Spontaneous symmetry breaking in QFT

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Foreword

In this short note we shall describe the process of symmetry breaking in Quantum Field Theory. We shall see that it leads to the appearance of massless bosons, the so-called Nambu-Goldstone bosons. We shall also see how the Higgs mechanism allows for the absorption of these bosons by the gauge bosons, giving them mass. This phenomenom is at the origin of the mass of the $W^{\pm}$ and Z^0 bosons in the Standard Model of particle physics. We shall describe precisely how the masses can be computed.

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1 What is symmetry breaking ?

In this first and introductory section we explain the concept of symmetry breaking. We shall see that it is a fundamental concept in physics, and that it leads to rather peculiar phenomena. But before one asks what is symmetry breaking in physics, one should ask what is symmetry in physics.

1.1 Symmetries

In physics, a symmetry is a transformation that leaves the laws of physics invariant. In other words, if a physical system is symmetric under a certain transformation, we shall say that this transformation is a *symmetry* of the system. Mathematically, this translates into an invariance of the action, or of the Lagrangian or Lagrangian density, under the transformation.

Let us make this more precise and denote by $\mathcal{L}$ the Lagrangian density (that we should now refer to simply as Lagrangian) of a system and S the associated action. Denoting by $\mathcal{C}$ the set of fields configurations, a symmetry of the system is a transformation $T : \mathcal{C} \longrightarrow \mathcal{C}$ such that

$$S[T(\mathcal{C})] = S[\mathcal{C}] \tag{1}$$

or equivalently

$$\mathcal{L}[T(\mathcal{C})] = \mathcal{L}[\mathcal{C}] \tag{2}$$

1.2 Canonical quantization

Moving from a field theory to a quantum field theory is a rather simple process. We should not detail it deeply here, but we shall give a brief overview of the process.

In a field theory, the fields are promoted to operators acting on a space called *Fock space*. This way the fields are quantized in terms of creation and annihilation operators that allow to create and destroy free particle states.

Let us consider the ground state of the system, denoted by $|\Omega\rangle$. The fields may have an arbitrary *vacuum expectation value* (VEV) in this state. However, canonical quantization only allows to have fields quantized around a VEV of 0. Hence if one where to have a field ϕ with a VEV $\langle\phi\rangle = v \neq 0$, one should rather work with the field $\varphi = \phi - v$, so that $\langle\varphi\rangle = 0$, whence the canonical quantization would make sense.

The apparition of this VEV and the fact that it may not be invariant under the transformation of the fields is the essence of symmetry breaking.

The point of view used here is the canonical quantization where fields are operators. In a path integral formalism, the same phenomenom would appear when considering fields as classical fields and using the fact that fields satisfy certain differential equation for the action or the effective action. We shall stick with the canonical point of view in the following, not that it would change much.

1.3 Symmetry breaking

Let us directly consider an example of symmetry breaking. We consider a single field described by the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - \mu^2\varphi^2\right] - \frac{\lambda}{4}\varphi^4$$

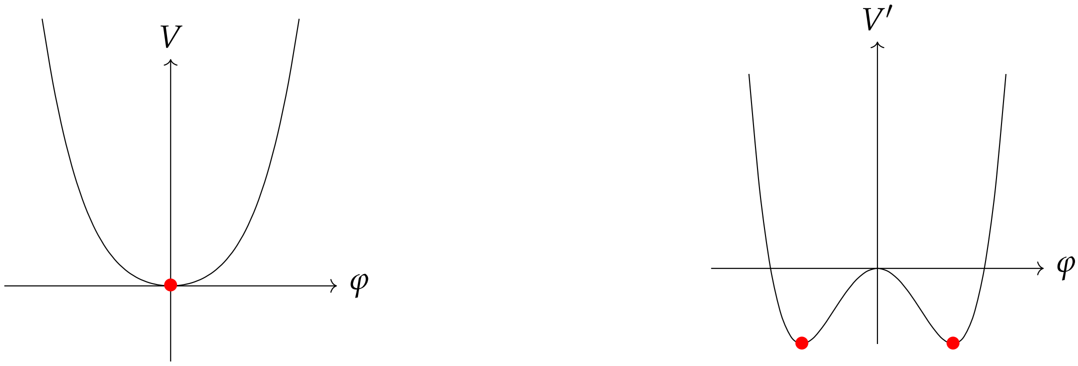
The derivative part is the kinetic part while the second part is the potential,

$$V(\varphi^2) = \frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4}\varphi^4$$

The Lagrangian is invariant under the transformation $\varphi \longrightarrow -\varphi$, which is a $\mathbb{Z}_2$ symmetry. (For the mathematically inclined, and foreseeing the next sections, we say that there is a $\mathbb{Z}_2$ symmetry group acting on the fields).

Let us plot this potential, for two different values of μ^2, depending on whether it is positive or negative. To keep some meaning to the notation μ^2 we rather plot,

$$V(\varphi) = \frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4}\varphi^4$$
$$V'(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4}\varphi^4$$



(a) $\mu^2 > 0$ (b) $\mu^2 < 0$

Figure 1: Shapes of the potential depending on the sign of μ^2

The red dots in Figure 1 refer to the minima of the potential *i.e.* the values of ϕ for which $V(\varphi)$ is minimal, which would then correspond to a ground state of the system.

Let us first take a look at the case $\mu^2 > 0$ describe by V in Figure 1a. In this case, the field minimum is at $\varphi = 0$, this means that as an operator, in the ground state, the field acquires the VEV $v = \langle\varphi\rangle = 0$. What is important to notice here is the fact that the action of $\mathbb{Z}_2$ on the VEV is trivial, *i.e.* $v = -v = 0$. This Lagrangian can then be quantified as usual and gives rise to a scalar particle (a meson) of mass μ. Nothing too fancy here. Let us rewrite the Lagrangian so that we can refer to it later,

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\mu^2\varphi^2 - \frac{\lambda}{4}\varphi^4,\ \mu^2 > 0 \tag{3}$$

Let us now consider the case of V' in Figure 1b. We now use the potential V'. In this case, there are two minimas of the potential, determined by the equation,

$$\frac{\mathrm{d}V}{\mathrm{d}\varphi} = 0 \iff \mu^2\varphi + \lambda\varphi^3 = 0 \iff \varphi = 0 \text{ or } \varphi\pm = \pm\sqrt{\frac{\mu^2}{\lambda}}$$

Only $\varphi_\pm$ correspond to minima of the potential. Hence the system will end up in one of those vacua in the ground state. Since the Lagrangian is invariant under the $\mathbb{Z}_2$ transformation, the physics should not depend on which minima we choose since $\phi_\pm = -\phi_\mp$.

