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# *Proximity Effects in Altermagnetic Systems*

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a bachelor thesis.

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# Acknowledgment

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# 1 Introduction



Superconductivity has today a wide range of application like transportation with the levitating trains, mostly implemented in China [31] and Japan around 2027 [10]. Important imaginary methods have emerged, like terahertz imaginary. This frequency range is between the normal usecases of optical components (400-800 THz) and electrical devices (up to the GHz range), giving the opportunity to picture a wider range of materials. Further nuclear magnetical resonance imaginary (NMRI) takes advantage of the response of the nuclear spin to an external magnetical field to provide a non intrusive spectroscopy technique. To provide stronger field and better imaging definition, superconducting magnets are used [13]. In fact they allow stable and strong fields up to around 24T [12], whereas conventional permanent magnets are limited around 2T [8].

Beside superconducting magnetic energy storage (SMES) can store around one and a hundred MW of energy, with a discharge time of a few seconds. This makes this technology interesting for the grid stabilisation, providing a very responsive energy storage system [9]. SMES is based on a superconducting coil within a cryostat that stores the energy in the coil and releases it when needed. Due to the superconducting nature of the coil, an energy conversion efficiency of 95% is observed [9].

Superconductors were proven to exhibit different characteristics in comparison to normal conductors as Meissner and Ochsenfeld discovered in 1933 [2]. They experimented tin Sn and lead Pb being subject to a magnetic field below a critical temperature  $T_c$ . They observed that the magnetic field lines on the surface of the superconductor were ordering in such a way, that no magnetical flux lines were able to penetrate the superconductor ( $\mathbf{B} = 0$  inside). This has been called the Meissner effect and presents the superconductors as perfect diamagnets.

Fist a few words about the notation. When having multiple indices under a symbol we omit the comma to make the notation more readable. For example,  $x_{ij}$  is the same as  $x_{i,j}$ . However when we have complexer indices such as  $x_{i-1,j}$ , we keep the comma between all the indices.

We are going to use Hilbert-space's operators. As often in physics they will be multiplied with the wavefunction. This is the case for the Hamiltonian operator for example. However the second quantisation operators work differently. We will give a proper definition when the moment comes.

## 2 Theoretical Background



In order to describe the superconductors we are going to introduce the second quantisation formalism based on [11]. This framework allows us to picture the wavefunction of a system using creation and annihilation operators over energy states of the system. This simplifies a lot the notation. The mathematical foundation of this formalism lays in the Hilbert space and requires as well its dual space. To picture the many-particles interaction we need as well to introduce the Fock space. The mathematical accuracy is not the goal of this work, so some derivation could be done with more rigor.

We want to work on fermionic systems. The formalism stays the same for bosons but the results are fundamentally different. One can mention the Pauli principle as an example which only applies on fermions. After some definition we are going to see how to derive it with the help of the second quantisation.

Please note that we aim to describe the operators with a  $\hat{\cdot}$ . Later the expression will however involve a dense notation, such that we will drop the hat for the readability. The reader will stay informed about the nature of each symbols.

### 2.1 Bosons and fermions

We consider without loss of generality the following hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_I. \quad (1)$$

It's a very generic definition that uses a single particle operator  $\hat{H}_0$  and the interaction operator  $\hat{H}_I$ . The single particle part sums up the kinetic and potential energy of each particles. The interaction part gives room for complex relations between the particles.

$$\hat{H}_0 = \sum_{i \in [N]} \hat{h}_i(x_i), \quad \hat{h}_i(x_i) = -\frac{\hbar^2}{2m} \nabla_i^2 + \hat{U}(x_i).$$

We introduce the notation  $[N] = \{n \in \mathbb{N} : n \leq N\}$ . The potential part may depend from the particle's position  $\mathbf{r}$  or spin  $s$ . We use  $x_i := (\mathbf{r}, s) \in \mathcal{X} \subseteq \mathbb{R}^3 \times \mathbb{S}$  to group this information. For example we have for an electron  $\mathbb{S} = -\frac{1}{2}, \frac{1}{2}$ .

Further we describe a quantum state that a particle can occupy with a wavefunction  $\phi_\nu(x)$ , which is related to a certain energy  $\epsilon_\nu \in \mathbb{R}$ . This energy depends on the wavevector and the spin of the particle:  $\nu = (\mathbf{k}, \sigma)$ . The fundamental equation of quantum mechanics relates the wave function with the hamiltonian using the energy of the state:

$$\hat{h}\phi_\nu(x) = \epsilon_\nu\phi_\nu(x)$$

The wave function lay in the Hilbert space  $\mathcal{H}$ . Therefore  $\phi_\nu(x)$  are eigenfunctions or -states of the Hamiltonian with eigenvalues  $\epsilon_\nu$ . Further the wavefunction should build an orthonormal basis:

$$\int_{\mathcal{X}} \phi_{\nu'}(x)\phi_\nu(x)dx = \delta_{\nu'\nu}.$$

$\nu$  and  $\nu'$  are two different states. We introduced here the Kronecker delta  $\delta_{\nu'\nu}$ , which is one when the two indices are equal, and zero otherwise. Because the spin  $s$  is not continuous, one can understand the integral in the following way:

$$\int_{\mathcal{X}} dx = \sum_{s \in \mathbb{S}} \int_{\mathbb{R}^3} d^3r$$

where  $\int_{\mathbb{R}^3} d^3r = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} dr_1 dr_2 dr_3$ . In other words we integrate over all possible states.

Now that we can picture a single particle, the next question is: How to describe many instances of this particle? This is done with a many-body wavefunction, that sums up all possible

combinations of wavefunctions in the system and should stay normalized. A combination is illustrated as the product of each possible wavefunction of the particle in a certain state. These particles can be swapped and therefore we need to consider all the combinations. We aim to work with fermions but give a quick insight with bosons for comparison. We admit having  $N \in \mathbb{N}$  particles in the system.

**Bosons** Bosons are more flexible because not restrained by the Pauli principle. Their many-particle wavefunction is symmetric (exponent  $S$ ) under swap of two particles.

$$\Phi^{(S)}(x_1, \dots, x_N) = \left( N! \prod_{\nu} (n_{\nu})! \right)^{-\frac{1}{2}} \sum_{P \in S_n} P \phi_{\nu_1}(x_1) \cdots \phi_{\nu_N}(x_N).$$

This represents an eigenfunction of the non interacting bosonic-Hamiltonian. We used  $n_{\nu}$ , which represents the number of particles in the state  $\nu$ . Therefore we usually call it the occupation number of the state  $\nu$ . For bosons this integer has no constraint in general. The permutation set  $S_n$  contains all the possible combinations of  $x_i$  in the state  $\nu_j$  for  $i, j \in \llbracket N \rrbracket$ .  $P$  describes a permutation in the  $x_i$ .

For example with 2 particles we have  $x_1$  and  $x_2 \in \mathcal{X}$  and the latter part of the expression reads  $\phi_{\nu_1}(x_1) \cdot \phi_{\nu_2}(x_2) + \phi_{\nu_1}(x_2) \cdot \phi_{\nu_2}(x_1)$  and is thanks to the prefactor, normalized. We now see, the permutation aims to describe each particle (an  $x_i$ ) in each possible state  $\nu_i$ .

**Fermions** Fermions are a bit different, their many-particle wavefunction is antisymmetric under swap of two particles. We denote it as

$$\Phi^{(A)}(x_1, \dots, x_N) = (N!)^{-\frac{1}{2}} \sum_{P \in S_n} \text{sgn}(P) \cdot P \phi_{\nu_1}(x_1) \cdots \phi_{\nu_N}(x_N).$$

which is an eigenfunction of the non interacting fermionic-Hamiltonian.  $\text{sgn}$  represents the signum function. Applied on a permutation  $P$ , it's one if  $P$  is even and minus one if  $P$  is odd. We already know that the Pauli principle implies that we can find up to one particle in each energy state. We therefore have  $n_{\nu} \in \{0, 1\}$ . The normalization factor is the same but the product over the  $n_{\nu}$  is always one, we have  $0! = 1$  and  $\prod 1 = 1$ .

At this point one might have recognised the formula of the determinant

$$\Phi^{(A)}(x_1, \dots, x_N) = (N!)^{-\frac{1}{2}} \det \begin{pmatrix} \varphi_{\nu_1}(x_1) & \cdots & \varphi_{\nu_1}(x_N) \\ \vdots & & \vdots \\ \varphi_{\nu_N}(x_1) & \cdots & \varphi_{\nu_N}(x_N) \end{pmatrix}.$$

We usually describe this expression as the Slater determinant. A determinant vanishes if two rows or columns are identical. If they are identical, we have two particles in the same state. This means that the probability of finding two fermions in the same state is zero. This is the Pauli principle. Only one or no particle may occupy each state. From this we get the constraint for  $n_{\nu}$ .

However following this method may lead to a major problem. The many-particle wave function of fermions is defined up to a sign. For instance if we consider two particles "having"  $x_1$  and  $x_2$ , we have two possible states  $\nu_1$  and  $\nu_2$ . Two possible solutions are

$$\begin{aligned} \Phi^{(A_1)} &= \frac{1}{\sqrt{2}} (\varphi_{\nu_1}(x_1) \varphi_{\nu_2}(x_2) - \varphi_{\nu_1}(x_2) \varphi_{\nu_2}(x_1)) \\ \text{or } \Phi^{(A_2)} &= \frac{1}{\sqrt{2}} (\varphi_{\nu_1}(x_2) \varphi_{\nu_2}(x_1) - \varphi_{\nu_1}(x_1) \varphi_{\nu_2}(x_2)) \\ &= -\Phi^{(A_1)}. \end{aligned}$$

This sign difference may lead to computation errors. To solve this we must give a labeling to our states when we count them, and keep it when we come to build the Slater determinant.

These bosonic and fermionic wavefunctions are eigenstates of the Hamiltonian  $\hat{H}_0$  and the corresponding eigenvalue  $E_{\nu}$  is given by summing the energy of each state times its occupation

number:  $E_\nu = \sum_\nu \epsilon_\nu n_\nu$ . The orthonormal property of the single-particle wavefunction propagates itself:

$$\int_{\mathcal{X}^N} \Phi_a^*(x_1, \dots, x_N) \Phi_b(x_1, \dots, x_N) d^N x = \delta_{ab}.$$

Therefore we can expand any many-particle wavefunction  $\Psi$  as the linear combination of these:

$$\Psi = \sum_a f_a \Phi_a(x_1, \dots, x_N)$$

where  $f_a$  is a coefficient and  $a$  a labeling.

What we just discussed is the so called first quantisation- or wave function formalism. Now we intend to introduce a shorter manner of describing our system.

## 2.2 The second quantisation

For a better description of the many-particle system we introduce a simpler notation. The second quantisation lays on three important objects. States, which are described as “kets”. We put any relevant information (e.g. quantum numbers) in the ket:  $|\mathbf{k}, \sigma, \dots\rangle$ . Then we need operators that act on these states to allow evolutions in the system. The second quantisation has two main operators. They can create and annihilates a state.

**States** In this section we describe a state as the number of particle that occupies each single-particle state. As explained before, we need to order the state:  $1 < 2 < \dots < N$ . We can then describe the wave function as follow  $|n_1, \dots, n_N\rangle$ .

Further the state where no particle are present is called the vacuum state, and we denote it as  $|0_{\nu_1}, \dots, 0_{\nu_N}\rangle = |0\rangle$ .

### 2.2.1 Second quantisation for fermions

It's now the time to define the operators for the fermionic case.

**Creation operator  $c_\nu^\dagger$**  The creation operator adds a particle in the concerned state and rephase it:

$$c_\nu^\dagger |n_1, \dots, n_\nu, \dots\rangle = (-1)^{\sum_{\mu < \nu} n_\mu} (1 - n_\nu) |n_1, \dots, n_\nu + 1, \dots\rangle$$

We intentionally discarded the hat on the  $c$  as said before. We notice the  $(1 - n_\nu)$  term, which avoids to create a particle in an already occupied state. This is the way we express the Pauli-principle. Further we can construct a state by applying this operator one after another on the vacuum state. To avoid the minus one adding a negative sign, we start from the end and add the particle backwards in the order of the state:

$$|n_1, \dots, n_N\rangle = (c_1^\dagger)^{n_1} \cdot \dots \cdot (c_N^\dagger)^{n_N} |0\rangle$$

**Annihilation operator  $c_\nu$**  Likewise the annihilation operator destroys a particle in the corresponding state. The operator reads

$$c_\nu |n_1, \dots, n_\nu, \dots\rangle = (-1)^{\sum_{\mu < \nu} n_\mu} (n_\nu) |n_1, \dots, n_\nu - 1, \dots\rangle.$$

We can easily recognise that due to the  $n_\nu$ -term, destroying a particle that doesn't exist gives zero, so it's only possible to annihilate existing particles. Now that we can make our system evolve, we can bring in some rules the operators yield.

The anticommutator of two operator reads  $[A, B]_+ = \{A, B\} := AB + BA$  and is an operator as well. We're going to stick with  $[A, B]_+$  since it's more consistent with the commutator notation  $[A, B]_-$  (or simply  $[A, B]$ ).

The following results are obtained by separating the  $\nu = \mu$  from the  $\nu \neq \mu$ . We must also say that the dagger  $\dagger$  should be understood as the complex transpose of the operator:  $(AB)^\dagger = B^\dagger A^\dagger$ .

$$[c_\nu, c_\mu]_+ = 0 \quad (\text{fer}_1)$$

$$[c_\nu^\dagger, c_\mu^\dagger]_+ = 0 \quad (\text{fer}_2)$$

$$[c_\nu^\dagger, c_\mu]_+ = \delta_{\mu, \nu} \quad (\text{fer}_3)$$

Fermionic operators “anticommute”. We can combine the creation and annihilation operators to count the number of particles in a state:

$$c_\nu^\dagger c_\nu |n_1, \dots, n_\nu, \dots\rangle = n_\nu |n_1, \dots, n_\nu, \dots\rangle.$$

From this we can define the number operator  $\hat{n}_\nu := c_\nu^\dagger c_\nu$  which we can combine in the total number operator

$$\hat{N} = \sum_\nu \hat{n}_\nu, \quad \text{where logically } N = \sum_\nu n_\nu.$$

If we apply the total number operator on the system wavefunction, we obtain the total number of particles. We kept the hat on the operator to avoid the confusion with the actual number  $n_\nu$  of particles.

## Second quantisation description of the single- and two- particle operators

The next step is to translate the Hamiltonian in our formalism. First we introduce two basis element  $|\Phi_\alpha\rangle$  and  $|\Phi_\beta\rangle$ , which can be many-particles eigenstate of the system. We can also call them Slater determinants. Further we introduce the probability of the configuration  $|\Phi_\alpha\rangle$  to scatter into the  $|\Phi_\beta\rangle$  due to the action of an operator  $A$  (momentum, potential, interactions,...). This is described by the matrix element  $\langle\Phi_\alpha|A|\Phi_\beta\rangle$  which involves the single particle states  $|\alpha_1\rangle, \dots, |\alpha_N\rangle$  of  $|\Phi_\alpha\rangle$  and  $|\beta_1\rangle, \dots, |\beta_N\rangle$  of  $|\Phi_\beta\rangle$ .

$$\langle\Phi_\alpha|A|\Phi_\beta\rangle = \sum_{ij} C_{ij} \langle\alpha_i|A|\beta_j\rangle$$

involving some constants  $C_{ij}$ . This describes the overlap of the two configurations, after that we modified  $|\Phi_\beta\rangle$  with  $A$ . On the right hand side (r.h.s) we introduced the bracket scalar product notation. The bra  $\langle\alpha|$  lives in the dual space of the Hilbert space. One reads it as the complex transpose of  $|\alpha\rangle$ .

We recall the single particle Hamiltonian we introduced earlier. Its second quantisation representation reads

$$\hat{H}_0 = \sum_{i \in \llbracket N \rrbracket} \hat{h}(x_i) \rightsquigarrow \sum_{\alpha\beta} \langle\alpha|\hat{h}|\beta\rangle c_\alpha^\dagger c_\beta \quad (2)$$

where  $\alpha$  and  $\beta$  are single-particle states of the system.  $c_\alpha^\dagger c_\beta$  tries to transfer a fermion from the state  $|\beta\rangle$  to the state  $|\alpha\rangle$ . However this transition can be restricted by physical laws. This is the role of the matrix element. It returns zero, if the operation does not bring  $|\alpha\rangle$  into  $|\beta\rangle$ .

The equivalent expression in the wavefunction formalism is

$$\langle\alpha|\hat{h}|\beta\rangle = \int \varphi_\alpha^*(x) \hat{h}(x) \varphi_\beta(x) dx \quad (3)$$

where  $x$  still represents the position and the spin of the particle.

$\hat{h}$  is a single particle operator, it means it acts on one particle at a time. Two states are going to be changed.  $|\alpha\rangle$  loses a particle and  $|\beta\rangle$  gains one. We say for instance, that the configuration before the scattering is  $|\Phi\rangle$  and after the scattering is  $|\Phi'\rangle$ . This means, if our two Slater determinants  $|\Phi\rangle$  and  $|\Phi'\rangle$  differs in more than two state. There are some scatterings that we can't describe. As the result the overlap must be zero.

With a better sentence we allow only two states to be modified. Otherwise the single-particle states differ and due to their orthogonal properties, we get a zero.

Similarly for the two-particle operator we have

$$\hat{H}_I = \frac{1}{2} \sum_{i \neq j \in \llbracket N \rrbracket} \hat{v}(x_i, x_j) \rightsquigarrow \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle\alpha\beta|\hat{v}|\gamma\delta\rangle c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta \quad (4)$$

involving a more nested overlap of the four states  $\alpha, \beta, \gamma$  and  $\delta$ :

$$\langle\alpha\beta|\hat{v}|\gamma\delta\rangle = \int \int \varphi_\alpha^*(x) \varphi_\beta^*(x') \hat{v}(x, x') \varphi_\gamma(x) \varphi_\delta(x') dx dx'. \quad (5)$$



This expression modifies four states, so that the overlap of two Slater determinant vanishes, if the determinant differs in at least four states. We kept the hats to be consistent with the definition of  $\hat{H}_0$ .

The l.h.s of the equation is the matrix element  $\langle \Phi_\alpha | \hat{v} | \Phi_\beta \rangle$  of the operator  $\hat{v}$ , which involves two basis state  $|\Phi_\alpha\rangle$  and  $|\Phi_\beta\rangle$ .

What we defined here, is the bridge between the first and the second quantisation. In later expressions with proper states we could compute each formalism separately, and notice that both lead to the same result. It is important to note that both Eq.4 and 5 build a scalar.

It exists as well a second quantisation for bosons, but this is not the aim of this thesis. For additional informations, please refer to [11].

### 2.2.2 Basis transformation

Until now we considered the wavefunction in a restricted basis  $\{\varphi_\alpha(x)\}$ . A wave function is defined as the overlap between the basis and the state:

$$\phi(x) = \langle x | \alpha \rangle.$$

Let us now define a new, more general operator that creates a state  $|\alpha\rangle = a_\alpha |0\rangle$  for fermions. (This works as well for bosons). We assume that we have another basis  $\{|\tilde{\alpha}\rangle\}$ . We now want to show, that the new operator  $a_\alpha$  can be expressed as a linear combination of the other operator  $a_{\tilde{\alpha}}$ . This will be a useful tool, allowing us to jump from a basis to another. We first notice the following identity relation:

$$\mathbb{I} = \sum_{\alpha} |\alpha\rangle \langle \alpha| = \sum_{\tilde{\alpha}} |\tilde{\alpha}\rangle \langle \tilde{\alpha}|,$$

this allow us to write

$$a_\alpha^\dagger |0\rangle = |\alpha\rangle = \sum_{\tilde{\alpha}} \mathcal{I} |\alpha\rangle = \sum_{\tilde{\alpha}} |\tilde{\alpha}\rangle \underbrace{\langle \tilde{\alpha} | \alpha \rangle}_{\text{scalar}} = \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle |\tilde{\alpha}\rangle.$$

Please note that we note  $|\tilde{\alpha}\rangle \langle \tilde{\alpha} | \alpha \rangle = |\tilde{\alpha}\rangle \langle \tilde{\alpha} | \alpha \rangle$ . Inverting the indices leads to the same result, which yields to the transformation rules

$$\begin{aligned} a_\alpha^\dagger &= \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle a_{\tilde{\alpha}}^\dagger \\ a_\alpha &= \sum_{\tilde{\alpha}} \langle \alpha | \tilde{\alpha} \rangle a_{\tilde{\alpha}}. \end{aligned}$$

Further we can use these relations to express a wavefunction in the basis of another wavefunction. We have

$$\phi_\alpha(x) = \langle x | \alpha \rangle = \langle x | \left( \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle |\tilde{\alpha}\rangle \right) = \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle \langle x | \tilde{\alpha} \rangle = \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle \varphi_{\tilde{\alpha}}(x).$$

Inverting  $\alpha$  and  $\tilde{\alpha}$  works as well:  $\varphi_{\tilde{\alpha}}(x) = \sum_{\alpha} \langle \alpha | \tilde{\alpha} \rangle \phi_\alpha(x)$ .

Moreover we can show that the basis transformation is unitary. This is an important feature because we can simplify the calculation by changing the basis, and the result would remain the same if we use unitary transformations. Such transformations plays a major role later in the thesis. We can save the  $\langle \tilde{\alpha} | \alpha \rangle$  in a matrix element  $D_{\tilde{\alpha}\alpha}$ , and prove that the matrix  $D$  is unitary. Here we need  $\langle \gamma | \beta \rangle = \langle \beta | \gamma \rangle$ .

$$\begin{aligned} \langle \tilde{\alpha} | \tilde{\beta} \rangle &= \sum_{\gamma} \langle \tilde{\alpha} | \gamma \rangle \langle \gamma | \beta \rangle = \sum_{\gamma} \langle \tilde{\alpha} | \gamma \rangle \langle \beta | \gamma \rangle^* \\ &= \sum_{\gamma} D_{\alpha\gamma} D_{\beta\gamma}^* = \sum_{\gamma} D_{\alpha\gamma} D_{\gamma\beta}^\dagger = (DD^\dagger)_{\alpha\beta}, \end{aligned}$$

where  $\langle \tilde{\alpha} | \tilde{\beta} \rangle = \delta_{\tilde{\alpha}\tilde{\beta}}$  such that  $DD^\dagger = \mathbb{I}$ . The matrix  $D$  is therefore unitary.

The last important step is to show that the basis transformation keeps the anti-, commutation relations. Let us for the seek of readability use the notation  $[A, B]_\xi = AB + \xi BA$  where  $\xi = -1$  for bosons and  $\xi = +1$  for fermions. We have for exemple, using  $[a_\alpha, a_{\alpha'}^\dagger]_\xi = \delta_{\alpha\alpha'}$

$$[a_{\tilde{\alpha}}, a_{\tilde{\alpha}'}^\dagger]_\xi = \sum_{\alpha\alpha'} \langle \tilde{\alpha} | \alpha \rangle \langle \alpha' | \tilde{\alpha}' \rangle [a_\alpha, a_{\alpha'}^\dagger]_\xi = \langle \tilde{\alpha} | \tilde{\alpha}' \rangle = \delta_{\tilde{\alpha}\tilde{\alpha}'}.$$

The first step follows after splitting the commutator in two parts, insert the transformation and recombine the new commutator. The last step involves the orthonormality of the basis. On the same way one can prove  $[a_{\tilde{\alpha}}, a_{\tilde{\alpha}'}] = [a_{\tilde{\alpha}}^\dagger, a_{\tilde{\alpha}'}^\dagger] = 0$ .

### 2.2.3 Field operators

Later in this thesis we are going to use field opertors to describe an order parameter of a superconductive system. These are creation and annihilation operators that are defined in the  $|x\rangle$ -space-spin basis regarding antoher basis  $\{|\alpha\rangle\}$ . We here give the state as an argument and not as an index anymore. Despite the notation, the following operators musn't be confused with a wavefunction.

$$\hat{\psi}^\dagger(x) = \sum_{\alpha} \langle \alpha | x \rangle a_{\alpha}^\dagger = \sum_{\alpha} \varphi_{\alpha}^*(x) a_{\alpha}^\dagger \quad (6)$$

$$\hat{\psi}(x) = \sum_{\alpha} \langle \alpha | x \rangle a_{\alpha} = \sum_{\alpha} \varphi_{\alpha}(x) a_{\alpha} \quad (7)$$

involving fermonic or bosonic operators  $a$ . We can then annihilation and create a particle at a spin-space location  $x$ . Because the  $a$  operators are involded, the commutations properties should be respected. Using the result we had for  $[a_\alpha, a_{\alpha'}]_\xi$ ,  $[a_\alpha, a_{\alpha'}^\dagger]_\xi$  and  $[a_\alpha^\dagger, a_{\alpha'}^\dagger]_\xi$  where  $\xi = -$  for the bosons and  $+$  for the fermions. We find the following commutation relations:

$$\begin{aligned} [\hat{\psi}(x), \hat{\psi}(x')]_\xi &= [\hat{\psi}(x)^\dagger, \hat{\psi}^\dagger(x')]_\xi = 0 \\ [\hat{\psi}(x), \hat{\psi}^\dagger(x')]_\xi &= \delta(x - x') \end{aligned}$$

In the last expression we obtain a  $\langle x | x' \rangle$  which can't be normalsize the  $\{|x\rangle\}$ -basis. Instead of a Korenker delta  $\delta_{xx'}$  we get a delta distribution  $\delta(x - x')$ . This can be justified because the wavefunction is normalsize and can be "seen" as well as a distribution. The goal is now to describe the Hamiltonian using this fields operators. Thefore we rebuild these operators in the Hamiltonian using a  $\{|x\rangle\}$  basis

$$\begin{aligned} \hat{H}_0 &= \int \hat{\psi}^\dagger(x) \hat{h}(x) \hat{\psi}(x) dx \\ \hat{H}_I &= \frac{1}{2} \int \int \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') \hat{v}(x, x') \hat{\psi}(x') \hat{\psi}(x) dx' dx \end{aligned}$$

We already know that the first and second quantisation are equivalent. We could now insert in the Hamiltonian the definition of the field operators defined in Eq.6 and 7. From this obtain a wave function description. Then we can just use the first to second quantisation translation at Eq.3 and 5 to get a the second quantised representation. The result is a generalisation of the expression with the  $a$  operators.

