Proposer-Builder Separation, Exclusive Order Flow, and Centralization in Blockchain*

Agostino Capponi[†], Ruizhe Jia[‡], and Sveinn Ólafsson [§]

Abstract

Maximal Extractable Value (MEV) has transformed the economic landscape of the Ethereum blockchain, favoring larger validators due to economies of scale. In response to the centralization driven by MEV, Ethereum introduced the Proposer-Builder Separation (PBS) framework, which separates the roles of block builders and proposers. We provide a game-theoretical analysis of block building that highlights the incentives driving the strategic allocation of order flow to builders, who then compete for block building rights auctioned off by validators. Our analysis demonstrates that the builder market tends toward centralization: larger builders, by attracting more order flow, enhance their market dominance in a self-reinforcing cycle, consistent with observed trends on the Ethereum blockchain. This centralization poses risk to the blockchain, such as weakened security through a single point of failure, and increasing ability of builders to extract rent from users and validators. Moreover, it disrupts the intended division of labor between builders and validators, jeopardizing the sustainability of the modular blockchain architecture that PBS aims to establish.

1 Introduction

In September 2022, the Ethereum network transitioned from Proof-of-Work (PoW) to Proof-of-Stake (PoS), through an upgrade known as the Merge. This significantly reduced the barriers to entry for securing the network, as operating a validating node under PoS is far less resource-intensive than mining blocks under PoW. Consequently, the fixed component of the block reward was reduced by around 90% (Ethereum Foundation (2023)). This, in turn, increased the importance of maximizing the variable component of the block reward,

*We are grateful to Zhiguo He, our discussant, as well as to the participants of the 2024 CBER Annual Conference and the a16z Summer Infra Mechanism Design Workshop.

[†]Columbia University, Department of Industrial Enginering and Operations Research, Email: ac3827@columbia.edu. Capponi gratefully acknowledges financial support from the Paradigm Policy Lab and the Ethereum Foundation.

[‡]Columbia University, Department of Industrial Enginering and Operations Research, Email: rj2536@columbia.edu.

[§]Stevens Institute of Technology, School of Business. Email: solafsso@stevens.edu.

i.e., the value derived from the selection and ordering of transactions included in a block, widely known as Maximal Extractable Value, or MEV (Daian et al. (2019)). Hence, while the Merge lowered the requirements for operating as a validator, it increased discrepancy in the staking returns attainable by validators, by leaving solo stakers and smaller staking service providers at a disadvantage from lacking the skills and infrastructure required to maximize block value. In other words, it jeopardized validator decentralization, and thus the blockchain's security, censorship resistance, and resilience to malicious behavior. To address these concerns, a structural change known as Proposer-Builder Separation (PBS) was introduced (Buterin (2021)). PBS delineates the roles of block proposing and block building, the former being the action of submitting a block of transactions for the approval of the network, and the latter being the action of transaction selection and ordering. Traditionally, blockchain validators are responsible for both tasks, but under PBS, they can opt to source blocks from a market of specialized builders. This change levels the playing field for validators by allowing them to focus solely on the standardized task of block proposal.

Under PBS, builders compete for the right to construct blocks for validators through an auction mechanism, where access to *exclusive order flow*—transactions that are only available to a specific builder—provides a competitive advantage. In response to this demand, private order flow channels, collecting transactions not broadcast over the public network, have emerged as a natural development, especially given the growing visibility and scrutiny surrounding MEV extraction. These channels offer benefits such as protection from certain types of MEV extraction, like frontrunning, and may even rebate users for the MEV generated by their transactions. Empirical evidence highlights the important role of private order flow in the builder competition on Ethereum. Thiery (2023) shows that while private transactions make up only about 30% of the total transaction count in Ethereum blocks, they account for 80% of the total value paid to builders. Yang et al. (2024) arrive at a similar conclusion using an extensive data set on Ethereum block auctions.

We develop a game-theoretical model of PBS, focusing on the role of exclusive order flow. The model involves three types of players. First, there are validators, who serve as block proposers under a PoS consensus protocol. When selected for this task based on their share of staked tokens, a validator sources a block from a market of specialized builders and proposes it to the network. The second type of players are builders, who compete for the right to construct blocks through an auction held by validators. A builder's success rate depends on its access to exclusive order flow, which determines its share of the block market—the proportion of blocks the builder constructs. Builders with superior access to order flow capture a larger market share. The third player is an order flow provider who allocates exclusive order flow to builders. The provider aims to ensure block inclusion for the allocated transactions while collecting fees from builders seeking to acquire order flow.[1]

[1]In practice, the order flow provider may use the fees collected from builders to rebate transaction originators for the value

In our model, events unfold across three stages. In the first stage, builders set fees to incentivize the order flow provider to route transactions to them, thereby increasing the value of their blocks. In the second stage, the order flow provider allocates transactions to builders based on their market shares and the fees they offer. In the third stage, the building competition takes place, determining the payoffs of both builders and validators. Under the PBS framework, the block auction mechanism operates as a first-price ascending auction. In this setup, the proposing validator receives a bid equivalent to the second-highest block value from all builders, while the winning builder earns the difference between the highest and second-highest block values. Notably, this implies that builders impose negative externalities on one another; the revenue of the winning builder is reduced by the efforts of other builders.

We characterize the equilibrium actions in our model using a backward induction approach. First, given the builders' fees, the order flow provider faces an optimal allocation problem. The allocation to a builder is primarily determined by the marginal gain from that builder, which depends on their market share, the fee they offer, and a parameter that reflects how much the provider values receiving rebates for order flow. At one extreme, the order flow provider bases its allocation solely on the builders' block inclusion probabilities; at the other extreme, the allocation is driven entirely by the earned fees.

Builders determine their fees by taking into account their impact on the subsequent order flow allocation. The resulting competition among builders exhibits *strategic complements*, as they compete for the same order flow and the same block-building opportunities. That is, competition drives up the fees offered by builders. Larger builders hold a natural competitive advantage due to their market share, which attracts more order flow. A less obvious but equally important factor is that larger builders also benefit more from acquiring order flow. This is because they face less competition compared to smaller builders, which increases their likelihood of capturing the value of order flow through successful block building. Smaller builders are unable to offset these disadvantages by raising their fees, resulting in larger builders dominating order flow acquisition.

These dynamics establish a feedback loop that intensifies *centralization* in the builder market: access to order flow boosts a builder's market share, which in turn draws even more order flow, thereby further increasing their market share. Notably, this cycle of centralization continues even when the order flow provider places a high priority on fees rather than the probability of block inclusion, because larger builders inherently derive more benefit from increased order flow.[2]

of their transactions.

[2]In practice, the promise of block inclusion is likely to outweigh the importance of fees. Yang et al. (2024) argue that the biggest challenge for new and emerging builders is their limited market share, which restricts their access to order flow. Some order flow providers further impose constraints on who can compete for their flow. For instance, MEV Blocker requires builders to have a minimum market share of 1% (see: `https://cow.fi/mev-blocker`). The relationship between order flow providers and builders is inherently untrusted, and market share is often seen as a key indicator of trustworthiness.

Block building demands sophisticated technology and advanced infrastructure. Since a builder's revenue is negatively affected by the block building efforts of its competitors, these costs naturally cap the number of builders that the market can support. Eventually, the market reaches a point where the expected block revenue falls short of covering operational costs. Combined with our findings on how order flow acquisition shapes competition among builders, this points toward an oligopolistic market, dominated by a small number of large builders.[3]

Our model of PBS assumes a distinct separation between builders and validators. Although this division appears to hold in practice,[4] the builder market is permissionless, and there are no formal restrictions preventing validators from also participating as builders. We examine the incentives for validators to become builders, which are driven by two main factors. Firstly, as block proposers, validators can increase their revenue by constructing blocks that exceed the value offered by the existing builder market. Secondly, when not serving as proposers, validators can generate additional revenue by competing with incumbent builders to construct blocks for the designated proposer. The potential for increased profits from becoming a builder is greater for larger validators, i.e., those with a larger share of staked tokens. This advantage stems from the fact that a proposing validator only needs to exceed the second-highest bid from another builder to secure extra revenue, while a non-proposing validator must exceed the highest bid. Consequently, the staked tokens provide validators with a natural competitive edge over other entities in entering the builder market.

The incentives for validators to become builders are particularly strong in scenarios characterized by limited competition among builders. This stems from the inherent tension between builders and validators over the division of block value; a builder's bid in a block auction reflects the portion of block value they are willing to share with validators. We find that a competitive builder market typically enhances the ability of validators to extract rent from builders through the auction process. Conversely, in a weak builder market, where there is less competition among builders, validators are able to extract less rent. This situation provides validators with a stronger incentive to enter the builder market themselves to claim a larger share of the block value, thereby directly benefiting from the reduced competition.

We show that a weak builder market is less effective at mitigating validator centralization. Specifically, such a market is less capable of narrowing the return disparity among validators. Consequently, our findings suggest that a weak builder market not only fails to achieve its objective of alleviating validator centralization but also compromises the modular structure of the blockchain as introduced by PBS. The clear separation of roles between builders and validators weakens, potentially leading to the emergence of builder-validator

[3] In fact, under PBS, a monopolistic builder could potentially emerge. This contrasts with Tullock rent-seeking contests in PoW and PoS systems, where a monopolistic outcome is not possible (Capponi et al. (2023)).

[4] According to statistics from the Ethereum Foundation, over 90% of blocks are sourced from the builder market, with no evidence of validators engaging in block building (for real-time analytics of the builder market, refer to `mevboost.pics`.

conglomerates.

In practice, competition among builders hinges on two main factors: access to order flow, which enables a builder to create blocks of higher value than its competitors, and the capacity to extract value from order flow. We find that even with equal access to order flow, a decentralized builder market is not guaranteed. If all builders receive identical order flow, even slight differences in their algorithmic efficiency can result in a highly centralized builder market. Therefore, we argue that maintaining some level of exclusivity in order flow access is crucial for fostering a competitive builder market. Essentially, if all builders have a chance to extract value from order flow that is exclusive to them, they can take turns constructing the most valuable block. This alternation is similar to the role that randomness plays in promoting decentralization in consensus mechanisms such as PoW and PoS.

We explore a variation of our model where validators are responsible for both building and proposing blocks, i.e., block construction occurs without the intermediation of builders. In this framework, we demonstrate that validator centralization naturally emerges, confirming concerns voiced prior to the Merge that led to the introduction of PBS. Our analysis shows that validator centralization is intrinsically tied to the consensus mechanism and would occur even without any initial order flow advantages for larger validators. The latter inherently acquire more order flow and achieve higher returns, largely because they are more frequently granted a monopoly over block production, which allows them to maximize the benefits from acquired order flow. Given that block building requires specialized skills that only professional validators possess, a market structure without an external builder market leads to significant validator centralization.

Literature Review. We examine the builder market under PBS, focusing particularly on the impact of exclusive order flow. Concerns regarding how exclusive order flow affects the builder market surfaced among practitioners and protocol designers before the Merge (see, e.g., Kilbourn (2022)). Recent empirical research, including studies by Thiery (2023), Yang et al. (2024), and Öz et al. (2024), supports the assertion that exclusive order flow significantly contributes to builder centralization. We build a game theoretical model, which provides analytical support for these claims. Our equilibrium analysis emphasizes the critical role that order flow acquisition plays in driving the level of centralization within the builder market.

A related study by Gupta et al. (2023) explores centralization in the competition between two PBS builders, who can exploit CEX-DEX arbitrage opportunities and capture privately auctioned transactions. They place special emphasis on integrated searcher-builders, whose advantages are further explored in Pai and Resnick (2024). In contrast to their work, we explore the transaction pipeline in a broader context, where builders extract rent from users but also lose rent to validators. Additionally, we focus on the interaction between strategic builders and order flow providers, exploring how the allocation of order flow is influenced by both the market shares of builders and the rebates they offer. The role of builder market shares has been demonstrated to be of critical practical importance, as evidenced by the findings in Yang, Nayak, and Zhang (2024) and similar studies.

Our study is related to the work by Bahrani et al. (2024). We demonstrate that differences in validator returns, assumed in their analysis, emerge in equilibrium from an order flow mechanism. This finding reinforces concerns about the risk of transitioning from PoW to PoS without simultaneously introducing a decentralizing mechanism such as PBS. Additionally, Bahrani et al. (2024) examine how a builder market can support validator decentralization. In their model, builders are exogenous (i.e., non-strategic) and assumed to be homogeneous, and thus with equal market shares. Our study, instead, focuses on the determinants of centralization in a market of heterogeneous, strategic builders.

Our study adds to the expanding body of research on centralization in blockchain networks, focusing on mining, staking, and related activities. Previous research, such as Capponi et al. (2023), has underscored the centralization risks in PoW mining, often due to economies of scale in mining technology investments. Additionally, other aspects of the mining supply chain have been explored, such as in Cong et al. (2020) where it is demonstrated that mining pools do not inevitably lead to centralization, contrary to conventional wisdom. Ferreira et al. (2023) view PoW blockchains as an industrial ecosystem of miners, mining pools, and mining hardware producers, and show how conglomerates may endogenously emerge and capture the governance of the blockchain. Rosu and Saleh (2021) argue that under a Proof-of-Stake (PoS) consensus protocol, the wealth of validators will grow at the same rate, irrespective of the size of their stake, thus preserving decentralization. Their study does not account for the impact of MEV, which has since become a significant component of block rewards. This is accounted for in Bahrani et al. (2024), who formulate PoS mining as a Tullock contest and show that validators with greater MEV-extraction ability choose a larger stake in the contest.

Our work also contributes to the literature on order flow allocation mechanisms in decentralized finance, an area with limited theoretical studies (see Gosselin and Chiplunka (2023) for an overview article). Our allocation mechanism falls under the category of exclusive batch auctions, and is an adaptation of payment for order flow (PFOF) in traditional finance, where retail order flow is privately directed by brokers to market makers for execution instead of being routed to public exchanges.[5] Theoretical studies on PFOF in traditional finance include Parlour and Uday (2003) and Ernst et al. (2024). On the empirical side, Schwarz et al. (2023) provide a recent analysis of PFOF's impact on execution quality in equity markets. We present a game-theoretical analysis of a generalized PFOF mechanism in which the fees offered by builders to attract

[5]PFOF has been practiced in the United States for over 30 years and has recently gained significant attention due to its key role in facilitating zero-commission trading. This model was popularized by Robinhood and subsequently adopted by major brokerages. For a recent overview of the history of PFOF and current industry practices, see Ernst and Spatt (2023).

order flow are endogenously determined. In our model, the allocation depends on both execution quality and the rebates offered. By contrast, in traditional PFOF the fees are exogenously set by brokers and are uniform across market makers, meaning they do not impact the allocation of order flow.

The remainder of the paper is structured as follows. In Section 2, we provide an overview of the blockchain transaction supply chain. Section 3 presents our model of PBS. Section 4 analyzes the impact of exclusive order flow on builder centralization. Section 5 studies the sustainability of PBS. Section 6 demonstrates how validator centralization arises in the absence of a builder market. Section 7 concludes. In Appendix A, we present an expanded model of the builder market and explore its implications for our findings. In Appendix B, we analyze the decentralizing effect of PBS on validators. In Appendix C, we present auxiliary results related to the allocation problem of the order flow provider. Proofs of all technical results can be found in Appendices D and E.

2 Institutional Details

In this section, we present an overview of the transaction supply chain—often referred to as the MEV supply chain—for PoS blockchains operating under the PBS framework.

At the core of transaction execution and settlement are validators, who are responsible for appending blocks of transactions to the blockchain. In each block slot, a validator is selected by the blockchain protocol to add a new block, with the selection probability being equal to the validator's share of staked tokens.

Traditionally, the role of validators encompasses both block building and block proposing. PBS aims to separate the dual roles of validators by introducing a new class of agents known as builders. The block construction process under PBS consists of the following steps:

1. **Block Building:** Specialized builders are responsible for constructing blocks of transactions. The transactions they use to assemble blocks can be divided into two main categories:

    - **Public Transactions:** These transactions are publicly visible in the network's mempool, which serves as a holding zone for pending transactions, i.e., transactions awaiting block inclusion.
    - **Private Transactions:** In contrast to public transactions, private transactions are routed from transaction originators (e.g., wallets and dApps) to builders without entering the public mempool. This process is facilitated by order flow providers that specialize in the allocation of order flow extraction rights.

    By controlling the content of a block, builders can maximize its value through strategic selection and ordering of transactions. The additional value extracted from transactions—beyond the transaction fees set by users to incentivize inclusion—is known as Maximal Extractable Value (MEV).

2. **Block Auction:** Builders compete for block space by participating in auctions held by validators. In these auctions, a builder's bid represents the compensation offered to validators for proposing the builder's block to the network. Builders thus face a tradeoff: a lower bid means that a smaller portion of the block's value (i.e., MEV) is shared with validators, but a lower bid also increases the risk of being outbid by other builders.

3. **Block Proposal:** PBS gives validators the option to propose the block from the builder with the highest bid. Alternatively, validators may fulfill their original dual role and propose blocks they have constructed themselves. Such self-building would enable them to keep all the block value rather than the partial value shared with them by builders.[6]

Figure 1 graphically illustrates the process described above. In Remark 2.1, we explore key properties of the implementation of PBS on the Ethereum blockchain.

Remark 2.1. The concept of PBS is currently implemented on Ethereum through MEV-Boost, an off-chain solution developed by Flashbots.[7] Because builders and proposers are mutually untrusted, the PBS block auction is intermediated by third party entities called relays, which (i) ensure that the promised bid is paid to validators, and (ii) protect builders against block unbundling attacks of validators. The long-term vision of Ethereum is to integrate PBS directly into the blockchain consensus protocol (i.e., enshrined PBS, or ePBS), which removes relays as a trusted and potentially centralized third party.

MEV-Boost became active on Ethereum with the Merge in September 2022. Since then, over 90% of blocks have consistently been produced via MEV-Boost block auctions. Furthermore, empirical analysis shows that block value is significantly higher for blocks sourced from builders, compared to self-built blocks, indicating that validators are unable to match the block value delivered through the auction (see, e.g., Wahrstätter (2024)).

3 Model of Proposer-Builder Separation (PBS)

In this section we present our model of PBS. We describe the agents of the model in Section 3.1. We provide the model timeline and the definition of equilibrium actions in Section 3.2. Finally, we characterize and

[6]Regardless of the block they choose to propose, validators also receive newly minted tokens for their service, known as a block subsidy. The value of this subsidy is fixed by the network protocol and does not depend on the content of the block.

[7]Flashbots is a research and development organization dedicated to mitigating the negative externalities that MEV imposes on blockchains (see: `https://boost.flashbots.net/`).

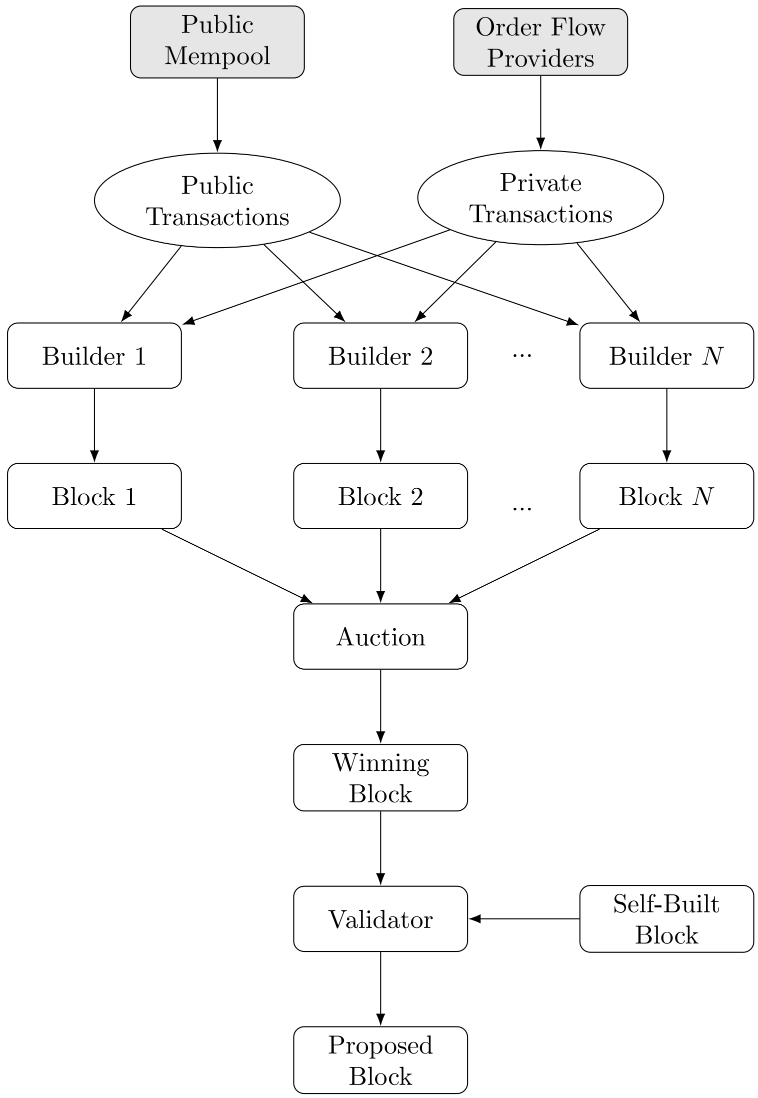

Figure 1: Block construction process under Proposer-Builder Separation (PBS).

analyze the equilibrium outcomes of the model in Section 3.3.