To make things clear let us suppose that the system chooses φ_+ as its VEV,

$$\langle\varphi\rangle = \varphi_+ = \sqrt{\frac{\mu^2}{\lambda}} = v$$

Then, as explained in the previous sections, one should work with the field $\phi = \varphi - v$, whence the Lagrangian becomes,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[(\partial\phi)^2 + \mu^2 v^2 + 2\mu^2\phi v + \mu^2\phi^2\right] - \frac{\lambda}{4}(v+\phi)^4 \\
&= \frac{\mu^4}{2\lambda} + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \mu^2\phi v - \frac{\lambda}{4}(v^4 + 4v^3\phi + 6v^2\phi^2 + 4v\phi^3 + \phi^4) \\
&= \frac{\mu^4}{2\lambda} - \frac{\lambda}{4}v^4 + \frac{1}{2}(\partial\phi)^2 + (\mu^2 v - \lambda v^3)\phi + \left(\frac{\mu^2}{2} - \frac{3}{2}\lambda v^2\right)\phi^2 - \lambda v\phi^3 - \frac{\lambda}{4}\phi^4 \\
&= \frac{\mu^4}{4\lambda} + \frac{1}{2}(\partial\phi)^2 - \mu^2\phi^2 - \sqrt{\lambda}\mu\phi^3 - \frac{\lambda}{4}\phi^4
\end{aligned}$$

Ignoring terms of order higher than 2, we find that the Lagrangian is of the form,

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\sqrt{2}\mu)^2\phi^2 + O(\phi^3) \tag{4}$$

A few things should be pointed out here. First, let us have a look at the constant term in the Lagrangian. What does it mean ? It means that the field ϕ has acquired a non-zero energy in the ground state. In fact, the Hamiltonian density is always defined as,

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

where π is the conjugate momentum to ϕ,

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

In this case it is easy to check that $\pi = \dot{\phi}$ and that the Hamiltonian density is then,

$$\mathcal{H} = \frac{1}{2}\left[(\partial\phi)^2 + (\sqrt{2}\mu)^2\phi^2\right] + \frac{\lambda}{4}\phi^4 - \frac{\mu^4}{4\lambda}$$

and so even when the field ϕ is at rest (in the ground state), there is an energy density $-\frac{\mu^4}{4\lambda}$ coming from the field φ. We shall just study energy differences in a quantum theory and so this constant can be ignored. Yet, let us notice that this constant should play a role in

General Relativity when any kind of energy must be included in the Einstein equations.

The second remark about this equation is the peculiar notation $\frac{1}{2}(\sqrt{2}\mu)^2$. To understand this one should compare (3) and (4). In fact, the latter now looks like the former but with a mass term for the field ϕ which is $\sqrt{2}\mu$. The particle has acquired mass through the symmetry breaking. This shall be a general feature of spontaneous symmetry breaking.

The last thing to emphasize is the apparition of polynomial terms in ϕ in (4). These represent interactions of the field with itself. The ϕ^3 and ϕ^4 terms correspond respectively to vertices with 3 and 4 ϕ mesons attached to them. The linear term proportional to ϕ disappeared and one can wonder why this is the case. This could very well have been predicted by the choices we made when we shifted our field. We claimed that the ground state of the field should be a stationnary point of the potential, that is to say,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\varphi}\right|_{v} = 0$$

Now see that the Lagrangian is polynomial in $\partial\varphi$ and φ so that in particular it can be written at any point a,

$$\mathcal{L}[\partial\varphi, \varphi] = K[\partial\varphi] + \sum_{n\geq 0} \frac{1}{n!} \left.\frac{\mathrm{d}^n V}{\mathrm{d}\varphi^n}\right|_{a} (\varphi - a)^n$$

Taking this formula at the specific point $a = v$, yields the result that in fact the term linear in φ vanishes. This is a very general feature of any Lagrangian polynomial in its fields, which are the Lagrangians one usually studies.

1.4 Combination of ground states

Our previous arguments omits the fact that in quantum mechanics, the system could be in some linear combination of ground states.

2 Global Symmetry Breaking

Let us now try to be more rigorous and describe exactly what is happening in the general context of symmetry breaking. In this section we should describe the phenomenom of breaking a *global symmetry*.

2.1 Mathematical framework

We shall be more precise than in the previous section and now introduce clearly what we mean by *transformations*. In general, one will hear physicists talk about a *group* of symmetry of the system.

The system is invariant under rotations $SO(3)$...
The Lagrangian in invariant under parity $\mathbb{Z}_2$...
The Standard Model has the symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Q$...

However, this is not entirely clear what this should mean. In order to understand that we should first introduce the notion of *representation* of a group.

To do so let us consider a group G. A represention of G is a pair (V, ρ) where V is a vector space and $\rho : G \longrightarrow GL(V)$ is a group morphism.
But what do these representations have to do with our Lagrangian ? In fact, these are precisely the kind of object that acts on the Lagrangian through the fields. Let us be more precise.

Let us consider a field Φ which is a huge vector of fields,

$$\Phi = (\varphi_1, \ldots, \varphi_N)$$

Then we already have our vector space, $V = \mathbb{R}^N$. A representation is then just a group morphism,

$$\rho : G \longrightarrow GL(V) = GL(\mathbb{R}^N) = GL_N(\mathbb{R})$$

It means the representation associates to each element $g \in G$ a matrix $M(g) \in GL_N(\mathbb{R})$. These association of matrix preserves the group structure, meaning that,

$$M(g_1 g_2) = M(g_1)M(g_2)$$

and that the identity is sent to the identity, $M(e) = \mathbb{1}$.

One can now translate the somewhat slippery physicist language into some very precise mathematical statements. For instance, if a physicist says,

The Lagrangian density is invariant under the symmetry group G.

what he means is,

There exists some representation ρ of G acting on the vector space of the fields Φ such that the Lagrangian density is invariant under the action of ρ.

