Symmetries in canonical Quantum Field Theory

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May 2, 2024

Foreword

This short note is designed to sum up the formalism under which symmetries are considered in the canonical formulation of Quantum Field Theory. It is widely inspired from Weinberg's discussion of symmetries in *The Quantum Theory of Fields : Foundations*.

In all of the document we use Einstein's summation convention, that is, we sum over repeated indices. All indices don't range over the same values, we summarize them here,

- $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$ are spacetime indices
- $i, j, k = 1, 2, 3$ are spatial indices
- $l, m, n, p = 1, 2, 3, \cdots$ are indices that run over the degrees of freedom of the fields
- $\alpha, \beta, \gamma, \delta = 0, 1$ are indices that run over the degrees of freedom of the fields in the string theory context ($\alpha = \tau, \sigma$)

We shall sometimes use the notation ∂_μ or ∂_α as a shorthand notation for $\frac{\partial}{\partial x^\mu}$ or $\frac{\partial}{\partial x^\alpha}$ respectively.

1 Lagrangian formalism

Let us first recall a few important features of the Lagrangian formalism. The Lagrangian is a functional

$$L[q(t), \dot{q}(t)]$$

It depends on generalized coordinates q_i, that we denoted simply with q in the above notation, that themselves evolve in time. These can be the positions along the three Cartesian axis for a single particle system. When the number of degree of freedom increases we can obtain a continuum so that every space point $\mathbf{x}$ has a degree of freedom. This is exaclty the paradigm of QFT, where the q_i becomes fields $\Psi^l(x)$. This way the Lagrangian reads,

$$L[\Psi(t), \dot{\Psi}(t)] \tag{1}$$

where the notation $\Psi(t)$ mean that the Lagrangian can n the values of the fields Ψ^l at any point of space $\mathbf{x}$ at time t.

The Lagrangian is not yet the main character of the story. In fact, the more fundamental quantity is the *action* defined as,

$$S[\Psi(t), \dot{\Psi}(t)] = \int \mathrm{d}t L[\Psi(t), \dot{\Psi}(t)] \tag{2}$$

Classically, the field configurations that are realised in nature (without considering the quantum fluctuations that shall be taken into account in the path integral formalism), are those that extremizes the action, that is,

$$\delta_{\delta\Psi} S \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(S[\Psi + \varepsilon \delta\Psi] - S[\Psi] \right) = 0 \tag{3}$$

We drop the subscript $\delta\Psi$ in the following for the sake of clarity.

This extremization is reflected on the Lagrangian through,

$$\begin{aligned}
\delta S[\Psi] &= \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left[\frac{\delta L}{\delta \Psi^l(x)} \delta\Psi^l(x) + \frac{\delta L}{\delta \dot{\Psi}^l(x)} \delta\dot{\Psi}^l(x) \right] \\
&= \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left[\frac{\delta L}{\delta \Psi^l(x)} \delta\Psi^l(x) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\delta L}{\delta \dot{\Psi}^l(x)} \right) \delta\Psi^l(x) \right] + \left[\int \mathrm{d}^3\mathbf{x} \frac{\delta L}{\delta \dot{\Psi}^l(x)} \delta\Psi^l(x) \right]_{t=-\infty}^{t=+\infty}
\end{aligned}$$

where we have integrated by parts the time derivative term. Making the boundary terms vanish (*e.g* by imposing that the field variation vaishes at $|t| = \infty$), we obtain the variation of the action,

$$\delta S[\Psi] = \int \mathrm{d}^4 x \left[\frac{\delta L}{\delta \Psi^l(x)} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\delta L}{\delta \dot{\Psi}^l(x)} \right) \right] \delta\Psi^l(x) \tag{4}$$

This must vanish whatever the variation $\delta\Psi^l$ is. This in turn implies the Euler Lagrange equations,

$$\frac{\delta L}{\delta \Psi^l(x)} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\delta L}{\delta \dot{\Psi}^l(x)} \right) = 0 \tag{5}$$

In most cases, one can go even further when the Lagrangian is obtained from a *Lagrangian density*,

$$L[\Psi(t), \dot{\Psi}(t)] = \int \mathrm{d}^3\mathbf{x} \mathcal{L}(\Psi^l(x), \partial_\mu \Psi^l(x)) \tag{6}$$