3.1 Model Agents

The model features three types of agents: validators, builders, and an order flow provider. The following three subsections describe the role of each of those agents.

3.1.1 Validators (Proposers)

There are $M \geq 1$ validators acting as block proposers under a Proof-of-Stake consensus protocol. The stake of validator i is denoted by $s_i \geq 0$, and the total stake of all validators by $S := \sum_{i=1}^{M} s_i$. In each block slot, validator i is selected to propose a block with probability equal to its stake share s_i/S. The proposed block is then appended to the blockchain by the network of validators.

Under PBS, the validator outsources the task of constructing the block to a market of block builders. The revenue of validator i is given by

$$\pi_i^{(v)} := \frac{s_i}{S}\mathbb{E}[b_{(2)}], \tag{3.1}$$

where $b_{(2)}$ is the second-highest block value from the builder market. This equals the validator's revenue because the competition among builders takes the form of an auction where the winning builder earns the spread $b_{(1)} - b_{(2)}$ between the two highest block values.

3.1.2 Builders

There are $N \geq 1$ builders that compete for block building rights auctioned off by the proposing validator. The block value of builder i is given by

$$b_i := X_i + \tilde{X}_i,$$

where the components X_i and $\tilde{X}_i$ are assumed to be of the form

$$X_i = \begin{cases} V_i, & \text{w.p. } p_{i,0}, \\ 0, & \text{w.p. } 1 - p_{i,0}, \end{cases} \qquad \tilde{X}_i = \begin{cases} \tilde{V}_i, & \text{w.p. } \tilde{p}_i, \\ 0, & \text{w.p. } 1 - \tilde{p}_i. \end{cases} \tag{3.2}$$

The binary structure of the block value distribution is motivated by the right-skewed nature of empirically observed block values, where a small fraction of blocks has a significantly greater value than the majority of blocks (see Wahrstätter (2024), who empirically analyzes the block construction market on Ethereum).

In the above expression, the first component is exogenous and comes from exclusive order flow that the builder already has access to. Specifically, builder i extracts value V_i from this order flow with probability $p_{i,0}$, a quantity we refer to as a value extraction probability. The second component is endogeneously determined and comes from additionally acquired exclusive order flow. Specifically, the value extraction probability $\tilde{p}_i := \tilde{p}_i(\alpha)$ depends on order flow allocated to builders by an order flow provider, which in turn depends on fees $\alpha := (\alpha_i)_{1\leq i\leq N}$ offered by builders to the provider.

We assume that the block values of different builders are independent, reflecting the fact that each builder's block value is derived from order flow that is exclusive to that builder. Similarly, V_i and $\tilde{V}_i$ are independent, positive random variables with finite means, and with the same distribution for $1 \leq i \leq N$. Lastly, block values are assumed to be independent of the random selection process that determines which validator is chosen to propose the block.

The PBS block building competition takes the format of a first-price ascending auction (see, e.g., Thiery (2023)). In an independent private value setting, this is equivalent to a second-price sealed-bid auction, under which truthful bidding is the dominant bidding strategy (Vickrey (1961)). Hence, the auction winner is the builder with the highest block value $b_{(1)}$, who receives the spread between $b_{(1)}$ and the second highest block value $b_{(2)}$. The objective function of builder i is thus given by

$$\pi_i(\alpha_i; \alpha_{-i}) := \mathbb{E}\big[(b_i - b_{(2)})^+\big] - \alpha_i \tilde{p}_i, \tag{3.3}$$

where the first term is the expected block revenue; note that the revenue is nonzero only if builder i constructs the most valuable block, i.e., if $b_i = b_{(1)}$. The second term $\alpha_i \tilde{p}_i$ is the cost of acquired order flow.

Remark 3.1. In relation to the block construction process described in Section 2, and depicted in Figure 1, we point out the following features of our model:

(i) Our focus is on the impact of exclusive order flow, setting aside order flow that is common to all builders, such as public mempool transactions. Exclusive order flow creates significant differences in the block values of builders, while common order flow inflates the block values of all builders, making it less critical in the competition between them.

(ii) The builder market is an option provided to validators that may instead choose to self-build blocks. For the purpose of this analysis, we assume that all validators source blocks from the builder market when acting as proposers, which aligns with the state of the Ethereum blockchain since the introduction of Proposer-Builder Separation (see Remark 2.1). We consider an extension of the model that accounts for validator self-building in Appendix B; we show that allowing for self-building does not qualitatively affect the results presented in this section.

Remark 3.2. In Appendix A, we extend the model of builders to a more general setting incorporating the following factors: (i) access of builders to public order flow (i.e., a common value component), (ii) ability of builders to extract value from order flow, and (iii) operational costs associated with block building. The model results would be qualitatively the same as for the model in this section, except that the presence of operational costs would exclude some builders from participating in the block building competition, because their expected revenue would not be sufficiently high to cover the operational costs.

3.1.3 Order Flow Provider

An order flow provider allocates order flow to builders. Let $v_i(\alpha) \geq 0$ represent the volume of order flow allocated to builder i, which depends on the fees α offered by the builders. Receiving order flow allows builder i to extract value with probability $\tilde{p}_i = \tilde{p}_i(\alpha)$, as defined in (3.2), and we assume this probability to be of the form $\tilde{p}_i(\alpha) = v_i(\alpha)$. This means that a unit of order flow leads to a unit increase in the probability. For this reason, we refer to $\tilde{p}_i$ as the order flow allocated to builder i.

Given fees α, the order flow provider determines the allocation $\tilde{p} := (\tilde{p}_i)_{1\leq i\leq N}$ that maximizes the objective function

$$\pi(\tilde{p};\alpha) := \sum_{i=1}^{N} \tilde{p}_i\big(\mathbb{P}(b_i) + \eta\alpha_i\big) - \frac{\gamma}{2}\sum_{i=1}^{N}\tilde{p}_i^2 - \frac{\kappa}{2}\Big(\sum_{i=1}^{N}\tilde{p}_i\Big)^2, \tag{3.4}$$

where $\eta \geq 0$, $\gamma \geq 0$, and $\kappa \geq 0$, are constants. We now explain the meaning of each of the three components of this objective function:

(i) The first component reflects the order flow provider's incentives to allocate order flow to builder i. These incentives are driven by the inclusion probability $\mathbb{P}(b_i) := \mathbb{P}(b_i = b_{(1)})$ and the fee α_i. The parameter $\eta \geq 0$ measures the relative importance of the fee. When $\eta = 0$, the provider considers only the inclusion probability of a builder, ignoring the fee. As η increases, greater emphasis is placed on the fee, shifting the provider's allocation strategy toward builders offering higher fees.

(ii) The second component reflects the order flow provider's aversion to concentration in its allocation, represented by the increasing marginal cost of allocating order flow to the same builder. For instance, if builders were homogeneous, the provider would prefer to distribute order flow across multiple builders. Such diversification reduces the risk of relying on a single builder, where on-chain inclusion of order flow becomes an "all-or-nothing" event.

(iii) The third component represents a soft capacity constraint, reflecting that the order flow provider has a limited amount of order flow to allocate. The convexity of this term constrains the magnitude of the total allocation to builders, $\sum_{i=1}^{N}\tilde{p}_i$. The parameter $\kappa > 0$ quantifies the severity of this constraint, with a higher κ indicating a tighter constraint.

The order flow allocation mechanism captures the concept of payment for order flow (PFOF) between builders and order flow providers, akin to traditional financial markets. In these markets, brokers preferentially direct retail order flows to market makers instead of exchanges. This order flow is pre-allocated wholesale to market makers, who are then accountable for its execution. Similarly, in our model, the value that builders derive from order flow is represented as a random variable, drawing a direct parallel to this traditional setup.

In traditional PFOF, order flow is allocated to market makers solely based on their recent performance (i.e., execution quality). However, it is not exclusively routed to the best performer, but rather distributed among multiple market makers. This approach avoids reliance on a single counterparty, and sustains competition by incentivizing market makers to improve their execution (Ernst et al. (2023)). In our model, this diversification motive is captured through a nonzero value of the parameter γ. Additionally, unlike in traditional payment for order flow where fees are externally determined by brokers and uniform across all market makers, our mechanism permits builders to set their own fees endogenously to attract order flow. Consequently, the allocation in our system depends on both execution quality (i.e., block success rate) and order flow rebates, introducing a competitive factor not present in traditional PFOF allocation.[8]

3.2 Model Equilibrium

Agents in our model strategically interact over three time periods labeled by $t = 1, 2, 3$. At time $t = 1$, builders determine their fees α offered to the order flow provider. At time $t = 2$, the provider allocates order flow $\tilde{p}$ to builders. At time $t = 3$, the building competition takes place, and the payoffs of builders and validators are determined.

This sequence of actions result in a noncooperative game with complete information, and we solve for the equilibrium outcome using the notion of subgame perfect Nash equilibrium. This means that we recover the equilibrium using backward induction with (i) builders internalizing the impact of their fees on the subsequent order flow allocation, and (ii) the order flow provider internalizing the impact of its allocation on the subsequent building competition; specifically, the block success probabilities $(\mathbb{P}(b_i))_{1\leq i\leq N}$ in the objective function (3.4) depend on the allocated order flow.

The above procedure is formalized in the following definition, and illustrated graphically in Figure 2.

Definition 3.3. *A subgame perfect equilibrium is a tuple $(\alpha^*, \tilde{p}^*)$ consisting of a fee profile $\alpha^* \in [0,\infty)^N$, and an order flow profile $\tilde{p}^* : [0,\infty)^N \mapsto [0,\infty)^N$ such that:*

(i) Optimal order flow allocation ($t = 2$). For any $\alpha \in [0,\infty)^N$ we have

$$\pi(\tilde{p}^*(\alpha); \alpha) = \sup_{\tilde{p}\geq 0} \pi(\tilde{p}; \alpha),$$

[8]Homogeneous, fixed fees uphold compliance with FINRA Rule 5310 by preventing brokers from routing orders based on financial incentives. Specifically, they eliminate the possibility of directing orders to a market maker offering higher payments for order flow at the expense of execution quality.

where $\pi(\cdot;\cdot)$ *has been defined in* (3.4)*.*

(ii) *Optimal fee selection* $(t = 1)$*. For* $1 \leq i \leq N$ *we have*

$$\pi_i\big(\alpha_i^*; \alpha_{-i}^*\big)|_{\tilde{p}=\tilde{p}^*(\alpha^*)} = \sup_{\alpha_i \geq 0} \pi_i\big(\alpha_i; \alpha_{-i}^*\big)|_{\tilde{p}=\tilde{p}^*(\alpha_i, \alpha_{-i}^*)},$$

where $\pi_i(\cdot;\cdot)$ *has been defined in* (3.3)*.*

For an equilibrium $(\alpha^*, \tilde{p}^*)$, the corresponding payoff of builder $1 \leq i \leq N$ is given by

$$\pi_i^* := \pi_i(\alpha_i^*; \alpha_{-i}^*)|_{\tilde{p}=\tilde{p}^*(\alpha^*)}, \tag{3.5}$$

and the payoff of validator $1 \leq i \leq M$ is given by

$$\pi_i^{*,(v)} := \pi_i^{(v)}|_{\tilde{p}=\tilde{p}^*(\alpha^*)}. \tag{3.6}$$

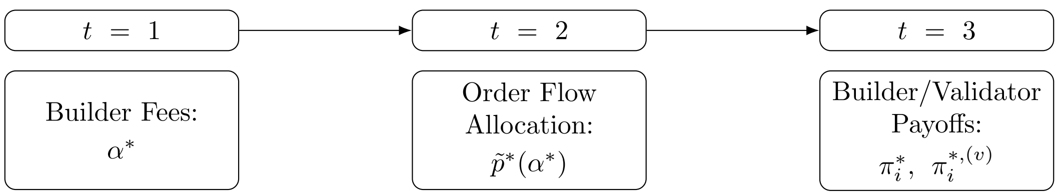

Figure 2: Graphical illustration of the model equilibrium in Definition 3.3. The equilibrium payoffs π_i^* and $\pi_i^{*,(v)}$ are defined in (3.5)–(3.6).

3.3 Model Equilibrium Outcomes

In this section, we characterize the subgame perfect equilibrium of the model. First, in Section 3.3.1, we solve for the equilibrium allocation of the order flow provider, given the fees set by the builders. Then, in Section 3.3.2, we derive the equilibrium fees for the builders.

Assumption 3.4. We consider large values of the parameters γ and κ in the objective function (3.4) of the order flow provider. Furthermore, to capture the relative importance of the provider's aversion to order flow concentration and its capacity constraint, we assume the following relationship between these parameters:

$$\kappa = \gamma^\beta,$$

where $\beta \geq 0$. For a given γ, larger values of β indicate a more capacity-constrained order flow provider. We separately analyze the cases $0 \leq \beta < 1$ and $\beta > 1$, as they result in different equilibrium structures.

Assumption 3.4 implies that the results in this section hold for large values of the parameters γ and κ. This corresponds to a small allocation regime, where the order flow allocation $\tilde{p}$ is small, which aligns with the empirical distribution of block values (see Section 3.1.2).[9]

3.3.1 Equilibrium Order Flow Allocation

We characterize the order flow allocation in the second stage of the game ($t = 2$). That is, we take the fees α of builders as given, and analyze the best response allocation $\tilde{p} = \tilde{p}(\alpha)$ of the order flow provider.

Recall from the relation between γ and κ in Assumption 3.4 that, as β increases from 0 to a larger than 1 value, we shift from a less capacity-constrained order flow provider to a more capacity-constrained one.

Order Flow Allocation if $0 \leq \beta < 1$. We first consider a less capacity constrained order flow provider, and introduce the notation.

$$P_{i,0} := \mathbb{P}(b_i)\big|_{\tilde{p}=0}, \tag{3.7}$$

which represents the initial block success probability of builder i, i.e., without any allocation of order flow. Additionally, for vectors $a, b \in \mathbb{R}^N$ that depend on the parameter γ, we introduce the notation $a \overset{\gamma}{\approx} b$ to represent the approximation $||a - b|| = o(\gamma^{-2})$.[10]

Proposition 3.5.

(i) For sufficiently large γ, there exists an equilibrium order flow allocation $\tilde{p}^*$. The equilibrium $\tilde{p}^*$ is such that $\tilde{p}_i^* > 0$ for $1 \leq i \leq N$.

(ii) The equilibrium order flow allocation in part (i) satisfies $\tilde{p}^* \overset{\gamma}{\approx} \hat{p}$, where $\hat{p} = (\hat{p}_i)_{1 \leq i \leq N}$ is the unique solution to a system of linear equations given by

$$a_{i,0} + \sum_{j=1}^{N} a_{i,j}\tilde{p}_i = 0, \quad 1 \leq i \leq N,$$

and the coefficients satisfy

$$a_{i,0} = \frac{\partial \pi}{\partial \tilde{p}_i}\Big|_{\tilde{p}=0} > 0, \qquad a_{i,i} = \frac{\partial^2 \pi}{\partial \tilde{p}_i^2}\Big|_{\tilde{p}=0} < 0, \qquad a_{i,j} = \frac{\partial^2 \pi}{\partial \tilde{p}_i \partial \tilde{p}_j}\Big|_{\tilde{p}=0} > 0.$$

Closed-form expressions for the coefficients in terms of the model parameters are given in (D.3).

[9] From the objective function (3.4), it can be seen that large values of γ and κ lead to smaller values of $\tilde{p}_i$. Formally, the proofs of Propositions 3.5 and 3.7 show that the equilibrium allocation is bounded by $1/\gamma$ and $1/\kappa$.

[10] In other words, $\frac{a-b}{\gamma^{-2}} \to 0$ as $\gamma \to \infty$.

Proposition 3.5 states that all builders have a positive order flow allocation, and yields the following relation between the allocation to different builders:

$$\tilde{p}_i^* \overset{\gamma}{\approx} -\frac{a_{i,0}}{a_{i,i}} + \sum_{j \neq i} \frac{a_{i,j}}{a_{i,i}} \tilde{p}_j^*.$$

The leading order term is positive, and depends on the slope $a_{i,0} > 0$ and concavity $a_{i,i} < 0$ of the order flow provider's objective function π with respect to the allocation $\tilde{p}_i$.

The coefficient of $\tilde{p}_j^*$ is negative, which signifies that there are negative externalities associated with allocating order flow to different builders. These externalities arise for two reasons. First, the order flow provider has limited capacity, so allocating to builder j restricts the flow available to builder i. Second, there is a feedback effect: allocating to builder j increases its block success probability, making it more attractive for further allocation, while simultaneously making allocation to builder i less desirable.

The following corollary presents explicit expressions for the equilibrium allocation in terms of the model parameters, derived by solving the system of equations in Proposition 3.5.

Corollary 3.6. *The equilibrium order flow allocation in Proposition 3.5 satisfies*

$$\tilde{p}_i^* \overset{\gamma}{\approx} \frac{1}{\gamma} a_{i,0} - \frac{1}{\gamma^{2-\beta}} \sum_{j=1}^{N} a_{j,0} + \frac{1}{\gamma^2} \sum_{j=1}^{N} b_{i,j} a_{j,0}, \tag{3.8}$$

where $a_{i,0} = P_{i,0} + \eta \alpha_i > 0$*, and* $b_{i,i} > 0$ *and* $b_{i,j} < 0$ *are given in (D.4).*

The key determinants of the order flow allocations are the quantities $(a_{i,0})_{1 \leq i \leq N}$, which represent the order flow provider's marginal gains of allocating to builders. As seen from the objective function (3.4), these gains depend on the builders' success probabilities $(P_{i,0})_{1 \leq i \leq N}$, and on the fees $(\alpha_i)_{1 \leq i \leq N}$ they offer. Consistently with intuition, the allocation to builder i is increasing in both the success probability $P_{i,0}$ and the fee α_i. Likewise, the externality from a competing builder j is negative and increasing in the builder j's success probability $P_{j,0}$ and its offered fee α_j, both captured through the marginal gain $a_{j,0}$. Expression (3.8) captures two types of externalities imposed on builder i by competing builders. First, a higher value of β, which indicates a more capacity-constrained order flow provider, intensifies the impact of its allocation to builder j on builder i. This effect is represented by the first summation term in (3.8). Second, allocating order flow to builder j decreases the block success probability for builder i, which in turn reduces the allocation received by them. This dynamic is reflected in the second summation term in (3.8).

<u>Order Flow Allocation if $\beta > 1$</u>. We now consider the case of a more capacity constrained order flow provider. In this case, order flow is only allocated to the builder offering the highest marginal gain to the order flow provider; recall that $a_{i,0} = P_{i,0} + \eta\alpha_i$ quantifies the marginal gain from allocating to builder i.

Proposition 3.7. *For sufficiently large γ, there exists a unique equilibrium allocation $\tilde{p}^*$ such that*

$$\tilde{p}_i^* = \begin{cases} \frac{a_{i,0}}{\gamma^\beta} + o(\gamma^{-\beta}), & i = i_0, \\ 0, & i \neq i_0, \end{cases}$$

where i_0 is the index of the builder with the largest value of $a_{i,0}$.

