Postulates of Quantum Physics

From linear algebra to quantum mechanics

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Introduction

In the labyrinth of quantum mechanics, where classical certainties give way to probabilistic intricacies, the foundational postulates emerge as guiding principles shaping the very fabric of reality on the smallest scales. This note embarks on an inquiry into the postulates that underpin the enigmatic world of quantum phenomena. With meticulous attention to theoretical detail, we aim to dissect and elucidate the fundamental principles governing quantum systems, offering an examination of concepts such as superposition.
As we embark on this intellectual journey, the objective is to provide a nuanced understanding of the theoretical scaffolding that defines the quantum landscape. This discourse transcends the boundaries of classical intuition, inviting readers to engage with the abstract foundations that have revolutionized the landscape of modern physics. Prepare for an exploration where precision meets ambiguity, as we navigate through the intricate postulates that delineate the behavior of particles at the quantum frontier. Welcome to an endeavor where the mysteries of quantum mechanics unfold with analytical rigor and intellectual depth.

Most of the discussions presnted in this short note are a simplified version of [4]. Other very interesting resources and point of views can be found in [3] and [2]. These discussions are more advanced but should enable the reader to appreciate the range of application and complexity of quantum mechanics. Finally, for the most mathematically inclined, the most precise discusions can be read in [5] and [1].

1 Superposition principle and Hilbert spaces

The starting point of quantum mechanics is the superposition principle. It is this mere fact that wraps up almost all the formalism of quantum mechanics. Not only the superposition principle makes the physics prettier and simpler, but it also allows to rely on the work of mathematicians that already set up all the framework that we shall use: *Hilbert spaces*.

1.1 Definition

The main difference between classical and quantum physics lies in the principle of superposition. In quantum mechanics, if we consider two states of the system, e_1 and e_2, the state $e_1 + e_2$ is also a possible state of the system (up to a normalization constant). This means that the states of a quantum system must be elements a *vector space*. In reality, for technical reasons that we won't detail here, one even assumes that they are elements of a *Hilbert space*. Let $\mathcal{H}$ denote this Hilbert space. This Hilbert space, generally considered as a complex vector space, naturally comes with a norm and a Hermitean product.

In the following, it is additionally assumed that this space is separable so that one can find a *Hilbert basis*, *i.e.* a set of vectors $\{e_i\}_{i\in I}$, where I is a countable set, verifying:

- Orthonormality: $(e_i, e_j) = \delta_{ij}$
- Completeness: $\forall\, e \in \mathcal{H},\ \exists\, (\lambda_i)_{i\in I} \in \mathbb{C}$ such that $e = \sum_{i\in I} \lambda_i e_i$ (equivalently one says that Vect $(\{e_i\}_{i\in I})$ is dense)

This Hilbert basis coincides with the classical orthonormal basis of a finite-dimensional vector space if $\mathcal{H}$ is finite-dimensional, which shall not be the case in general. Moreover, the

vectors e_i are indeed independent in the sense that any finite subfamily of $\{e_j\}_{j\in J}$, $J \subset I$, is linearly independent.

1.2 *Kets* and *bras*

From now on, we denote $|\phi\rangle$ the elements of $\mathcal{H}$. This notation is called *ket*. To each vector $|\phi\rangle$, one associates a continuous linear form $\langle\phi|$, read *bra* and defined by:

$$\begin{array}{rccc} \langle\phi| : & \mathcal{H} & \longrightarrow & \mathcal{H} \\ & |\psi\rangle & \longmapsto & \langle\phi|\left(|\psi\rangle\right) = \left(|\phi\rangle, |\psi\rangle\right) = \langle\phi|\psi\rangle \end{array},$$

where we denoted the inner product $\langle\cdot|\cdot\rangle$. The standard notation for the space of linear forms is $\mathcal{H}^*$, that's the one that shall be used in the following.

Note that $\langle\phi| = |\phi\rangle^\dagger$, so that in terms of vectors it corresponds simply to transposing and conjugating the vector $|\phi\rangle$. Let us already give an example for this. Suppose one works in a 3-dimensional Hilbert space. This is just equivalent to working in $\mathbb{C}^3$ (namely these two spaces are isomorphic), so that one can then denote his vectors

$$|\varphi\rangle = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

To fix the ideas, let us work with two simple vectors $|\psi\rangle = \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} 1 \\ 1 \\ i \end{pmatrix}$. Then, by definition,

$$\langle\phi|\varphi\rangle = \left(|\phi\rangle, |\varphi\rangle\right) = 1^* z_1 + 1^* z_2 + i^* z_3 = z_1 + z_2 - i z_3,$$

so that $\langle\phi| = \begin{pmatrix} 1 & 1 & -i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ i \end{pmatrix}^\dagger$. This proves that the *bras* are just the transconjugate of the *kets*.

