EE 559 HW 13

Q1

a) $p(\underline{x} \mid s_i) = N(\underline{x}, \underline{m}_i, \underline{\underline{\Sigma}}_i)$ $\quad i = 1, 2$

$p(s_1) = p(s_2) = 0.5$

For 2 classes,

$$P(\underline{x} \mid S_1)\, P(S_1) \underset{S_2}{\overset{S_1}{\gtrless}} P(\underline{x} \mid S_2)\, P(S_2)$$

$$g_k(\underline{x}) = -\frac{1}{2} \ln |\underline{\underline{\Sigma}}_k| - \frac{1}{2} (\underline{x} - \underline{m}_k)^T \underline{\underline{\Sigma}}_k^{-1} (\underline{x} - \underline{m}_k) + \ln P(S_k)$$

Now,

putting, $g_i(x) = g_j(x)$;

$$-\frac{1}{2} \ln |\underline{\underline{\Sigma}}_1| - \frac{1}{2} (\underline{x} - \underline{m}_1)^T \underline{\underline{\Sigma}}_1^{-1} (\underline{x} - \underline{m}_1) + \ln P(S_1)$$
$$= -\frac{1}{2} \ln |\underline{\underline{\Sigma}}_2| - \frac{1}{2} (\underline{x} - \underline{m}_2)^T \underline{\underline{\Sigma}}_2^{-1} (\underline{x} - \underline{m}_2) + \ln P(S_2)$$

$$\therefore \frac{1}{2} \ln \left| \frac{\underline{\underline{\Sigma}}_2}{\underline{\underline{\Sigma}}_1} \right| - \frac{1}{2} (\underline{x} - \underline{m}_1)^T \underline{\underline{\Sigma}}_1^{-1} (\underline{x} - \underline{m}_1) + \frac{1}{2} (\underline{x} - \underline{m}_2)^T \underline{\underline{\Sigma}}_2^{-1} (\underline{x} - \underline{m}_2) = 0$$

$$\therefore \ln \left| \frac{\underline{\underline{\Sigma}}_2}{\underline{\underline{\Sigma}}_1} \right| - \left[\underline{x}^T \underline{\underline{\Sigma}}_1^{-1} \underline{x} - \underline{x}^T \underline{\underline{\Sigma}}_1^{-1} \underline{m}_1 - \underline{m}_1^T \underline{\underline{\Sigma}}_1^{-1} \underline{x} + \underline{m}_1^T \underline{\underline{\Sigma}}_1^{-1} \underline{m}_1 \right]$$
$$+ \left[\underline{x}^T \underline{\underline{\Sigma}}_2^{-1} \underline{x} - \underline{x}^T \underline{\underline{\Sigma}}_2^{-1} \underline{m}_2 - \underline{m}_2^T \underline{\underline{\Sigma}}_2^{-1} \underline{x} + \underline{m}_2^T \underline{\underline{\Sigma}}_2^{-1} \underline{m}_2 \right] = 0$$

$$\therefore \ln\left|\frac{\underline{\underline{\Sigma}}_2}{\underline{\underline{\Sigma}}_1}\right| - \underline{x}_1^T \underline{\underline{\Sigma}}_1^{-1} \underline{x}_1 + \underline{x}_2^T \underline{\underline{\Sigma}}_2^{-1} \underline{x}_2 + 2\underline{x}^T\left[\underline{\underline{\Sigma}}_1^{-1}\underline{m}_1 - \underline{\underline{\Sigma}}_2^{-1}\underline{m}_2\right]$$

$$- \underline{m}_1^T \underline{\underline{\Sigma}}_1^{-1} \underline{m}_1 + \underline{m}_2^T \underline{\underline{\Sigma}}_2^{-1} \underline{m}_2 = 0$$