Recall from Corollary 3.6 that the negative externalities from order flow allocation are increasing in the parameter β, for $0 \leq \beta < 1$, indicating a more capacity constrained order flow provider. As the capacity constraint becomes increasingly binding ($\beta > 1$), the externalities grow strong enough to potentially restrict the allocation to a single builder. This means that even a low level of builder heterogeneity can result in a significant discrepancy in order flow acquisition, where only one builder ends up receiving order flow.

Remark 3.8. In Appendix C, we demonstrate that our conclusions are robust with respect to the assumed relationship $\kappa = \gamma^\beta$ between γ and κ. We show that as the ratio κ/γ grows, the allocation of order flow becomes more concentrated among fewer builders. This finding is consistent with the results discussed in this section: a low κ/γ ratio, corresponding to $\beta < 1$, leads to a distribution of order flow across all builders (see Prop. 3.5), whereas a high κ/γ ratio, indicative of $\beta > 1$, results in order flow being allocated to a single builder (see Prop. 3.7).

3.3.2 Equilibrium Fees of Builders

We now consider the first stage of the game ($t = 1$) and characterize the equilibrium fees offered by builders to the order flow provider. Following our analysis in Section 3.3.1, where we analyzed the second stage of the game ($t = 2$), we separately examine the cases $0 \leq \beta < 1$, and $\beta > 1$.

<u>Equilibrium Fees if $0 \leq \beta < 1$</u>. We introduce the quantities

$$\mu_{i,0} := \frac{\partial}{\partial \tilde{p}_i} \mathbb{E}\big[\big(b_i - b_{(2)}\big)^+\big]\Big|_{\tilde{p}=0} > 0, \qquad \mu_{i,j,0} := -\frac{\partial}{\partial \tilde{p}_j} \mathbb{E}\big[\big(b_i - b_{(2)}\big)^+\big]\Big|_{\tilde{p}=0} > 0, \tag{3.9}$$

which are the sensitivities of expected block revenue with respect to the acquisition of order flow. The quantity $\mu_{i,0}$ represents builder i's marginal gain from acquiring order flow, while $\mu_{i,j,0}$ represents builder

i's marginal loss from builder j acquiring order flow.[11] These quantities measure the value of order flow to the builders, making them crucial in determining their order flow acquisition strategies.

Proposition 3.9. *For sufficiently large γ, the competition between builders is a game of strategic complements, i.e., the objective functions of builders are concave and satisfy*

$$\frac{\partial^2 \pi_i}{\partial \alpha_j \partial \alpha_i}\Big|_{\tilde{p}=\tilde{p}^*(\alpha)} > 0, \quad i \neq j.$$

Moreover, the equilibrium fees α^* satisfy

$$\alpha_i^* = \begin{cases} \frac{1}{2}\tilde{a}_{i,0} + \frac{1}{2}\frac{1}{\gamma^{1-\beta}}\big(\sum_{j\neq i}\tilde{a}_{i,j,0} + \sum_{j\neq i}\alpha_j^*\big) + O(\gamma^{-1}), & \tilde{a}_{i,0} > 0, \\ 0, & \tilde{a}_{i,0} \leq 0, \end{cases}$$

where $\tilde{a}_{i,0} := \mu_{i,0} - \frac{P_{i,0}}{\eta}$ and $\tilde{a}_{i,j,0} := \mu_{i,j,0} + \frac{P_{j,0}}{\eta}$. The equilibrium fee α_i^* of builder i is increasing in both the marginal gain $\mu_{i,0}$ and the marginal loss $\mu_{i,j,0}$, and decreasing in the block inclusion probability $P_{i,0}$.

As stated in the Proposition 3.9, the game between builders exhibits *strategic complements*, which implies that an increase in the fee offered by builder j is associated with an increase in the fee of builder i. This is because builders compete for the same order flow, and for the same block building opportunities. By increasing its fee, a builder acquires more order flow at the expense of its competitors, increasing the value of its blocks, which prompts its competitors to raise their fees as well. In other words, competition between builders pushes up the fees earned by the order flow provider.

The primary determinants of a builder's fee are its block inclusion probability $P_{i,0}$ and the marginal gain $\mu_{i,0}$; disregarding second-order terms, the expression for the fee given in Proposition 3.9 satisfies

$$\alpha_i^* \sim \frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+,$$

where x^+ denotes the positive part of x. A builder adjusts its fee upwards if order flow has a positive impact on its revenue, i.e., if $\mu_{i,0}$ is large. Interestingly, the rebate fee offered to the order flow provider decreases with the builder's block success probability $P_{i,0}$. At first glance, this may appear counterintuitive, but it reflects the builder's competitive position: a high success probability makes the builder more attractive to the order flow provider. As a result, the builder can attract order flow without offering a high fee.

Finally, it is important to note that the fees set by builders increase with the parameter η, which captures the importance the order flow provider places on receiving fees. Notably, when η is small, the fees set by

[11]The marginal loss is positive, indicating that builder i's block revenue decreases with order flow acquisition of builder j.

all builders are zero, indicating that fees do not significantly influence the allocation of order flow in this scenario.

Equilibrium Fees if $\beta > 1$. Recall from Proposition 3.7 that, in this case, order flow is allocated to a single builder. The next proposition demonstrates that this builder is the one with the highest initial block success probability, i.e., the largest value of $P_{i,0}$.

Proposition 3.10. *Let i_0 and j_0 denote the indices of the builders with the highest and second-highest block success probabilities, P_{i_0} and P_{j_0}, respectively. Builder i_0 is the only builder that acquires order flow and its equilibrium fee satisfies*

$$\alpha_{i_0}^* = \max\left\{\frac{1}{2}\big(\mu_{i_0,0} - \frac{P_{i_0,0}}{\eta}\big)^+, \Big(\mu_{j_0,0} + \mu_{j_0,i_0,0} - \frac{P_{i_0,0} - P_{j_0,0}}{\eta}\Big)^+\right\} + o(\gamma^{-1}).$$

The largest builder in terms of market share, i.e., the one with the highest initial block success probability, sets a fee that strategically deters other builders from increasing their fees to a level that would enable them to capture the order flow. Specifically, the first term within the maximum in the formula for $\alpha_{i_0}^*$ represents the profit-maximizing fee that builder i_0 would choose in the absence of competition with other builders, but the second term represents the smallest fee required to outbid the second-largest builder, and thereby also smaller builders. This terms reflects how the fee level required to price out other builders depends on key characteristics of the second largest builder j_0. That is, the fee increases with (i) $\mu_{j_0,0} > 0$, the benefit j_0 gains from acquiring order flow; (ii) $\mu_{j_0,i_0,0}$, the marginal loss j_0 incurs when i_0 secures order flow, because the loss of a competing builder j_0 is associated with a gain of builder i_0; and (iii) $P_{j_0,0}$, which measures the attractiveness of builder j_0 to the order flow provider.

As explained next, the competitive dynamics in order flow acquisition result in an *arms race* between builders. While only one builder eventually secures the order flow, the competition drives the winning builder to set a fee that may surpass the optimal profit-maximizing level it would select if it were not facing competition. In fact, the profit of the winning builder may be lower than it would be in a scenario without competitive order flow acquisition. To illustrate this, consider a situation where the difference in initial block success probabilities between builders i_0 and j_0 is small. Under these conditions, according to the formula in the proposition, the fee set by the winning builder satisfy the following expression,[12]

$$\alpha_{i_0}^* \approx \mu_{i_0,0} + \mu_{i_0,j_0,0},$$

[12]This approximation follows from $P_{i_0,0} \approx P_{j_0,0}$, from which it follows that $\mu_{i_0,0} \approx \mu_{j_0,0}$ and $\mu_{j_0,i_0,0} \approx \mu_{i_0,j_0,0}$.

which exceeds the marginal benefit $\mu_{i_0,0}$ that builder i_0 derives from order flow acquisition. This implies that the profit of builder i_0 after acquiring order flow satisfies:

$$\pi^*_{i_0}\big|_{\tilde{p}=\tilde{p}^*} < \pi^*_{i_0}\big|_{\tilde{p}=0}.$$

Despite this reduction in profit, builder i_0 chooses to acquire order flow in order to prevent competing builders from capturing it. If a competing builder were to succeed, builder i_0 would would suffer from an even greater profit reduction.

4 Order Flow Acquisition and Centralization in Block Building

Building on the equilibrium results presented in the previous section, we explore the effects of order flow acquisition on the builder market. In Section 4.1, we analyze how order flow acquisition influences the market shares of builders. In Section 4.2, we consider measures of centralization among builders.

4.1 Block Success Probabilities

The share of the builder market captured by a particular builder corresponds to its block success probability, i.e., the fraction of successfully constructed blocks attributed to that builder. The change in a builder's market share due to order flow acquisition is therefore given by

$$\Delta^*_i := \mathbb{P}(b_i)\big|_{\tilde{p}=\tilde{p}^*} - P_{i,0},$$

where $P_{i,0} := \mathbb{P}(b_i)\big|_{\tilde{p}=0}$ is the builder's initial block success probability, and $\mathbb{P}(b_i)\big|_{\tilde{p}=\tilde{p}^*}$ is the block success probability accounting for the equilibrium order flow acquisition $\tilde{p}^* = \tilde{p}^*(\alpha^*)$.

Evaluating the impact of order flow acquisition on block success probabilities requires accounting for two counteracting effects. First, a builder's order flow acquisition increases its success probability, as added order flow boosts its capacity to produce high-value blocks. Conversely, the success probability diminishes when competing builders also enhance their order flow, which dilutes the potential for success of others.

Builder i's initial order flow level is captured by its initial value extraction probability $p_{i,0}$, given in Eq. (3.2). Moreover, there is a monotonic relation between order flow levels and block success probabilities, that is, if $p_{i,0} > p_{j,0}$, then $P_{i,0} > P_{j,0}$ (see Lemma D.2-(i)). The next proposition shows that the change in market share Δ^*_i is increasing in the initial order flow level $p_{i,0}$, which means that larger builders expand their market share. Furthermore, since the block success probabilities of all builders sum to one, this implies that larger builders expand their market share at the expense of smaller builders.[13]

Proposition 4.1. *Let γ large enough so that an equilibrium order flow allocation $\tilde{p}^* = \tilde{p}^*(\alpha^*)$ exists (as guaranteed by Propositions 3.7 and 3.9). The following statements then hold:*

(i) The allocation $\tilde{p}^*$ is such that $\tilde{p}^*_i > \tilde{p}^*_j$ if $p_{i,0} > p_{j,0}$.

(ii) The change in block success probabilities satisfies $\Delta^*_i = \tilde{\Delta}^*_i(1 + O(||p_0||)$, where $(\tilde{\Delta}^*_i)_{1\leq i\leq N}$ is such that $\tilde{\Delta}^*_i > \tilde{\Delta}^*_j$ if $p_{i,0} > p_{j,0}$.

The first part of the proposition, states that the order flow acquisition of builders is increasing in their initial order flow levels. Initial order flow advantages are amplified for two main reasons. First, a higher order flow level leads to a larger block inclusion probability, making the builder more attractive to order flow providers. Second, builders with higher order flow levels benefit more from acquired order flow. This is because such builders face less competition relative to other builders, making them more likely to win block auctions and thus better positioned to capitalize on acquired order flow. This is reflected in our model, as these builders have a higher marginal revenue from acquiring additional order flow (see Lemma D.2-(ii)).

In summary, our findings reveal a cyclical pattern leading to centralization: builders with greater initial order flow have higher block inclusion probabilities, which attracts even more order flow, further enhancing their success rates. Notably, this holds even if the parameter η in (3.4) is large, indicating that the order flow provider prioritizes the fees it receives when allocating order flow. In this scenario, the order flow provider does not have an inherent preference for larger builders. However, larger builders still derive greater benefit from acquired order flow, which is a sufficient incentive for them to outpace smaller builders in terms of order flow acquisition, and thus also in terms of market share.

4.2 Measures of Builder Centralization

The results in Section 4.1 demonstrate that order flow acquisition amplifies the market share of larger builders at the expense of smaller ones, leading to an increasingly uneven distribution of market share. A direct corollary to Proposition 4.1 is that the disparity in the market shares of any two builders is higher after order flow acquisition. In particular, this implies that a smaller set of builders is needed to collectively capture a given share of the market, such as 50%. Such a threshold value is a commonly used metric for quantifying the level of decentralization in a blockchain network.[14]

[13] Proposition 4.1-(ii) assumes that the initial value extraction probabilities $(p_{i,0})_{1\leq i\leq N}$ are small. This is in line with the discussion following Assumption 3.4, stating that small values are most relevant for value extraction probabilities, as they are consistent with empirical block value distributions.

[14] This is sometimes referred to as the Nakamoto coefficient, and it has been used in prior studies to quantify centralization in blockchain networks (Srinivasan and Lee (2017)).

Corollary 4.2. *Let i and j be two builders such that $P_{i,0} > P_{j,0}$. The change in their block success probabilities satisfies $\tilde{\Delta}_i^* > \tilde{\Delta}_j^*$.*

An alternative measure of centralization is the Herfindahl-Hirschman index, a common measure of market concentration of an industry, which is given by the sum of squared market shares of each firm. This index is given by the sum of the squared market shares of each builder

$$HH := \sum_{i=1}^{N} (\mathbb{P}(b_i))^2 \in (0,1].$$

A larger value of HH indicates a more centralized market, and the change in HH due to order flow acquisition is given by

$$\Delta_{HH}^* := HH\big|_{\tilde{p}=\tilde{p}^*} - HH\big|_{\tilde{p}=0}.$$

The following analysis confirms that the centralization effect identified in Prop. 4.1 is consistent across different measures used to quantify concentration. Specifically, we demonstrate that order flow acquisition leads to an increase in HH.

Corollary 4.3. *The Herfindahl-Hirschman index of the builder market satisfies $\Delta_{HH}^* = \tilde{\Delta}_{HH}^*(1 + O(||p_0||)$, where $\tilde{\Delta}_{HH}^* > 0$, which indicates an increased centralization.*

5 Is Proposer-Builder Separation Sustainable?

Our model of PBS in Section 3 assumes a clear separation between builders and validators. That is, the set of builders and the set of validators are disjoint. We now consider whether this blockchain modularity is sustainable by analyzing the incentives for validators to become builders, i.e, to acquire exclusive order flow and compete with the incumbent builders. We do this by considering the strategy space where a single validator is allowed to become a builder, and in Section 5.1 we introduce the objective function of the *builder-validator*. In Section 5.2 we present the equilibrium order flow allocation in the presence of a builder-validator, and analyze the benefit the validator derives from becoming a builder.

5.1 Objective Function of a Builder-Validator

We assume the block value of the builder-validator to be of the form

$$b_i^{(v)} := \tilde{X}_i^{(v)}, \qquad \tilde{X}_i^{(v)} = \begin{cases} V_i^{(v)}, & \text{w.p. } \tilde{p}_i^{(v)}, \\ 0, & \text{w.p. } 1 - \tilde{p}_i^{(v)}, \end{cases}$$

where the value extraction probability $\tilde{p}_i^{(v)} \geq 0$ is from acquired order flow. Note that the block value formulation here is consistent with that in Eq. (3.2), with the primary difference being the absence of initial order flow. This is because we are considering a validator who is not currently a builder, but is transitioning to become one.

The objective function of the builder-validator is then given by

$$\pi_i^{(v)}(\alpha_i^{(v)}; \alpha) := \frac{s_i}{S}\mathbb{E}\big[\max\{b_i^{(v)}, b_{(2)}\}\big] + \big(1 - \frac{s_i}{S}\big)\mathbb{E}\big[(b_i^{(v)} - b_{(2)})^+\big] - \alpha_i^{(v)}\tilde{p}_i^{(v)}, \tag{5.1}$$

where $\alpha_i^{(v)} \geq 0$ is the fee offered for order flow by the builder-validator, and $\alpha = (\alpha_i)_{1 \leq i \leq N}$ are the fees offered by the incumbent builders. As before, $b_{(2)}$ is the second largest block value from the set of block builders, which now includes the builder-validator.

This objective function incorporates two revenue components. First, when selected by the protocol to serve as a block proposer (with a probability of s_i/S), the validator's payoff is the greater of its own block value or the second highest block value from the builder market. Second, when not selected by the protocol (with a probability of $1 - s_i/S$), the validator acts as a builder and earns the difference between the value of its own block and the second highest block value among all builders.

These revenue components highlight the dual incentives for validators to engage in building. First, by acting as their own builders, validators can potentially enhance the value of their blocks beyond what the market typically offers. Second, by competing with other builders, they can generate additional revenue by constructing blocks for other validators.

5.2 Equilibrium Order Flow Allocation

In Section 3.3, we characterized a subgame perfect equilibrium consisting of fees offered by builders to an order flow provider, and the corresponding order flow allocated to them. In this section, we extend the characterization of the equilibrium to include the order flow acquisition of a builder-validator whose objective function is given by (5.1). The objective function of the order flow provider is then given by

$$\pi(\tilde{p}, \tilde{p}_i^{(v)}; \alpha, \alpha_i^{(v)}) := \sum_{i=1}^{N} \tilde{p}_i \big(\mathbb{P}(b_i) + \eta\alpha_i\big) + \tilde{p}_i^{(v)} \eta \alpha_i^{(v)} - \frac{\gamma}{2}\Big(\sum_{i=1}^{N} \tilde{p}_i^2 + (\tilde{p}_i^{(v)})^2\Big) - \frac{\kappa}{2}\Big(\sum_{i=1}^{N} \tilde{p}_i + \tilde{p}_i^{(v)}\Big)^2,$$

which is identical to the objective function of the order flow provider in Section 3.1.3, with the exception that order flow is now additionally allocated to a builder-validator offering fee $\alpha_i^{(v)}$. Note that from the perspective of the order flow provider, the validator has no initial share of the builder market, and can thus only acquire order flow by offering a fee.

The following proposition states that the incentive to become a builder is stronger for larger validators, i.e., the ones with a larger stake.

Proposition 5.1. *For sufficiently large values of γ, a subgame perfect equilibrium exists within the PBS model that incorporates a builder-validator. Furthermore:*

(i) The order flow allocation $\tilde{p}_i^{*,(v)}$ to the builder-validator is increasing in the stake share s_i/S.

(ii) The validator's profit increase from becoming a builder is increasing in the stake share s_i/S.

To understand the increased benefit of larger validators, it can be noted that the expected revenue of a validator from building blocks is given by

$$\Delta_i^{(v)} := \frac{s_i}{S}\mathbb{E}\big[(b_i^{(v)} - \tilde{b}_{(2)})^+\big] + \big(1 - \frac{s_i}{S}\big)\mathbb{E}\big[(b_i^{(v)} - \tilde{b}_{(1)})^+\big],$$

where $\tilde{b}_{(1)}$ and $\tilde{b}_{(2)}$ are the two highest block values of incumbent builders (i.e., excluding the block of the builder-validator). This indicates that the revenue change is increasing in the validator's stake share s_i/S, because the revenue increase is greater for block slots where the validator is chosen as a proposer:

$$\mathbb{E}\big[(b_i^{(v)} - \tilde{b}_{(2)})^+\big] > \mathbb{E}\big[(b_i^{(v)} - \tilde{b}_{(1)})^+\big].$$

When chosen by the protocol, the builder-validator only needs to beat the second best builder to earn additional revenue beyond that of a pure validator, but when not chosen by the protocol, the validator needs to beat the best builder. In other words, there is an advantage associated with having a larger stake and thus a greater chance of being selected as a block proposer.

The formula for $\Delta_i^{(v)}$ also highlights how the incentive to become a builder is decreasing in the "strength" of the builder market, because the expected revenue is decreasing in the two highest block values $\tilde{b}_{(1)}$ and $\tilde{b}_{(2)}$ of the builder market. Intuitively, if the builder market is competitive, resulting in larger values of

$\tilde{b}_{(1)}$ and $\tilde{b}_{(2)}$, then the builder-validator is less likely to build blocks that are at least as valuable as those blocks, which reduces the incentive to become a builder. Corollary 5.2 confirms this intuition by viewing a more competitive builder market as one with a greater number of builders (i.e., a larger N), each of whom constructs more valuable blocks (i.e., higher value extraction probabilities $(p_{i,0})_{1\le i\le N}$).

