Problem 2

I think that the most distinguishing difference between a human generated sequence and a computer-generated sequence is that:

- For computers, writing a new digit – either 1 or 0 has equal probability. That is each digit has a 0.5 chance of being 1 or 0.
- In the case of humans, it is generally not so. The probability of a digit becoming 0 or 1 is dependent on the previous digits. So if the first digit is 1, then the probability of the second digit being 1 becomes less than 0.5 and so on. This is because we tend to not repeat digits after a certain interval. Computers don't have such preconceived notions.

So, I think that some good features will be:

- Number of times 0 occurs 3 times in a row, 4 times in a row, and so on till 20 times in a row.
- Number of times 1 occurs 3 times in a row, 4 times in a row, and so on till 20 times in a row.
- Number of times 0 and 1 occur consecutively like 01.
- Number of times 0 and 1 occur consecutively like 10.

Page No.
Date

Q3 (1) $z_n(u) = y_n(u) + q_n(u)$
$= h_0 x_n(u) + h_1 x_{n-1}(u) + h_2 x_{n-2}(u) + q_n(u)$

$h_0 = 1 \qquad h_1 = 0.5 \qquad h_2 = 0.25$

x_n - sequence of iid Gaussians
$q_n(u)$ - iid, Gaussian with $\mu = 0$ and $\sigma^2 = \sigma_q^2$

$SNR = 1/\sigma_q^2$

$V_n(u) = [\, x_n(u), \; x_{n-1}(u), \; x_{n-2}(u) \,]^T$
$\hat{Z}_n = W_{Lmmse}^T V_n$
$W_{Lmmse} = R_V^{-1} V_{V2}$

(a) $E\{V_n(u)\, Z_n(u)\}$
$= E\{V_n(u) \cdot (y_n(u) + q_n(u))\}$
$= E\{V_n(u) \cdot y_n(u) + V_n(u) \cdot q_n(u)\}$

Now, for random variables X and Y, we can write,
$E(X+Y) = E(X) + E(Y)$
So here,

$E\{V_n(u) \cdot y_n(u)\} + E\{V_n(u) \cdot q_n(u)\}$

As independent standard gaussian, so,
$= E\{V_n(u) \cdot y_n(u)\} + E\{V_n(u)\} \cdot E\{q_n(u)\}$
$= E\{V_n(u) \cdot y_n(u)\} + E\{V_n(u)\} \cdot \mu$

As $\mu = 0$, so,
$= E\{V_n(u) \cdot y_n(u)\}$

So, $E\{V_n(u) \cdot Z_n(u)\} = E\{V_n(u) \cdot y_n(u)\}$

(b) $R_{xz}(m)$

$= E[x_n(u) \cdot z_{n+m}(u)]$

$= E[x_n(u) \cdot \{h_0 x_{n+m}(u) + h_1 x_{n+m-1}(u) + h_2 x_{n+m-2}(u) + q_{n+m}(u)\}]$

$= E\{h_0 x_n(u) x_{n+m}(u) + h_1 x_n(u) x_{n+m-1}(u) + h_2 x_n(u) x_{n+m-2}(u) + x_n(u) q_{n+m}(u)\}$

$= h_0 E\{x_n(u) x_{n+m}(u)\} + h_1 E\{x_n(u) x_{n+m-1}(u)\} + h_2 E\{x_n(u) x_{n+m-2}(u)\}$
$+ E\{x_n(u) q_{n+m}(u)\}$

$\because q \sim N(0, \sigma_q^2)$
and q iid so, last term is 0

Substituting $m = 0$,

$= h_0 E\{x_n^2(u)\} + h_1 E\{x_n(u) x_{n-1}(u)\} + h_2 E\{x_n(u) x_{n-2}(u)\}$

Here, we know, $x_n \longrightarrow$ iid $N(0,1)$

$= h_0$ $\quad \because$ last two terms become 0

Substituting $m = 1$,

$= h_0 E\{x_n(u) x_{n+1}(u)\} + h_1 E\{x_n^2(u)\} + h_2 E\{x_n(u) x_{n-1}(u)\}$

$= h_1$ $\quad \because$ first and last terms become 0

Similarly, substituting $m = 2$,

$= h_2$

So, $R_{xz}[m] = E[x_n(u) z_{n+m}(u)] = h_m$

$R_{xy}[m] = E[x_n(u) y_{n+m}(u)]$

$R_{xz}[m] = E[x_n(u) z_{n+m}(u)] = E[x_n(u) \cdot \{y_{n+m}(u) + q_{n+m}(u)\}]$

$= E[x_n(u) y_{n+m}(u)] + E[\underbrace{x_n(u) q_{n+m}(u)}_{0}]$

$= R_{xy}[m]$

Hence Proved.

Page No.
Date

(c) $R_{vn} = E[V_n(u) \cdot V_n^T(u)]$

$= E\left\{ \begin{bmatrix} x_n(u) \\ x_{n-1}(u) \\ x_{n-2}(u) \end{bmatrix} [x_n(u) \;\; x_{n-1}(u) \;\; x_{n-2}(u)] \right\}$

$= E\begin{bmatrix} x_n^2 & x_n x_{n-1} & x_n x_{n-2} \\ x_n x_{n-1} & x_n^2 & x_{n-1} x_{n-2} \\ x_n x_{n-2} & x_{n-1} x_{n-2} & x_n^2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\because \; E[x_n^2] = \sigma^2 = 1$

$E[x_n x_{n-1}] = 0 = E[x_n x_{n-2}]$

Hence R_{vn} is not a function of n.