## 2.3 Case of study: the interactive electron gas

As we are later going to study, the electron are allowed to interact with each other. The most simple interacting system involving electron is the electron-gas. During these interaction processes a photon is usaly carying the momentum transfert of the scattering from one electron to another. We're going to use the formalism we introduced to find a second quantisation representation of the interacting Hamiltonian.

The system we're studing is a cube of side length  $L$  with volumn  $\Omega = L^3$  containing  $N$  electrons. Further we consider a periodic boundary condition for an arbitrary position vector  $\mathbf{r} = (x, y, z)$ :

$$(L, y, z) = (0, y, z) , \quad (x, L, z) = (x, 0, z) , \quad (x, y, L) = (x, y, 0).$$

We use the general form of the Hamiltonian introduced in the beginning under the Eq.1. We consider a Coulomb potential and a kinetic energy term.

$$\hat{H}_0 = \hat{T} + \hat{U} = - \sum_{i \in \llbracket N \rrbracket} \frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i \in \llbracket N \rrbracket} \hat{u}(x_i) \quad (8)$$

$$\hat{H}_I = \hat{V} = \sum_{i \neq j \in \llbracket N \rrbracket} \hat{v}(x_i, x_j) \quad (9)$$

Where the pairwise potential is just a Coulomb potential  $\hat{v}(x_i, x_j) = \frac{e^2}{4\pi\epsilon_0|x_i - x_j|} = v(|x_i - x_j|)$ . We aim to describe the Hamiltonian in the momentum space, which is more convenient for the second quantisation. Therefore we first need to find an expression for the wavevector  $\mathbf{k}$ . We describe a state  $\alpha = (\mathbf{k}, \sigma)$  at a spin-space coordinate  $x = (\mathbf{r}, s)$ . First we take a plane wave solution of the Schrödinger equation:

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k} \cdot \mathbf{r}} \chi_{\sigma}(s)$$

The periodicity of the system allows us to write for each dimension  $\psi(x=0) = \psi(x=L)$ . Using these boundary conditions we obtain  $1 = e^{ik_x L}$  in all directions. The result reflects itself in the wavevector  $\mathbf{k}$ , which becomes quantised:

$$\mathbf{k} = \frac{2\pi}{L} (n_x, n_y, n_z), \quad n_i \in \mathbb{Z}.$$

As we saw earlier, the eigenfunctions build a complete basis:

$$\begin{aligned} \int \psi_{\alpha'}^*(x) \psi_{\alpha}(x) dx &= \int \sum_s \psi_{\alpha'}^*(x) \psi_{\alpha}(x) d^3 r \\ &= \frac{1}{\Omega} \int e^{i\mathbf{r} \cdot (\mathbf{k} - \mathbf{k}')} \underbrace{\sum_s \delta_{s\sigma} \delta_{s\sigma'}}_{\delta_{\sigma\sigma'}} d^3 r \\ &= \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'} = \delta_{\alpha\alpha'}. \end{aligned}$$

The integral over the exponential function diverges if  $\mathbf{k} \neq \mathbf{k}'$ , so we use the case  $\mathbf{k} = \mathbf{k}'$  and set the integral to zero otherwise. The kinetic energy operator is a single-particle operator. We have according to Eq.2:

$$T = \sum_{\alpha\alpha'} \langle \alpha | \frac{\hat{\mathbf{p}}^2}{2m} | \alpha' \rangle c_{\alpha}^{\dagger} c_{\alpha'}.$$

with  $\hat{\mathbf{p}}$  the impulse operator in all space directions:  $\hat{\mathbf{p}} = i\hbar\nabla$ . To compute this expression we use its first quantised form (see Eq.3) involving the  $\delta_{\sigma\sigma'}$  trick we introduced in the derivation of the complete basis. We also use  $\mathbf{k}' = \mathbf{k}$  and hide the  $\frac{1}{\Omega}$  is the  $\delta_{\mathbf{k}\mathbf{k}'}$ . The result reads

$$T = \sum_{\alpha, \alpha'} \delta_{\alpha\alpha'} \frac{\hbar \mathbf{k}^2}{2m} c_{\alpha}^{\dagger} c_{\alpha} = \sum_{\alpha} \underbrace{\frac{\hbar \mathbf{k}^2}{2m}}_{=: \epsilon_{\mathbf{k}}} c_{\alpha}^{\dagger} c_{\alpha}. \quad (10)$$

We recognise the occupation number operator  $\hat{n}_{\alpha} = c_{\alpha}^{\dagger} c_{\alpha}$  and the energy of the state  $\epsilon_{\mathbf{k}}$ . This variable plays a central role later. We obtain a quite meaningful result, the non-interacting energy part of the system is the product of the energy of a state with the number of particle within that state, summed over all states. The single-particle operator engage a  $\hat{u}$ -term as well. This can be used by a single-particle potential such as an external electric field in the case of the electrons. We will ignore this term.

For the interaction potential we have a two-particles operator. This is described by Eq.4 and requires Eq.5 to be solved:

$$\hat{V} = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \hat{v} | \gamma\delta \rangle c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta}.$$

We can first work on the matrix element using  $v(\mathbf{r}, \mathbf{r}') = v(\mathbf{r} - \mathbf{r}')$ :

$$\langle \alpha\beta | \hat{v} | \gamma\delta \rangle = \frac{1}{\Omega^2} \delta_{\sigma_{\alpha}\sigma_{\gamma}} \delta_{\sigma_{\beta}\sigma_{\delta}} \int \int e^{i\mathbf{r} \cdot (\mathbf{k}_{\gamma} - \mathbf{k}_{\alpha})} e^{i\mathbf{r}' \cdot (\mathbf{k}_{\delta} - \mathbf{k}_{\beta})} v(\mathbf{r} - \mathbf{r}') d\mathbf{r}' d\mathbf{r}.$$

After making a substitution  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ , adding and subtracting a  $(\mathbf{k}_\gamma - \mathbf{k}_\alpha)\mathbf{r}'$ , we obtain:

$$\langle \alpha\beta | v | \gamma\delta \rangle = \frac{1}{\Omega^2} \delta_{\sigma_\alpha \sigma_\gamma} \delta_{\sigma_\beta \sigma_\delta} \underbrace{\int e^{-i\mathbf{R}(\mathbf{k}_\alpha - \mathbf{k}_\delta)} v(\mathbf{R}) d\mathbf{R}}_{v_{\mathbf{k}_\delta - \mathbf{k}_\alpha}} \underbrace{\int e^{i\mathbf{r}'(\mathbf{k}_\gamma - \mathbf{k}_\beta + \mathbf{k}_\delta - \mathbf{k}_\alpha)} d\mathbf{r}'}_{=\delta_{\mathbf{k}_\gamma - \mathbf{k}_\beta + \mathbf{k}_\delta, \mathbf{k}_\alpha}}.$$

Here we see that the first underbraced term is the Fourier transform of a Coulomb potential  $v$  and the second term is a Kroneker delta. From this we derive a combuersom equation

$$\hat{V} = \frac{1}{2\Omega} \sum_{\substack{\mathbf{k}_\alpha \mathbf{k}_\beta \mathbf{k}_\gamma \mathbf{k}_\delta \\ \sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\delta}} \delta_{\sigma_\alpha \sigma_\gamma} \delta_{\sigma_\beta \sigma_\delta} \delta_{\sigma_\alpha \sigma_\gamma} \delta_{\sigma_\beta \sigma_\delta} \delta_{\mathbf{k}_\alpha, \mathbf{k}_\gamma - \mathbf{k}_\beta + \mathbf{k}_\delta} v_{\mathbf{k}_\alpha - \mathbf{k}_\delta} c_{\mathbf{k}_\alpha \sigma_\alpha}^\dagger c_{\mathbf{k}_\beta \sigma_\beta}^\dagger c_{\mathbf{k}_\gamma \sigma_\gamma} c_{\mathbf{k}_\delta \sigma_\delta}.$$

We can sum over the  $\mathbf{k}_\alpha$ , rename  $\sigma_\alpha \rightarrow \sigma$  and  $\sigma_\beta \rightarrow \sigma'$  and sum up over  $\sigma_\gamma$  and  $\sigma_\delta$  to simplify the Kroneker deltas.

$$\hat{V} = \frac{1}{2\Omega} \sum_{\sigma \sigma'} \sum_{\mathbf{k}_\beta \mathbf{k}_\gamma \mathbf{k}_\delta} v_{\mathbf{k}_\gamma - \mathbf{k}_\beta} c_{\mathbf{k}_\delta + \mathbf{k}_\gamma - \mathbf{k}_\beta, \sigma}^\dagger c_{\mathbf{k}_\beta \sigma'}^\dagger c_{\mathbf{k}_\gamma \sigma'} c_{\mathbf{k}_\delta \sigma'},$$

where using  $\mathbf{k}_\alpha = \mathbf{k}_\gamma - \mathbf{k}_\beta + \mathbf{k}_\delta$  from the Kroneker delta we get  $v_{\mathbf{k}_\alpha - \mathbf{k}_\delta} = v_{\mathbf{k}_\gamma - \mathbf{k}_\beta + \mathbf{k}_\delta - \mathbf{k}_\delta}$ . Then we can introduce the following variable transformations:

$$\mathbf{k}_\delta \rightarrow \mathbf{k}, \quad \mathbf{k}_\gamma \rightarrow \mathbf{k}', \quad \mathbf{k}_\beta \rightarrow \mathbf{k}' - \mathbf{q}$$

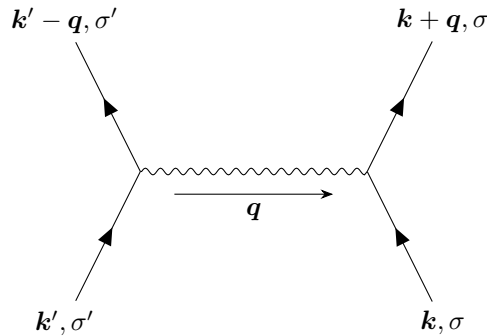
which yields

$$\begin{aligned} \mathbf{k}_\delta + \mathbf{k}_\gamma - \mathbf{k}_\beta &= \mathbf{k} + \mathbf{q}, \\ \mathbf{k}_\gamma - \mathbf{k}_\beta &= -\mathbf{q} \end{aligned}$$

and we finally get our second quantised interaction operator:

$$\hat{V} = \frac{1}{2\Omega} \sum_{\mathbf{q}} v_{\mathbf{q}} \sum_{\substack{\mathbf{k} \sigma \\ \mathbf{k}' \sigma'}} c_{\mathbf{k} + \mathbf{q}, \sigma}^\dagger c_{\mathbf{k}' - \mathbf{q}, \sigma'}^\dagger c_{\mathbf{k}' \sigma'} c_{\mathbf{k} \sigma}.$$

This describes an electron transferring a momentum  $\mathbf{q}$  to another electron. The formula tells that we first kill both electrons as they are before the interaction. Then we create two in the new states, where one lost some momentum that has been transferred the other electron. Here the  $v$ -term modualtes the strengh of the process. The Coulomb force is known to degrow with the inverse of the distance  $\mathbf{r} - \mathbf{r}'$  between the electrons, but do not depend on their respective positions  $\mathbf{r}$  and  $\mathbf{r}'$ . This exchange is therfore invariant under translations. The following diagramm is a good illustration of this process.



**Figure 2:** The interaction of two electrons modulated by a photon of momentum  $\mathbf{q}$ . The leftmost starts in the state  $(\mathbf{k}', \sigma')$  and gives a momentum  $\mathbf{q}$  to the rightmost electron. The time is represented on the horizontal axis and the space on the vertical one.

This closes the introduction theory on the second quantisation. We saw how we can express integrals over wavefunctions as bra-kets scalarproducts. We introduce two operators that create and annihilate particles in a certain state. Using this formalism we were able to compute the

Hamiltonian of the system in the momentum space. We found that the non interacting-part relies on the energy of each state times their occupation number. In the last part we showed that an interaction between two electrons can be described as a momentum transfert between them, which is modulated by the fourier transform of the Coulomb potential.

The goal of this thesis is to study how the superconductivity behaves in the proximity of an altermagnet. Over the next chapter we are going to explain how a superconductive state works. The altermagnet will be presented in a later part of this work.

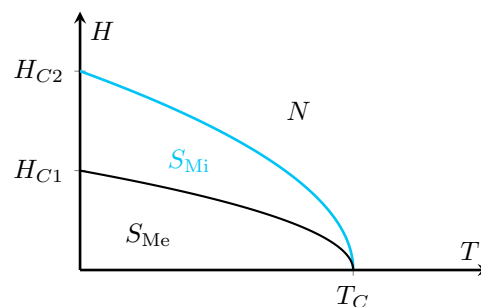
### 3 Superconductivity



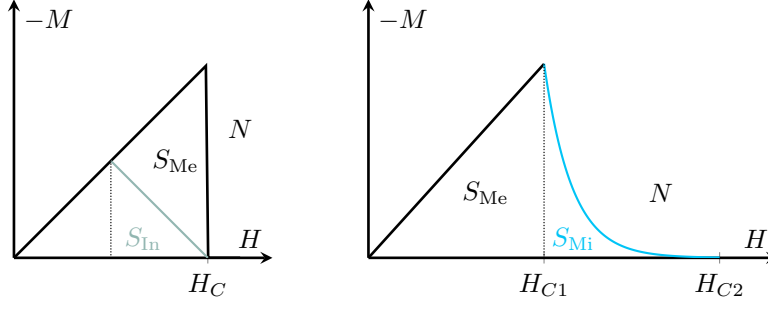
Superconductivity can be illustrated as a phase transition of a metal when its temperature lays under a critical temperature  $T_C$ . In the superconductive state the material become a perfect diamagnet and its resistivity vanishes. Let us apply a magnetic field on superconductive metal, we will observe some shielding currents that arise on its surface. They are labeled as the Faraday lines. If we take a spherical material, the density of these lines is bigger on the equator than on the poles but this depends on the shape of the metal. We have as well an internal magnetic field, that cancels exactly the one that is applied outside. The magnetic susceptibility in the body is then  $\chi = -1$ .

A second property that found multiple technical use cases is the friction-less flow of a current in the metal. The current can stream for a very long time without losing energy. In the first chapter of [6] Fossheim and Sudbø even calculated a significant decay of this current that lays above the age of the universe. The superconductive state is also described as a Meissner state.

Suppose that we now heat the material at the critical temperature  $T_c$ , some fluctuation effects arise and break the superconductive state. We usually distinguish type I and type II superconductors. The type I superconductors lose abruptly their magnetisation over  $T_c$ . Type II have a mix of ordinary and superconductive properties at intermediary temperatures. In this mixed state the magnetisation slowly decreases while heating the metal, until we can't measure any Meissner state anymore over  $T_C$ . Above an intermediary field strength, we observe as well some vortices and flux lines that pierce the material.



**Figure 3:** The phase diagram of a superconductive material regarding the temperature and the external applied magnetic field strength. Type I superconductor can only be in the Meissner state  $S_{Me}$  or in the normal state  $N$ . Type II superconductor additionally have mixed state  $S_{Mi}$  between the two. The critical values depend on the material. Increasing the field strength lowers the critical temperature.



**Figure 4:** The left plot shows how the magnetisation of a type I grows with the external field strength to cancel the applied field inside the material. Above the critical field strength  $H_C$  the material becomes a normal metal and the field flows through it. If the material has a non zero demagnetisation factor, the field starts to penetrate the body at an intermediary field strength, leading to an intermediate (not mixed!) state  $S_{In}$ . The field penetrates in form of flux lines, along the normal to the surface. On the other hand we have a figure that after showing the cancelation of the field inside the body, highlights the presence of a mixed state in a type II superconductor. The field penetrates the material in the form of vortecies, and finally let all the external field stream in the material.

Durring the break of the superconductive state more and more magnetic field flows inside of the material. Asuming that some particles are responsible for the superconductivity, the field achieve to penetrate where we observe a lower density of these particles.

The Meissner state is a thermodynamical state. This means that we can completely identify it with a set of variable that we are going to derive. We can show that the free energy of the superconductive state is higher than the normal state. This results in a lower entropy compared to the normal state ([6] chap. 1.6). Along with the derivation of the superconductivity, we are going to follow closely the work of Fossheim and Sudbø [6] from chapter 2 to 4. For readability reason we set the redacted Planck constant  $\hbar = 1$  in the following. Further they added the chemical potential to the energy of the state such that now  $\epsilon_k + \mu \rightarrow \epsilon_k$ , which was originaily defined at Eq.10.

The Hamiltonian of the system is described by the solid state physics. We consider the energy of the electrons and the ions in a lattice and the interaction between them resulting in the following, very generic expression of the Hamiltonian:

$$H = H_{e^-e^-} + H_{e^-ion} + H_{ion ion}.$$

Each term consist of a kinetik and potential energy term. For a more mathematical approach we consider a system of  $N$  electrons and  $L$  ions and define:

$$\begin{aligned} H_{e^-e^-} &= \sum_{i \in \llbracket N \rrbracket} \frac{p_i^2}{2m} + \sum_{ij \in \llbracket N \rrbracket} V_{\text{Coulomb}}^{e^-e^-}(\mathbf{r}_i - \mathbf{r}_j), \\ H_{ion ion} &= \sum_{i \in \llbracket M \rrbracket} \frac{p_i^2}{2M} + \sum_{ij \in \llbracket L \rrbracket} V_{\text{Coulomb}}^{\text{ion-ion}}(\mathbf{R}_i - \mathbf{R}_j), \\ H_{e^-ion} &= \sum_{i \in \llbracket N \rrbracket, j \in \llbracket L \rrbracket} V_{\text{Coulomb}}^{e^-ion}(\mathbf{r}_i - \mathbf{R}_j). \end{aligned}$$

We have  $m$  and  $M$  as the mass of the electron and the ion.  $\mathbf{r}$  and  $\mathbf{R}$  are the position of the electron and the ion. The ion-ion potential freezes the ions into the lattice. First we are going to introduce some concepts by describing a non-interacting electron and then improve it to include the interactions. Usely the potential of a lattice is periodic. If so, according to the Bloch theorem, the wavefunction of a particle moving in the system is a plane wave modulated by a periodic function. Eigenstates of the corresponding Hamiltonian are called Bloch state.

### 3.1 The non-interacting electron gas

In this case of study the Hamiltonian only include a kinetic term

$$H = \sum_{\mathbf{k}, \sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma}. \quad (11)$$

We assume that it exist a the ground state  $|0\rangle$ , where the system is filled up with a certain amount of electron until the Fermi-energy  $\epsilon_F$  is reached. Associated with this energy we find a wave vector  $\mathbf{k}_F$ , the Fermi-momentum. The set of enery up to  $\epsilon_F$  is called the Fermi-sea, as an analogy to the level zero of the topographic maps. Put into a mathematical form, the Fermi-sea is defined as:

$$\hat{n}_{\mathbf{k}, \sigma} |0\rangle = \Theta(\epsilon_F - \epsilon_{\mathbf{k}}) |0\rangle. \quad (12)$$

We introduced here a very useful tool called the Heavyside-step function which is defined as:

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}, \quad \Theta(-x) = 1 - \Theta(x). \quad (13)$$

This means that we find no particles that have an energy heigher than the Fermi-energy ( $\mathbf{k} > \mathbf{k}_F$ ).

Now we want to study how the electron can scatter in different states. The function that we're using is called the propagator, and gives the probailty to find the particle in the state  $|\mathbf{k}', \sigma\rangle$  at a time  $t'$  knowing it at  $|\mathbf{k}, \sigma\rangle$  and  $t$ . An important fact is that without interaction, the particle shouldn't scatter in another state due to energy conservation. Therefore

$$G_0(\mathbf{k}, \mathbf{k}', t' - t) = G_0(\mathbf{k}, t' - t) \delta_{\mathbf{k}, \mathbf{k}'},$$

which is zero if the wave-vectors between the two timepoint differs. We observe that only the past time  $t' - t$  is relevant. This is due to the time independence of the Hamiltonian. We are going to use the representation in the frequency space, using a Fourier-transformation.

$$G_0(\mathbf{k}, \omega) = \int_{\mathbb{R}} e^{i\omega t} G_0(\mathbf{k}, t) dt = \frac{1}{\omega - \epsilon_{\mathbf{k}} + i\delta_{\mathbf{k}}} \quad (14)$$

where  $\delta_{\mathbf{k}} = \delta \cdot \text{sgn}(\epsilon_{\mathbf{k}} - \epsilon_F)$  involving a very small, non-zero number  $\delta$ . We observe that this analytical function has a pole given by

$$\begin{aligned} \omega - \epsilon_{\mathbf{k}} + i\delta_{\mathbf{k}} &= 0 \\ \iff \omega &= \epsilon_{\mathbf{k}} - i\delta_{\mathbf{k}}. \end{aligned}$$

where we denote  $i$  as the imaginary unit to avoid confusion with the index  $i$ . The momentum  $\omega$  gives the so called spectrum of the excitation from the unique-particle system. Please remeber that we set  $\hbar = 1$ . The imaginary part serves as a damping term and is inversly proportional to the lifetime of the particle. As a result of the absence of scattering,  $\delta$  is a small number due to the infinitely long lifetime of the electrons.

Further the propagator yields important informations on the system, when considering the integration over its different arguments. First we take the imaginary part of the propagator, called the single particle spectral weight  $A(\mathbf{k}, \omega)$ .

$$\begin{aligned} A(\mathbf{k}, \omega) &= -\frac{1}{\pi} \mathcal{I}m [G_0(\mathbf{k}, \omega)] = \frac{1}{\pi} \frac{\delta_{\mathbf{k}}}{(\omega - \epsilon_{\mathbf{k}})^2 + \delta_{\mathbf{k}}^2} \\ &= \delta(\omega - \epsilon_{\mathbf{k}}) \end{aligned} \quad (15)$$

which informs us about the occupation of a state  $|\mathbf{k}\rangle$  with frequency  $\omega$ . We can find a form for the momentum distribution  $n(\mathbf{k})$ :

$$n(\mathbf{k}) = \int A(\mathbf{k}, \omega) d\omega, \quad (16)$$

and for the density of state:

$$D(\omega) = \int A(\mathbf{k}, \omega) d^3k, \quad \text{or for discontinuous state } \sum_{\mathbf{k}} A(\mathbf{k}, \omega). \quad (17)$$

These equations picture the non itneracting electron gas. Now that we put the groundstones, we can complexify the model.

### 3.2 Fermi-Liquid: The interacting case

Now that we've described the non interacting system, let us complexify the model by introducing the interactions. In an earlier section we saw how

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k},\sigma} + \sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} V_{\mathbf{k}\mathbf{k}',q} c_{\mathbf{k}-\mathbf{q},\sigma}^\dagger c_{\mathbf{k}+\mathbf{q},\sigma'} c_{\mathbf{k},\sigma} c_{\mathbf{k}',\sigma'} \quad (18)$$

represent the pairwise interaction of multiple electrons and their respective energy.