1.3 Basis and decomposition of a state

Keeping the previous notations, it is straightforward to determine the coefficients involved in the decomposition of a $|\phi\rangle$ on a Hilbert basis of $\mathcal{H}$. For example, let $|\psi_k\rangle$, $k \in \mathbb{N}$, be such a basis. It readily follows that:

$$|\phi\rangle = \sum_{k\in\mathbb{N}} \frac{\langle\psi_k|\phi\rangle}{\langle\psi_k|\psi_k\rangle} |\psi_k\rangle.$$

If the chosen basis is normalized, one also has the relation $\langle\psi_k|\psi_k\rangle = 1$.

Let us sketch an example for such a decomposition. This time we work with a 2-dimensional Hilbert space, that is identified with $\mathbb{C}^2$. One has to choose a basis for this space. Let us not be too whimsical and take the simplest one, namely the *canonical basis*,

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is fairly easy to check that $\langle \psi_i | \psi_i \rangle = 1$. Now if one denotes any vector $|\phi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ it is obvious that it can be decomposed as,

$$|\phi\rangle = z_1 \, |\psi_1\rangle + z_2 \, |\psi_2\rangle \, .$$

Any basis is as good as any other, so that one can change basis and take a more complicated one. To understand how this works, we choose,

$$|\varphi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |\varphi_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

One can easily check that these indeed form a basis for $\mathbb{C}^2 \simeq \mathcal{H}$. These vectors are not normalized anymore. Let us normalize them (it is explained why the normalization is important in the next paragraph), it yields $\langle \varphi_i | \varphi_i \rangle = \sqrt{2}$. Let us compute the projection of our general vector $|\phi\rangle$ on this basis,

$$\langle \varphi_1 | \phi \rangle = z_1 + z_2 \quad \text{and} \quad \langle \varphi_2 | \phi \rangle = z_1 - z_2,$$

so that finally,

$$|\phi\rangle = \frac{1}{\sqrt{2}} \left[(z_1 + z_2) \, |\varphi_1\rangle + (z_1 - z_2) \, |\varphi_2\rangle \right] .$$

Changing basis is just a matter of expressing the new vectors in the canonical basis. It may seem cumbersome and useless, but as we shall see by the end of this note, when it comes to studying physical quantities, it may be important to choose a specific basis.

1.4 Normalization of physical states

Not all vectors in the Hilbert space are physical. Because of the probabilistic interpretation (see Born's rule), one defines physical state vectors as the vectors in $\mathcal{H}$ with norm 1, *i.e.*, $|\psi\rangle$ such that $\langle \psi | \psi \rangle = 1$. It is easy to normalize any vector to render it physical. For instance, if one works again with the vector $|\varphi_1\rangle$, that is non-normalized, one can transform it into an eligible physical state by defining, $|\Phi_1\rangle = |\varphi_1\rangle / \sqrt{2}$, so that $\langle \Phi_1 | \Phi_1 \rangle = 1$. This means that in general, we don't loose anything by working with non-normalized vectors and normalized them only by the end of our computations.

2 Measurements or why is Quantum Mechanics hard ?

The Hilbert space structure of quantum mechanics opens up many possibilities to act on the states. In this section, we rely on linear algebra to relate physical quantities (*observables*) and linear maps (*operators*).

2.1 Observables and operators

2.1.1 Definition

Now that we have equipped ourselves with a Hilbert space $\mathcal{H}$ in which we find our state vectors, it is natural to define the operators acting on this Hilbert space, i.e., the linear maps

$$\begin{array}{rccc} \widehat{O}: & \mathcal{H} & \longrightarrow & \mathcal{H} \\ & |\psi\rangle & \longmapsto & \widehat{O}\left(|\psi\rangle\right) = \widehat{O}\,|\psi\rangle \end{array}$$

which are mathematically called endomorphisms of $\mathcal{H}$ and denoted by $\mathcal{L}(\mathcal{H})$. For a finite-dimensional Hilbert space, these operators are in bijection with square matrices $\mathcal{M}(\dim\mathcal{H}, \mathbb{C})$. In infinite dimensions, these *matrices* naturally take on an infinite (countable) size, and we shall continue to call them matrices even though they are of infinite dimension. Sometimes operators are denoted with a *hat* on them, $\widehat{O}$.

