Introduction to ϕ^4 theory

Quantum Field Theory

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Introduction and Outline

1. Feynman rules for ϕ^4 theory
2. Regularization
3. Renormalization

I. Feynman rules for ϕ^4 theory

Lagrangian

- Consider the Lagrangian density for a real scalar field ϕ :

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

- Equations of motion:

$$(\Box + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \tag{2}$$

- In the interaction picture, the field operator is given by:

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(a_{\mathbf{p}}e^{-ipx} + a_{\mathbf{p}}^{\dagger}e^{ipx}\right) \tag{3}$$

- One computes amplitude using the Dyson formula for the S-matrix:

$$S = T\exp\left(-i\int d^4x\mathcal{H}_I(x)\right) \tag{4}$$

- with the interaction Hamiltonian density:

$$\mathcal{H}_I(x) = \frac{\lambda}{4!}\phi^4(x) \tag{5}$$

Example of amplitude

- Initial state $|p\rangle$, final state $|q\rangle$
- Compute at first order in λ, $\langle q| S |p\rangle$
- Wick's theorem yields two kinds of terms,

$$\langle q|\phi\phi\phi\phi|p\rangle$$

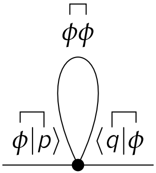
$$\langle q|\phi\phi\phi\phi|p\rangle$$

- The different contractions are given by:
  - $\phi(x)\phi(y) = \Delta_F(x-y)$
  - $\phi(x)|p\rangle = e^{-ipx}$
  - $\langle q|\phi(x) = e^{+iqx}$
  - $\langle q|p\rangle = (2\pi)^3 2E_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{q})$

Feynman diagrams

- To each term in the Wick's expansion can be associated a Feynman diagram
- At each spacetime point $x, y, z, \ldots$ we associate a vertices
- Contractions between fields are represented by lines joining the vertex at which the fields are attached
- Contractions with creation and annihilation operators join external lines to vertices

Example of a diagram at first order in λ:$\langle q|\phi\phi\phi\phi|p\rangle$



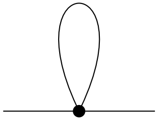
Feynman rules

- The other way around, to each diagram is associated an amplitude $\mathcal{M} \in \mathbb{C}$
- This amplitude is computed with Feynman rules :
  1. Each vertex comes with a factor $(-i\lambda)$ and a Dirac delta $(2\pi)^4\delta^{(4)}(p_{in}^v - p_{out}^v)$
  2. Each internal line corresponds to a propagator $\frac{i}{p^2-m^2+i\epsilon}$
  3. The internal lines are integrated over all 4-momenta with measure $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
  4. There is an overall Dirac delta conservation $(2\pi)^4\delta^{(4)}(p_{in} - p_{out})$
  5. Divide by the symmetry factor for each diagram

Symmetry factors

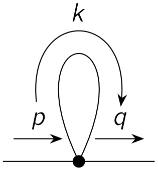
- In the Lagrangian the coupling is $\frac{\lambda}{4!}$ but we use λ in the Feynman rules
- For example, in the Dyson expansion, the term $\langle q|\phi\phi\phi\phi|p\rangle$ can be obtained by 12 different contractions
- The symmetry factor is then,

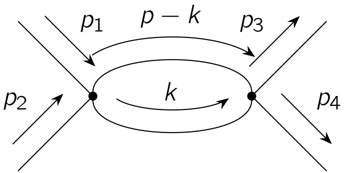
$$D^{-1} = \frac{12}{4!} = \frac{1}{2} \quad \text{for}$$



- Three important rules
  1. Every propagator joined to the same vertex $D_i = 2$
  2. Every pair of vertices joined by n propagators, $D_i = n!$
  3. $D = \prod_i D_i$

Two interesting diagrams





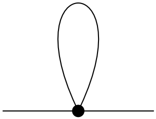
$$\mathcal{M} = \frac{(-i\lambda)}{2}(2\pi)^4\delta^{(4)}(p-q) \times \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}$$

$$\mathcal{M} = \frac{(-i\lambda)^2}{2}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4) \times \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} \times \frac{i}{(p-k)^2-m^2+i\epsilon}$$

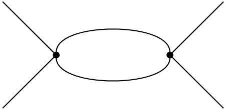
II. Regularization

Divergences

The two previous diagrams diverge, the integral over k is not convergent.



$$\longrightarrow \quad \mathcal{M} \sim \int \frac{\mathrm{d}^4 k}{k^2} \sim \int k \mathrm{d}k$$



$$\longrightarrow \quad \mathcal{M} \sim \int \frac{\mathrm{d}^4 k}{k^4} \sim \int \frac{\mathrm{d}k}{k}$$

- Both amplitudes diverge
- What to do with them ?
- Are there other divergent diagrams ?

