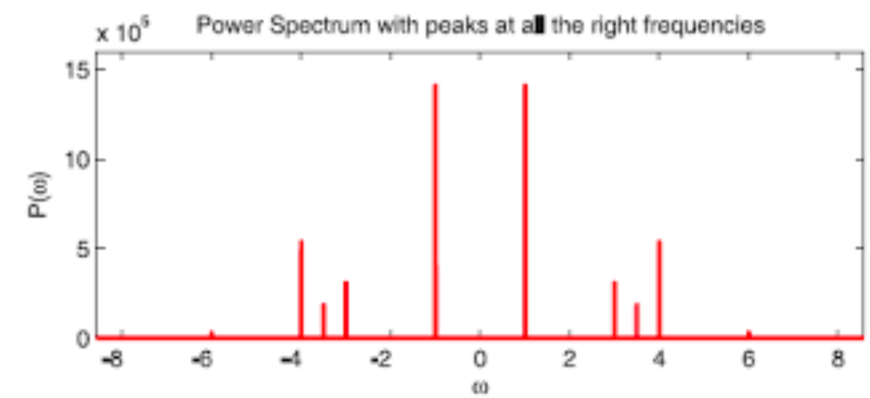
Question of the Day

Can you play a song from knowing its DFT or power spectrum?



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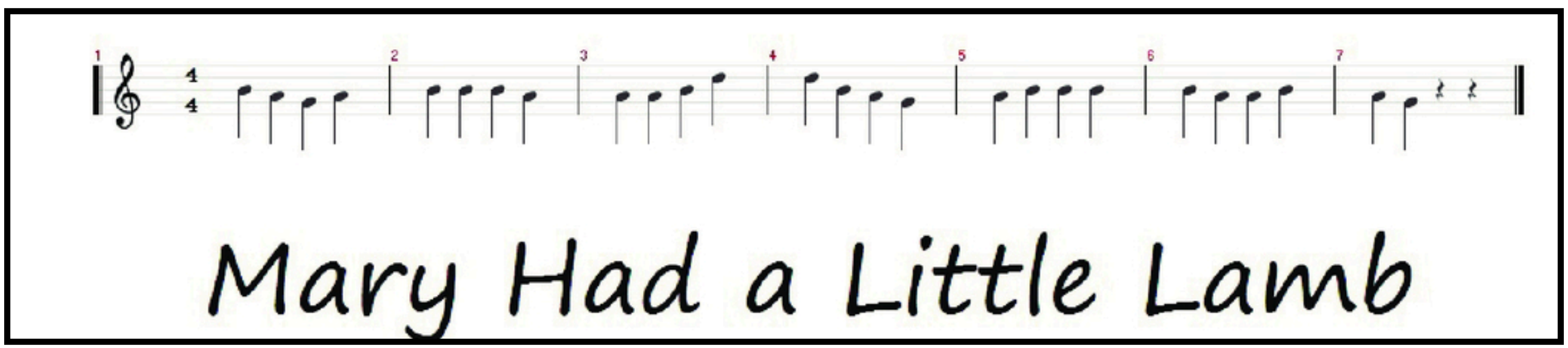
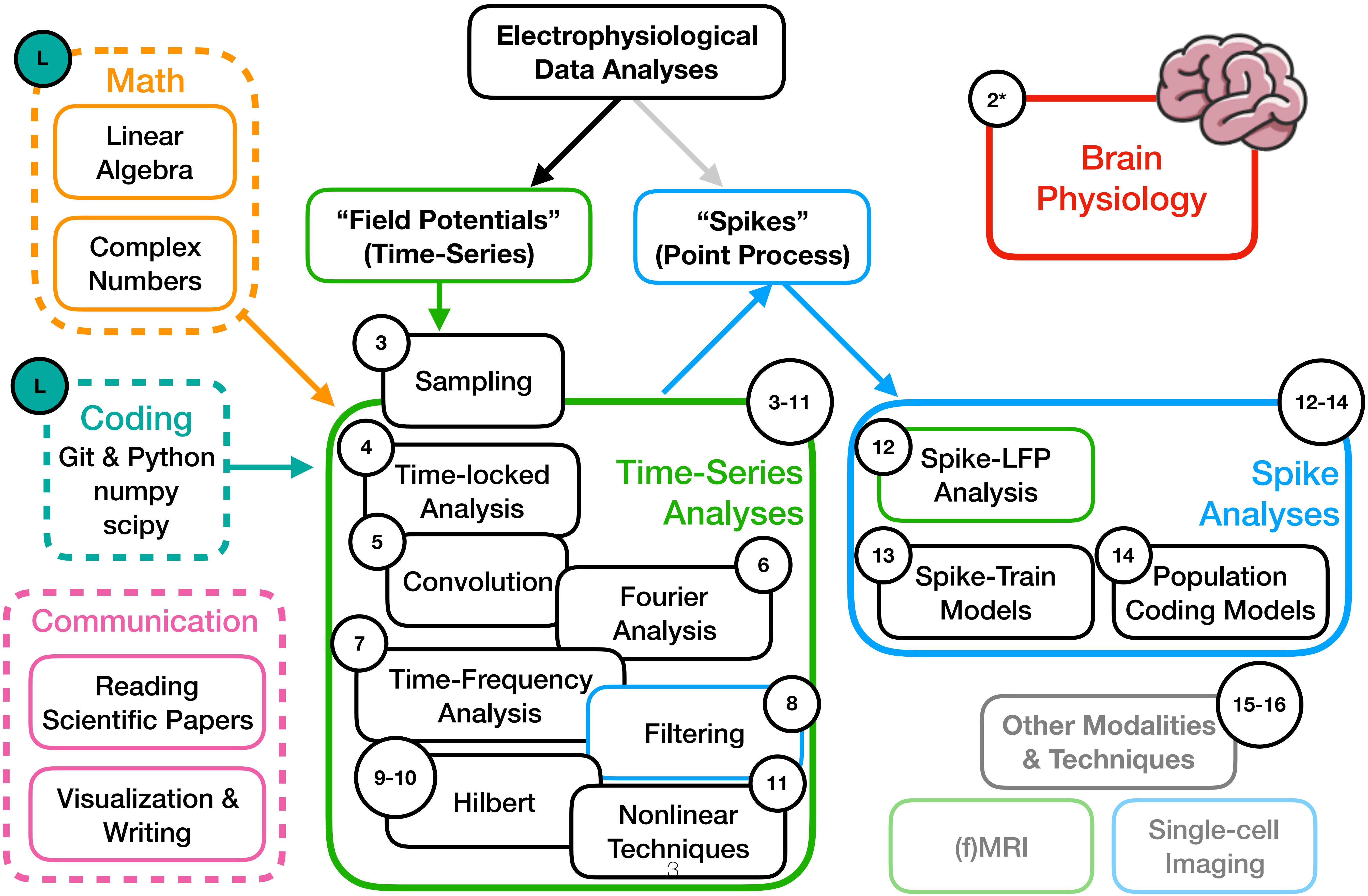


Fourier Part II & Time-Frequency Analysis

Lecture 7

July 11, 2019

Course Outline: Road Map



Goals for Today

1. Some Fourier properties

2. Computing frequency from wave number

3. Time-frequency analysis

Handy Properties of DFT

Linearity

Orthogonality

Periodicity

Duality

(Circular) Convolution Theorem

Power Conservation (Parseval's Theorem)

Linearity

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \quad \textbf{(Eq.1)}$$

if $F\{x(n)\}_k = X_k,\ F\{y(n)\}_k = Y_k$

$$F(\{ax(n) + by(n)\})_k = \qquad aX_k + bY_k$$

Because the **dot product is a linear** operation.

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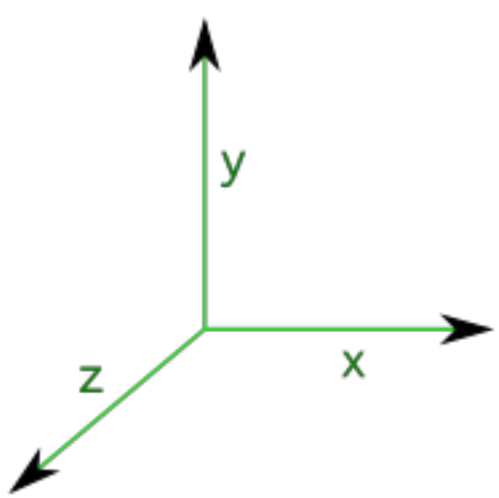
Orthogonality

$$
\begin{aligned}
X_k &= \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)],
\end{aligned}
\qquad \textbf{\textit{(Eq.1)}}
$$

for wave numbers k and m:

$$dot(e^{\frac{i2\pi n}{N}k}, e^{\frac{i2\pi n}{N}m}) \qquad = 0 \ \mathit{if} \ k \neq n, = 1 \ \mathit{if} \ k = n$$



The complex exponentials of all k combine to form an N-dimensional orthogonal basis.

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \quad \textbf{(Eq.1)}$$

for wave numbers k and m:

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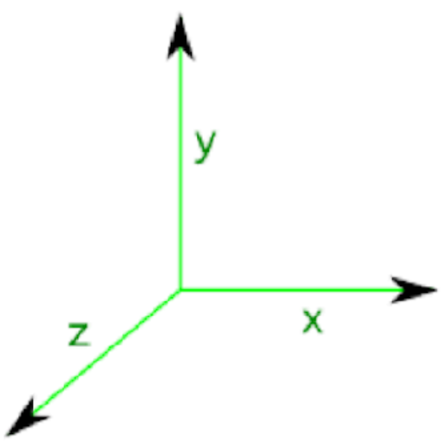


vector

$$e^{i\frac{2\pi}{N}k[0,1,2\ldots N-1]}$$

$$\Rightarrow [e^{i\frac{2\pi}{N}0}, [\;]^1, [\;]^2 \ldots]$$

Periodicity

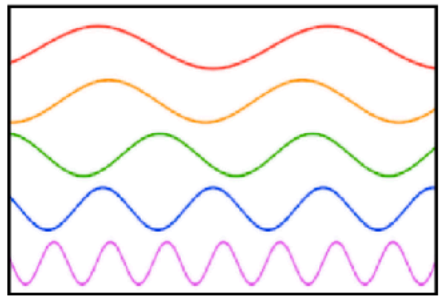
From yesterday:

Fourier Decomposition: represent an **infinite-length** signal as the summation of a number of **infinite-length** cosine and sine waves.

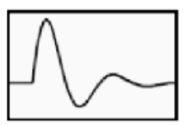
$cos(\omega t)$

$sin(\omega t)$

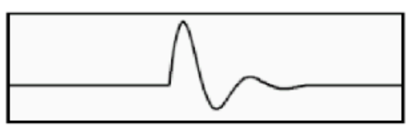
$for\ t \in (-\infty, \infty)$



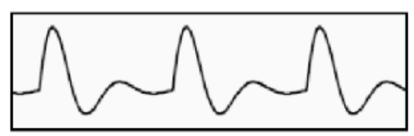
But real life signals are always **finite**.



extend as aperiodic signal (with 0s)



extend as periodic signal



Also from yesterday:

$\{\mathbf{x_n}\} := x_0, x_1, \ldots, x_{N-1}$ into another sequence of complex numbers, $\{\mathbf{X_k}\} := X_0, X_1, \ldots, X_{N-1}$, which is defined by

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \end{aligned} \qquad \textit{(Eq.1)}$$

We can measure the contribution of a single frequency, f, but which frequencies to include so we can fully decompose the signal?

The answer is in the equation: **k goes from 0 to N-1**

You need as many frequencies as you have time points!

