Lecture 2: Convex sets

August 28, 2008

Outline

- Review basic topology in $\mathbb{R}^n$
- Open Set and Interior
- Closed Set and Closure
- Dual Cone
- Convex set
- Cones
- Affine sets
- Half-Spaces, Hyperplanes, Polyhedra
- Ellipsoids and Norm Cones
- Convex, Conical, and Affine Hulls
- Simplex
- Verifying Convexity

Topology Review

Let $\{x_k\}$ be a sequence of vectors in $\mathbb{R}^n$

Def. The sequence $\{x_k\} \subseteq \mathbb{R}^n$ *converges* to a vector $\widehat{x} \in \mathbb{R}^n$ when

$$\|x_k - \widehat{x}\| \text{ tends to 0 as } k \to \infty$$

- Notation: When $\{x_k\}$ converges to a vector $\widehat{x}$, we write $x_k \to \widehat{x}$
- The sequence $\{x_k\}$ converges $\widehat{x} \in \mathbb{R}^n$ if and only if for each component i: the i-th components of x_k converge to the i-th component of $\widehat{x}$

$$|x_k^i - \widehat{x}^i| \text{ tends to 0 as } k \to \infty$$

Open Set and Interior

Let $X \subseteq \mathbb{R}^n$ be a nonempty set

Def. The set X is *open* if for every $x \in X$ there is an open ball $B(x, r)$ that entirely lies in the set X, i.e.,

for each $x \in X$ there is $r > 0$ s.th. for all z with $\|z - x\| < r$, we have $z \in X$

Def. A vector x_0 is an *interior point* of the set X, if there is a ball $B(x_0, r)$ contained entirely in the set X

Def. The *interior* of the set X is the set of all interior points of X, denoted by $\int(X)$

- How is $\int(X)$ related to X?
- Example $X = \{x \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 > 0\}$

$$\int(X) = \{x \in \mathbb{R}^2 \mid x_1 > 0,\ x_2 > 0\}$$

$\int(S)$ of a probability simplex $S = \{x \in \mathbb{R}^n \mid x \succeq 0,\ e'x = 1\}$

Th. For a convex set X, the *interior* $\int(X)$ *is also convex*

Closed Set

Def. The *complement* of a given set $X \subseteq \mathbb{R}^n$ is the set of all vectors that do not belong to X:

$$\text{the complement of } X = \{x \in \mathbb{R}^n \mid x \notin X\} = \mathbb{R}^n \setminus X$$

Def. The set X is *closed* if its complement $\mathbb{R}^n \setminus X$ is open

- Examples: $\mathbb{R}^n$ and $\emptyset$ (both are open and closed)
  $\{x \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 > 0\}$ is open or closed?
  hyperplane, half-space, simplex, polyhedral sets?
- The *intersection* of *any* family of closed set is closed
- The *union* of a *finite* family of closed set is closed
- The *sum* of two closed sets is *not necessarily closed*
  - Example: $C_1 = \{(x_1, x_2) \mid x_1 = 0,\ x_2 \in \mathbb{R}\}$
    $$C_2 = \{(x_1, x_2) \mid x_1 x_2 \geq 1,\ x_1 \geq 0\}$$
    $C_1 + C_2$ is not closed!
  - Fact: *The sum of a compact set and a closed set is closed*

Closure

Let $X \subseteq \mathbb{R}^n$ be a nonempty set

Def. A vector $\hat{x}$ is a *closure point* of a set X if there exists a sequence $\{x_k\} \subseteq X$ such that $x_k \to \hat{x}$

Closure points of $X = \{(-1)^n/n \mid n = 1, 2, \ldots\}$, $\hat{X} = \{1 - x \mid x \in X\}$?

- Notation: The set of closure points of X is denoted by $\mathrm{cl}(X)$

- What is relation between X and $\mathrm{cl}(X)$?

Th. A set is closed if and only if it contains its closure points, i.e.,

$$X \text{ is closed iff } \mathrm{cl}(X) \subset X$$

Th. *For a convex set, the closure* $\mathrm{cl}(X)$ *is convex*

Boundary

Let $X \subseteq \mathbb{R}^n$ be a nonempty set

Def. The *boundary of* the set X is the set of closure points for both the set X and its complement $\mathbb{R}^n \setminus X$, i.e.,

$$\mathrm{bd}(X) = \mathrm{cl}(X) \cap \mathrm{cl}(\mathbb{R}^n \setminus X)$$

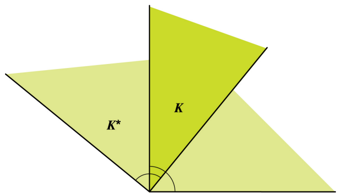
- Example $X = \{x \in \mathbb{R}^n \mid g_1(x) \leq 0, \ldots, g_m(x) \leq 0\}$. Is X closed? What constitutes the boundary of X?