(d) $r_n = E\{ v_n(u) \, y_n(u) \}$

$= E\{ v_n(u) \, z_n(u) \}$ — From (a)

$= E\left\{ \begin{bmatrix} x_n(u) \\ x_{n-1}(u) \\ x_{n-2}(u) \end{bmatrix} [h_0 x_n(u) + h_1 x_{n-1}(u) + h_2 x_{n-2}(u) + q_n(u)] \right\}$

$= E\begin{bmatrix} h_0 x_n^2 + h_1 x_n x_{n-1} + h_2 x_n x_{n-2} + x_n q_n \\ h_0 x_n x_{n-1} + h_1 x_{n-1}^2 + h_2 x_{n-1} x_{n-2} + x_{n-1} q_n \\ h_0 x_n x_{n-2} + h_1 x_{n-1} x_{n-2} + h_2 x_{n-2}^2 + x_{n-2} q_n \end{bmatrix}$

$= \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix}$

(e) $w_{lmmse} = R_v^{-1} r_{vz}$

$SNR_1 = 3 = 1/\sigma_q^2$ $\quad \therefore \sigma_q^2 = 1/3$

$R_v^{-1} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So,

$$w_{Lmmse} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix}$$

It would be same for both SNRs of 3 dB and 10 dB

(f) $LMMSE = E\{[y(u) - w^T x(u)]^2\}$

$$= \sigma_z^2 - R_{vz}^T R_v^{-1} R_{vz}$$

$$= E[(z_n)^2] - [1 \quad 0.5 \quad 0.25] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix}$$

$\boxed{\because E(z_n) = 0}$

$$= E[(z_n)^2] - [1 \quad 0.5 \quad 0.25] \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix}$$

$$= E[(z_n)^2] - 1.3125$$

Now,

$$E(z_n) = E(h_0 x_n(u) + h_1 x_{n-1}(u) + h_2 x_{n-2}(u) + q_n(u))$$
$$= h_0 E[x_n(u)] + h_1 E(x_{n-1}(u)] + h_2 E[x_{n-2}(u)] + E[q_n(u)]$$
$$= 0$$

$$E[(z_n)^2] = E[(y_n(u) + q_n(u))^2]$$
$$= E[y_n^2(u)] + E[q_n^2(u)] + 2E[y_n(u)] \underbrace{E[q_n(u)]}_{0}$$
$$= E[y_n^2(u)] + \sigma_q^2 \qquad \because E[q_n(u)] = 0$$
$$= E[(h_0 x_n + h_1 x_{n-1} + h_2 x_{n-2})^2] + \sigma_q^2$$
$$= E\left[\begin{array}{l} h_0^2 x_n^2 + h_1^2 x_{n-1}^2 + h_2^2 x_{n-2}^2 + 2h_0 h_1 x_n x_{n-1} \\ \quad + 2h_0 h_2 x_n x_{n-2} + 2h_1 h_2 x_{n-1} x_{n-2} \end{array}\right] + \sigma_q^2$$
$$= h_0^2 + h_1^2 + h_2^2 + \sigma_q^2 \qquad = 1.3125 + \sigma_q^2$$

$\because$ $E[x_n^2] = 1$

$E[x_n x_{n-1}] = 0$

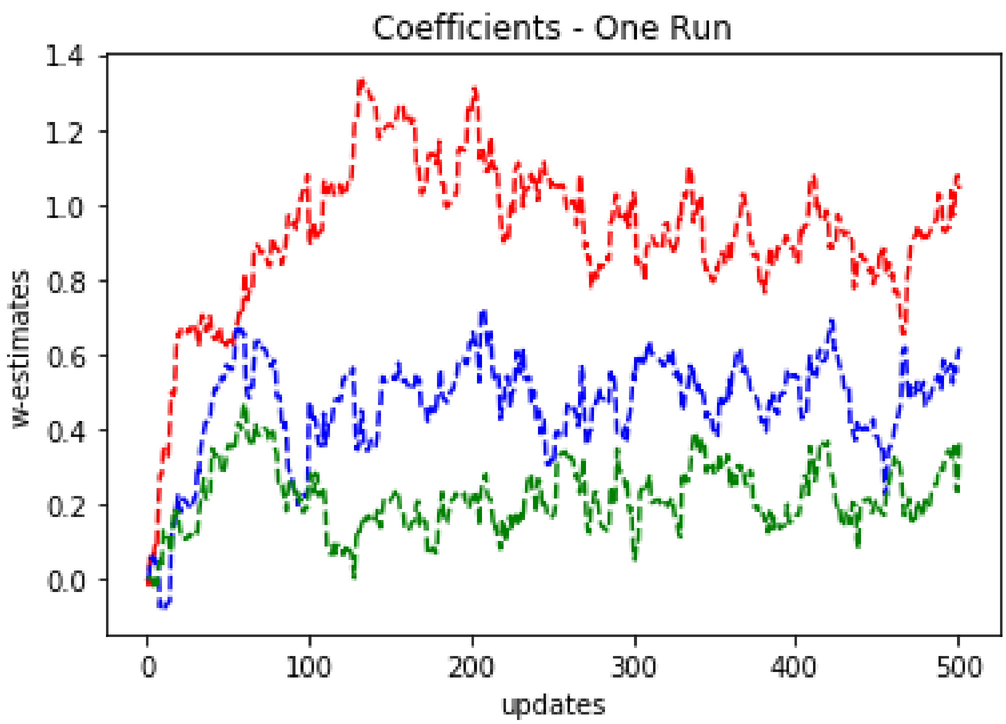
So, $LMMSE = \sigma_q^2 + 1.3125 - 1.3125$
$= \sigma_q^2$

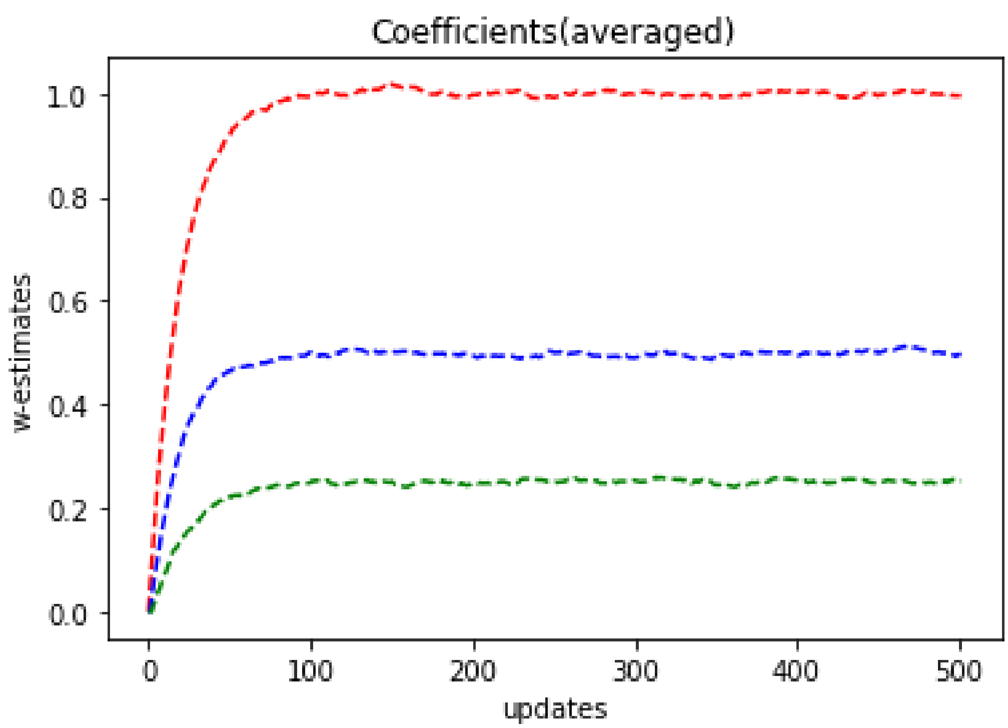
$\therefore LMMSE = \begin{cases} \frac{1}{3} & \text{for } SNR = 3\,dB \\ \frac{1}{10} & \text{for } SNR = 10\,dB \end{cases}$