To extend the model we now want to introduce two new quantities, the propagator  $G$  and the one-particle irreducible self-energy  $\Sigma$ . The propagator maps in the complex space and gives the probability amplitude of finding a the particle in the state  $|\mathbf{k}, \sigma\rangle$  at a time  $t$ . On the other hand  $\Sigma = \Sigma_R + i\Sigma_I$  contains the lifetime of the particle in this state, and the shift of the particle's energy due to the interaction with the surroundings. The framework defines the non-interacting energy of the particle as  $\epsilon_{\mathbf{k}}$ . When put in an interacting system, the spectrum shifts and becomes  $\tilde{\epsilon}_{\mathbf{k}} = \epsilon_{\mathbf{k}} + \Sigma_R$ . Due to the interactions, the particle then has a much smaller lifetime.  $\Sigma_I$  is antiproportional to the particle's lifetime  $\tau_{\mathbf{k}}$ . As said,  $\Sigma_I$  is very small in the non interacting case. These two quantities are linked through the Dyson equation, which reads

$$(G(\mathbf{k}, \omega))^{-1} = (G_0(\mathbf{k}, \omega))^{-1} - \Sigma(\mathbf{k}, \omega). \quad (19)$$

One can use a Fourier-transformation to switch from the time representation to the frequency representation  $\omega$ . Reordering the equation and using the result from 14 we obtain

$$G(\mathbf{k}, \omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} - \Sigma}.$$

This function has a pole at  $\omega = \epsilon_{\mathbf{k}} + \Sigma_R + i\Sigma_I$ , where in the none interacting case  $\omega = \epsilon_{\mathbf{k}} + i\Sigma_I$ . This expression makes sense, the particle spectrum is now shifted due to its finite lifetime, as a result of the interaction. The pole yields to an expression for the spectrum

$$\omega - \epsilon_{\mathbf{k}} - (\Sigma_R(\mathbf{k}, \omega) + i\Sigma_I(\mathbf{k}, \omega)) = 0 \quad (20)$$

In complex analysis the order of a pole is given as  $n$  if  $f(z)$  is meromorphic and has a pole at  $z_0$  where

$$(z - z_0)^n f(z)$$

is also meromorphic in the neighbourhood of  $z_0$ . In our case we're interested in the 0.th order of the pole and first ignore pole's imaginary part. What we get is an expression for the energy shift:

$$\omega = \epsilon_{\mathbf{k}} + \Sigma_R(\mathbf{k}, \epsilon_{\mathbf{k}}) = \tilde{\epsilon}_{\mathbf{k}}.$$

If we take into account a tiny imaginary part of  $\Sigma$  we obtain a shifted pole. Performing a Taylor expansion of  $\Sigma_R$  in the neighbourhood of  $\omega = \tilde{\epsilon}_{\mathbf{k}}$  will help us.

$$\Sigma_R(\mathbf{k}, \omega) = \Sigma_R(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}) + (\omega - \tilde{\epsilon}_{\mathbf{k}}) \left. \frac{\partial \Sigma_R}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}} + \mathcal{O}(\omega^2)$$

We aim to compute to the first order in  $\Sigma_I$  such that we evaluate it at  $\omega = \tilde{\epsilon}_{\mathbf{k}}$ . Starting from Eq.20, and after inserting the Taylor expansion for  $\Sigma_R$ , we obtain

$$\begin{aligned} \omega - \underbrace{(\epsilon_{\mathbf{k}} + \Sigma_R(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}))}_{=\tilde{\epsilon}_{\mathbf{k}}} - (\omega - \tilde{\epsilon}_{\mathbf{k}}) \left. \frac{\partial \Sigma_R}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}} - i\Sigma_I(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}) &= 0 \\ \iff (\omega - \tilde{\epsilon}_{\mathbf{k}}) \left( 1 - \left. \frac{\partial \Sigma_R}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}} \right) - i\Sigma_I(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}) &= 0. \end{aligned} \quad (21)$$

We define the residue of the propagator as

$$z_{\mathbf{k}} = \frac{1}{1 - \left. \frac{\partial \Sigma}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}}}, \quad (22)$$


which we can insert in the inverse lifetime of the electron occupying the state  $|\mathbf{k}, \sigma\rangle$

$$\frac{1}{\tau_{\mathbf{k}}} = -z_{\mathbf{k}} \Sigma_I(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}).$$



This justifies the statement  $\Sigma_I \propto 1/\tau_{\mathbf{k}}$ . This residue is a decreasing function of the energy, which means that its influence is more important for low energies. An interpretation could be that the slow moving electrons have less time to interact with their fast homologues, which results in a longer lifetime.

We recall once again the difference of the propagators to conclude this section:

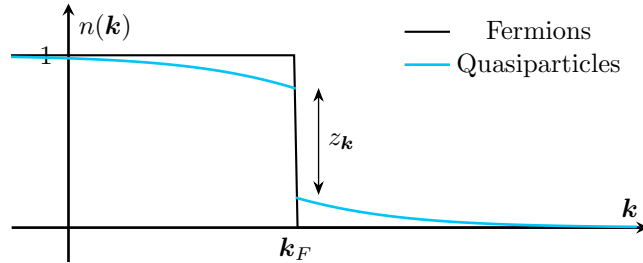
Free electron		Interacting electron
$G_0(\mathbf{k}, \omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} + i\delta_{\mathbf{k}}}$		$G(\mathbf{k}, \omega) = \frac{1}{\omega - \tilde{\epsilon}_{\mathbf{k}} + i\frac{1}{\tau_{\mathbf{k}}}}$

Interacting electrons represent a degraded version of the free-electron case, characterized by a quasiparticle weight  $z_{\mathbf{k}} < 1$ .

**Quasi-particles** The main question we have now is: How does the residue look like on the Fermi surface? We put the context of a low energy electron, close to the Fermi-surface and after an interaction happend. It turns out, that if there is a  $z_{\mathbf{k}_F} > 0$  we find a precise low energy single particle excitation. This excitation is very close to the exact eigenfunction of an non-interacting Hamiltonian. We call this state akin to the free electron, a quasi-particle.

This is it, a Fermi-liquid is a system of interacting electrons and quasi-particles. These quasi-particles are not eigenstates of the interacting Hamiltonian anymore. We can't consider them as an electron like excitation in the interaction context. The interactions allow to scatter some states in and out of the Bloch-State.

The above expression for the momentum distribution  $n(\mathbf{k})$  at Eq.16 can be plotted, and we recognise a gap of  $z_{\mathbf{k}_F} < 1$ . This property of the Fermi liquid is a central difference with the non interacting system. Back then the momentum distribution had a discontinuity from one to zero when crossing the Fermi surface (Eq.12).



**Figure 5:** The momentum distribution in a Fermi liquid for fermions and quasi-particles. The quasi-particles have a smaller gap, allowing to have energies beyond the fermi surface. This gap has exactly the magnetude of the quasiparticle's weigth defined in Eq.22.

An important feature to mention, that is the key understanding of the Fermi-liquid, is the on-to-one correspondance of quantum numbers between the interacting and non-interacting electrons. This revealed itself to be very correct for metals. This means we can start with simple Hamiltonian like Eq.11 and add new particles by introducing some perturbations. The obtained quasi-particles are a result of the excitation of the non-interacting system. We then observe this one-to-one correspondance.

The last question one can ask is: Why does this approximation works so well in the most case? The answer lays in the reference system we choosed to perturb around. Keeping in mind the Pauli-principle, we notice that the added particles are restricted in their scattering. This is so strong, that it almost cancels out the "switching-on" of the interactions. It's a good decision to perturb around the free system. Due to the fact that we are limited in the number of ground states, thermodynamic quantites moves in a smooth manner when perturbing.

We want to emphasize one last time that the Fermi-liquid formalism work because of the momentum restrictions due to the Pauli exclusion principle.

**Repulsive interactions** In condensed matter physics the dominant effect is the repulsive interactions between the electrons due to the coulomb potential. We already described it in the Hamiltonian  $H_{e-e}$  using some pairwise interactions in Eq.18. Therefore the goal is now to find an expression for the Coulomb potential.

As we can see in Eq.18 the system is described in the momentum-space. For this reason we need to consider the Fourier-transform of the real-space potential. We start with the coulomb potential which is predominant in the solid state physics. However the integral is going to diverge for  $r \rightarrow 0$ . To solve this problem We introduce the Yukawa potential which exponentially modulates the coulomb potential:

$$V_\lambda(r) = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} e^{-r\lambda}, \quad (23)$$

This is a solid approximation since we can easily fourier transform it and let  $\lambda \rightarrow 0$  in the result. [Result in appendix?/internet source](#). This yields to

$$V_{\mathbf{k},\mathbf{k}',\mathbf{q}} = V_{\text{el}}(\mathbf{q}) := \frac{1}{4\pi\epsilon_0} \frac{2\pi e^2}{q^2}, \quad (24)$$

with  $\mathbf{q}$  the momentum transfert during the interaction. We see that in the same way Coulomb potential in its real space form doesn't depend on the respective positions but only on the distance, this expression is independent of the initial states. The Fermi-liquid remains stable to the repulsive processes, because the accessed energy level are all avoid the surface.

However as we're going to see in the next section, attractive interactions also take place due to an exchange of phonons between two electrons. This will be an additional step towards the description of the Meissner state.

### 3.3 Instability due to attractive Interactions

We are going to show how the attractive interactions destabilise the Fermi-liquid. This is done by proving that new ground states open. Leon Cooper introduced a very specific context of attraction, that has a strong influence on the stability. Taking this exemple we are going to make clear that attractive interactions can in some cases exceed the Coulomb force, resulting in unexpected new eigenstates.

Let's assume we have an impotent Fermi sea, where the electrons are considered non interactive. Adding electrons requires to place them above the sea. The exotic context is due to the interactions. They can only interact if they are within a small frequency cover  $\omega_0$  over the Fermi surface, one on each side, facing them through the complete surface. If this is not the case the interaction vanishes.

Fossheim and Sudbø derived a method in [6] from p.67 to 69. They conclude with the fact, that allowing such interaction leads to a total energy of the interacting system  $E$  smaller than  $2\epsilon_F$ . This means that the attraction of the electrons shift them in a state that lays under the Fermi sea. Further they showed that if the Fermi sea vanishes (one could take electrons out of the system), then this attractive pairing disappears. The same is observed as we approach the classical limit. By forming a pair the electron share their fermionic properties and act as bosons. These pairs are named Cooper pairs by the eponymous physicist. The Pauli principle doesn't rule their energies anymore.

We now understand why attractive processes create instabilities in the liquid. An energy gap opens below the energy surface, reflecting the energy needed to break the new formed Cooper pairs. The shell  $\omega_0$  is the maximum frequency delivering the "adhesive" that allows the electron to pair. Now that we showed the influence of attractive interactions, we seek some candidate processes that are attractive.

### 3.4 Attractive forces mediated by phonons

As known from condensed matter physics, the lattice can have some intern oscillations called phonons resulting from the spring coupling between the ions. Now we can imagine that due to the Coulomb interactions, an electron can shift an ions producing a phonon. If this phonon travels and influences another electron on it way, we result in an effective electron-electron interaction thanks to the phonon. A similar case would be the exchange of a photon between two electrons. We are going to show how this exchange can lead to an attractive interaction.

In a dense lattice the ions move much slowly around their equilibrium positions than the lights electrons who pass by. The electrons move the charges of the ions, inducing in a small dipol moment. A second electron that also passes in the surrounding is going to feel the dipol moment, and will be attracted. Then the ion shifts back in its new equilibrium position, and the dipol moment vanishes long after the first electron passed.

Morover due to the Coulomb interactions, the electrons aim to put as much distance as they can between them, in a minimal amount of time. Therefore we can say that the  $\mathbf{k}$ -quantum number should be opposite between the two electron. If we target to put these concepts in a mathematical form, we use our previous Hamiltonian, and add an electron-phonon interaction term.

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k},\sigma}^{\dagger} c_{\mathbf{k},\sigma} + \sum_{\mathbf{k},\sigma,\mathbf{k}',\sigma'} V_{\mathbf{k}\mathbf{k}',\mathbf{q}} c_{\mathbf{k}-\mathbf{q},\sigma}^{\dagger} c_{\mathbf{k}+\mathbf{q},\sigma'}^{\dagger} c_{\mathbf{k},\sigma} c_{\mathbf{k}',\sigma'} + V_{e-\text{phonon}}$$

where  $V_{e-\text{phonon}}$  usely depends on the sum picturing the phonons-modes  $\lambda$ . These modes are similar to the oscillations modes we have in a  $\text{CO}_2$ -molecule for exemple. The expression that forms the phonon-depend potential in momentum space reads

$$V_{e-\text{phonon}} = \sum_{\mathbf{k},\mathbf{q},\sigma} M_{\mathbf{q}} (a_{-\mathbf{q}}^{\dagger} + a_{\mathbf{q}}) c_{\mathbf{k}+\mathbf{q},\sigma}^{\dagger} c_{\mathbf{k},\sigma}. \quad (25)$$

$M_{\mathbf{q}} (a_{-\mathbf{q}}^{\dagger} + a_{\mathbf{q}})$  is a matrix element of the coupling between the electron and the phonon. The  $a_{\mathbf{q}}$  and  $a_{\mathbf{q}}^{\dagger}$  are annihilation and creation operators of a phonon with wavevector  $\mathbf{q}$ . Further researches have shown that [11] [why does this commutator hold? More precise explanation needed or just bc we know that phonons are bosons?](#)

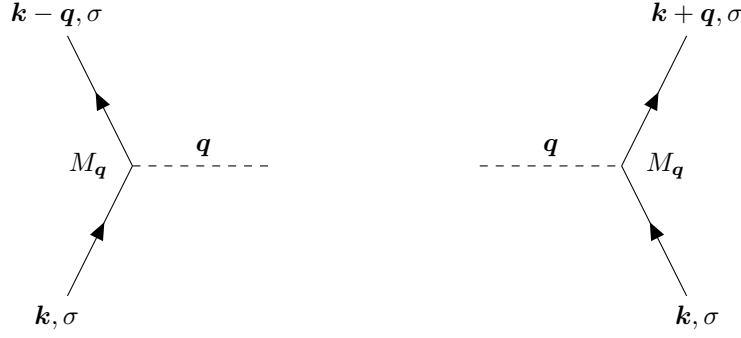
$$[a_{\mathbf{q}}, a_{\mathbf{q}'}^{\dagger}] = \delta_{\mathbf{q},\mathbf{q}'}$$

and therefore phonons act like bosons. Their number is however not conserved in a solid. The matrix element  $M$  is a function of the eigenfrequency of the phonon  $\omega_{\mathbf{q},\lambda}$  and the fourier transform  $\tilde{V}$  of the electrostatic potential  $V_{\lambda}(\mathbf{q})$  between the electron and the phonon of mode  $\lambda$ , if included.

$$M_{\mathbf{q},\lambda} = i(\mathbf{q} \cdot \boldsymbol{\xi}_{\lambda}) \sqrt{\frac{\hbar}{2M\omega_{\mathbf{q},\lambda}}} \tilde{V}_{\lambda}(\mathbf{q}).$$

$M$  can't be realy computed due to its complexity, we hold it as a parameter here. This pairing is much weaker than the electron-photon interaction. An other important fact is that for  $\mathbf{q} \rightarrow 0$  the matrix element  $M$  vanishes.  $M$  is proportional to  $\mathbf{q}$  which illustrate how the electron-phonon interaction happens between a point charge and a dipol.

The goal is now the implicitly express the phonon exchange with an effective electron-electron process. If we consider two diagrams, one aiming to describe the absorption of a phonon and one the emission of a phonon, we can combine them to get a new effective interaction like the photon exchange case.



**Figure 6:** The emission (left) and absorption (right) digramm of a phonon of wavevector  $\mathbf{q}$  by an electron. The matrix element  $M_{\mathbf{q}}$  gives the probability amplitude of emitting or absorbing the phonon. This plays a role in the Fermi golden rule, when it comes to give the transition rate of the scattering of the electron in its new state.

If we represent this interaction by linking both  $\mathbf{q}$ -edges, i.e. the emitted phonon is absorbed by another electron, the energy of phonon is simply defined as

$$H_{\text{phonon}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}.$$

We can write a propagator which describes the dashed line in a connected context similarly to the photon case described in Fig.2.

$$D_0(\mathbf{q}, \omega) = \frac{2\omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2 + i\eta},$$

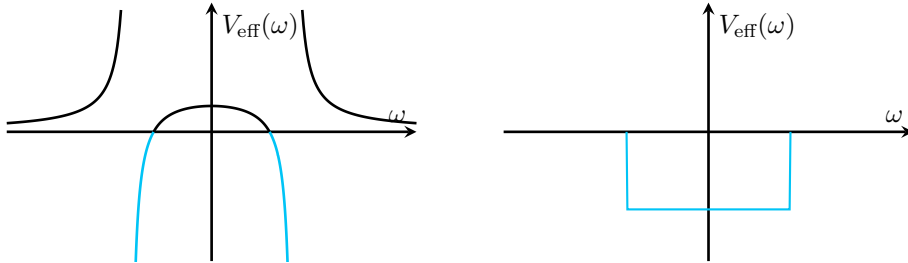
involving a very small quantity  $\eta$ . From this we obtain an expression for the phonon-mediated interaction of two electrons. This is performed using  $V_{\text{eff}}^{(ph)}(\mathbf{q}, \omega) = \text{Re}(|M_{\mathbf{q}}|^2 D_0(\mathbf{q}, \omega))$  and discarding the second order  $\eta$  term which is very small. We get:

$$V_{\text{eff}}^{(ph)}(\mathbf{q}, \omega) = \frac{2|M_{\mathbf{q}}|^2 \omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2}, \quad (26)$$

with  $\mathbf{q}$  the momentum transfert. We can now use a more complete potential in the Hamiltonian involving both the electrostatic (Eq.24) and effective phonon-mediated interactions (Eq.26):

$$V_{\text{eff}}(\mathbf{q}, \omega) = V_{\text{el}}(\mathbf{q}) + V_{\text{eff}}^{(ph)}(\mathbf{q}, \omega) = \frac{1}{4\pi\epsilon_0} \frac{2\pi e^2}{\mathbf{q}^2} + \frac{2|M_{\mathbf{q}}|^2 \omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2}. \quad (27)$$

Here we have reached a very important point. This new potential can be in some case negative, which means it results in an attractive interaction between the electrons. In other words, in some cases the phonon exchange can be attractive and even overcome the strong repulsive coulomb potential.

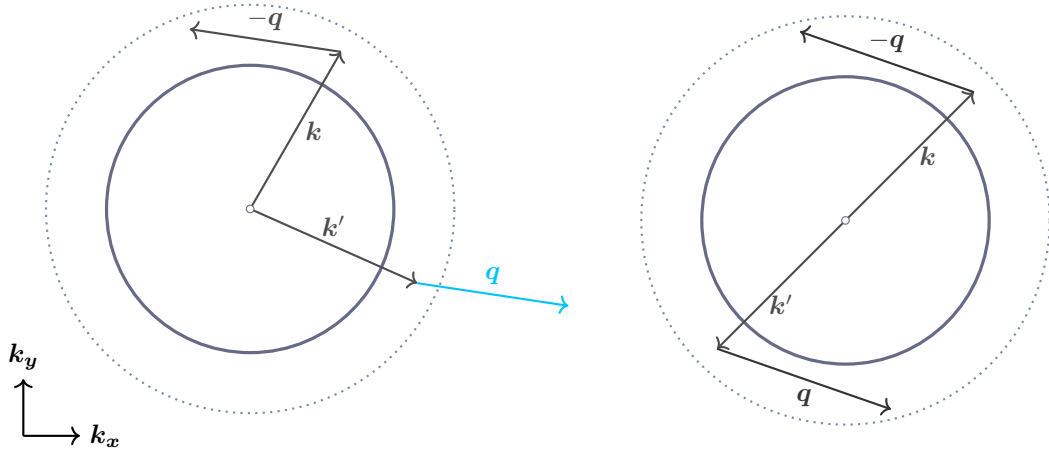


**Figure 7:** Left the course of the phonon-mediated effective potential  $V_{\text{eff}}$  defined in Eq.27. For some values of  $\omega$  the potential is attractive. On the right is a simplified version of the potential that revealed itself to be very accurate with the experiments.

### 3.5 Contraction of the effective Hamiltonian

We want to restrict ourselves in the case where the effective Hamiltonian is attractive. This happens in a small shell around the Fermi-surface. If we want to maximise the phase space for

the scattering, the state before and after the scattering have to be in this shell. A good idea is to consider that the two electrons have opposite wavevectors. The following figure illustrates this process.



**Figure 8:** The scattering process of two electrons with opposite wavevectors  $\mathbf{k}$  and  $\mathbf{k}'$ . We have a momentum transfer  $\mathbf{q}$  between them. The thick line illustrates the Fermi-surface and the dotted one the thin shell. As we see, if the electrons have opposite wavevectors, and the  $\mathbf{k}$  electron scatters into the shell, then the  $\mathbf{k}'$  electron scatters in the shell as well. This is not always the case if the wavevectors don't agree, as we can see on the left figure. The right figure points out the maximisation process. With opposite wavevectors, we get the largest possibility spectrum for the scattering.

Further the attractivity is a short range effect, so if we want to consider it, we must think that the electrons are very close to each other. This requires the electrons to have opposite spins due to the Pauli principle. Latter we are going to introduce the tight binding formalism. In this context we want the electrons to be on the same lattice site to be able to form a pair. This shell approximation turns out to be a good model.

Now we allow us to rename some variables:

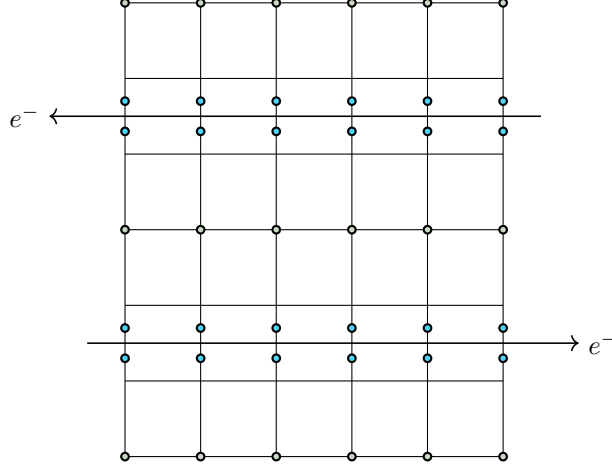
$$\mathbf{k} + \mathbf{q} \longrightarrow \mathbf{k}, \quad \mathbf{k} \longrightarrow \mathbf{k}'.$$

The Hamiltonian that follows from these transformations is called the BCS-reduced Hamiltonian and first found by Bardeen, Cooper and Schrieffer in 1957 [5].

$$H = \sum_{\mathbf{k}, \sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^{\dagger} c_{\mathbf{k}, \sigma} + \sum_{\mathbf{k}, \mathbf{k}', \sigma} V_{\mathbf{k} \mathbf{k}'} c_{\mathbf{k}, \sigma}^{\dagger} c_{-\mathbf{k}, \bar{\sigma}}^{\dagger} c_{-\mathbf{k}', \bar{\sigma}} c_{\mathbf{k}', \sigma}, \quad (28)$$

where  $\bar{\sigma}$  is the opposite spin of  $\sigma$ . The two rightmost operators destroy the electron with opposite wavevectors and spin before the interaction and creates two new with opposite wavevectors, including the loss/gain of momentum, and opposite spin after the interaction. To convince oneself, we can just look at the expression before the variable transformation. The potential  $V_{\mathbf{k} \mathbf{k}'}$  that modulates the strength of this pairing is now a matrix element that acts if the wavevectors are close to the Fermi-surface. The electrons have to move in opposite directions with opposite spins. Due to the retardation processes we introduced earlier, there remains a distortion in the lattice long after the electrons passed. Due to the inducing dipol moment, the other electron is attracted towards the distortion. As we also saw, the coulomb repulsion causes a colinear displacement, close to the distortion that its homologue provoked. This phenomenon is called the Cooper-pairing and is a coupling that happens in momentum space.

The reader might want to see a picture of what is happening in the real space. To achieve a representation two simple statements are enough. First, the Coulomb force aims to maximize the distance between the electrons in minimal time. This is achieved by moving them collinearly in opposite directions. Second, due to the retardation process, it's energetically more favourable for the electron to move along the distortion of the lattice. The result is illustrated in Fig.9.



**Figure 9:** A lattice of ions, where two electrons propagate with opposite directions.

Here the moved ions in blue are shifting back to their equilibrium position long after the passage of the second electron. However it is important to remember, that when the second electron passes, it experiences almost the same distortion everywhere. An interesting fact is that these interactions are the source of the superconductivity, but also the main origin of resistivity in clean materials.

### 3.6 On our way to the BCS-theory

After this introduction on the phonon coupling between the electrons in the momentum space, or Cooper-pairing, we aim to describe the energy of the superconductor in a mean-field approach. The goal of it is to reduce the description involving neighbours, to an on-site representation in the mean field of the other sites. Therefore we are going to describe a one-body problem which is easier to compute. As known, mean-field approaches require self-consistent equations to be comprehensive.

The first step is to introduce the following expectation values:

$$b_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \quad (29)$$

$$b_{\mathbf{k}}^{\dagger} = \langle c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger} \rangle \quad (30)$$

which lead to a new expression for the  $c$  operators:

$$c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} = b_{\mathbf{k}} + \underbrace{c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - b_{\mathbf{k}}}_{\delta_{b_{\mathbf{k}}}}, \quad (31)$$

where we can see the  $\delta_{b_{\mathbf{k}}}$  as a deviation, or fluctuation term. If we introduce it back into the BCS-reduced Hamiltonian of Eq.??, we can write

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} (b_{\mathbf{k}}^{\dagger} + \delta_{b_{\mathbf{k}}}^{\dagger}) (b_{\mathbf{k}} + \delta_{b_{\mathbf{k}}}).$$

We can compute the product of the two terms in parenthesis and forget the terms in  $\mathcal{O}(\delta_{b_{\mathbf{k}}}^2)$ , because the fluctuation are small. We then obtain the following expression

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}, \mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left( b_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + b_{\mathbf{k}'}^{\dagger} c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger} - b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}'} \right).$$

The next step is to define the superconduction gap parameter  $\Delta$ , which is a key thermodynamic variable to the description of the Cooper-pairs:

$$\Delta_{\mathbf{k}} := - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}}^{\dagger}, \quad (32)$$

$$\Delta_{\mathbf{k}}^{\dagger} := - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}'} . \quad (33)$$

This brings our Hamiltonian in another form:

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left( \Delta_{\mathbf{k}}^\dagger c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + \Delta_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger - b_{\mathbf{k}}^\dagger \Delta_{\mathbf{k}}^\dagger \right). \quad (34)$$

We took the liberty to split the sum, rename the  $\mathbf{k}'$  to  $\mathbf{k}$  and recombine the sum. We notice that this form involves a lot of creation and annihilation terms, and this is not is rather uncommon for an effective non-interacting electron gaz. We remember that we aim a one particle description in the mean field of its neighbours. This complexity will lead to some difficulties to express the quasi-particle spectrum. A good solution is to rotate the basis of the  $c$  operators, to land in a basis that diagonalises the Hamiltonian and therefore minimises the number of operators.

