Second quantization

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Second quantization allows to write any operator acting on the Fock space as a polynomial of creation and annihilation operators. It is useful to write operators in a more compact form. A famous result states that for single particle operators, the second quantized form is the same as the first quantized form. More precisley, if A is a single particle operator, one may expand it as,

$$A = \sum_{i,j} A_{ij} \left| i \right\rangle \left\langle j \right|$$

and the scond quantized operator can be written as

$$A = \sum_{i,j} A_{ij} a_i^\dagger a_j$$

where $a_i^\dagger$ and a_i are the creation and annihilation operators.

It is not obvious what this boils down to when considering rather peculiar combination such as, $A = a_l a_k^\dagger$. In the following we have a look to the way this works.

To do this, we just use the previous theorem, but first we need to express A in the correct fashion. We have (with a slight abuse of notation),

$$A = a_l a_k^\dagger = \sum_{i,j} \left\langle i \right| a_l a_k^\dagger \left| j \right\rangle \left| i \right\rangle \left\langle j \right|$$

Now using commutation relations, we have,

$$a_l a_k^\dagger = \delta_{lk} + a_k^\dagger a_l$$

and we can write,

$$A = \sum_{i,j} \left\langle i \right| \left(\delta_{lk} + a_k^\dagger a_l \right) \left| j \right\rangle \left| i \right\rangle \left\langle j \right|$$

And finally, one writes,

$$A = \sum_{i,j} \left(\delta_{lk} \delta_{ij} + \delta_{ik} \delta_{jl} \right) \left| i \right\rangle \left\langle j \right|$$

And so the second quantized version of A is,

$$A = \sum_{i,j} \left(\delta_{lk} \delta_{ij} + \delta_{ik} \delta_{jl} \right) a_i^\dagger a_j$$

This can be recast as,

$$A = \delta_{lk} \sum_{i} a_i^\dagger a_i + a_k^\dagger a_l$$

One can show easily that the first term is the number operator, that is the second quantized version of the identity operator, $I = \sum_i |i\rangle \langle i|$.

One could have guessed the following form by directly writing A (in its first quantized form) as,

$$A = \delta_{lk} I + a_k^\dagger a_l$$

And then the operation of second quantization is linear and allows to second quantized each term appearing in A.