In such cases, one can compute the variation on the Lagrangian in the following fashion,

$$\delta L = \int \mathrm{d}^3\mathbf{x} \left[\frac{\partial \mathcal{L}}{\partial \Psi^l} \delta\Psi^l(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l)} \partial_\mu \delta\Psi^l(x) \right]$$

Splitting the derivatives into spatial and time derivative, one writes,

$$\begin{aligned}
\delta L &= \int \mathrm{d}^3\mathbf{x} \left[\frac{\partial \mathcal{L}}{\partial \Psi^l(x)} \delta\Psi^l(x) + \frac{\partial \mathcal{L}}{\partial(\nabla \Psi^l(x))} \delta\nabla\Psi^l(x) + \frac{\partial \mathcal{L}}{\partial(\dot{\Psi}^l(x))} \delta\dot{\Psi}^l(x) \right] \\
&= \int \mathrm{d}^3\mathbf{x} \left[\left\{ \frac{\partial \mathcal{L}}{\partial \Psi^l(x)} - \nabla \frac{\partial \mathcal{L}}{\partial(\nabla \Psi^l(x))} \right\} \delta\Psi^l(x) + \frac{\partial \mathcal{L}}{\partial(\dot{\Psi}^l(x))} \delta\dot{\Psi}^l(x) \right]
\end{aligned}$$

where we once again integrated by part and dropped the boundary terms. This short calculation proves that,

$$\frac{\delta L}{\delta \Psi^l(x)} = \frac{\partial \mathcal{L}}{\partial \Psi^l(x)} - \nabla \frac{\partial \mathcal{L}}{\partial(\nabla \Psi^l(x))} \quad \text{and} \quad \frac{\delta L}{\delta \dot{\Psi}^l(x)} = \frac{\partial \mathcal{L}}{\partial(\dot{\Psi}^l(x))}$$

This way, the Euler Lagrange equations (5) can be written as,

$$\frac{\partial \mathcal{L}}{\partial \Psi^l(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l(x))} = 0 \tag{7}$$

where we've gathered back the spatial and time derivatives into a single notation ∂_μ to have a Lorentz invariant form of the equations.

2 Symmetries

In this section we define symmetries and study them in three different frameworks, depending on the system under study and more precisely on the presence of an action, a Lagrangian or a Lagrangian density.

2.1 Symmetries of the action

Formally, a symmetry of the action is a transformation of the fields that leaves the action invariant. Let us introduce a few notations. Let us call $\mathcal{F}$ the space of fields configuration. The actions is then seen as a map,

$$S : \mathcal{F} \to \mathbb{R}$$

With these notations, a symmetry is a map f,

$$f : \mathcal{F} \to \mathcal{F}$$

such that the action is invariant under it,

$$(f^* S)[\Psi] \equiv S[f(\Psi)] = S[\Psi] \tag{8}$$

where $f^* S$ is the pullback of the action under the transformation f.

More specifically we will be interested in the infinitesimally local symmetries,

$$f(\Psi) = \Psi^l + i\varepsilon \mathcal{G}^l$$

where G^l is also a field and ε is a small constant parameter. The action is invariant under this transformation if,

$$\delta S[\Psi] = 0 \tag{9}$$

even when the fields don't extremize the action (otherwise the statement is abvious following the arguments of the previous section).

If one considers ε as a continuous parameter, one would expect the variation of the action to be in the form,

$$\delta S = -\int \mathrm{d}^4 x J^\mu(x) \partial_\mu \varepsilon$$

where the minus sign is conventional. In fact, this yields, $\delta S = 0$ when ε is a constant.

Working again with a non constant, infinitesimal parameter $\varepsilon(x)$ and supposing that the fields follow the equations of motion, one can write,

$$0 = \delta S = -\int \mathrm{d}^4 x J^\mu(x) \partial_\mu \varepsilon(x) = \int \mathrm{d}^4 x \partial_\mu J^\mu(x) \varepsilon(x)$$

This must be true whatever $\varepsilon(x)$ is, so that the integrand must vanish,

$$\partial_\mu J^\mu(x) = 0 \tag{10}$$

This is the *current conservation* equation.