This can be equated to :

$$\underline{x}^T \underline{\underline{\Sigma}}_2^{-1} \underline{x} - \underline{x}^T \underline{\underline{\Sigma}}_1^{-1} \underline{x} + \underline{w}^T \underline{x} + w_0 = 0$$

where,

$$w_0 = \ln\left|\frac{\underline{\underline{\Sigma}}_2}{\underline{\underline{\Sigma}}_1}\right| - \underline{m}_1^T \underline{\underline{\Sigma}}_1^{-1} \underline{m}_1 + \underline{m}_2^T \underline{\underline{\Sigma}}_2^{-1} \underline{m}_2$$

$$w^T = 2\left[\underline{\underline{\Sigma}}_1^{-1}\underline{m}_1 - \underline{\underline{\Sigma}}_2^{-1}\underline{m}_2\right]^T$$

$$\Rightarrow \quad \underline{X} + \underline{w}^T \underline{x} + w_0 = 0$$

So, $\underline{X} = \underline{x}^T \underline{\underline{\Sigma}}_2^{-1} \underline{x} - \underline{x}^T \underline{\underline{\Sigma}}_1^{-1} \underline{x}$

Hence, decision rule :

$$\boxed{\underline{X} + \underline{w}^T \underline{x} \underset{S_2}{\overset{S_1}{\gtrless}} -w_0}$$

b) $\underline{m}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\underline{m}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\underline{\underline{\Sigma}}_1 = \underline{\underline{\Sigma}}_2 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}$$

To determine decision rule,

$$\underline{X} + \underline{w}^T \underline{x} + w_0 \underset{S_2}{\overset{S_1}{\gtrless}} 0$$

So,

$$\underline{X} = \underline{x}^T \underline{\underline{\Sigma}}_2^{-1} \underline{x} - \underline{x}^T \underline{\underline{\Sigma}}_1^{-1} \underline{x}$$

$$= [x_1 \quad x_2] \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [x_1 \quad x_2] \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 0$$

$$\underline{w}^T \underline{x} = 2 \left[\underline{\underline{\Sigma}}_1^{-1} \underline{m}_1 - \underline{\underline{\Sigma}}_2^{-1} \underline{m}_2 \right]^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2 \left[\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2 [-0.5 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -x_1 + 2x_2$$

$$w_0 = \ln \left| \frac{\underline{\underline{\Sigma}}_2}{\underline{\underline{\Sigma}}_1} \right| - \underline{m}_1^T \underline{\underline{\Sigma}}_1^{-1} \underline{m}_1 + \underline{m}_2 \underline{\underline{\Sigma}}_2^{-1} \underline{m}_2$$

$$= -[1 \quad 1] \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1 \quad -1] \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 1$$