Physics usually goes the other way around. Instead of starting with some fields (or particles since those are equivalent), one starts with a group structure and the requirement that physics should be invariant under the transformations of this group.

Then to understand which kind of mathematical objects can be realised physically, one only needs to look at the representations of the group. To this respect the classification of the representations of groups is extremely important. We shall not detail further on this here, and we will only use the results relevant to our case when we need them.

In physics we usually consider the case of the so-called *Lie groups*. A Lie group is a group that is also a smooth manifold and for which the multiplication and inversion are smooth maps.

Such a group (as a manifold) is locally isomorphic to $\mathbb{R}^n$ for some n. Otherwise said, the group can be described by a vector of n parameters that determines uniquely the group element locally.

As a smooth manifold, a Lie group has a so-called *tangent space* at the identity. This is the space of all possible velocities at the identity. This space is a vector space and is denoted by

$\mathfrak{g}$. The elements of this space are called *generators* of the group (the physicist call them generators, but these are actually called the Lie algebra elements). Such a Lie algebra can always be embedded in some matrix algebra, $\mathfrak{g} \subset \mathfrak{gl}_k(\mathbb{R}) = M_k(\mathbb{R})$ for some k. Then what is called the dimension of the group is the dimension of the Lie algebra (it is also the dimension of the group as a manifold, *i.e.* n using our previous notations).

The Lie algebra naturally comes with a *Lie bracket* that is a bilinear map that satisfies the Jacobi identity and is anticommutative. This bracket is usually denoted by $[\cdot, \cdot]$. The action of the Lie bracket is completely determined by its action on a basis $\{T^a\}$ of $\mathfrak{g}$ by the so-called *structure constants*,

$$[T^a, T^b] = i f^{abc} T^c$$

As physicist we chose the T's to be hermitean which is the reason for the presence of an i in the previous formula. However, mathematicans may make a different choice and the i may not appear.

A representation of a group induces naturally a representation of the Lie algebra, that is to say a linear map, $\rho_* : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}(V)$.

2.2 An example

Let us now consider an example with two fields φ_1 and φ_2. The Lagrangian that we consider is

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi_1)^2 + (\partial\varphi_2)^2\right] + \frac{1}{2}\mu^2(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2$$

The potential is given by,

$$V(\varphi_1, \varphi_2) = -\frac{1}{2}\mu^2(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2$$

This potential is the so-called *Mexican hat* represented in Figure 2.

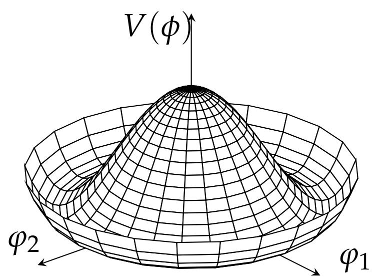


Figure 2: *Mexican hat* potential

We see that the Lagrangian has an $SO(2)$ symmetry (actually $O(2)$ but it is of now concern to us for now) for which the considered representation is the identity representation. Recall that

$$SO(2) = \left\{ R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \ \theta \in [0, 2\pi) \right\}$$

This means that the Lagrangian is invariant under the transformation,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

This symmetry is called a *global symmetry* since the transformation is the same at every point in space.

In the context of symmetry breaking, we suppose that the field $\Phi = (\varphi_1, \varphi_2)$ has a VEV,

$$\langle\Phi\rangle = \begin{pmatrix} \langle\varphi_1\rangle \\ \langle\varphi_2\rangle \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

v_1 and v_2 must be chosen such that the potential is minimized. This means that we must have,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\varphi_1}\right|_{v_1,v_2} = \left.\frac{\mathrm{d}V}{\mathrm{d}\varphi_2}\right|_{v_1,v_2} = 0$$

This translates into the equation

$$v_1^2 + v_2^2 = v^2$$

with $v = \sqrt{\frac{\mu^2}{\lambda}}$.

Without any loss of generality, we can choose $v_1 = v$ and $v_2 = 0$. In fact, suppose we work with the field Φ, the $SO(2)$ symmetry tells use that we might as well work with $\Phi' = R_\theta\Phi$ for some θ. Choosing $\theta = -\arctan(v_1/v_2)$, we find that,

$$\langle\Phi'\rangle = R_\theta\,\langle\Phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{1+\left(\frac{v_1}{v_2}\right)^2}} & \frac{\frac{v_2}{v_1}}{\sqrt{1+\left(\frac{v_1}{v_2}\right)^2}} \\ -\frac{\frac{v_2}{v_1}}{\sqrt{1+\left(\frac{v_1}{v_2}\right)^2}} & \frac{1}{\sqrt{1+\left(\frac{v_1}{v_2}\right)^2}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

We can safely set $v_1 = v$ and $v_2 = 0$ and that's what we shall do in the following.

Expanding the fields around the VEV,

$$\varphi_1 = v_1 + \phi_1,\ \varphi_2 = v_2 + \phi_2,$$

we can now write the Lagrangian into the new quantizable fields ϕ_1 and ϕ_2,

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \mu^2\phi_1^2 + O(\|\phi\|^3) \tag{5}$$

where we denoted $\phi = (\phi_1, \phi_2)$.

This proves that the symmetry breaking has given mass to one of the fields, namely ϕ_1. However ϕ_2 remained massless. This is a general feature that we should explore in the upcoming sections.

One can wonder what would have happened if we did not choose the VEV to be along the ϕ_1-axis. In fact, the Lagrangian would have been more involved. However, it would not have been diagonalized meaning that the fields would have been mixed up and one could not see clearly the terms propotionnal to ϕ_1^2 and ϕ_2^2 independently (there would be some $\phi_1\phi_2$ term popping up). This does not mean that both fields ϕ_1 and ϕ_2 would have aqcuired mass, but rather that the mass terms would have been mixed up. In order to find the fields that are massive, one would have to diagonalize the Hamiltonian and so the Lagrangian. This would correspond precisely to applying the rotation in (2.2) to the fields and we would have found the same result than in equation (5).