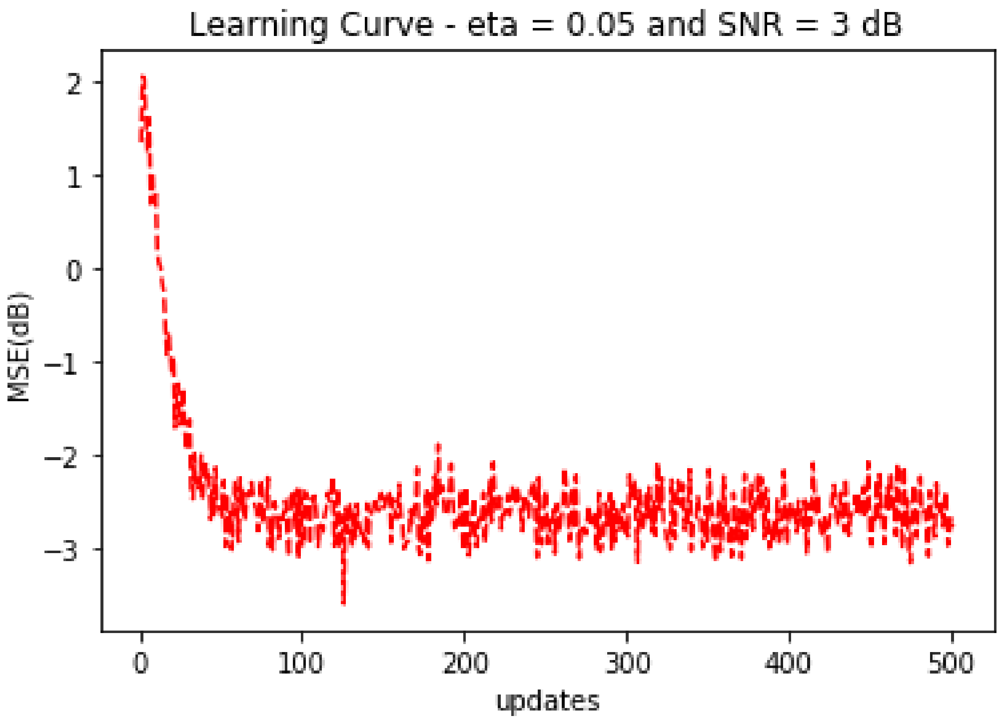
Problem 3

3.2 (b)

