The Path Integral Formalism

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Avant propos

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An *easy* introduction

In this section expose the basics of the path integral formalism. This section is somewhat out of the scope of this note. In fact, the reader more aquainted to mathematical formalism could probably start directly from the next section. Though, in this section we expose the main idea that led to the path integral formalism and the basics of the mathematical formalism. With that being said, the readers who only wish to grasp the main ideas of the topic could stick to this section only and still get a feeling for its applications to quantum mechanics. However, for more advanced applications such as quantum field theory, one would probably have to go through the maths to catch on everything.

1 Path integral unleashed

Contrary to the previous section, this section is specifically designed to describe the path integral formalism as precisely as possible. Thus, the reader should be prepared to go through nasty computations and non trivial mathematical material. Disclaimers said, it should still be mentioned that going through this whole section precisely is extremely rewarding for the understanding of Quantum Mechanics and more advanced topics in quantum field theory, particle physics and condensed matter physics (and, as far as I can tell, pretty much any physics fields !).

This section is organised as follow. First we will give a few reminders of the canonical formalism. In fact, the path integral is more easily introduced with the canonical formalism kept in mind. Note that it could be somewhat derived dubiously and be given as a machinery to derive Feynman rules. However, the point of view we will adpot here is to make the introduction of the path integral as natural as possible. Hence we will start from quantum mechanics and will only have to take a small leap to apply our results to quantum field theory thanks to the canonical formalism.

We will mostly be walking in Weinberg's steps in this section.

1.1 A few reminders on canonical formalism

1.2 Path integral for bosonic particles

As previously stated our starting point will be the canonical formalism which provided us with a set of operator Q_a associated with their conjugate momenta P_a such that,

$$[Q_a, P_b]_- = i\delta_{ab} \tag{1}$$

$$[Q_a, Q_b]_- = [P_a, P_b]_- = 0 \tag{2}$$

For now we restrict ourselves to variables satisfying commutation relations rather than anti-commutation relations, as indicated by the subscript $-$. With that in mind, we are describing bosonic particles.

As the Q_a all commutes between each other, we can diagonalize them in a single basis that we will denote $|q\rangle$, such that

$$Q_a |q\rangle = q_a |q\rangle \tag{3}$$

where we've denoted $\{q_a\}$ the eignevalues of the Q_a. Furthermore, we can take these eigenvectors to be orthonormal (because the Q_a are hermitian), that is to say,

$$\langle q'|q\rangle = \prod_a \delta(q'_a - q_a) \tag{4}$$

With this normalization the completeness relation reads,

$$1 = \int \prod_a \mathrm{d}q_a \,|q\rangle\,\langle q| \tag{5}$$

The same argument applies for the set of operators P_a and we build a basis denoted $|p\rangle$, such that,

$$P_a\,|p\rangle = p_a\,|p\rangle \tag{6}$$

$$\langle p'|p\rangle = \prod_a \delta(p'_a - p_a) \tag{7}$$

$$1 = \int \prod_a \mathrm{d}p_a \,|p\rangle\,\langle p| \tag{8}$$

As in quantum mechanics, it is fairly easy to show that this basis can be decomposed one into the other in the way,

$$\langle q|p\rangle = \prod_a \frac{e^{ip_a q_a}}{\sqrt{2\pi}} \tag{9}$$

As is usually the case, we settle in the Heisenberg Picture of quantum mechanics and so the operators Q_a and P_a evolve at a time t following the equations,

$$Q_a(t) = \exp(iHt)Q_a\exp(-iHt) \tag{10}$$

$$P_a(t) = \exp(iHt)P_a\exp(-iHt) \tag{11}$$

We can then denote their eigenstates by

$$|q;t\rangle = \exp(iHt)\,|q\rangle \tag{12}$$

$$|p;t\rangle = \exp(iHt)\,|p\rangle \tag{13}$$

These states don't correpond to the states $|q\rangle$ and $|p\rangle$, eveloved to a further time t in the Schrödinger picture. These states still obviously satisfty orthnormality and completeness. The scalar product $\langle q;t|p;t\rangle$ also remains unchanged compared to the scalar product $\langle q|p\rangle$.

We now focus on computing the quantity $\langle q';t'|q;t\rangle$, that is to say the amplitude to find the system in state $|q';t'\rangle$ at time t', knowing that it was in state $|q;t\rangle$ at time t.

In order to do that we first compute the infinitesimal [1] amplitude

1. This shouldn't be understood as if this quantity were small. It wouldn't even make sens as it's a complexe number ! By *infinitesimal* we really mean that it's the amplitude over an infinitesimal amount of time.

1.3 Quelques rappels du formalisme canonique

1.4 L'intégrale de chemin pour les bosons

1.5 L'intégrale de chemin pour les fermions

2 Théorie quantique des champs : diagrammes et règles de Feynman

Maintenant que nous avons posé le cadre de notre étude, nous étudions comment retrouver l'approche diagrammatique de Feynman à partir du formalisme de l'intégrale de chemin. Pour ce faire nous modifions astucieusement notre Lagrangien en ajoutant des termes de source,

$$\mathcal{L}' = \mathcal{L} + J(x)\phi(x) \tag{14}$$

où $\mathcal{L}$ correspond au Lagrangien complet de notre théorie, à savoir le Lagrangien libre et le Lagrangien d'interaction. Par exemple, avec un seul champ scalaire ϕ et une théorie ϕ^4 on a typiquement,

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] - \frac{\lambda}{4!}\phi^4 \tag{15}$$

Les termes J correspondent à des termes de source permettant de créer où annihiler des particules ϕ. Ils sont inspirés astucieusement de la physique statistique. En physique statistique, la quantité d'intérêt est proportionnelle à la mesure $\int \mathrm{d}E e^{-\beta E}$ et on peut en retrouver tous ses moments en dérivant par rapport au paramètre β. L'idée sera ici la même où l'on utiliser une *dérivée fonctionnelle* pour dériver par rapport à $J(x)$ pour obtenir les différents propagateurs.

Ainsi l'intégrale de chemin qui nous intéresse réellement est plutôt

$$Z(J) = \int D\varphi \; e^{\frac{i}{\hbar}\int \mathrm{d}^d x [\mathcal{L} + J(x)\phi(x)]} \tag{16}$$

3 Propagateurs et valeurs moyennes

4 Symétries

5 Physique statistique

5.1 Rotations de Wick

5.2 Physique statistique en dimension $d + 1$