Corollary 5.2. *The profit increase of a validator from becoming a builder is decreasing in both the number of builders N and the value extraction probabilities $(p_{i,0})_{1\le i\le N}$.*

This above result reflects the inherent tension that exists between builders and validators, stemming from the fact that the value of the proposed block is divided between the winning builder and the proposing validator. Formally, under a clear division of labor between builders and validators (i.e., without any builder-validators), the total revenue extracted by builders and validators in a given block slot is

$$\Pi := \mathbb{E}[b_{(1)} - b_{(2)}], \qquad\qquad \Pi^{(v)} := \mathbb{E}[b_{(2)}].$$

This suggests that a more competitive builder market reduces the rent builders can extract (i.e., narrows the margin $b_{(1)} - b_{(2)}$), while it increases the rent extracted by validators (i.e., increases $b_{(2)}$). This is confirmed by the following proposition.

Proposition 5.3. *Consider a market of N builders, each with value extraction probability p_0. Furthermore, assume that $\lambda_N := \mathbb{E}[V_{(1)} - V_{(2)}]$ is constant or decreasing in N, where $V_{(1)}$ and $V_{(2)}$ are the largest and second largest values of $V_1, \ldots, V_N$ in (3.2).*

(i) The revenue of builders, Π, is decreasing in N and p_0, and the revenue of validators, $\Pi^{(v)}$, is increasing in N and p_0.

(ii) The revenue share of builders, $\Pi/(\Pi + \Pi^{(v)})$, is decreasing in N and p_0, and the revenue share of validators, $\Pi^{(v)}/(\Pi + \Pi^{(v)})$, is increasing in N and p_0.

In a weak builder market, the rent extracted by validators is not only smaller in absolute terms but also constitutes a lesser fraction of the total rent captured by both builders and validators. This underscores how a weak builder market strengthens the incentives for validators to become builders, and thus jeopardizes the separation between builders and validators. In other words, the modular structure of a PBS blockchain is vulnerable to disruptions in a weak builder market, potentially leading to the emergence of builder-validator conglomerates.

The technical condition on λ_N in Proposition 5.3 dictates how builder revenue is impacted by competition among them. The condition is satisfied if the block value distribution of builders, i.e., the distribution of the value they extract from order flow, does not exhibit excessively heavy tails; if it does, the total revenue of builders could actually increase with a greater number of builders, due to an increased probability of extreme block values. The exact conditions on the tail-heaviness of the block value random variables $V_1, \ldots, V_N$ are discussed in Remark E.1, following the proof of Proposition 5.3 in Appendix E.

5.3 Which Validators Are Likely to Become Builders?

Under PBS, there is limited overlap between the skillsets required to be a builder and a validator. Validators stake tokens and run nodes to secure the network, making sure appended blocks follow the rules of the network and contain no invalid or conflicting transactions. Builders use sophisticated strategies to acquire order flow and order transactions to maximize block value. The complexity of these strategies introduces significant operational costs that a prospective builder-validator must overcome. The findings in Section 5.2 suggest that larger validators stand to gain the most from becoming builders, as they are more likely to offset the costs associated with vertical integration.

In addition to operational costs, Yang et al. (2024) utilize data from Ethereum block auctions to argue that a considerable entry barrier for new builders is associated with block subsidization. Specifically, new builders with limited or no market share are unable to acquire order flow, and thus unable to build competitive blocks to gain market share. As a result, they resort to the strategy of subsidizing block value to win the block auction, which may be viable due to the right-skewed distribution of block value (see Section 3.1.2).[15]

It can be argued that a larger stake share reduces the expected cost of block subsidization for a builder becoming a validator. This reduction occurs because when selected to serve as a block proposer, the revenue sacrificed by the builder-validator through block subsidization is represented by $(b_i^{(v)} - \tilde{b}_{(2)})^+$. Conversely, when not serving as a block proposer, the cost of block subsidization is $(b_i^{(v)} - \tilde{b}_{(1)})^+$.[16] The cost therefore diminishes when the validator is selected to serve as proposer. Consequently, validators, by virtue of their stake, possess a competitive advantage over other entities in entering the builder market through block subsidization. This advantage is more pronounced for larger validators, who are selected as proposers more frequently.

[15] The presence of block subsidizing can be inferred from block profitability data presented by Flashbots at `relayscan.io`.
[16] As before, $\tilde{b}_{(1)}$ and $\tilde{b}_{(2)}$ represent the highest and second-highest block values of incumbent builders.

6 Block Construction in the Absence of PBS

In previous sections, we have shown that the economic incentives of market participants may lead to the emergence of a centralized builder market (see Section 4). While validator centralization is alleviated or even avoided under PBS, as all validators source blocks from the same market of builders (see Appendix B), a centralized market of builders can emerge.

We now analyze the degree of centralization among validators in the absence of PBS. That is, we consider the alternative scenario where validators are responsible for block building, as opposed to auctioning off block building rights to builders. This process is visualized in Figure 3, which displays the block construction process in the absence of PBS (compare with Figure 1, which displays the process under PBS).

In Section 6.1, we introduce the objective function of validators in this setting and analyze their incentive to acquire order flow. In Section 6.2, we characterize the equilibrium order flow allocation, and in Section 6.3 we study how this impacts validator centralization.

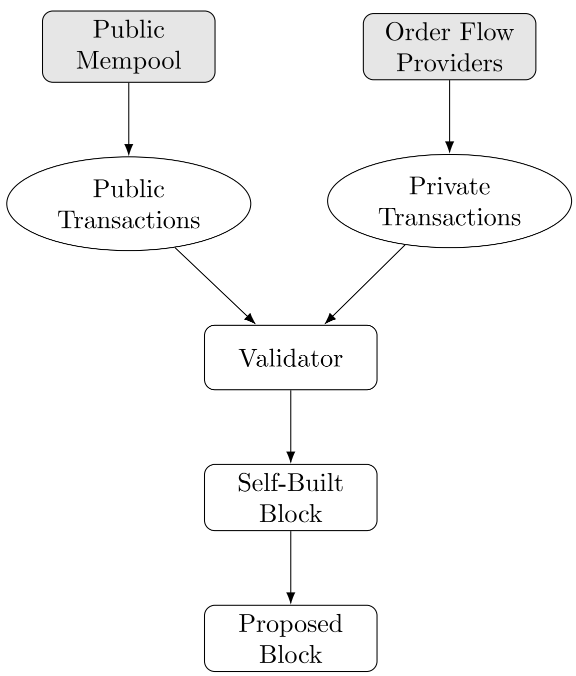

Figure 3: Block construction process in the absence of Proposer-Builder Separation (PBS).

6.1 Objective Function of Validators

As in Section 3, there are $M \geq 1$ validators operating under a Proof-of-Stake consensus protocol. Furthermore, validator i has stake $s_i \geq 0$, the total stake of all validators is $S := \sum_{i=1}^{M} s_i$, and validator i is selected by the protocol to serve as a block proposer with probability equal to its stake share s_i/S.

In the absence of builders, validators are responsible for block construction, which gives them an incentive to acquire order flow. Building on the analysis of builder block value in Section 3 (refer to Eq. (3.2)), we model the block value for validators using a similar form[17]

$$b_i^{(v)} := X_i^{(v)} + \tilde{X}_i^{(v)},$$

where the components $X_i^{(v)}$ and $\tilde{X}_i^{(v)}$ are given by

$$X_i^{(v)} = \begin{cases} V_i^{(v)}, & \text{w.p. } p_{i,0}^{(v)}, \\ 0, & \text{w.p. } 1 - p_{i,0}^{(v)}, \end{cases} \qquad \tilde{X}_i^{(v)} = \begin{cases} \tilde{V}_i^{(v)}, & \text{w.p. } \tilde{p}_i^{(v)}, \\ 0, & \text{w.p. } 1 - \tilde{p}_i^{(v)}. \end{cases}$$

As before, the first component is exogenous and comes from exclusive order flow that the validator already has access to, while the second component is endogeneously determined and comes from additionally acquired order flow, which in turn depends on fees $\alpha^{(v)} := (\alpha_i^{(v)})_{1 \le i \le M}$ offered by validators to an order flow provider. The objective function of validator i is then given by

$$\pi_i^{(v)}(\alpha_i^{(v)}; \alpha_{-i}^{(v)}) := \frac{s_i}{S}\mathbb{E}[b_i^{(v)}] - \alpha_i^{(v)} \tilde{p}_i^{(v)}, \tag{6.1}$$

where the first term represents block revenue, and the second term the cost of acquired order flow.

Remark 6.1. The block revenue of validators in the absence of builders differs structurally from the block revenue of builder under PBS (see Eq. (3.3)). First, builders win the block building competition only by producing the most valuable block, whereas validators are given block proposal rights by the protocol with probability equal to their stake share. Second, the block revenue of builders is squeezed by competition with other builders, whereas validators do not compete with other validators; they have a monopoly when chosen by the protocol to build a block, and therefore keep the entire block value.

6.2 Equilibrium Order flow Allocation

Section 3.3 characterized a subgame perfect equilibrium under PBS, consisting of fees offered by builders for order flow, and the corresponding order flow allocated to them. Here, a subgame perfect equilibrium is characterized in the same way, where order flow is allocated to validators with objective function (6.1). The

[17] Because we are considering a model of validators that actively serve as block builders, this generalizes the model of validators in Section 5.1 by allowing them to have initial order flow levels.

objective function of the order flow provider is given by

$$\pi(\tilde{p}^{(v)}; \alpha^{(v)}) := \sum_{i=1}^{M} \tilde{p}_i^{(v)} \big(P_{i,0}^{(v)} + \eta \alpha_i^{(v)}\big) - \frac{\gamma}{2} \sum_{i=1}^{M} (\tilde{p}_i^{(v)})^2 - \frac{\kappa}{2} \big(\sum_{i=1}^{M} \tilde{p}_i^{(v)}\big)^2,$$

which is analogous to the objective function of the order flow provider in Section 3.1.3, with the exception that the block success probability $\mathbb{P}(b_i)$ of builders becomes equal to the stake share $P_{i,0}^{(v)} := s_i/S$ of validators. This is because validators are assigned to propose blocks by the protocol, and this probability is fully determined by their stake share.

The following result states how the equilibrium fees offered by validators, and the order flow allocated to them, depends on their size as measured by their stake shares.

Proposition 6.2. *For large enough γ, a subgame perfect equilibrium $(\alpha^{*,(v)}, \tilde{p}^{*,(v)})$ exists. Furthermore:*

(i) The equilibrium fee $\alpha_i^{*,(v)}$ and order flow allocation $\tilde{p}_i^{*,(v)}$ are proportional to the stake share s_i/S.

(ii) The order flow allocation $\tilde{p}_i^{*,(v)}$ is independent of the initial order flow level $p_{i,0}^{(v)}$.

The fee paid by a validator and the acquired order flow are proportional to the validator's stake share. This is because a validator's probability of capitalizing on order flow through block construction is equal to the validator's stake share. In other words, larger validators, i.e., those with a larger stake, acquire more order flow simply because there is a higher likelihood of them benefiting from it by being granted a monopoly over block construction as block proposers.

Notably, the allocation to a validator is independent of the validator's initial order flow $p_{i,0}^{(v)}$. This is again because the validator's ability to capitalize on acquired order flow is only determined by its stake, and not impacted by prior order flow levels. This is in contrast to the order flow acquisition of builders, which was increasing in their initial order flow levels (see Proposition 4.1), because of the reinforcing nature of existing and acquired order flow in the block building competititon.

6.3 Validator Returns

The return-per-stake of validator i is defined by

$$r_i^{(v)} := \frac{\pi_i^{(v)}}{s_i},$$

and the change in returns due to order flow acquisition is given by

$$\Delta r_i^{*,(v)} := r_i^{*,(v)} - r_{i,0}^{(v)}.$$

In the above equation, $r_{i,0}^{(v)} := r_i^{(v)}\big|_{\tilde{p}^{(v)}=0}$ is the initial return without order flow acquisition, and $r_i^{*,(v)} := r_i^{(v)}\big|_{\tilde{p}^{(v)}=\tilde{p}^{*,(v)}}$ is the return accounting for the equilibrium order flow acquisition. The following corollary to Proposition 6.2 shows that validators with larger stakes observe a greater return boost through order flow acquisition compared to smaller validators.

Corollary 6.3. *Let i and j be two validators such that $s_i > s_j$. Then the change in their return-per-stake due to order flow acquisition satisfies $\Delta r_i^{*,(v)} > \Delta r_j^{*,(v)}$.*

Crucially, larger validators tend to outperform and outgrow their smaller counterparts in terms of returns, even when all validators begin with equal initial order flow levels, suggesting an uneven playing field developing over time.[18] Hence, validator centralization naturally arises from heterogeneity in network stakes. The PBS framework modifies the incentive structure by reducing or eliminating the need for validators to maximize block value themselves, instead providing them with the alternative option to source competitive blocks externally.

The main take-away is that, in the absence of a builder a market, centralization occurs at the validator level. This concern was the driving motive behind the introduction of PBS. It is worth pointing out that the analysis in this section underestimates validator centralization in the absence of PBS, because it does not account for the fact that typically only professional validators have the capability to maximize block value effectively. As discussed in Section 5.3, block building requires investment in infrastructure and human capital that is simply not viable for smaller validators. Indeed, most validators lack the technical sophistication required to optimize block building beyond merely selecting the highest-fee transactions from the public mempool. Therefore, in the scenario described in this section, professional validators would gradually accumulate a larger stake in the network, resulting in a degree of centralization similar to what is currently observed in the builder market.

7 Conclusions

Proposer-Builder Separation (PBS) was introduced by the Ethereum Foundation to decouple the roles of block construction and block proposal. Under PBS, the former is carried out by a set of specialized builders, and the latter, as customary, by the network validators.

We have built a game-theoretical model that accounts for the interaction between block builders and order flow providers under PBS. Block proposers auction off construction rights to builders competing for

[18] This argument uses the fact that the initial return $r_{i,0}^{(v)}$ is proportional to the initial order flow level $p_{i,0}^{(v)}$.

block inclusion. Builders enhance their competitive advantage by exclusive agreements with an order flow provider, thus increasing their likelihood of winning the block auction.

Our findings indicate that competitive order flow acquisition and economies of scale foster centralization in the builder market, potentially leading to a small number of dominant builders. Specifically, larger builders, who have better access to and derive greater benefit from order flow, tend to increase their market share at the expense of smaller builders, creating a self-reinforcing cycle leading to centralization. Such centralization poses significant risks to blockchain neutrality, censorship resistance, and resilience against malicious activities. Moreover, it facilitates rent extraction of builders from users, through reduced rebates to the order flow provider, and from validators, through diminished block rewards.

Builder centralization raises critical questions about the effectiveness and sustainability of the PBS framework. While PBS aims to mitigate validator centralization, our analysis reveals that a less competitive builder market diminishes this effect. Furthermore, a centralized builder market elevates the incentives for validators to become builders, thereby undermining the division of labor between builders and validators, and compromising the modular structure that PBS seeks to establish within the blockchain ecosystem.

Despite these shortcomings of the builder market, we argue that the introduction of PBS was essential to prevent validator centralization. Our analysis demonstrates that validator centralization naturally occurs in scenarios where validators are responsible for both block construction and block proposal.

Our model highlights the threat of centralization among builders, in particular because smaller builders do not have sufficient market share to attract order flow. In practice, an important strategy of small and emerging builders is to overbid for blocks in order to gain market share and thus attract order flow (see Section 5.3). This mechanism could be captured in an extended two-period model, where in the first period builders subsidize blocks to gain market share (i.e., operate at a loss) in order to attract order flow and be profitable in the second period.

While we consider a PFOF mechanism for the order flow allocation to builders, we conjecture that different order flow auction (OFA) mechanisms should not qualitatively impact our main results. This claim is supported by our results showing that a primary driver of centralization is the synergy between existing and acquired order flow. Builders with initial order flow advantages are in a better position to acquire additional order flow, as they derive greater benefits from it. This synergy is inherent to the nature of the block auction and would influence builders similarly across different order flow mechanisms.[19]

Our research findings highlight the importance of identifying strategies that can mitigate the centralizing effects of exclusive order flow on the builder market. One promising approach involves protocol changes

[19] A taxonomy of such mechanisms is provided in Gosselin and Chiplunka (2023), where PFOF is categorized under exclusive batch auctions.

that reduce the dependence of future acquired order flow on existing order flow. For instance, implementing distributed or decentralized block building would allow multiple builders to collaboratively construct a single block. Proposed models include sequential block auctions, where the block is incrementally built by various builders, as suggested by Buterin (2022). This method effectively increases the frequency of block production, thereby reducing the order flow advantages enjoyed by larger builders and enabling smaller builders, who may specialize in niche order flow opportunities, to participate in block building without having to outcompete large, centralized builders for entire blocks.

While exploring the technical feasibility of partial block-building is still an ongoing area of research, these methods hold considerable promise for decentralizing the builder market by lowering entry barriers for smaller builders to derive value from order flow. A notable example of partial block building involves separate auctions for "top-of-block" and "rest-of-block" building rights, instead of auctioning the rights to construct the entire block. This technique, proposed by Gupta et al. (2023) and Monnot (2023), aims to diminish the dominance of builders with superior skills in CEX-DEX arbitrage (top-of-block) in controlling the order flow auctions for the remainder of the block.

References

Maryam Bahrani, Pranav Garimidi, and Tim Roughgarden. 2024. Centralization in Block Building and Proposer-Builder Separation. arXiv:2401.12120. http://arxiv.org/abs/2401.12120

Vitalik Buterin. 2021. Proposer/Block Builder Separation-Friendly Fee Market Designs. Ethereum Research. https://ethresear.ch/t/proposer-block-builder-separation-friendly-fee-market-designs/9725

Vitalik Buterin. 2022. Decentralizing the Builder Role. 2022 MEV-sbc Workshop.

Agostino Capponi, Sveinn Ólafsson, and Humoud Alsabah. 2023. Proof-of-Work Cryptocurrencies: Does Mining Technology Undermine Decentralization? *Management Science* 69, 11 (2023), 6417–7150.

Lin William Cong, Zhiguo He, and Jiasun Li. 2020. Decentralized Mining in Centralized Pools. *The Review of Financial Studies* 34, 3 (2020), 1191–1235.

Philip Daian, Steven Goldfeder, Tyler Kell, Yunqi Li, Xueyuan Zhao, Iddo Bentov, Lorenz Breidenbach, and Ari Juels. 2019. Flash Boys 2.0: Frontrunning, Transaction Reordering, and Consensus Instability in Decentralized Exchanges. arXiv:1904.05234. https://arxiv.org/abs/1904.05234

Thomas Ernst, Andrey Malenko, Chester Spatt, and Jian Sun. 2023. What Does Best Execution Look Like? https://microstructure.exchange/papers/BrokerRouting.pdf Working Paper.

Thomas Ernst and Chester Spatt. 2023. Payment for Order Flow and the Retail Trading Experience. Wharton Initiative on Financial Policy & Regulation. White Paper.

Thomas Ernst, Chester Spatt, and Jian Sun. 2024. Would Order-By-Order Auctions Be Competitive? *Journal of Finance, forthcoming* (2024).

Ethereum Foundation. 2023. How The Merge impacted ETH supply - Ethereum Roadmap. https://ethereum.org/en/roadmap/merge/issuance/

Daniel Ferreira, Jin Li, and Radoslawa Nikolowa. 2023. Corporate Capture of Blockchain Governance. *The Review of Financial Studies* 36, 4 (2023), 1364–1407.

Stephane Gosselin and Ankit Chiplunka. 2023. Frontier Research. https://frontier.tech/the-orderflow-auction-design-space

Tivas Gupta, Mallesh M. Pai, and Max Resnick. 2023. The Centralizing Effects of Private Order Flow on Proposer-Builder Separation. In *5th Conference on Advances in Financial Technologies (AFT 2023)*. IEEE, 20:1–20:15.

Quintus Kilbourn. 2022. Order Flow, Auctions, and Centralization: A Warning. Flashbots. `https://writings.flashbots.net/order-flow-auctions-and-centralisation`

Barnabé Monnot. 2023. PEPC FAQ. `https://efdn.notion.site/PEPC-FAQ-0787ba2f77e14efba771ff2d903d67e4`

G. Mudholkar, Y. Chaubey, and L. Tian. 2009. Extremes, Extreme Spacings and Tail Lengths: An Investigation for Some Important Distributions. *Calcutta Statistical Association Bulletin* 61, 1-4 (2009), 243–266.