In quantum mechanics, there is a specific type of operators that is of particular interest to us because they represent measurable physical quantities. We call these operators *observables* and define them as Hermitian operators, i.e., operators O such that

$$\widehat{O}^\dagger = \widehat{O},$$

where $\cdot^\dagger$ denotes the *adjoint* (or *transconjugate*).

2.1.2 Basis and matrix representation

Let us choose a basis $\mathcal{B} = \{|\psi_k\rangle\}$ of the physical Hilbert space. Consider an arbitrary operator A acting on $\mathcal{H}$. One can always describe the operator A by calculating its *matrix elements* on the basis $\mathcal{B}$. That is to say, one computes the coefficients,

$$A_{ij} = (|\psi_i\rangle\,, A\,|\psi_j\rangle),$$

where the notation $(\cdot,\cdot)$ was used for the Hermitean product. Consequently, A acts on a state $|\psi\rangle = \sum_k a_k\,|\psi_k\rangle$ in the form,

$$A\,|\psi\rangle = \sum_k a_k A\,|\psi_k\rangle = \sum_{kl} a_k A_{lk}\,|\psi_l\rangle\,.$$

It is then natural to write,

$$A = \sum_{ij} A_{ij}\,|\psi_i\rangle\,\langle\psi_j|\,.$$

Let us return to Hermitian operators and denote by O such an operator. All the results described for A remain true for O, and this allows to take some freedom with the notations. Similarly, one can write,

$$O = \sum_{ij} O_{ij}\,|\psi_i\rangle\,\langle\psi_j|\,.$$

However, since O is Hermitian it verifies $O^*_{ji} = O_{ij}$. This simple fact allows one to write,

$$O_{ij} = (|\psi_i\rangle, O|\psi_j\rangle) = (|\psi_i\rangle O^\dagger, |\psi_j\rangle) = \langle\psi_i| O |\psi_j\rangle,$$

where the ambiguity in the last term of this equation has been removed, which would be ill-defined if O were not Hermitian.

Let us give an example to explain this more clearly. We turn back yet again to a 2-dimensional Hilbert space with the canonical basis so that we work with states, $|\phi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Operator are just given by matrices. Let us define two operators,

$$A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad \text{and} \quad O = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

In this case, A is not Hermitean whereas O is. It means that one can unwarily use the *braket* notations for O but that they should be very careful when we use it for A. Furthermore, for instance O can be written as,

$$O = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + i|\psi_1\rangle\langle\psi_2| - i|\psi_2\rangle\langle\psi_1|.$$

Hermitian operators also have the peculiarity of possessing real eigenvalues, which turns out to be fundamental when associating them with measurable physical quantities. The relationship between a Hermitian operator and a physical observable is as follows: to every measurable physical quantity $\mathcal{O}$, we associate a Hermitian operator $O \in \mathcal{L}(\mathcal{H})$. Additionally, note that this relationship is one-way; each physical observable is associated with a Hermitian operator, but not all Hermitian operators necessarily correspond to physical observables.

2.1.3 More involved Hilbert spaces

When the Hilbert space is finite dimensional, any operator can be written as a matrix. However, finite dimensional Hilbert spaces don't appear so often in quantum mechanics. A very classical example of Hilbert space is $L^2(\mathbb{R})$, the space of square-integrable wavefunctions. On this space, operators are acting on functions and should yield square integrable functions. An example of operator is the derivative (which represents momentum in quantum mechanics). It is far from true that the derivative of any $L^2(\mathbb{R})$ function is in $L^2(\mathbb{R})$ but we should overlook this issue as this discussion is only here to give some intution about operators.

2.2 Measurements

If it were not for the *measurement postulates*, quantum mechanics would be a fairly straightforward theory. However, because of *Born's rule*, that we shall detail further in a few sentences, quantum systems behave in an unintuitive manner. The postulate goes as follows.

Assuming the system is in a state $|\psi\rangle \in \mathcal{H}$, the measurement of the physical observable (meaning any physical measurable quantity) $\mathcal{O}$ in this state necessarily yields a value $o_i \in \text{Sp } O \subset \mathbb{R}$ (*i.e* an eigenvalue of O). This justifies retroactively our choice of Hermitean operators for the observables.