X_k is infinite-length
(there are infinite # of coefficients)

k goes from 0 to N-1
(there are N coefficients)

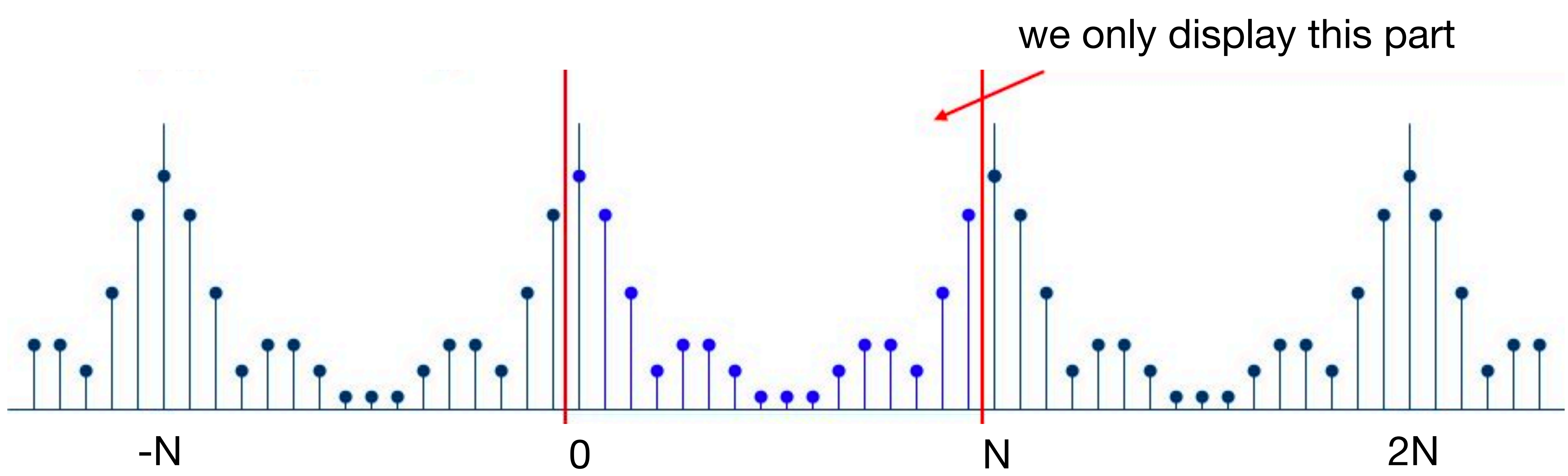
WTF?

Periodicity

Answer:

X_k is periodic, with period N.

-> k repeats itself after every N



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Periodicity

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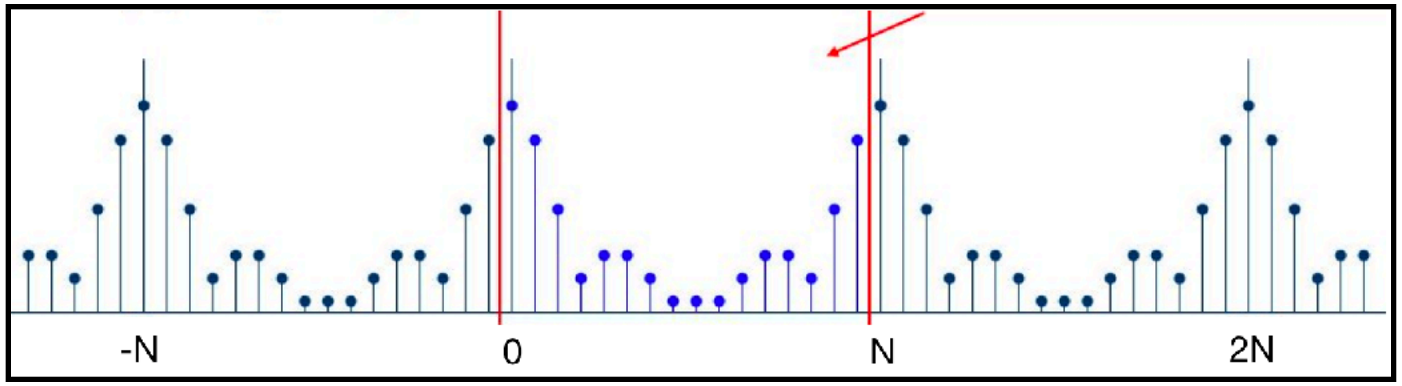
Time Domain

Frequency Domain



DFT transforms an infinite-length periodic signal in time
to infinite-length periodic signal in frequency

Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

Inverse Fourier Transform

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \cdot e^{i2\pi kn/N}$$

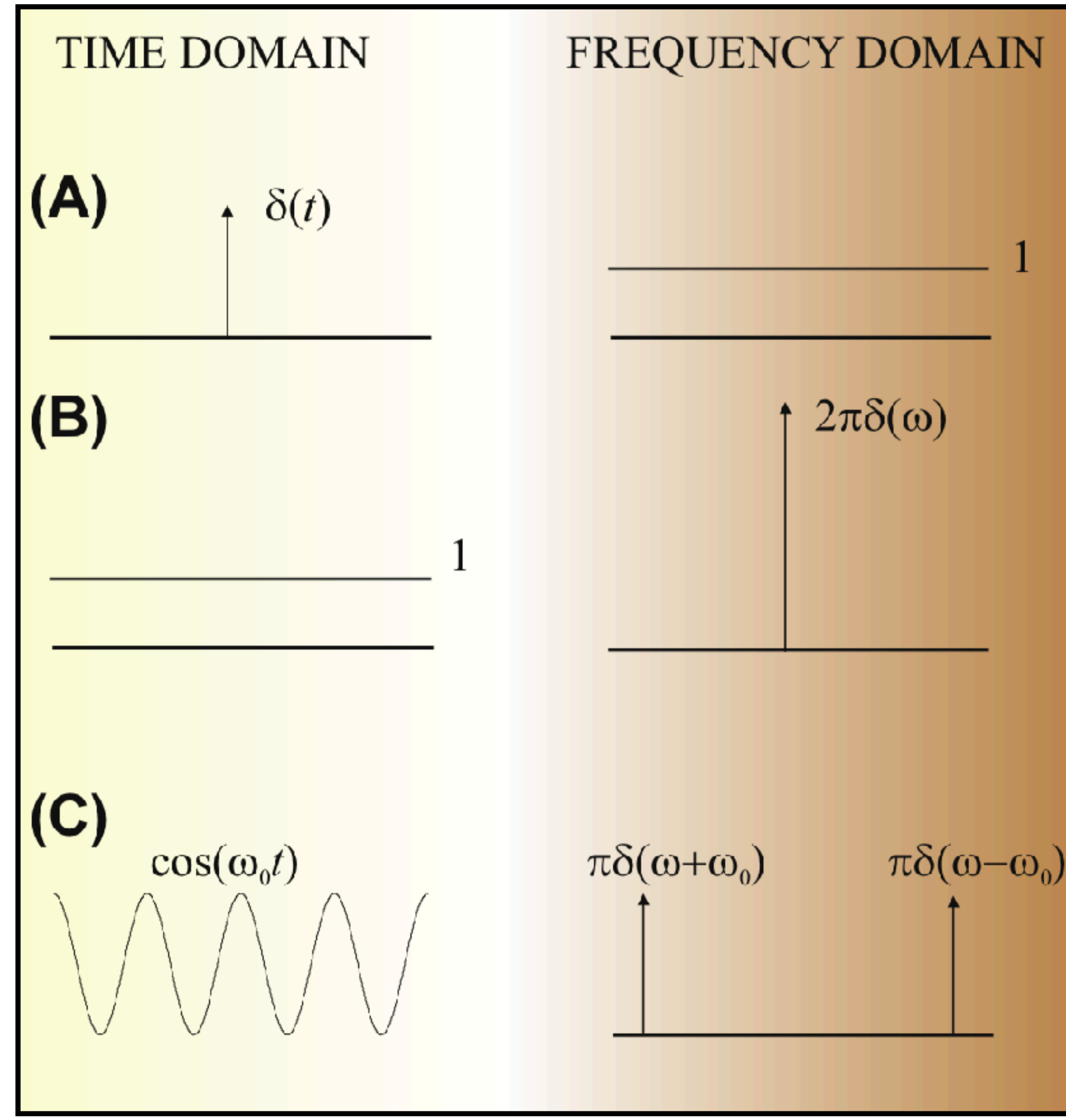
They are identical!
(up to a constant and sign)

Implication: as far as your digital computer is concerned, the sequence of number can be in time or frequency domain.

delta & (co)sines have a special relationship

TABLE 6.1 Examples of Fourier Transform Pairs

Time/Spatial Domain $f(t)$	Frequency Domain $F(\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$



(Circular) Convolution Theorem

Convolution in time domain is equivalent to multiplication in frequency domain, and the converse is true as well (duality).

x: input
h: system's IRF
y: output

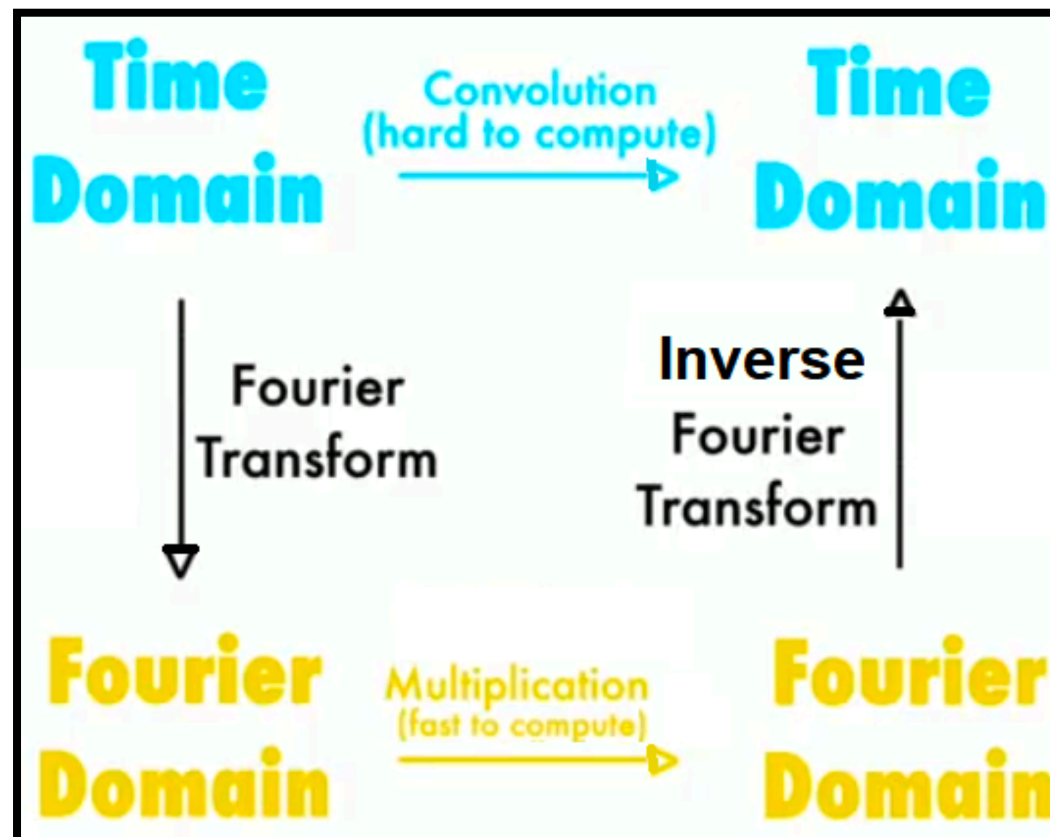
$$x(t) \circledast h(t) = y(t)$$
$$X(f)H(f) = Y(f)$$

IRF: [2,-1,-1,0]