So,

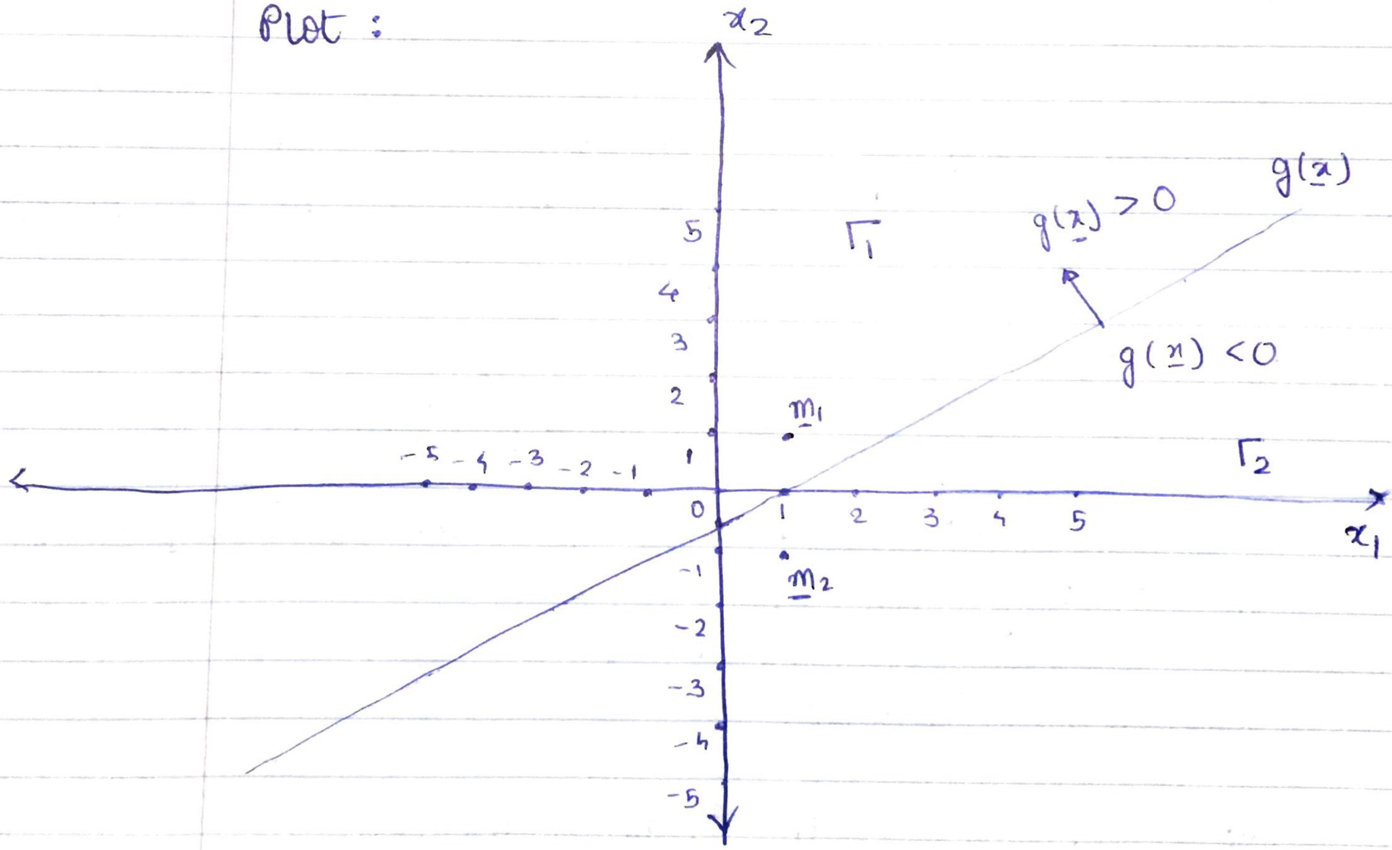
$$\underline{X} + \underline{w}^T \underline{x} + w_0 \underset{S_2}{\overset{S_1}{\gtrless}} 0$$

$$\therefore \; -x_1 + 2x_2 + 1 \underset{S_2}{\overset{S_1}{\gtrless}} 0$$

Decision Boundary : $g(x) = 0$

$\Rightarrow -x_1 + 2x_2 + 1 = 0$

Plot :



Q2.

Decision rule:

$$P(s_1) \prod_{j=1}^{2} p(x_j | s_1) \underset{s_2}{\overset{s_1}{\gtrless}} P(s_2) \prod_{j=1}^{2} P(x_j | s_2)$$

This is because,

$$P(\underline{x} | s_i) = \prod_{j=1}^{D} p(x_j | s_i) \quad \forall i = 1, \cdots, c$$

So,

$$\because p(x_j | s_k) = \frac{1}{\sqrt{2\pi}\, \sigma_j(k)} \exp\left[\frac{-\frac{1}{2}(x_j - m_j(k))^2}{\sigma_j(k)^2} \right]$$

$$p(s_1) = P(s_2)$$

$$\sigma_1^{(1)^2} = \sigma_1^{(2)^2} = 1$$

$$\sigma_2^{(1)^2} = \sigma_2^{(2)^2} = 2.25$$

$\therefore$ Decision Rule:

$$\prod_{j=1}^{2} \frac{1}{\sqrt{2\pi}\, \sigma_j^{(1)}} \exp\left[-\frac{1}{2} \frac{(x_j - m_j^{(1)})^2}{\sigma_j^{(1)2}} \right]$$

$$\gtrless \prod_{j=1}^{2} \frac{1}{\sqrt{2\pi}\, \sigma_j^{(2)}} \exp\left[\frac{-1}{2} \frac{(x_j - m_j^{(2)})^2}{\sigma_j^{(2)2}} \right]$$