The statement is usually known as *Noether's theorem*. It states that for every continuous symmetry of the action, there is a current that is conserved when the equations of motion are verified.
More interestingly, we have access to a conserved charge, that is a quantity Q such that $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$. This charge is defined as,

$$Q = \int \mathrm{d}^3\mathbf{x} J^0(x) \tag{11}$$

where $J^0(x)$ is the time component of the current.
Let us verify that this quantity is in fact conserved,

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\mathbf{x} J^0(x) \\
&= \int \mathrm{d}^3\mathbf{x} \partial_0 J^0(x) \\
&= -\int \mathrm{d}^3\mathbf{x} \partial_i J^i(x) \\
&= -\int_{S^\infty} \mathrm{d}S\mathbf{n} \cdot J^i(x) \\
\frac{\mathrm{d}Q}{\mathrm{d}t} &= 0
\end{aligned}$$

where we used the current conservation equation (10) in third line, Stokes theorem in the fourth line and assumed that the current vanished on the sphere at infinity S^∞.

2.2 Symmetries of the Lagrangian

The symmetries of the Lagrangian are a bit more subtle but provide more information. The Lagrangian is a functional of the fields and their time derivatives. The symmetries of the Lagrangian are defined as the transformations of the fields that leave the Lagrangian invariant,

$$\delta L[\Psi, \dot{\Psi}] = 0 \tag{12}$$

An actual symmetry of the system is only a symmetry of the action, and so one needs a weaker condition,

$$\delta L[\Psi, \dot{\Psi}] = \varepsilon \frac{\mathrm{d}G}{\mathrm{d}t} \tag{13}$$

where G is a functional of the fields and their time derivatives.
Let us check that this is enough to have a symmetry of the action. Under such a transformation,

$$\begin{aligned}
\delta S &= \int \mathrm{d}t \delta L \\
&= \varepsilon \int \mathrm{d}t \frac{\mathrm{d}G}{\mathrm{d}t} \\
&= [G(t)]_{t=-\infty}^{t=+\infty} \\
&= 0
\end{aligned}$$

where we once again assumed that the boundary terms vanish.

Let us now express variation of the action under a transformation

$$\Psi^l(x) \to \Psi^l(x) + i\varepsilon(t)\mathcal{G}^l(x)$$

where we made the parameter ε depend on time t but not on spatial coordinates $\mathbf{x}$. Ultimately, we shall consider variation of the actions when ε is constant, but making it a function of t will help us for now. The variation of the action reads,

$$\delta S = i\int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left[\frac{\delta L}{\delta\Psi^l(x)}\varepsilon(t)\mathcal{G}^l(x) + \frac{\delta L}{\delta\dot{\Psi}^l(x)}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\varepsilon(t)\mathcal{G}^l(x)\Big)\right]$$

The *invariance* of the Lagrangian for a constant ε reads,

$$\varepsilon\frac{\mathrm{d}G}{\mathrm{d}t} = \delta L = i\int \mathrm{d}^3\mathbf{x}\left[\frac{\delta L}{\delta\Psi^l(x)}\varepsilon\mathcal{G}^l(x) + \frac{\delta L}{\delta\dot{\Psi}^l(x)}\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}^l(x)\right]$$

which can be recast as,

$$\frac{\mathrm{d}G}{\mathrm{d}t} = i\int \mathrm{d}^3\mathbf{x}\left[\frac{\delta L}{\delta\Psi^l(x)}\mathcal{G}^l(x) + \frac{\delta L}{\delta\dot{\Psi}^l(x)}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}^l(x)\right] \tag{14}$$

whence the variation of the action becomes,

$$\delta S = \int \mathrm{d}t \left[\varepsilon(t)\frac{\mathrm{d}G}{\mathrm{d}t} + i\int \mathrm{d}^3\mathbf{x}\frac{\delta L}{\delta\dot{\Psi}^l(x)}\mathcal{G}^l(x)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right]$$

After an n-th integration by parts dropping boundary terms, one gets,

$$\delta S = \int \mathrm{d}t \left[i\int \mathrm{d}^3\mathbf{x}\frac{\delta L}{\delta\dot{\Psi}^l(x)}\mathcal{G}^l(x) - G(t)\right]\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{15}$$

This directly gives us the expression for the time component of the current,

$$J^0(x) = -i\frac{\delta L}{\delta\dot{\Psi}^l(x)}\mathcal{G}^l(x) + \delta^{(3)}(\mathbf{x})G(t) \tag{16}$$

Knowing the variation of the Lagrangian, we then get access to the expression of the time component of the current. One can check that the charge,

$$Q = \int \mathrm{d}^3\mathbf{x}J^0(x) = \int \mathrm{d}^3\mathbf{x}\left[-i\frac{\delta L}{\delta\dot{\Psi}^l(x)}\mathcal{G}^l(x) + \delta^{(3)}(\mathbf{x})G(t)\right]$$

is conserved using the equation of motions (5) and the invariance of the Lagrangian (14).

