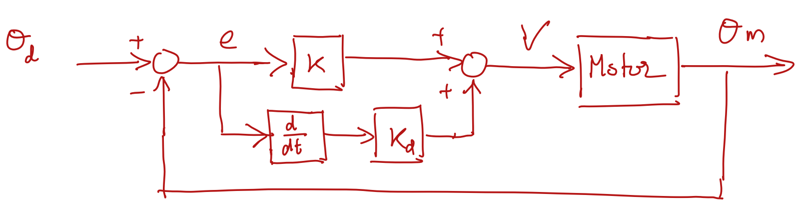
Another approach to answering T1 is to determine q_{1d} and q_{2d} (desired values) for each time instant using x_d and y_d (desired values) in (3) (like in first-level answer); but then compute torques using (6) using these q_{1d} & q_{2d} and their derivatives. Works better if dynamic effects are dominant. Still need feedback control.

Motor control : what needs to be done at motor level to control torque τ or achieve set point tracking.

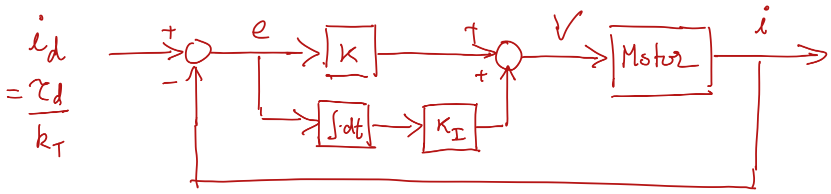
Motor has two sets of dynamics,
mechanical dynamics & electrical dynamics

But can start with simpler approach without considering motor dynamics

PD / PI / PID control



Position control with motor angle measured and used as feedback signal



Torque control with motor current measured and used as feedback signal

Motors

Control loop

- Brushed DC
- BLDC
- Stepper
- Servo
- AC synchronous / induction
  - asynchronous

Multi-level block diagram

Hardware loop

- Microcontroller
- Motor driver
- Power supply
- Sensors
- Motors
  - Gearbox
  - Encoder
  - Current sensor

No gearbox?

High voltage v/s Low voltage

Backdrivability

Inertia / torque implication of gearbox

Schematic Block Diagram