Dual Cone

Let K be a nonempty cone in $\mathbb{R}^n$

Def. The *dual cone* of K is the set K^* defined by

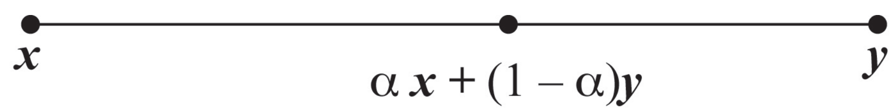
$$K^* = \{z \mid z'x \geq 0 \text{ for all } x \in K\}$$



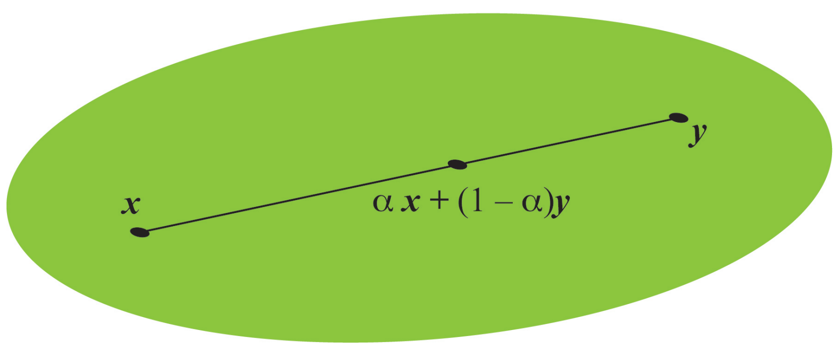
- The dual cone K^* is a *closed convex cone* even when K is neither closed nor convex
- Let S be a subspace. Then, $S^* = S^\perp$.
- Let C be a closed convex cone. Then, $(C^*)^* = C$.
- For an arbitrary cone K, we have $(K^*)^* = \mathrm{cl}(\mathrm{conv}(K))$.

Convex set

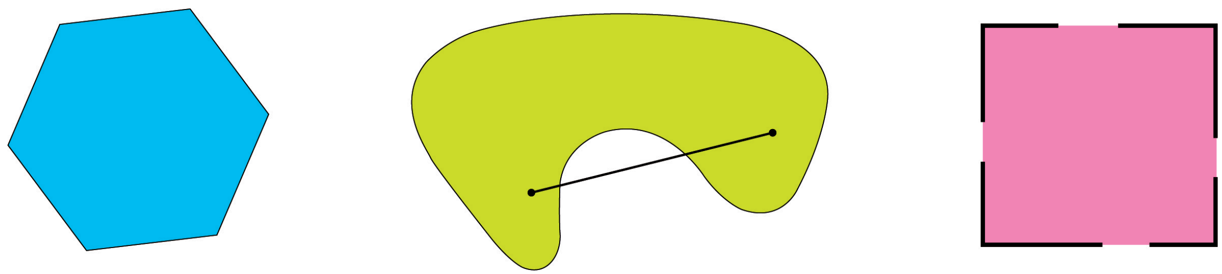
- A *line segment* defined by vectors x and y is the set of points of the form $\alpha x + (1-\alpha)y$ for $\alpha \in [0,1]$



- A set $C \subset \mathbb{R}^n$ is *convex* when, with any two vectors x and y that belong to the set C, the line segment connecting x and y also belongs to C



Examples

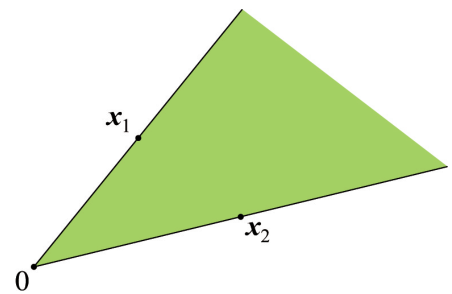


Which of the following sets are convex?