PTO

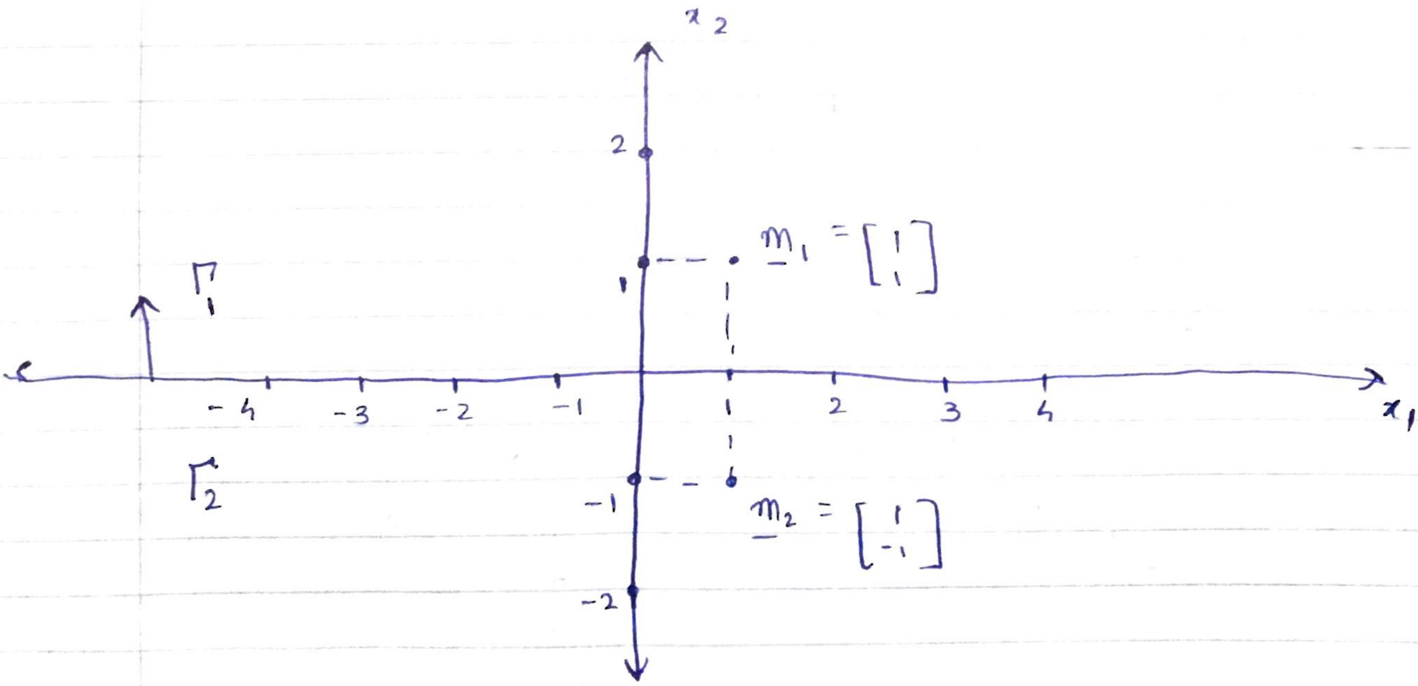
$$\therefore \prod_{j=1}^{2} \exp\left[\frac{-1}{2}\frac{(x_j - m_j^{(1)})^2}{\sigma_j^{(1)2}}\right] \gtrless \prod_{j=1}^{2} \exp\left[\frac{-1}{2}\frac{(x_j - m_j^{(2)})^2}{\sigma_j^{(2)2}}\right]$$

$$\therefore \boxed{\sum_{j=1}^{2}\left[\frac{-1}{2}\frac{(x_j - m_j^{(1)})^2}{\sigma_j^{(1)2}}\right] \underset{S_2}{\overset{S_1}{\gtrless}} \sum_{j=1}^{2}\left[\frac{-1}{2}\frac{(x_j - m_j^{(2)})^2}{\sigma_j^{(2)2}}\right]}$$