2.2.1 Born's rule

The postulate is actually even stronger than this previous statement as it also yields the probability of finding each eigenvalue of the operator. However, in order to make this precise one needs to introduce a few more notations.

Consider a physical observable $\mathcal{O}$ and its associated Hermitean operator O. As O is Hermitean, it is diagonalizable (this comes right from mathematicians and is a utmost fundamental theorem called *spectral theorem*), and hence, there exists an orthogonal basis of eigenvectors for O. Let o_i be the eigenvalues of O and

$$g_i = \dim \ker(O - o_i \mathrm{id}_{\mathcal{H}}).$$

g_i is known as the *degeneracy* of the eigenvalue o_i. It corresponds to the number of independent states that have the same eigenvalue o_i.
One can build up an orthogonal basis for O in the form of

$$\mathcal{B} = \left\{ \left| \phi_i^l \right\rangle \right\}_{i,\, 1 \leq l \leq g_i},$$

such that $\mathcal{B}_i = \mathrm{Vect} \left\{ \left| \phi_i^l \right\rangle \right\}_{1 \leq l \leq g_i}$ forms a basis of $\ker(O - o_i \mathrm{id}_{\mathcal{H}})$ (we shall refer to these subspaces as *proper subspaces*). Orthogonality is expressed by,

$$\left\langle \phi_i^l \middle| \phi_j^k \right\rangle \propto \delta_{ij} \delta_{lk}.$$

Now consider an arbitrary state $|\psi\rangle$. It can be decomposed as,

$$|\psi\rangle = \sum_{i,l} a_i^l \left| \phi_i^l \right\rangle,$$

where $a_i^l = \frac{\langle \phi_i^l | \psi \rangle}{\langle \phi_i^l | \phi_i^l \rangle}$ (note that we have not committed to the normalization of the $\left| \phi_i^l \right\rangle$).

A measurement of the observable $\mathcal{O}$ yields the value o_i with the probability,

$$P(o_i) = \sum_{l=1}^{g_i} \left| \left\langle \phi_i^l \middle| \psi \right\rangle \right|^2.$$

This is known as *Born's rule*. For the probabilistic interpretation to be valid, it is necessary that,

$$\sum_i P(o_i) = 1.$$

This can be expressed as,

$$\begin{aligned}
\sum_i \sum_{l=1}^{g_i} \left| \left\langle \phi_i^l \middle| \psi \right\rangle \right|^2 &= \sum_{il} \left| a_i^l \right|^2, \\
&= \sum_{il} \left| \frac{\langle \phi_i^l | \psi \rangle}{\langle \phi_i^l | \phi_i^l \rangle} \right|^2, \\
&= \sum_{il} \frac{\langle \phi_i^l | \psi \rangle \langle \phi_i^l | \psi \rangle^*}{\left| \langle \phi_i^l | \phi_i^l \rangle \right|^2}, \\
&= \langle \psi | \left(\sum_{il} \frac{\left| \phi_i^l \right\rangle \left\langle \phi_i^l \right|}{\left| \langle \phi_i^l | \phi_i^l \rangle \right|^2} \right) | \psi \rangle, \\
&= \langle \psi | \psi \rangle = 1.
\end{aligned}$$

where we have used the result (easy to prove),

$$\sum_{il} \frac{\left|\phi_i^l\right\rangle \left\langle\phi_i^l\right|}{\left|\left\langle\phi_i^l|\phi_i^l\right\rangle\right|^2} = \mathrm{id}_{\mathcal{H}}.$$

This probabilistic interpretation explains why physical states were chosen with a unit norm.

2.2.2 Wavepacket collapse

Quantum mechanics gets even weirder because a measurement doesn't leave the system invariant. In fact, if an observer measures an observable of a state at some time t, right after the measure the *wavepacket collapses*, meaning that we know exactly in which state the system is just afterwards the measure. Using the previous notations, if the measured value is o_n. The state of the system right after the measure is given by $|\psi'\rangle$

$$\left|\psi'\right\rangle = \frac{1}{\sqrt{P(o_n)}} \sum_{j=1}^{g_n} \left\langle \phi_n^j \middle| \psi \right\rangle \left|\phi_n^j\right\rangle.$$

The interpretation for this postulate is that if one was to measure again the same physical quantity an infinitesimally small amount of time after the first measure, they would expect it to be exactly the same. Hence, the state $|\psi\rangle$ is projected onto the proper subspace associated to the eigenvalue o_n.