Burak Öz, Danning Sui, Thomas Thiery, and Florian Matthes. 2024. Who Wins Ethereum Block Building Auctions and Why?. In *6th Conference on Advances in Financial Technologies*.

Mallesh M. Pai and Max Resnick. 2024. Structural Advantages for Integrated Builders in MEV-Boost. In *Financial Cryptography and Data Security*. Springer Nature, 128–132.

Christine Parlour and Rajan Uday. 2003. Payment for Order Flow. *Journal of Financial Economics* 68, 3 (2003), 379–411.

Ioanid Rosu and Fahad Saleh. 2021. Evolution of Shares in a Proof-of-Stake Cryptocurrency. *Management Science* 67, 2 (2021), 661–672.

Christopher Schwarz, Brad Barber, Xing Huang, Philippe Jorion, and Terrance Odean. 2023. The 'Actual Retail Price' of Equity Trades. `https://ssrn.com/abstract=455730` Working Paper.

Balaji Srinivasan and Leland Lee. 2017. Quantifying Decentralization. `https://news.earn.com/quantifying-decentralization-e39db233c28e`

Thomas Thiery. 2023. Empirical analysis of Builders' Behavioral Profiles (BBPs). Ethereum Research. `https://ethresear.ch/t/empirical-analysis-of-builders-behavioral-profiles-bbps/16327`

William Vickrey. 1961. Counterspeculation, Auctions and Competitive Sealed Tenders. *Journal of Finance* 16 (1961), 8–39.

Toni Wahrstätter. 2024. Is it worth using MEV-Boost? Ethereum Research. `https://ethresear.ch/t/is-it-worth-using-mev-boost/19753`

Sen Yang, Kartik Nayak, and Fan Zhang. 2024. Decentralization of Ethereum's Builder Market. arXiv:2405.01329. https://arxiv.org/pdf/2405.01329

A Extended Model of Builders

We consider three practical extensions of the models of builders presented in Section 3, and analyze how they augment the equilibrium builder dynamics:

(i) First, we extend the block value equation (3.2) for the builder to include a value component extracted from public order flow, i.e., transactions broadcast to public mempools, and thus accessible by all builders.

(ii) Second, we assume that different builders have different levels of technical sophistication, i.e., the ability to extract value from order flow is not the same for all builders. This implies that builders differ across two dimensions: access to order flow and building ability. In the model of Section 3, we focus on the former because it is of primary importance in practice.[20]

(iii) Third, we account for the fact that builders incur an operational cost associated with block building. This includes the cost of infrastructure and human capital needed to optimize block value.

In Appendix A.1, we introduce the objective function of builders that incorporates the above extensions. In Appendix A.2, we consider the competition between builders, in particular how the operational cost of builders acts as an entry barrier, resulting in only a subset of builders being active participants in the builder market. Furthermore, we study how the set of active builders depends on both the access to order flow of builders, and their building ability.[21]

A.1 Objective Function of Builders

To emphasize the primary economic forces derived from the model extensions, we focus on the competition between builders. That is, we consider a set of builders with exogenously determined value extraction probabilities $(p_i)_{1\leq i\leq N}$, and abstract away from the order flow provider whose allocation alters those extraction probabilities.

To incorporate the effect of public order flow and block building ability, the block value of builders is assumed to be of the form

$$b_i := e_i(X + X_i), \qquad X_i = \begin{cases} V_i, & \text{w.p. } p_i, \\ 0, & \text{w.p. } 1 - p_i. \end{cases} \tag{A.1}$$

[20] See, for example, Thiery (2023), whose empirical analysis shows that private order flow represents around 30% of the block content on Ethereum, but accounts for 80% of the value paid to builders. Similar results regarding the importance of order flow on Ethereum are observed in Yang et al. (2024).

[21] Proofs of the results in this section can be found in Appendix E.4.

This extends the block value equation (3.2) through the component X, which captures extractable value from public order flow that is common to all builders; X is a nonnegative random variable, independent of the exclusive component X_i of any builder, and observable by all builders. Additionally, this extends the specification in Eq. (3.2) by including the parameter $0 < e_i \leq 1$, which quantifies the ability of builder i to extract value from order flow. A larger value of e_i means that a builder is able to capture a larger fraction of extractable value. Finally, to account for the operational costs of builders, we modify their objective function in Eq. (3.3) to be of the form

$$\pi_i\big(\xi_i;\xi_{-i}\big) := \begin{cases} \mathbb{E}\big[(b_i - b_{(2)})^+|\xi\big] - K, & \xi_i = 1, \\ 0, & \xi_i = 0. \end{cases} \tag{A.2}$$

In the above expression, $K \geq 0$ is the cost associated with block building, and ξ_i is the decision variable of builder i: $\xi_i = 1$ if builder i participates in the block building competition, and $\xi_i = 0$ otherwise. Intuitively, a builder only participates in block building if its expected block revenue exceeds the operational cost. An equilibrium profile of builders' participation is then a vector $\xi^* \in \{0,1\}^N$ such that

$$\sup_{\xi_i \in \{0,1\}} \pi_i(\xi_i, \xi^*_{-i}) = \pi_i(\xi^*_i; \xi^*_{-i}), \quad 1 \leq i \leq N.$$

A.2 Set of Active Builders

The number of active builders is endogenously determined and given by

$$n := \sum_{i=1}^{N} \xi_i.$$

The following proposition states that a higher operational cost leads to fewer active builders by eliminating those whose block revenue is insufficient to recover costs. Together with our centralization results in Section 4, this suggests that the big-get-bigger effect of exclusive order flow would simultaneously lead to a reduction in the number of active builders. In particular, there is no inherent mechanism that prevents an equilibrium outcome where a single monopolistic builder prevails.

In the proposition we make the assumption that a builder with a higher value extraction probability p_i also has a higher building ability e_i. This assumption is purely for simplicity of exposition and does not qualitatively impact the results. This is because the key driver of our findings is that competition among builders compresses their revenue from block auctions. Specifically, a more crowded builder market reduces the incentive for new builders to enter.

Proposition A.1. *Consider the objective functions* $(\pi_i)_{1\le i\le N}$ *in (A.2), and assume that* $p_i \ge p_j$ *if and only if* $e_i \ge e_j$*.*

(i) An equilibrium ξ^* *exists for* $(\pi_i)_{1\le i\le N}$*.*

(ii) Denote by $\underline{n}^*$ *and* $\bar{n}^*$ *the smallest and largest equilibrium number of active builders for* $(\pi_i)_{1\le i\le N}$*. Then* $0 \le \underline{n}^* \le \bar{n}^* \le N$*, and the lower and upper bounds* $\underline{n}^*$ *and* $\bar{n}^*$ *are both decreasing in the operational cost* K*. In particular,* $\underline{n}^* = \bar{n}^* = N$ *for* K *small enough, and* $\underline{n}^* = \bar{n}^* = 0$ *for* K *large enough.*

Note that because builders impose negative externalities on each other, the equilibrium number of active builders may not be uniquely determined. Intuitively, any active builder reduces the expected revenue of other builders, making entry less feasible. However, the possible range of active builders, $[\underline{n}^*, \bar{n}^*]$, is unambiguously decreasing in the operational cost. In particular, if the cost is small enough, then all builders are active, as in Section 3.

By considering the case of homogeneous builders, we can shed light on how the number of active builders depends on exclusive order flow levels.

Corollary A.2. *Assume builders are such that* $p_i \equiv p$ *and* $e_i \equiv e$*. The number of active builders* n^* *is then uniquely determined. Furthermore, there exists* $0 < \bar{p} \le 1$ *such that* n^* *is increasing in* p*, for* $0 \le p \le \bar{p}$*.*

Notably, the number of active builders is generally increasing in the value extraction probability p.[22] That is, increased access to exclusive order flow results in a higher number of active builders. Indeed, exclusive order flow is precisely what gives a builder an edge over its competitors, and thus increases its block revenue.

The previous result indicates that builders having access to exclusive order flow is key to them capturing a share of the builder market and recouping their operational cost. At the same time, heterogeneity in access to order flow leads to disparities in market shares among builders. Therefore, one could conjecture that equalizing access to order flow across all builders would enhance parity within the market. This would be tantamount to $p_i \approx 0$ in (A.1), indicating that no builder has exclusive order flow. Somewhat surprisingly, this would lead to a highly skewed market, because a marginally more sophisticated builder might capture a disproportionately large share of the block market.

Proposition A.3. *Assume that* $p_i \equiv p$*, and let* i_0 *be the builder with the highest sophistication:*

$$e_{i_0} > \max_{i \ne i_0} e_i.$$

Then the block success probability of builder i_0 *satisfies* $\mathbb{P}(b_{i_0} = b_{(1)}) \to 1$*, as* $p \to 0$*.*

[22] Recall from Section 3.1.2 that small values of p are consistent with the empirical distribution of block value.

This occurs because if all builders have access to the same order flow pool, then the builder with the highest technical sophistication always extracts the most value and builds the winning block. This highlights how block building becomes close to an all-or-nothing competition if builders only differ from each other in terms of technical ability. Hence, while builders deviate from each other across two dimensions—access to order flow and technical ability—the former is crucial for maintaining a competitive builder market and, thus, sustaining decentralization in building; the randomness in the value extracted from exclusive order flow effectively allows builders to alternately construct the most valuable block.

Remark A.4. The PBS building mechanism can be compared to PoW and PoS, which are decentralized consensus protocols such that a given share of committed resources (hash rate in PoW, stake in PoS) guarantees a given share of built blocks. PBS can be described as "Proof-of-MEV", because the ability to extract the most rent from the system gives the privilege to construct blocks.

PoW and PoS are less susceptible to winner-takes-all outcomes than PBS. In these systems, two validators with similar hashing capabilities or similar levels of stake tend to construct comparable fractions of all blocks. In contrast, within a PBS framework, a builder will construct no blocks if another builder, even marginally more sophisticated, exists and there is limited access to exclusive order flow.

B Impact of PBS on Validator Centralization

In the PBS model presented in Section 3, it is assumed that all validators commit to proposing blocks from the builder market. By construction, this implies that all validators have the same return-per stake. That is, the quantity

$$r_i^{(v)} := \frac{\pi_i^{(v)}}{s_i}, \tag{B.1}$$

where $\pi_i^{(v)}$ is given in (3.1), is the same for all validators. It follows that validator decentralization is achieved under PBS in the sense that all validators grow at the same rate.

In practice, the builder market is an option provided to validators, who may instead choose to self-build blocks when serving as proposers (see the institutional details in Section 2). In Appendix B.1, we consider how validator decentralization is impacted by both validator self-building and the strength of the builder market.

B.1 Builders and (Self-Building) Validators

We consider a set of $N \geq 1$ builders with exogenously determined value extraction probabilities $(p_i)_{1\leq i\leq N}$, and block values of the form (A.1), allowing for both public and exclusive block value components, as well as heterogeneous block building ability parameters $(e_i)_{1\leq i\leq N}$.

Additionally, we consider $M \geq 1$ validators, where validator i has stake $s_i \geq 0$, and the total stake of all validators is $S = \sum_{i=1}^{M} s_i$. Furthermore, the self-building block value of validator i is denoted by $b_i^{(v)}$, and is assumed to be a nonnegative random variable that is independent of the block values of builders.

When serving as a proposer, validator i makes a profit-maximizing decision between using a self-built block and selecting a block from the builder market. The profit of validator i is therefore given by

$$\pi_i^{(v)} := \frac{s_i}{S}\mathbb{E}\big[\max\{b_i^{(v)}, b_{(2)}\}\big],$$

and the corresponding return-per-stake is given in (B.1).

The following proposition states that a validator with superior self-building ability achieves higher returns.[23] However, the difference in the returns of any two validators is decreasing in the strength of the builder market, as measured by the number of builders N, their value extraction probabilities $(p_i)_{1\leq i\leq N}$, and their ability parameters $(e_i)_{1\leq i\leq N}$. This extends a result in Bahrani et al. (2024), which shows that for a market of N homogeneous builders, the return discrepancy of validators is decreasing in N.

Proposition B.1. *The return-per-stake of validators i and j are such that:*

(i) If $b_i^{(v)}$ first-order stochastically dominates $b_j^{(v)}$, then $r_i^{(v)} \geq r_j^{(v)}$.

(ii) The return difference $r_i^{(v)} - r_j^{(v)}$ is decreasing in N, $(p_i)_{1\leq i\leq N}$, and $(e_i)_{1\leq i\leq N}$.

The proposition implies that the discrepancy in validator returns is smaller under PBS than without it, where validators are limited to using their own blocks. In that sense, centralization among validators is alleviated by the introduction of a builder market, but the extent of the reduction depends on the strength of the builder market.

Intuitively, for the builder market to be effective, the block value $b_{(2)}$ needs to consistently higher than the block value of validators. At one extreme, a monopolistic builder market has no impact on validator returns, because in that case $b_{(2)} = 0$. Conversely, if the value of blocks constructed by builders consistently dominates validator blocks, i.e., if $b_{(2)} \geq b_i^{(v)}$, then decentralization among validators is achieved in that

[23]For real-valued random variables X_1 and X_2, we say that X_1 first-order stochastically dominates X_2 if for any $x \in \mathbb{R}$ we have $\mathbb{P}(X_1 \leq x) \leq \mathbb{P}(X_2 \leq x)$.

they all grow at the same rate. The latter situation is akin to the current state of the Ethereum blockchain, where the block value of builders with exclusive order flow exceeds the block value of validators using public order flow, meaning that validators largely experience the same return-per-stake (see Remark 3.1).

C Order Flow Allocation: Supplementary Results

In Section 3.3.1, we considered the allocation of order flow to builders under Assumption 3.4. The first part of the assumption states that the parameters γ and κ in the objective function (3.4) are large, leading to a small allocation regime, consistent with the empirical distribution of block value.

The second part of the assumption specifies that the allocation problem is analyzed under a defined parametric relationship $\kappa = \gamma^{\beta}$. To examine the robustness of our results with respect to this part of the assumption, we now consider the order flow allocation problem for arbitrary values of γ and κ.

To facilitate our analysis, we assume that the order flow provider does not internalize its impact on the success probabilities of builders. That is, the order flow provider views a builder's success probability, $\mathbb{P}(b_i)$, as a constant equal to $P_{i,0} = \mathbb{P}(b_i)|_{\tilde{p}=0}$. This simplifies the optimization problem of the order flow provider considerably, by removing the feedback loop where allocating order flow to a builder has a positive effect on its success probability, making further allocation to the builder more attractive. We remark that the model results under the assumption of no internalization can be viewed as a reliable approximation of the corresponding results under the more general model. This is because in the regime considered where the order flow allocation is relatively small, there is a limited difference between whether the order flow provider internalizes its impact on block success probabilities or not.

Proposition C.1 below confirms that the intuition delivered by our results in Section 3.3.1 extends to the case where γ and κ are allowed to take on arbitrary values. In particular, we point out the following aspects of Proposition C.1:

- First, the leading order term of the allocation to builder i depends on the marginal gain $a_{i,0}$, and coincides with the leading order term in Section 3.3.1 (see Cor. 3.6 and Prop. 3.7).

- Second, the externality from builder $j \neq i$ depends on the marginal gain $a_{j,0}$, and its magnitude is increasing in the ratio κ/γ; this is also consistent with the results in Section 3.3.1 (see Cor. 3.6, where a larger value of β corresponds to a larger value of κ/γ).

- Third, a larger value of κ/γ leads to increased concentration in allocation, i.e., fewer builders acquiring order flow. This interpolates between our result in Cor. 3.6, which shows that if β is small (i.e., if κ/γ

is small), then order flow is allocated to all builders, and Prop. 3.7, which shows that if β is large (i.e., if κ/γ is large), then order flow is allocated to a single builder.

Altogether, larger values of κ/γ lead to stronger allocation externalities, and order flow becoming concentrated among a smaller number of builders. This parallels our results in Section 3.3.1, where β plays the role of κ/γ, and small values of β lead to allocation to all builders, while large values of β lead to allocation to a single builder.

Proposition C.1. *Assume the order flow provider does not internalize its impact on the success probabilities of builders, and assume builders to be ordered in increasing order of $a_{i,0} = P_{i,0} + \eta\alpha_i$. Then, a unique equilibrium allocation $\tilde{p}^*$ exists for any values of γ and κ such that $\gamma + \kappa > 0$.*

(i) If $\gamma > 0$, the equilibrium $\tilde{p}^*$ is given by

$$\tilde{p}_i^* = \begin{cases} \frac{1}{\gamma}\left(a_{i,0} - \frac{\kappa/\gamma}{1+n\kappa/\gamma}\sum_{j=1}^{n} a_{j,0}\right), & 1 \leq i \leq n^*, \\ 0, & n^* < i \leq N, \end{cases}$$

where n^* is the equilibrium number of builders that receive positive allocation. The value of n^* is decreasing in κ/γ; for κ/γ small enough we have $n^* = N$, and for κ/γ large enough we have $n^* = 1$.

(ii) If $\gamma = 0$, the equilibrium $\tilde{p}^*$ is given by

$$\tilde{p}_i^* = \begin{cases} \frac{a_{i,0}}{\kappa}, & i = 1, \\ 0, & i \neq 1, \end{cases}$$

thus only the builder with the highest marginal gain receives a positive allocation.

Proof: See Appendix D.2. □

D Proof of Results in Section 3.3

Section D.1 contains two lemmas that are used in the proofs of the main results of Sections 3.3.1 and 3.3.2, which can be found in Sections D.2 and D.3, respectively.

D.1 Auxiliary Lemmas

The first part of Lemma D.1 shows that a builder's block success probability is increasing in the builder's value extraction probability, and decreasing in the extraction probabilities of other builders.

For the second part of the lemma, note from (3.9) that the marginal gain $\mu_{i,0}$ and marginal loss $\mu_{i,j,0}$ satisfy $\mu_{i,0} = \mu_i|_{\tilde{p}=0}$ and $\mu_{i,j,0} = \mu_{i,j}|_{\tilde{p}=0}$, where

$$\mu_i := \frac{\partial}{\partial \tilde{p}_i}\mathbb{E}\big[(b_i - b_{(2)})^+\big], \qquad \mu_{i,j} := -\frac{\partial}{\partial \tilde{p}_j}\mathbb{E}\big[(b_i - b_{(2)})^+\big]. \tag{D.1}$$

The lemma then states that the expected revenue of a builder is increasing in the builder's value extraction probability, and decreasing in the extraction probabilities of other builders.

Lemma D.1.

(i) The block success probability of builder i is linear in $\tilde{p}_i$ and $\tilde{p}_j$, and satisfies

$$\frac{\partial}{\partial \tilde{p}_i}\mathbb{P}(b_i) \in (0,1), \qquad \frac{\partial}{\partial \tilde{p}_j}\mathbb{P}(b_i) \in (-1,0).$$

(ii) The sensitivities of the revenue of builder i to $\tilde{p}_j$ and $\tilde{p}_j$ satisfy

$$\mu_i > 0, \qquad \mu_{i,0} > 0, \qquad \mu_{i,j} > 0, \qquad \mu_{i,j,0} > 0.$$

Proof of Lemma D.1: The block success probability of builder i can be written as

$$\mathbb{P}(b_i) = \tilde{p}_i \mathbb{P}(b_i|\tilde{X}_i > 0) + (1-\tilde{p}_i)\mathbb{P}(b_i|\tilde{X}_i = 0),$$

and the derivative w.r.t. $\tilde{p}_i$ is given by

$$\frac{\partial}{\partial \tilde{p}_i}\mathbb{P}(b_i) = \mathbb{P}(b_i|\tilde{X}_i > 0) - \mathbb{P}(b_i|\tilde{X}_i = 0) \in (0,1).$$

The derivative is positive because $b_i|(\tilde{X}_i > 0)$ stochastically dominates $b_i|(\tilde{X}_i = 0)$, and b_k is independent of $\tilde{X}_i$ for $k \neq i$. The block success probability of builder i can also be written as

$$\mathbb{P}(b_i) = \tilde{p}_j \mathbb{P}(b_i|\tilde{X}_j > 0) + (1-\tilde{p}_j)\mathbb{P}(b_i|\tilde{X}_j = 0),$$

and the derivative w.r.t. $\tilde{p}_j$ becomes

$$\frac{\partial}{\partial \tilde{p}_j}\mathbb{P}(b_i) = \mathbb{P}(b_i|\tilde{X}_j > 0) - \mathbb{P}(b_i|\tilde{X}_j = 0) \in (-1,0).$$

The derivative is negative because $b_j|(\tilde{X}_j > 0)$ stochastically dominates $b_j|(\tilde{X}_j = 0)$, and b_k is independent of $\tilde{X}_j$ for $k \neq j$.