2.3 Symmetries of the Lagrangian density

It is even more interesting when one has a symmetry of the Lagrangian density. Similarly, a symmetry of the Lagrangian density is not only an invariance, but rather a transformation of the fields such that,

$$\delta\mathcal{L}(\Psi(x), \partial_\mu\Psi(x)) = \varepsilon\partial_\mu G^\mu$$

This ensures that the action is invariant under the transformation (one can check this in the previous fashion).

Let us now consider a transformation of the fields,

$$\Psi^l(x) \to \Psi^l(x) + i\varepsilon(x)\mathcal{G}^l(x)$$

where we now make ε vary along every spacetime coordinate. Yet again, ultimately we shall consider the case where ε is constant. The variation of the action reads,

$$\delta S = i \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \Psi^l(x)} \varepsilon(x)\mathcal{G}^l(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l(x))} \partial_\mu \left(\varepsilon(x)\mathcal{G}^l(x) \right) \right]$$

For a constant ε, the invariance of the Lagrangian reads,

$$\varepsilon \partial_\mu G^\mu = \delta\mathcal{L} = i \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \Psi^l(x)} \varepsilon\mathcal{G}^l(x) + \varepsilon \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l(x))} \partial_\mu \mathcal{G}^l(x) \right]$$

That can be recast as,

$$\partial_\mu G^\mu = i \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \Psi^l(x)} \mathcal{G}^l(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l(x))} \partial_\mu \mathcal{G}^l(x) \right] \tag{17}$$

Using this relation, one can rewrite the variation of the action as,

$$\delta S = \int \mathrm{d}^4x \left[i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l(x))} \mathcal{G}^l(x) - G^\mu(x) \right] \partial_\mu \varepsilon(x) \tag{18}$$

This gives us the expression of the current,

$$J^\mu(x) = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l(x))} \mathcal{G}^l(x) + G^\mu(x) \tag{19}$$

It is easy to input the shape of J^0 in Q and check that it is conserved.

2.4 Summary table

Let us summarize the results of the previous sections in the following table,

Symmetry	Variation	Quantity	Formula
Symmetry of S	$\delta S = 0$	Charge Q	$Q = \int \mathrm{d}^3\mathbf{x} J^0$
Symmetry of L	$\delta L = \mathrm{d}_t G$	Time component of the current J^0	$J^0 = -i \frac{\delta L}{\delta \Psi^l(x)} \mathcal{G}^l(x) + \delta^{(3)}(\mathbf{x}) G(t)$
Symmetry of $\mathcal{L}$	$\delta\mathcal{L} = \partial_\mu G^\mu$	Current density J^μ	$J^\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^l(x))} \mathcal{G}^l(x) + G^\mu(x)$

3 An aside in String Theory

This section is somewhat out of the scope of this note as it tackles the field of string theory. It is just interesting to see how it goes when one deals with space variables depending on two parameters.

3.1 A quick introduction to string theory

String theory is just the statement that the trajectories of particles are not determined by a single parameter τ (*e.g* the proper time) but by two parameters σ and τ. The string is then a two-dimensional object that evolves in a D-dimensional spacetime (in fact it is no longer possible to assume only a 4-dimensional spacetime for the theory to be consistent). One does not talk anymore about a wordline parametrized by τ but rather about a worldsheet parametrized by (τ, σ). τ keeps the meaning of a proper time for the particle whereas σ manifests itself as an internal degree of freedom that we usually parametrized over $[0, \pi]$.

The string is parametrized by the coordinates $X^\mu(\sigma, \tau)$ where μ runs over the D dimensions of spacetime. In the most general form, the action can then be taken to be the integral of a Lagrangian over the worldsheet,

$$S[X] = \int \mathrm{d}\sigma \mathrm{d}\tau L(X^\mu(\sigma,\tau), \partial_\alpha X^\mu(\sigma,\tau)) \tag{20}$$