$$\therefore \frac{(x_1-1)^2}{1} + \frac{(x_2-1)^2}{2.25} \underset{S_2}{\overset{S_1}{\gtrless}} \frac{(x_1-1)^2}{1} + \frac{(x_2+1)^2}{2.25}$$

$$\therefore (x_2-1)^2 \underset{S_2}{\overset{S_1}{\gtrless}} (x_2+1)^2$$

$$\therefore x_2 \underset{S_2}{\overset{S_1}{\gtrless}} 0$$



Q3

a) $\Delta(\underline{x})$

N_K training sample $\underline{x}_i^{(k)}$ of class S_K
$k = 1, 2, \cdots c$

$$P_n(\underline{x}) = \frac{1}{n} \sum_{i=1}^{n} \Delta(\underline{x} - \underline{x}_i)$$

To find decision rule on Bayes Minimum Error criterion,

$$P_n(\underline{x} \mid S_K) = \frac{1}{N_K} \sum_{K=1}^{N_K} \Delta(\underline{x} - \underline{x}_K)$$

Let $\hat{p}(S_K) = \frac{N_K}{N}$

To get decision rule,

$$\hat{p}(\underline{x} \mid S_i)\,\hat{p}(S_i) > \hat{p}(\underline{x} \mid S_j)\,\hat{p}(S_j)$$

$$\therefore \frac{1}{N_i} \sum_{i=1}^{N_i} \Delta(\underline{x} - \underline{x}_i) \frac{N_i}{N} > \frac{1}{N_j} \sum_{j=1}^{N_j} \Delta(\underline{x} - \underline{x}_j) \frac{N_j}{N}$$

$$\Rightarrow \frac{1}{N} \sum_{i=1}^{N_i} \Delta(\underline{x} - \underline{x}_i) > \frac{1}{N} \sum_{j=1}^{N_j} \Delta(\underline{x} - \underline{x}_j)$$

$$\therefore \sum_{i=1}^{N_K} \Delta(\underline{x} - \underline{x}_i^{(k)}) > \sum_{i=1}^{N_j} \Delta(\underline{x} - \underline{x}_i^{(j)})$$

$$\therefore \boxed{\sum_{i=1}^{N_K} \Delta(\underline{x} - \underline{x}_i^{(k)}) > \sum_{i=1}^{N_j} \Delta(\underline{x} - \underline{x}_i^{(j)}) \quad j \neq k}$$

b) $C = 2$

$\Lambda_{kj} \qquad k = 1, 2 \qquad j = 1, 2$

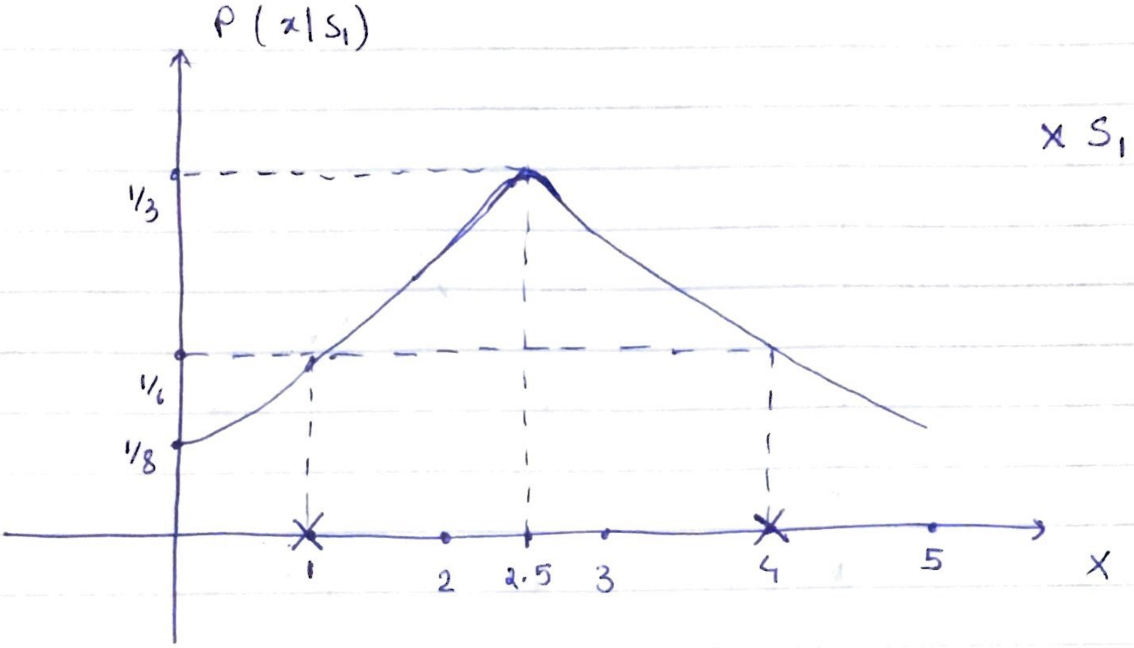
decision rule:

