# Modelling Turbulence in 1D Channel Flow with the k-ε Model

Tammo Dukker & Kumar Abinash Mishra

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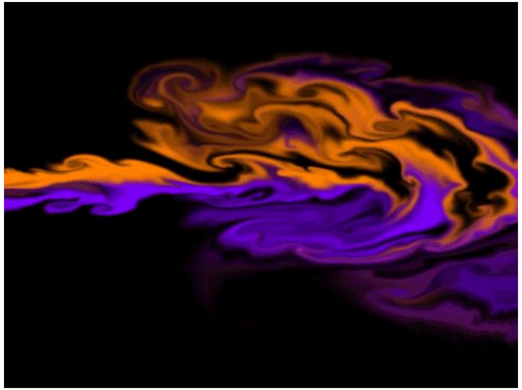
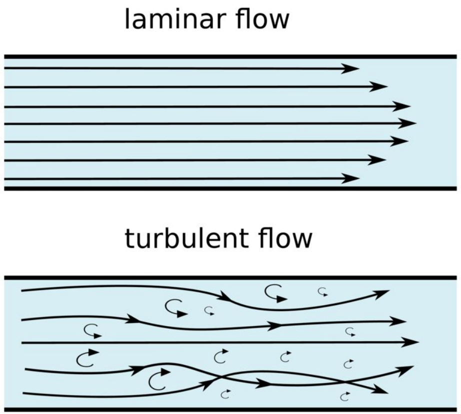
# Introduction



- Motivation: Fluid majors
- K - ε is one the most popular models to account for turbulence effects on mean flow
  - two transport equations: k & ε
- Channel flow: flow between 2 plates
- Goal: Numerically solve for turbulent fluid flow and compare to simpler models & benchmarks from literature

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# Theoretical Background: RANS

- Starting point is the NS equation (F=ma)

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_j u_i) = \frac{-1}{\rho}\frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u_i}{\partial x_j})$$

- Account for turbulence: Reynolds average

$$u_i = \underbrace{U_i}_{\text{Mean part}} + \underbrace{u_i'}_{\text{Fluctuating}} \Longrightarrow \text{Time average}$$

RANS equation

$$\frac{\partial U_i}{\partial t} + \frac{\partial}{\partial x_j}(U_j U_i) = \frac{-1}{\rho}\frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j}(\nu \frac{\partial U_i}{\partial x_j} \boxed{- \overline{u_i' u_j'})}$$

Reynolds Stress

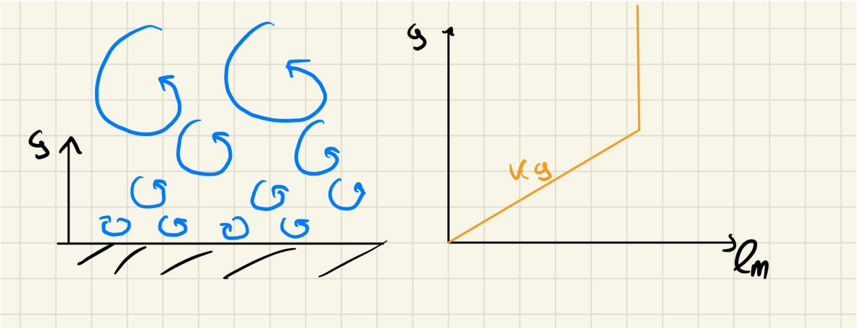
- RANS turbulence modelling revolves around modelling this Reynolds stress term (closing of the eq)

# Theoretical Background: Turbulence Models

- Boussinesq hypothesis: analogy with viscous stress
  - Viscous stress: $\tau = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{x_i})$
  - Reynolds Stress: $-\rho\overline{u_i'u_j'} = \mu_t(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{x_i}) - \frac{2}{3}\rho k \delta_{ij}$

- So $\mu_t$ needs to be calculated to close the equations
  - Zero equation models: Prandtl mixing length

$$\mu_t = l_m^2 \left|\frac{\partial U}{\partial y}\right|$$



# Theoretical Background: k-ε

- Turbulence convects and is created & destroyed
  - Two equation Models: k-ε
- The equations: nonlinear, connected PDEs

$$\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho k \vec{U}) = \nabla \cdot (\frac{\mu_t}{\sigma_k} \nabla k) + 2\mu_t S_{ij} \cdot S_{ij} - \rho \epsilon$$