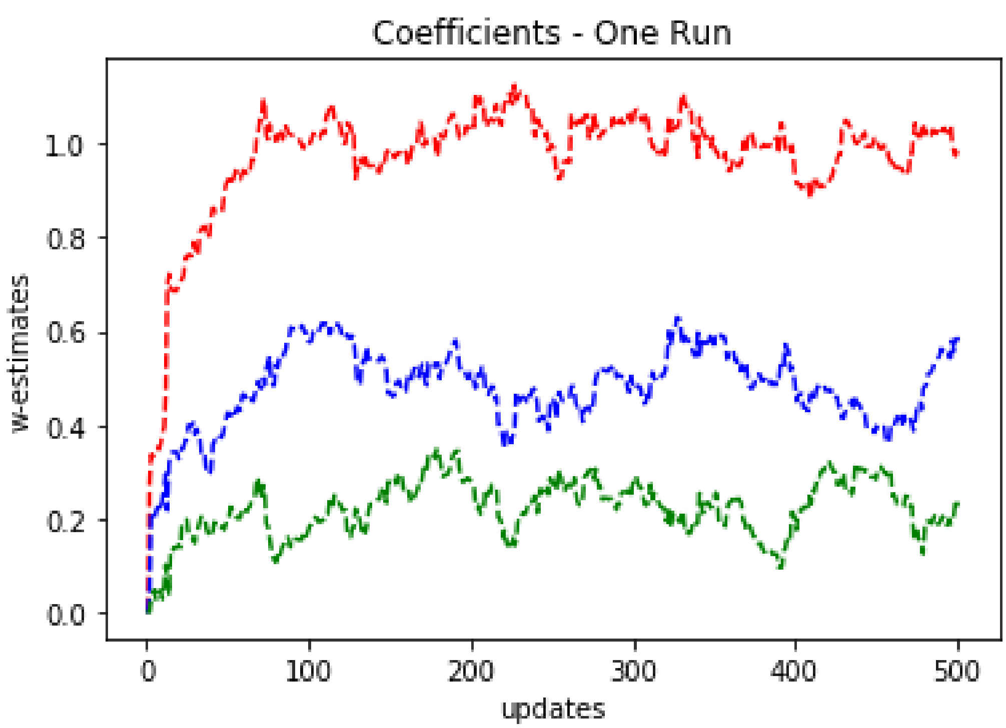
For eta = 0.05 and SNR = 3 dB,

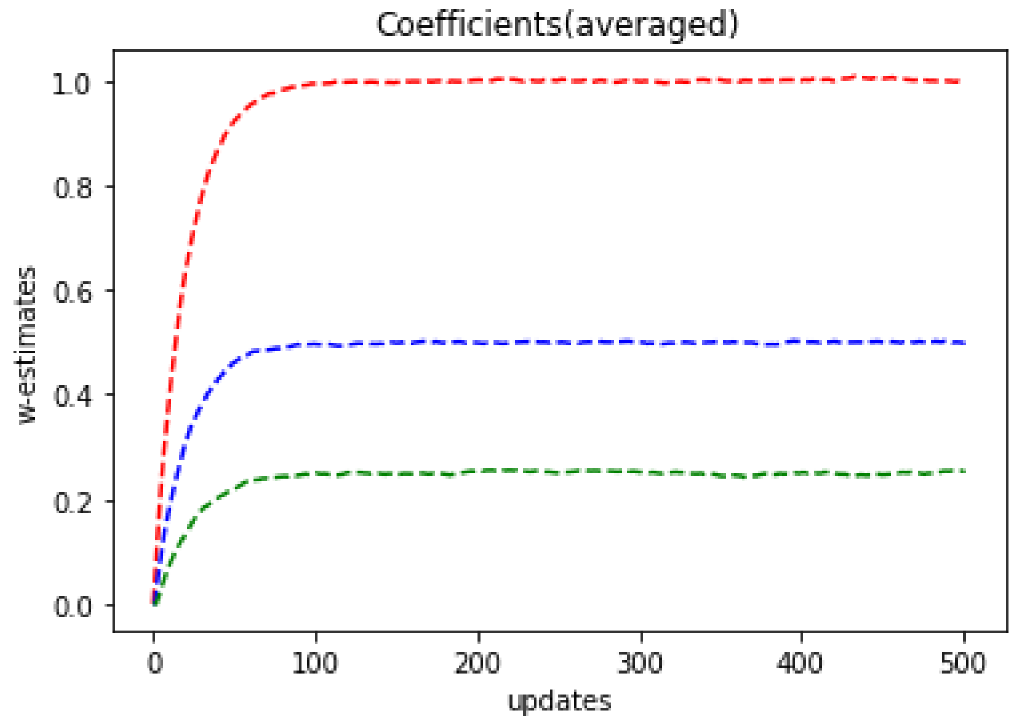


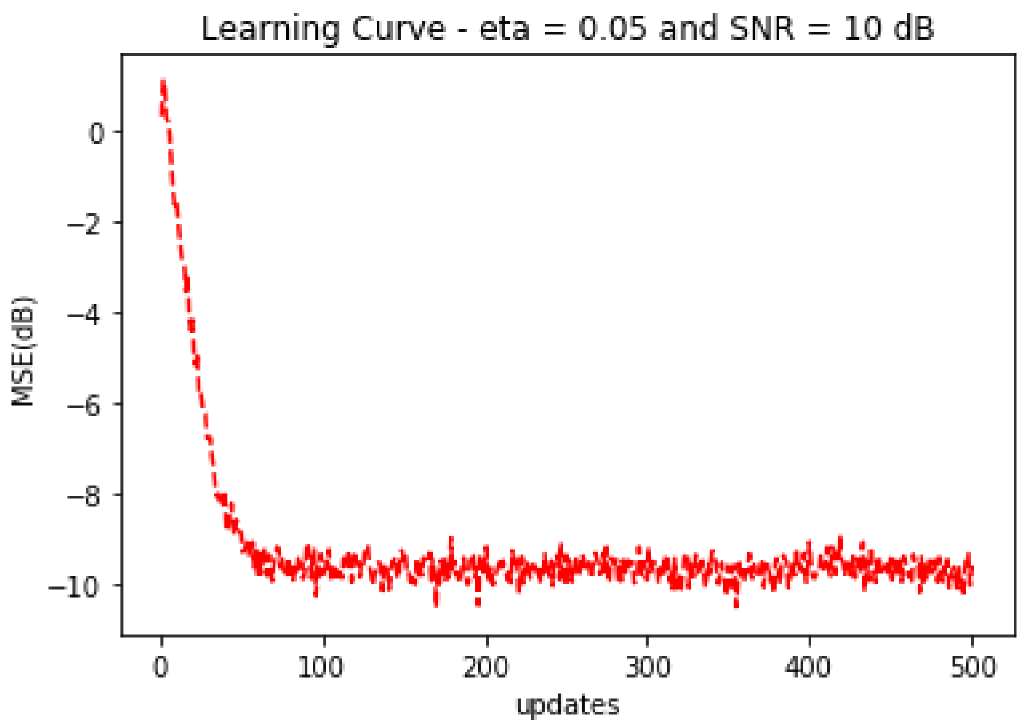




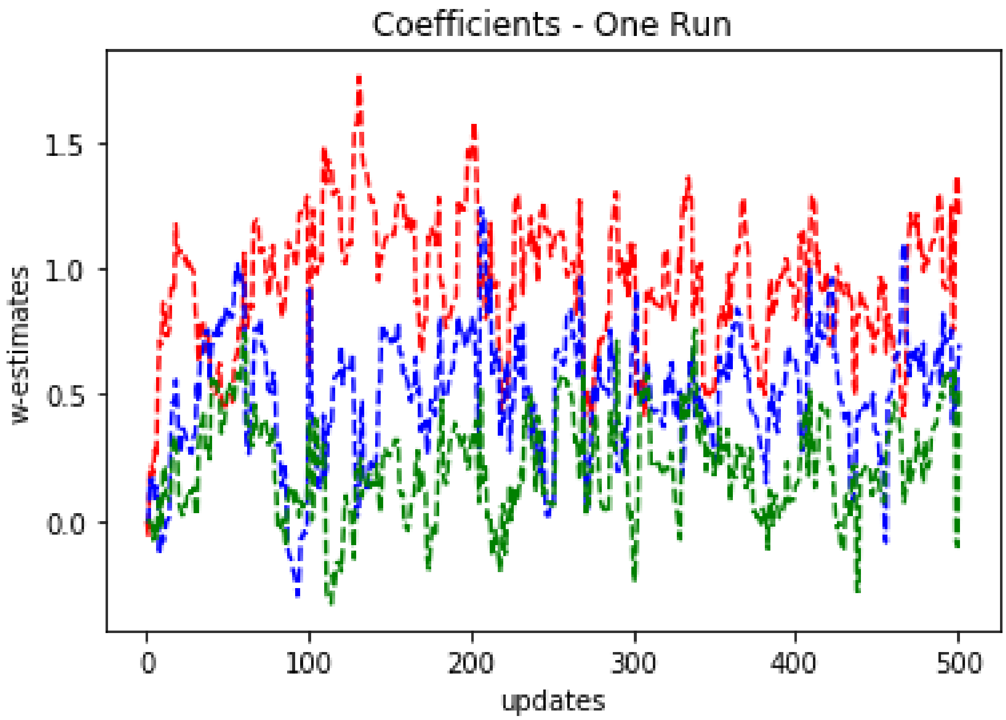
For eta = 0.05 and SNR = 10 dB,