3.2 Equations of motion

In a fashion similar to the classical mechanics, one derives the equation of motion by supposing that the action is extremized under a small variation δX^μ. This yields,

$$\begin{aligned} 0 = \delta S &= \int \mathrm{d}\sigma \mathrm{d}\tau \left[\frac{\delta L}{\delta X^\mu(\sigma,\tau)} \delta X^\mu(\sigma,\tau) + \frac{\delta L}{\delta(\partial_\alpha X^\mu(\sigma,\tau))} \partial_\alpha \delta X^\mu(\sigma,\tau) \right] \\ &= \int \mathrm{d}\sigma \mathrm{d}\tau \left[\frac{\delta L}{\delta X^\mu(\sigma,\tau)} \delta X^\mu(\sigma,\tau) - \partial_\alpha \left(\frac{\delta L}{\delta(\partial_\alpha X^\mu(\sigma,\tau))} \right) \delta X^\mu(\sigma,\tau) \right] \\ &+ \int \mathrm{d}\sigma \mathrm{d}\tau \partial_\alpha \left(\frac{\delta L}{\delta(\partial_\alpha X^\mu(\sigma,\tau))} \delta X^\mu(\sigma,\tau) \right) \end{aligned}$$

Taking the boundary terms to vanish, one obtains the equations of motion,

$$\frac{\delta L}{\delta X^\mu(\sigma,\tau)} - \partial_\alpha \left(\frac{\delta L}{\delta(\partial_\alpha X^\mu(\sigma,\tau))} \right) = 0 \tag{21}$$

3.3 Symmetries of the Lagrangian

Let us now turn to the symmetries of the Lagrangian. Consider an infinitesimal transformation of the coordinates,

$$X^\mu(\sigma,\tau) \to X^\mu(\sigma,\tau) + i\varepsilon \mathcal{G}^\mu(\sigma,\tau) \tag{22}$$

where ε is a small constant parameter. Suppose this transformation leaves the Lagrangian invariant $\delta L = 0$ (actually one could study transformations such that $\delta L = \partial_\alpha G^\alpha$).

Taking ε to be a function of τ, one can compute the variation of the action,

$$\delta S = i \int \mathrm{d}\sigma \mathrm{d}\tau \left[\frac{\delta L}{\delta X^\mu(\sigma,\tau)} \varepsilon(\tau) \mathcal{G}^\mu(\sigma,\tau) + \frac{\delta L}{\delta(\partial_\alpha X^\mu(\sigma,\tau))} \partial_\alpha (\varepsilon(\tau) \mathcal{G}^\mu(\sigma,\tau)) \right]$$

However, the invariance of the Lagrangian reads (for a constant ε),

$$0 = \delta L = \frac{\delta L}{\delta X^\mu(\sigma,\tau)} i\varepsilon \mathcal{G}^\mu(\sigma,\tau) + i\varepsilon \frac{\delta L}{\delta(\partial_\alpha X^\mu(\sigma,\tau))}\partial_\alpha \mathcal{G}^\mu(\sigma,\tau)$$

so that,

$$\frac{\delta L}{\delta X^\mu(\sigma,\tau)} \mathcal{G}^\mu(\sigma,\tau) + \frac{\delta L}{\delta(\partial_\alpha X^\mu(\sigma,\tau))}\partial_\alpha \mathcal{G}^\mu(\sigma,\tau) = 0 \tag{23}$$

Using this last equation in the variation of the action, one gets,

$$\delta S = i \int \mathrm{d}\sigma \mathrm{d}\tau \frac{\delta L}{\delta(\partial_\tau X^\mu(\sigma,\tau))} \mathcal{G}^\mu(\sigma,\tau)\partial_\tau \varepsilon \tag{24}$$

This in turn proves that

$$Q = \int \mathrm{d}\sigma \frac{\delta L}{\delta(\partial_\tau X^\mu(\sigma,\tau))} \mathcal{G}^\mu(\sigma,\tau) \tag{25}$$

is convserved. This can be checked easily using the equation of motion and computing $\frac{\mathrm{d}Q}{\mathrm{d}\tau}$.

This concludes our very short trip in the world of String Theory. This quick computations just emphasized how powerful the framewrok we developped is. It is indeed very general and can be applied to a wide range of systems and theories.

4 Conserved charges and Lie algebras structure

Conserved charges are often said to be the *generators* of the symmetries. In this section, we unveil what this means mathematically and prove this statement.
From now on, we shall only consider transformations of the fields that depend on the time only through the canonical fields,

$$\mathcal{G}^l(\mathbf{x},t) = \mathcal{G}^l(\mathbf{x},\Psi(t))$$

where $\Psi(t)$ is the field configuration at time t, that is a shorthand notation for $(\Psi^n(t))_n$.