Input:
x(n) = [1,0,3,7,2,5]

deltas

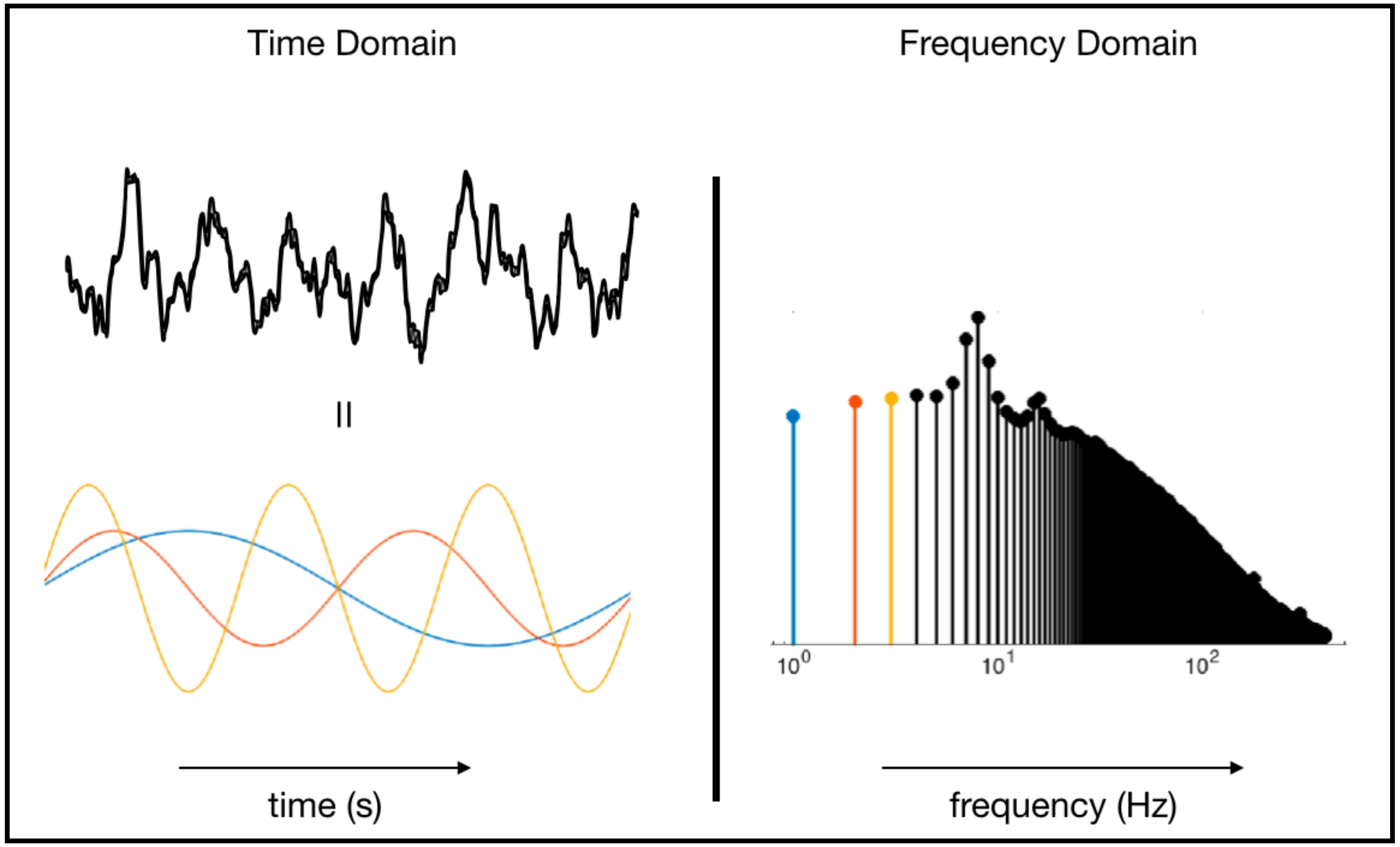
n =	0	1	2	3	4	5	6	7	8	9	10
1	2	-1	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0						
3	0	0	6	-3	-3	0					
7	0	0	0	14	-7	-7	0				
2	0	0	0	0	4	-2	-2	0			
5	0	0	0	0	0	10	-5	-5	0	0	
	2	-1	5	11	-6	1	-7	-5	0	0	0



Parseval's Theorem

Power is conserved in from time and frequency domain.

$$\boxed{\sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_k|^2}$$



Fourier Transform allows us to examine the power at each frequency.

"Spectral Analysis"

Parseval's Theorem

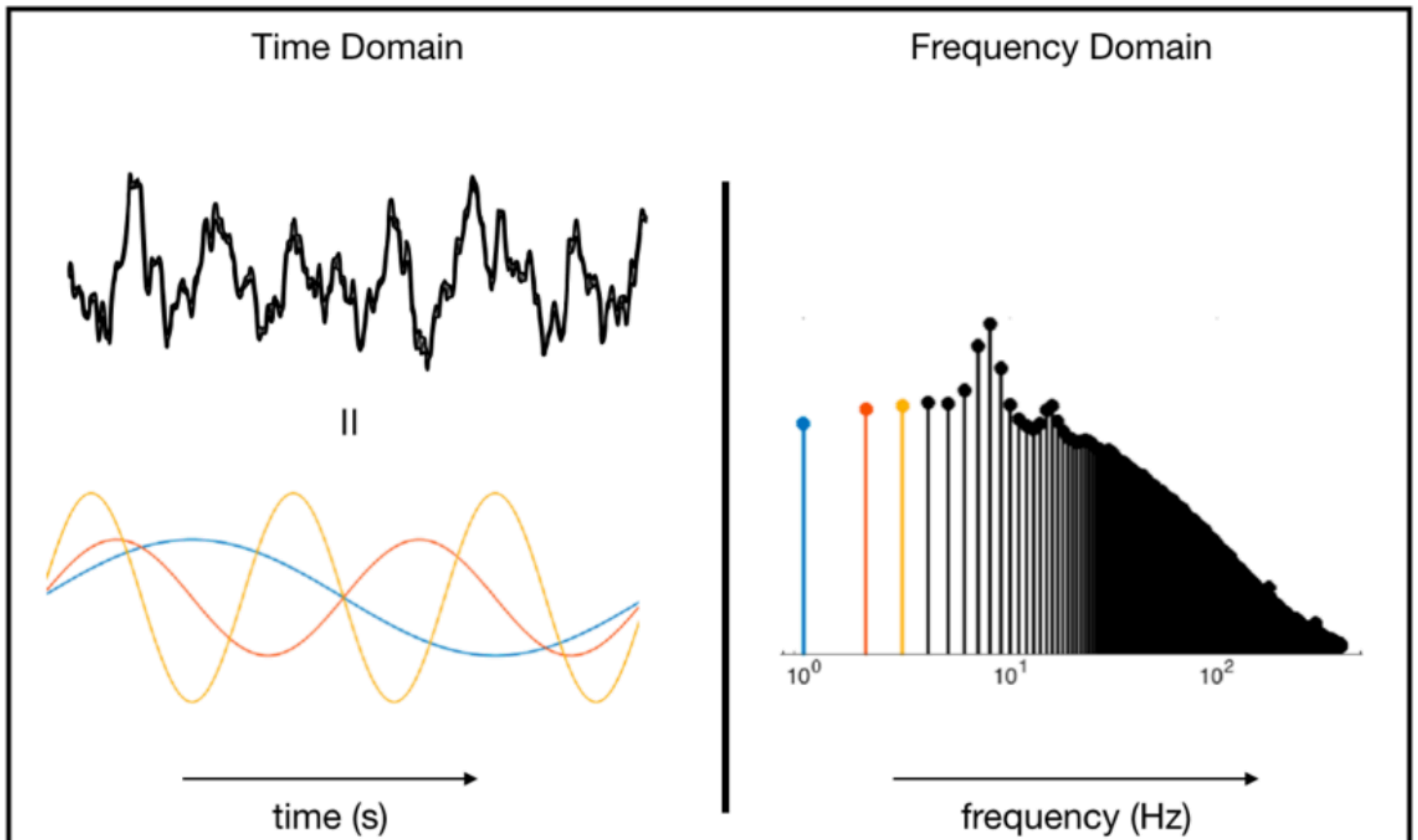
Power is conserved in from time and frequency domain.

$P(t) = V^2$

$$\boxed{\sum_{n=0}^{N-1}|x_n|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X_k|^2} = \sum_{n=0}^{N-1}(x_n - \cancel{\bar{x}})^2$$

$\boxed{\text{var}(x)}$

mean of x



Fourier Transform allows us to examine the power at each frequency.

"Spectral Analysis"

Handy Properties of DFT

Linearity

Orthogonality

Periodicity

Duality

(Circular) Convolution Theorem

Power Conservation (Parseval's Theorem)

Goals for Today

1. Some Fourier properties

2. Computing frequency from wave number

3. Time-frequency analysis

How Many Frequencies?

$\{\mathbf{x_n}\} := x_0, x_1, \ldots, x_{N-1}$ into another sequence of complex numbers,
$\{\mathbf{X_k}\} := X_0, X_1, \ldots, X_{N-1}$, which is defined by

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \end{aligned} \qquad \textbf{\textit{(Eq.1)}}$$

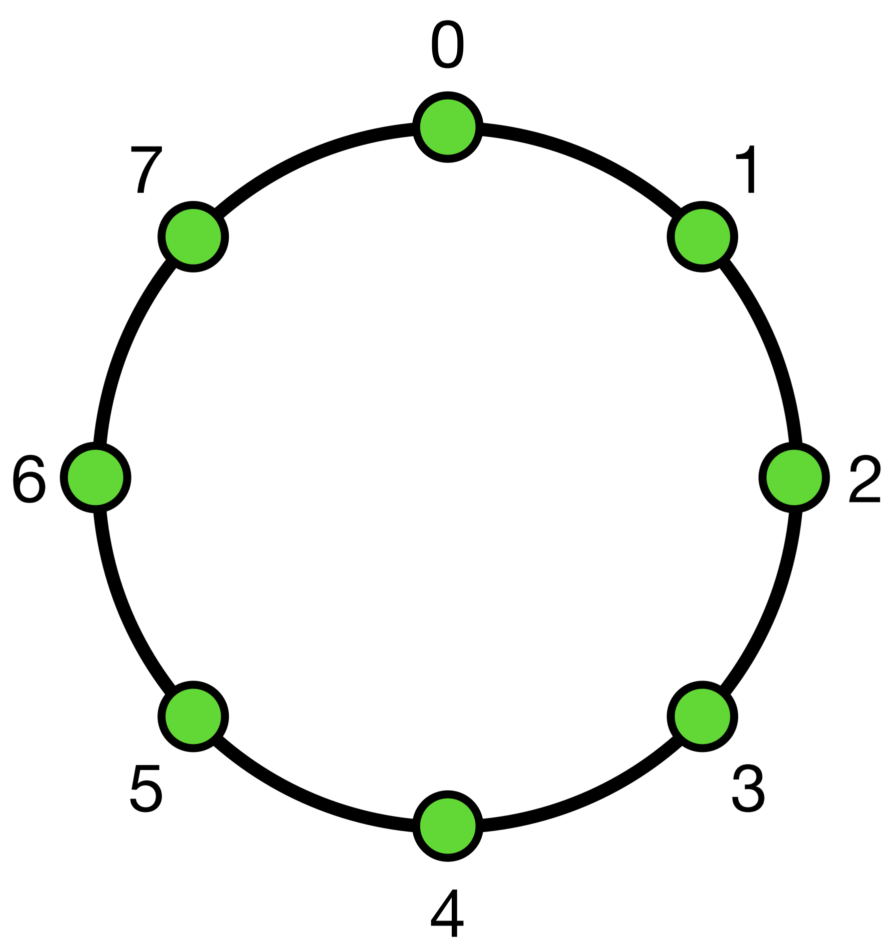
We can measure the contribution of a single frequency, f, but which frequencies to include so we can fully decompose the signal?