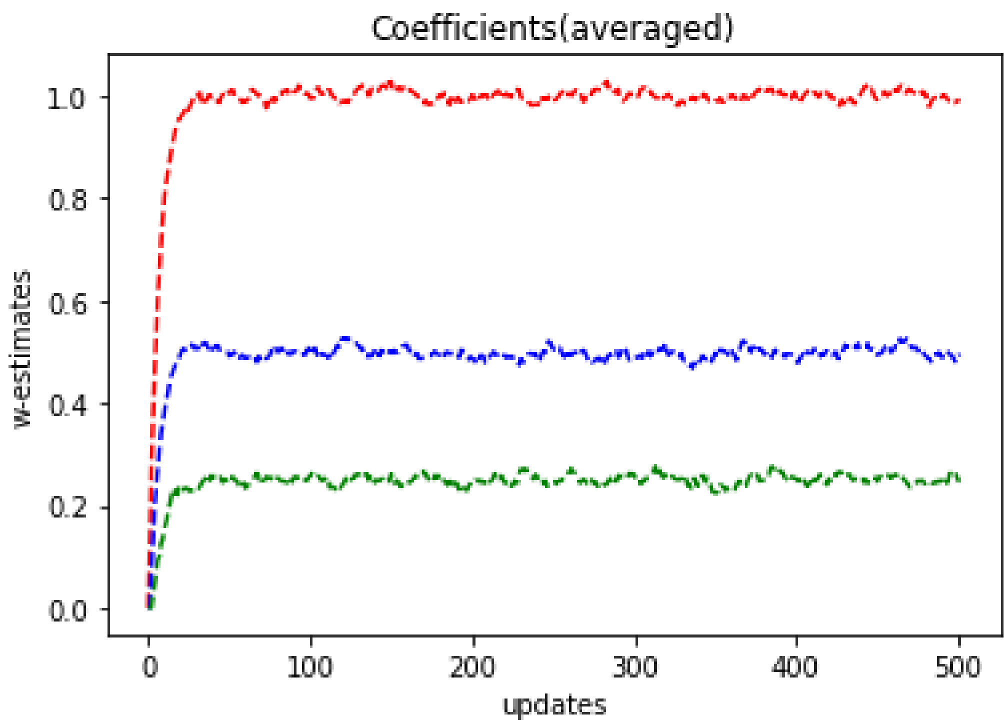


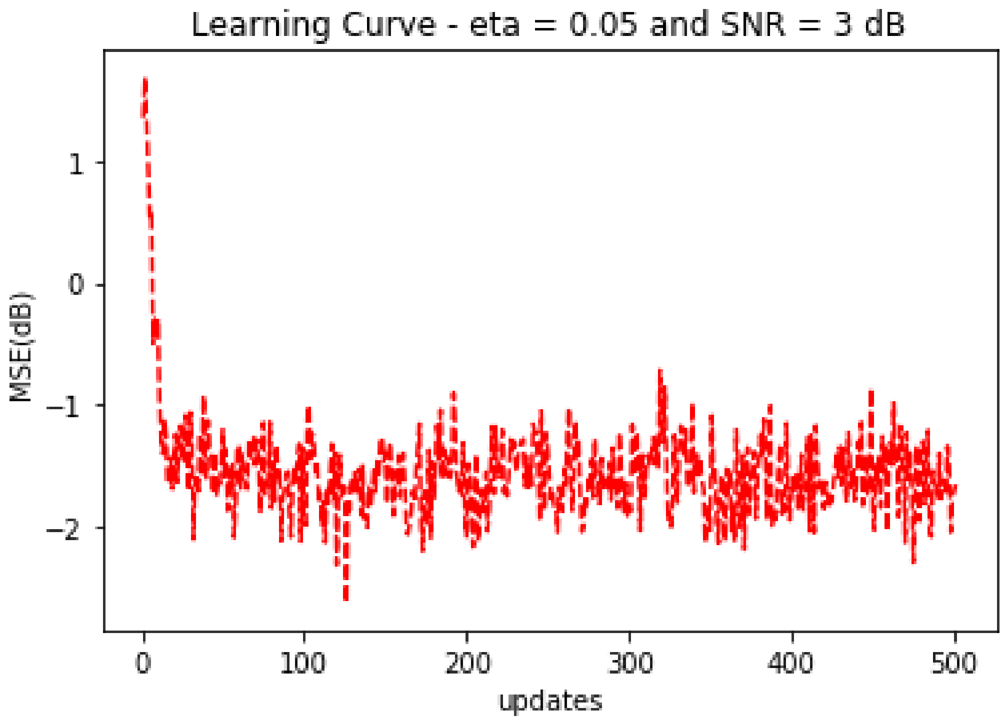




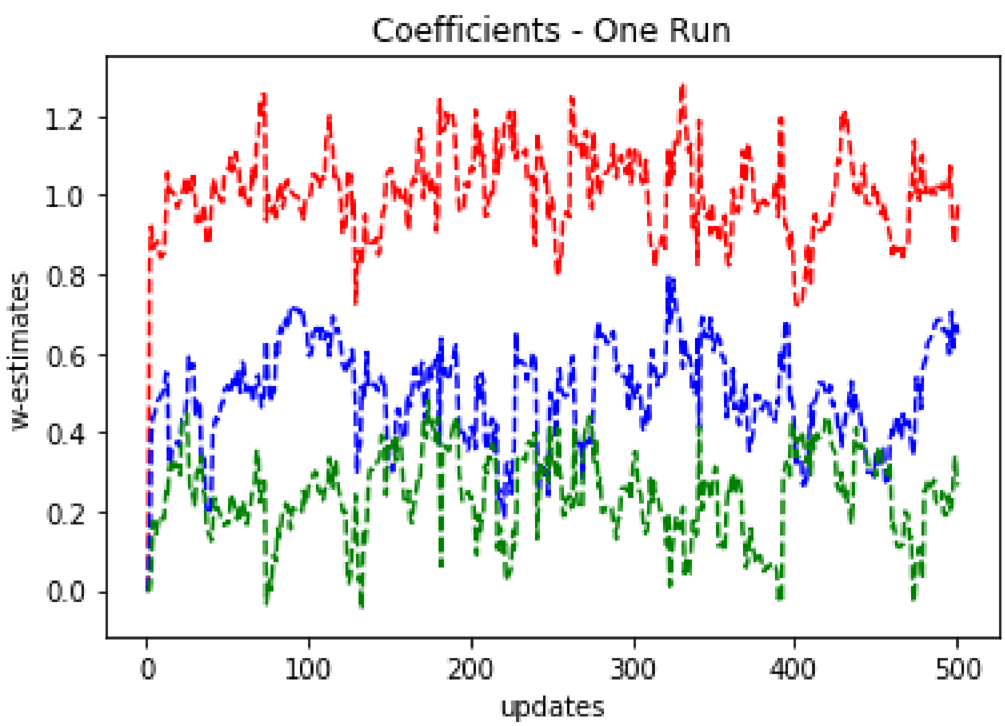
For eta = 0.15 and SNR = 3 dB,

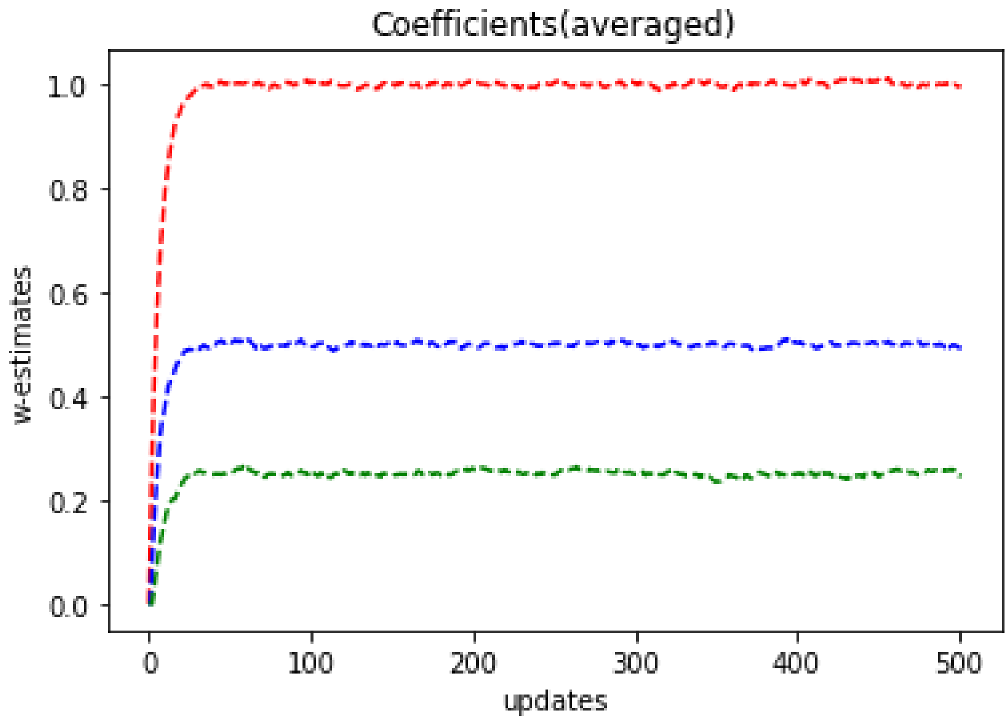