Transport by Convection

Transport by Diffusion

Production

Dissipation

$$\frac{\partial \rho \epsilon}{\partial t} + \nabla \cdot (\rho \epsilon \vec{U}) = \nabla \cdot (\frac{\mu_t}{\sigma_\epsilon} \nabla \epsilon) + C_{1\epsilon} \frac{\epsilon}{k} 2\mu_t S_{ij} \cdot S_{ij} - C_{2\epsilon} \frac{\epsilon^2}{k}$$

Solve in domain

$$\mu_t = \rho C_\mu \frac{k^2}{\epsilon}$$

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# Numerical Implementation

- Apply Assumptions: 1D, steady state & fully developed

$$\cancel{\frac{\partial u_i}{\partial t}} + \cancel{\frac{\partial}{\partial x_j}(u_j u_i)} = \frac{-1}{\rho}\frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j}\right)$$

$$\cancel{\frac{\partial(\rho k)}{\partial t}} + \cancel{\nabla\cdot(\rho k \vec{U})} = \nabla\left(\frac{\mu_t}{\sigma_k}\nabla k\right) + 2\mu_t S_{ij}\cdot S_{ij} - \rho\epsilon$$

$$\cancel{\frac{\partial \rho\epsilon}{\partial t}} + \cancel{\nabla\cdot(\rho\epsilon\vec{U})} = \nabla\left(\frac{\mu_t}{\sigma_\epsilon}\nabla\epsilon\right) + C_{1\epsilon}\frac{\epsilon}{k}2\mu_t S_{ij}\cdot S_{ij} - C_{2\epsilon}\frac{\epsilon^2}{k}$$

$$\frac{d}{dy}\left[(\mu+\mu_t)\frac{du}{dy}\right] = \frac{dp}{dx}$$

$$\frac{d}{dy}\left[\frac{\mu_t}{\sigma_k}\frac{dk}{dy}\right] + \mu_t\left|\frac{du}{dy}\right|^2 = \rho\epsilon$$

$$\frac{d}{dy}\left[\frac{\mu_t}{\sigma_\epsilon}\frac{d\epsilon}{dy}\right] + C_{1\epsilon}\frac{\epsilon}{k}\mu_t\left|\frac{du}{dy}\right|^2 = C_{2\epsilon}\rho\frac{\epsilon^2}{k}$$

- Discretize the equations: Momentum equation with finite volume & k-ε with finite difference
  - System of nonlinear equations: requires iteration

# Numerical Implementation: Finite Volume

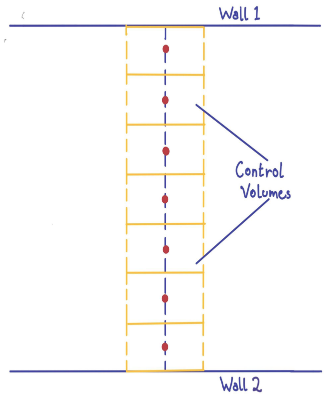
$$\frac{d}{dy}[(\mu + \mu_t)\frac{du}{dy}] = \frac{dp}{dx}$$

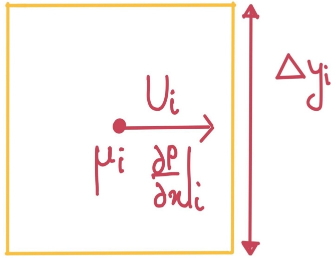
Integrate over a cell and apply Gauss

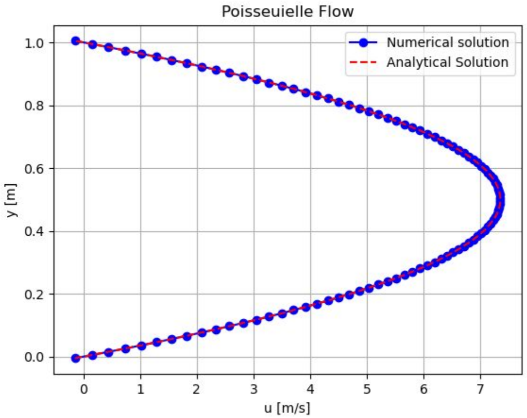
$$[(\mu + \mu_t)\frac{du}{dy}]_s^n = \frac{dp}{dx}|_p \Delta x$$