The answer is in the equation: **k goes from 0 to N-1**

You need as many frequencies as you have time points!

Wave Number to Frequency

$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N)$$

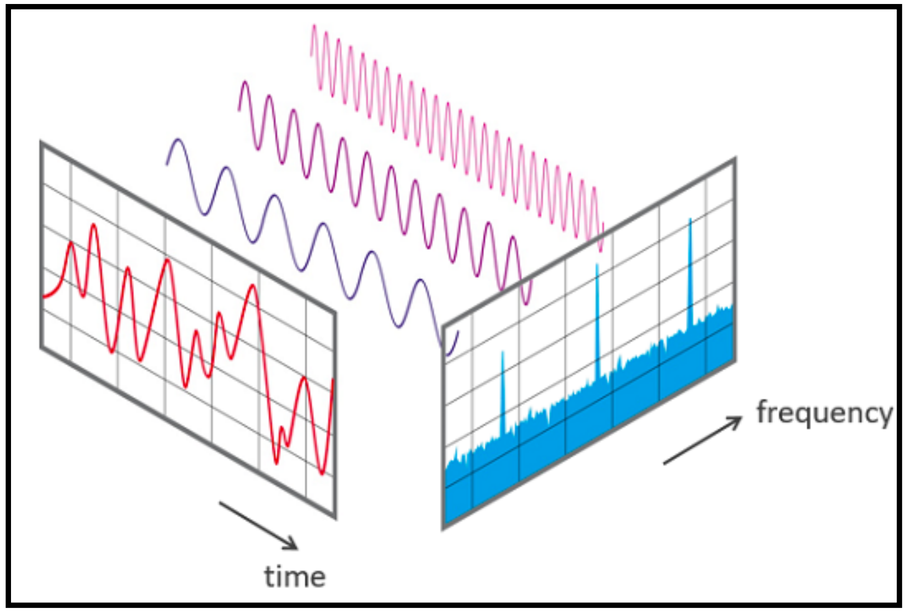


Suppose N = 8

-> k = 0, 1, … 7

for a given k, n = 0, 1, … 7

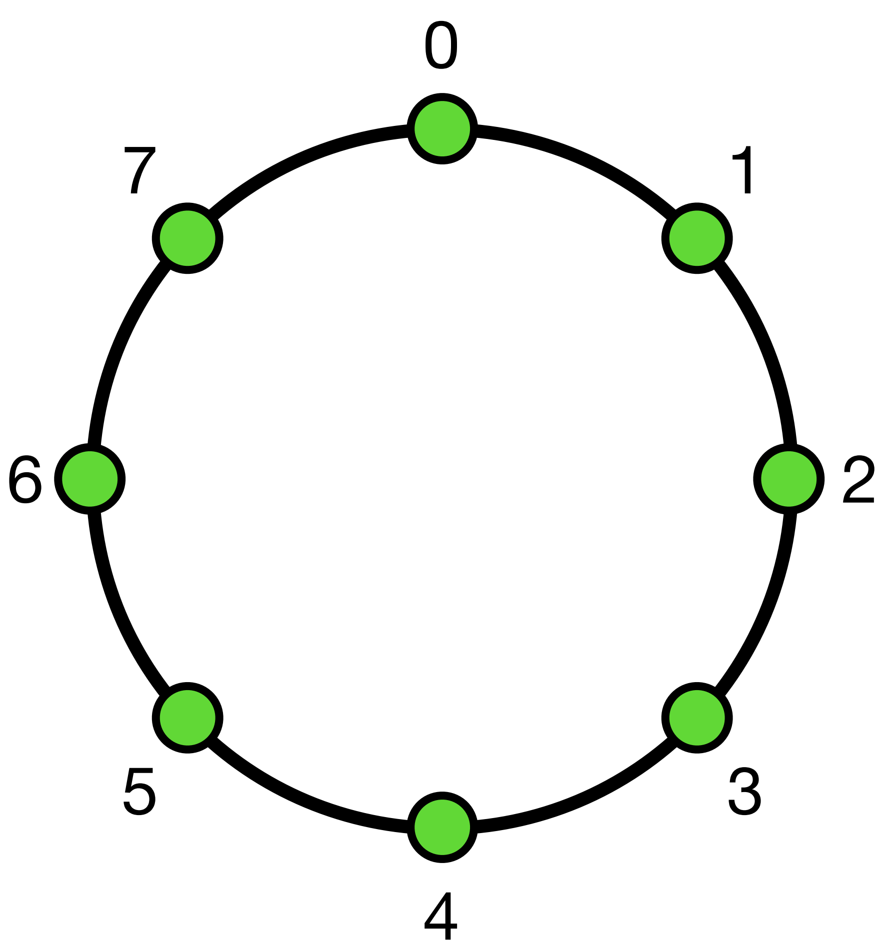
The Real Question: What frequency does the wave number **k** correspond to in Hz?



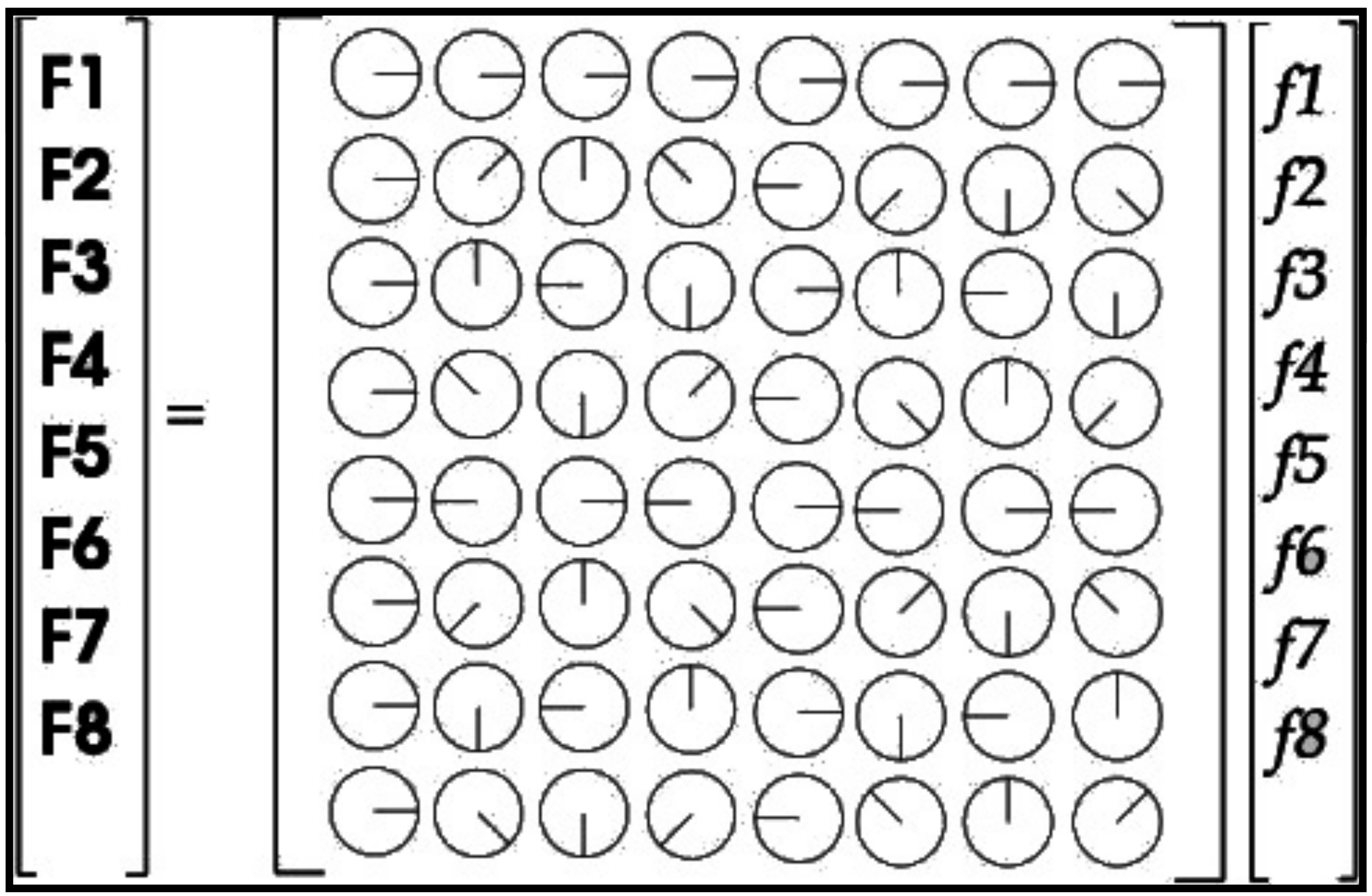
Wave Number to Frequency

At the risk of confusing
you even more...

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)]$$



k*n



Wave Number to Frequency

$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N)$$

The Real Question: What frequency does the wave number **k** correspond to in Hz?

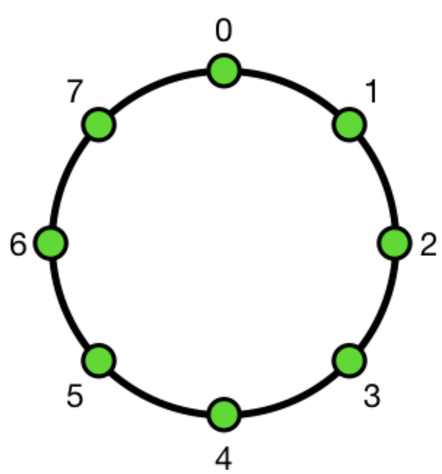
Time Domain Variables	**Frequency Domain Variables**
dt = sampling interval	k = wave number
T = signal duration	**df = frequency interval**

fs = sampling rate
N = # of samples / frequencies

Wave Number to Frequency

$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N)]$$

The Real Question: What frequency does the wave number **k** correspond to in Hz?



wave k goes around the circle k times after N steps (k cycles)

N steps spans T seconds

so **wave k** goes through k cycles in T seconds

what's its frequency?

max f $= \frac{N-1}{T}$

next one: $\frac{N}{T} = fs$

$$freq(k) = \frac{k}{T} = \frac{k}{N\ dt} = \frac{k}{N\frac{1}{fs}} \qquad df = \frac{1}{T}$$

Wave Number to Frequency

$$freq(k) = \frac{k}{T} = \frac{k}{N\,dt} = \frac{k}{N\frac{1}{fs}} \qquad df = \frac{1}{T}$$