Coefficients - One Run
w-estimates
1.5
1.0
0.5
0.0
0
100
200
300
400
500
updates

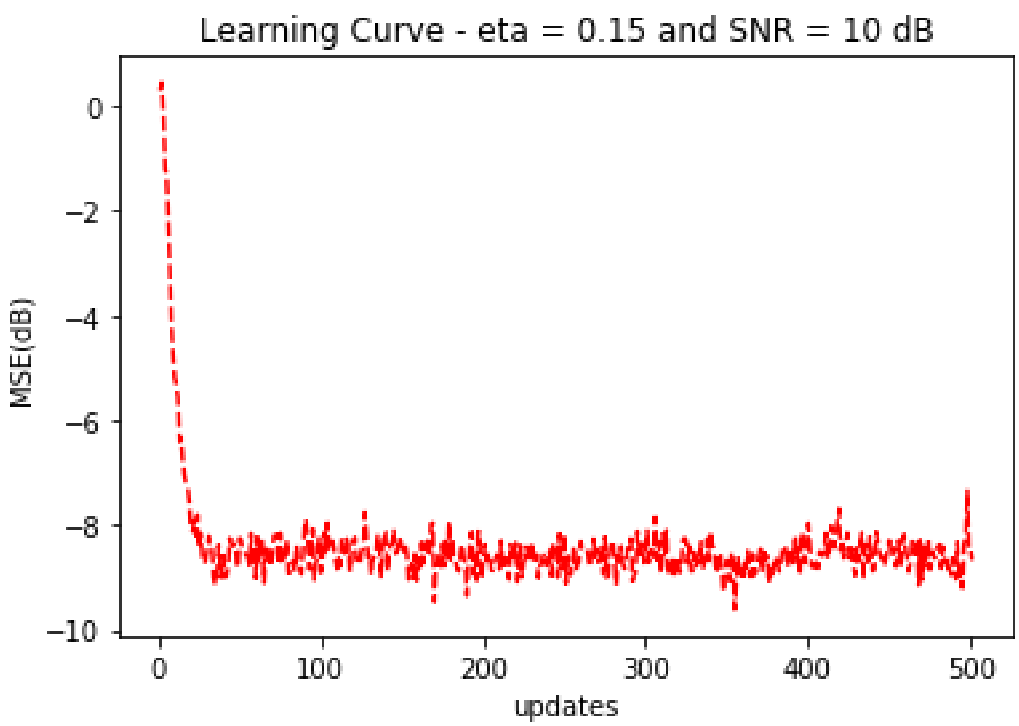

Coefficients(averaged)
w-estimates
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
updates



