HW I Solutions

EE 588: Optimization for the Information and Data Sciences

University of Southern California

Release Date: September 17, 2018 Solutions by: Mahdi Soltanolkotabi

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2.12 Which of the following sets are convex?

(a) A *slab*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

(b) A *rectangle*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \dots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, i.e., $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

(d) The set of points closer to a given point than a given set, i.e.,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbb{R}^n$.

(e) The set of points closer to one set than another, i.e.,

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

Solution

(a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

(d) This set is convex because it can be expressed as

$$\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\},$$

i.e., an intersection of halfspaces. (For fixed y, the set

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

is a halfspace; see exercise 2.9).

(e) In general this set is not convex, as the following example in $\mathbb{R}$ shows. With $S = \{-1, 1\}$ and $T = \{0\}$, we have

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\} = \{x \in \mathbb{R} \mid x \le -1/2 \text{ or } x \ge 1/2\}$$

which clearly is not convex.

(f) This set is convex. $x + S_2 \subseteq S_1$ if $x + y \in S_1$ for all $y \in S_2$. Therefore

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y),$$

the intersection of convex sets $S_1 - y$.

(g) The set is convex, in fact a ball.

$$\begin{aligned} &\{x \mid \|x - a\|_2 \le \theta \|x - b\|_2\} \\ &\qquad = \{x \mid \|x - a\|_2^2 \le \theta^2 \|x - b\|_2^2\} \\ &\qquad = \{x \mid (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \le 0\} \end{aligned}$$

If $\theta = 1$, this is a halfspace. If $\theta < 1$, it is a ball

$$\{x \mid (x - x_0)^T (x - x_0) \le R^2\},$$

with center x_0 and radius R given by

$$x_0 = \frac{a - \theta^2 b}{1 - \theta^2}, \qquad R = \left(\frac{\theta^2 \|b\|_2^2 - \|a\|_2^2}{1 - \theta^2} + \|x_0\|_2^2 \right)^{1/2}.$$

■

2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \dots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbb{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

(a) $\alpha \le \mathbb{E} f(x) \le \beta$, where $\mathbb{E} f(x)$ is the expected value of $f(x)$, i.e., $\mathbb{E} f(x) = \sum_{i=1}^n p_i f(a_i)$. (The function $f : \mathbb{R} \to \mathbb{R}$ is given.)

(b) $\mathbf{prob}(x > \alpha) \le \beta$.

(c) $\mathbb{E}|x^3| \le \alpha \mathbb{E}|x|$.

(d) $\mathbb{E} x^2 \le \alpha$.

(e) $\mathbb{E} x^2 \ge \alpha$.

(f) $\mathbf{var}(x) \le \alpha$, where $\mathbf{var}(x) = \mathbb{E}(x - \mathbb{E}x)^2$ is the variance of x.

(g) $\mathbf{var}(x) \ge \alpha$.

Solution We first note that the constraints $p_i \geq 0$, $i = 1, \dots, n$, define halfspaces, and $\sum_{i=1}^n p_i = 1$ defines a hyperplane, so P is a polyhedron.

The first five constraints are, in fact, linear inequalities in the probabilities p_i.

(a) $\mathbb{E} f(x) = \sum_{i=1}^n p_i f(a_i)$, so the constraint is equivalent to two linear inequalities

$$\alpha \leq \sum_{i=1}^n p_i f(a_i) \leq \beta.$$

(b) $\mathbf{prob}(x \geq \alpha) = \sum_{i:\, a_i \geq \alpha} p_i$, so the constraint is equivalent to a linear inequality

$$\sum_{i:\, a_i \geq \alpha} p_i \leq \beta.$$

(c) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i(|a_i^3| - \alpha |a_i|) \leq 0.$$

(d) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i a_i^2 \leq \alpha.$$

(e) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i a_i^2 \geq \alpha.$$

The first five constraints therefore define convex sets.