The second part of the lemma is shown in exactly the same way, by first conditioning on $\tilde{X}_i$ to show that $\mu_i > 0$, and then conditioning on $\tilde{X}_j$ to show that $\mu_{i,j} < 0$. The signs of $\mu_{i,0}$ and $\mu_{i,j,0}$ then follow by the definition of those quantities. $\square$

Part (i) of Lemma D.2 states that the initial block success probabilities $P_{i,0}$, defined in (3.7), have the same ordering as the initial value extraction probabilities $p_{i,0}$. Parts (ii) and (iii) state that a larger builder has a larger marginal gain from acquiring order flow, and a larger marginal loss from the order flow acquisition of other builders. Part (iv) states that a larger builder has a larger marginal loss from the order flow acquisition of a smaller builder, compared to the smaller builder in the opposite scenario.

Lemma D.2. *For i and j such that $p_{i,0} \geq p_{j,0}$, and $k \neq i, j$, we have the following relations:*

$$\text{(i) } P_{i,0} \geq P_{j,0}, \qquad \text{(ii) } \mu_{i,0} \geq \mu_{j,0}, \qquad \text{(iii) } \mu_{i,k,0} \geq \mu_{j,k,0}, \qquad \text{(iv) } \mu_{i,j,0} \geq \mu_{j,i,0}.$$

Proof of Lemma D.2: Part (i) follows from the fact that $b_i|_{\tilde{p}=0}$ stochastically dominates $b_j|_{\tilde{p}=0}$.

For part (ii), we introduce the notation $\mathbb{E}^{(0)}[\cdot]$ for $\mathbb{E}[\cdot]\big|_{\tilde{p}=0}$, and write

$$\mu_{i,0} = \mathbb{E}^{(0)}[(b_i - b_{(2)})^+|\tilde{X}_i > 0] - \mathbb{E}^{(0)}[(b_i - b_{(2)})^+|\tilde{X}_i = 0].$$

Denoting by $b_{(1,-i)}$ and $b_{(1,-j)}$ the largest block values outside of blocks i and j, respectively, we then have

$$\begin{aligned}\mu_{i,0} &= \mathbb{E}^{(0)}[(X_i + \tilde{V}_i - b_{(1,-i)})^+] - \mathbb{E}^{(0)}[(X_i - b_{(1,-i)})^+] \\ &\geq \mathbb{E}^{(0)}[(X_j + \tilde{V}_j - b_{(1,-j)})^+] - \mathbb{E}^{(0)}[(X_j - b_{(1,-j)})^+] = \mu_{j,0}.\end{aligned}$$

The inequality follows from X_i stochastically dominating X_j, V_i and V_j having the same distribution, $b_{(1,-j)}$ stochastically dominating $b_{(1,-i)}$, and $b_{(1,-i)}$ and $b_{(1,-j)}$ being independent of $X_i+\tilde{V}_i$ and $X_j+\tilde{V}_j$, respectively.

For part (iii), note that we can write

$$\begin{aligned}\mu_{i,k,0} &= -\big(\mathbb{E}^{(0)}[(b_i - b_{(2)})^+|\tilde{X}_k > 0] - \mathbb{E}^{(0)}[(b_i - b_{(2)})^+|\tilde{X}_k = 0]\big) \\ &\geq -\big(\mathbb{E}^{(0)}[(b_j - b_{(2)})^+|\tilde{X}_k > 0] - \mathbb{E}^{(0)}[(b_j - b_{(2)})^+|\tilde{X}_k = 0]\big) = \mu_{j,k,0}.\end{aligned}$$

The inequality follows from b_i stochastically dominating b_j, and $b_{k'}$ being independent of $\tilde{X}_k$ for $k' \neq k$.

For part (iv), we similarly have

$$\begin{aligned}\mu_{i,j,0} &= -\big(\mathbb{E}^{(0)}[(b_i - b_{(2)})^+ | \tilde{X}_j > 0] - \mathbb{E}^{(0)}[(b_i - b_{(2)})^+ | \tilde{X}_j = 0]\big) \\ &\geq -\big(\mathbb{E}^{(0)}[(b_j - b_{(2)})^+ | \tilde{X}_i > 0] - \mathbb{E}^{(0)}[(b_j - b_{(2)})^+ | \tilde{X}_i = 0]\big) = \mu_{j,i,0},\end{aligned}$$

using that b_i stochastically dominates b_j, and that b_k is independent of $\tilde{X}_j$ for $k \neq j$. $\square$

D.2 Proofs of Results in Section 3.3.1

Proof of Proposition 3.5: We divide the proof into several steps.

Step 1: show that we can assume that $0 \leq \tilde{p}_i < K/\gamma$, where $K < \infty$ is a constant. The derivative of the objective function π with respect to $\tilde{p}_i$ satisfies

$$\frac{\partial \pi}{\partial \tilde{p}_i} = \mathbb{P}(b_i) + \eta\alpha_i + \sum_{j=1}^{N} \tilde{p}_j \frac{\partial}{\partial \tilde{p}_i}\mathbb{P}(b_j) - \gamma\tilde{p}_i - \kappa\sum_{j=1}^{N}\tilde{p}_j \leq 1 + \eta\alpha_i + \tilde{p}_i(1-\gamma), \tag{D.2}$$

where the inequality uses Lemma D.1-(i). It follows that there exists a constant $K > 0$ such that $\tilde{p}_i \leq K/\gamma$ for γ large enough. Otherwise $\partial_{\tilde{p}_i}\pi$ is negative, which cannot be the case for an interior equilibrium point.

Step 2: Show that there exists an equilibrium. We begin by showing that the objective function π is concave in each of its arguments. Taking the derivative of (D.2) w.r.t. $\tilde{p}_i$ gives

$$\frac{\partial^2 \pi}{\partial \tilde{p}_i^2} = \frac{\partial}{\partial \tilde{p}_i}\mathbb{P}(b_i) + \frac{\partial}{\partial \tilde{p}_i}\mathbb{P}(b_i) + \sum_{j=1}^{N}\tilde{p}_j\frac{\partial^2}{\partial \tilde{p}_i^2}\mathbb{P}(b_j) - \gamma - \kappa \leq 2 - \gamma,$$

where the inequality follows from Lemma D.1-(i). This quantity is negative for large enough γ, and existence of equilibrium then follows by the concavity of π in each of its arguments, the continuity of π in each of its arguments, and Step 1, which established that the action space for $\tilde{p}_i$ can be assumed to be compact.

Step 3: Show that $\tilde{p}_i > 0$ in equilibrium for all i. For $\tilde{p}_i = 0$, the first-order condition (D.2) becomes

$$\frac{\partial \pi}{\partial \tilde{p}_i}\Big|_{\tilde{p}_i=0} = \mathbb{P}(b_i)\big|_{\tilde{p}_i=0} + \eta\alpha_i + \sum_{j\neq i}\tilde{p}_j\frac{\partial}{\partial \tilde{p}_i}\mathbb{P}(b_j)\big|_{\tilde{p}_i=0} - \kappa\sum_{j\neq i}\tilde{p}_j.$$

By Step 1 and Lemma D.1-(i), it follows that $\partial_{\tilde{p}_i}\pi|_{\tilde{p}_i=0} > 0$ for large enough γ, which, in turn, implies that $\tilde{p}_i > 0$ has to hold in equilibrium.

Step 4: Develop an approximation for the equilibrium. By applying a first-order expansion around $\tilde{p} = 0$ to $\partial_{\tilde{p}_i}\pi$ in (D.2), it follows that any equilibrium has to satisfy $a_0 + A\tilde{p} + e = 0$, where $a_0 = (a_{i,0})_{1\leq i\leq N}$,

$A = (a_{i,j})_{1\le i,j\le N}$, and $e = (e_i)_{1\le i\le N}$. The vector e satisfies $||e|| = o(||\tilde{p}||)$, and the elements of a_0 and A are given by

$$\begin{aligned} a_{i,0} &= \frac{\partial \pi}{\partial \tilde{p}_i}\Big|_{\tilde{p}=0} = \mathbb{P}(b_i)\big|_{\tilde{p}=0} + \eta\alpha_i > 0, \\ a_{i,i} &= \frac{\partial^2 \pi}{\partial \tilde{p}_i^2}\Big|_{\tilde{p}=0} = 2\partial_{\tilde{p}_i}\mathbb{P}(b_i)\big|_{\tilde{p}=0} - \gamma - \kappa < 0, \\ a_{i,j} &= \frac{\partial^2 \pi}{\partial \tilde{p}_i \partial \tilde{p}_j}\Big|_{\tilde{p}=0} = \partial_{\tilde{p}_j}\mathbb{P}(b_i)\big|_{\tilde{p}=0} + \partial_{\tilde{p}_i}\mathbb{P}(b_j)\big|_{\tilde{p}=0} - \kappa < 0, \end{aligned} \tag{D.3}$$

where the sign of $a_{i,j}$ follows from Lemma D.1-(i).

Consider now the system of equations $a_0 + A\tilde{p} = 0$, and decompose the matrix A as

$$A = B - I_\gamma - \mathbf{1}_\kappa = \gamma\big(\gamma^{-1}B - (I + \mathbf{1}_{\kappa/\gamma})\big),$$

where I_γ is a diagonal matrix with the i-th element equal to γ, $\mathbf{1}_\kappa$ and $\mathbf{1}_{\kappa/\gamma}$ are matrices with all entries equal to κ and κ/γ, respectively, and the matrix $B = (b_{i,j})_{1\le i,j\le N}$ is given by

$$b_{i,j} = a_{i,j} + \kappa, \quad i \ne j, \qquad\qquad b_{i,i} = a_{i,i} + \gamma + \kappa. \tag{D.4}$$

The matrix $\mathbf{1}_{\kappa/\gamma}$ is a rank one matrix, so by the Sherman-Morrison formula we have

$$(I + \mathbf{1}_{\kappa/\gamma})^{-1} = I - \frac{\kappa}{\gamma}\frac{\mathbf{1}\mathbf{1}^T}{1 + \frac{\kappa}{\gamma}\mathbf{1}^T\mathbf{1}} = I - \mathbf{1}_{\frac{\kappa/\gamma}{1+N\kappa/\gamma}}.$$

The matrix B is bounded, by Lemma D.1-(i), and the set of non-singular matrices is open, so $\gamma^{-1}B-(I+\mathbf{1}_{\kappa/\gamma})$ is invertible for large enough γ. The system of equations $a_0 + A\tilde{p} = 0$ thus has a unique solution

$$\tilde{p} = -A^{-1}a_0 = (I_\gamma + \mathbf{1}_\kappa - B)^{-1}a_0. \tag{D.5}$$

We now show that the solution (D.5) approximates an equilibrium. Any equilibrium $\tilde{p}^*$ has to satisfy

$$\tilde{p}^* = -A^{-1}(a_0 + e^*) = (I_\gamma + \mathbf{1}_\kappa - B)^{-1}(a_0 + e^*),$$

where $e^* = e|_{\tilde{p}=\tilde{p}^*}$. Using $||I_\gamma + \mathbf{1}_\kappa - B||^{-1} = O(\gamma^{-1})$, and $||e^*|| = o(||\tilde{p}||) = o(\gamma^{-1})$, we then have

$$||\tilde{p} - \tilde{p}^*|| = ||(I_\gamma + \mathbf{1}_\kappa - B)^{-1}e^*|| = o(\gamma^{-2}).$$

Finally, from $\tilde{p}^* = -A^{-1}(a_0 + e^*)$ we have that any equilibrium $\tilde{p}^*$ satisfies

$$\tilde{p}_i^* = -\frac{a_{i,0}}{a_{i,i}} - \sum_{j \neq i} \frac{a_{i,j}}{a_{i,i}} \tilde{p}_j^* - \frac{e_i^*}{a_{i,i}} = -\frac{a_{i,0}}{a_{i,i}} - \sum_{j \neq i} \frac{a_{i,j}}{a_{i,i}} \tilde{p}_j^* + o(\gamma^{-2}),$$

where we used $a_{i,i} = O(\gamma)$ and $e_i^* = o(\gamma^{-1})$. □

Proof of Corollary 3.6: From (D.5) it follows that the solution to the system of equations in Proposition 3.5 can be written as

$$\tilde{p} = \frac{1}{\gamma}\big(I + \mathbf{1}_{\kappa/\gamma} - \frac{1}{\gamma}B\big)^{-1} a_0 = \frac{1}{\gamma} a_0 + \frac{1}{\gamma}\big(\frac{1}{\gamma}B - \mathbf{1}_{\kappa/\gamma}\big) a_0 + o(\gamma^{-2}).$$

Using $k/\gamma = \gamma^{\beta-1}$, the expansion for $\tilde{p}_i$ becomes

$$\tilde{p}_i = \frac{a_{i,0}}{\gamma} - \frac{1}{\gamma^{2-\beta}} \sum_{j=1}^{N} a_{j,0} + \frac{1}{\gamma^2} \sum_{j=1}^{N} b_{i,j} a_{j,0} + o(\gamma^{-2}),$$

where $b_{i,i} > 0$ and $b_{i,j} < 0$ for $j \neq i$. □

Proof of Proposition 3.7: As in the proof of Proposition 3.5, we divide the proof into several steps. We omit the proofs of Steps 1 and 2 because they are identical to the proofs of Steps 1 and 2 in Proposition 3.5.

Step 1: We can assume that $0 \leq \tilde{p}_i < K/\kappa$, where $K < \infty$ is a constant.

Step 2: There exists an equilibrium.

Step 3: $\tilde{p}_i > 0$ in equilibrium for exactly one i. Recall from (D.2) that

$$\frac{\partial \pi}{\partial \tilde{p}_i} = \mathbb{P}(b_i) + \eta\alpha_i + \sum_{j=1}^{N} \tilde{p}_j \frac{\partial}{\partial \tilde{p}_i} \mathbb{P}(b_j) - \gamma \tilde{p}_i - \kappa \sum_{j=1}^{N} \tilde{p}_j.$$

First observe that $\partial_{\tilde{p}_i} \pi|_{\tilde{p}=0} > 0$ holds for at least one i, so $\tilde{p}_i > 0$ has to hold for at least one builder in equilibrium. We now show that for large enough γ, the first-order condition cannot simultaneously be satisfied for two builders indexed by i and j. Assume without loss of generality that

$$d := \mathbb{P}(b_i)\big|_{\tilde{p}=0} + \eta\alpha_i - (\mathbb{P}(b_j)\big|_{\tilde{p}=0} + \eta\alpha_j) > 0.$$

We will show that if $\partial_{\tilde{p}_i} \pi = 0$, then $\partial_{\tilde{p}_j} \pi = 0$ cannot hold. To that end, note that for any $\epsilon > 0$ we can, using Step 1, Lemma D.1-(i), and $\kappa = \gamma^\beta$, find γ large enough for the following to hold for $k = i, j$:

$$\left|\mathbb{P}(b_k) - \mathbb{P}(b_k)|_{\tilde{p}=0}\right| < \epsilon, \qquad \gamma\tilde{p}_k < \epsilon, \qquad \sum_{l=1}^{N} \tilde{p}_l \frac{\partial}{\partial\tilde{p}_k}\mathbb{P}(b_l) < \epsilon.$$

Using that, the definition of d, and the expression for $\partial_{\tilde{p}_i}\pi$, we obtain

$$0 = \frac{\partial\pi}{\partial\tilde{p}_i} > \frac{\partial\pi}{\partial\tilde{p}_j} + d - 5\epsilon.$$

This holds for any $\epsilon > 0$, and it follows that $\partial_{\tilde{p}_j}\pi < 0$. One can similarly show that if $\partial_{\tilde{p}_j}\pi = 0$, then $\partial_{\tilde{p}_i}\pi > 0$ for large enough γ. It follows that the first-order condition can hold for at most one builder, which is the builder with the largest value of $P_{i,0} + \eta\alpha_i$.

Step 4: Approximating an equilibrium. For large enough γ, we have $\tilde{p}_{i_0} > 0$ and $\tilde{p}_{-i_0} = 0$, where $i_0 = \arg\max_{1\leq i\leq N}(P_{i,0} + \eta\alpha_i)$. Using $\tilde{p}_{-i_0} = 0$ and Lemma D.1-(i), the expression for $\partial_{\tilde{p}_{i_0}}\pi$ becomes

$$\frac{\partial\pi}{\partial\tilde{p}_{i_0}} = \mathbb{P}(b_{i_0})\big|_{\tilde{p}=0} + \eta\alpha_{i_0} + \big(2\partial_{\tilde{p}_{i_0}}\mathbb{P}(b_{i_0})\big|_{\tilde{p}=0} - \gamma - \kappa\big)\tilde{p}_{i_0}.$$

Setting the above to zero yields

$$\tilde{p}_{i_0} = \frac{\mathbb{P}(b_{i_0})|_{\tilde{p}=0} + \eta\alpha_{i_0}}{\gamma + \kappa - 2\partial_{\tilde{p}_{i_0}}\mathbb{P}(b_{i_0})\big|_{\tilde{p}=0}} = \frac{\mathbb{P}(b_{i_0})|_{\tilde{p}=0} + \eta\alpha_{i_0}}{\gamma^\beta} + o(\gamma^{-\beta}), \tag{D.6}$$

where we used $\kappa = \gamma^\beta$, where $\beta > 1$. □

Proof of Proposition C.1: If the order flow provider does not internalize its impact on the block success probabilities, the objective function (3.4) becomes

$$\pi(\tilde{p};\alpha) = \sum_{i=1}^{N} \tilde{p}_i\big(P_{i,0} + \eta\alpha_i\big) - \frac{\gamma}{2}\sum_{i=1}^{N}\tilde{p}_i^2 - \frac{\kappa}{2}\Big(\sum_{i=1}^{N}\tilde{p}_i\Big)^2.$$

We begin by noting that if $\gamma = 0$ and $\kappa > 0$, the equilibrium allocation is given by $\tilde{p}_1^* = a_{1,0}/\kappa$, and $\tilde{p}_{-1}^* = 0$.

If $\gamma > 0$, we first derive the equilibrium allocation without restricting $\tilde{p}$ to be nonnegative. The first-order condition can be written in matrix form as $A\tilde{p} = a_0$, where $A = I_\gamma + \mathbf{1}_\kappa$ and $a_0 = P_0 + \eta\alpha$. The matrix A is invertible and by the Sherman-Morrison formula we have

$$A^{-1} = \frac{1}{\gamma}(I + \mathbf{1}_{\kappa/\gamma})^{-1} = \frac{1}{\gamma}\big(I - \mathbf{1}_{\frac{\kappa/\gamma}{1+N\kappa/\gamma}}\big).$$

It follows that the unique equilibrium allocation is given by

$$\tilde{p}_i^* = \frac{1}{\gamma}\Big(a_{i,0} - \frac{\kappa/\gamma}{1+N\kappa/\gamma}\sum_{j=1}^{N} a_{j,0}\Big).$$

Now we consider the case where $\tilde{p}$ is restricted to be nonnegative. It is easy to see that (i) if $\tilde{p}_i > 0$ in equilibrium, then $\tilde{p}_j > 0$ will also hold for $j < i$, and (ii) there will be at least one positive allocation in equilibrium, i.e., $\tilde{p}_1 > 0$. It follows that the unique equilibrium allocation is given by

$$\tilde{p}_i^* = \begin{cases} \frac{1}{\gamma}\big(a_{i,0} - \frac{\kappa/\gamma}{1+n\kappa/\gamma}\sum_{j=1}^{n} a_{j,0}\big), & 1 \leq i \leq n, \\ 0, & n < i \leq N, \end{cases}$$

where n is the largest number from 1 to N such that $a_{n,0} - \frac{\kappa/\gamma}{1+n\kappa/\gamma}\sum_{j=1}^{n} a_{j,0} > 0$. It is clear that the value of n is decreasing in the ratio κ/γ, because the function $x \mapsto x/(1+nx)$ is increasing for $x \geq 0$. Furthermore, for κ/γ small enough we have $n = N$, and for κ/γ large enough we have $n = 1$. □

D.3 Proofs of Results in Section 3.3.2

The following lemma contains technical results that are used in the proof of Proposition 3.9. For part (ii) of the lemma, we define

$$\mu_i^* := \mu_i\big|_{\tilde{p}=\tilde{p}^*}, \qquad \mu_{i,j}^* := \mu_{i,j}\big|_{\tilde{p}=\tilde{p}^*}, \tag{D.7}$$

where μ_i and $\mu_{i,j}$ are the marginal gain and loss of order flow acquisition, defined in (D.1). We also recall the definitions of $\mu_{i,0}$ and $\mu_{i,j,0}$ in (3.9), given by $\mu_{i,0} := \mu_i|_{\tilde{p}=0}$ and $\mu_{i,j,0} := \mu_{i,j}|_{\tilde{p}=0}$.