2.2.3 A simple example

Let us sketch an example for this. Suppose we work in a 3-dimensional Hilbert space with a basis, identified with the canonical basis of $\mathbb{C}^3$, in which some operators O and A are diagonal so that one can write them as,

$$O = \begin{pmatrix} o_1 & 0 & 0 \\ 0 & o_2 & 0 \\ 0 & 0 & o_3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}.$$

Suppose the system is in some state,

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle).$$

where the $|\psi_i\rangle$ are the eigenvectors for O and A. One can compute the probability for measuring o_1 and a_1 for instance. o_1 is non-degenerate and so the formula simplifies greatly,

$$P(o_1) = |\langle\psi_1|\phi\rangle|^2 = \left|\frac{1}{\sqrt{2}} \cdot 1\right|^2 = \frac{1}{2}.$$

The state just after the measure of o_1 is given by,

$$\left|\phi'\right\rangle = \frac{1}{\sqrt{1/2}} \left\langle\psi_1|\phi\right\rangle |\psi_1\rangle = |\psi_1\rangle.$$

The case of a_1 is a bit trickier as a_1 is a degenerate eigenvalue. We simply follow the rules,

$$P(a_1) = |\langle\psi_1|\phi\rangle|^2 + |\langle\psi_2|\phi\rangle|^2 = 2 \cdot \left|\frac{1}{\sqrt{2}} \cdot 1\right|^2 = 1.$$

And the state remains unchanged as is shown below,

$$|\phi'\rangle = \frac{1}{\sqrt{1}}(\langle\psi_1|\phi\rangle\,|\psi_1\rangle + \langle\psi_2|\phi\rangle\,|\psi_2\rangle) = |\phi\rangle\,.$$

2.3 Commutator and Complete Set of Commuting Observables

2.3.1 Commutator

In quantum physics, the postulates indicate that measuring a physical observable $\widehat{A}$ amounts to transforming the state of our physical system $|\psi\rangle$ into the state $|\psi'\rangle = \widehat{A}\,|\psi\rangle$.[What we say here is actually not quite true. We'd better say that the information concerning the state after the measure of the observable $\mathcal{A}$ is contained in the ket $|\psi'\rangle$.]. Thus, successive measurement of physical observables $\widehat{A}$ and $\widehat{B}$ should not yield the same result depending on whether we apply $\widehat{A}$ or $\widehat{B}$ first. In fact, for this to be the case, it would generally be necessary for $\widehat{A}\widehat{B} = \widehat{B}\widehat{A}$. Thinking in terms of matrices, it is obvious that this isn't always the case! That is why we define one of the most important object in quantum physics, the *commutator*.

Let $\widehat{A}$ and $\widehat{B}$ be two operators $\mathcal{H} \to \mathcal{H}$. Their commutator is defined by

$$\left[\widehat{A}, \widehat{B}\right] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}.$$

Now mathematics give us a very important result,

$$\textit{if } \left[\widehat{A}, \widehat{B}\right] = 0\textit{, then } \widehat{A} \textit{ and } \widehat{B} \textit{ have a common basis of eigenvectors.}$$

2.3.2 Complete Set of Commuting Observables

A Complete Set of Commuting Observables (CSCO) is a set of observables $E = \{\widehat{A}, \widehat{B}, \ldots, \widehat{Z}\}$ whose commutators are all zero (the observables commute pairwise) and whose n-tuple $(a, b, \ldots, z)$ of respective eigenvalues of each operator defines a unique quantum state that will then be denoted by $|a, b, \ldots, z\rangle$ and such that

$$\begin{cases} \widehat{A}\,|a, b, \ldots, z\rangle & = a\,|a, b, \ldots, z\rangle \\ \widehat{B}\,|a, b, \ldots, z\rangle & = b\,|a, b, \ldots, z\rangle \\ \quad\vdots & \quad\vdots \\ \widehat{Z}\,|a, b, \ldots, z\rangle & = z\,|a, b, \ldots, z\rangle \end{cases}$$

The definition as it stands is rather mathematical and somewhat obscure, but the idea is quite simple. We wish to find enough observables that commute, meaning their values can be determined at the same time, to determine the state of the system uniquely.