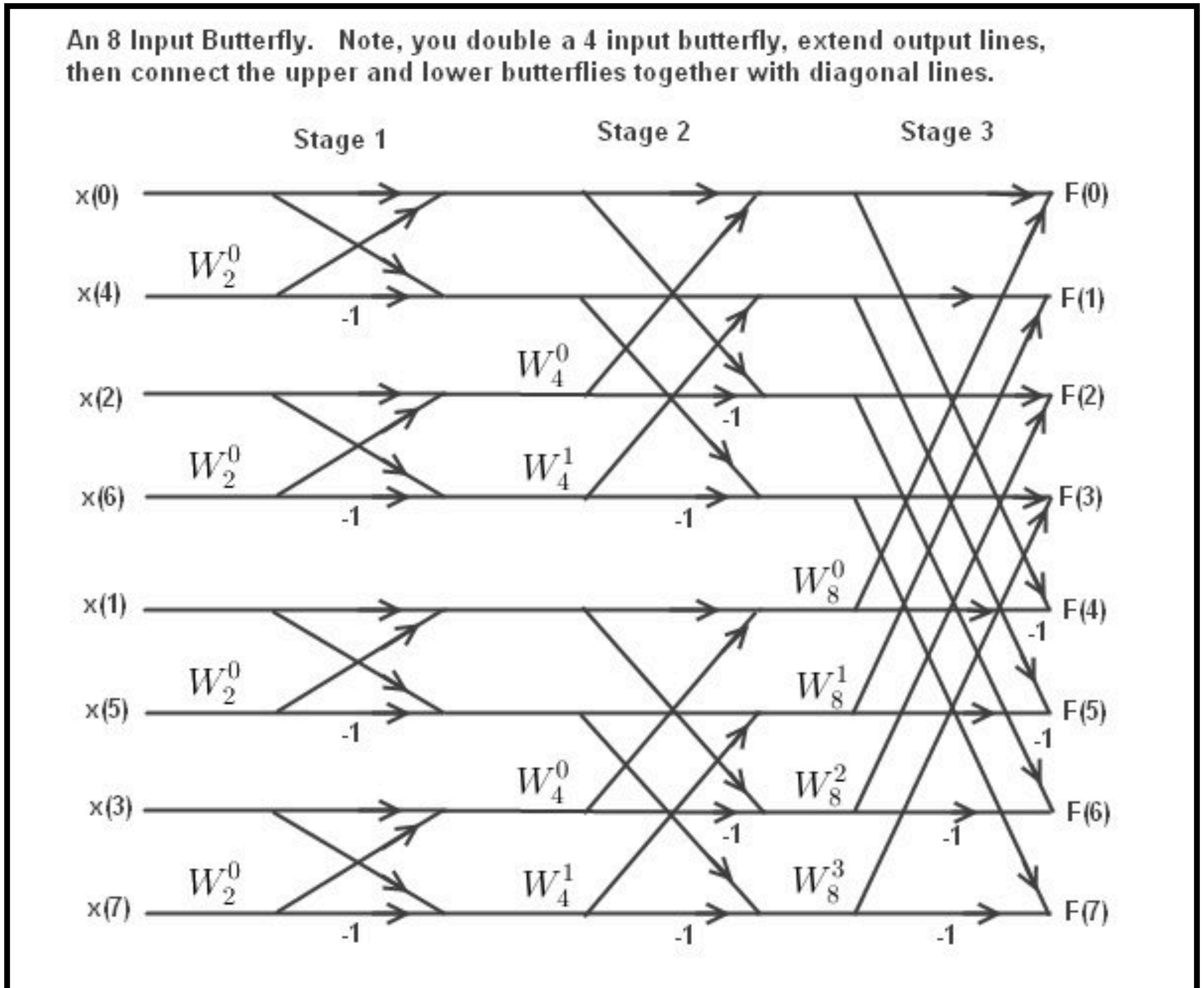
WvD Chapter 6

TABLE E6.4 DFT Time- and Frequency Domain Parameters

T (s)	s (samples/s) fs	Δf (Hz) df	F (Hz)
10	10,000	—	—
—	—	1	50
—	—	0.5	10
5	—	0.2	—
5	100	—	—
—	1000	—	500
—	10,000	1	—

Note on DFT & FFT

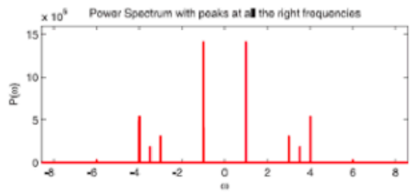
Fast Fourier Transform (FFT) is a clever algorithm that takes advantage of symmetries and recurrence in the data. It does **not** implement DFT as you do.



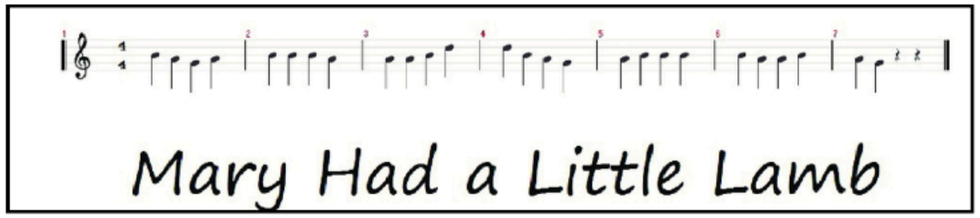
1. Some Fourier properties
2. Computing frequency from wave number
3. Time-frequency analysis

Time-Frequency Analysis

Can you play a song from knowing its DFT or power spectrum?



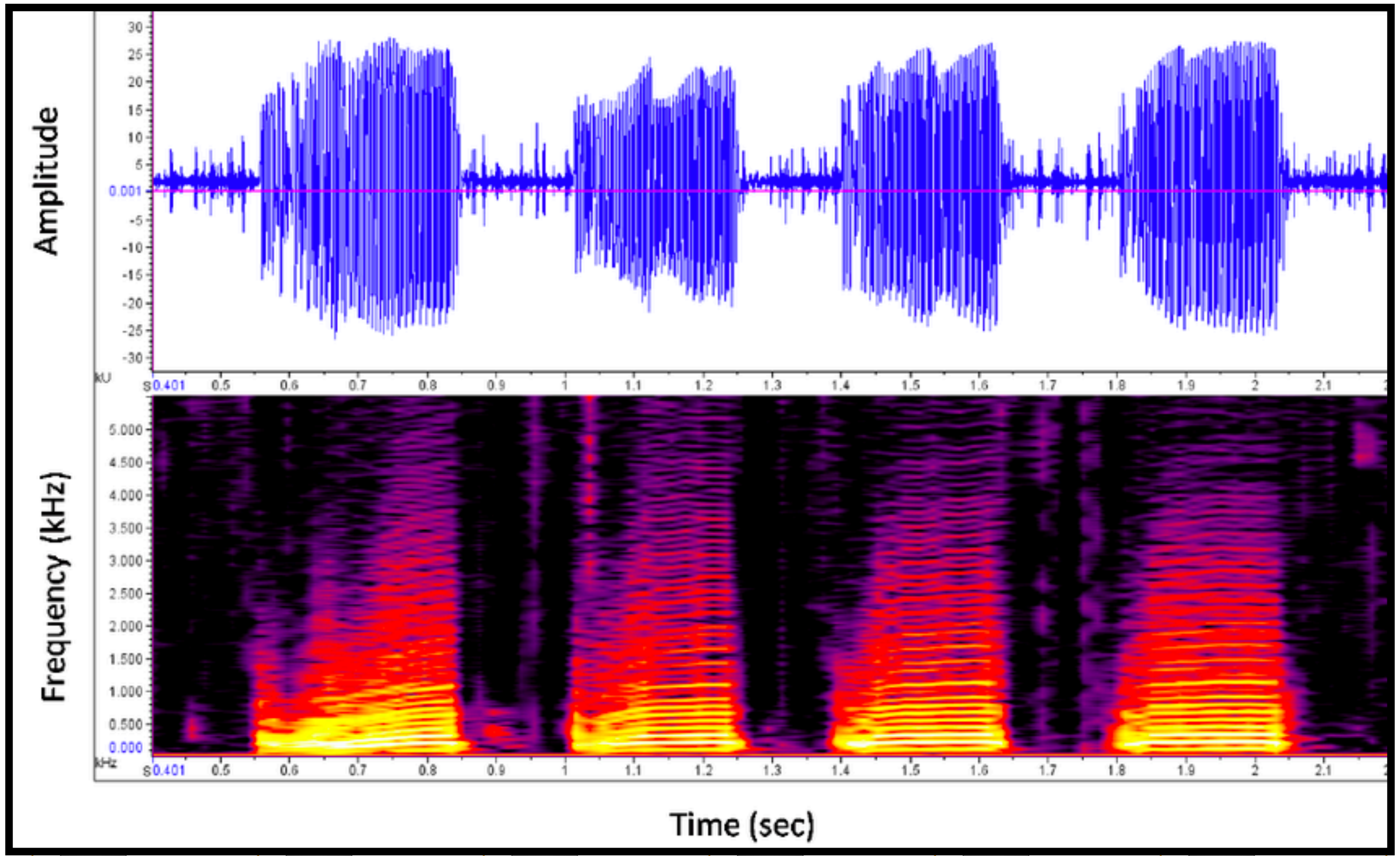
G A B D



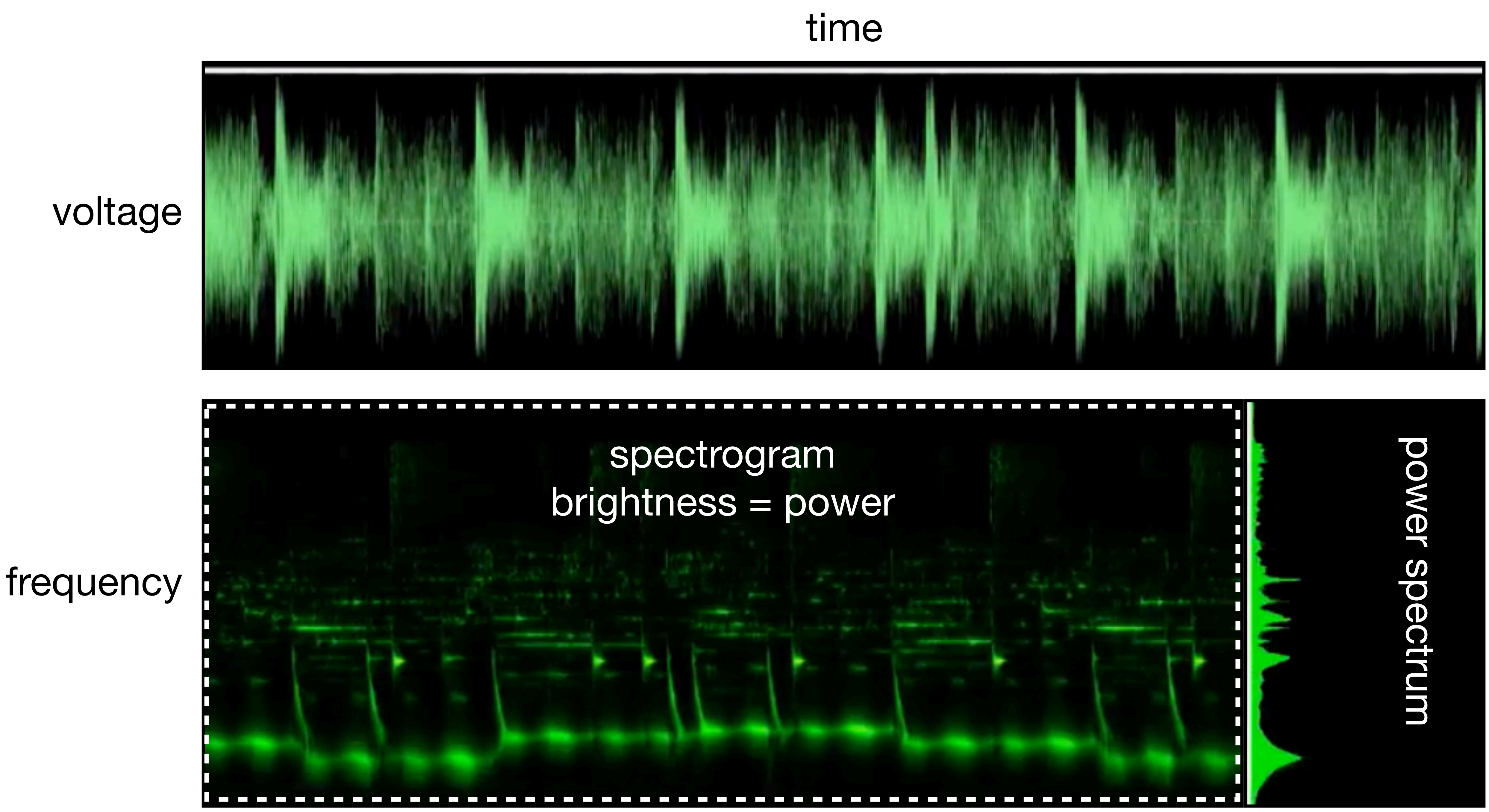
Looking at the DFT or power spectrum is like averaging over all the pictures in a movie, or all the notes in a song.