For eta = 0.15 and SNR = 10 dB,




Coefficients(averaged)
w-estimates
updates
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500


Learning Curve - eta = 0.15 and SNR = 10 dB
MSE(dB)
updates
0
−2
−4
−6
−8
−10
0
100
200
300
400
500

3.2 (c)

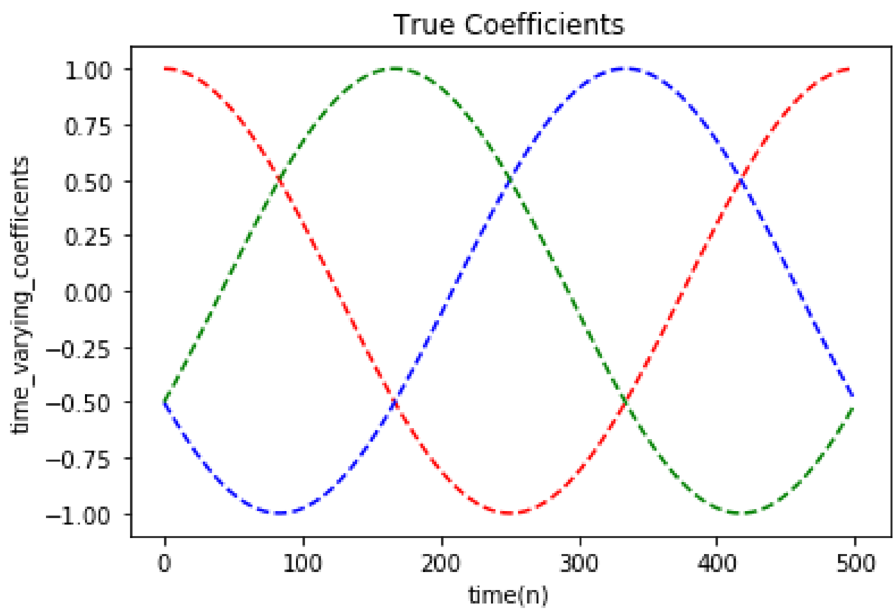
The MSE for these learning curves are comparable to the LMMSE found in the analytical part above. For the SNR = 3 dB, the MSE is almost equal to the LMMSE. For SNR = 10 dB, the MSE is nearly equal to the LMMSE.

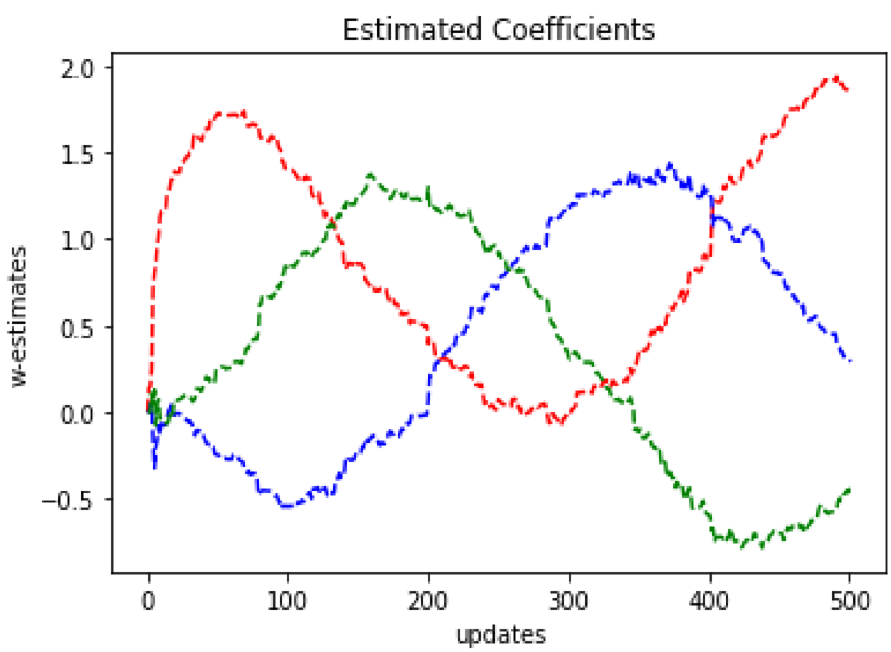
3.2 (d)

For SNR = 3dB, at eta = 0.25, we start getting divergent MSE.

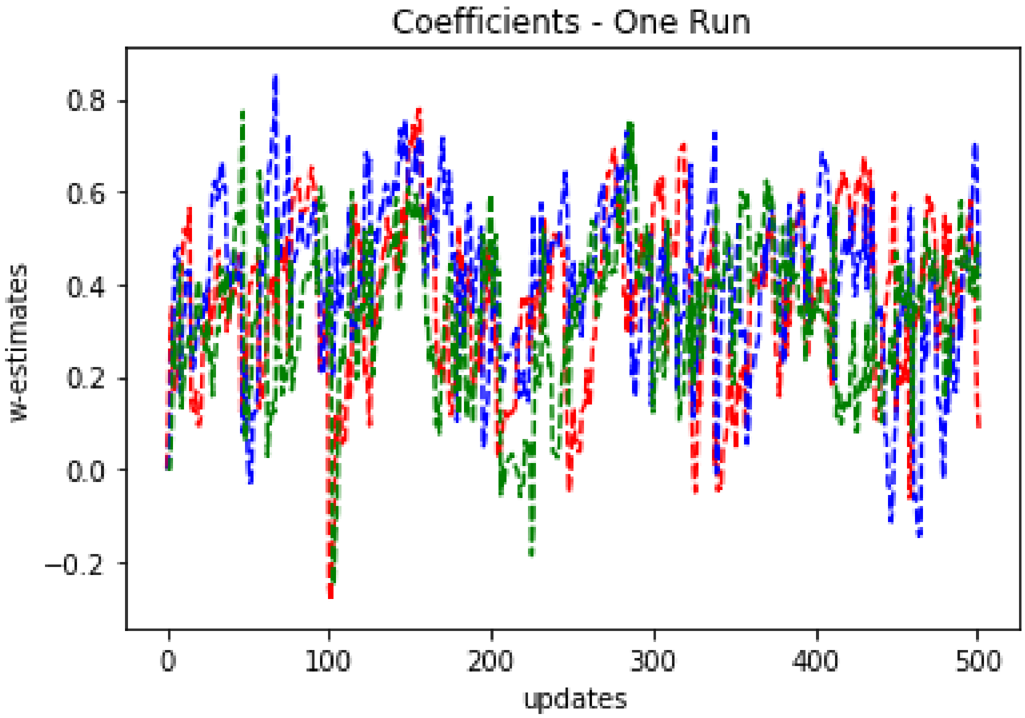
For SNR = 10dB, at eta = 0.25, we start getting divergent MSE.