Degree of divergence

Only a few diagrams diverge, can be seen by computing the superficial degree of divergence:

- The superficial degree of divergence is given by:

$$D = 4L - 2I$$

- where L is the number of loops and I is the number of internal lines
- *Euler's theorem* states that

$$L = I - V + 1$$

- Each vertex is attached to 4 lines, internal line are attached twice and external lines once,

$$4V = 2I + B_E$$

$$D = 4 - B_E$$

- Converge if $D > 0$
- Diagrams with more than 4 external lines converge, only need to regularize the two previous ones

Regularization

- Regularize the theory by introducing a cutoff Λ

  "Cutting off our ignorance"

- All integrals are now over $d^4 p/(2\pi)^4$ with $|p| < \Lambda$
- The cutoff is removed at the end of the calculation by taking the limit $\Lambda \to \infty$
- Other possible regularization:
  - *Pauli-Villars regularization* :
  - *Dimensional regularization* :
  - *Lattice regularization* :
- What does it mean physically ?

III. Renormalization

Meson-Meson scattering

- The amplitude is given by:

$$\mathcal{M} = -i\lambda + \frac{1}{2}(-i\lambda)^2 \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - m^2 + i\epsilon}$$

- That can be computed to be equal to, (s, t, u are Mandelstam vasriables, see Backup)

$$\mathcal{M} = -i\lambda + iC\lambda^2 \left[\ln\left(\frac{\Lambda^2}{s}\right) + \ln\left(\frac{\Lambda^2}{t}\right) + \ln\left(\frac{\Lambda^2}{u}\right) \right]$$

- Depends on λ and Λ, which are both non-physical, $\mathcal{M}$ cannot depend on any of them, what if we change Λ ?
- Make λ change accordingly so $\mathcal{M}$ does not change,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\Lambda} = 6\lambda^2 C$$

Measuring physical quantities

- An experimentalist studying meson-meson scattering measures, $\mathcal{M} = -i\lambda_P$ at given energy s_0, t_0, u_0
- It means that,

$$-i\lambda = -i\lambda_P - iC\lambda_P^2 \left[\ln\left(\frac{\Lambda^2}{s_0}\right) + \ln\left(\frac{\Lambda^2}{t_0}\right) + \ln\left(\frac{\Lambda^2}{u_0}\right) \right]$$

- So that,

$$\mathcal{M} = -i\lambda_P + iC\lambda_P^2 \left[\ln\left(\frac{s_0}{s}\right) + \ln\left(\frac{t_0}{t}\right) + \ln\left(\frac{u_0}{u}\right) \right]$$

- It does not depend on Λ anymore, only on the measurable physical constant λ_P

Renormalization

- Express amplitudes in terms of physical quantities
- Parameters in the Lagrangian are infinite and non-physical, they are expressed as the sum of two terms,

$$\beta = \beta_P + \delta\beta$$

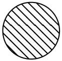
- For our theory, mass and coupling constant are renormalized as,

$$\begin{aligned} m =& m_P + \delta m \\ \lambda =& \lambda_P + \delta\lambda \end{aligned}$$

Diagrammatic interpretation : Propagator

- 1PI diagrams : cannot be separated into two disconnected diagrams by cutting a single line
- 1PI self-energy :

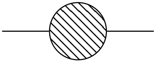
$$-i\Sigma(p) = \sum \begin{pmatrix} \text{All 1PI diagrams} \\ \text{with 2 external legs} \end{pmatrix} =$$