- The space $\mathbb{R}^n$
- A *line* through two given vectors x and y
$$l(x, y) = \{z \mid z = x + t(y - x), \quad t \in \mathbb{R}\}$$
- A *ray* defined by a vector x
$$\{z \mid z = \lambda x, \ \lambda \ge 0\}$$
- The *positive orthant* $\{x \in \mathbb{R}^n \mid x \succeq 0\}$ ($\succeq$ componentwise inequality)
- The set $\{x \in \mathbb{R}^2 \mid x_1 > 0, \ x_2 \ge 0\}$
- The set $\{x \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$

Cone

A set $C \subset \mathbb{R}^n$ is a *cone* when, with every vector $x \in C$, the ray $\{\lambda x \mid \lambda \geq 0\}$ belongs to the set C



- A cone may or may not be convex
- Examples: $\{x \in \mathbb{R}^n \mid x \succeq 0\}$ $\{x \in \mathbb{R}^2 \mid x_1 x_2 \geq 0\}$

For a two sets C and S, the sum $C + S$ is defined by
$C + S = \{z \mid z = x + y,\ x \in C,\ y \in S\}$ (the order does nor matter)

Convex Cone Lemma: A cone C is convex if and only if $C + C \subseteq C$
Proof: Pick any x and y in C, and any $\alpha \in [0, 1]$. Then, αx and $(1 - \alpha)y$ belong to C because... . Using $C + C \subseteq C$, it follows that ... Reverse: Let C be convex cone, and pick any $x, y \in C$. Consider $1/2(x + y)$...

Affine Set

A set $C \subset \mathbb{R}^n$ is a *affine* when, with every two distinct vectors $x, y \in C$, the line $\{x + t(y - x) \mid t \in \mathbb{R}\}$ belongs to the set C

- An affine set is always convex
- A *subspace* is an affine set

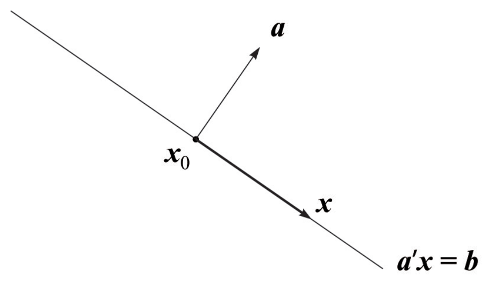
A set C is affine if and only if C is a translated subspace, i.e.,

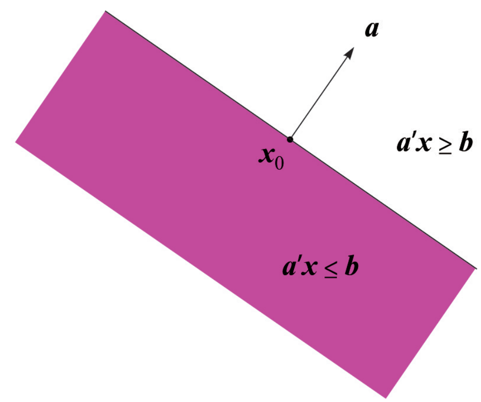
$$C = S + x_0 \quad \text{for some subspace } S \text{ and some } x_0 \in C$$

Dimension of an affine set C is the dimension of the subspace S

Hyperplanes and Half-spaces

Hyperplane is a set of the form $\{x \mid a'x = b\}$ for a nonzero vector a





Half-space is a set of the form $\{x \mid a'x \leq b\}$ with a nonzero vector a

The vector a is referred to as the *normal vector*

- A hyperplane in $\mathbb{R}^n$ divides the space into two half-spaces

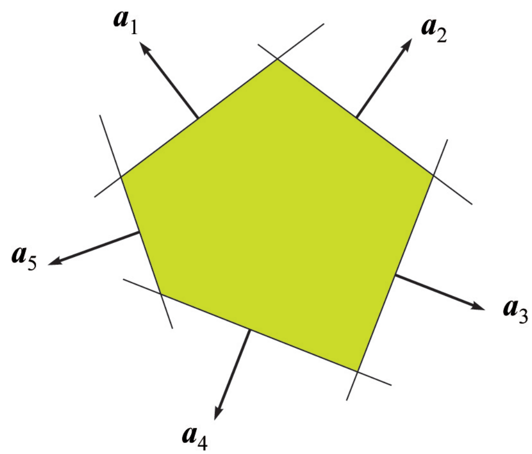
$$\{x \mid a'x \leq b\} \quad \text{and} \quad \{x \mid a'x \geq b\}$$

- Half-spaces are convex
- Hyperplanes are convex and affine

Polyhedral Sets

A *polyhedral* set is given by finitely many linear inequalities

$$C = \{x \mid Ax \preceq b\} \quad \text{where } A \text{ is an } m \times n \text{ matrix}$$



- Every polyhedral set is convex
- *Linear Problem*

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Bx \leq b, \;\; Dx = d \end{aligned}$$

The constraint set $\{x \mid Bx \leq b, \;\; Dx = d\}$ is polyhedral.