$$(\Lambda_{21} - \Lambda_{11})\, P(\underline{x} \mid S_1)\, P(S_1) > (\Lambda_{12} - \Lambda_{22})\, P(\underline{x} \mid S_2)\, P(S_2)$$

$$\therefore\ (\Lambda_{21} - \Lambda_{11}) \sum_{i=1}^{N_1} \Delta(\underline{x} - x_i^{(1)}) \underset{S_2}{\overset{S_1}{\gtrless}} (\Lambda_{12} - \Lambda_{22}) \sum_{i=1}^{N_2} \Delta(\underline{x} - x_i^{(2)})$$

Q4

a)

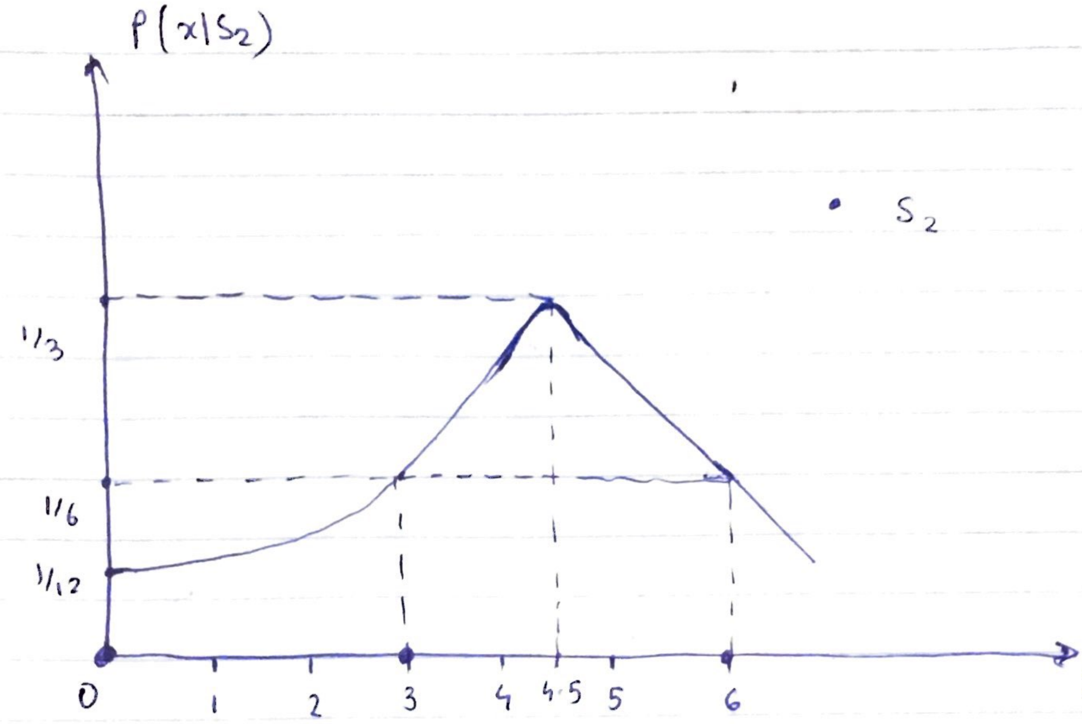


$$P(x|S_1) = \frac{k/n}{V}$$

$$= \frac{2/2}{V} = \frac{1}{V}$$

$$= \begin{cases} \dfrac{1}{2|x-4|} & , \ x < 2.5 \\[2ex] \dfrac{1}{2|x-1|} & , \ x \geq 2.6 \end{cases}$$