2.3.3 A simple example

Let us give some simple examples. Let us consider yet again a 3-dimensional Hilbert space with canonical basis, so that we have three operators, A, B and C defined by,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using block multiplication (or direct computation), one can show that $AB = BA$ and $CB = BC$. However, let us compute AC,

$$AC = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since A and C are symmetric, it follows that $(AC)^T = C^T A^T = CA$ then,

$$CA = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that both $\{A, B\}$ and $\{C, B\}$ could be perfectly suitable CSCO but $\{A, B, C\}$ is not.

2.4 Probability measures

The discussion about Born's rule draw a deep relationship between quantum mechanics and probabilities. So much that we can actually define a probability measure for any state $|\psi\rangle$ that we denote P_ψ. P_ψ is defined on any set of eigenvalues of commuting operators and allows us to pull back the whole mathematical theory of probabilities to quantum mechanics.

2.4.1 Mean value

The first thing we can do is define the mean value of an operator, that can now be seen as a random variable, in a set $|\psi\rangle$. Mathematically this reads,

$$\langle A \rangle_\psi = \mathbb{E}_\psi[A],$$

where the subscript ψ means that we are taking the expectancy under the probability measure P_ψ. This rather mathematical definition allows us to formally define everything but it's not so useful in practice. Let us compute precisely the expectancy (we consider a simple non degenerate case here, but the result is the same for the degenrate case and carrying out the

computations is a very good exercise).

$$\begin{aligned}
\langle A\rangle_\psi &= \sum_n P_\psi(a_n) a_n, \\
&= \sum_n |\langle\psi|\phi_n\rangle|^2 a_n, \\
&= \sum_n (\langle\psi|\phi_n\rangle)^* \langle\psi|\phi_n\rangle a_n, \\
&= \sum_n \langle\phi_n|\psi\rangle \langle\psi|\phi_n\rangle a_n, \\
&= \sum_n \langle\psi| a_n |\phi_n\rangle \langle\phi_n|\psi\rangle \quad \text{since } a_n \text{ is a scalar and thus commutes with bras and kets,} \\
&= \sum_n \langle\psi| \widehat{A} |\phi_n\rangle \langle\phi_n|\psi\rangle \quad \text{since } \widehat{A}|\phi_n\rangle = a_n|\phi_n\rangle, \\
&= \langle\psi| \widehat{A} \left(\sum_n |\phi_n\rangle \langle\phi_n|\right) |\psi\rangle \quad \text{using the closure formula.}
\end{aligned}$$

Finally we obtained the useful result we've been craving,

$$\langle A\rangle_\psi = \langle\psi| \widehat{A} |\psi\rangle .$$

2.4.2 Standard deviation

Mathematicians would then go on defining any kind of n-th order momenta but we shall be more secular and limit ourselves to standard deviation. We define the squared standard deviation (variance) of $\widehat{A}$ in the state $|\psi\rangle$ as

$$\sigma_\psi^2\left(\widehat{A}\right) = \mathbb{E}_\psi\left[(A - \mathbb{E}_\psi[A])^2\right] = \left\langle A^2\right\rangle_\psi - \langle A\rangle_\psi^2 .$$

2.4.3 Heisenberg's inequality

This leads to a well-known and fundamental property known as *Heisenberg's inequality*. Let $\widehat{A}, \widehat{B}$ be two operators and $|\psi\rangle \in \mathcal{H}$, then one can show

$$\sigma_\psi\left(\widehat{A}\right)\sigma_\psi\left(\widehat{B}\right) \geq \frac{1}{2}\left|\left\langle\left[\widehat{A},\widehat{B}\right]\right\rangle_\psi\right| .$$

2.4.4 A simple example

Let us work with a 2-dimensional Hilbert space identified with $\mathbf{C}^2$. Working in the canonical basis, let us define to operators,

$$A = \begin{pmatrix} 1 & 1-i \\ 1+i & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} .$$

A and B are Hermitean operators. Let us choose a state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle).$$

We can compute, the mean values of A and B,

$$\begin{aligned}\langle A\rangle_\psi = \langle\psi| A |\psi\rangle &= \frac{1}{2}(\langle\psi_1| + \langle\psi_2|)A(|\psi_1\rangle + |\psi_2\rangle) = \frac{1}{2}\left(A_{11} + A_{12} + A_{21} + A_{22}\right), \\ &= \frac{1}{2}(1 + i + 1 - i + 1 + 1), \\ &= 2,\end{aligned}$$

$$\begin{aligned}\langle B\rangle_\psi = \langle\psi| B |\psi\rangle &= \frac{1}{2}(\langle\psi_1| + \langle\psi_2|)B(|\psi_1\rangle + |\psi_2\rangle) = \frac{1}{2}\left(B_{11} + B_{12} + B_{21} + B_{22}\right), \\ &= \frac{1}{2}(i - i + 1 + 1), \\ &= 1.\end{aligned}$$