Central difference

A**u** = b

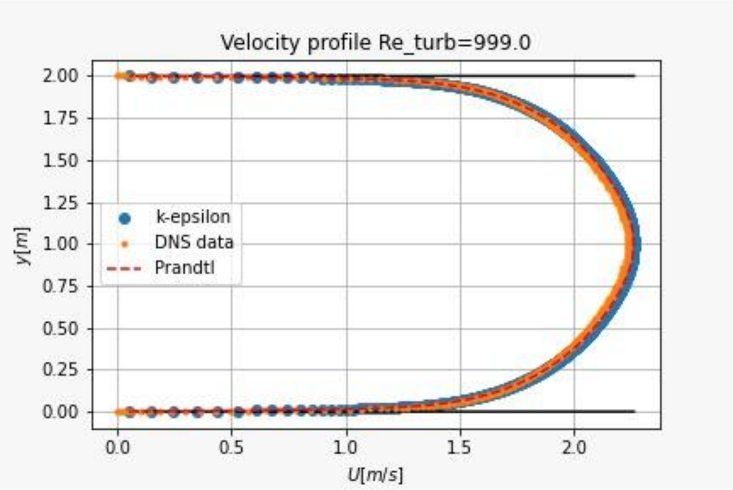


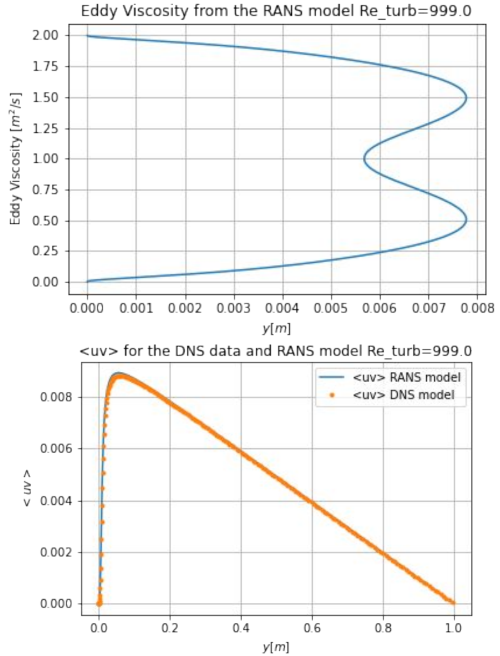



# Results: Velocity Profile

- Our numerical results our compared to the John Hopkins Turbulence Database of DNS data (1)



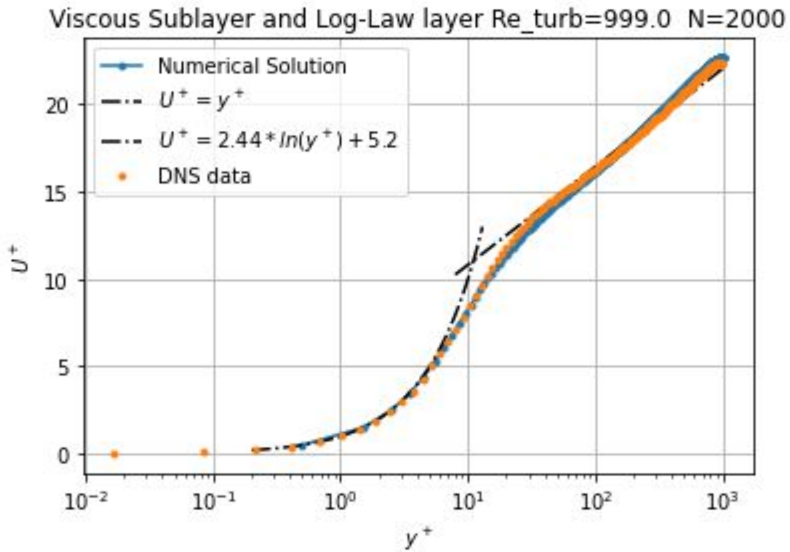

Velocity Profiles Compared



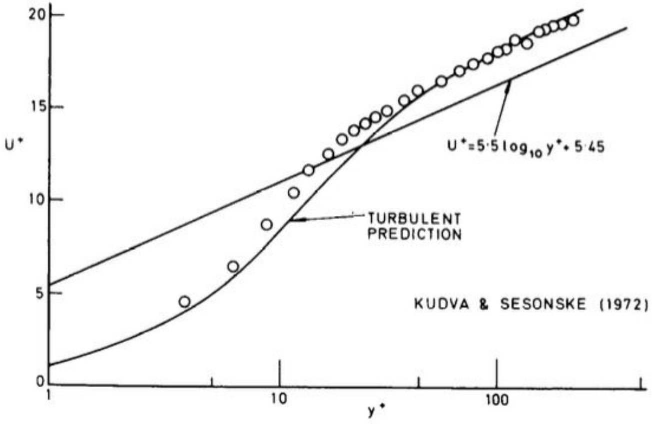

Total viscosity, Reynolds Stress distribution in channel

# Results: Log-Law

- Log law describes universal velocity profile near a wall
- It is a very common benchmarking test
- Van Driest damping addition!