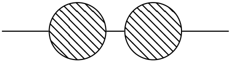
- True propagator:

$$G_P(p) =$$

$$+$$



$$+$$



$$+ \cdots$$

$$= G_0(p) + G_0(p)(-i\Sigma(p))\, G_0(p) + \ldots$$

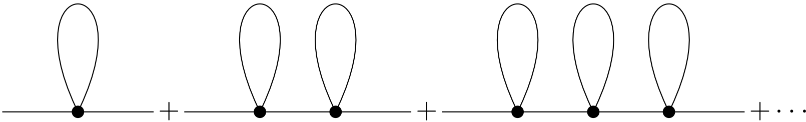
Hence,

$$G_P(p) = \frac{1}{p^2 - m^2 - \Sigma(p) + i\epsilon}$$

Compute $\Sigma(p)$ in orders of λ

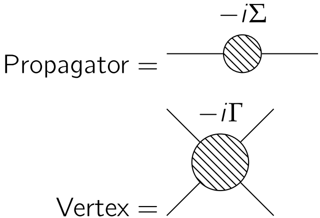
At first order, in λ,

$$-i\Sigma^{(1)}(p) =$$ 

$$G_P^{(1)} =$$ 

What needs to be renormalized ?

- Power counting tells us that only diagrams with 2 and 4 external legs need to be renormalized
- These correspond to vertex function and self energy,



Conclusion 1/2

- ϕ^4 theory is a simple theory from which we learn many things
- Toy model for deriving Feynman rules and computing amplitudes
- The presence of diverging diagrams enforces a regularization procedure followed by renormalization
- Physical parameters are the result of substracting two infinite quantities
- Parameter λ is not physical, it depends on the cutoff Λ
- General for most theories in QFT, can be studied through the renormalization group

Conclusion 2/2

- What is ϕ^4 in real life ?
- It is the Higgs bosons field !
- One can compute (in a more involved way) its mass renormalization
- Hierarchy problem
- Supersymmetry ? (probably not imo)

Backup

Contractions

- In ϕ^4 theory, one will need to compute three kinds of contractions
  - $\phi(x)\phi(y) = \Delta_F(x-y)$
  - $\phi(x)|p\rangle = \langle 0|\,\phi(x)a^\dagger_{\mathbf{p}}\,|0\rangle = e^{-ipx}$
  - $\langle p|\phi(x) = \langle 0|\,a_{\mathbf{p}}\phi(x)\,|0\rangle = e^{+ipx}$
- These can be shown using the Fourier transform of the field operator

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx}\right)$$

- The first one comes from the fact that $\Delta_F(x-y) = \langle 0|\,T\phi(x)\phi(y)\,|0\rangle = [\phi_-(x), \phi_+(y)]$
- The other ones are easy computations

Euler's theorem

Euler's theorem

For a connected graph, Euler's theorem holds, that is to say

$$V - E + L = 1 \tag{6}$$

where V is the number of vertices, E the number of edges and L the number of loops.

Proof

Suppose we have a graph with n vertices. If it's a tree, the statement is trivial. Otherwise, there must be some cycle. Let us remove one edge from such a cycle. The new graph, now has $n-1$ edges and n vertices and $n-1$ loops. We can repeat this operation until we have a tree and we see that the quantity, $L-E$ is invariant under this operation (after k steps). At the end of the procedure, we obtain a tree $G=(E,V')$ $(L'=0)$ for which the result is obvious, $V-E'=1$. Since $E'=E-L$, one obtains the desired result.

Other regularizations

- Pauli-Villars : Replace

$$\frac{i}{p^2 - m^2 + i\epsilon} \to \frac{i}{p^2 - m^2 + i\epsilon} - \frac{i}{p^2 - M^2 + i\epsilon}$$

  The integral goes as $|k|^{-4}$ when $M \ll |k|$ and is convergent. The theory is recovered in the limite $M \to \infty$

- Lattice regularization : Replace spacetime by a lattice of parameter a making momentum quantized and discrete of order a^{-1}. The theory is recovered in the limit $a \to 0$

- Dimensional regularization : It is based on the fact that,

$$\int_0^\infty \frac{k^{D-1}\mathrm{d}k}{[k^2 + \Delta^2]^2} = \frac{1}{2}\Delta^{n-D/2}\frac{\Gamma(n - D/2)\Gamma(D/2)}{\Gamma(n)}$$

  is well-defined when D is not even. One then does the computation with $D = 4 - 2\varepsilon$ and recovers the theory when $\varepsilon \to 0$.