Let us sketch a slightly more involved case before turning to the utmost general framework. We shall now consider n fields $= \Phi = (\varphi_1, \ldots, \varphi_n)$ and the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}(\partial\varphi_i)^2 + \frac{1}{2}\mu^2\sum_{i=1}^{n}\varphi_i^2 - \frac{\lambda}{4}\left(\sum_{i=1}^{n}\varphi_i^2\right)^2$$

This Lagrangian has an $SO(n)$ symmetry (or $O(n)$). The VEV is then in general

$$\langle\Phi\rangle = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where is the same fashion than before we chose to set all the components of the VEV apart from the first one to zero. Once again v is determined by minimizing the potential,

$$v^2 = \frac{\mu^2}{\lambda}$$

The Lagrangian can then be written in terms of the new fields

$$\phi_i = \varphi_i - v\delta_{i1}$$

and we find that the Lagrangian is of the form,

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}\sum_{i=1}^{n}(\partial\phi_i)^2 - \mu^2\phi_1^2 + O(\|\phi\|^3) \tag{6}$$

Once again only one field has acquired mass.

2.3 Goldstone's theorem

We now setup the most general framework possible for symmetry breaking. We suppose working with a Lagrangian $\mathcal{L}$ that admits some Lie group G as a symmetry group. Let us suppose that $\mathcal{L}$ depends on some fields that we pack up into a vector $\Phi \in V$, V being some vector space. The symmetry group G will act on the fields Φ through a representation ρ that can be identified as a matrix (since we said Φ is a vector). The invariance of the Lagrangian means that for any $g \in G$,

$$\mathcal{L}[\Phi] = \mathcal{L}[\rho(g)\Phi]$$

In the ground state of the system $|\Omega\rangle$, the field Φ will have a VEV,

$$\langle\Phi\rangle = v$$

Symmetry breaking occurs when some v is not invariant under the action of G. Let us be more precise and denote by G_v the isotropy group of v,

$$G_v = \{g \in G \mid \rho(g)v = v\}$$

It is easy to show that G_v is a subgroup of G. And we shall even assume that G_v is a proper subgroup of G, *i.e.* $G_v \subsetneq G$. Note that this last assumption implies that $v \neq 0$. To simplify notations we shall denote $H = G_v$. One can show that H is a closed Lie subgroup of G. In the context of physics, we shall say that the symmetry is broken from G to H.

A Lie group is a manifold and as such can be described by some real parameters (locally). The number of parameters is determined by the dimension of the manifold or the dimension of the Lie algebra.

A nice feature of Lie groups is that they can be described by the exponential map,

$$\exp : \mathfrak{g} \longrightarrow G$$

where $\mathfrak{g}$ is the Lie algebra of G. One would love to state that any element of G can be written as the exponential of some element of the Lie algebra. This is not always true but it is true for some neighbourhood of the identity.

The exponential map of a connected Lie group is surjective and so the previous statement would hold for such a group. It is always surjective in some neighbourhood of the identity but is not even surjective on the connected component of the identity.

The most general statement is that the exponential map *generates* the identity component of the group. This means that any element of the identity component can be written as the finite product of exponential of some elements of the Lie algebra.

Let us denote by G_e the identity component of G. Then, for any $x \in G_e$, there exists some vectors $X_i \in \mathfrak{g}$ such that,

$$x = \exp(X_1)\exp(X_2)\dots\exp(X_k)$$

Let us denote by G_x the connected component of G containing x. The translation map defined by,

$$\begin{aligned} L_x :\ G_e &\longrightarrow G_x \\ y &\longmapsto xy \end{aligned}$$

is smooth and invertible with inverse L_x^{-1} so that G_x is isomorphic to G_e.

This in turn means that if $y \in G_x \neq G_e$, then

$$y = xx^{-1}y = xL_x^{-1}(y) = x\exp(Y_1)\exp(Y_2)\dots\exp(Y_k)$$

for some $Y_i \in \mathfrak{g}$. This explains why we can only focus on the identity component of the group and include he other component if necessary.

From now on we shall work with elements of the group that can be written as exponentials of elements of the Lie algebra,

$$G \ni g = \exp(X),\ X \in \mathfrak{g}$$

However, as emphasized before, we don't really care about G itself but rather about its representation ρ. A representation of the group induces a representation on the Lie algebra, $\rho_* : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}(V)$. The main feature of this induced representation is that it is compatible with the exponential map,

$$\rho(\exp(X)) = \exp(\rho_*(X))$$

So this means that when we consider the action of group elements on the field, $\rho(\exp X)\Phi$, we might as well consider the action of the induced representation ρ_*. That's what we shall do from now on.

As explained before, the Lie algbera is determined by a basis of generators T^a that satisfy some commutations relation rules,

$$\left[T^a, T^b\right] = if^{abc}T^c$$

where f^{abc} are the structure constants of the Lie algebra. The field are then acted upon by the representations of these T^a, $\rho_*(T^a)$. Hence, the transformation of fields we are considering are of the form,

$$\Phi \longrightarrow \exp(\theta_a \rho_*(T^a))\Phi$$

In the following, and when there are no ambiguity we shall lighten the notation and denote $\rho_*(T^a)$ by T^a.

Let us now call to some utmost famous theorem that is, the *Noether's theorem*. This theorem states that for any continuous symmetry of the Lagrangian, there exists a conserved charge. This means that when the Lagrangian is invariant under the action of some Lie group G, there exists exactly $\dim G$ conserved charges. These charges are the generators of the group in the sense that they represent the Lie algebra on the Hilbert space of the theory. One can picture it this way, when we transform the fields we transform the state of the Hilbert space. This transformation of state should follow the group law and should then be represented by a Lie algebra. The generators of the transformation are denoted by Q^a and satisfy the same commutation relations,

$$\left[Q^a, Q^b\right] = if^{abc}Q^c$$

Now that we have set up our framework, we shall state and prove the Goldstone's theorem.

Goldstone's theorem

Let $\mathcal{L}$ be a Lagrangian that is invariant under the action of some Lie group G Let H be the isotropy group of the VEV of the field. Then, the number of massless bosons in the spectrum of the theory is $\dim G - \dim H$.

These massless bosons are referred to as *Nambu-Goldstone bosons*.