(f) The constraint

$$\mathbf{var}(x) = \mathbb{E} x^2 - (\mathbb{E} x)^2 = \sum_{i=1}^n p_i a_i^2 - (\sum_{i=1}^n p_i a_i)^2 \leq \alpha$$

is not convex in general. As a counterexample, we can take $n = 2$, $a_1 = 0$, $a_2 = 1$, and $\alpha = 1/5$. $p = (1, 0)$ and $p = (0, 1)$ are two points that satisfy $(x) \leq \alpha$, but the convex combination $p = (1/2, 1/2)$ does not.

(g) This constraint is equivalent to

$$\sum_{i=1}^n p_i a_i^2 - (\sum_{i=1}^n p_i a_i)^2 = b^T p - p^T A p \geq \alpha,$$

where $b_i = a_i^2$ and $A = aa^T$. We write this as

$$p^T A p - b^T p + \alpha \leq 0.$$

This defines a convex set, since the matrix aa^T is positive semidefinite.

■

2.24 *Supporting hyperplanes.*

(a) Express the closed convex set $\{x \in \mathbb{R}^2_+ \mid x_1x_2 \geq 1\}$ as an intersection of halfspaces.

Solution The set is the intersection of all supporting halfspaces at points in its boundary, which is given by $\{x \in \mathbb{R}^2_+ \mid x_1x_2 = 1\}$. The supporting hyperplane at $x = (t, 1/t)$ is given by

$$x_1/t^2 + x_2 = 2/t,$$

so we can express the set as

$$\bigcap_{t>0} \{x \in \mathbb{R}^2 \mid x_1/t^2 + x_2 \geq 2/t\}.$$

■

(b) Let $C = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbb{R}^n$, and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

Solution $s^Tx \geq s^T\hat{x}$ for all $x \in C$ if and only if

$$\begin{array}{ll} s_i < 0 & \hat{x}_i = 1 \\ s_i > 0 & \hat{x}_i = -1 \\ s_i = 0 & -1 < \hat{x}_i < 1. \end{array}$$

■

2.28 *Positive semidefinite cone for $n = 1,\ 2,\ 3$.* Give an explicit description of the positive semidefinite cone $\mathcal{S}^n_+$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1,\ 2,\ 3$. To describe a general element of $\mathcal{S}^n$, for $n = 1,\ 2,\ 3$, use the notation

$$x_1, \qquad \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

Solution A symmetric matrix X is positive semidefinite if and only if all principal minors (determinants of symmetric submatrices) are nonnegative. For $n = 1$ the condition is just $x_1 \geq 0$. For $n = 2$ the condition is

$$x_1 \geq 0, \qquad x_3 \geq 0, \qquad x_1x_3 - x_2^2 \geq 0.$$

For $n = 3$ the condition is

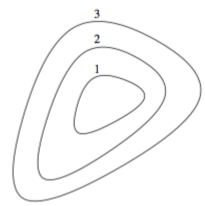
$$x_1 \geq 0, \qquad x_4 \geq 0, \qquad x_6 \geq 0, \qquad x_1x_4 - x_2^2 \geq 0, \qquad x_4x_6 - x_5^2 \geq 0, \qquad x_1x_6 - x_3^2 \geq 0$$

and

$$x_1x_4x_6 + 2x_2x_3x_5 - x_1x_5^2 - x_6x_2^2 - x_4x_3^2 \geq 0.$$

■

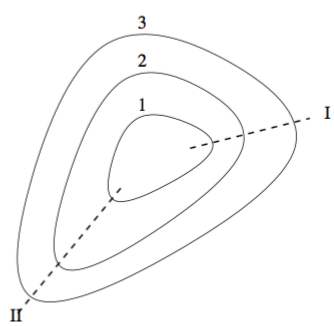
3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.



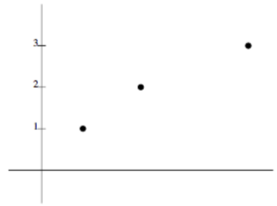
Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.

Solution The first function could be quasiconvex because the sublevel sets appear to be convex. It is definitely not concave or quasiconcave because the superlevel sets are not convex.