This, in turn, implies that we consider transformation of the Lagrangian such that,

$$\delta L[\Psi,\dot{\Psi}] = \varepsilon \frac{\mathrm{d}G[\Psi(t)]}{\mathrm{d}t} \tag{26}$$

In most cases, $G = 0$, and that's the case we will be studying here. In the general cases, extra terms may appear in our derived expressions and this shall blur what we are trying to show here.

4.1 Conjugate momenta

In the canonical formalism that leads to the canonical quantization of QFT, one introduces the conjugate momenta,

$$\Pi_n(\mathbf{x},t) = \frac{\delta L}{\delta \dot{\Psi}^n(\mathbf{x},t)} \tag{27}$$

and the equips the fields with the canonical commutation relations at equal time t,

$$[\Psi^n(\mathbf{x},t),\Pi_m(\mathbf{y},t)] = i\delta^n_m\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{28}$$

All the other commutation relations vanish. For our purposes we focus only on bosonic fields, that is to say ones with commutation relations rather than anticommutation relations. The case of fermionic fields can be studied likewise.

We then express all of our quantities in terms of the canonical fields and their conjugate momenta.

4.2 Conserved charges as generators

Let us first explain why we usually call the conserved charges the generators of the symmetries. To do so we should compute the commutator of the conserved charge and the canonical fields. Since the charge is conserved we can take it at the same time as the field. Let us recall the expression of the conserved charge,

$$Q = -i\int \mathrm{d}^3\mathbf{x}\frac{\delta L}{\delta\dot{\Psi}^n(\mathbf{x},t)}\mathcal{G}^n(\mathbf{x},\Psi(t))$$

Using the expression for the conjugate momenta introduced above, it yields,

$$Q = -i\int \mathrm{d}^3\mathbf{x}\Pi_n(\mathbf{x},t)\mathcal{G}^n(\mathbf{x},\Psi(t)) \tag{29}$$

It is now straightforward to compute the interesting commutator,

$$\begin{aligned}
[Q,\Psi^m(\mathbf{y},t)] &= -i\int \mathrm{d}^3\mathbf{x}\left[\Pi_n(\mathbf{x},t)\mathcal{G}^n(\mathbf{x},\Psi(t))\Psi^m(\mathbf{y},t) - \Psi^m(\mathbf{y},t)\Pi_n(\mathbf{x},t)\mathcal{G}^n(\mathbf{x},\Psi(t))\right] \\
&= -i\int \mathrm{d}^3\mathbf{x}\left[\Pi_n(\mathbf{x},t)\Psi^m(\mathbf{y},t) - \Psi^m(\mathbf{y},t)\Pi_n(\mathbf{x},t)\right]\mathcal{G}^n(\mathbf{x},\Psi(t)) \\
&= -i\int \mathrm{d}^3\mathbf{x}(-i)\delta^n_m\delta^{(3)}(\mathbf{x}-\mathbf{y})\mathcal{G}^n(\mathbf{x},\Psi(t)) \\
&= -\mathcal{G}^m(\mathbf{y},\Psi(t))
\end{aligned}$$

In the second equality we used the fact $\mathcal{G}^n$ depends only on the fields and not on the conjugate momenta which implies that it commutes with Ψ^m. In the third equality we used the canonical commutation relations (28).

This result explains why Q is said to *generate* the symmetry,

$$[Q,\Psi^m(\mathbf{y},t)] = -\mathcal{G}^n(\mathbf{y},\Psi(t)) = i\frac{\mathrm{d}\Psi^n}{\mathrm{d}\varepsilon} \tag{30}$$

Computing the commutator of the conserved charge and the conjugate momenta is a bit more involved. Let us do it there as we shall need it later on.