Let us now prove this statement. To do so, we shall use Noether's theorem, which states that the conserved charges can be cast as,

$$Q^a = \int \mathrm{d}^3x\ J^0_a$$

These are taken as Hermitean operators (this can always be achieved by a redefinition of the current by a factor of $\pm i$).
Let us order the generators of the group such that the first $\dim H$ generators are generators of H, the remaining $\dim G - \dim H$ are the other generators of G.
Since the vacuum state $|\Omega\rangle$ is invariant under H, one finds,

$$Q^a |\Omega\rangle = 0,\ \forall a \in [\![1, \dim H]\!]$$

In fact consider a general transformation belonging to H,

$$\exp\left(i\sum_{a=1}^{\dim H}\theta_a Q^a\right)|\Omega\rangle = |\Omega\rangle$$

Taking each θ_a to be zero one by one and expadning the exponential around zero as,

$$\exp(i\varepsilon X) = 1 + i\varepsilon X + O(\varepsilon^2)$$

yields the result.

Let us now consider the transformation of the vacuum state under the action of a generator of $G \setminus H$. We suppose that H it the largest subgroup of G that keeps the vacuum state invariant, so that if $Q_a \in \mathfrak{g} \setminus \mathfrak{h}$, then

$$\exp(i\theta Q_a)|\Omega\rangle \neq |\Omega\rangle$$

In fact, if this wouldn't be the case, one could extend H's Lie algebra to $\mathfrak{h} \cup \mathbb{R}Q_a$, thus yielding a larger group. This in turn implies that,

$$Q_a|\Omega\rangle \neq 0$$

Furthermore, since $Q_a \in \mathfrak{g} \setminus \mathfrak{h}$, it cannot let the ground state invariant,

$$Q_a|\Omega\rangle \neq |\Omega\rangle$$

Taking Heisenberg's equation of motion (in QFT one works with Heisenberg's point of view rather than Schrödinger's),

$$\frac{\mathrm{d}Q^a}{\mathrm{d}t} = \frac{i}{\hbar}[H, Q^a]$$

one finds that,

$$[H, Q^a] = 0$$

since the Q^a are conserved charges. This effectively means that the vacuum state is degenerate. In fact, let us assume $H|\Omega\rangle = 0$ (one can always achieve that by a shift in energy), then,

$$HQ^a|\Omega\rangle = Q^a H|\Omega\rangle = 0$$

And so one is able to find $\dim G - \dim H$ linearly independent states that are also vacua.

Let us now consider the states,

$$|s_a\rangle = \int \mathrm{d}^3x\, e^{-ik\cdot x}\, J_a^0(x,t)\,|\Omega\rangle$$

It can be shown that these states have momentum k (we show this in Appendix A).

When $k \to 0$, the energy of the states $|s_a\rangle$ tends to zero, which basically proves that these are massless states.

This proof is not as rigorous as one would like it to be. In fact, there is a fmous theorem called the *Fabri-Picasso theorem* that states that the conserved charges operators are well defined only if $Q|\Omega\rangle = 0$. This can be resolved but it is a bit more involved.

A second interesting proof is purely algebraic and comes back to the definition of the ground states. Consider the field Φ and it VEV v. Let us consider an infinitesimal transformation of the field,

$$\Phi \longrightarrow \exp(i\theta_a T^a)\Phi \simeq \Phi + i\theta_a T^a \Phi$$

The Lagrangian reads,

$$\mathcal{L}[\Phi] = \text{Derivative terms} + V(\Phi)$$

Taking the field to be constant, the derivative terms vanish and the potential reads,

$$V(\Phi) = V(\phi + i\theta_a T^a \Phi)$$

One can expand this equation in term of a Taylor series to find that,

$$V(\Phi) = V(\Phi) + i\theta_a \left.\frac{\partial V}{\partial \Phi_i}\right|_{\Phi} T^a_{ij}\Phi_j + O(\|\theta\|^2)$$

so that,

$$\left.\frac{\partial V}{\partial \Phi_i}\right|_{\Phi} T^a_{ij}\Phi_j = 0$$

We can now differentiate with respect to the value taken everywhere by the field and set $\Phi = v$ to find that,

$$0 = T^a_{il}\left.\frac{\partial V}{\partial \Phi_i}\right|_{v} + \left.\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j}\right|_{v} T^a_{jl} v_l$$

Now let us use the fact that v is a minimum of the potential,

$$\left.\frac{\partial V}{\partial \Phi_i}\right|_{v} = 0$$

and we find that,

$$\left.\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j}\right|_{v} T^a_{jl} v_l = 0$$

This means that the *mass matrix* $\left(\left.\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j}\right|_{v}\right)_{ij}$ has as many zero eigenvectors as generators that don't annihilate v. This proves that there are $\dim G - \dim H$ massless bosons. The other bosons are massive, with masses given by positive eigenvalues of the mass matrix.

A slight loophole has been used before since we never proved that the mass matrix is positive semi-definite. This can be shown by considering the second order term in the Taylor expansion and using the fact that v is a minimum of the potential.

3 Breaking a Gauge Symmetry

Let us now turn to the concept of gauge symmetry. We shall first define what a gauge symmetry and explain how it arises in physics. We shall then consider an example of gauge symmetry breaking and see how the Higgs mechanism can be used to give mass to the gauge bosons. We shall generalize this example to any kind of groups. Finally we will have a look at the Standard Model Higgs boson.

3.1 What is gauge symmetry ?

In the previous section we enforced a Lagrangian to be symmetric under certain transformations, the so-called global symmetries. It meant that one would just stack up the fields without caring about the spacepoint these were defined at and upon the action of a group through a representation, the fields would be transformed just as a vector multiplied by a matrix.

But transformation could be more general and depend on the spacepoint. This means that the fields are not just vectors but rather sections of some bundle over the spacetime. This is the case for the so-called gauge symmetries. Cutting the mathematical jargon, a gauge symmetry is a symmetry that depends on the point in spacetime. Namely, if we denote Φ the fields, they would transform under a representation of the group G as,

$$\Phi(x) \longrightarrow \rho(g, x)\Phi(x)$$

where $g \in G$ and x is the spacetime point.

It is far less easy to be invariant under such a transformation. In fact, let us have a quick look into some complex scalar field theory for which the Lagrangian is given by

$$\mathcal{L} = (\partial\varphi)^2 - m^2\varphi^2$$

This Lagrangian is invariant under the transformation,

$$\varphi(x) \longrightarrow e^{i\alpha}\varphi(x)$$

but not under the transformation,

$$\varphi(x) \longrightarrow e^{i\alpha(x)}\varphi(x)$$

The reason for this is that the derivative operator ∂ does not commute with the exponential. This means that the derivative operator is not invariant under the transformation. In order to make the Lagrangian invariant, one has to introduce a new field, the so-called gauge field, that will cancel out the variation of the derivative operator. This is the essence of gauge symmetry.