Computing the squares of the operators will allow us to compute the standard deviation of A and B,

$$A^2 = \begin{pmatrix} 3 & 2(1-i) \\ 2(1+i) & 3 \end{pmatrix} = I + 2A \quad \text{and} \quad B^2 = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} = 2B.$$

So that the mean values of the squared operators can be computed as,

$$\begin{aligned}\left\langle A^2\right\rangle_\psi &= \langle\psi| A^2 |\psi\rangle = \langle\psi| (I + 2A) |\psi\rangle = 1 + 2\langle A\rangle_\psi = 5, \\ \left\langle B^2\right\rangle_\psi &= \langle\psi| B^2 |\psi\rangle = \langle\psi| 2B |\psi\rangle = 2\langle B\rangle_\psi = 2.\end{aligned}$$

And the standard deviation is expressed as,

$$\begin{aligned}\sigma_\psi\left(\widehat{A}\right) &= \sqrt{\langle A^2\rangle_\psi - \langle A\rangle_\psi^2} = \sqrt{5 - 2^2} = 1, \\ \sigma_\psi\left(\widehat{B}\right) &= \sqrt{\langle B^2\rangle_\psi - \langle B\rangle_\psi^2} = \sqrt{2 - 1^2} = 1.\end{aligned}$$

To check that Heisenberg's inequality is verified, one has to compute the commutator of A and B,

$$[A, B] = AB - BA = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}.$$

This way, the expectancy of $[A, B]$ in the state $|\psi\rangle$ is simply 0 and Heisenberg's inequality is trivially verified.

3 Schrödinger's equation

It may seem quite surprising that on an introductory note to quantum mechanics, Schrödinger shall appear only in the third and last section. Schrödinger's equation is probably the most iconic postulate of quantum mechanics, so much that any *boomer* would not spend a minute before blabbering it when one would mention the mystic name of *quantum mechanics*.

Schrödinger's equation is the equivalent of Newton's law. It rules the system but it doesn't not exempt anybody from defining what are refrence frames, forces, vectors before using it. It's the same for Schrödinger's equation, we needed Hilbert spaces and operator formalism to make sense of this equation.

3.1 Schrödinger's equation

Schrödinger's equation is only a mere evolution principle. It tells us how a state $|\psi\rangle \in \mathcal{H}$ should evolve in time.

Every system possesses a particular physical quantity called energy. The operator associated with it is the Hamiltonian, denoted $\widehat{H}$. The time evolution of $|\psi(t)\rangle$ is given by the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi\rangle = \widehat{H} |\psi\rangle .$$

Actually the only tricky part about Schrödinger's equation is to determine H. Fortunately, this goes far beyond the scope of this short note. At a very basic level, one could argue that it only consists in taking the classical energy and promoting everybody to operators. This might work in very simple cases, but most physicists (or at least theoreticians) would probably have a heart attack reading this.

3.2 Stationnary evolution

For a stationnary Hamiltonian (*i.e* when the Hamiltonian does not depend on time) it is very easy to find the form of the evolving state $|\psi\rangle$ by expanding it on an eigenbasis of $\widehat{H}$. Since $\widehat{H}$ is Hermitian it has an orthonormal basis that diagonalizes it, which is denoted $\{|\phi_n\rangle\}_{n\in\mathbb{N}}$. This basis does not depend on time and verifies, $\widehat{H} |\phi_n\rangle = E_n |\phi_n\rangle$, where E_n are the eigenenergies (eigenvalues of $\widehat{H}$ that are not necessarily distinct so that degeneracies are hidden under the rug here).