Lemma D.3.

(i) The sensitivities of the equilibrium order flow $\tilde{p}^*$ in Proposition 3.5 with respect to α satisfy

$$\partial_{\alpha_i}\tilde{p}_i^* = \frac{\eta}{\gamma} - \frac{\eta}{\gamma^{2-\beta}} + O(\gamma^{-2}), \qquad \partial_{\alpha_i}\tilde{p}_j = -\frac{\eta}{\gamma^{2-\beta}} + O(\gamma^{-2}).$$

Furthermore, the second derivatives of $\tilde{p}^*$ with respect to α satisfy

$$\partial^2_{\alpha_i}\tilde{p}_i^* = o(\gamma^{-2}), \qquad \partial^2_{\alpha_i}\tilde{p}_j^* = o(\gamma^{-2}), \qquad \partial^2_{\alpha_i\alpha_j}\tilde{p}_i^* = o(\gamma^{-2}).$$

(ii) The marginal gain μ_i and loss $\mu_{i,j}$ of order flow acquisition satisfy

$$\mu_i^* = \mu_{i,0} + O(\gamma^{-1}), \qquad \mu_{i,j}^* = \mu_{i,j,0} + O(\gamma^{-1}),$$

where μ_i^* and $\mu_{i,j}^*$ are defined in (D.7), and $\mu_{i,0}$ and $\mu_{i,j,0}$ are defined in (3.9). Furthermore,

$$\partial_{\alpha_i}\mu_i\big|_{\tilde{p}=\tilde{p}^*} = O(\gamma^{-(2-\beta)}), \qquad \partial_{\alpha_i}\mu_{i,j}\big|_{\tilde{p}=\tilde{p}^*} = O(\gamma^{-1}).$$

Proof of Lemma D.3: From Step 4 in the proof of Proposition 3.5 we have that any equilibrium $\tilde{p}^*$ satisfies

$$\tilde{p}^* = (I_\gamma + \mathbf{1}_\kappa - B)^{-1}(a_0 + e^*), \tag{D.8}$$

where $e^* = (e_j^*)_{1\le j\le N}$ is given by

$$e_j^* = \mathbb{P}(b_j)\big|_{\tilde{p}=\tilde{p}^*} - \mathbb{P}(b_j)\big|_{\tilde{p}=0} - \sum_{j'=1}^N \tilde{p}_{j'}^*\partial_{\tilde{p}_{j'}}\mathbb{P}(b_j)\big|_{\tilde{p}=0} + \sum_{j'=1}^N \tilde{p}_{j'}^*\partial_{\tilde{p}_j}\mathbb{P}(b_{j'})\big|_{\tilde{p}=\tilde{p}^*} - \sum_{j'=1}^N \tilde{p}_{j'}^*\partial_{\tilde{p}_j}\mathbb{P}(b_{j'})\big|_{\tilde{p}=0}.$$

By following the same steps as in the proof of Lemma D.1, we can write $\partial_{\alpha_i}e_j^* = \sum_{j'=1}^N \xi_{i,j,j'}\partial_{\alpha_i}\tilde{p}_{j'}^*$, where

$$\xi_{i,j,j'} = \sum_{k=1}^N \xi_{i,j,j'}^{(k)}\tilde{p}_k^* + R_{i,j,j'}.$$

In the above equation, the coefficients $\xi_{i,j,j'}^{(k)}$ are exogenous constants, independent of γ, and the remainder term is bounded and satisfies $R_{i,j,j'} = O(\gamma^{-2})$. We can use the above to write

$$\partial_{\alpha_i}e^* = \xi\partial_{\alpha_i}\tilde{p}^*,$$

where ξ is an $N\times N$ matrix where the j-th line is given by $(\xi_{i,j,j'})_{1\le j'\le N}$.

Taking the derivative w.r.t. α_i of both sides of equation (D.8) gives

$$\partial_{\alpha_i}\tilde{p}^* = (I_\gamma + \mathbf{1}_\kappa - B)^{-1}(\partial_{\alpha_i}a_0 + \partial_{\alpha_i}e^*) = (I_\gamma + \mathbf{1}_\kappa - B)^{-1}(\partial_{\alpha_i}a_0 + \xi\partial_{\alpha_i}\tilde{p}^*),$$

which can be rewritten as

$$\partial_{\alpha_i}\tilde{p}^* = \gamma^{-1}\big(I + \mathbf{1}_{\kappa/\gamma} - \gamma^{-1}B - \gamma^{-1}\xi\big)^{-1}\partial_{\alpha_i}a_0,$$

where we used that $I_\gamma + \mathbf{1}_\kappa - B - \xi$ is invertible for large enough γ; this follows from $I_\gamma + \mathbf{1}_\kappa - B$ being invertible, in addition to the matrix ξ being bounded in γ. We then have

$$\partial_{\alpha_i}\tilde{p}^* = \gamma^{-1}\partial_{\alpha_i}a_0 + \gamma^{-1}\big(\gamma^{-1}B + \gamma^{-1}\xi - \mathbf{1}_{\kappa/\gamma}\big)\partial_{\alpha_i}a_0 + o(\gamma^{-2}) = \gamma^{-1}\partial_{\alpha_i}a_0 + \gamma^{-(2-\beta)}\partial_{\alpha_i}a_0\mathbf{1} + o(\gamma^{-2}),$$

and the results for $\partial_{\alpha_i}\tilde{p}_i^*$ and $\partial_{\alpha_i}\tilde{p}_j^*$ follow. Taking the second derivative of (D.8) w.r.t. α_i and following the same steps above yields the results for the second derivatives of $\tilde{p}_i^*$ and $\tilde{p}_j^*$.

The relation between μ_i^* and $\mu_{i,0}$ in the second part of the lemma follows from writing

$$\mu_i = \mathbb{E}[(b_i - b_{(2)})^+|\tilde{X}_i > 0] - \mathbb{E}[(b_i - b_{(2)})^+|\tilde{X}_i = 0],$$

and using Step 1 in the proof of Proposition 3.5, which states that $||\tilde{p}|| = O(\gamma^{-1})$. The identity $\partial_{\alpha_i}\mu_i^* = o(\gamma^{-1})$ then follows from the above expression for μ_i, which is independent of $\tilde{p}_i$, and part (i) of the lemma. The results for $\mu_{i,j}^*$ are obtained in the same way. $\square$

Proof of Proposition 3.9: For the objective function (3.3) of builder i, we have for $\tilde{p}^* = \tilde{p}^*(\alpha)$,

$$\frac{\partial \pi_i}{\partial \alpha_i}\Big|_{\tilde{p}=\tilde{p}^*} = \mu_i^*\frac{\partial \tilde{p}_i^*}{\partial \alpha_i} - \sum_{j\neq i}\mu_{i,j}^*\frac{\partial \tilde{p}_j^*}{\partial \alpha_i} - \tilde{p}_i^* - \alpha_i\frac{\partial \tilde{p}_i^*}{\partial \alpha_i},$$

with μ_i^* and $\mu_{i,j}^*$ are defined in (D.7). The second derivative w.r.t. α_i then satisfies

$$\frac{\partial^2 \pi_i}{\partial \alpha_i^2}\Big|_{\tilde{p}=\tilde{p}^*} = \mu_i^*\frac{\partial^2 \tilde{p}_i^*}{\partial \alpha_i^2} + \frac{\partial \mu_i^*}{\partial \alpha_i}\frac{\partial \tilde{p}_i^*}{\partial \alpha_i} - \sum_{j\neq i}\big(\frac{\partial \mu_{i,j}^*}{\partial \alpha_i}\frac{\partial \tilde{p}_j^*}{\partial \alpha_i} + \mu_{i,j}^*\frac{\partial^2 \tilde{p}_j^*}{\partial \alpha_i^2}\big) - 2\frac{\partial \tilde{p}_i^*}{\partial \alpha_i} - \alpha_i\frac{\partial^2 \tilde{p}_i^*}{\partial \alpha_i^2} = -2\frac{\partial \tilde{p}_i^*}{\partial \alpha_i} + o(\gamma^{-2}),$$

where the second equality follows from Lemma D.3. Hence, the objective function is concave (and continuous) for large enough γ, where the sign of the second derivative follows from Lemma D.3. An equilibrium thus exists because the domain of α_i can be assumed to be bounded; α_i is bounded from below by zero, and the reasoning used in Step 1 in the proof of Prop. 3.5 can be used to bound the domain of α_i from above.

Next, using Corollary 3.6 and Lemma D.3, the derivative of the objective function w.r.t. α_i satisfies

$$\frac{\partial \pi_i}{\partial \alpha_i}\Big|_{\tilde{p}=\tilde{p}^*} = \frac{\eta}{\gamma}\Big(\mu_{i,0} - \frac{P_{i,0}}{\eta} - 2\alpha_i\big(1 - \frac{1}{\gamma^{1-\beta}}\big) + \frac{1}{\gamma^{1-\beta}}\Big(\sum_{j\neq i}\mu_{i,j,0} - \mu_{i,0} + \frac{1}{\eta} + \sum_{j\neq i}\alpha_j\Big)\Big) + e_i,$$

where $e_i = O(\gamma^{-2})$. The first-order condition of builder i can then be written as

$$2\alpha_i\big(1 - \frac{1}{\gamma^{1-\beta}}\big) = \mu_{i,0} - \frac{P_{i,0}}{\eta} + \frac{1}{\gamma^{1-\beta}}\Big(\sum_{j\neq i}\mu_{i,j,0} - \mu_{i,0} + \frac{1}{\eta} + \sum_{j\neq i}\alpha_j\Big) + e_i\frac{\gamma}{\eta},$$

which yields

$$\begin{aligned} 2\alpha_i &= \mu_{i,0} - \frac{P_{i,0}}{\eta} + \frac{1}{\gamma^{1-\beta}}\Big(\sum_{j\neq i}\mu_{i,j,0} - \mu_{i,0} + \frac{1}{\eta} + \sum_{j\neq i}\alpha_j + \mu_{i,0} - \frac{P_{i,0}}{\eta}\Big) + e_i\frac{\gamma}{\eta} + O(\gamma^{-2(1-\beta)}) \\ &= \mu_{i,0} - \frac{P_{i,0}}{\eta} + \frac{1}{\gamma^{1-\beta}}\Big(\sum_{j\neq i}\big(\mu_{i,j,0} + \frac{P_{j,0}}{\eta}\big) + \sum_{j\neq i}\alpha_j\Big) + O(\gamma^{-1}). \end{aligned}$$

It follows that for γ large enough, an equilibrium solution has to satisfy

$$\alpha_i^* = \begin{cases} \frac{1}{2}\tilde{a}_{i,0} + \frac{1}{2}\frac{1}{\gamma^{1-\beta}}\big(\sum_{j\neq i}\tilde{a}_{i,j,0} + \sum_{j\neq i}\alpha_j^*\big) + O(\gamma^{-1}), & \tilde{a}_{i,0} > 0, \\ 0, & \tilde{a}_{i,0} \leq 0. \end{cases}$$

Finally, similar to the approach used to demonstrate the concavity of π_i, we can apply Lemma D.3 to express the following:

$$\frac{\partial^2\pi_i}{\partial\alpha_i\partial\alpha_j}\Big|_{\tilde{p}=\tilde{p}^*} = -\frac{\partial\tilde{p}_i^*}{\partial\alpha_j} + O(\gamma^{-2}),$$

where the leading order term is positive, so the mixed derivative is positive for large enough values of γ. □

Proof of Proposition 3.10: Proposition 3.7 states that for large enough γ, we have $\tilde{p}_i^*(\alpha) > 0$ for the builder with the largest value of $a_{i,0} = P_{i,0} + \eta\alpha_i$, and $\tilde{p}_i^*(\alpha) = 0$ for other builders. Moreover, Eq. (D.6) in the proof of Proposition 3.7 shows that the positive allocation satisfies

$$\tilde{p}_i^*(\alpha) = \frac{P_{i,0} + \eta\alpha_i}{\gamma^\beta + \gamma - 2P'_{i,0}},$$

where $P'_{i,0} := \partial_{\tilde{p}_i}P(b_i)|_{\tilde{p}=0}$. To analyze the equilibrium fees of builders, consider first the case of two builders i and j such that $P_{i,0} < P_{j,0}$, and note that

$$a_{i,0} \geq a_{j,0} \quad \Longleftrightarrow \quad \alpha_i \geq \bar{\alpha}_{i,j}(\alpha_j) := \alpha_j + \frac{P_{j,0} - P_{i,0}}{\eta} > \alpha_j.$$

For a given α_j, and γ large enough, the profit of builder i thus satisfies

$$\pi_i(\alpha_i;\alpha_j) = \begin{cases} \pi_{i,0} - \tilde{p}_j^*(\alpha_i,\alpha_j)\mu_{i,j,0}, & \alpha_i \leq \bar{\alpha}_{i,j}(\alpha_j), \\ \pi_{i,0} + \tilde{p}_i^*(\alpha_i,\alpha_j)(\mu_{i,0}-\alpha_i), & \alpha_i > \bar{\alpha}_{i,j}(\alpha_j), \end{cases}$$

where $\pi_{i,0} := \pi_i|_{\tilde{p}=0}$, and we used that $\mu_{i,j} = \mu_{i,j,0}$ for $\tilde{p}_{-j} = 0$, and $\mu_i = \mu_{i,0}$ for $\tilde{p}_{-i} = 0$. It follows that the profit change from choosing $\alpha_i > \bar{\alpha}_{i,j}(\alpha_j)$, rather than $\alpha_i \leq \bar{\alpha}_{i,j}(\alpha_j)$, is given by

$$\Delta_{i,j}(\alpha_i;\alpha_j) := \tilde{p}_i^*(\alpha_i,\alpha_j)(\mu_{i,0}-\alpha_i) + \tilde{p}_j^*(0,\alpha_j)\mu_{i,j,0}, \quad \alpha_i > \bar{\alpha}_{i,j}(\alpha_j).$$

For $\alpha_i = \bar{\alpha}_{i,j}(\alpha_j)$, we have

$$\Delta_{i,j}(\bar{\alpha}_{i,j}(\alpha_j);\alpha_j) = \frac{P_{i,0}+\eta\bar{\alpha}_{i,j}(\alpha_j)}{\gamma^\beta+\gamma-2P'_{i,0}}\big(\mu_{i,0}-\bar{\alpha}_{i,j}(\alpha_j)\big) + \frac{P_{j,0}+\eta\alpha_j}{\gamma^\beta+\gamma-2P'_{j,0}}\mu_{i,j,0},$$

and it follows that

$$\Delta_{i,j}(\bar{\alpha}_{i,j}(\alpha_j);\alpha_j) = 0 \quad \Longleftrightarrow \quad \bar{\alpha}_{i,j}(\alpha_j) = \mu_{i,0} + \frac{\gamma^\beta+\gamma-2P'_{i,0}}{\gamma^\beta+\gamma-2P'_{j,0}}\mu_{i,j,0}.$$

From the definition of $\bar{\alpha}_{i,j}(\alpha_j)$, it follows that the above identity is satisfied if $\alpha_j = \tilde{\alpha}_{j,i}$, and $\tilde{\alpha}_{j,i} > 0$, where

$$\tilde{\alpha}_{j,i} := \Big(\mu_{i,0} + \frac{\gamma^\beta+\gamma-2P'_{i,0}}{\gamma^\beta+\gamma-2P'_{j,0}}\mu_{i,j,0} - \frac{P_{j,0}-P_{i,0}}{\eta}\Big)^+.$$

Moreover, $\Delta_{i,j}(\alpha_i;\tilde{\alpha}_{j,i}) < 0$ for $\alpha_i > \bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i})$. To see that, first note that $\Delta_{i,j}(\alpha_i;\tilde{\alpha}_{j,i})$ can be shown to be decreasing in α_i for $\alpha_i > \frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+$. Then, observe that

$$\bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i}) = \mu_{i,0} + \frac{\gamma^\beta+\gamma-2P'_{i,0}}{\gamma^\beta+\gamma-2P'_{j,0}}\mu_{i,j,0} \geq \frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+.$$

Hence, because $\Delta_{i,j}(\bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i});\tilde{\alpha}_{j,i}) = 0$, we have $\Delta_{i,j}(\alpha_i;\tilde{\alpha}_{j,i}) < 0$ for $\alpha_i > \bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i})$. If $\tilde{\alpha}_{j,i} \leq 0$, then $\Delta_{i,j}(\bar{\alpha}_{i,j}(\alpha_j);\alpha_j) \leq 0$ for $\alpha_j = 0$. Furthermore, for $\alpha_j = \tilde{\alpha}_{j,i} = 0$,

$$\bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i}) = \frac{P_{j,0}-P_{i,0}}{\eta} > \mu_{i,0} + \frac{\gamma^\beta+\gamma-2P'_{i,0}}{\gamma^\beta+\gamma-2P'_{j,0}}\mu_{i,j,0} \geq \frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+,$$

and thus again $\Delta_{i,j}(\alpha_i;\tilde{\alpha}_{j,i}) < 0$ for $\alpha_i \geq \bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i})$.

We have shown that if $\alpha_j = \tilde{\alpha}_{j,i}$, then it is suboptimal for builder i to raise its fee sufficiently to capture the order flow. It is also suboptimal for builder j to lower its fee from $\alpha_j = \tilde{\alpha}_{j,i} > 0$, if $\alpha_i = \bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i})$. Doing so would result in builder i capturing the order flow, and a change in the profit of builder j equal to

$$\begin{aligned}
-\tilde{p}_j^*(\alpha)(\mu_{j,0} - \tilde{\alpha}_{j,i}) - \tilde{p}_i^*(\alpha)\mu_{j,i,0} &< -\tilde{p}_i^*(\alpha)(\mu_{i,0} - \bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i})) - \tilde{p}_j^*(\alpha)\mu_{i,j,0} + o(\gamma^{-\beta}) \\
&= -\Delta_i(\bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i}); \tilde{\alpha}_{j,i}) + o(\gamma^{-\beta}) \\
&= o(\gamma^{-\beta}),
\end{aligned}$$

where $\alpha = (\bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i}), \tilde{\alpha}_{j,i})$, and the inequality follows from $\bar{\alpha}_{i,j}(\tilde{\alpha}_{j,i}) > \tilde{\alpha}_{j,i}$ and Lemma D.2. In the above analysis, we assumed $P_{i,0} < P_{j,0}$. One can similarly show that if $P_{i,0} > P_{j,0}$, then builder j observes a profit increase by lowering its fee and losing the order flow to builder i.