The transformation involves two new fermionic operator  $\eta$  and  $\gamma$  that therefore respect  $\mathfrak{Fet}_1$  to  $\mathfrak{Fet}_3$  and reads in a matrix form:

$$\begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \eta_{\mathbf{k}} \\ \gamma_{\mathbf{k}} \end{pmatrix}. \quad (35)$$

along with the conjugate transpose of each component of the l.h.s vector, as well in a matrix-vector form:

$$\begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger \\ c_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \eta_{\mathbf{k}}^\dagger \\ \gamma_{\mathbf{k}}^\dagger \end{pmatrix}. \quad (36)$$

We can reintroduce these into the Hamiltonian of Eq.34. Some of the multiplications involve  $\cos(\theta)^2 - \sin(\theta)^2 = \cos(2\theta)$  and  $\cos(\theta)^2 + \sin(\theta)^2 = 1$ . Further due to the anticommutations we have  $\eta\gamma^\dagger = -\gamma^\dagger\eta$  and so on for  $\gamma\eta^\dagger = -\eta^\dagger\gamma$ . We obtain the following expression

$$\begin{aligned} H &= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \mathbf{1} + \Delta_{\mathbf{k}} b_{\mathbf{k}}^\dagger \\ &+ \sum_{\mathbf{k}} \left[ \epsilon_{\mathbf{k}} \cos(2\theta) - \cos(\theta) \sin(\theta) \left( \Delta_{\mathbf{k}}^\dagger + \Delta_{\mathbf{k}} \right) \right] \eta_{\mathbf{k}}^\dagger \eta_{\mathbf{k}} \\ &- \sum_{\mathbf{k}} \left[ \epsilon_{\mathbf{k}} \cos(2\theta) - \sin(\theta) \cos(\theta) \left( \Delta_{\mathbf{k}}^\dagger + \Delta_{\mathbf{k}} \right) \right] \gamma_{\mathbf{k}}^\dagger \eta_{\mathbf{k}} \\ &- \sum_{\mathbf{k}} \left[ \Delta_{\mathbf{k}} \cos(\theta)^2 - \Delta_{\mathbf{k}}^\dagger \sin(\theta)^2 + 2\epsilon_{\mathbf{k}} \cos(\theta) \sin(\theta) \right] \eta_{\mathbf{k}}^\dagger \gamma_{\mathbf{k}} \\ &- \sum_{\mathbf{k}} \left[ \Delta_{\mathbf{k}} \cos(\theta)^2 - \Delta_{\mathbf{k}}^\dagger \sin(\theta)^2 + 2\epsilon_{\mathbf{k}} \cos(\theta) \sin(\theta) \right] \gamma_{\mathbf{k}}^\dagger \eta_{\mathbf{k}}. \end{aligned} \quad (37)$$

The goal is to diagonalise the Hamiltonian in the  $(\gamma, \eta)$  basis. Therefore we have to choose  $\theta$ , such that the terms with  $\gamma^\dagger\eta$  and  $\eta^\dagger\gamma$  vanish. A difficulty that we may encounter along with this idea is that  $\Delta$  is an order parameter and own a complex phase fluctuation. In other words one can write  $\Delta = |\Delta|e^{i\tau}$  where  $\tau$  has some fluctuations, but we are going to ignore them.

We can set

$$\Delta_{\mathbf{k}} = \Delta_{\mathbf{k}}^\dagger \quad \text{and} \quad \tan(2\theta) = -\frac{\Delta_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}$$

and introduce two new variables called the coherence factors:

$$v_{\mathbf{k}}^2 := \sin(\theta)^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}} \right), \quad (38)$$

$$u_{\mathbf{k}}^2 := \cos(\theta)^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}} \right). \quad (39)$$

along with a new energy  $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$ . These factors play an important role in NMR as well as in the ultra sound propagation in superconductors. Cooper and Schrieffer made some correct predictions in an advanced many-body system. This is labeled as one of the greatest achievements in condensed matter physics in the 20th century [6]. They are going as well to play a central role in the simulations later.

We obtain a Hamiltonian that looks very similar to a free fermion quasiparticle gas:

$$H = \underbrace{\sum_{\mathbf{k}} \left( \epsilon_{\mathbf{k}} + \Delta_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right)}_{=: H_0 \text{ constant mean-field term}} + \underbrace{\sum_{\mathbf{k}} E_{\mathbf{k}} \left( \eta_{\mathbf{k}}^{\dagger} \eta_{\mathbf{k}} - \gamma_{\mathbf{k}}^{\dagger} \gamma_{\mathbf{k}} \right)}_{\text{Spinless fermion system with two types of fermions of energies } E_{\mathbf{k}} \text{ and } -E_{\mathbf{k}}}. \quad (40)$$

This expression forgets the interactions to describe two types of particles in a mean field. As [6] described it in chapter 3, p.83-84, one can note that we don't have any spins involved. This is due to the fact that we describe the quasiparticles as a linear combination of electrons and holes with opposite spins (see Eq.35). Therefore talking about degrees of freedom the two electron-hole singlets replace the  $\uparrow, \downarrow$  degrees of freedom, These are therefore preserved.

The energy  $E_{\mathbf{k}}$  present a gap in the energy of the quasiparticles when looking at the fermi-surface  $\epsilon_{\mathbf{k}} = 0$  (we remember that  $\mu$  is present in  $\epsilon_{\mathbf{k}}$ ). This is due to the fact that the expectation values we introduced in 29 and 30 do not vanish. The gap is a result of the Cooper-pairing. These expectation values are the order parameters of the Meissner state and shouldn't be confused with the superconducting gap  $\Delta$ . They can however be zero both at the same time.

This free particle context is a good opportunity to introduce the grand canonical ensemble of the Hamiltonian. Even if we could give the direct result, deriving this ensemble involves a lot of important steps, so for the seek of completeness, we are going to go step to step through it.

We define the particle number operator  $N = \sum_{\mathbf{k}} (n_{\eta\mathbf{k}} + n_{\gamma\mathbf{k}})$ , that counts the quasiparticles of both class. Further the possible states  $\mathbf{k}$  are in a "continuous" set  $\mathfrak{K} = \{\mathbf{k}_1, \mathbf{k}_2, \dots\}$ . The set  $\{n_{\mathbf{k}}\}$  gathers the different occupation numbers of all the states  $\mathbf{k}$  under one configuration. The configurations are stored in an ensemble  $\mathcal{C}$ . Further we consider fermions, so the occupation numbers of the particle type  $\eta$  or  $\gamma$  in state  $\mathbf{k}$  are labeled  $n_{\eta\mathbf{k}}$  and  $n_{\gamma\mathbf{k}}$  and equals either 0 or 1. In the following we assume that  $E_{\mathbf{k}}$  has no  $\mu$ -dependence such that we make the  $N$  operator appear.

$$\begin{aligned} Z_G &= \text{Tr} \left( e^{-\beta(H+\mu N)} \right) \\ &= \sum_{\{n_{\mathbf{k}}\} \in \mathcal{C}} e^{-\beta H_0} \langle \{n_{\mathbf{k}}\} | \exp \left( \sum_{\mathbf{k}'} -\beta E_{\mathbf{k}'} \eta_{\mathbf{k}'}^{\dagger} \eta_{\mathbf{k}'} + \beta E_{\mathbf{k}'} \gamma_{\mathbf{k}'}^{\dagger} \gamma_{\mathbf{k}'} \right) \exp(-\beta \mu N) | \{n_{\mathbf{k}}\} \rangle \\ &= \sum_{\{n_{\eta\mathbf{k}}\}} \sum_{\{n_{\gamma\mathbf{k}}\}} e^{-\beta H_0} \exp \left( \sum_{\mathbf{k}} -\beta \left[ E_{\mathbf{k}} \eta_{\mathbf{k}}^{\dagger} \eta_{\mathbf{k}} + \mu \right] + \beta \left[ E_{\mathbf{k}} \gamma_{\mathbf{k}}^{\dagger} \gamma_{\mathbf{k}} - \mu \right] \right) \\ &= e^{-\beta H_0} \sum_{\substack{n_{\eta\mathbf{k}_1}, \\ n_{\eta\mathbf{k}_2}, \dots}} \sum_{\substack{n_{\gamma\mathbf{k}_1}, \\ n_{\gamma\mathbf{k}_2}, \dots}} \exp \left( \sum_{\mathbf{k}} -\beta \left[ E_{\mathbf{k}} n_{\eta\mathbf{k}} + \mu \right] + \beta \left[ E_{\mathbf{k}} n_{\gamma\mathbf{k}} - \mu \right] \right) \\ &= e^{-\beta H_0} \sum_{\substack{n_{\eta\mathbf{k}_1}, \\ n_{\eta\mathbf{k}_2}, \dots}} \prod_{\mathbf{k}} \exp(-\beta [E_{\mathbf{k}} n_{\eta\mathbf{k}} + \mu]) \sum_{\substack{n_{\gamma\mathbf{k}_1}, \\ n_{\gamma\mathbf{k}_2}, \dots}} \prod_{\mathbf{k}} \exp(\beta [E_{\mathbf{k}} n_{\gamma\mathbf{k}} - \mu]) \\ &= e^{-\beta H_0} \sum_{n_{\eta\mathbf{k}_1}} \exp(-\beta [E_{\mathbf{k}_1} n_{\eta\mathbf{k}_1} + \mu]) \sum_{n_{\eta\mathbf{k}_2}} \exp(-\beta [E_{\mathbf{k}_2} n_{\eta\mathbf{k}_2} + \mu]) \dots \\ &\quad \sum_{n_{\gamma\mathbf{k}_1}} \exp(\beta [E_{\mathbf{k}_1} n_{\gamma\mathbf{k}_1} - \mu]) \sum_{n_{\gamma\mathbf{k}_2}} \exp(\beta [E_{\mathbf{k}_2} n_{\gamma\mathbf{k}_2} - \mu]) \dots \\ &= e^{-\beta H_0} \prod_{\mathbf{k}} \left( 1 + e^{-\beta(E_{\mathbf{k}} + \mu)} \right) \left( 1 + e^{\beta(E_{\mathbf{k}} - \mu)} \right) \end{aligned}$$

Now we reinsert the  $\mu$ -dependence in  $E_{\mathbf{k}}$  and can therefore set  $\mu$  to zero in the above expression. This partition function becomes useful if we want to derive the free energy of the system. It will help us to solve the self consistency equation for the gap.

According to [27] p.99 the free energy can be derived from the partition function as:

$$F = -\frac{1}{\beta} \ln(Z_G) = H_0 - \frac{1}{\beta} \sum_{\mathbf{k}} + \ln(1 + e^{-\beta E_{\mathbf{k}}}) \ln(1 + e^{\beta E_{\mathbf{k}}}).$$



Following [6] we obtain in the limit of low temperatures ( $\beta \rightarrow \infty$ ):

$$\begin{aligned}
F &= H_0 + \sum_{\mathbf{k}} E_{\mathbf{k}} \theta(-E_{\mathbf{k}}) + E_{\mathbf{k}} \theta(E_{\mathbf{k}}) \\
&= H_0 + \sum_{\mathbf{k}} E_{\mathbf{k}} - E_{\mathbf{k}} \theta(E_{\mathbf{k}}) + E_{\mathbf{k}} \theta(E_{\mathbf{k}}) \\
&= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} + \Delta_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} E_{\mathbf{k}},
\end{aligned}$$

where we used the properties from the Heavyside step-function described in 13. The minimization of the free energy  $F$  should self-consistently determine the gap  $\Delta_{\mathbf{k}}$ . This is a statement that depends neither on the momentum space structure, nor on the attractivity of the potential [6].  $E_{\mathbf{k}}$  has an implicit  $\Delta_{\mathbf{k}}$ -dependence. We compute the derivation  $\frac{\partial F}{\partial \Delta_{\mathbf{k}}}$  and search the argument of the zero-postion. Derivating one  $\mathbf{k}$  from the sum is enough. We demand the following to be satisfied:

$$\frac{\partial F}{\partial \Delta_{\mathbf{k}}} = 0, \quad \frac{\partial F}{\partial \Delta_{\mathbf{k}}^{\dagger}} = 0. \quad (41)$$

This leads to [Appendix?](#)

$$b_{\mathbf{k}}^{\dagger} = \Delta_{\mathbf{k}} \underbrace{\frac{\tanh(\beta E_{\mathbf{k}}/2)}{E_{\mathbf{k}}}}_{\chi(\mathbf{k})},$$

defining  $\chi(\mathbf{k})$  as the pair-susceptibility, and gives how capable the system is to create Cooper-pairs [6]. If we allow us to relabel  $\mathbf{k} \rightarrow \mathbf{k}'$ , then multiply on both sides with  $-\sum_{\mathbf{k}} V_{\mathbf{k}\mathbf{k}'}$  and introduce 33, we obtain the self-consistent equation for the gap. This usely designated as the BCS gap equation.

$$\Delta_{\mathbf{k}} = - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \Delta_{\mathbf{k}'} \frac{\tanh(\beta E_{\mathbf{k}'}/2)}{E_{\mathbf{k}'}}. \quad (42)$$

Fossheim and Sudbø [6] emphasize that this equation is not limited to phonon mediated interaction, as long we don't specify the potential. Further Cooper-pairs can condensate, because in the pair, the spin of the electrons adds up to zero. The pair is not ruled anymore by the Pauli principle and can take part into the superconducting condensate. [a few words on the Bose-Einstein condensate?](#) Even if the formation temperature of Cooper-pairs in higher than the critical temperature  $T_C$ , these mean field approach makes this temperatures agree.

As we introduced earlier the gap has a complex phase fluctuation. This phase is hard to vary in good metals because the number of carrier electrons is high. On the other side in poor metals, the fluctuations that break the pair are more easily reched, i.e. at a lower temperature than in good metal. Superconductivity is therefore more stable in metals of good value.

In this section we saw how to reduce the many body Hamiltonian to a single-body problem in the means field of its neighbours. The system that follows from this is a non interacting gas of electrons and holes. This involves the introduction of the superconducting gap parameter that we can thanks to the statistical mechanics formalism express a in a self-consistent way.

### 3.7 Generalized gap equation, s-wave and d-wave superconductivity

As we introduced in an earlier section, the formalism follows from the attractive phonon exchange but can be generalized. The superconductivity is the result of the pairing of the electrons into Cooper-pairs. These pairs shift into a condensate and take part to a coherent matter-wave. Making a link to the coherent light wave, superconductivity can be seen as the *analogos* of lasers like Fossheim and Sudbø highlight it in [6].

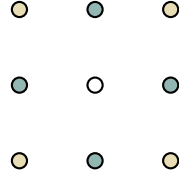
To achieve such generlisation we let the formalism be open to other kind of interactions and don't restrict it to the thin shell around the Fermi-surface. The potential matrix-element can therefor take a complexer form than introduced in Eq.27.

We recall that the electrons move into a lattice. We can asume that this crystal owns some symmetries that are reflected in its crysalographic (complete) basis  $\{g_{\eta}(\mathbf{k})\}$ . Similar to the Bloch

state, we could imagine that the symmetries emphasize some physical quantities. Asuming this, we propose a form for the potential that takes advantage of the basis.

$$V_{\mathbf{k}\mathbf{k}'} = \sum_{\eta} \lambda_{\eta} g_{\eta}(\mathbf{k}) g_{\eta}(\mathbf{k}')$$

$\eta$  is also often labeled as the channel [6]. Asuming we have a square lattice,



with interaction strength  $U/2$  on site, for nearest neighbours  $2V$  and  $4W$  for second-nearest neighbours. We can for each site describe their position with a shift vector to the central site. Using

$$f(k, k') = \cos(k) \cos(k') + \sin(k) \sin(k'),$$

fourier transforming these displacements leads to the following:

$$V_{\mathbf{k}\mathbf{k}'} = U + 2V (f(k_x, k'_x) + f(k_y, k'_y)) \\ + 4W (f(k_x, k'_x) \cdot f(k_y, k'_y))$$

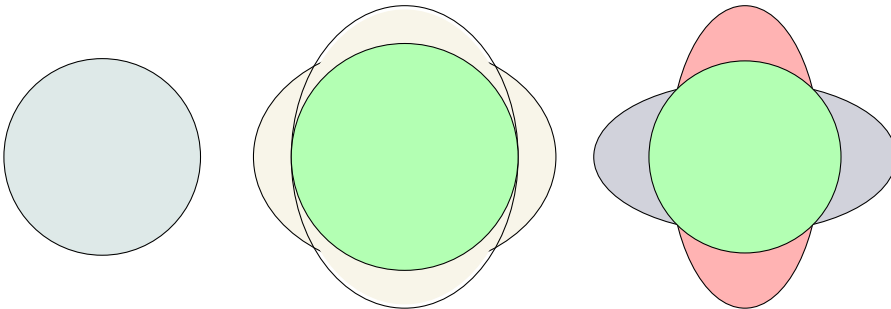
We can then introduce the basis-functions  $\{g_{\eta}(\mathbf{k})\}$  we talked about a few paragraphs ago.

$$\begin{aligned} g_1(\mathbf{k}) &= \frac{1}{2\pi} \\ g_2(\mathbf{k}) &= \frac{1}{2\pi} (\cos(k_x) + \cos(k_y)) && \text{(s-wave)} \\ g_3(\mathbf{k}) &= \frac{1}{2\pi} \cos(k_x) \cos(k_y) \\ g_4(\mathbf{k}) &= \frac{1}{2\pi} (\cos(k_x) - \cos(k_y)) && \text{(d-wave)} \\ g_5(\mathbf{k}) &= \frac{1}{2\pi} \sin(k_x) \sin(k_y) \end{aligned}$$

from which involves the following lambdas  $\lambda_1 = 2U\pi^2$ ,  $\lambda_2 = \lambda_4 = 4V\pi^2$ ,  $\lambda_3 = \lambda_5 = 4W\pi^2$  and for greater  $\eta$  ge set  $\lambda_{\eta \geq 6} = 0$ .

For  $\eta \in \{1, 2\}$ , the corresponding  $g$  is the identity under all symmetries of  $C_{v4}$  and for  $\eta \in \{3, 4, 5\}$  we observe a change of sign under  $\pi/2$  rotations. The same behaviour is found in the spherical harmonics for the respective quantum numbers  $l = 0$  and  $l = 2$ . From this we can deduce the use of the terms s-wave and d-wave.

S-wave is isotropic in the momentum space whereas the d-wave has a different countour for its part with positive phase



Further we can reintroduce the BCS gap equation 42 and obtain a lattice-dependent form for

the gap:

$$\begin{aligned}
\Delta_{\mathbf{k}} &= - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \Delta_{\mathbf{k}'} \chi(E_{\mathbf{k}}) \\
&= - \sum_{\eta \in \llbracket 5 \rrbracket} \lambda_{\eta} g_{\eta}(\mathbf{k}) \underbrace{\sum_{\mathbf{k}'} g_{\eta}(\mathbf{k}') \Delta_{\mathbf{k}'} \chi(E_{\mathbf{k}'})}_{=:\Delta_{\eta}/\lambda_{\eta}} \\
&= \sum_{\eta \in \llbracket 5 \rrbracket} \Delta_{\eta} g_{\eta}(\mathbf{k}).
\end{aligned} \tag{43}$$

This is a nice form to have as Fossheim and Sudbø [6] point out p.92. The gap is a linear combination of the physical quantities  $g_{\eta}(\mathbf{k})$ . The newly introduced  $\Delta_{\eta}$  are independent from the wavevector  $\mathbf{k}$  but are function of the temperature. We can express these in the basis of another  $\eta'$  ( $\lambda_{\eta \geq 6} = 0$ ):

$$\begin{aligned}
\Delta_{\eta} &= \sum_{\eta' \in \llbracket 5 \rrbracket} \Delta_{\eta'} \mathcal{M}_{\eta, \eta'} \\
\mathcal{M}_{\eta, \eta'} &= - \lambda_{\eta} \sum_{\mathbf{k}} g_{\eta}(\mathbf{k}) g_{\eta'}(\mathbf{k}) \chi(E_{\mathbf{k}}).
\end{aligned}$$

These are numerically cumbersome to compute, but taking  $U > 0$ ,  $V < 0$  and  $W = 0$  in the square lattice we obtain no attraction [means  \$V\_{\mathbf{k}\mathbf{k}'} \eta=1 > 0\$ ? or 0](#) in the  $\lambda_1$ -channel but attraction in the s- and d-wave channels. This tells us a lot like [6] analysed it. First the gap is highly associated to the Fourier transform of the wavefunction of the Cooper-pairs [Which variable describes their wavefunction?](#) Further the gap accommodates itself to be zero in the channel where the wavefunction is non-zero [“for zero separation between the electrons”](#). The s- and d-wave channels are preferred in order to avoid the on-site Coulomb repulsion (i.e. the Coulomb force between the two members of the pair). This can be put in comparison with the special case of the phonon pairing. In the latest the electrons used retardation processes to cancel out the repulsion. They avoided themselves in time. Here the avoidance takes place in space. [\(How to recognise the angular momentum coupling here?\)](#)

The manifestation of the retardation effects are actually a direct consequence of restricting ourselves to a thin shell around the Fermi-surface. Logically if we have a Cooper pair formation that doesn't follow this restriction, the whole Brillouin zone should be taken into account to compute the gap.

Further we can express the Fourier transform of the gap as  $\Delta(\mathbf{r}) = \sum_{\mathbf{k}} \Delta_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}}$  which leads  $\Delta(0) = \sum_{\mathbf{k}} \Delta_{\mathbf{k}}$ . However a repulsive interaction is obtained for  $\Delta(0) \neq 0$  so we need to find  $\Delta_{\mathbf{k}}$ s whose sum satisfies this. This can be done using the s-wave and d-wave channels, i.e.  $\Delta_{\mathbf{k}} = \Delta_0(T) g_2(\mathbf{k})$  and  $\Delta_{\mathbf{k}} = \Delta_0(T) g_4(\mathbf{k})$ . We already motivated superconductivity and its difficulty to maintain due to the phase fluctuations above the (freezing) critical temperature  $T_c$ . However superconductivity involving d-wave channels is believed [6] p.92 to be found in materials with high  $T_c$ .  $\Delta_{\mathbf{k}} = \Delta_0(T) g_4(\mathbf{k})$  is therefore a good case of study. [maybe find a usecase of d wave + cite](#)

### Density of Meissner states

We imagine some fluctuation described by  $g(\mathbf{k})$  that only depend on  $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$ . Further we assume  $\Delta$  being  $\mathbf{k}$ -independent for now. We can introduce the normal density of states.

$$D_n(\epsilon) = \frac{1}{N} \sum_{\mathbf{k}} \delta(\epsilon - \epsilon_{\mathbf{k}})$$

with  $N$  the number of Fourier modes. This is the same as the number of lattice sites for us [6] p.93.

$$\sum_{\mathbf{k}} g(\mathbf{k}) = \int D_n(\epsilon) g\left(\sqrt{\epsilon^2 - \Delta^2}\right) d\epsilon$$

as well as a variable transformation  $E = \sqrt{\epsilon^2 + \Delta^2}$  which leads to

$$dE = \frac{\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} d\epsilon \iff d\epsilon = \frac{E}{\epsilon} dE,$$

$$\sum_{\mathbf{k}} g(\mathbf{k}) = \int D_n(\epsilon) g(E) \frac{E}{\epsilon} dE.$$

Further we have  $\frac{\partial \epsilon}{\partial E} = \frac{E}{\sqrt{E^2 - \Delta^2}} = \frac{E}{\epsilon}$  so that

$$\sum_{\mathbf{k}} g(\mathbf{k}) = \int g(E) \underbrace{D_n(\epsilon) \frac{\partial \epsilon}{\partial E}}_{D_s(E)} dE. \quad (\text{take total or partial derivative?})$$

Then we use a common method in solid state physics to simplify the expression. We assume that  $D_n(\epsilon)$  is slowly varying so that  $D_n(\epsilon) \approx D_n(0)$  where  $\epsilon$  is measured relative to the Fermi-surface. Hopefully we don't have any sharpness around the Fermi-surface [6] p.93. We can therefore exclude the Van Hove singularities close to the Fermi-surface.

$$\begin{aligned} \frac{D_s(E)}{D_n(0)} \frac{E}{\sqrt{E^2 - \Delta^2}} &= \frac{E}{\sqrt{E^2 - \Delta^2}} \frac{D_n(\epsilon)}{D_n(0)} \\ &= \frac{E}{\sqrt{E^2 - \Delta^2}} \Theta(E - |\Delta|) \end{aligned}$$

We use the Heaviside step-function because the last term is either one or zero. This is our final expression for the density of the Meissner state when the gap is  $\mathbf{k}$ -independent. For the  $\mathbf{k}$  dependence of the gap we can use the spectral weight introduced in Eq.15. In analogy to the propagator or Green function introduced through the Dyson equation at Eq.19 for the electrons, we can define one for the superconducting condensate that involves time-dependent fermionic operators.

$$G(\mathbf{k}, t; \sigma) = -i \langle 0 | c_{\mathbf{k}\sigma}(t) c_{\mathbf{k}\sigma}^\dagger(0) | 0 \rangle$$

involving the BCS superconducting ground state  $|0\rangle$  which is an eigenstate of the Hamiltonian of Eq.40. As before we can consider the Fourier transform of the propagator  $G(\mathbf{k}, \omega; \sigma)$

$$G(\mathbf{k}, \omega; \sigma) = \frac{1}{2\pi} \int e^{i\omega t} G(\mathbf{k}, t; \sigma) dt.$$

We recall that Cooper pairs are made of two electrons with opposite spin. We can first take a look at the density of state of the pair-member with  $\sigma = \uparrow$ . However we here are only considering spin singlet pairing, so proving that the spectral weight  $A$  is spin independent should show that computing it for  $\sigma = \uparrow$  is enough as Fossheim and Sudbø argued in [6] p.94.