$$\begin{aligned}
[Q,\Pi_m(\mathbf{y},t)] &= -i\int \mathrm{d}^3\mathbf{x}\left[\Pi_n(\mathbf{x},t)\mathcal{G}^n(\mathbf{x},\Psi(t))\Pi_m(\mathbf{y},t) - \Pi_m(\mathbf{y},t)\Pi_n(\mathbf{x},t)\mathcal{G}^n(\mathbf{x},\Psi(t))\right] \\
&= -i\int \mathrm{d}^3\mathbf{x}\left[\Pi_n(\mathbf{x},t)\Pi_m(\mathbf{y},t) - \Pi_m(\mathbf{y},t)\Pi_n(\mathbf{x},t)\right]\mathcal{G}^n(\mathbf{x},\Psi(t)) \\
&+ (-i)\int \mathrm{d}^3\mathbf{x}\Pi_n(\mathbf{x},t)[\mathcal{G}^n(\mathbf{x},\Psi(t)),\Pi_m(\mathbf{y},t)] \\
&= -i\int \mathrm{d}^3\mathbf{x}\Pi_n(\mathbf{x},t)[\mathcal{G}^n(\mathbf{x},\Psi(t)),\Pi_m(\mathbf{y},t)]
\end{aligned}$$

We now prove an important relationship. consider a function $H(\Psi(t), \Pi(t))$. Let us prove that,

$$[H, \Psi^n(\mathbf{x}, t)] = -i\frac{\delta H}{\delta \Pi_n(\mathbf{x}, t)} \quad \text{and} \quad [H, \Pi_n(\mathbf{x}, t)] = i\frac{\delta H}{\delta \Psi^n(\mathbf{x}, t)} \tag{31}$$

To prove these we shall consider a monomial in Ψ and Π and compute the commutator. Then one would assume that general H can be written as sum of monomials (which might not always be true but holds for a large class of functions). To make this as simple as possible we shall denote, $\Psi^n(\mathbf{x}, t) = \Psi^a(t)$ and $\Pi_n(\mathbf{x}, t) = \Pi_a(t)$, where the index a encapsulates both the space variable and fields indices. Hence δ^a_b is the product of two delta functions. Since all of the fields are taken at the same time t, we drop the variable t for the computations.

As stated, we assume that

$$H(\Psi, \Pi) = \prod_a \Psi^a \prod_b \Pi_b = \Psi^{a_1} \dots \Psi^{a_t} \Pi_{b_1} \dots \Pi_{b_s}$$

We can always arrange H in this fashion using the commutation relations (28). Let us for instance compute the product of H with Ψ^c

$$\begin{aligned}
H\Psi^c &= \Psi^{a_1} \dots \Psi^{a_t} \Pi_{b_1} \dots \Pi_{b_s} \Psi^c \\
&= \prod_{j=1}^{t} \Psi^{a_j} \left[\Pi_{b_1} \dots \Psi^c \Pi_{b_s} - i\Pi_{b_1} \dots \Pi_{b_{s-1}} \delta^c_{b_s} \right] \\
&= \prod_{j=1}^{t} \Psi^{a_j} \left[\Psi^c \Pi_{b_1} \dots \Psi^c \Pi_{b_s} - i \sum_{i=1}^{s} \prod_{l \neq i} \Pi_{b_l} \delta^c_{b_i} \right] \\
&= \Psi^c H - i\frac{\delta H}{\delta \Pi_c}
\end{aligned}$$

Hence the expected result.

Computing the other commutation relation is the same and yields the second relation of (31).

Coming back to the commutator of the conserved charge and Π_m, we finally find that,

$$[Q, \Pi_m(\mathbf{y}, t)] = \int \mathrm{d}^3\mathbf{x} \Pi_n(\mathbf{x}, \Psi(t)) \frac{\delta \mathcal{G}^n(\mathbf{x}, \Psi(t))}{\delta \Psi^m(\mathbf{y}, t)} \tag{32}$$

4.3 Conserved charges represent the Lie algebra of the symmetry group

Suppose we study some fields that transform under a certain symmetry group G, that is a Lie group. As we understand above, only infinitesimal transformation are interesting. For a Lie group, infinitesimal transformation are parametrized by the Lie algebra $\mathfrak{g}$ of the group. Infinitesimally, the fields transform as,

$$\Psi^n(x) \to \Psi^n(x) + i\varepsilon^a (T_a)^n_m \Psi^m(x) \tag{33}$$

where a runs over the dimensionality of the Lie algebra (which is also the dimensionality of the Lie group as a smooth manifold) $\dim \mathfrak{G}$ and T_a is a basis of the Lie algebra in the corresponding representation (these can be seen as matrices). These have constant structures defineds by,

$$[T_a, T_b] = i f^c_{ab} T_c \tag{34}$$

The fields must furnish a representation of the symmetry group, and it is sufficient to study the Lie algebra of the group. This fact is rather peculiar since one wouldn't expect that the Lie algebra encapsulate the full information of the symmetry group since the relationship between Lie groups and Lie algebra is not one-to-one (for instance it is easy to check that $\mathfrak{su}(2) = \mathfrak{so}(3)$). However, one can show that in quantum physics only projective representations matter, these are fully determined by the Lie algebra.