Our Numerical results compared to DNS (1)




Log Law plot from original paper (2)

Introduction

Theoretical Background

Numerical Implementation

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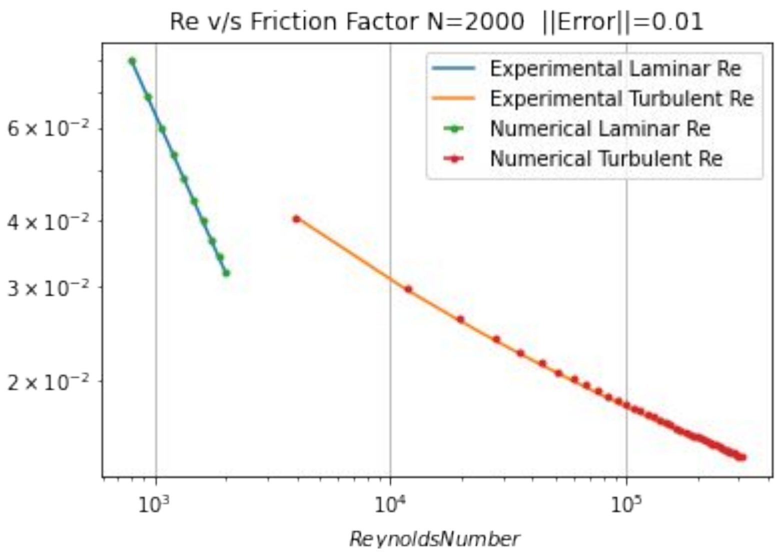
# Results: Friction Factor

- Measure for pressure drop. Different for Laminar & Turbulent regime




## Darcy Friction Factor

- Laminar Flow

$$f_e = \frac{64}{Re}$$

- Turbulent Flow

$$\frac{1}{\sqrt{f_e}} = -2log\left(\frac{\epsilon}{3.7D_H} + \frac{2.51}{Re\sqrt{f_e}}\right)$$

where,

$$Re = \frac{\overline{U}D_H}{\nu}$$

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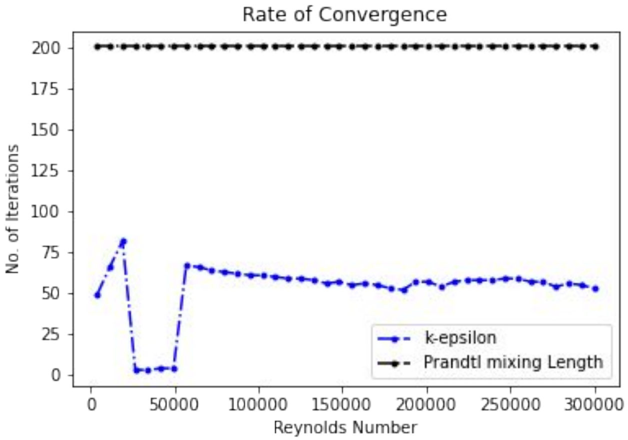
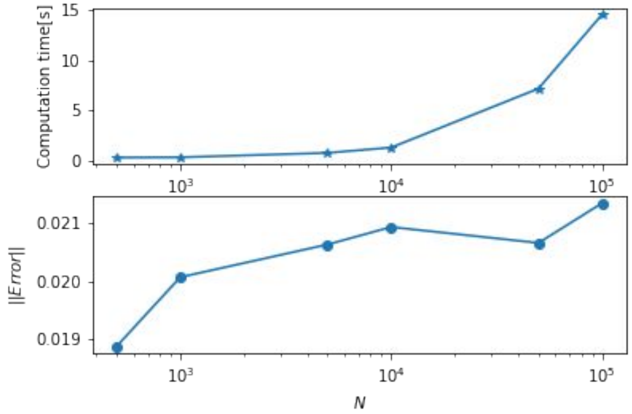
# Discussion

- Performance of Code
  - We used sparse matrices to improve performance



# Discussion: Future Scope

- Try out different versions of k-ε: low Re & High Re
- Implementing wall functions to reduce no. of cells
- Try different geometries: jets, backstep flow

# References


- (1) : John Hopkins Turbulence Database. Data obtained from the JHTDB at http://turbulence.pha.jhu.edu
- (2): Launder, B. E., & Spalding, D. B. (1983). The numerical computation of turbulent flows. In *Numerical prediction of flow, heat transfer, turbulence and combustion* (pp. 96-116). Pergamon.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# Questions?