So without loss of generality we set  $\sigma = \uparrow$  and reintroduce the rotation of the basis of the  $c$  operators we performed in Eqs.35 and 36.

$$\begin{aligned} G(\mathbf{k}, t; \uparrow) &= -i \langle 0 | \left( \cos(\theta) \eta_{\mathbf{k}}^\dagger(t) - \sin(\theta) \gamma_{\mathbf{k}}^\dagger(t) \right) \\ &\quad \cdot \left( \cos(\theta) \eta_{\mathbf{k}}(0) - \sin(\theta) \gamma_{\mathbf{k}}(0) \right) | 0 \rangle \end{aligned}$$

The goal of these rotated operators was to diagonalise the hamiltonian. It follows

$$\langle 0 | \eta_{\mathbf{k}}^\dagger \gamma_{\mathbf{k}}^\dagger | 0 \rangle = 0 = \langle 0 | \eta_{\mathbf{k}} \gamma_{\mathbf{k}} | 0 \rangle$$

and therefore

$$\begin{aligned} G(\mathbf{k}, t; \uparrow) &= -i \cos(\theta)^2 \langle 0 | \eta_{\mathbf{k}}^\dagger(t) \eta_{\mathbf{k}}(0) | 0 \rangle \\ &\quad - i \sin(\theta)^2 \langle 0 | \eta_{\mathbf{k}}^\dagger(t) \gamma_{\mathbf{k}}(0) | 0 \rangle. \end{aligned} \quad (44)$$

We can nicely split the propagator in the sum of free  $\eta$ - and  $\gamma$ -particles in the superconducting state. Employing the coherence factors of Eqs.39 and 38 in the Fourier transform of we obtain

$$G(\mathbf{k}, \omega; \uparrow) = \frac{u_{\mathbf{k}}^2}{\omega - E_{\mathbf{k}} + i\delta_{\mathbf{k}}} + \frac{v_{\mathbf{k}}^2}{\omega + E_{\mathbf{k}} - i\delta_{\mathbf{k}}}.$$

From this we achieve an expression for the spectral weight:

$$A(\mathbf{k}, \omega; \uparrow) = u_{\mathbf{k}}^2 \delta(\omega - E_{\mathbf{k}}) + v_{\mathbf{k}}^2 \delta(\omega + E_{\mathbf{k}}).$$

This value is spin independent and using the arguments we've just given in such a case, we find

$$\begin{aligned} D_s(\omega) &= \frac{1}{N} \sum_{\mathbf{k}} A(\mathbf{k}, \omega) \\ &= \frac{1}{N} \sum_{\mathbf{k}} (u_{\mathbf{k}}^2 \delta(\omega - E_{\mathbf{k}}) + v_{\mathbf{k}}^2 \delta(\omega + E_{\mathbf{k}})). \end{aligned}$$

The density of state of the superconductive condensate  $D_s$  is also spin independent since we're considering spin-singlet pairings. This formula should be used when computing simulations of the superconducting state.

### 3.8 Transition temperature and energy gap

The goal of this discussion will be to derive a universal ratio between  $\Delta$  and the critical temperature. In the last section we already introduced some expressions for  $\Delta_{\mathbf{k}}(T)$  and  $V_{\mathbf{k}\mathbf{k}} = V$ . Let us consider the simplest case where the potential stays constant  $V_{\mathbf{k}\mathbf{k}} = V$ .

The phonon modulated interaction has a cover  $\omega_0 = \omega_D$ , that is the Debye-frequency. Inserting this back to the BCS gap equation 42, we see that the gap loses its  $\mathbf{k}$ -dependence and results as the identity when applying the symmetries ruling the crystal:

$$1 = V \sum_{\mathbf{k}'} \frac{\tanh(\beta E_{\mathbf{k}'})}{2E_{\mathbf{k}'}}.$$

This equation can be easily solved for  $T = T_C$  or  $T = 0$ .

Considering  $T$  approaching  $T_C$  from below, we can assume that the gap vanishes. We replace the  $\mathbf{k}$ -sum by an integral over the normal density of state  $D_n(\epsilon)$ . Due to the shell the sum occurs in a tiny volume around the Fermi-Surface, so that  $D_n(\epsilon)$  is evaluated close to the surface. We assume that in this neighbourhood  $D_n$  varies slowly, such that avoid some van Hove singularities we simply approximate  $D_n(\epsilon) \rightarrow D_n(0)$ . We choose zero because  $\epsilon$  is counted relative to the surface in our early thoughts. When introducing  $\lambda = V D_n(0)$  we obtain:

$$\begin{aligned} 1 &= \lambda \int_0^{\omega_D} \frac{\tanh(\beta\epsilon/2)}{\epsilon} d\epsilon \\ &= \lambda \ln \left( \frac{2e^{\gamma} \beta \omega_D}{\pi} \right). \end{aligned}$$

using  $\gamma := \lim_{m \rightarrow \infty} \left( \sum_{l \in \llbracket m \rrbracket} 1/l - \ln(m) \right)$  the Euler-Mascheroni constant. More details are provided in [6] p.88-89. We obtain [check the exponent](#)

$$k_B T_C \approx 1.13 \cdot \omega_D e^{-1/\lambda}$$

For  $T \rightarrow 0$  the gap equation takes a simpler form to solve:

$$1 = V \sum_{\mathbf{k}'} \frac{1}{2E_{\mathbf{k}'}} = \lambda \int_0^{\omega_D} \frac{1}{\sqrt{\epsilon^2 + \Delta^2}} d\epsilon$$

and leads to

$$\Delta(T = 0) = 2\omega_D e^{-1/\lambda}$$

according to the same source. We see how these expressions are closely dependent on  $\lambda$ . Moreover we can interpret the essential singularity at  $\lambda \rightarrow 0$  as following: The attractive processes are singular perturbations of the non interacting electron gas.  $m$  is very demanding to solve even for simple metals and is a function of multiple small details of the system. We aim here to acquire a qualitative knowledge. Let us now bring the ratio

$$\frac{2\Delta(T = 0)}{k_B T_C} = \frac{2\pi}{e^{\gamma}} \approx 3.52$$

which is a universal ratio and does not depend anymore on the properties of the material. Knowing the critical temperature one can know the gap at 0K.

In this long section we have derived how the Cooper pairs arise from the coupling of two electrons of opposite spin and momentum. Using a mean field approach and a rotation of the basis we were able to reduce the many-body Hamiltonian to a single-body problem. This allowed us to derive a self-consistent solution for the energy gap that the coupling of the electrons left behind. Then we have seen how the gap can be expressed in a basis of the crystallographic symmetries. Finally we derived an expression for the density of the Meissner state.

Now that we have a better understanding of the superconductivity, we can move on to the next section where we will present a method to self-consistently solve the gap. This will be done by diagonalising the Hamiltonian and reading the superconductivity from the eigenvalues and -vectors. To do so we need to discretize the system, by introducing the tight binding model.

## 4 Bogoliubov-de Gennes Formalism



The Bogoliubov-de Gennes transformation allows us to express the Hamiltonian in a diagonal way and express our quantities by looking at the eigenvectors of the Hamiltonian. The resulting matrix is expressed in a huge space and is very sparse.

To give a taste of it, it will allow us to rewrite our Hamiltonian as following

$$H = E_0 - \frac{1}{2} \tilde{c}^\dagger \tilde{H} \tilde{c}, \quad (45)$$

involving  $\tilde{c} = (\hat{c}_1, \dots, \hat{c}_N)$ , where each  $\hat{c}_i$  is a vector containing the creation and annihilation operators of a lattice site  $i$ :  $\hat{c}_i = (c_{i\uparrow}, c_{i\downarrow}, c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger)$ .

As we see we just describe each site with the four possible  $c$ -operators. This means for each lattice site, we have a  $4 \times 4$ -submatrix that reflects the possible combinations of creation and annihilation operators of both spins. For the readability we are going to drop the comma between the site and spin indices.

For exemple if one has (without loss of generality) a chemical potential at the site  $i$ , then the Hamiltonian is discribed in the following way:

$$H_{\mu i} = - \sum_{\sigma} \mu_i c_{i\sigma}^\dagger c_{i\sigma}.$$

If we want to discribe it in therm of  $\hat{c}_i$  we have:

$$H_{\mu, i} = \begin{pmatrix} c_{i\uparrow}^\dagger \\ c_{i\downarrow}^\dagger \\ c_{i\uparrow} \\ c_{i\downarrow} \end{pmatrix} \cdot \mu_i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{i\uparrow} \\ c_{i\downarrow} \\ c_{i\uparrow}^\dagger \\ c_{i\downarrow}^\dagger \end{pmatrix}$$

Please not that in later discussions we will discuss about how the matrix must exhibit a certain symmetry to respect the fermionic relations and have real eigenvalues. This was just an exemple.

Depending on the interaction we wish to describe, we can figure out what combination of operators we want and design the  $4 \times 4$  matrix accordingly. To achieve a full description of the system we can consider the interaction between two site  $i, j$  as a  $4 \times 4$  matrix involving the  $\hat{c}_i^\dagger$  and  $c_j$  operators. Then we can build a huge matrix  $\tilde{H}$  based on  $4 \times 4$  matrices at  $\tilde{H}_{ij}$ . The vectors we multiply this matrix with are jsut the  $\hat{c}_i^\dagger$  and  $c_j$  operators stack above one and other forming the above-introduced  $\tilde{c}$  vector. As a result, one gets the first formula introduced in this section at Eq.45. We can then compute the eigenvalues and -vectors to express the quantities we're interested in. This is what we call the Bogoliubov-de Gennes transformation.

Now that the motivation is clear, we need to bring our Hamiltonian in a form that involves the fermionic operators  $c_{i\sigma}$  and  $c_{i\sigma}^\dagger$ . This section follows closely the derivation of J. A. Ouassou in Ref. [19]. Some details of this derivation are also found in Ref.[30].

### 4.1 Tigh Binding Model

Our goal is now to fix our particles on lattice sites and describe their interactions. We are thefore going to translate our wavefunction formalism in an on-site plus nearest neighbour description.

For the generalities, we asume we have the Hamiltonian in the second quantisation formalism:

$$H = \sum_{\sigma\sigma'} \int \phi_\sigma^\dagger(\mathbf{r}) H_{\sigma\sigma'}(\mathbf{r}) \psi_{\sigma'}(\mathbf{r}) d^3r \\ + \sum_{\sigma\sigma'} \int \int \phi_\sigma^\dagger(\mathbf{r}) \phi_{\sigma'}^\dagger(\mathbf{r}') V_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') \phi_{\sigma'}(\mathbf{r}') \phi_\sigma(\mathbf{r}) d^3r' d^3r.$$

We introduce a basis of so called Wannier orbitals  $w(\mathbf{r} - \mathbf{R}_i)$  with  $\mathbf{R}_i$  an atom location. There are multiple way to define these orbitals [32], but the most simple ones [4] is enough here. These describe where the electrons evolve around the atoms and could be large in the neighbourhood of  $\mathbf{R}_i$  and vanishes when the distance tends to infinity. They are therefore called “localised”. The orbital can build a complete basis and verify the orthonormality condition:

$$\int w^*(\mathbf{r} - \mathbf{R}_i)w(\mathbf{r} - \mathbf{R}_j)d^3r = \delta_{ij}.$$

From this property we can define some field operators in this basis, based on creation and annihilation operators acting on a lattice site  $i$ :

$$\phi_\sigma(\mathbf{r}) := \sum_i w(\mathbf{r} - \mathbf{R}_i)c_{i\sigma}, \quad \phi_\sigma^\dagger(\mathbf{r}) := \sum_i w^*(\mathbf{r} - \mathbf{R}_i)c_{i\sigma}^\dagger \quad (46)$$

which is not a continuous description anymore. Inserting these operators back into our above Hamiltonian and using the orthonormality allows us to have an on site/nearest neighbour description of the Hamiltonian. Taking for instance the first part of the Hamiltonian, we have:

$$\begin{aligned} H &= \sum_{\sigma\sigma'} \int \psi_\sigma^\dagger(\mathbf{r})H_{\sigma\sigma'}(\mathbf{r})\psi_{\sigma'}(\mathbf{r})d^3r \\ &= \sum_{ij\sigma\sigma'} c_{i\sigma}^\dagger c_{j\sigma'} \int w^*(\mathbf{r} - \mathbf{R}_i)H_{\sigma\sigma'}(\mathbf{r})w(\mathbf{r} - \mathbf{R}_j)d^3r \\ &:= \sum_{i\sigma\sigma'} \epsilon_i^{\sigma\sigma'} c_{i\sigma}^\dagger c_{i\sigma'} - \sum_{\langle ij \rangle \sigma\sigma'} t_{ij}^{\sigma\sigma'} c_{i\sigma}^\dagger c_{j\sigma'} + .. \end{aligned}$$

In the last line we include a local energy term  $\epsilon$  and the so called hopping term  $t_{ij}$ , which is the interaction with the nearest neighbour sites  $j$  of  $i$ . For a more precise description one could consider more neighbour. The spin dependent term can be used to describe spin orbit coupling or spin-flip processes.

We now aim to define the useful processes for this thesis using this formalism.

#### 4.1.1 Non-interacting electrons

The two main components of the non-interacting system Hamiltonian  $H_N$  are the chemical potential  $\mu_i$  which is specific to each site and the hopping term  $t_{ij}$ . The chemical potential is modulated by the number of particles on the site  $i$  and the hopping term gives the amplitudes of moving an electron from site  $i$  to  $j$ . We assume it as spin-independent here.

$$H_N = - \sum_{i\sigma} \mu_i c_{i\sigma}^\dagger c_{i\sigma} - \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma}, \quad (47)$$

where  $\langle ij \rangle$  is a commonly-used notation to sum over  $i$  and its nearest neighbours  $j$ , skipping  $i = j$ . We label it the normal Hamiltonian.

The hopping amplitude can be computed from the overlap of the orbitals under a kinetic operator  $-\nabla^2/(2m)$ , which explains the meaning “hopping”:

$$\begin{aligned} t_{ij} &= - \int w^*(\mathbf{r} - \mathbf{R}_i) \frac{\nabla^2}{2m} w(\mathbf{r} - \mathbf{R}_j) d^3r \\ &= + \frac{1}{2m} \int (\nabla w(\mathbf{r} - \mathbf{R}_i))^* (\nabla w(\mathbf{r} - \mathbf{R}_j)) d^3r. \end{aligned}$$

We used a partial integration considering the boundary conditions of the Wannier orbitals  $w(\pm\infty) = 0$ . Therefore one part of the partial integration vanishes and we integrate/differentiate the integrands in the other integral, leading to two  $\nabla$ s. Further we see that  $t_{ij} = t_{ji}^*$  by swapping the two integrands.



### 4.1.2 Superconductivity

Previous study of ours on the superconductivity have led us to the following Hamiltonian:

$$H_S = - \int U(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) d^3r$$

on which we can apply a mean field approximation  $\Delta(\mathbf{r}) = U(\mathbf{r}) \langle \psi_{\uparrow}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \rangle$ . This yields to a common BCS-Hamiltonian for regular superconductors.

$$H_S = - \int \left( \Delta(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\uparrow}^{\dagger}(\mathbf{r}) + \Delta(\mathbf{r})^* \psi_{\uparrow}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \right) d^3r.$$

we see that the second integrand is just the complex conjugate of the first one. To spare some place, we are going to focus ourselves on the first one and denoted its homologue with *h.c.* “hermitian conjugate”. We insert Eq.46 in our last expression and obtain:

$$\begin{aligned} H_S &= - \sum_{ij} c_{i\downarrow}^{\dagger} c_{i\uparrow}^{\dagger} \int \Delta(\mathbf{r}) w^*(\mathbf{r} - \mathbf{R}_i) w^*(\mathbf{r} - \mathbf{R}_j) d^3r + \text{h.c.} \\ &:= - \sum_{ij} \Delta_{ij} c_{i\downarrow}^{\dagger} c_{i\uparrow}^{\dagger} + \text{h.c.} \end{aligned}$$

$\Delta(\mathbf{r})$  is an order parameter and doesn't vary to much in the coherence length [of the sustem, the condensate?](#), which is much bigger than the atomic length. Therefore we can say that the orbitals vary faster than the gap [\[Jabir2021\]](#). Moreover these orbitals are picked in the neighbourhood of the atomic location  $\mathbf{R}_i$  and  $\mathbf{R}_j$ . Achieving the integral we get  $\Delta_{ij} = \Delta_i \delta_{ij}$ . We can from then reintroduce the *h.c.* and we get

$$H_S = - \sum_i \Delta_i c_{i\downarrow}^{\dagger} c_{i\uparrow}^{\dagger} + \Delta_i^* c_{i\uparrow} c_{i\downarrow}. \quad (48)$$

We're however missing the mean field term  $E_0$ :

$$E_0 = \int U \langle \psi_{\downarrow}^{\dagger} \psi_{\uparrow}^{\dagger} \rangle \langle \psi_{\uparrow} \psi_{\downarrow} \rangle d^3r = \int U \frac{\Delta^*}{U} \frac{\Delta}{U} d^3r = \int \frac{|\Delta|^2}{U} d^3r.$$

After applying the tight binding formalism we get:

$$E_0 = \sum_i \frac{|\Delta_i|^2}{U},$$

which is a term we can add to the Hamiltonian in Eq.???. From these equations we have the final Hamiltonian for the superconducting system:

$$H = E_0 + H_N + H_S.$$

## 4.2 A more symmetric Hamiltonian

As we introduced it while motivating the Bogoliubov-de Gennes formalism, we aspire to describe each state as a vector-matrix-vector product of

$$\hat{c}_i = \left( c_{i\uparrow}, c_{i\downarrow}, c_{i\uparrow}^{\dagger}, c_{i\downarrow}^{\dagger} \right).$$

However using the form we have in the superconducting (Eq.??) and normal (Eq.47) Hamiltonian will later not act as a fermionic operator upon the transformation we're about to do. We need to rewrite the Hamiltonian in a more symmetric way to later respect the anticommutation relations.

**The chemical potential** term can be expressed using the anticommutation relations of the fermionic operators  $[c_{i\sigma}^{\dagger}, c_{i\sigma}]_+ = 1$ :

$$\sum_{i\sigma} \mu_i c_{i\sigma}^{\dagger} c_{i\sigma} = \frac{1}{2} \sum_{i\sigma} \mu_i \left( c_{i\sigma}^{\dagger} c_{i\sigma} - c_{i\sigma} c_{i\sigma}^{\dagger} + 1 \right) \quad (49)$$

The trick we used is quite straight forward but not obvious. Taking two different states  $\alpha$  and  $\gamma$  we have:

$$c_\alpha^\dagger c_\gamma = \frac{1}{2} c_\alpha^\dagger c_\gamma + \frac{1}{2} c_\alpha^\dagger c_\gamma = \frac{1}{2} c_\alpha^\dagger c_\gamma + \underbrace{\frac{1}{2} c_\alpha^\dagger c_\gamma + \frac{1}{2} c_\gamma c_\alpha^\dagger - \frac{1}{2} c_\gamma c_\alpha^\dagger}_{\frac{1}{2} [c_\alpha^\dagger, c_\gamma]_+ = \frac{1}{2} \delta_{\alpha\gamma}}. \quad (\text{\texttt{Tr1}})$$

**The hopping term** can in the same way be expressed as:

$$\sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} = \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} (c_{i\sigma}^\dagger c_{j\sigma} - c_{j\sigma} c_{i\sigma}^\dagger).$$

We can also take the liberty to reorder the indicies and use the fact that  $t_{ij} = t_{ji}^*$ :

$$\sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} = \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - t_{ij} c_{i\sigma} c_{j\sigma}^\dagger = \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - t_{ij}^* c_{i\sigma} c_{j\sigma}^\dagger. \quad (50)$$

**The superconducting term** for its part takes the form:

$$\sum_i \Delta_i c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger + \Delta_i^* c_{i\uparrow} c_{i\downarrow} = \frac{1}{2} \sum_i \Delta_i (c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger - c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger) + \Delta_i^* (c_{i\uparrow} c_{i\downarrow} - c_{i\downarrow} c_{i\uparrow}). \quad (51)$$

We then finish this section by using Eq.49, 50 and 51 in the Hamiltonian and obtain the following form:

$$\begin{aligned} H = E_0 - \frac{1}{2} \sum_{i\sigma} \mu_i (c_{i\sigma}^\dagger c_{i\sigma} - c_{i\sigma} c_{i\sigma}^\dagger) \\ - \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - t_{ij}^* c_{i\sigma} c_{j\sigma}^\dagger \\ - \frac{1}{2} \sum_i \Delta_i (c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger - c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger) + \Delta_i^* (c_{i\uparrow} c_{i\downarrow} - c_{i\downarrow} c_{i\uparrow}). \end{aligned} \quad (52)$$

The constant term  $\frac{1}{2} \sum_{i\sigma} \mu_i$  of the normal Hamiltonian just vanished in the  $E_0$ . [right?](#) We can now rewrite the Hamiltonian in a more compact way:

$$H = E_0 - \frac{1}{2} \sum_{i,j} \hat{c}_i^\dagger \hat{H}_{ij} \hat{c}_j \quad (53)$$

where the on site matrix reads

$$\hat{H}_{ij} = \begin{pmatrix} \mu_i \mathbb{I}_2 \delta_{ij} + t_{ij} & -i\sigma_2 \Delta_i \delta_{ij} \\ i\sigma_2 \Delta_i^* \delta_{ij} & -\mu_i \mathbb{I}_2 \delta_{ij} - t_{ij}^* \end{pmatrix} = \begin{pmatrix} H_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -H_{ij}^* \end{pmatrix} \quad (54)$$

where we use  $\mathbb{I}_n$  as an  $n$ -dimensional identity matrix. We haven't explicitly removed the  $t_{ij}$  if we're not considering nearest neighbours. At this point it's interesting to note that if we wish to build some periodic boundary conditions, it might be the case that a site on a side is neighbour with a site on the other side.

We can further compress our  $\hat{c}_i$  operator by introducing

$$\check{c} = (\hat{c}_1, \dots, \hat{c}_N)$$

along with the system Hamiltonian-matrix  $\check{H}_{ij}$  that orders the  $\hat{H}_{ij}$  matrices together. Please note that  $\check{H}_{ij}$  corresponds to an entry of  $\hat{H}_{ij}$ . The later maps two sites  $i, j$  to a  $4 \times 4$  matrix while the second concatenate all the matrices.

In  $\check{H}_{ij}$  taking the entries  $a = 4(i-1) + 1$  to  $4(i-1) + 4$  and  $b = 4(j-1) + 1$  to  $4(j-1) + 4$  corresponds to  $\hat{H}_{ab}$  if  $a$  is the site  $i$  and  $b$  the site  $j$ . This allows us to rewrite the Hamiltonian 53 as:

$$H = E_0 - \frac{1}{2} \check{c}^\dagger \check{H} \check{c}. \quad (55)$$

### 4.3 Eigenvalues

We now have a look at the following eigenvalue problem, which later helps from the diagonalization of the Hamiltonian:

$$\check{H}\check{\chi}_n = E_n\check{\chi}_n, \quad (56)$$

$n$  runs over the number of the eigenvalue and  $\check{\chi}_n$  is the corresponding eigenvector. We can decompose the  $\check{\chi}_n$  to reflect each lattice site:  $\check{\chi}_n = (\hat{\chi}_{n1}, \dots, \hat{\chi}_{nN})$ . This means  $\chi_{ni}$  refers to a  $4 \times 4$  block, i.e. the one on the submatrix we had earlier talked about. Therefore this  $\chi_{ni}$  contains four values, grouped in two vectors of length two, one for each spin:  $\chi_{n,i} = (u_{ni}, v_{ni})$ . Further  $u_{ni} = (u_{ni\uparrow}, u_{ni\downarrow})$  couples to the two first components  $(c_{i\uparrow}, c_{i\downarrow})$  we had in  $\hat{c}$  and similarly  $v_{ni} = (v_{ni\uparrow}, v_{ni\downarrow})$  to the two last components  $(c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger)$  of the four operator  $\hat{c}$ . These values represent the coherence factors we defined in Eqs.38 and 39.

We can simplify the eigenvalue problem by only taking a look at a site  $i$ . We then sum up over  $i$ :

$$\sum_{j \in \llbracket N \rrbracket} \hat{H}_{ij} \hat{\chi}_{nj} = E_n \hat{\chi}_{ni}.$$

To generalize we can sum over the rows of  $\check{H}_{ij}$  with the components of  $\check{\chi}_n$ . We remember that  $\check{H}_{ij}$  represent a complex scalar and  $\hat{H}_{ij}$  is a  $4 \times 4$  matrix with complex entries. So it follows by reintroducing Eq.54 the following set of equations:

$$\begin{cases} \sum_{j \in \llbracket N \rrbracket} H_{ij} u_{nj} + \Delta v_{nj} = & E_n u_{nj} \\ \sum_{j \in \llbracket N \rrbracket} \Delta^\dagger u_{nj} - H_{ij}^* v_{nj} = & E_n v_{nj}. \end{cases} \xrightarrow{(1)} \begin{cases} \sum_j H_{ij} u_{nj} + \Delta v_{nj} = & E_n u_{nj} \\ \sum_j H_{ij} v_{nj}^* + \Delta^\dagger u_{nj}^* = & -E_n v_{nj}^*. \end{cases} \quad (57)$$

Where in (1) we took the conjugate of the second equation and used  $\Delta^\dagger = -\Delta^*$ . This is an important result, because it shows that if  $\check{\chi}_n = (u_{n1}, v_{n1}, u_{n2}, v_{n2}, \dots)$  is an eigenvector with eigenvalue  $E_n$ , then so should be  $(v_{n1}^*, u_{n1}^*, v_{n2}^*, u_{n2}^*, \dots)$  with the eigenvalue  $-E_n$ .

This leads to a symmetry in the energy spectrum of  $H = E_0 \pm \frac{1}{2} \check{c}^\dagger \check{H} \check{c}$ . This flexibility allows us to choose the version of  $H$  with the positive sign, which is more commonly used.