Since this is not a course on gauge theory, we shall not delve into the details and just accept that in order to make the theory covariant again one just needs to introduce new vectors fields that will cancel out the variation of the fields under the transformation. These field are introduced by the use of covariant derivatives,

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + igA_\mu$$

where g is the coupling constant and A_μ are the gauge fields.

These fields transform in the adjoint representation of the group G, that is,

$$A_\mu(x) \longrightarrow gA_\mu(x)g^{-1} - \frac{i}{g}(\partial_\mu g)g^{-1}$$

In a very general way, the original Lagrangian reads,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - V(\Phi)$$

and is promoted to a gauge invariant Lagrangian,

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}\big(F^{\mu\nu}F_{\mu\nu}\big) + \frac{1}{2}D_\mu\Phi^\dagger D^\mu\Phi - V(\Phi)$$

where the kinematics of the gauge fields is given by the field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

and theTr refers to the sum over all gauge fields. This should be detailed in Section 3.3.

3.2 An example of gauge symmetry breaking

In order to understand what happens, let us first consider an example with the simplest possible gauge group, that is $U(1)$. The globally symmetric Lagrangian reads

$$\mathcal{L} = (\partial_\mu\varphi)^\dagger(\partial^\mu\varphi) + \mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2$$

The dimension of the group is $\dim U(1) = 1$ so that to obtain a gauge invariant Lagrangian, one has to introduce a gauge field A_μ and a covariant derivative,

$$D_\mu = \partial_\mu + igA_\mu$$

The gauge invariant Lagrangian reads,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) + \mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2$$

where the field strength tensor is given by,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Note that in this last equation the commutator does not appear since the group is abelian.

The gauge field and the complex scalar field interact through the covariant derivatives. In fact, if we expand it out, we find that,

$$(D_\mu\varphi)^\dagger(D^\mu\varphi) = \partial_\mu\varphi^\dagger\partial^\mu\varphi + igA_\mu(\varphi^\dagger\partial^\mu\varphi - \partial^\mu\varphi^\dagger\varphi) + g^2A_\mu A^\mu\varphi^\dagger\varphi$$

This Lagrangian is invariant under the transformations,

$$\begin{aligned}
\varphi(x) &\longrightarrow e^{i\alpha(x)}\varphi(x) \\
A_\mu(x) &\longrightarrow e^{i\alpha(x)}A_\mu(x)e^{-i\alpha(x)} + \partial_\mu(e^{i\alpha(x)})e^{-i\alpha(x)} = A_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x)
\end{aligned}$$

It is easy to check that these transformations represent the gauge group $U(1)$.

3.3 The Higgs mechanism

3.4 The Standard Model Higgs Boson

ramener par une transformation de jauge globale de $O(N)$). On aurait alors trouvé

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}\sum_{i=1}^{N}(\partial\phi_i)^2 - \mu^2\phi_1 + O(|\phi|^3)$$

et de la même manière seul un des bosons (champs scalaires) aurait acquis la masse $\sqrt{2}\mu$. Ce résultat très général s'énonce sous la forme du théorème de Goldstone:

Théorème 3.4.1. (Théroème de Goldstone)
On considère un Lagrangien admettant G pour groupe de symétrie continu et son état fondamental (vide) qui n'admet que $H \subset G$, sous groupe de G, comme groupe de symétrie. C'est à dire que $\mathcal{L}(\varphi)$ est invariant pour φ se transformant sous une représentation de G et que $|0\rangle$ l'état fondamental est invariant sous l'action d'une représentation de H.
On note $n(G)$ et $n(H)$ les générateurs respectivement de G et H alors il apparait $n(G) - n(H)$ bosons sans masse dits de Nambu-Goldstone.

Proof. Pour montrer ce théorème on utilise une approche de quantification canonique. Par le théorème de Noether, l'invariance du Lagrangien sous la représentation d'un groupe G implique la présence de $n(G)$ charges conservées. Il y a $n(G) - n(H)$ charges $\widehat{Q}_\alpha$, $\alpha \in [\![1, n(G) - n(H)]\!]$ telles que $\widehat{Q}_\alpha |0\rangle \neq 0$ (pour le formalisme canonique les charges conservées sont aussi des générateurs du groupe de symétrie, si le vide est n'est pas invariante sous action des éléments $G - H$ alors pour Q générateur de G mais pas de H, $e^{i\theta Q} |0\rangle \neq |0\rangle$ et donc $Q |0\rangle \neq 0$).
Soit $\alpha \in [\![1, n(G) - n(H)]\!]$.
Le formalisme canonique nous place sous le point de vue de Heisenberg et il vient alors

$$\frac{\mathrm{d}\widehat{Q}_\alpha}{\mathrm{d}t} = \frac{i}{\hbar}\left[\widehat{H}, \widehat{Q}_\alpha\right]$$

Ainsi la conservation des charges implique $\left[\widehat{H}, \widehat{Q}_\alpha\right] = 0$.
Choisissons le vide tel que $\widehat{H} |0\rangle = |0\rangle$. On a alors $\left[\widehat{H}, \widehat{Q}_\alpha\right] |0\rangle = \widehat{H}\widehat{Q}_\alpha |0\rangle = 0$. D'où $\widehat{Q}_\alpha |0\rangle$ a même énergie que $|0\rangle$ mais est bien un état différent.
Pour le formalisme canonique,

$$\widehat{Q}_\alpha = \int \mathrm{d}^D x \; J^0_\alpha(x,t), \;\; \forall t$$

Considérons alors l'état

$$|s_\alpha\rangle = \int \mathrm{d}^D x \; e^{-ik\cdot x} \; J^0_\alpha(x,t) \, |0\rangle \tag{7}$$

a pour impulsion k (facile à montrer en utilisant les opérateurs impulsions dans le formalisme canonique).
Quand $k \to 0$, $|s_\alpha\rangle \to \widehat{Q}_\alpha |0\rangle$ qui a pour énergie 0 (ou du moins la même que celle du vide) et donc $|s_\alpha\rangle$ correspond bien a une particule scalaire sans masse. □