P.T.O



$$p(x|S_2) = \frac{k/n}{V} = \frac{2/2}{V} = \frac{1}{V}$$

$$= \begin{cases} \dfrac{1}{2|x-6|}, & x < 4.5 \\[2ex] \dfrac{1}{2|x-3|}, & x \geq 4.5 \end{cases}$$

b) $\hat{P}(S_1) = \frac{2}{4} = \frac{1}{2}$

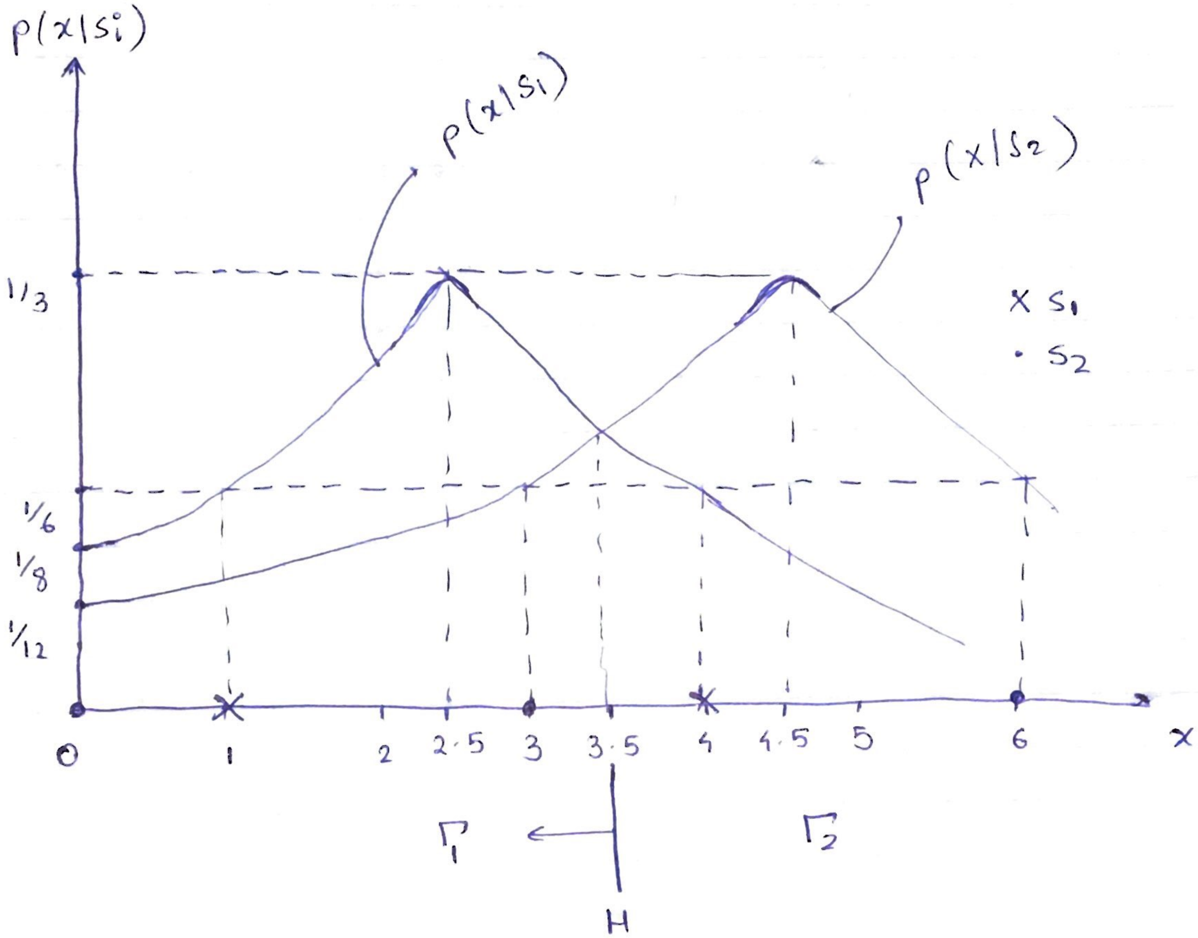
$\hat{P}(S_2) = \frac{2}{4} = \frac{1}{2}$