### 4.4 Diagonalization

Our goal is now to express the Hamiltonian relative to its energy eigenvalues, which is more practical to work with. As we have seen in the last section, eigenvectors  $\chi_n$  allows us to compute the energies.

First we define a matrix from all the eigenstates  $\check{X} = [\check{\chi}_{\pm 1}, \dots, \check{\chi}_{\pm 2N}]$ . We then introduce a matrix  $\check{D} = \text{diag}(E_{\pm 1}, \dots, E_{\pm 2N})$  with the eigenvalues sorted on the diagonal. The order in which the eigenvectors appear is not important but  $\check{\chi}_i$  should be on the same column index as  $E_i$  not to lose track of which eigenvector is linked with which eigenvalue. Then, assuming that the Hamiltonian can be diagonalised, we can write it as:

$$\check{H} = \check{X} \check{D} \check{X}^{-1} = \check{X} \check{D} \check{X}^\dagger \quad (58)$$

where we used the fact that .

The eigenvalues corresponds to the energies of the system, this means they should be real because they correspond to a physical quantity. The above expression is equivalent to saying that  $\check{H}$  is hermitian. Hermitian matrices are known to have real eigenvalues and can be represented

with a unitary matrix  $\check{X}$ . [cite?](#) We can then transform the Hamiltonian with  $\check{c} := \check{X}\tilde{\gamma}$

$$\begin{aligned}
H &= E_0 + \frac{1}{2}\check{c}^\dagger \check{H} \check{c} = E_0 + \frac{1}{2}\tilde{\gamma}^\dagger \check{X}^\dagger \check{H} \check{X} \tilde{\gamma} \\
&= E_0 + \frac{1}{2}\tilde{\gamma}^\dagger \underbrace{\check{X}^\dagger \check{X}}_{=\mathbb{I}} \check{D} \underbrace{\check{X}^{-1} \check{X}}_{=\mathbb{I}} \tilde{\gamma} \\
&= E_0 + \frac{1}{2}\tilde{\gamma}^\dagger \check{D} \tilde{\gamma} \\
&= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}} E_n \gamma_n^\dagger \gamma_n
\end{aligned}$$

where  $\mathcal{N} = \{\pm n : n \in \llbracket N \rrbracket\}$ . Reagrranging the transformation of  $\check{c}$ , we get  $\gamma = \check{X}^\dagger \check{c}$ . Now that we've made the structure of the involved variables clear in the last section, we find the expression of the  $\gamma$  wich is  $2N$ -dimensional. A component of  $\gamma$  is:

$$\begin{aligned}
\gamma_n &= \sum_i \left( u_{ni\uparrow}^* c_{i\uparrow} + v_{ni\uparrow}^* c_{i\uparrow}^\dagger + u_{ni\downarrow}^* c_{i\downarrow} + v_{ni\downarrow}^* c_{i\downarrow}^\dagger \right) \\
&= \sum_{i\sigma} \left( u_{ni\sigma}^* c_{i\sigma} + v_{ni\sigma}^* c_{i\sigma}^\dagger \right),
\end{aligned}$$

and due to the symmerty we saw erlier,

$$\gamma_{-n} = \sum_{i\sigma} \left( v_{ni\sigma} c_{i\sigma} + v_{ni\sigma}^* c_{i\sigma}^\dagger \right)$$

for  $n \in \llbracket N \rrbracket$ . We now take a look at the conjugate transpose of  $\gamma_{-n}$ . Because scalar are dimension  $1 \times 1$  we have  $(uc^\dagger)^\dagger = (c^\dagger)^\dagger u^\dagger = cu^* = u^*c$  and it follows:

$$\gamma_{-n}^\dagger = \sum_{i\sigma} \left( v_{ni\sigma}^* c_{i\sigma}^\dagger + u_{ni\sigma}^* c_{i\sigma} \right) = \gamma_n.$$

Using this we can link each  $\gamma_i$  to the corresponding eigenvalue  $E_i$ :  $\gamma_n$  to the corresponding eigenvalue  $E_n$  and  $\gamma_{-n}$  to the corresponding eigenvalue  $E_{-n} = -E_n$ . We recall that we had  $2N$  degrees of freedom  $c_{i\sigma}$  due to the two possible spins, but after the transformation we get  $4N$  degrees into  $\hat{c}_i$  nevertheless. But because our energies  $E_n$  and  $E_{-n}$  are realted to eachother, we can keep only the positive  $2N$  eigenvalues and this maintain the total number of degree of freedom.

We can now apply this statement into the expression. First we can split the sum over the  $n \in \mathcal{N}$  in two parts:  $\mathcal{N}_+ = \{n \in \mathcal{N} : n > 0\}$  and  $\mathcal{N}_- = \{n \in \mathcal{N} : n < 0\}$ .

$$\begin{aligned}
H &= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n + \frac{1}{2} \sum_{n \in \mathcal{N}_-} E_n \gamma_n^\dagger \gamma_n \\
&= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_{-n} \gamma_{-n}^\dagger \gamma_{-n} \\
&= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n - \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_{-n}^\dagger \gamma_{-n} \\
&= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n - \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n \gamma_n^\dagger \\
&= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n (\gamma_n^\dagger \gamma_n - \gamma_n \gamma_n^\dagger).
\end{aligned}$$

We used the energy symmetry  $E_{-n} = -E_n$ ,  $\gamma_{-n}^\dagger = \gamma_n$ , as well as  $\gamma_{-n} = \gamma_n^\dagger$ .

Using this knowledge, we can express a final formula for the Hamiltonian by using the anti-commutation properties of the fermionic  $\gamma$ -operators:  $[\gamma_n^\dagger, \gamma_n]_+ = 1$ , so using the trick  $\Im \mathfrak{r}1$  and

bringing the  $\frac{1}{2}$  prefactor in the sum:

$$H = E_0 - \sum_{n \in \llbracket N \rrbracket} E_n \left( \gamma_n^\dagger \gamma_n - \frac{1}{2} \right). \quad (59)$$

This is the final form of the Hamiltonian in the Bogoliubov-de Gennes formalism. As a user one should build the Hamiltonian and computes its eigenvalues,-vector and use them to compute some physical quantites.

## 4.5 Physical Quantities

### 4.5.1 Superconducting Gap

We already covered how the superconducting gap  $\Delta$  is a relevant property of the Meissner state. We now aim to use the mean field theorie in order to find the gap. This requires a self consistency equation, which we can be derived from the Hamiltonian.

The gap was defined as  $\Delta(\mathbf{r}) := U(\mathbf{r}) \langle \psi_\uparrow(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \rangle$ . Back to the tight binding formalism, the gap now depends on the lattice site  $i$  and reads  $\Delta_i = U_i \langle c_{i\uparrow} c_{i\downarrow} \rangle$  and we can express  $c_{i\sigma}$  in terms of the  $\gamma$ -operators:

$$\begin{aligned} c_{i\sigma} &= \sum_{n \in \mathcal{N}} u_{ni\sigma} \gamma_n \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma} \gamma_n + u_{-n,i\sigma} \gamma_{-n} \quad (60) \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma} \gamma_n + v_{ni\sigma}^* \gamma_n^\dagger \end{aligned} \quad \begin{array}{c} \circ \\ \hline \circ \end{array} \quad \begin{aligned} c_{i\sigma}^\dagger &= \sum_{n \in \mathcal{N}_+} (u_{ni\sigma} \gamma_n)^\dagger + (v_{ni\sigma}^* \gamma_n^\dagger)^\dagger \\ &= \sum_{n \in \mathcal{N}_+} \gamma_n^\dagger u_{ni\sigma}^\dagger + \gamma_n (v_{ni\sigma}^*)^\dagger \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma}^* \gamma_n^\dagger + v_{ni\sigma} \gamma_n \end{aligned} \quad (61)$$

where we used the symmetry of the eigenvectors. We can now compute expectation value involved in the gap by taking the thermal average and the quantum expectation value:

$$\begin{aligned} \langle c_{i\uparrow} c_{i\downarrow} \rangle &= \sum_{n,m \in \mathcal{N}_+} \langle (u_{ni\uparrow} \gamma_n + v_{ni\uparrow}^* \gamma_n^\dagger) (u_{mi\downarrow} \gamma_m + v_{mi\downarrow}^* \gamma_m^\dagger) \rangle \\ &= \sum_{n,m \in \mathcal{N}_+} \langle (u_{ni\uparrow} u_{mi\downarrow} \gamma_n \gamma_m + u_{ni\uparrow} v_{mi\downarrow}^* \gamma_n \gamma_m^\dagger + v_{ni\uparrow}^* u_{mi\downarrow} \gamma_n^\dagger \gamma_m + v_{ni\uparrow}^* v_{mi\downarrow}^* \gamma_n^\dagger \gamma_m^\dagger) \rangle \\ &\stackrel{(*)}{=} \sum_{n \in \mathcal{N}_+} \langle u_{ni\uparrow} v_{ni\downarrow}^* \gamma_n \gamma_n^\dagger \rangle + \langle v_{ni\uparrow}^* u_{ni\downarrow} \gamma_n^\dagger \gamma_n \rangle \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{ni\downarrow}^* \langle \gamma_n \gamma_n^\dagger \rangle + v_{ni\uparrow}^* u_{ni\downarrow} \langle \gamma_n^\dagger \gamma_n \rangle \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{ni\downarrow}^* (1 - f(E_n)) + v_{ni\uparrow}^* u_{ni\downarrow} f(E_n) \end{aligned} \quad (62)$$

where  $f$  is the Fermi-Dirac distribution. In  $(*)$  we notice no  $\gamma\gamma$  or  $\gamma^\dagger\gamma^\dagger$  terms in the Hamiltonian, so their expectation value is zero <sup>1</sup>.

The expectation value  $\langle a \hat{A} \rangle_\Phi$  of a scalar times an operator reads  $\langle \Phi | a \hat{A} | \Phi \rangle_\Phi = a \langle \Phi | \hat{A} | \Phi \rangle_\Phi = a \langle \hat{A} \rangle_\Phi$ . To convince onselves, we just take a look at the first quantisation expression of this braket. This result leads to the self consistency equation:

$$\Delta_i = U_i \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{ni\downarrow}^* (1 - f(E_n)) + u_{ni\downarrow} v_{ni\uparrow}^* f(E_n) \quad (63)$$

We plan to solve this equation numerically, inserting some guess in the Hamiltonian, diagonalize it, update  $\Delta_i$  and reinsert it into H. We should repeat this until we reach a fixpoint or the desired accuracy.

□ \_\_\_\_\_ □

<sup>1</sup>This is like the expectation value of killing twice a fermion in a state. It is not possible, because we can't annihilate a state that has a possession number of zero. And in the same way due to the Pauli-principle we can't have more than one particle in the same state, so  $\langle \gamma^\dagger \gamma^\dagger \rangle = 0$ . We are not finding these terms in the Hamiltonian. Here we additionally removed the indices, in fact the diagonalization also takes place on the indices so that we end just with  $n$ . The Hamiltonian is diagonal in  $\gamma\gamma^\dagger$  and  $\gamma^\dagger\gamma$  right?

### 4.5.2 Currents

Using the charge conservation and the Heisenberg picture we are going to derive an expression for the current in the lattice. The system we are taking into account is two dimensional.

The charge conservation reads

$$\partial_t \rho_i = -\nabla \cdot \mathbf{j}_i$$

and identifies the time variation of the charge density on the site  $i$  as the negative divergence of the current density. Now, performing some transformations, we bring this expression in a more useful form. The goal here is to integrate on both side over our two-dimensional surface  $\Omega$ . For the charge density this yields to the charge at a site:

$$\int_{\Omega} \partial_t \rho_i d\mathbf{r} = \partial_t Q_i.$$

For the current density we can use the Gauß law to change the integration set:

$$\int_{\Omega} \nabla \cdot \mathbf{j}_i d\mathbf{r} = \int_{\partial\Omega} \mathbf{j}_i \cdot \mathbf{n} = \sum_n J_{i,n} a = \sum_n I_{i,n}$$

where  $\partial\Omega$  is the boundary of  $\Omega$ . The normal vector  $\mathbf{n}$  points in the 2D-plane, outward from the boundary. Assuming we have a square lattice, we can assign to each lattice site a square unit cell with side length  $a$ . The sum over  $n$  happens to be over all the side. Now introducing the Heisenberg picture with  $\hbar = 1$  we get

$$\partial_t Q_i = i[H, Q_i].$$

Finally we can introduce the second quantisation in the charge:

$$Q_i = \sum_{\sigma} c_{i\sigma}^{\dagger} c_{i\sigma} = \sum_{\sigma} n_{i,\sigma}$$

which is quite trivial, summing over all the particle at a site leads the charge of the site. After putting all together, this yields

$$I_i^{+x} + I_i^{+y} + I_i^{-x} + I_i^{-y} = -i \left[ H, \sum_{\sigma} n_{i,\sigma} \right] \quad (64)$$

This means the last step to perform is to compute the commutator of the different terms of the Hamiltonian with the charge at a site  $i$ .

We remind here that our Hamiltonian contains a chemical potential, a hopping, a superconducting and an antiferromagnetic term.

**The hopping term** We set remember the use of a constant hopping amplitude  $t_{ij} = t$ .

$$\left[ \sum_{\langle ij \rangle \sigma} c_{i\sigma}^{\dagger} c_{j\sigma}, \sum_{\sigma} n_{l\sigma} \right] = \sum_{\langle ij \rangle \sigma \sigma'} c_{i\sigma}^{\dagger} c_{j\sigma} n_{l\sigma'} - n_{l\sigma'} c_{i\sigma}^{\dagger} c_{j\sigma}$$

We can then introduce a useful trick that involves the commutator  $[n_{\mu}, c_{\nu}] = -\delta_{\mu\nu} c_{\mu}$

$$\begin{aligned} c_{i\sigma}^{\dagger} c_{j\sigma} n_{i,\sigma'} &= c_{i\sigma}^{\dagger} \left( \underbrace{c_{j\sigma} n_{l\sigma'} - n_{l\sigma'} c_{j\sigma}}_{-[n_{l\sigma'}, c_{j\sigma}]} + n_{l\sigma'} c_{j\sigma} \right) \\ &= c_{i\sigma}^{\dagger} (\delta_{\sigma'\sigma} \delta_{lj} c_{i\sigma'} + n_{l\sigma'} c_{j\sigma}). \end{aligned}$$

Following the same schema we derive the other part of the commutator. Here the expressions involves  $[n_{\mu}, c_{\nu}^{\dagger}] = \delta_{\mu\nu} c_{\mu}^{\dagger}$ :

$$\begin{aligned} n_{l\sigma'} c_{i\sigma}^{\dagger} c_{j\sigma} &= \left( \underbrace{n_{l\sigma'} c_{i\sigma}^{\dagger} - c_{i\sigma}^{\dagger} n_{l\sigma'}}_{[n_{l\sigma'}, c_{i\sigma}^{\dagger}]} + c_{i\sigma}^{\dagger} n_{l\sigma'} \right) c_{j\sigma} \\ &= (\delta_{\sigma'\sigma} \delta_{li} c_{j\sigma'}^{\dagger} + c_{i\sigma}^{\dagger} n_{l\sigma'}) c_{j\sigma} \end{aligned}$$

After substracting the second term from the first one we are left with

$$\begin{aligned} \left[ \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma}, \sum_{\sigma} n_{i\sigma} \right] &= \sum_{\langle ij \rangle \sigma \sigma'} \delta_{\sigma' \sigma} \left( \delta_{lj} c_{i\sigma}^\dagger c_{j\sigma'} - \delta_{li} c_{i\sigma'}^\dagger c_{j\sigma} \right) \\ &= \frac{1}{2} \sum_{i \delta \sigma \sigma'} \delta_{\sigma' \sigma} \left( \delta_{l, i+\delta} c_{i\sigma}^\dagger c_{i+\delta, \sigma'} - \delta_{li} c_{i\sigma'}^\dagger c_{i+\delta, \sigma} \right) \end{aligned}$$

Because of the squared lattice we can summerise the neighbour set  $\langle ij \rangle$  to  $\{i + \delta_x, i - \delta_x, i + \delta_y, i - \delta_y\}$  involving  $\delta_{\text{axis}}$  the displacemnt from the site to neighbour one along the given axis. This is abstracted in  $i + \delta$ . We obtain after summing up over the  $\sigma'$  and  $i$ , and writing explicitey everything we obtain

$$\begin{aligned} &= \frac{1}{2} \sum_{\sigma} (c_{l-\delta_x, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l+\delta_x, \sigma}) + (c_{l+\delta_x, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l-\delta_x, \sigma}) \\ &\quad + (c_{l-\delta_y, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l+\delta_y, \sigma}) + (c_{l+\delta_y, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l-\delta_y, \sigma}), \end{aligned} \quad (65)$$

where the one half factor avoids summing twice over the nearest neighbours. This proportional to the current at a site  $l$ . The current of one side of the square unit cell is represnted by a pair of  $c_{l\pm\delta_x, \sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l\pm\delta_x, \sigma}$  where the same displacement  $\delta_x$  is involved. This will be an important consideration when coming to the altermagnet.

**Chemical potentail term** For the chemical potential term it is useful to introduce that the commutator between two number operator vanishes. Since the charge and the chenical potential operators involves only number operator, we find that this term dont't take part to the current.

**Superconducting term** The superconduting term has a particular behaviour. If one can sovl the gap (self consistently), this term doesn't contribute to the current as we are going to see. We first form the commutator between the Hubbard term and the charge operator:

$$\left[ \sum_i \left( \Delta_i c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + \Delta_i^* c_{i\uparrow} c_{i\downarrow} \right), \sum_{\sigma} n_{l, \sigma} \right]$$

again make the already introded commutator appear, we obtain

$$\begin{aligned} &= \sum_{i\sigma} \delta_{il} \left( \Delta_i^\dagger (\delta_{\sigma\downarrow} + \delta_{\sigma\uparrow}) c_{i\downarrow} c_{i\uparrow} - \Delta_i (\delta_{\sigma\downarrow} + \delta_{\sigma\uparrow}) c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \right) \\ &= 2 \left( \Delta_i^\dagger c_{l\downarrow} c_{l\uparrow} - \Delta_i c_{l\uparrow}^\dagger c_{l\downarrow}^\dagger \right) \end{aligned}$$

This expression mixes creation and annihilation operators and makes it hard to recognise a current. One can view this term as a source [need citation?](#) meaning we havw a new term  $\mathcal{C}_i$  that appears in the equation

$$-\partial_t Q_i = \mathcal{C}_i + \sum_n I_{i,n}.$$

To know the contribution to the current we can investigate the rate of charge generation of this term. This is achieved by taking the quantum expectation and the thermal average of the system.

$$2 \left( \Delta_l^\dagger \langle c_{l\downarrow} c_{l\uparrow} \rangle - \Delta_l \langle c_{l\uparrow}^\dagger c_{l\downarrow}^\dagger \rangle \right) = \frac{2}{U_l} \left( -\Delta_l^\dagger \Delta_l + \Delta_l \Delta_l^\dagger \right)$$

This is achieved using the anticommutation relation of the fermionic operators Eq. [3fr1](#) and Eq. [3fr2](#). We have as well  $\Delta_l^\dagger = U_i^* \langle (c_{l\uparrow} c_{l\downarrow})^\dagger \rangle = U_i \langle c_{l\downarrow}^\dagger c_{l\uparrow}^\dagger \rangle$ . The dagger operator is asumed to be inserted into the expectation value using the linearity of the integral defining  $\langle \cdot \rangle$ .

Solving  $\Delta$  self-consistently will lead to  $\Delta \Delta^\dagger = \Delta^\dagger \Delta$  and therefore we find that the value vanishes.

**Altermagnetic term** As we introduced it in a previous discussion, the altermagnetic term is more complicated than the last one treated and needs more work. In essence we describe an advanced hopping term, that changes regarding of the hopping axis. We can first bring the commutator where the matrix element  $(\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'}$  is a scalar:

$$\begin{aligned} \left[ \sum_{\langle ij \rangle \sigma \sigma'} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \cdot c_{i\sigma}^\dagger c_{j\sigma'}, \sum_{\tilde{\sigma}} n_{l,\tilde{\sigma}} \right] &= \sum_{\substack{\langle ij \rangle \\ \sigma \sigma' \tilde{\sigma}}} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left( c_{i\sigma}^\dagger c_{j\sigma'} n_{l\tilde{\sigma}} - n_{l\tilde{\sigma}} c_{i\sigma}^\dagger c_{j\sigma'} \right) \\ &= \sum_{\substack{\langle ij \rangle \\ \sigma \sigma' \tilde{\sigma}}} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \delta_{\sigma\tilde{\sigma}} \left( \delta_{lj} c_{i\sigma}^\dagger c_{j\tilde{\sigma}} - \delta_{li} c_{i\tilde{\sigma}}^\dagger c_{j\sigma'} \right) \\ &= \frac{1}{2} \sum_{i\delta\sigma\sigma'} (\mathbf{m}_{i,i+\delta} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left( \delta_{l,i+\delta} c_{i\sigma}^\dagger c_{i+\delta,\sigma} - \delta_{li} c_{i\sigma}^\dagger c_{i+\delta,\sigma'} \right) \end{aligned}$$

Using the same transformation we made earlier to introduce the commutator between  $n, c$  and  $n, c^\dagger$  we obtain after summing over the  $\tilde{\sigma}$  introducing a new set made of  $\delta$ s as we did before. The summation over the  $\delta$  results in the following [combersom](#) expression

$$\begin{aligned} &= \sum_{\sigma} \left[ \sum_{\sigma'} (\mathbf{m}_{l,l+\delta_x} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left( c_{l-\delta_x,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l+\delta_x,\sigma'} \right) \right. \\ &\quad + \sum_{\sigma'} (\mathbf{m}_{l,l-\delta_x} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left( c_{l+\delta_x,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l-\delta_x,\sigma'} \right) \\ &\quad + \sum_{\sigma'} (\mathbf{m}_{l,l+\delta_y} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left( c_{l-\delta_y,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l+\delta_y,\sigma'} \right) \\ &\quad \left. + \sum_{\sigma'} (\mathbf{m}_{l,l-\delta_y} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left( c_{l+\delta_y,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l-\delta_y,\sigma'} \right) \right] \end{aligned} \quad (66)$$

When it will come to identify the current on each side of the unit cell, we will have to take into account the displacement  $\delta$  that is involved in the  $c_{l\pm\delta_x,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l\pm\delta_x,\sigma'}$ . This has to be consistent with what we already did for the hopping term. This means each current flowing through one side will be dependent on the spin hopping on both directions of the side's axis. This will become clear in the derivation.

**Side currents** Each unit cell was introduced as a square. We can from the last derived equation identify the contribution of the current on each side of the cell. We have a side on each axis direction  $+x, +y, -x, -y$  as introduced in Eq.64. Due to the symmetric properties of the current, we can abstract the notation for  $r \in \{x, y\}$ . Using Eq.65 and Eq.66 we can write the current on each side  $r$  as follows. We also reintroduce the minus of every summand of the Hamiltonian (due to their attractive nature) we let aside for the commutator relations. This multiplies with the minus one from the Heisenberg picture resulting in a plus sign. Further taking  $i = -1/i$  we get:

$$(I_i^{\pm r})_{hop} = -\frac{t}{2i} \sum_{\sigma} c_{i\pm\delta_r,\sigma}^\dagger c_{i\sigma} - c_{i\sigma}^\dagger c_{i\pm\delta_r,\sigma} \quad (67)$$

$$(I_i^{\pm r})_{AM} = -\frac{1}{2i} \sum_{\sigma\sigma'} c_{i\pm\delta_r,\sigma}^\dagger c_{i\sigma} (\mathbf{m}_{i,i\mp\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} - c_{i\sigma}^\dagger c_{i\pm\delta_r,\sigma'} (\mathbf{m}_{i,i\pm\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \quad (68)$$

In this sense the hopping and altermagnetic term are very similar from nature. However the altermagnetic term is scaled by the hopping amplitude  $t$  on each summand, which shows a spin-dependent behaviour.

However we're not quite finished. The way we defined the matrix element, is to be isotropic on each axis. By doing so we can assume that  $(\mathbf{m}_{i,i-\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} = (\mathbf{m}_{i,i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'}$  so that:

$$(I_i^{\pm r})_{AM} = -\frac{1}{2i} \sum_{\sigma\sigma'} (\mathbf{m}_{i,i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left( c_{i\pm\delta_r,\sigma}^\dagger c_{i\sigma} - c_{i\sigma}^\dagger c_{i\pm\delta_r,\sigma'} \right). \quad (69)$$



**Total currents** Until now we are able to describe how the current flows through each face of the unit cell. For this we can simply split the term we derived. From this we now aim to derive the current flows on each axis. This can easily be done assuming that the current flowing in the  $-x$  direction subtracted from the current in the positive  $x$  direction forms the total current in  $\mathbf{e}_x$ . From this we get for  $r \in \{x, y\}$ :

$$I_i^r = I_i^{+r} - I_i^{-r}.$$

The real current that we can measure can be obtained by taking the quantum expectation value and the thermal average of the currents. Further we also introduce the BdG-transformed operators with the eigenvalues 60 and 61.