You know **what notes** are in the song, but not **when** they are played.

We need time-frequency analysis, or **spectrogram**.

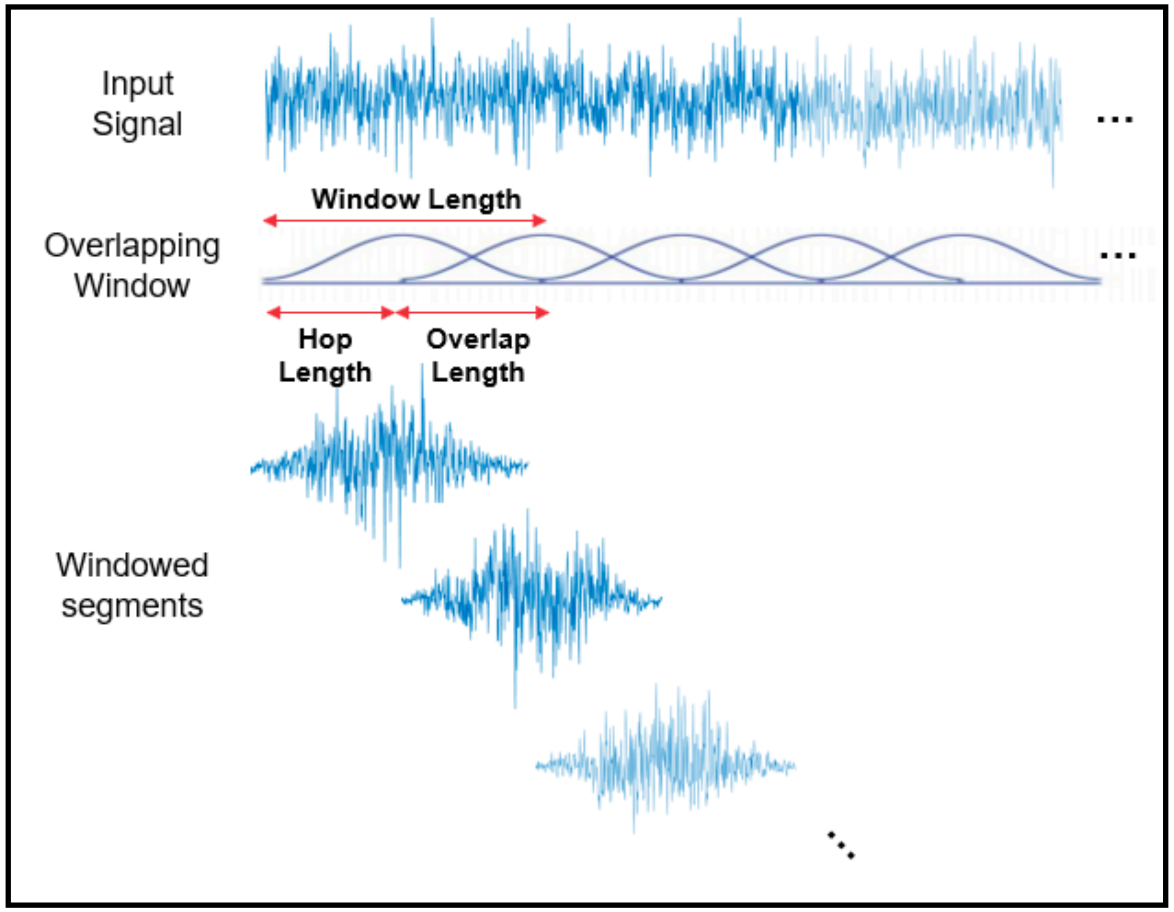


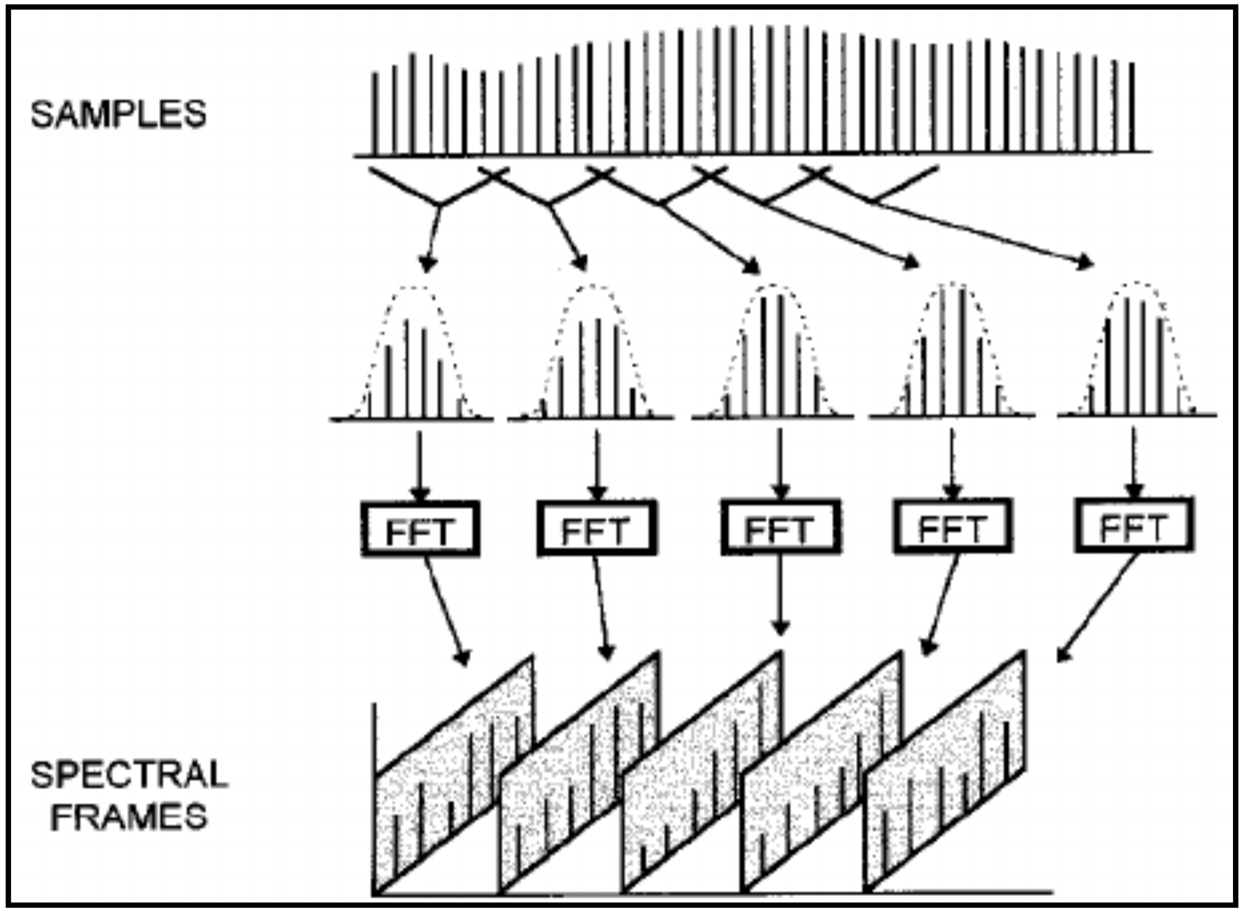
Time-Frequency Analysis

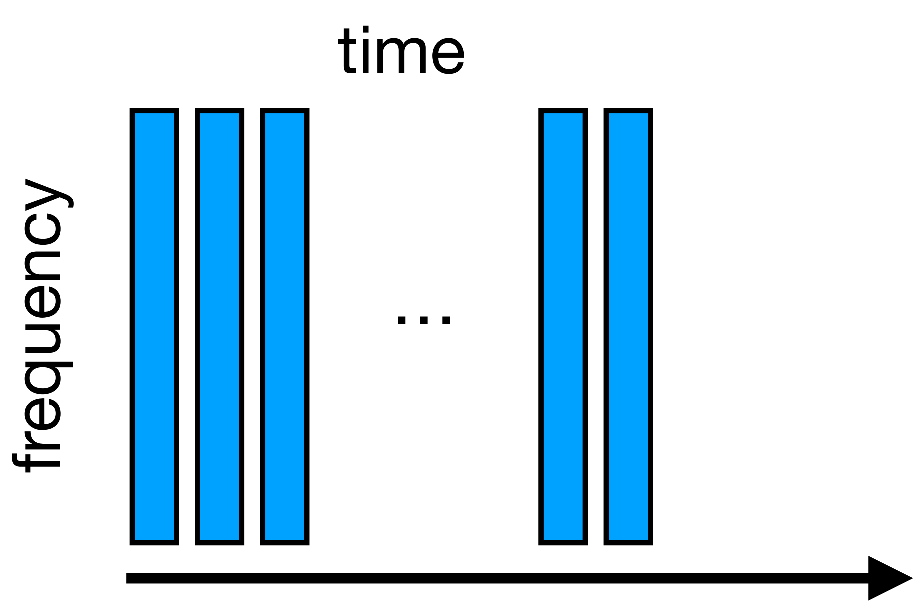


Sliding Window Analysis

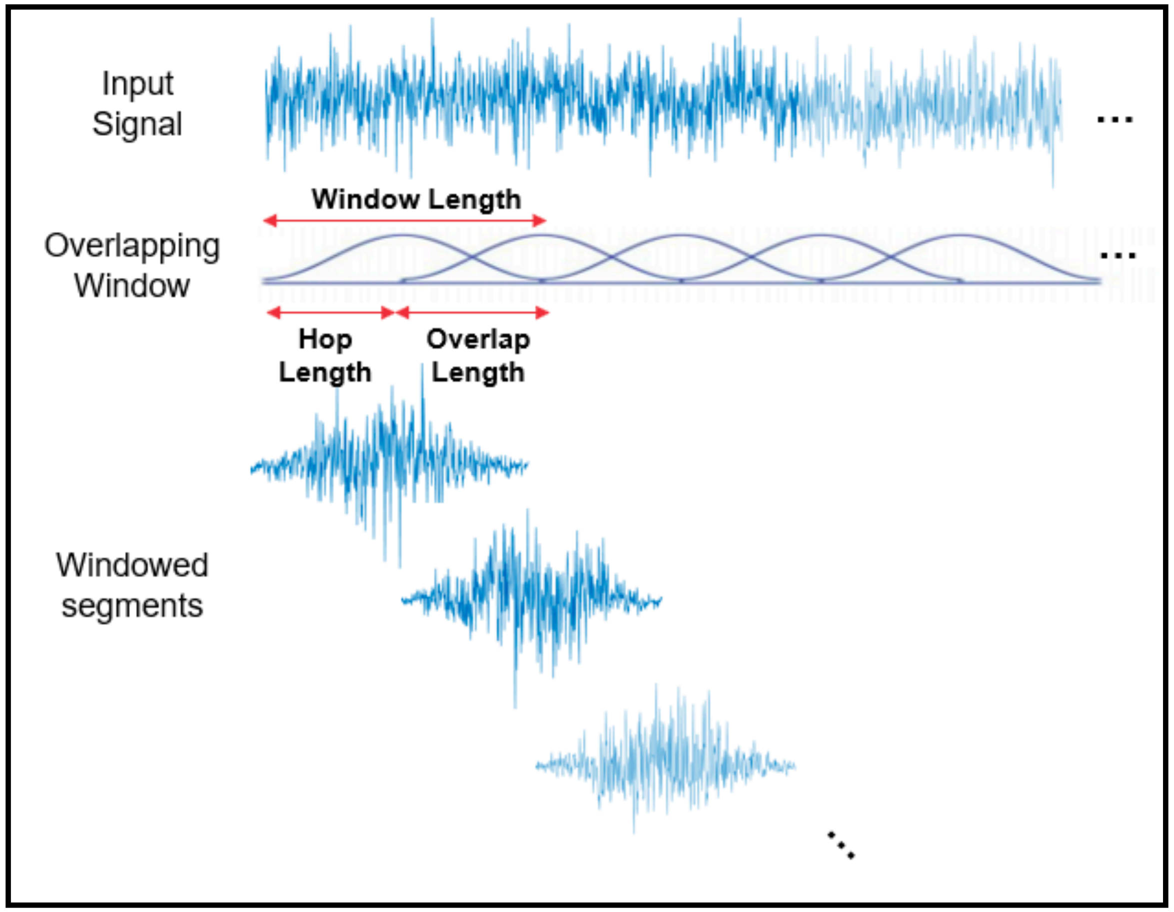
Short Time Fourier Transform (STFT)

