HW IV

EE 588: Optimization for the information and data sciences

University of Southern California

Assigned on: October 10, 2018 Due date: beginning of class on October 24, 2018

The following exercises from the text: 6.2, 6.6, 6.9, 7.8 and the following additional exercise.

- *Total variation image interpolation.* A grayscale image is represented as an $m \times n$ matrix of intensities U^{orig}. You are given the values U^{orig}_{ij}, for $(i,j) \in \mathcal{K}$, where $\mathcal{K} \subset \{1,\ldots,m\} \times \{1,\ldots,n\}$. Your job is to *interpolate* the image, by guessing the missing values. The reconstructed image will be represented by $U \in^{m\times n}$, where U satisfies the interpolation conditions $U_{ij} = U^{\text{orig}}_{ij}$ for $(i,j) \in \mathcal{K}$.

  The reconstruction is found by minimizing a roughness measure subject to the interpolation conditions. One common roughness measure is the ℓ_2 variation (squared),

  $$\sum_{i=2}^{m}\sum_{j=2}^{n}\left((U_{ij} - U_{i-1,j})^2 + (U_{ij} - U_{i,j-1})^2\right).$$

  Another method minimizes instead the *total variation*,

  $$\sum_{i=2}^{m}\sum_{j=2}^{n}\left(|U_{ij} - U_{i-1,j}| + |U_{ij} - U_{i,j-1}|\right).$$

  Evidently both methods lead to convex optimization problems.

  Carry out ℓ_2 and total variation interpolation on the problem instance with data given in `tv_img_interp.m`. This will define `m`, `n`, and matrices `Uorig` and `Known`. The matrix `Known` is $m \times n$, with (i,j) entry one if $(i,j) \in \mathcal{K}$, and zero otherwise. The mfile also has skeleton plotting code. (We give you the entire original image so you can compare your reconstruction to the original; obviously your solution cannot access U^{orig}_{ij} for $(i,j) \notin \mathcal{K}$.)

- *Piecewise-linear fitting.* In many applications some function in the model is not given by a formula, but instead as tabulated data. The tabulated data could come from empirical measurements, historical data, numerically evaluating some complex expression or solving some problem, for a set of values of the argument. For use in a convex optimization model, we then have to fit these data with a convex function that is compatible with the solver or other system that we use. In this problem we explore a very simple problem of this general type.

  Suppose we are given the data (x_i, y_i), $i = 1,\ldots,m$, with $x_i,\ y_i \in \mathbb{R}$. We will assume that x_i are sorted, i.e., $x_1 < x_2 < \cdots < x_m$. Let $a_0 < a_1 < a_2 < \cdots < a_K$ be a set of fixed knot points, with $a_0 \leq x_1$ and $a_K \geq x_m$. Explain how to find the convex piecewise linear function f, defined over $[a_0, a_K]$, with knot points a_i, that minimizes the least-squares fitting criterion

$$\sum_{i=1}^{m} (f(x_i) - y_i)^2.$$

You must explain what the variables are and how they parametrize f, and how you ensure convexity of f.

Hints. One method to solve this problem is based on the Lagrange basis, $f_0, \ldots, f_K$, which are the piecewise linear functions that satisfy

$$f_j(a_i) = \delta_{ij}, \quad i, j = 0, \ldots, K.$$

Another method is based on defining $f(x) = \alpha_i x + \beta_i$, for $x \in (a_{i-1}, a_i]$. You then have to add conditions on the parameters α_i and β_i to ensure that f is continuous and convex.

Apply your method to the data in the file `pwl_fit_data.m`, which contains data with $x_j \in [0, 1]$. Find the best affine fit (which corresponds to $a = (0, 1)$), and the best piecewise-linear convex function fit for 1, 2, and 3 internal knot points, evenly spaced in $[0, 1]$. (For example, for 3 internal knot points we have $a_0 = 0,\ a_1 = 0.25,\ a_2 = 0.50,\ a_3 = 0.75,\ a_4 = 1$.) Give the least-squares fitting cost for each one. Plot the data and the piecewise-linear fits found. Express each function in the form

$$f(x) = \max_{i=1\ldots,K} (\alpha_i x + \beta_i).$$

(In this form the function is easily incorporated into an optimization problem.)