For the sake of readability we are going to stick with this  $r$ -notation. In fact the total currents takes a disproportionate size on the page so we abstract a bit. Further due to the linearity of the commutator one can split the current in different terms. We first start with the derivation of the physical current that comes from the hopping term.

$$\langle I_i^r \rangle_{\text{hop}} = \frac{t}{2i} \left[ \sum_{\sigma} \langle c_{i-\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle - \langle c_{i\sigma}^{\dagger} c_{i-\delta_r, \sigma} \rangle - \langle c_{i+\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle + \langle c_{i\sigma}^{\dagger} c_{i+\delta_r, \sigma} \rangle \right]$$

and after introducing the BdG-transformed operators we obtain for the first term:

$$\begin{aligned} \langle c_{i-\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle &= \sum_{n, m \in \mathcal{N}_+} u_{n, i-\delta_r, \sigma}^* u_{ni\sigma} \underbrace{\langle \gamma_n^{\dagger} \gamma_m \rangle}_{\delta_{mn} f(E_n)} + u_{n, i-\delta_r, \sigma}^* v_{mi\sigma}^* \underbrace{\langle \gamma_n^{\dagger} \gamma_m^{\dagger} \rangle}_0 \\ &\quad + v_{n, i-\delta_r, \sigma} u_{mi\sigma} \underbrace{\langle \gamma_n \gamma_m \rangle}_0 + v_{n, i-\delta_r, \sigma} v_{mi\sigma}^* \underbrace{\langle \gamma_n \gamma_m^{\dagger} \rangle}_{\delta_{mn} (1-f(E_n))}. \end{aligned}$$

In the same way, we obtain for the other terms:

$$\begin{aligned} -\langle c_{i\sigma}^{\dagger} c_{i-\delta_r, \sigma} \rangle &= - \sum_{n \in \mathcal{N}_+} u_{ni\sigma}^* u_{n, i-\delta_r, \sigma} f(E_n) + v_{ni\sigma} v_{n, i-\delta_r, \sigma}^* (1 - f(E_n)), \\ -\langle c_{i+\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle &= - \sum_{n \in \mathcal{N}_+} u_{n, i+\delta_r, \sigma}^* u_{ni\sigma} f(E_n) + v_{n, i+\delta_r, \sigma} v_{ni\sigma}^* (1 - f(E_n)), \\ \langle c_{i\sigma}^{\dagger} c_{i+\delta_r, \sigma} \rangle &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma}^* u_{n, i+\delta_r, \sigma} f(E_n) + v_{ni\sigma} v_{n, i+\delta_r, \sigma}^* (1 - f(E_n)). \end{aligned}$$

Here we can recognise a very useful relation that is going to simplify everything. We have  $\langle c_{i\pm\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle = \langle c_{i\sigma}^{\dagger} c_{i\pm\delta_r, \sigma} \rangle^*$ . We can then recall a useful relation that we can use in these expressions. Have  $z$  a complex we obtain  $\text{Im}(z) = \frac{1}{2i}(z - z^*)$ . Further  $\text{Im}(z) + \text{Im}(z') = \text{Im}(z + z')$ . After summing up the terms we obtain the following expression for current of the hopping term along the axis  $r$ :

$$\begin{aligned} \langle I_i^r \rangle_{\text{hop}} &= t \cdot \text{Im} \left[ \sum_{\sigma} \sum_{n \in \mathcal{N}_+} f(E_n) u_{ni\sigma} (u_{n, i-\delta_r, \sigma}^* - u_{n, i+\delta_r, \sigma}^*) \right. \\ &\quad \left. + (1 - f(E_n)) v_{ni\sigma}^* (v_{n, i-\delta_r, \sigma} - v_{n, i+\delta_r, \sigma}) \right]. \end{aligned} \tag{70}$$

And inside the atom magnet we add the following term using the same derivation method. The matrix element  $(\mathbf{m}_{i, i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'}$  is a constant regarding the states-brackets. In this sense the quantum expectation and thermal average leave this quantity unchanged. Finally we obtain the exact same result than a  $t$  hopping but with a prefactor. This is consistent with the observation already made.

$$\langle I_i^r \rangle_{\text{AM}} = \frac{1}{2i} \sum_{\sigma\sigma'} (\mathbf{m}_{i, i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left[ \langle c_{i-\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle - \langle c_{i\sigma}^{\dagger} c_{i-\delta_r, \sigma'} \rangle - \langle c_{i+\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle + \langle c_{i\sigma}^{\dagger} c_{i+\delta_r, \sigma'} \rangle \right]$$

which after using the same derivation method as before yields:

$$\begin{aligned}
\langle I_i^T \rangle_{\text{AM}} = \frac{1}{2i} & \left[ \sum_{\sigma\sigma'} (\boldsymbol{m}_{i,i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \right. \\
& \sum_{n \in \mathcal{N}_+} f(E_n) \left[ u_{ni\sigma} (u_{n,i-\delta_r,\sigma}^* - u_{n,i+\delta_r,\sigma}^*) + u_{ni\sigma}^* (u_{n,i+\delta_r,\sigma'} - u_{n,i-\delta_r,\sigma'}) \right] \\
& \left. + (1 - f(E_n)) \left[ v_{ni\sigma}^* (v_{n,i-\delta_r,\sigma} - v_{n,i+\delta_r,\sigma}) + v_{ni\sigma} (v_{n,i+\delta_r,\sigma'}^* - v_{n,i-\delta_r,\sigma'}^*) \right] \right].
\end{aligned} \tag{71}$$

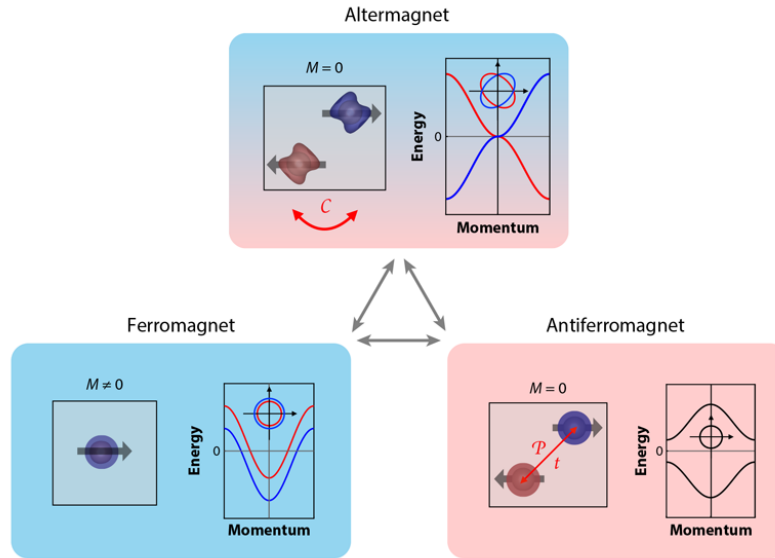
## 5 Altermagnetism



### 5.1 Introduction and overview

The reader might already be familiar with ferromagnets and all the regular magnetic models. Taking into account the spin and wave vector of each particle in a site, one can derive some symmetries under transform operations. For exemple a ferromagnet is symmetric under spin-flip (also called time reversal) and a rotation. In the exemple of the antiferromagnetism, we have two sub-lattice with opposite spins. In such systems the spin compensate eachother resulting in a null magnetisation. The system is symmetric under spin-flip and translation and was theoretised in 1948 by Louis Néel as the Néel antiferromagnets [3]. Allowing more complexity one could imagine multiple ions in a unit cell linked by rotation, screw, etc symmetries. The half ions of the cell owns an opposite spin from the other half. The spectra compoente eachother resulting in a zero net magnetisation. Moreover these mediums keep same electron spectra under such transformations, making them the same as Néel's antiferromagnets. These material are labeled as zero  $q$  antiferromagnets. This section summerises the historical work of [29].

Altermagnets implement two or more sub-lattice which are not related between eachother by translation or inversion. Since these operations are known to conserve the electrons spectra, we observe a band splitting of the spectra with respect to the spins. These class of material was pinpointed in 2019 [14]. The symmerty that actualy links these sublattices is a rotation, along with a spin-polarisation in the momentum space linked with a rotation as well [22]. We observe an alternating sign in the spin-splitted momenta, as well as an alternating spin polarisation in the real space, which leads to the name altermagnetism [21].



**Figure 10:** The altermagnet shares similar properties with the antiferromagnets like two sublattices of opposite spin, resulting in a zero net magnetisation. Whereas the antiferromagnet's sublattices are linked by a translation  $t$  and a space inversion  $\mathcal{P}$ , in the altermagnet one can link each sublattice with a rotation, here labelled as  $\mathcal{C}$ . We observe as well an alternating sign in the energy spectra of both altermagnet's sublattices [22][33].

These properties put the altermagnet in between the conventional dicotomy of ferromagnets and antiferromagnet. In fact the altermagnet compensates its magnetical moments taking advantage of the two sublattices, like the antiferromagnet does. On the other hand it shares similar properties with the ferromagnets, such as anomalous Hall effect, spin current [17], magneto-electric effect [24], tunneling magnetoresistance [23] among others [21].

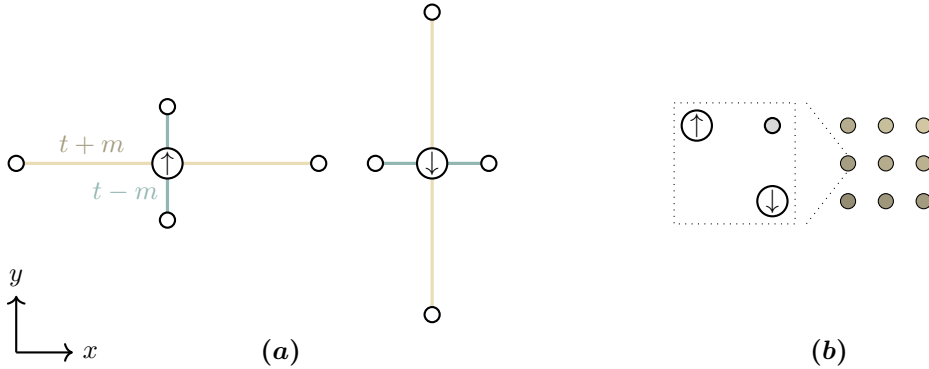
Now that we have a better understanding of altermagnetism we can compare it with another

class of material, where the sublattices aren't linked with the conventional crystal symmetries. As a consequence summing up the spin might not result in a trivial expression like the antiferromagnetic material. In fact the overall spin projection might almost be zero but not exactly. However studying half metal and insulator leads to different result. Half metal can be seen as an insulator in one spin channel but not both [29]. The Quin Luttinger's theorem, showed that such substance must have an integer number of Bohr spin magnetic moment. Therefore structures with a small net magnetisation see these quantity be floored to zero.

A few material seems to exhibit this properties. 1D material won't work because of lack of rotation possibilities. A diverse list of two and three dimensional materials were listed by [22] such as semi-metals with a  $\text{Cr}_2\text{O}$  monolayer [26], a variety of insulators like perovskite oxide  $\text{LaMnO}_3$  [20]  $\text{CaCrO}_3$  [18], ferrite  $\text{Fe}_2\text{O}_3$  [21], or organic insulators  $\kappa\text{-Cl}$  [15]. We find as well the chalcogenide  $\text{MnTe}$  semi-conductor [21].

On the search of practical application, researchers have found (an exhaustive list is provided in [29]) that domain like tunneling magnetoresistance (TMR) are limited under the properties of the material. For instance the actual use of ferromagnets limits the frequency to the gigahertz range [check the values at https://www.nature.com/articles/158670a0](https://www.nature.com/articles/158670a0). This has to deal with the ferromagnetic resonance that connects the magnetisation with electromagnetic waves. The altermagnets could be used in the terahertz range. Mentioning that TMR is a key component of the magnetic random access memory (MRAM) [7][25], the reader can easily estimate the performance improvement such upgrade could deliver in a computer.

After this introduction on altermagnets, we intend to find a way to implement these into our system. In a more formal way, we can distinguish two types of altermagnet. In the first type, the altermagnet's lattice site have different distance to the neighbours depending on the linking axis and the spin of the particle in the site. This is illustrated in Fig.11 (a). On the other hand we can consider a unit cell being unsymmetric in its ion position. We consider a square lattice where each unit cell has a non magnetic ion and two magnetic ion with opposite spin. Please consider the second schema (b).



**Figure 11:** Altermagnet of type 1 and 2.

## 5.2 Symmetries

After giving an introduction on the geometric background of altermagnet we can now focus on the symmetries of the material. In this section we are going to focus on two simple transformation. The inversion  $P$  and the spin flip (time reversal)  $T$  which are broadly discussed in the literature [22][21].

$$\begin{aligned} P : \mathbf{r} &\mapsto -\mathbf{r} & \mathbb{R}^3 &\rightarrow \mathbb{R}^3, \\ T : t &\mapsto -t \hat{=} \sigma \mapsto -\sigma & \mathbb{R} &\rightarrow \mathbb{R}. \end{aligned}$$

We can first assume that the energy  $\epsilon$  of the particle (or band structure) depends on the spin  $\sigma$  and the wave vector  $\mathbf{k}$ . Using  $\mathbf{p} = d\mathbf{r}/dt$  and the de Broglie hypothesis  $\mathbf{p} = \hbar\mathbf{k}$  [1] leads to:

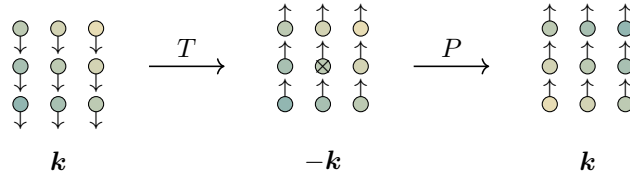
$$\begin{aligned} P(\epsilon(\mathbf{k}, \sigma)) &= \epsilon(-\mathbf{k}, \sigma) \\ T(\epsilon(\mathbf{k}, \sigma)) &= \epsilon(-\mathbf{k}, -\sigma) \end{aligned}$$

We observe the  $PT$  operation  $P \circ T$ . If the system is  $PT$  symmetric one could get  $\epsilon(\mathbf{k}, \sigma) = P \circ T(\epsilon(\mathbf{k}, \sigma))$ . Assuming it's the case we can write

$$\epsilon(\mathbf{k}, \sigma) = P \circ T(\epsilon(\mathbf{k}, \sigma)) = \epsilon(\mathbf{k}, \downarrow)$$

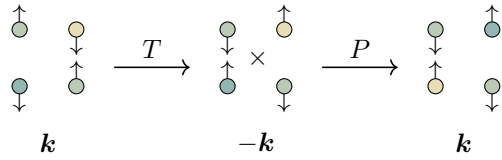
and therefore the  $PT$ -symmetric systems are spin-degenerated. We have the same energy for a given momentum at two opposite spins. Reciprocally this means that the existence of  $\epsilon(\mathbf{k}, \sigma)$  is followed by an observation of  $\epsilon(\mathbf{k}, -\sigma)$ .

**A few examples** We can imagine having a small lattice section for the sake of readability. We consider a ferromagnetic lattice where we apply the space inversion in the middle where the black cross is. The arrows represent the spin and the overall sign of the wave vector  $\mathbf{k}$  is given under each state. Moreover we use the shading to keep track of the real space location of the site. This representation is in the real space.

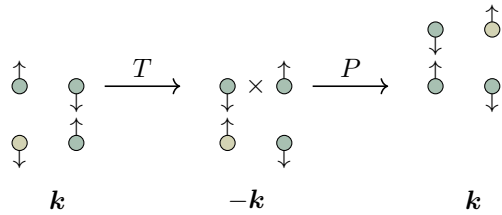


The relevant variables for the symmetry in the spectrum are the spin projection and the momentum. We see however that the spins are flipped before and after the  $PT$ -operation. Letting the spins point downwards again will require to use an additional operation. Therefore the ferromagnet is not  $PT$ -symmetric and we observe no spin degeneracy as the equation shows it.

The antiferromagnet has a spin switch in every direction. Here when choosing a point to apply our operations to, we see that neither a lattice point nor a point in the middle of two sites lead to a  $PT$  symmetry. As we see the spins point in different directions.



However if we apply the inversion in the inbetween, we observe something different.



This point we apply the  $P$  transformation around makes the antiferromagnets having a  $PT$  symmetry. We see that the lattice sites have moved but only the spin and the wave vector take part to the band  $\epsilon$ . We therefore have a degeneracy in the energy spectrum. Finding at least one point where the  $PT$  symmetry works makes the system  $PT$  symmetric.

If we now take a look at an altermagnet pictured in Fig.11, we have two different models to test. In the first one (a) the effect of the time reversal  $T$  makes the hopping change  $m \rightarrow -m$ . Then we find no point to invert the system around so that this first model doesn't have a  $PT$  symmetry. For the second one (b) we can first invert all spins but afterwards inverting the space doesn't bring back the system to the original configuration. In fact neither applying

the  $P$  after  $T$  on the lattice point nor on the space inbetween leads to a the lattice we began with. In both cases the degenrancy is lifted, we have dissimilar energies for the different spin-orientation. This is the reason for the band splitting we discussed in the introducing paragraph.

**Beyond the scope** We refer as spin-orbit coupling (SOC) the interaction between the spin of the electron and the orbital motion. When SOC is disgarded we observe a rotation symmetry around the spins. A rotation  $R$  can turn the spin back after the application of  $T$  resulting in a  $RT$  symmetry.  $RT(\epsilon(\mathbf{k}, \sigma)) = \epsilon(-\mathbf{k}, \sigma)$  must equal  $\epsilon(\mathbf{k}, \sigma)$ .

If we now consider a colinear magnet without SOC, we have a  $RT$  but no  $PT$  symmetry. Under such circumstances the spin-splitting must be symmetric under  $\epsilon(-\mathbf{k}, \sigma) = \epsilon(\mathbf{k}, \sigma)$  wich describes an even band structure. Such systems are called inversion symeric altermagnets.

### 5.3 Implementation of an altermagnet

After introducing the basic properties of an altermagnet we now aim to describe this system with the already introduced formalism. We are going to consider the first model involving a spin-dependent hopping. The Hamiltonian is given by [28] as:

$$H_{AM} = - \sum_{\langle i,j \rangle \sigma \sigma'} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma \sigma'} c_{i\sigma}^\dagger c_{j\sigma'}$$

involving  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$  the Pauli matrices.  $\mathbf{m}_{ij}$  is the actual spin dependent hopping term. If the connection line between the two sites  $i$  and  $j$  lays on the  $\mathbf{e}_x$  axis, we have  $\mathbf{m}_{ij} = m(0, 0, 1)$  and  $\mathbf{m}_{ji} = m(0, 0, -1)$  along  $\mathbf{e}_y$ .  $m$  is the desired hopping amplitude, and  $\mathbf{m}_{ij}$  scales and masks the Pauli matrices.  $\sigma_3$  puts  $m$  on the diagonal an takes care of the sign switch. Now having the normal metal t-hopping and this term leads to the  $t \pm m$  we described in Fig.11 (a).

Since the BdG formalism uses fermonic operators, we again need to bring the matrix in a way that reflects the fermionic properties regarding to the  $\tilde{c}$  opertor. We can remake use of Eq.  $\mathfrak{Tr}1$ :

$$H_{AM} = - \sum_{\langle ij \rangle \sigma \sigma'} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma \sigma'} \frac{1}{2} \left( c_{i\sigma}^\dagger c_{j\sigma'} - c_{i\sigma'} c_{j\sigma}^\dagger \right).$$

For the last summand we splitted the sum, exchanged the labeling of the states  $i$  and  $j$  and recombined the sums. However due to the matrix element in the front that we want to factorise with, we can't exchange the labbelings of  $\sigma$  and  $\sigma'$ . Bringing this Hamiltonian in the Nambu formalism we ontain by defining  $(\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma \sigma'} = \mathcal{M}_{\sigma \sigma'}$  the following Hamiltonian:

$$H_{ij}^{(AM)} = \frac{1}{2} \begin{pmatrix} -\mathcal{M} & \mathcal{O} \\ \mathcal{O} & \mathcal{M}^T \end{pmatrix}, \quad \text{where} \quad \mathcal{M} = \begin{pmatrix} \mathcal{M}_{\uparrow\uparrow} & \mathcal{M}_{\uparrow\downarrow} \\ \mathcal{M}_{\downarrow\uparrow} & \mathcal{M}_{\downarrow\downarrow} \end{pmatrix}.$$

Making use of this matrix in an altermagnetic lattice point, we can generate the according eigenvectors and -values. We can then study how the physical quantities behave in this material.

## 6 Simulations and results



In the last chapters we introduced the theoretical background of superconductivity and altermagnets. The goal of this thesis is to use numerical methods to solve self-consistently the superconductive gap. We remind here that all the code is available on a GitHub repository at <https://github.com/Tamwyn001/Thesis>.

### 6.1 Methodology

We're here going to simulate the s-wave superconductivity in a two-dimensional lattice, in a superconductor and a junction of a superconductor and an altermagnet. The orientation of the interface can play some role in the shape of the gap, which makes the study of a straight and diagonal interface a case of study of particular interest. Further we will illustrate how the phase gradient of the gap can induce a current in the superconductor. The shape of the d-wave gap parameter is also of interest. However we missed the time to make a comprehensive study of it. The de Josephson effect in a SC-AM-SC trilayer is in the code as well but needs to be reworked.

The repository contains the matlab code (.m) used to generate the data in the form of (coordinate, value) files (.dat). Gnuplot scripts (.gp) allow us to produce some plots that we can then directly link to the LaTeX source code of this thesis. Finally some powershell scripts (.ps1) automate the gnuplot-latex process.

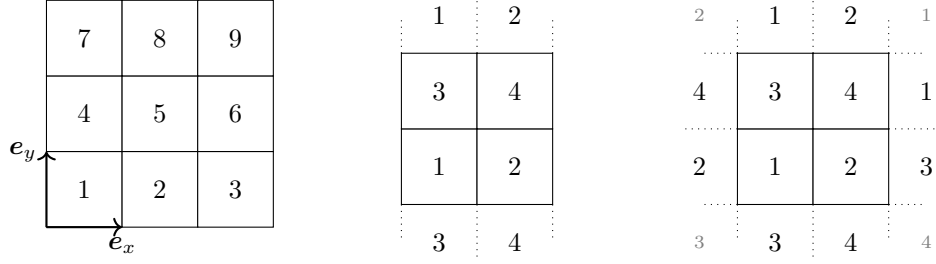
The matlab code contains two important files. The `system.m` file is a class that describes which material we are using for the lattice, the physical context like the temperature and the chemical potential as well as the use or not of periodic boundary conditions. It contains also the Hamiltonian of the system and keeps track of all lattice sites.

The `LatticeSites.m` describes the position of a lattice site, stores its neighbours and monitors the lattice site specific physical quantities. These quantities can be the superconducting gap or the current, etc.

We introduced earlier the need of a self-consistent solution for the gap to correctly describe the Hamiltonian. This is achieved in the `GapEquation.m` file which is the main script we run. We start with a guess for  $\Delta_i$  and introduce it in the Hamiltonian. This guess can depend on the material, or the site if we aim to introduce some phase shift at specific locations. We then derive a new site-dependent value using Eq.63 for all sites and reinsert it into the Hamiltonian. This requires to compute the eigenvectors and values of the Hamiltonian. This loop will be performed until the difference of each gap with its previous value reaches a threshold. We then have a proper Hamiltonian and can start to derive some values like the current.

Matlab is a program that works well with matrices so it is a preferred choice. However later in the process the fact that Matlab works by values and not by reference makes a lot of complications. In other words instead of passing a reference to a location a variable is stored at, we just copy it and pass it over. Here is an example: We have for each site up to four neighbours. When computing the current divergence, we need the current of the neighbours. If we were to use a reference method, then the lattice sites will be fixedly defined in the memory and we can read it from all the places. However since we work by values, we need to send the values in each copy of the site. So we have an array storing the lattice sites in the system, and another for each lattice point storing its neighbours. However these sites *are the same entities*. Working by value means we need to set explicitly the value everywhere a site is stored. Working by reference would have just needed to set the value where the site is stored in the memory, allowing all the references that point to this location to be implicitly updated. In this sense a wiser choice would have been C++ with an appropriated diagonalization library.

The site coordinate is stored as  $(x, y)$ -tuple as well as an index  $i$  that iterates over  $y$ -row and then starts again at the beginning of the next row. This means that with different boundary conditions we get the following shape for the system:



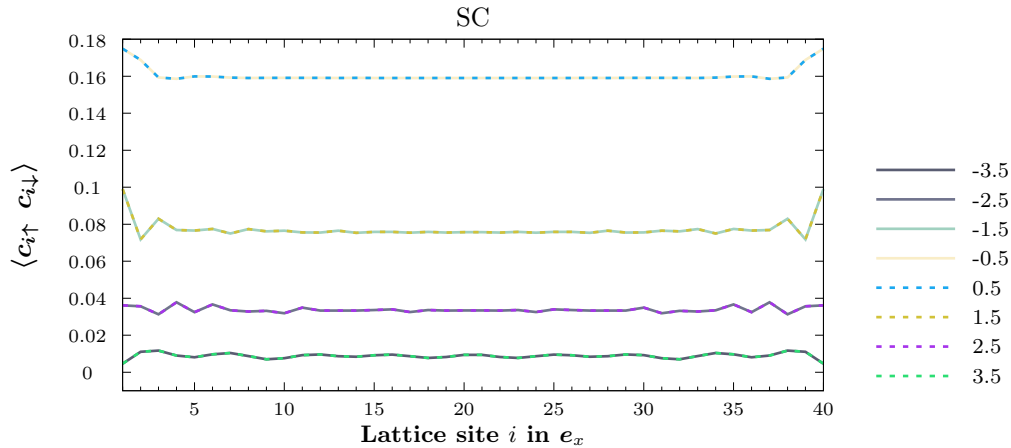
**Figure 12:** The sites indexing method as well as a representation of the different boundary conditions.

Left we have a  $3 \times 3$ -system. The others are  $2 \times 2$  with vertical periodic boundary conditions (PBC) and on the right with both vertical and horizontal PBC.