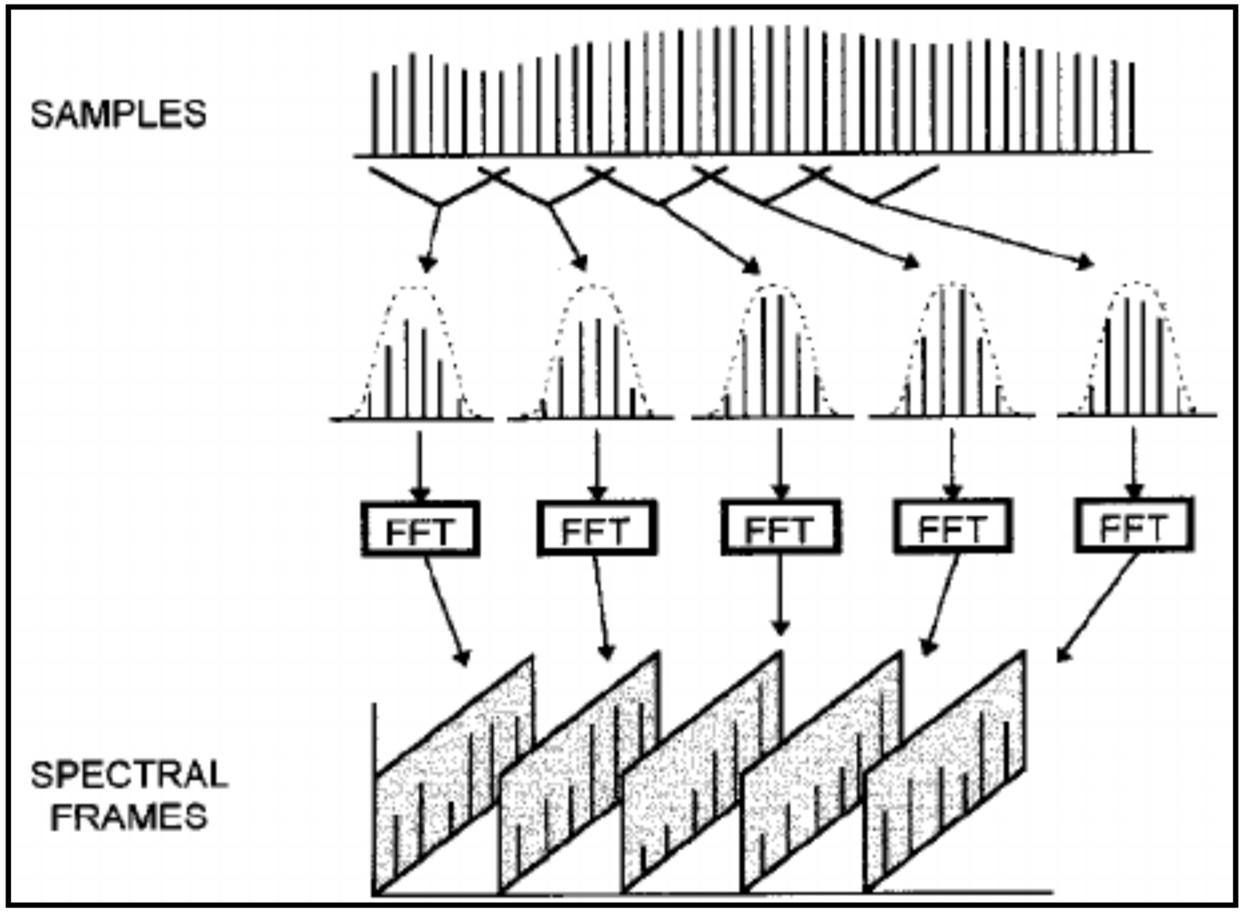


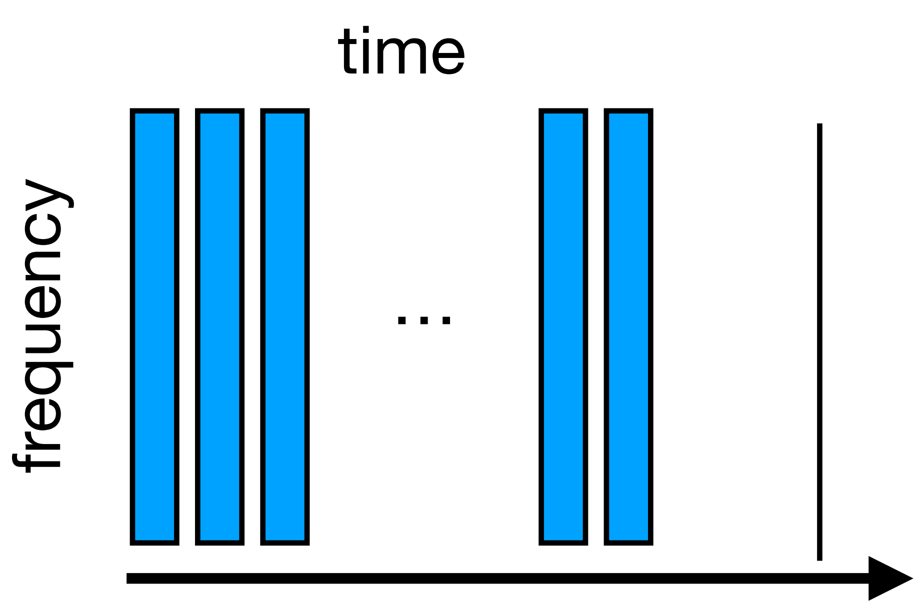




Short Time Fourier Transform (STFT)





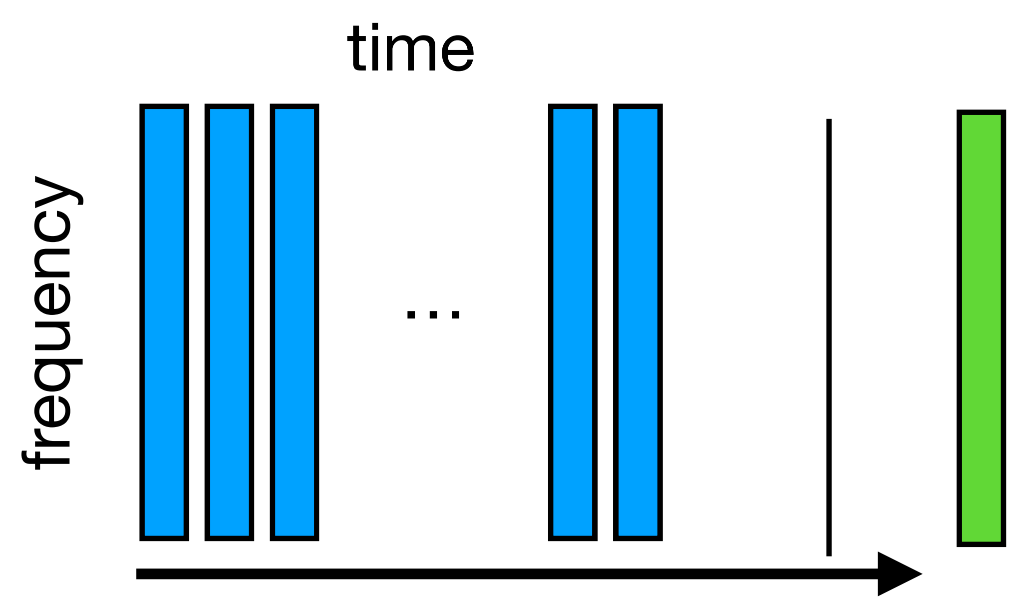


average across time

Power Spectral Density (PSD)