At time $t = 0$, the state is $|\psi(0)\rangle = \sum_{n=0}^{\infty} a_n(0) |\phi_n\rangle$. We seek the expression for $|\psi(t)\rangle$. At any time, it is possible to write (because we can expand any state on a basis for $\mathcal{H}$)):

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} a_n(t) |\phi_n\rangle .$$

Injecting this expression into the Schrödinger equation, we have:

$$i\hbar \frac{d}{dt} |\psi\rangle = \widehat{H} |\psi\rangle ,$$
$$i\hbar \sum_n \frac{da_n(t)}{dt} |\phi_n\rangle = \sum_n a_n(t) E_n |\phi_n\rangle .$$

Since $|\phi_n\rangle$ is a basis, for all $n \in \mathbb{N}$:

$$i\hbar \frac{da_n(t)}{dt} = a_n(t)E_n,$$

which has a unique solution given by:

$$a_n(t) = a_n(0)e^{-\frac{iE_n t}{\hbar}}.$$

Therefore, we have the utmost important relation:

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} a_n(0)e^{-\frac{iE_n t}{\hbar}} |\phi_n\rangle .$$

3.3 A simple example

Let us sketch an example for how this can work out in practice. We work in a 2-dimensional Hilbert space, identified with $\mathbb{C}^2$, for which we denote the canonical basis $|\psi_1\rangle$ and $|\psi_2\rangle$. In such a basis, the Hamiltonian is constant given by,

$$H = \begin{pmatrix} \varepsilon_0 & V \\ V & \varepsilon_0 \end{pmatrix}.$$

We assume that the system is in a state,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle).$$

Let us compute the explicit form of the state at any further time t. The first thing that should be kept in mind is that we can compute very easily the time evolution of a state if it is expressend in an *eigenbasis* of H. Hence our first step is to diagonalize H. To do so we compute the eigenvalues of H,

$$\det(XI_2 - H) = \begin{vmatrix} X - \varepsilon_0 & -V \\ -V & X - \varepsilon_0 \end{vmatrix} = (X - \varepsilon_0)^2 - V^2 = (X - \varepsilon_0 + V)(X - \varepsilon_0 - V).$$

Now we solve for the eigenvectors $|\phi_1\rangle$ associated with $\varepsilon_0 - V$ and $|\phi_2\rangle$ associated with $\varepsilon_0 + V$. For instance, let us solve for $|\phi_1\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, it verifies,

$$\begin{aligned} H|\phi_1\rangle = (\varepsilon_0 - V)|\phi_1\rangle &\Leftrightarrow \begin{pmatrix} \varepsilon_0\alpha_1 + V\alpha_2 \\ \varepsilon_0\alpha_2 + V\alpha_1 \end{pmatrix} = (\varepsilon_0 - V)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \\ &\Leftrightarrow \begin{cases} \varepsilon_0\alpha_1 + V\alpha_2 &= (\varepsilon_0 - V)\alpha_1 \\ \varepsilon_0\alpha_2 + V\alpha_1 &= (\varepsilon_0 - V)\alpha_2 \end{cases}, \\ &\Leftrightarrow \alpha_2 = -\alpha_1. \end{aligned}$$

One can thus pick the normalized state $|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$. Similarly one can show that we can use $|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. It follows that $|\psi(0)\rangle = |\phi_2\rangle$.

Now that $|\psi(0)\rangle$ is expanded on a basis of eigenvectors of H, and we can use our previous result to compute the evolution in time of $|\psi(t)\rangle$, it is given by,

$$|\psi(t)\rangle = e^{-i(\varepsilon_0 - V)t/\hbar} |\phi_2\rangle .$$

Now imagine we were to measure the energy of the system at any time t, we would necessarily get an eigenvalue of H, that is to say either, $\varepsilon_0 - V$ or $\varepsilon_0 + V$. Let us compute the probability to measure either of these energies at any time t.

$$P_{\varepsilon_0 - V}(t) = |\langle \phi_2 | \psi(t) \rangle|^2 = 1,$$

so that necessarily,

$$P_{\varepsilon_0 + V}(t) = 0.$$

It means that we know exactly the energy of the system at any time t. This purely comes from the fact that the system at time $t = 0$ was prepared in an eigenstate of H.

Conclusion

In drawing our discourse on quantum mechanics to a close, we find ourselves standing at the threshold of an intricate and fascinating realm. The postulates we've scrutinized constitute the foundational principles that govern the behavior of matter at the quantum scale.
This analytical exploration sought to shed light on the theoretical underpinnings of the quantum landscape, revealing a complex and nuanced reality that defies classical intuition.
As we bring this intellectual journey to a conclusion, it is evident that the postulates we've dissected are not mere theoretical constructs but rather the guiding principles shaping our understanding of the microcosmic universe.
This formal reflection underscores the continuous pursuit of knowledge, emphasizing the importance of grappling with the mysteries inherent in the quantum world. As we close this chapter, let us recognize that our comprehension of quantum mechanics remains a work in progress, an evolving narrative that invites further exploration and scholarly inquiry.

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