c) Bayes Minimum Error Classifier

$$p(x|S_1) \cdot P(S_1) \underset{S_2}{\overset{S_1}{\gtrless}} p(x|S_2) \cdot P(S_2)$$

⇒ $p(x|s_1) \underset{s_2}{\overset{s_1}{\gtrless}} p(x|s_2)$ ⇒ decision boundary

$x = 3.5$

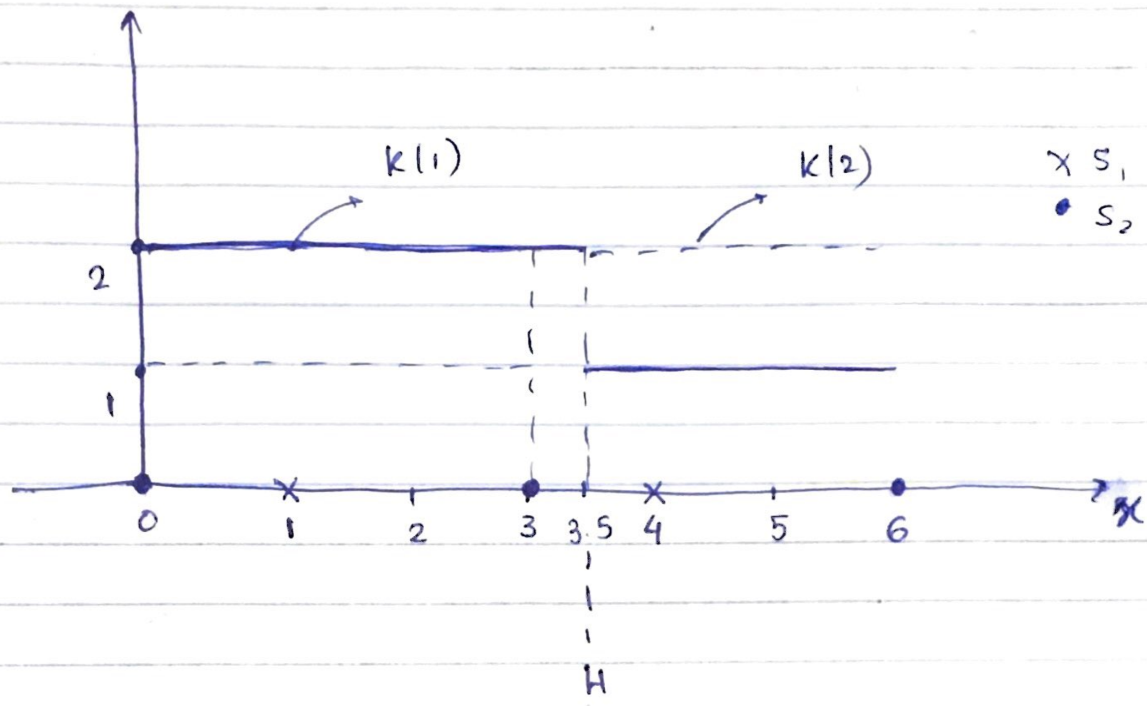


d) <u>Discriminative KNN classifier</u>

Decision boundary ⇒ $x = 3.5$

If $x = 3.25$, then $x \in S_1$

If $x = 3.75$, then $x \in S_2$



K(1)
K(2)
× S1
• S2
2
1
0
1
2
3
3.5
4
5
6
x
H