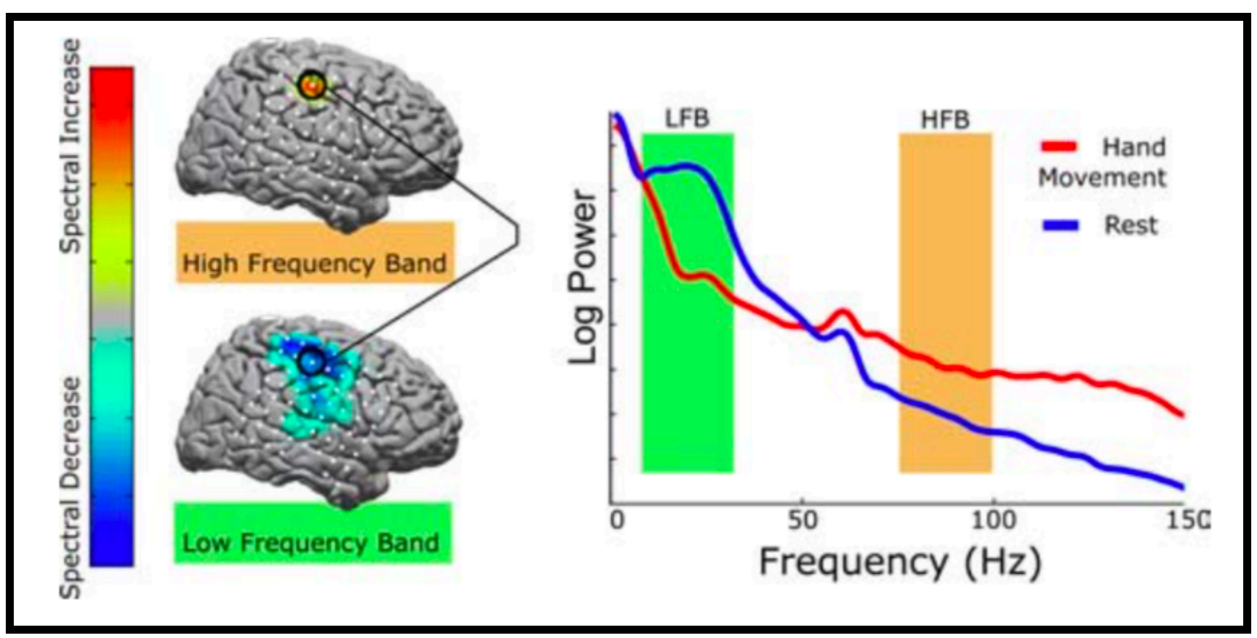
Power Spectral Density

Welch's (overlapping windows) **or**
Bartlet's (non-overlapping windows)
Method



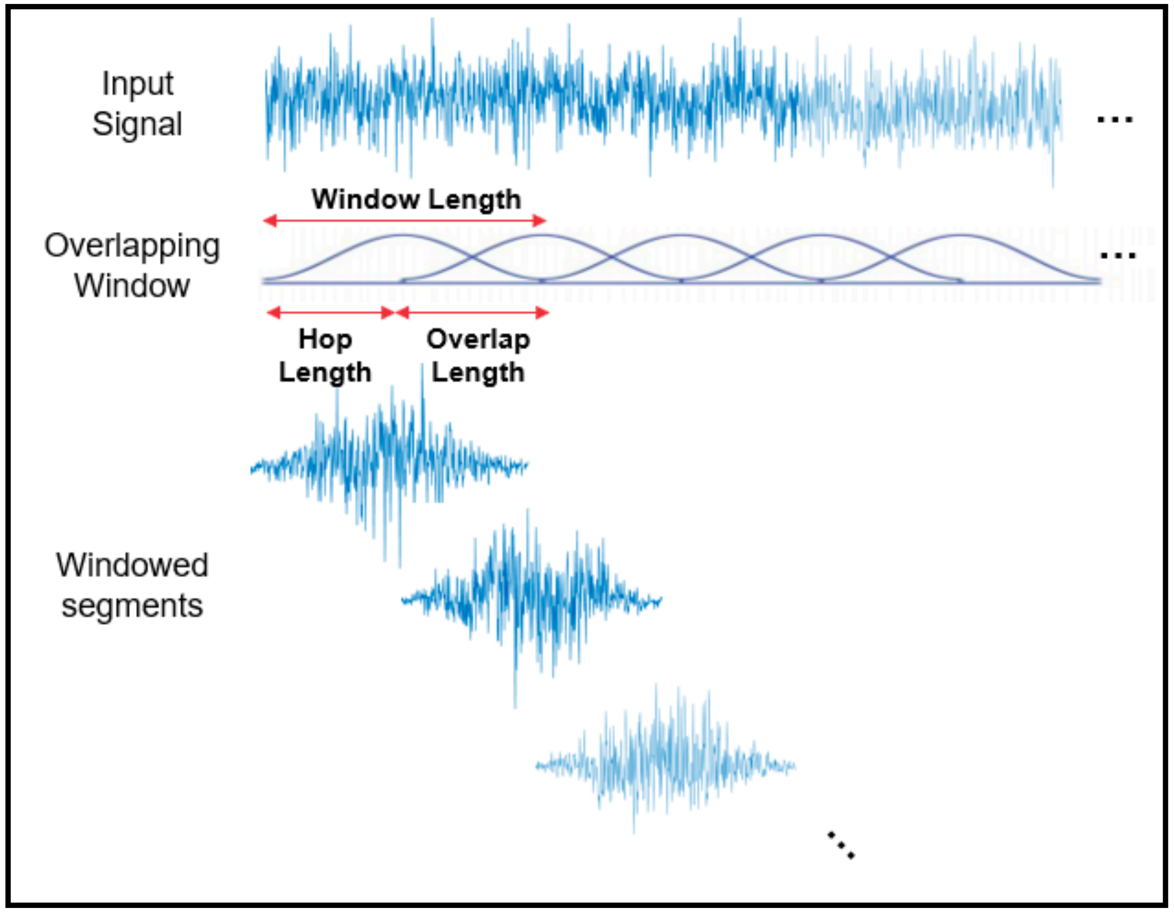
average across
time

Power Spectral Density (PSD)



Smooth estimate of the power spectrum.

Time-Frequency Resolution Tradeoff

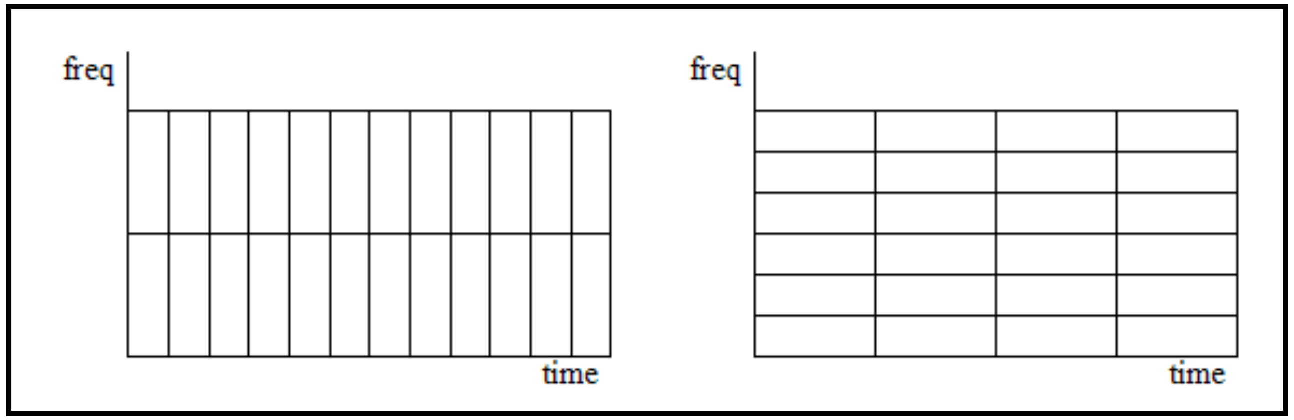


$$freq(k) = \frac{k}{T} = \frac{k}{N\,dt} = \frac{k}{N\frac{1}{fs}}$$

$$df = \frac{1}{T}$$

If your data was collected over a total of 60 seconds, and divide it into 12 windows, you can now estimate power every 5 seconds.

But what is your frequency resolution now?



1. Some Fourier properties
2. Computing frequency from wave number
3. Time-frequency analysis

https://tinyurl.com/cogs118c-att

Midterm Info

1 full sized cheatsheet on plain white (or lined) paper

60 min long: before or after reading discussion?

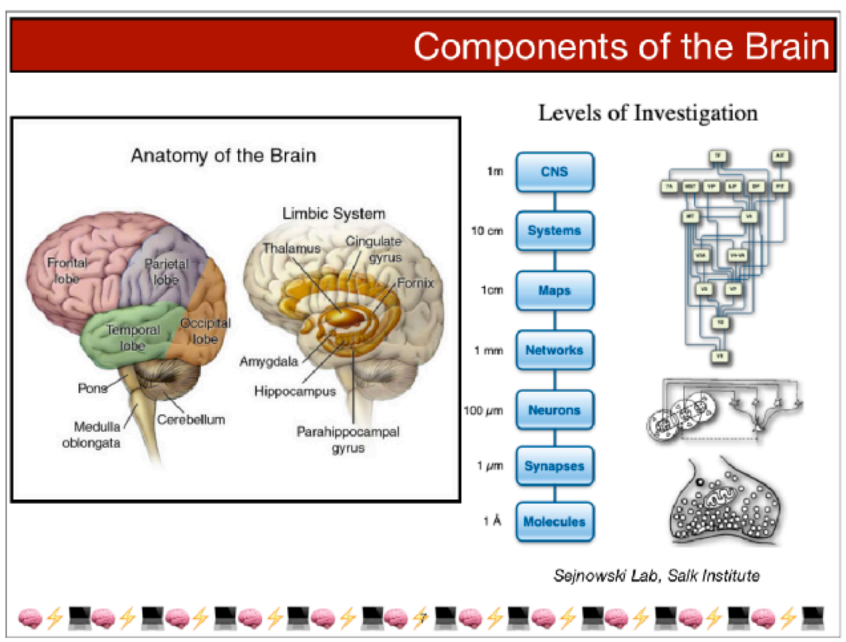
multiple choice, short answers, and math

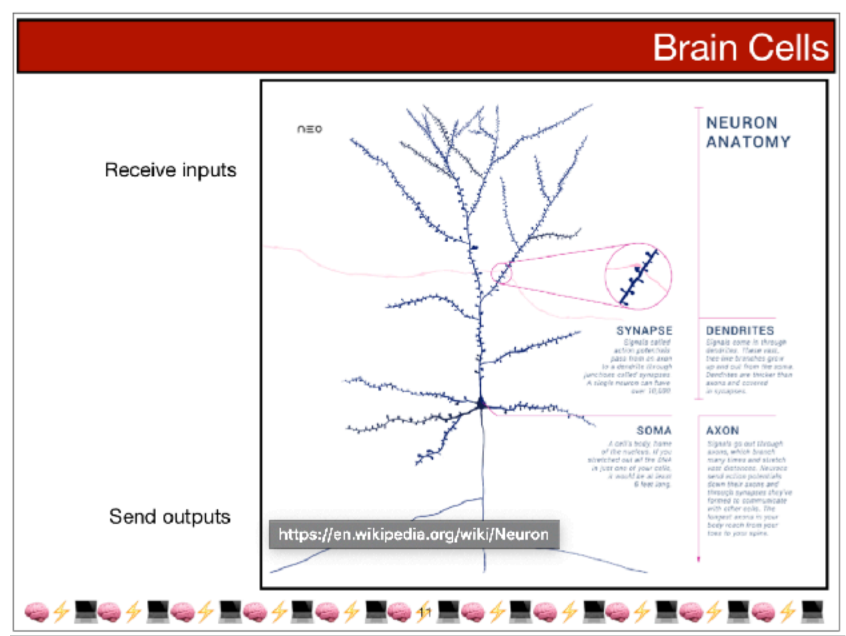
L2 - L7

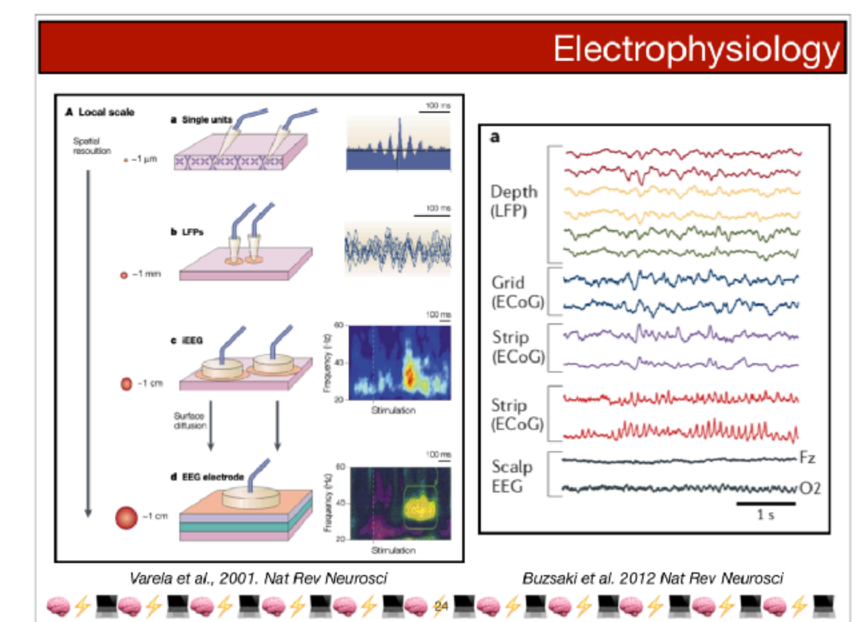
Midterm Info

L2 - physiology

Know the stuff on the highlighted slights & notes
Anatomy, physiology, spatiotemporal scales of methods
Different electrophysiology methods & scales







Midterm Info

L3 - sampling
digitization & sampling
ADC resolution calculations
time resolution & Nyquist rate

L4 - noise
ERP paradigm + reading!
A1 material
ms, rms, SNR calculations
sources of noise

L5 - LTI system
LTI properties
simple convolution calculation
definitions (delta, LTI, etc)

L6 - Fourier
family of FT
A2 material
complex exponential
dot product
wavenumber & frequency

L7 - Fourier
Fourier properties
time & freq variable calculations

definitions, calculations applications, & examples

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