Ellipsoids

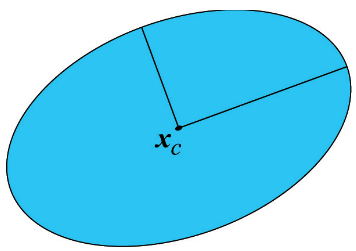
Let A be a square $(n \times n)$ matrix.

- A is *positive semidefinite* when $x'Ax \geq 0$ for all $x \in \mathbb{R}^n$
- A is *positive definite* when $x'Ax > 0$ for all $x \in \mathbb{R}^n, \; x \neq 0$

An *ellipsoid* is a set of the form

$$\mathcal{E} = \{x \mid (x - x_0)'P^{-1}(x - x_0) \leq 1\}$$

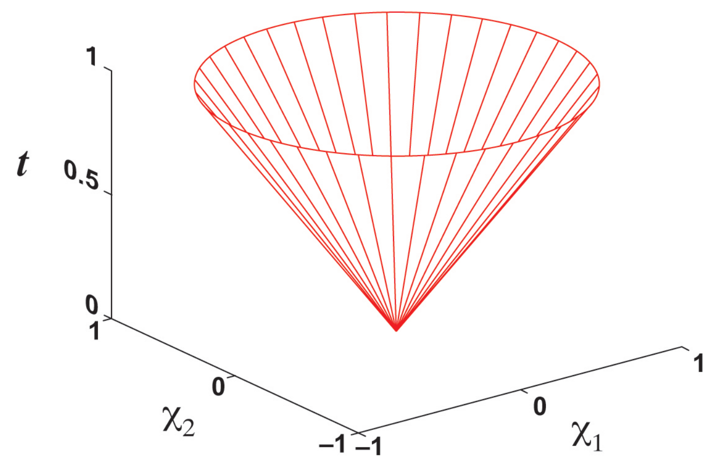
where P is symmetric and positive definite



- x_0 is the center of the ellipsoid $\mathcal{E}$
- A ball $\{x \mid \|x - x_0\| \leq r\}$ is a special case of the ellipsoid $(P = r^2 I)$
- Ellipsoids are convex

Norm Cones

A *norm cone* is the set of the form
$C = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \le t\}$



- The norm $\|\cdot\|$ can be any norm in $\mathbb{R}^n$
- The norm cone for Euclidean norm is also known as *ice-cream cone*
- Any norm cone is convex

Convex and Conical Hulls

A *convex combination* of vectors $x_1, \ldots, x_m$ is a vector of the form

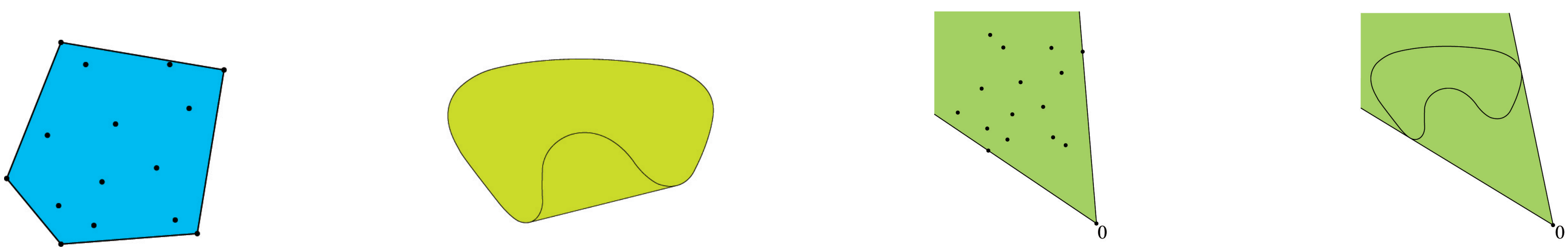
$$\alpha_1 x_1 + \ldots + \alpha_m x_m \qquad \alpha_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{m} \alpha_i = 1$$