3.3

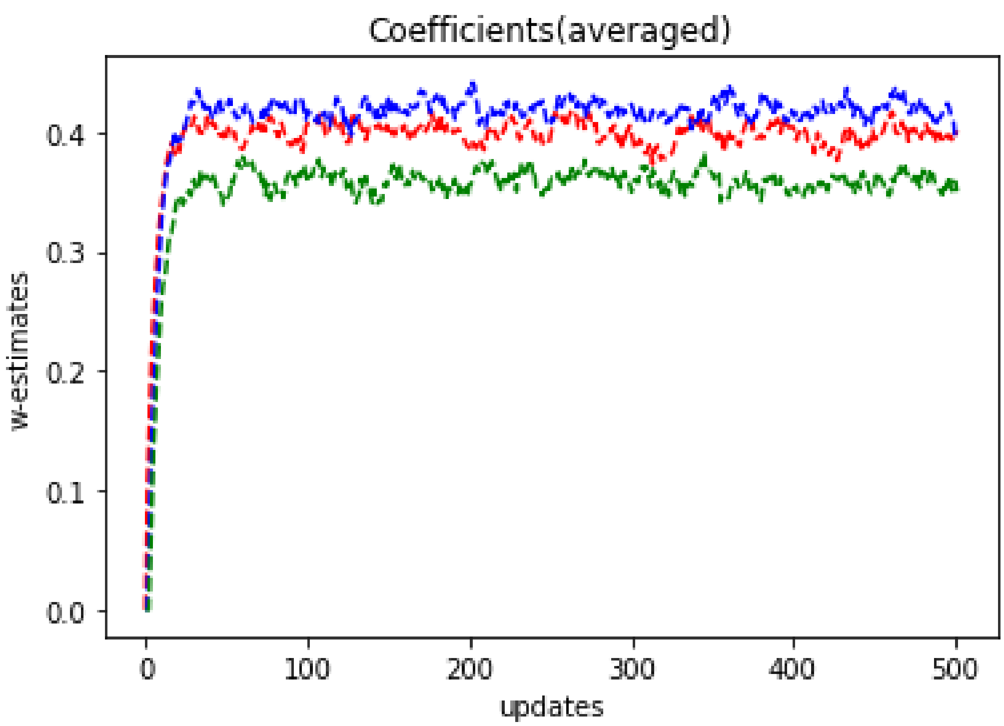




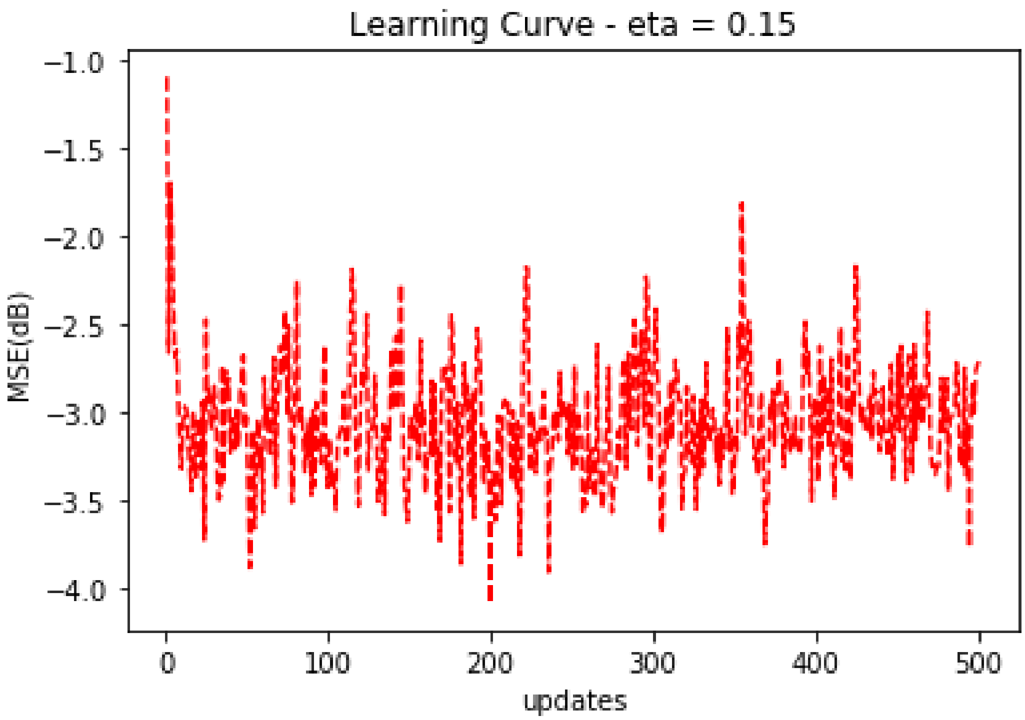
3.4 (a)



Coefficients - One Run
w-estimates
0.8
0.6
0.4
0.2
0.0
−0.2
0
100
200
300
400
500
updates



Coefficients(averaged)
w-estimates
0.4
0.3
0.2
0.1
0.0
0
100
200
300
400
500
updates



3.4 (b)

Rvn : [[0.99587395 -0.00186886 0.00205905]

[-0.00186886 0.99408593 -0.00194828]

[0.00205905 -0.00194828 0.99201923]]

Rn : [[0.39892994 0.38404952 0.36914017]]

LLSE : 0.33275257

For me, The LLSE is lower than the LMS learning curve after convergence.