Taking each ε^a in (33) to be 0 except for one, it is easy to see that there are dim G conserved current and charges. We shall see that the charges, that we have proven to generate the symmetries in the previous section, also furnish a representation of the Lie algebra.
Let us denote the currents by J^μ_a. They are conserved,

$$\partial_\mu J^\mu_a = 0 \tag{35}$$

Denoting the charges by $\mathcal{T}_a$, they are defined as,

$$\mathcal{T}_a = \int \mathrm{d}^3\mathbf{x} J^0_a \tag{36}$$

Supposing again that we work with a Lagrangian for which G is a symmetry group, one writes,

$$\mathcal{T}_a = -i \int \mathrm{d}^3\mathbf{x} \Pi_n(x) (T_a)^n{}_m \Psi^m(x) \tag{37}$$

These charges are conserved and so we can take them at any time t. In the calculation, one should then take them at the same time at which the fields are defined, since we only have commutation relations at equal times (28). For this reason, in the following when we write x, we suppose that t is the same for all fields and conjugate momenta.

One can easily compute the commutation relations using the results of the previous paragraph

$$[\mathcal{T}_a, \Psi^n(y)] = -(T_a)^n{}_m \Psi^m(y) \tag{38}$$
$$[\mathcal{T}_a, \Pi_n(y)] = +(T_a)^m{}_n \Pi_m(y) \tag{39}$$
$$\tag{40}$$

The first one comes straight from the relation derived in the previous paragraph. The second one is obtained by noticing that $\frac{\delta F^m[\mathbf{x},\Psi(t)]}{\delta \Psi^n(y)} = (T_a)^m{}_n$.

Using this relations, we can compute the commutation relations between the charges,

$$\begin{aligned}
[\mathcal{T}_a, \mathcal{T}_b] &= -i \int \mathrm{d}^3\mathbf{x}\, [\Pi_n(x)(T_a)^n{}_m \Psi^m(x), \mathcal{T}_b] \\
&= -i \int \mathrm{d}^3\mathbf{x} (T_a)^n{}_m \left\{\Pi_n(x)[\Psi^m(x), \mathcal{T}_b] + [\Pi_n(x), \mathcal{T}_b]\Psi^m(x)\right\} \\
&= -i \int \mathrm{d}^3\mathbf{x} (T_a)^n{}_m \left\{\Pi_n(x)(\mathcal{T}_b)^m{}_l \Psi^l(x) - (\mathcal{T}_b)^l{}_n \Pi_l(x)\Psi^m(x)\right\} \\
&= -i \int \mathrm{d}^3\mathbf{x} \left\{\Pi_n(x)(\mathcal{T}_a\mathcal{T}_b)^n{}_l \Psi^l(x) - \Pi_l(x)(\mathcal{T}_b\mathcal{T}_a)^l{}_m \Psi^m(x)\right\} \\
&= -i \int \mathrm{d}^3\mathbf{x} \left\{\Pi_n(x)([\mathcal{T}_a, \mathcal{T}_b])^n{}_l \Psi^l(x)\right\} \\
[\mathcal{T}_a, \mathcal{T}_b] &= i f^c_{ab} \mathcal{T}_c
\end{aligned}$$

This proves that the charges $\mathcal{T}_a$ furnish a representation of the Lie algebra of the symmetry group.

A representation of a Lie algebra maps elements of the Lie algebra to the set of endormorphisms of a vector space. One can ask which Hilbert space this acts on. This is juste the Hilbert space of the quantum theory (typically some Fock space).

Conclusion

This study has explained the importance of symmetries in physics and the bridge them provide to move from mathematical descriptions to physical interpretations. We have seen how symmetries can be used to derive conserved charges and how these charges are related to the Lie algebra of the symmetry.

This note is a very brief introduction to the subject and many aspects have been left out. Most importantly, we never tackled the path integral formulation of QFT. This is a very powerful tool that allows to compute correlation functions and scattering amplitudes. It is also a very elegant way to introduce symmetries in the theory. However, the way quantities are conserved and symmmetries appear is not as straightforward as what we juste provided here, since path integral includes fluctuations of fields.