It follows from the above that if $P_{i,0} > P_{j,0}$, the only equilibrium fees for large enough γ are given by

$$\begin{aligned}
\alpha_i^* = \max\Big\{\tilde{\alpha}_{i,j}, \frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+\Big\} &= \max\Big\{\Big(\mu_{j,0} + \frac{\gamma^\beta + \gamma - 2P'_{j,0}}{\gamma^\beta + \gamma - 2P'_{i,0}}\mu_{j,i,0} - \frac{P_{i,0} - P_{j,0}}{\eta}\Big)^+, \frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+\Big\} \\
&= \max\Big\{\Big(\mu_{j,0} + \mu_{j,i,0} - \frac{P_{i,0} - P_{j,0}}{\eta}\Big)^+, \frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+\Big\} + O(\gamma^{-\beta}),
\end{aligned}$$

where $\frac{1}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+$ is the profit-maximizing fee of builder i in the absence of other builders, and

$$\alpha_j^* = \bar{\alpha}_{j,i}(\alpha_i^*) = \Big(\alpha_i^* + \frac{P_{i,0} - P_{j,0}}{\eta}\Big)^+.$$

Generalizing the above analysis to more than two builders, it follows that the only equilibrium fees satisfy

$$\begin{aligned}
\alpha_{i_0}^* &= \max\Big\{\max_{j \neq i_0} \tilde{\alpha}_{i_0,j}, \frac{1}{2}\big(\mu_{i_0,0} - \frac{P_{i_0,0}}{\eta}\big)^+\Big\} \\
&= \max\Big\{\Big(\mu_{j_0,0} + \mu_{j_0,i_0,0} - \frac{P_{i_0,0} - P_{j_0,0}}{\eta}\Big)^+, \frac{1}{2}\big(\mu_{i_0,0} - \frac{P_{i_0,0}}{\eta}\big)^+\Big\} + O(\gamma^{-\beta}),
\end{aligned}$$

where i_0 and j_0 are the builders with the largest and second largest values of $P_{i,0}$, respectively, and

$$\alpha_j^* = \bar{\alpha}_{j,i_0}(\alpha_{i_0}^*) = \Big(\alpha_{i_0}^* + \frac{P_{i_0,0} - P_{j,0}}{\eta}\Big)^+, \quad j \neq i_0.$$

The second equality in the expression for $\alpha_{i_0}^*$ follows from Lemma D.2. □

E Other Technical Proofs

E.1 Proofs of Results in Section 4

Proof of Proposition 4.1: For part (i), we combine the expression for the order flow $\tilde{p}_i^*$ in (3.8), and the expression for the fee α_i^* in Proposition 3.9, yielding

$$\tilde{p}_i^* = \frac{P_{i,0} + \frac{\eta}{2}\big(\mu_{i,0} - \frac{P_{i,0}}{\eta}\big)^+}{\gamma} + o(\gamma^{-1}),$$

which is increasing in both $P_{i,0}$ and $\mu_{i,0}$. The result then follows by Lemma D.2.

For part (ii), the change in block success probabilities satisfies $\Delta_i^* = \Delta_i|_{\tilde{p}=\tilde{p}^*}$, where

$$\Delta_i := \mathbb{P}(b_i) - \mathbb{P}(b_i)\big|_{\tilde{p}=0} = \tilde{p}_i(c_i - c_{-i}) + \sum_{j\neq i} \tilde{p}_j(c_j - c_{-j}) + O(||\tilde{p}||^2),$$

where we have defined

$$c_i := \mathbb{P}(b_i|\tilde{X}_i > 0)\big|_{\tilde{p}=0}, \quad c_{-i} := \mathbb{P}(b_i|\tilde{X}_i = 0)\big|_{\tilde{p}=0}, \quad c_j := \mathbb{P}(b_i|\tilde{X}_j > 0)\big|_{\tilde{p}=0}, \quad c_{-j} := \mathbb{P}(b_i|\tilde{X}_j = 0)\big|_{\tilde{p}=0}.$$

We may write

$$\begin{aligned}
c_i - c_{-i} &= p_{i,0} + \frac{1}{2}\sum_{k\neq i} p_{k,0} + (1 - \sum_{k=1}^{N} p_{k,0}) - p_{i,0} - (1 - \sum_{k=1}^{N} p_{k,0})\frac{1}{N} + O(||p_0||^2) \\
&= 1 - \frac{1}{N} - \frac{1}{2}p_{i,0} - \big(\frac{1}{2} - \frac{1}{N}\big)P + O(||p_0||^2),
\end{aligned}$$

where we let $P := \sum_{k=1}^{N} p_{k,0}$, and

$$c_j - c_{-j} = \frac{1}{2}p_{i,0} - \big(p_{i,0} + (1 - \sum_{k=1}^{N} p_{k,0})\frac{1}{N}\big) + O(||p_0||^2) = -\frac{1}{N} - \frac{1}{2}p_{i,0} + \frac{1}{N}P + O(||p_0||^2).$$

Combining the above yields $\Delta_i^* = \tilde{\Delta}_i^*\big(1 + O(||p_0||)\big) + O(\gamma^{-2})$, where

$$\tilde{\Delta}_i^* = \tilde{p}_i^*\big(1 - \frac{1}{N}\big) - \frac{1}{N}\sum_{j\neq i} \tilde{p}_j^*,$$

and the result then follows from the first part of the proposition. $\square$

Proof of Corollary 4.3: Using $\mathbb{P}(b_i)\big|_{\tilde{p}=\tilde{p}^*} = P_{i,0} + \Delta_i^*$, we have

$$\Delta_{HH}^* = \sum_{i=1}^{N} \left(2\Delta_i^* P_{i,0} + (\Delta_i^*)^2\right) = 2\sum_{i=1}^{N} \tilde{\Delta}_i^* P_{i,0}(1 + O(||p_0||) + O(\gamma^{-2}) =: \tilde{\Delta}_{HH}^*(1 + O(||p_0||)) + O(\gamma^{-2}),$$

where the second equality follow from $\Delta_i^* = \tilde{\Delta}_i^*(1 + O(||p_0||) + O(\gamma^{-2})$, where $\tilde{\Delta}_i^* = O(\gamma^{-1})$. Hence,

$$\frac{1}{2}\tilde{\Delta}_{HH}^* \geq P_{k-1,0}\sum_{i=1}^{k-1}\tilde{\Delta}_i^* + P_{k,0}\sum_{i=k}^{N}\tilde{\Delta}_i^* = \sum_{i=1}^{k-1}\tilde{\Delta}_i^*(P_{k-1,0} - P_{k,0}) > 0,$$

where we used that there exists $1 \leq k \leq N$ such that $\tilde{\Delta}_i^* > 0$ for $1 \leq i < k$, and $\tilde{\Delta}_i^* < 0$ for $k \leq i \leq N$. $\square$

E.2 Proof of Results in Section 5

Proof of Proposition 5.1: The solution methodology of Section 3.3 extends to the case where order flow is allocated to validators in addition to builders. In particular, from Corollary 3.6 it follows that the order flow allocated to the builder-validator satisfies

$$\tilde{p}_i^{*,(v)} = \frac{a_{i,0}^{(v)}}{\gamma} + O(\gamma^{-(2-\beta)}),$$

where $a_{i,0}^{(v)} := \eta\alpha_i^{(v)}$. Furthermore, by Proposition 3.9, the equilibrium fee of the builder-validator satisfies

$$\alpha_i^{*,(v)} = \frac{1}{2}\tilde{a}_{i,0}^{(v)} + O(\gamma^{-(1-\beta)}),$$

where $\tilde{a}_{i,0}^{(v)} := \mu_{i,0}^{(v)}$, and

$$\mu_{i,0}^{(v)} := \frac{\partial}{\partial\tilde{p}_i^{(v)}}\Big(\frac{s_i}{S}\mathbb{E}\big[\max\{b_i^{(v)}, b_{(2)}\}\big] + \big(1 - \frac{s_i}{S}\big)\mathbb{E}\big[(b_i^{(v)} - b_{(2)})^+\big]\Big)\Big|_{\tilde{p}=\tilde{p}_i^{(v)}=0}.$$

Let $\tilde{b}_{(1)}$ and $\tilde{b}_{(2)}$ be the two highest block values of all builders, excluding the builder-validator. For the first term in the expression for $\mu_{i,0}^{(v)}$ we then have

$$\begin{aligned}\mu_{i,1}^{(v)} := \frac{\partial}{\partial\tilde{p}_i^{(v)}}\mathbb{E}^{(0)}[\max\{b_i^{(v)}, b_{(2)}\}] &= \mathbb{E}^{(0)}[\max\{b_i^{(v)}, b_{(2)}\}|\tilde{X}_i^{(v)} > 0] - \mathbb{E}^{(0)}[\max\{b_i^{(v)}, b_{(2)}\}|\tilde{X}_i^{(v)} = 0] \\ &= \mathbb{E}^{(0)}[\tilde{b}_{(2)} + (b_i^{(v)} - \tilde{b}_{(2)})^+|\tilde{X}_i^{(v)} > 0] - \mathbb{E}^{(0)}[\tilde{b}_{(2)}|\tilde{X}_i^{(v)} > 0] \\ &= \mathbb{E}^{(0)}[(b_i^{(v)} - \tilde{b}_{(2)})^+|\tilde{X}_i^{(v)} > 0],\end{aligned}$$

where $\mathbb{E}^{(0)}[\cdot] := \mathbb{E}[\cdot]\big|_{\tilde{p}=\tilde{p}_i^{(v)}=0}$, and for the second term we have

$$\begin{aligned}\mu_{i,2}^{(v)} := \frac{\partial}{\partial \tilde{p}_i^{(v)}}\mathbb{E}^{(0)}[(b_i^{(v)} - b_{(2)})^+] &= \mathbb{E}^{(0)}[(b_i^{(v)} - b_{(2)})^+|\tilde{X}_i^{(v)} > 0] - \mathbb{E}^{(0)}[(b_i^{(v)} - b_{(2)})^+|\tilde{X}_i^{(v)} = 0] \\ &= \mathbb{E}^{(0)}[(b_i^{(v)} - \tilde{b}_{(1)})^+|\tilde{X}_i^{(v)} > 0].\end{aligned}$$

Hence, the fee of the builder-validator satisfies

$$\alpha_i^{*,(v)} = \frac{1}{2}\Big(\frac{s_i}{S}\mathbb{E}^{(0)}\big[(b_i^{(v)} - \tilde{b}_{(2)})^+|\tilde{X}_i^{(v)} > 0\big] + \big(1 - \frac{s_i}{S}\big)\mathbb{E}^{(0)}\big[(b_i^{(v)} - \tilde{b}_{(1)})^+|\tilde{X}_i^{(v)} > 0\big]\Big) + O(\gamma^{-(1-\beta)}).$$

For large enough γ, the fee $\alpha_i^{*,(v)}$ is increasing in the stake share s_i/S because $\mu_{i,1}^{(v)} > \mu_{i,2}^{(v)}$, and it follows that the same holds for the allocation $\tilde{p}_i^{*,(v)}$.

Next, we analyze the excess profit of the builder-validator from becoming a builder, which satisfies

$$\begin{aligned}&\frac{s_i}{S}\mathbb{E}^*[\max\{b_i^{(v)}, \tilde{b}_{(2)}\}] + \big(1 - \frac{s_i}{S}\big)\mathbb{E}^*[(b_i^{(v)} - \tilde{b}_{(1)})^+] - \alpha_i^{*,(v)}\tilde{p}_i^{*,(v)} - \frac{s_i}{S}\mathbb{E}^*[\tilde{b}_{(2)}] + o(\gamma^{-1}) \\ &= \frac{s_i}{S}\mathbb{E}^*[(b_i^{(v)} - \tilde{b}_{(2)})^+] + \big(1 - \frac{s_i}{S}\big)\mathbb{E}^*[(b_i^{(v)} - \tilde{b}_{(1)})^+] - \alpha_i^{*,(v)}\tilde{p}_i^{*,(v)} + o(\gamma^{-1}) \\ &= \tilde{p}_i^{*,(v)}\Big(\frac{s_i}{S}\mathbb{E}[(b_i^{(v)} - \tilde{b}_{(2)})^+|\tilde{X}_i^{(v)} > 0] + \big(1 - \frac{s_i}{S}\big)\mathbb{E}[(b_i^{(v)} - \tilde{b}_{(1)})^+|\tilde{X}_i^{(v)} > 0]\Big) - \alpha_i^{*,(v)}\tilde{p}_i^{*,(v)} + o(\gamma^{-1}) \\ &= \frac{1}{2}\tilde{p}_i^{*,(v)}\Big(\frac{s_i}{S}\mu_{i,1}^{(v)} + \big(1 - \frac{s_i}{S}\big)\mu_{i,2}^{(v)}\Big) + o(\gamma^{-1}),\end{aligned} \tag{E.1}$$

where $\mathbb{E}^*[\cdot] := \mathbb{E}[\cdot]\big|_{\tilde{p}=\tilde{p}^*,\tilde{p}_i^{(v)}=\tilde{p}_i^{*,(v)}}$. This quantity is increasing in the stake share s_i/S, because $\mu_{i,1}^{(v)} > \mu_{i,2}^{(v)}$, and because $\tilde{p}_i^{*,(v)}$ is increasing in s_i/S. □

Proof of Corollary 5.2: The result follows from Eq. (E.1) for the profit change of a validator from becoming a builder, and the fact that $\tilde{p}_i^{*,(v)}$, $\mu_{i,1}^{(v)}$, and $\mu_{i,2}^{(v)}$, are decreasing in both N and $(p_{i,0})_{1\le i\le N}$. □

Proof of Proposition 5.3: The total revenue of validators, $\mathbb{E}[b_{(2)}]$, is increasing in both N and p_0 because $b_{(2)}$ is first-order stochastically increasing in both N and p_0.

The total revenue of builders satisfies

$$\mathbb{E}[b_{(1)} - b_{(2)}] = \sum_{k=0}^{N}\mathbb{P}\big(Y^{(N,p_0)} = k\big)\mathbb{E}[V_{(1,k)} - V_{(2,k)}],$$

where $Y^{(N,p_0)} \sim Bin(N, p_0)$, and $V_{(1,k)}$ and $V_{(2,k)}$ are the largest and second largest values in the set $\{V_1, \ldots, V_k\}$. From the fact that $Y^{(N,p_0)}$ is first-order stochastically increasing in both N and p_0, it follows that $\mathbb{E}[b_{(1)} - b_{(2)}]$ is decreasing (increasing, constant) in N if λ_N is decreasing (increasing, constant) in N.

The revenue share of builders is decreasing in N and p_0 if λ_N is decreasing in N, because in that case $\mathbb{E}[b_{(1)} - b_{(2)}]$ is decreasing in N and p_0, and $\mathbb{E}[b_{(1)}]$ increasing in N and p_0. It then follows that the revenue share of validators is increasing in N and p_0 if λ_N is decreasing in N. $\square$

Remark E.1. For the exponential distribution, one can show that λ_N in Proposition 5.3 is constant, and for the half-normal distribution, λ_N is decreasing.[24] Beyond such special cases, general results for the behavior of λ_N for finite values of N are not currently available. However, the following asymptotic results for large values of N are available (Mudholkar et al. (2009)):

(i) λ_N is asymptotically constant for a class of distributions including the gamma distribution (and thus the exponential distribution); these distributions are said to have *medium extreme spacing*;

(ii) λ_N is asymptotically decreasing for a class of distributions including the normal distribution; these distributions are said to have *short extreme spacing*;

(iii) λ_N is asymptotically increasing for a class of distributions including the Pareto distribution; these distributions are said to have *long extreme spacing.*

E.3 Proof of Results in Section 6

Proof of Proposition 6.2: The solution methodology of Section 3.3 extends to the case where order flow is allocated to validators instead to builders. By Proposition 3.9, the equilibrium fee of validator i satisfies

$$\alpha_i^{*,(v)} = \frac{(\tilde{a}_{i,0}^{(v)})^+}{2} + O(\gamma^{-(1-\beta)}) = \frac{1}{2}\big(\mu_{i,0}^{(v)} - \frac{P_{i,0}^{(v)}}{\eta}\big)^+ + O(\gamma^{-(1-\beta)}) = \frac{1}{2}\frac{s_i}{S}\big(\mathbb{E}[\tilde{V}_i^{(v)}] - \frac{1}{\eta}\big)^+ + O(\gamma^{-(1-\beta)}),$$

where in the second equality we used

$$\mu_{i,0}^{(v)} := \frac{\partial}{\partial \tilde{p}_i^{(v)}}\Big(\frac{s_i}{S}\mathbb{E}[b_i^{(v)}]\Big)\Big|_{\tilde{p}^{(v)}=0} = \frac{s_i}{S}\mathbb{E}[\tilde{V}_i^{(v)}].$$

From Corollary 3.6 it then follows that the order flow allocated to validator i satisfies

$$\tilde{p}_i^{*,(v)} = \frac{a_{i,0}^{(v)}}{\gamma} + O(\gamma^{-(2-\beta)}) = \frac{s_i/S + \eta\alpha_i^{*,(v)}}{\gamma} + O(\gamma^{-(2-\beta)}) = \frac{1}{\gamma}\frac{s_i}{S}\Big(1 + \frac{\eta}{2}\big(\mathbb{E}[\tilde{V}_i^{(v)}] - \frac{1}{\eta}\big)^+\Big) + O(\gamma^{-(2-\beta)}).$$

The quantities $\alpha_i^{*,(v)}$ and $\tilde{p}_i^{*,(v)}$ are proportional to s_i because $\mathbb{E}[\tilde{V}_i^{(v)}]$ is independent of i, and both quantities are independent of the initial orer flow level $p_{i,0}^{(v)}$. $\square$

[24] If $X \sim \mathcal{N}(0, \sigma^2)$, for some $\sigma > 0$, then $Y = |X|$ is said to follow a half-normal distribution.

Proof of Corollary 6.3: Letting $\mu := \mathbb{E}[V_i^{(v)}] = \mathbb{E}[\tilde{V}_i^{(v)}]$, we have

$$r_i^{*,(v)} = \frac{1}{s_i}\big(\frac{s_i}{S}(p_{i,0}^{(v)} + \tilde{p}_i^{*,(v)})\mu - \tilde{p}_i^{*,(v)}\alpha_i^{*,(v)}\big) = r_{i,0}^{(v)} + \tilde{p}_i^{*,(v)}\big(\frac{\mu}{S} - \frac{\alpha_i^{*,(v)}}{s_i}\big),$$

where we used $r_{i,0}^{(v)} = p_{i,0}^{(v)}\mu/S$. From Proposition 6.2 it follows that the second term in the above expression is increasing in s_i, and thus that $\Delta r_i^{*,(v)}$ is increasing in s_i. □

E.4 Proofs of Results in Appendix A

Proof of Proposition A.1: Assume without loss of generality that builders are ordered such that $p_1 \geq \cdots \geq p_N$, and thus $e_1 \geq e_2 \geq \cdots \geq e_N$. For $K \geq 0$, the existence of an equilibrium can then be established by iteratively considering the building revenue in a contest between the first k builders. Formally, for $k = 0, 1, \ldots, N$, let $\xi^{(k)}$ be such that $\xi_i^{(k)} = 1$ for $1 \leq i \leq k$, and $\xi_i^{(k)} = 0$ for $k < i \leq N$. If $\pi_N(\xi_N^{(N)}; \xi_{-N}^{(N)}) > 0$, then $\xi^{(N)}$ is an equilibrium. Otherwise, $\xi^{(k_0)}$ is an equilibrium, where k_0 is the smallest $0 \leq k < N$ such that $\pi_{k+1}(\xi_{k+1}^{(k+1)}; \xi_{-(k+1)}^{(k+1)}) < 0$. This uses the fact that builders are ordered in decreasing order of p_i and e_i, which implies that $\pi_i(\xi_i^{(k)}; \xi_{-i}^{(k)}) \leq \pi_j(\xi_j^{(k)}; \xi_{-j}^{(k)})$ for $i < j$ and any $0 \leq k \leq N$.

The statements about the lower and upper bounds $\underline{n}^*$ and $\bar{n}^*$ are established using the same logic. □

Proof of Corollary A.2: Assume without loss of generality that $e = 1$. If there are n active builders, the expected revenue of each one can be written as

$$\mu^{(n)} := p\sum_{k=0}^{n-1}\binom{n}{k}p^k(1-p)^{n-1-k}\mu_{k+1} = p\sum_{k=0}^{n-1}\mathbb{P}(Y^{(n-1)} = k)\mu_{k+1},$$

where $Y^{(n-1)} \sim Bin(n-1, p)$, and $\mu_{k+1} := \mathbb{E}[(V_{k+1} - \max_{1\leq i\leq k} V_i)^+]$. Note that $Y^{(n)}$ stochastically dominates $Y^{(n-1)}$, and $\max_{1\leq i\leq k} V_i$ stochastically dominates $\max_{1\leq i<k} V_i$, which implies $\mu_{k+1} \leq \mu_k$. It follows that $\mu^{(n+1)} \leq \mu^{(n)}$. Hence, builders enter until $\mu^{(n)}$ is smaller than the operational cost K.

We now consider the dependence of $\mu^{(n)}$ on p. Standard calculations yield

$$\frac{\partial\mu^{(n)}}{\partial p} = \mu_1 - 2p\big((n-1)\mu_1 - n\mu_2\big) + o(p).$$

Hence, for $1 \leq n \leq N$, the expected revenue $\mu^{(n)}$ is increasing in p for small p. It follows that the number of active builders is increasing in p for small p. □