Vérifions alors que les résultats obtenus sont cohérents. Rappelons d'abord que $n(O(N)) = n(SO(N)) = \frac{1}{2}N(N-1)$ et $n(U(N)) = N^2$ et $n(SU(N)) = N^2 - 1$.
Ainsi pour un Lagrangien de symétrie interne globale $O(N)$, l'état fondamental n'est plus

symétrique que par $O(N-1)$ (tous les champs qui valent 0 peuvent être interchangés), et donc il apparait $n(O(N)) - n(O(N-1)) = N-1$ bosons sans masse, ce qui est bien le résultat que nous avions trouvé.
Reprenons le cas $N = 2$. Dans le Lagrangien obtenu finalement (1.6), le champs massife ϕ_1 ne peut pas s'éloigner facilement de sa valeur fondamental $\phi_1 = 0$ à cause du terme de masse qui montre directement que cela couterait de l'énergie au système. Cependant, qu'en est-il pour ϕ_2 ? Etudions, du point de vue de l'intégrale de chemin, les fluctuations statistiques de ϕ_2. Comme $\langle\phi_2(x)\rangle = 0$, la quantité à étudier est à $\left\langle\phi_2(x)^2\right\rangle$. Par homogénéité, on peut l'étudier pour $x = 0$.

$$\begin{aligned}
\left\langle \phi_2(0)^2 \right\rangle &= \frac{1}{Z}\int^{\Lambda} D\ \phi e^{iS[\phi]}\phi_2(0)^2 \\
&= \lim_{x\to 0}\frac{1}{Z}\int^{\Lambda} D\phi\ e^{iS[\phi]}\phi_2(x)\phi_2(0) \\
&= \lim_{x\to 0} G(x,0) \\
&= \lim_{x\to 0}\int^{\Lambda}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{e^{ikx}}{k^2}
\end{aligned}$$

où l'on a noté $Z = \int^{\Lambda} D\phi\ e^{iS[\phi]}$. L'intégrale a été renormalisée à une échelle d'énergie Λ et l'on a reconnu les propagateur d'un champ scalaire $G(x,0)$.
Ainsi, grâce à Λ, il ne peut pas y avoir de problème de convergence UV. Toutefois, comme ϕ_2 est sans masse, le propagateur (de la forme $\frac{1}{k^2+\mu^2}$ pour une particule massive) peut diverger à faibles énergies.
Pour $d > 2$, la convergence est assurée. Cependant pour $d \leq 2$, les fluctuations de ϕ_2 autour de sa valeur d'équilibre deviennent infinies. C'est une conséquence du plus large théorème de Coleman-Mermin-Wagner qui indique qu'il n'y a pas de brisure spontanée de symétrie à l'équilibre possible en dimension $d \leq 2$.

4 Mécanisme de Higgs-Anderson

On considère désormais travailler sur une théorie jaugée en introduisant un champ vectoriel A^μ sans masse. La prescription de couplage minimal indique

$$\partial_\mu\varphi \longrightarrow D_\mu\varphi = \left(\partial_\mu - ieA_\mu\right)\varphi \tag{8}$$

Le Lagrangien devient alors

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \left(D^\mu\varphi\right)^\dagger\left(D_\mu\varphi\right) + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2 \tag{9}$$

En écrivant $\varphi = \rho e^{i\theta}$, ce Lagrangien est désormais invariant par transformation de jauge locale $U(1)$:$\begin{cases}\theta & \longrightarrow & \theta+\alpha(x) \\ A_\mu & \longrightarrow & A_\mu + \frac{1}{e}\partial_\mu\alpha\end{cases}$. Avec ces deux nouveaux champs réels ρ et θ, le Lagrangien peut se réécrire,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \rho^2\left(\partial_\mu\theta - eA_\mu\right)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4$$

En absorbant le champ θ dans le champ de jauge A_μ, on obtient un nouveau champ $B_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta$. Remarquons que B_μ est invariant par transformation de jauge et $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu$ et l'on écrit finalement

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \rho^2 e^2 B^\mu B_\mu + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4 \tag{10}$$

Effectuons désormais la brisure spontanée de symétrie en écrivant $\rho = \frac{1}{\sqrt{2}}(\chi + v)$, avec $v = \sqrt{\frac{\mu^2}{\lambda}}$. (Encore une fois on est libre de choisir le point θ=0 comme état fondamental, mais remarquons de toute façon que le champ θ a été absorbé dans le champ B_μ, cachant alors la symétrie $U(1)$).
Le Lagrangien devient alors

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}e^2v^2B_\mu^2 + \frac{1}{2}(\partial\chi)^2 + e^2v\chi B_\mu^2 + \frac{1}{2}e^2\chi^2B_\mu^2 - \mu^2\chi^2 - \sqrt{\lambda}\mu\chi^3 - \frac{\lambda}{4}\chi^4 \quad (11)$$

où l'on a enlevé le terme constant, qui ne sert à rien pour la suite de la discussion.
Ainsi, on remarque que comme précédemment le meson χ a acquis la masse $\sqrt{2}\mu$ (terme $-\frac{1}{2}(\sqrt{2}\mu)\chi^2$), mais le champ vectoriel B_u a aussi acquis une masse $M = ev$ (terme $+\frac{1}{2}(ev)^2B_\mu B^\mu$, le signe est l'opposé du champ scalaire, c'est le bon signe pour un champ vectoriel).
Le boson de Nambu-Goldstone θ a été absorbé par le champ vectoriel B_μ et ce dernier a acquis une masse. N'a-t-on alors pas perdu de degrés de liberté ?
Au début nous avions deux degrés de liberté associés au champ scalaire complexe φ et deux degrés de libertés associés au champ vectoriel sans masse A_μ. A la fin nous avons un champ scalaire réel χ, soit un degré de liberté, et un champ massif vectoriel B_μ, soit 3 degrés de liberté.

$$\begin{array}{c} 2\text{ (champ scalaire complexe)} \ + \ 2\text{ (champ vectoriel sans masse)} \\ \big\downarrow \\ 1\text{ (champ scalaire réel)} \ + \ 3\text{ (champ vectoriel massif)} \end{array}$$

Nous avons donc le bon nombre de degrés de liberté.