Mandelstam variables

- In the case of meson-meson scattering, the Mandelstam variables are defined as

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

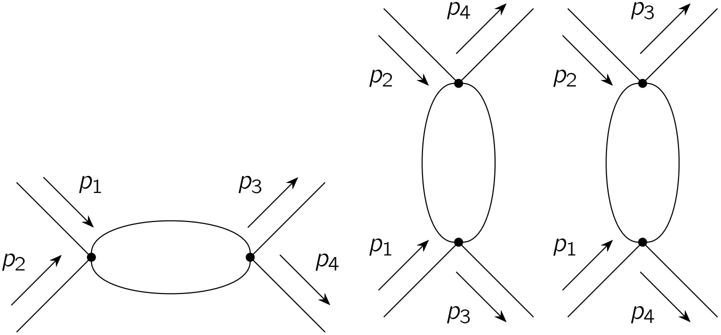
- They are related by

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

- They are Lorentz invariant, s, t, u are the same in all frames
- They are useful to describe the kinematics of a scattering process

Total amplitude

The total amplitude at order 2 in λ $\mathcal{M}^{(2)}$ comes from the 3 diagrams :



$$\mathcal{M}^{(2)} = iC\lambda^2 \left[\ln\left(\frac{\Lambda^2}{s}\right) + \ln\left(\frac{\Lambda^2}{t}\right) + \ln\left(\frac{\Lambda^2}{u}\right) \right]$$

λ as a function of λ_P

Denote $L(s,t,u) = \ln\left(\frac{\Lambda^2}{s}\right) + \ln\left(\frac{\Lambda^2}{t}\right) + \ln\left(\frac{\Lambda^2}{u}\right)$. Since $\mathcal{M} = -i\lambda_P = -i\lambda + iC\lambda^2 L(s_0,t_0,u_0) + O(\lambda^3)$, one gets,

$$\begin{aligned}
-i\lambda &= -i\lambda_P - iC\lambda^2 L(s_0,t_0,u_0) + O(\lambda^3) \\
&= -i\lambda - iC\lambda_P^2 L(s_0,t_0,u_0) + O(\lambda_p^3)
\end{aligned}$$

λ as a function of Λ

Let us start from $\mathcal{M}$ at order λ^2:

$$\mathcal{M} = -i\lambda + iC\lambda^2 L(\Lambda) \tag{7}$$

We want to make $\mathcal{M}$ independent of Λ. We can do that by changing λ to $\lambda(\Lambda)$. We find the evolution of λ using the fact that $\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}\ln\Lambda} = 0$. We find

$$\begin{aligned}
\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\Lambda} &= 2C\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\Lambda}\lambda L(\Lambda) + 6C\lambda^2 \\
&= 2\lambda C\left[2C\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\Lambda}\lambda L(\Lambda) + 6C\lambda^2\right] + 6C\lambda^2
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\Lambda} = 6\lambda^2 C$$

Computing $G_P(p)$

$$
\begin{aligned}
G_P(p) &= G_0(p) + G_0(p)(-i\Sigma(p))\,G_0(p) + \ldots \\
&= G_0(p) \sum_{n>0} (-i\Sigma(p)\,G_0(p))^n \\
&= G_0(p) \frac{1}{1 + i\Sigma(p)\,G_0(p)}
\end{aligned}
$$

with $G_0(p) = \frac{i}{p^2 - m^2 + i\epsilon}$, we find

$$
G_P(p) = \frac{i}{p^2 - m^2 - \Sigma(p) + i\epsilon} \tag{8}
$$

Vertex renormalization

Vertex renormalization goes like propagator renormalization.

$$-i\Gamma(s_0, t_0, u_0) = \sum \begin{pmatrix} \text{All connected diagrams} \\ \text{with 4 external legs} \end{pmatrix}$$

To first order in λ,

$$\begin{aligned} -i\Gamma^{(1)}(s_0, t_0, u_0) &= -i\lambda_P \\ &= -i\lambda + iC\lambda^2 \left[\ln\left(\frac{\Lambda^2}{s_0}\right) + \ln\left(\frac{\Lambda^2}{t_0}\right) + \ln\left(\frac{\Lambda^2}{u_0}\right) \right] \end{aligned}$$