The *convex hull* of a set X is the set of all convex combinations of the vectors in X, denoted $\mathrm{conv}(X)$

A *conical combination* of vectors $x_1, \ldots, x_m$ is a vector of the form

$$\lambda_1 x_1 + \ldots + \lambda_m x_m \qquad \text{with } \lambda_i \geq 0 \text{ for all } i$$

The *conical hull* of a set X is the set of all conical combinations of the vectors in X, denoted by $\mathrm{cone}(X)$



Affine Hull

An *affine combination* of vectors $x_1, \ldots, x_m$ is a vector of the form

$$t_1 x_1 + \ldots + t_m x_m \qquad \text{with } \sum_{i=1}^m t_i = 1,\ t_i \in \mathbb{R} \text{ for all } i$$

The *affine hull* of a set X is the set of all affine combinations of the vectors in X, denoted $\mathrm{aff}(X)$

The *dimension* of a set X is the dimension of the affine hull of X

$$\dim(X) = \dim(\mathrm{aff}(X))$$

Simplex

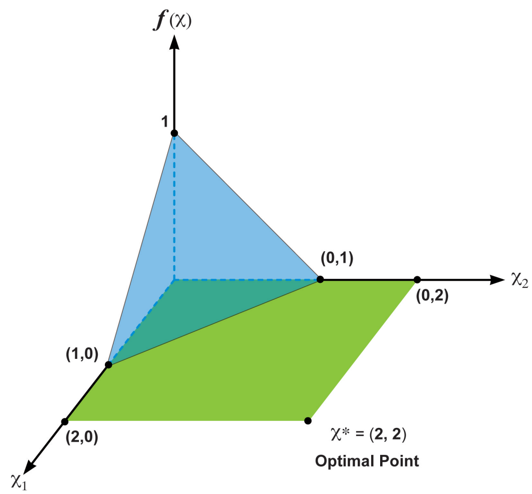
A *simplex* is a set given as a convex combination of a finite collection of vectors $v_0, v_1, \ldots, v_m$:

$$C = \text{conv}\{v_0, v_1 \ldots, v_m\}$$

The dimension of the simplex C is equal to the maximum number of linearly independent vectors among $v_1 - v_0, \ldots, v_m - v_0$.

Examples

- Unit simplex $\{x \in \mathbb{R}^n \mid x \succeq 0,\ e'x \leq 1\}, \quad e = (1, \ldots, 1)$, dim -?
- Probability simplex $\{x \in \mathbb{R}^n \mid x \succeq 0,\ e'x = 1\}$, dim -?



Practical Methods for Establishing Convexity of a Set

Establish the convexity of a given set X

- The set is one of the "recognizable" (simple) convex sets such as polyhedral, simplex, norm cone, etc
- Prove the convexity by directly applying the definition
  For every $x, y \in X$ and $\alpha \in (0, 1)$, show that $\alpha x + (1 - \alpha)y$ is also in X
- Show that the set is obtained from one of the simple convex sets through an operation that preserves convexity

Operations Preserving Convexity

Let $C \subseteq \mathbb{R}^n$, $C_1 \subseteq \mathbb{R}^n$, $C_2 \subseteq \mathbb{R}^n$, and $K \subseteq \mathbb{R}^m$ be convex sets. Then, the following sets are also convex:

- The *intersection* $C_1 \cap C_2 = \{x \mid x \in C_1 \text{ and } x \in C_2\}$
- The *sum* $C_1 + C_2$ of two convex sets
  - The *translated* set $C + a$
- The *scaled* set $tC = \{tx \mid x \in C\}$ for any $t \in \mathbb{R}$
- The *Cartesian product* $C_1 \times C_2 = \{(x_1, x_2) \mid x_1 \in C_1,\ x_2 \in C_2\}$
- The *coordinate projection* $\{x_1 \mid (x_1, x_2) \in C \text{ for some } x_2\}$
- The *image* AC under a linear transformation $A : \mathbb{R}^n \mapsto \mathbb{R}^m$:
$$AC = \{y \in \mathbb{R}^m \mid y = Ax \text{ for some } x \in C\}$$
- The *inverse image* $A^{-1}K$ under a linear transformation $A : \mathbb{R}^n \mapsto \mathbb{R}^m$:
$$A^{-1}K = \{x \in \mathbb{R}^n \mid Ax \in K\}$$