5 Spectre de masse des bosons

On se place dans un cas plus général avec un Lagrangien admettant un groupe G comme groupe de symmétrie interne, avec $\varphi \in \mathbb{R}^{\dim R}$ un champ qui se transforme selon une réprésentation R de ce groupe. On suppose la théorie jaugée avec des champs A_μ^a, avec $1 \leq a \leq n(G)$. On suppose qu'il y a brisure spontanée de symétrie et l'état fondamental n'est symétrique que pour un sous groupe $H \subset G$.
Ainsi le Lagrangien fait intervenir un terme de la forme $\frac{1}{2}\left(D_\mu\varphi\right)^\dagger(D^\mu\varphi)$, avec

$$D_\mu\varphi = \partial_\mu\varphi - igA_\mu^aT^a(R)\varphi \quad (12)$$

où l'on a noté $T^a(R)$ les matrices des générateurs du groupe G dans la représentation R. Elles sont hermitiennes. Sous brisure de symétrie spontanée, φ prend une valeur moyenne non nulle $v \in \mathbb{R}^{\dim R}$ et le terme $\frac{1}{2}\left(D_\mu\varphi\right)^\dagger(D^\mu\varphi)$, donne naissance à un terme

$$\frac{1}{2}g^2v^\dagger T^a(R)^\dagger T^b(R)vA_\mu^aA^{\mu b} \quad (13)$$

Soit alors $\mu \in \mathcal{M}_{n(G)}(\mathbb{R})$ la matrice carrée définie par

$$\mu^{ab} = \frac{1}{2}g^2\left\langle T^a(R)v\ ,\ T^b(R)v\right\rangle \quad (14)$$

μ est hermitienne : $\left(\mu^{ab}\right)^* = \mu^{ba}$. A ce titre, elle admet des valeurs propres réelles. De plus, μ est positive.
En effet, soit $x = x^a \in \mathbb{C}^{n(G)}$

$$
\begin{aligned}
x^\dagger \mu x &= \frac{1}{2} g^2 \sum_{a,b} x_a^* \left\langle T^a(R)v \; , \; T^b(R)v \right\rangle x_b \\
&= \frac{1}{2} g^2 \left\langle \sum_a x_a T^a(R)v \; , \; \sum_b x_b T^b(R)v \right\rangle \\
&= \frac{1}{2} g^2 \left\langle Xv \; , \; Xv \right\rangle \\
&= \frac{1}{2} g^2 \|Xv\|^2 \geq 0
\end{aligned}
$$

où l'on a noté $X = \sum_a x_a T^a(R)$.
Ainsi, les valeurs propres de μ sont positives ou nulles.
Comme μ est hermitienne, par le théorème spectral, elle est diagonalisable en base unitaire. C'est à dire qu'il existe, $m_1, \ldots, m_{n(G)}$ et $U \in U(n(G))$ tels que $\mu = U^\dagger \; \mathrm{diag}(m_1, \ldots, m_{n(G)}) \; U$.
Ainsi, trouver les masses des bosons suite à la brisure de symétrie spontanée revient simplement à diagonaliser la matrice μ. Les masses sont alors les m_i et la transformation de jauge à faire sur les champs A_μ pour les faire apparaître explicitement les termes de masse dans le Lagrangien est donnée par la matrice unitaire U. Quand le Lagrangien est écrit sous cette forme on parle de jauge unitaire.
On peut montrer facilement que μ a $n(H)$ valeurs propres nulles. En effet, notons les générateurs de $n(H)$ les $T^c(R)$ pour $c \in [\![n(G) - n(H) + 1, n(G)]\!]$. Comme l'état fondamental est invariant par H, il vient $T^c(R)v = 0$, $\forall c \in [\![n(G) - n(H) + 1, n(G)]\!]$. Ainsi la matrice μ se met sous la forme

$$
\mu = \left(\begin{array}{c|c} \mu_{G-H} & \mathbb{O}^{(n(G)-n(H)) \times n(H)} \\ \hline \mathbb{O}^{(n(H) \times (n(G)-n(H)))} & \mathbb{O}^{n(H) \times n(H)} \end{array} \right) \tag{15}
$$

où $\mathbb{O}$ désigne la matrice nulle des dimensions correspondantes.
Ainsi, il est évident que μ admet 0 pour valeur propre avec multiplicité $n(H)$, ce qui correspond aux $n(H)$ bosons qui restent (logiquement) sans masse.

Ainsi, si l'on compte désormais les degrés de liberté du système, nous en avions initialement $\dim R$ pour le champ φ et $2n(G)$ pour les bosons sans masse. A la fin nous en avons $\dim R - (n(G) - n(H))$ pour le champ scalaire (ce qui montre d'ailleurs qu'il faut que $\dim R \geq n(G) - n(H)$ pour que cette brisure de symétrie ait lieu), $3(n(G) - n(H))$ pour les bosons qui ont acquis une masse et $2n(H)$ pour ceux qui sont restés sans masse. Soit alors $\dim R + 2n(G)$ comme initialement.

A Goldstone modes have momentum k

We prove the result that states that Goldstone modes really do have momentum k.
Let us work with some Goldstone mode, $|s_a\rangle$,

$$|s_a\rangle = \int \mathrm{d}^3x\, e^{-ik\cdot x}\, J^0_a(x,t)\,|\Omega\rangle$$

Acting with the momentum operator $\mathbf{P}$ on this state,

$$\begin{aligned}
\mathbf{P}\,|s_a\rangle &= \int \mathrm{d}^3x\, e^{-ik\cdot x}\, \mathbf{P}J^0_a(x,t)\,|\Omega\rangle \\
&= \int \mathrm{d}^3x\, e^{-ik\cdot x}\, \left[\mathbf{P}, J^0_a(x,t)\right]|\Omega\rangle \\
&= \int \mathrm{d}^3x\, e^{-ik\cdot x}\, (-i)\nabla J^0_a(x,t)\,|\Omega\rangle \\
&= \int \mathrm{d}^3x\, e^{-ik\cdot x}\, \mathbf{k}J^0_a(x,t)\,|\Omega\rangle \\
&= \mathbf{k}\,|s_a\rangle
\end{aligned}$$

which proves the result. In the second line we used that the action of the momentum operator on a vacuum state is always 0. In the third line we used the fact that the momentum operator is the generator of translations. Finally in the fourth line we used an integration by parts.