It is also not convex, for the following reason. We plot the function values along the dashed line labeled I.



Along this line the function passes through the points marked as black dots in the figure below. Clearly along this line segment, the function is not convex.



If we repeat the same analysis for the second function, we see that it could be concave (and therefore it could be quasiconcave). It cannot be convex or quasiconvex, because the sublevel sets are not convex. ■

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$.
Solution Strictly convex, and therefore quasiconvex. Also quasiconcave but not concave. ■

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
Solution The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is neither positive semidefinite nor negative semidefinite. Therefore, f is neither convex nor concave. It is quasiconcave, since its superlevel sets

$$\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$$

are convex. It is not quasiconvex. ■

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.
Solution The Hessian of f is

$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2/(x_1^2) & 1/(x_1 x_2) \\ 1/(x_1 x_2) & 2/x_2^2 \end{bmatrix} \succeq 0$$

Therefore, f is convex and quasiconvex. It is not quasiconcave or concave. ■

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.
Solution The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$

which is not positive or negative semidefinite. Therefore, f is not convex or concave.
It is quasiconvex and quasiconcave (i.e., quasilinear), since the sublevel and superlevel sets are halfspaces. ■

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

Solution f is convex, as mentioned before. This is easily verified by working out the Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} 2/x_2 & -2x_1/x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix} = (2/x_2) \begin{bmatrix} 1 \\ -x_1/x_2 \end{bmatrix} \begin{bmatrix} 1 & -x_1/x_2 \end{bmatrix} \succeq 0.$$

Therefore, f is convex and quasiconvex. It is not concave or quasiconcave. ■

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$.

Solution Concave and quasiconcave. The Hessian is

$$\begin{aligned}\nabla^2 f(x) = & \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & (1-\alpha)(-\alpha)x_1^\alpha x_2^{-\alpha-1} \end{bmatrix} \\ =& \alpha(1-\alpha)x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} -1/x_1^2 & 1/x_1x_2 \\ 1/x_1x_2 & -1/x_2^2 \end{bmatrix} \\ =& -\alpha(1-\alpha)x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} 1/x_1 \\ -1/x_2 \end{bmatrix} \begin{bmatrix} 1/x_1 \\ -1/x_2 \end{bmatrix}^T \\ \preceq & 0.\end{aligned}$$

f is not convex or quasiconvex. ■

- **Additional Exercise**

Dual cones in $\mathbb{R}^2$. Describe the dual cone for each of the following cones.

(a) $K = \{0\}$.

(b) $K = \mathbb{R}^2$.

(c) $K = \{(x_1, x_2) \mid |x_1| \leq x_2\}$.

(d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

Solution

(a) $K^* = \{y \mid y^T x \geq 0 \text{ for } x = 0\} = \mathbb{R}^2$.

(b) $K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in \mathbb{R}^2\}$. If $y \neq 0 \in K^*$, then $y^T(-y) = -||y||^2 \geq 0$, which shows that K^* contains no nonzero element. Therefore, $K^* = \{0\}$.

(c) $K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in \mathbb{R}^2 \text{ such that } |x_1| \leq x_2\}$. If we choose $(x_1, x_2) = (-1, 1)$ and $(x_1, x_2) = (1, 1)$, then we get $-y_2 \leq y_1 \leq y_2$, or equivalently $|y_1| \leq y_2$. Moreover, if $|y_1| \leq y_2$, then $y_2 x_2 \geq |x_1 y_1|$ for all (x_1, x_2) with $|x_1| \leq x_2$. Whence, $y^T x = y_1 x_1 + y_2 x_2 \geq 0$. Therefore, $K^* = K$.

(d) $y^T x \geq 0 \Leftrightarrow x_1(y_1 - y_2) \geq 0$ for all $x_1 \in \mathbb{R} \Leftrightarrow y_1 = y_2$. Hence, $k^* = \{(y_1, y_2) \mid y_1 = y_2\}$

■