We want to maximize the definition of the gap in the lattice. With the achitecture we have (Intel i7 11th gen, 8GB RAM), an upperbound of  $40 \times 20 = 1200$  sites was achieved. Matlab being an interpreter, this could be optimized by using a compiled language. In the literature we find square lattices of  $100 \times 100$  sites [16].

## 6.2 Results

We start with the simplest system, a superconductor involving some vertical periodic boundary conditions. We use an attractive potential of  $U = -2$  for the on site superconductivity in the superconductor and zero everywhere else.



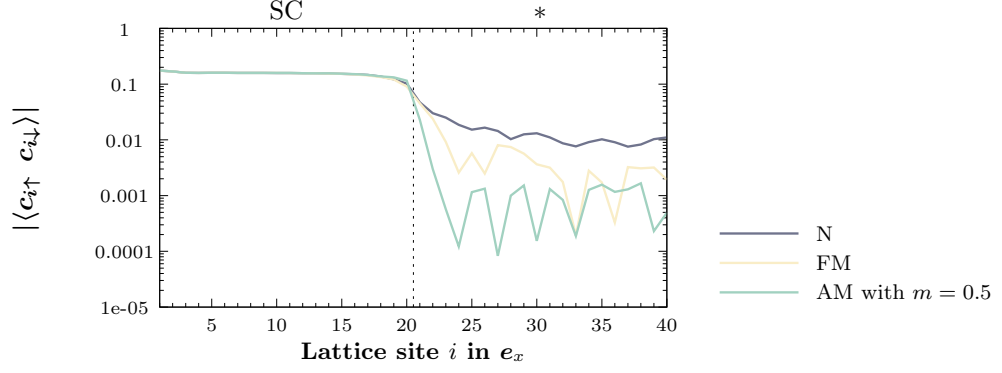
**Figure 13:** The expectation value of the c operators taking part into the gap as  $\Delta_i = U_i \langle c_{i\uparrow} c_{i\downarrow} \rangle$ . The vertical periodic boundary condition makes each site on the same  $x$ -coordinate having the same value. We can then make an average on each of these columns and plot the result.

The superconducting gap lays around  $0.01 \cdot U$  and  $0.16 \cdot U$  eV? for the range of selected chemical potential. Further we observe a clear symmetry in the Cooper pairs distribution around the level  $\mu = 0$ . The overlap of the Fermi surface with the s-wave seems to be the same for  $\pm\mu$ . Further we observe some oscillations on the left and right sides. On these locations the sites have only three neighbours, there are open boundaries, the rest is vacuum. One can see this lack of neighbours as impurities in the system. These oscillations may be caused by Andreev bound states [Bobkov'2024], that may form because the quasiparticles are reflected at the boundary. These can interfere with the Cooper pairs and cause the oscillations in the spatial representation of the energy gap.

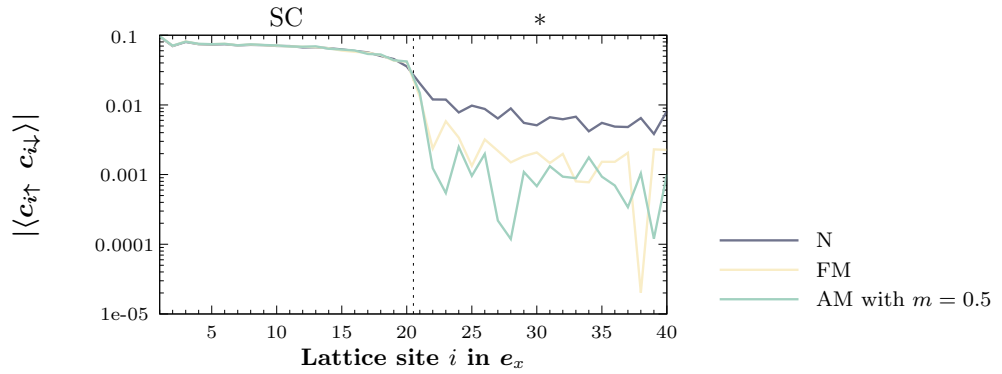
From this we can complexify the system by adding a different material on the right side of the superconductor. The most simple one is a normal metal (N) that has a hopping  $t$  of 1 in every direction. Then we can replace it with an alternating hopping  $t + m$  depending on a  $\uparrow\uparrow$



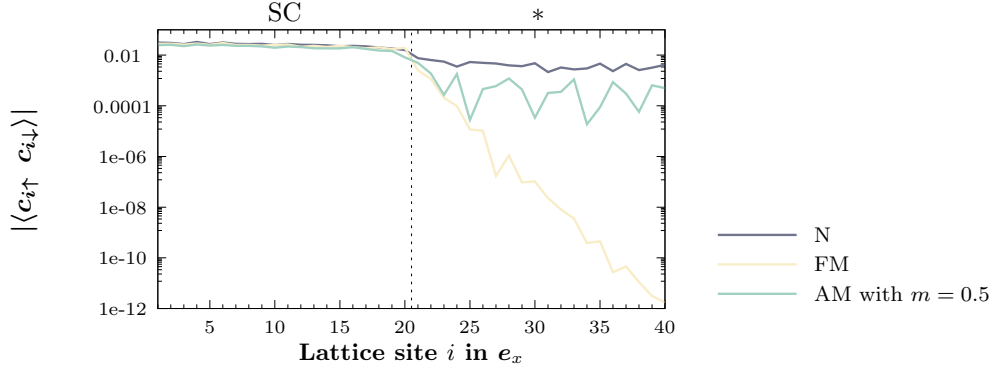
interaction and a  $t - m$  depending on a  $\downarrow\downarrow$  interaction. This is a ferromagnetic material (FM). Making the sign of  $m$  alternating if we have an interaction along the  $x$  or the  $y$  axis describes an altermagnet (AM). Here we set  $m = 0.5$  and study different results for the chemical potential  $\mu < 0$  and compare the AM with the others. \* represent a placeholder for the material a curve describes.



**Figure 14:** Evolution of the gap in the  $x$  direction for a junction of SC-N SC-FM and SC-AM at  $\mu = -0.5$ .



**Figure 15:** Evolution of the gap in the  $x$  direction for a junction of SC-N SC-FM and SC-AM at  $\mu = -1.5$ .



**Figure 16:** Evolution of the gap in the  $x$  direction for a junction of SC-N SC-FM and SC-AM at  $\mu = -2.5$ .

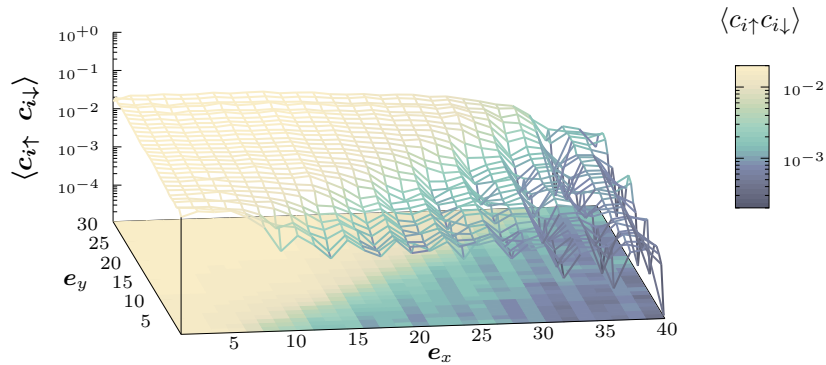
We can compare to the normal metal case. We have a clear decay of the gap after entering the normal metal. At this point study that were already made tend to have an exponential decay of the gap [16]. Having a logarithmic scale we would expect a more straight line but this is still a reasonable exponential decay. A way to approach the line would be to increase the resolution of the lattice, as well as simulating a more squared one. Further we see that both the FM and the AM add some oscillations in the decay.

The ferromagnet seems to follow closely the normal metal for  $\mu = -0.5, -1.5$ . It is interesting to note a really nice exponential decay when having a chemical potential of  $-2.5$  modulated by oscillations. For the AM the initial decay is stronger in the sites near the interface than in the FM. Then the decay seems to follow the FM. We observe about the same number of oscillations in the FM and the AM. For  $\mu = -0.5$  we observe a clean line in the first four sites of the AM. Then we have oscillations and the Cooper-pairs are more present than expected regarding the initial decay. The reason might be that the first oscillation combined with the initial decay make the line reach its second deepest point (around 10–4) and looks therefore very small compared to the rest.

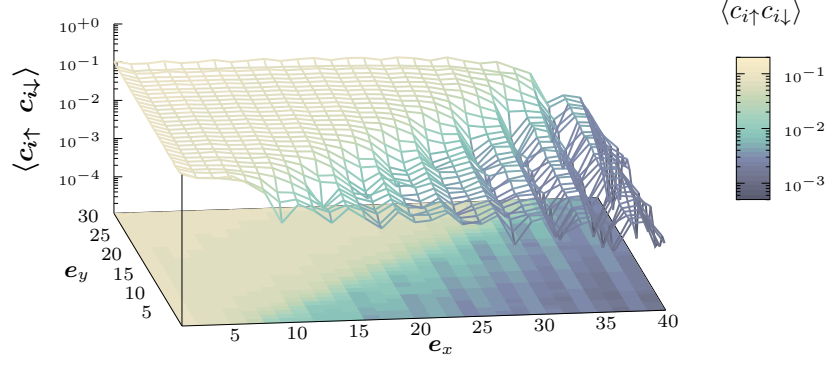
The oscillation amplitude seems to have increased as well, we observe nearly two orders of magnitude in their amplitude

The maximal oscillation amplitude seems to be the same in every context with a value around an order of magnitude.

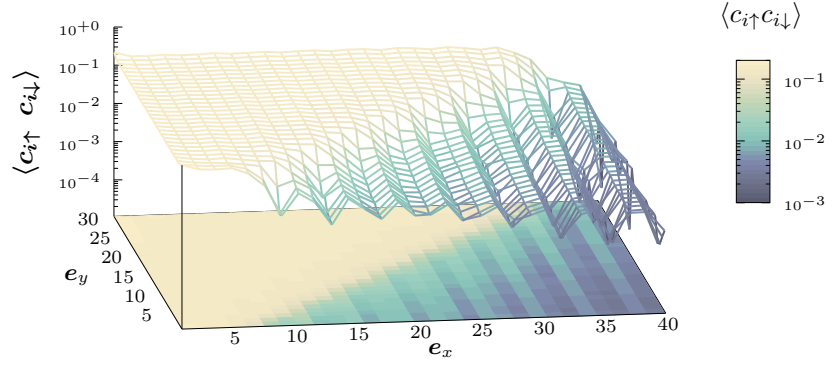
We now simulate how does the s-wave superconductivity behave on a diagonal interface. We focus on three different values of  $\mu < 0$ .



**Figure 17:**  $\mu = -2.5$



**Figure 18:**  $\mu = -1.5$



**Figure 19:**  $\mu = -0.5$

The superconductivity fills uniformly the superconductor. It's magnitude seems to grow with while  $|\mu|$  get closer to zero. Once again the oscillating exponential decay is present. We count up to eight oscillations for the three cases of study, whose frequency is around three sites in all configurations. The oscillations takes only place along the  $x$  axis but the exponential decay seems to point along the normal of the interface, making the (40,0) region the less populated in Cooper pairs. The lower the  $|\mu|$ , the less noise we observe in the plots. Similarly to the straight interface the decay is strong in the begging and then slows down when approaching the farest regions from the interface.

#### [A side view to comapre the oscilaltion amplitude.](#)

In the same way than before, this increase in Cooper pairs formation with degrowing  $|\mu|$  is due to the shape of the fermi surface. For  $\mu = -2.5$  we see that the populated region do not especialy fully fill the states occupied by the s-wave. For  $\mu = -1.5$  the populated region is larger more states can be found in the s-wave. For  $\mu = -0.5$  we observe a factor two increase in the number of Cooper pairs, while we have a factor ten between the first two cases. This is most likely due to the fact that the fermi surface at  $\mu = -1.5$  was already very close to the shape of the s-wave in the momentum space. Therefore the majority of the newly accessible states are outside the s-wave range, resulting in a smaller increase in the number of Cooper pairs when compared to the case of  $\mu = -2.5$ .

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## A An advanced superconductivity

### A.1 Site dependent potential

In this section we are going to highlight how the neighbour-depending potential can lead to a new kind of superconductivity. Among others we are going to discuss the  $d$ - wave superconductivity which shares similarities with more exotic magnets, such as the altermagnet. Indeed the spatial repartition of the phase of the gap seems to have the same geometry as the spin polarisation of the altermagnet [22] [figure?](#).

The derivations we are going to make are closely based on the work of A.H. Mjøs and J. Linder in [16].

The Hamiltonian beeing the groundstone of this disscussion, we're going to beggin with it. It differs slightly from the symmertic one we derived in Eq.52. This extended version contains a neighbour-depending potential  $H_V$  that is going to produce the new superconductive  $\Delta$ -part of the Hamiltonian. This potential is as well attractive such that  $V > 0$ , similar to the BCS theorie.

$$H = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} - \mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} - \frac{V}{2} \sum_{\langle ij \rangle \sigma} n_{i\sigma} n_{j\bar{\sigma}}. \quad (72)$$

Here the  $\bar{\sigma}$ -notation means the opposite spin of  $\sigma$ . In other words the attraction finds only place between particles of opposite spin, mirroring the formation process of Cooper-pairs. We have a factor of one half to avoid counting twice the neighbours.

Trained eyes will recognise that  $H_V$  is not quadratic in the creation and annihilation operators which makes the Hamiltonian impossible to write in the BdG-formalism. For this reason we can use the so called Hartree-Fock mean field approximation  $H_V \rightarrow H_V^{HF}$  defined as:

$$H_V^{HF} = -\frac{1}{2} \sum_{\langle ij \rangle \sigma} V_{ij} \left( F_{ij}^{\sigma\bar{\sigma}} c_{j\bar{\sigma}}^\dagger c_{i\sigma}^\dagger + \text{h.c.} + |F_{ij}^{\sigma\bar{\sigma}}|^2 \right), \quad (73)$$

involving the pairing amplitude  $F_{ij}^{\sigma\bar{\sigma}} = \langle c_{i\sigma} c_{j\bar{\sigma}} \rangle$  that we already introduced. The  $V_{ij}$  are the neighbour-depending potential. The last term is a constant energy term, which we can discard during the diagonalization process. If one wanted to compute the free energy this must be included there. From this point simplifications can be made to reach out final Hamiltonian.

Using the fermionic property  $[c_{i\sigma}^\dagger, c_{j\sigma}^\dagger]_+ = 0$  we have  $c_{i\sigma}^\dagger c_{j\sigma}^\dagger = -c_{j\sigma}^\dagger c_{i\sigma}^\dagger$  which leads to  $\langle c_{i\sigma}^\dagger c_{j\sigma}^\dagger \rangle = -\langle c_{j\sigma}^\dagger c_{i\sigma}^\dagger \rangle$ . We can use this in the last step of the following simplification:

$$\begin{aligned} H_V^{HF} &= -\frac{1}{2} \sum_{\langle ij \rangle \sigma} V_{ij} \left( F_{ij}^{\sigma\bar{\sigma}} c_{j\bar{\sigma}}^\dagger c_{i\sigma}^\dagger + \text{h.c.} \right) \\ &= -\frac{1}{2} \sum_{\langle ij \rangle} V_{ij} \left( F_{ij}^{\uparrow\downarrow} c_{j\downarrow}^\dagger c_{i\uparrow}^\dagger + F_{ij}^{\downarrow\uparrow} c_{j\uparrow}^\dagger c_{i\downarrow}^\dagger + \text{h.c.} \right) \\ &= -\sum_{\langle ij \rangle} V_{ij} \left( F_{ij}^{\uparrow\downarrow} c_{j\downarrow}^\dagger c_{i\uparrow}^\dagger + \text{h.c.} \right) \end{aligned} \quad (74)$$

Using the raltion  $F_{ij}^{\uparrow\downarrow} = -F_{ij}^{\downarrow\uparrow}$  and the symmerty of the potential  $V_{ij} = V_{ji}$  we can add the two terms up and remove the one half factor. The Hamiltonian is now ready to be diagonalized but first, we are going to discuss which advantages a Fourier transform of the Hamiltonian could bring us.

In an homogenous material we can consider a lattice and imagine some periodic boundary conditions in all directions. There is a translation invariance. In heterostructures however the material may vary, let's asume without loss of generality, in a direction. As we now, a periodic signal is a good candiade for a Fourier transform, which we can then express in a finite set of coefficient. This is very handfull. However the direction we want to transform on has to show a periodicty. For this reason an homogenous two-dimensional lattice involves a Fourier transformation in two space dimensions while a heterostructure we transform only in the direction where the material is the same. For instance a multilayer material in the  $x$ -direction

can be described by combining a real space description in the  $x$ -axis combined to a Fourier transformation in the  $y$ -direction. In this thesis we are going to focus ourselves on a multilayer material in the  $x$ -direction.

## A.2 In the vertical periodic boundary setup

The description of the creation and annihilation operators with a step in the momentum space for the  $y$ -direction can be expressed as follow:

$$c_{xy\sigma} = \frac{1}{\sqrt{N_y}} \sum_{k_y} c_{xk_y\sigma} e^{ik_y y} \quad (75)$$

$$c_{xy\sigma}^\dagger = \frac{1}{\sqrt{N_y}} \sum_{k_y} c_{xk_y\sigma}^\dagger e^{-ik_y y} \quad (76)$$

How do we find an expression for  $k_y$ ? Well the periodicity (for ex. in the  $c$  operator) yields  $c_{xy\sigma} = c_{x,y+N_y,\sigma}$  this means the following condition must be fulfilled:

$$\begin{aligned} c_{xy\sigma} = c_{x,y+N_y,\sigma} &\Leftrightarrow e^{ik_y y} = e^{ik_y(y+N_y)} \Leftrightarrow e^{ik_y N_y} = 1 \\ &\Leftrightarrow k_y = \frac{2\pi n}{N_y}. \end{aligned}$$

We know that the momentum index should cover the entire first Brillouin zone. This covers the momentum from  $-\pi/a$  to  $\pi/a$  where  $a$  is the lattice constant. Further due to the  $2\pi/a$  periodicity we have the same  $k_y$  at  $-\pi/a$  and  $\pi/a$ . For this reason we need to map  $k_y$  in  $[-\pi/a; \pi/a)$ . This means we have  $n \in [-N_y/2; N_y/2 - 1]$  including 0. For  $n = 4$  we then have  $k_y \in \{-\pi, -\pi/2, 0, \pi/2\}$ .

For the readability we are going to use  $k_y \rightarrow k$ . Further to an index  $i$  can associate  $(x, y)$  and to  $j$ ,  $(x', y')$ . We first want to transform the hopping term:

$$H_{\text{hop}} = - \sum_{\langle ij \rangle \sigma} t_{ij} \sum_{kk'} c_{xk\sigma}^\dagger c_{x'k'\sigma} e^{i(k'y' - ky)}$$

Here we can use the neighbour shift properties  $y' = y + \delta_y$  where  $\delta_y = \pm 1$ . Doing so we have

$$e^{i(k'y' - ky)} = e^{i(k'(y + \delta_y) - ky)} = e^{i(k' - k)y} e^{ik'\delta_y}$$

No we need to express the neighbourhood sum. Precily we have

$$\begin{aligned} H_{\text{hop}} &= - \sum_{\langle ij \rangle \sigma} t_{ij} \sum_{kk'} c_{xk\sigma}^\dagger c_{x'k'\sigma} e^{i(k-k')y} e^{ik'\delta_y} \\ &= - \sum_{xy\sigma} \sum_{kk'} \left( \underbrace{t_{x,x+1} c_{xk\sigma}^\dagger c_{x+1k'\sigma} e^{ik' \cdot (0)}}_{+x \text{ hopping, no } \delta_y} + \underbrace{t_{x,x-1} c_{xk\sigma}^\dagger c_{x-1k'\sigma} e^{ik' \cdot (0)}}_{-x \text{ hopping, no } \delta_y} \right. \\ &\quad \left. + \underbrace{t_{x,y-1} c_{xk\sigma}^\dagger c_{xk'\sigma} e^{ik' \cdot (-1)}}_{\delta_y = -1} + \underbrace{t_{x,y+1} c_{xk\sigma}^\dagger c_{xk'\sigma} e^{ik' \cdot (1)}}_{\delta_y = 1} \right) e^{i(k-k')y} \end{aligned}$$

As we see the  $y$  direction is now expressed in the  $k$ -index, which is unique for each lattice  $y$ -slice. The information is then conserved. We know that system have different material on the  $x$  axis. This means  $t_{x,y-1} = t_{x,x} = t_{x,y+1}$  because the material are isotropic in  $y$ . Every material has a different hopping term nevertheless. Beside we can use the following relation  $1/N_y \sum_y e^{i(k-k')y} = \delta_{kk'}$ . Perfroming both expression leads after a summation over  $k'$  to

$$H_{\text{hop}} = - \sum_{xk\sigma} t_{x,x+1} c_{xk\sigma}^\dagger c_{x+1k\sigma} + t_{x,x-1} c_{xk\sigma}^\dagger c_{x-1k\sigma} + t_{x,x} c_{xk\sigma}^\dagger c_{xk\sigma} (e^{ik} + e^{-ik})$$

And now we can reintroduce an arbitray second coordinate  $x'$  to describe the neighbours.

$$H_{\text{hop}} = - \sum_{xx'k\sigma} t_{x,x'} c_{xk\sigma}^\dagger c_{x'k\sigma} (\delta_{x+1,x'} + \delta_{x-1,x'} + \delta_{x,x'} 2 \cos(k)) \quad (77)$$

The chemical potential term is more easily given. In fact the number operator yields to use two operators  $c^\dagger c$  at a same coordinate  $i$ .

$$H_\mu = -\mu \sum_{xkk'\sigma} c_{xk\sigma}^\dagger c_{xk'\sigma} e^{i(k-k')y} = - \sum_{xx'k\sigma} \mu c_{xk\sigma}^\dagger c_{xk'\sigma} \delta_{xx'} \quad (78)$$

We finally have for the terms involving  $c_{xx'\sigma}^\dagger c_{xx'\sigma}$ :

$$H_{\text{hop}} + H_\mu = \sum_{xx'k\sigma} \epsilon_{xx'k\sigma} c_{xk\sigma}^\dagger c_{x'k\sigma}$$

using

$$\epsilon_{xx'k\sigma} = -t_{xx'} (\delta_{x+1,x'} + \delta_{x-1,x'}) - (t_{xx'} 2 \cos(k) + \mu) \delta_{xx'}$$

Moving on to the potential term we have to introduce a new notation. The goal is to describe a displacement in a direction from a point. So assuming we follow the indexing introduced in Fig.12:  $i \pm \hat{x} = i \pm 1$  and  $i \pm \hat{y} = i \pm N_x$ . For the brevity we use  $f(a) = V_{ia} F_{ia} c_{a\downarrow}^\dagger c_{i\uparrow}^\dagger$ . Still assuming that  $V_{ij} > 0$  and that the interaction is attractive, we have: [!! sign check for V and in frotn oh Hv!](#)

$$\begin{aligned} H_V &= - \sum_{\langle ij \rangle} V_{ij} F_{ij} c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger + \text{h.c.} \\ &= - \sum_i f(i-1) + f(i+1) + f(i+N_x) + f(i-N_x) + \text{h.c.} \end{aligned}$$

We can now insert our Fourier transformation introduced in Eq.76 and Eq.75 to obtain

$$\begin{aligned} H_V &= - \sum_{xy} \frac{1}{N_y} \sum_{kk'} \left( V_{x,x+1} F_x^{x+} c_{x+1,k,\downarrow}^\dagger c_{xk'\uparrow}^\dagger + V_{x,x-1} F_x^{x-} c_{x+1,k,\downarrow}^\dagger c_{xk'\uparrow}^\dagger \right. \\ &\quad \left. + V_{xx} (F_x^{y+} e^{-ik} - F_x^{y-} e^{ik}) c_{xk\downarrow}^\dagger c_{xk'\uparrow}^\dagger \right) e^{-i(k+k')y} + \text{h.c.} \end{aligned}$$

Defining a more general form for the summand involving two site  $i$  and  $j$ :

$$\begin{aligned} F_{xx'k} &= V_{x,x'} (F_x^{x+} \delta_{x+1,x'} + F_x^{x-} \delta_{x-1,x'} \\ &\quad + (F_x^{y+} e^{-ik} - F_x^{y-} e^{ik}) \delta_{xx'}) , \end{aligned}$$

where we got  $k' = -k$  from the sum over the  $y$ -direction. We can rewrite the expression of  $H_V$  as

$$H_V = - \sum_x \sum_{kk'} F_{xx'k} c_{xk\uparrow}^\dagger c_{x',-k,\downarrow}^\dagger + F_{xx'k}^* c_{x',-k,\downarrow} c_{xk\uparrow}.$$

Now using  $H_{\text{hop}}$ ,  $H_\mu$  and  $H_V$  we can write the full Hamiltonian as [\(add the formula for the altermagnet as well. Like Jacob said: We can reduce to a 2x2\)](#)

$$\begin{aligned} H &= \sum_{xx'k} D_{xk}^\dagger H_{xx'k} D_{x'k} \\ &= \sum_{xx'k} \begin{pmatrix} c_{xk\uparrow}^\dagger & c_{x,-k,\downarrow} \end{pmatrix} \begin{pmatrix} \epsilon_{xx'k\uparrow} & F_{xx'k} \\ F_{xx'k}^* & -\epsilon_{xx'k\downarrow} \end{pmatrix} \begin{pmatrix} c_{x'k\uparrow} \\ c_{x',-k,\downarrow}^\dagger \end{pmatrix}. \end{aligned} \quad (79)$$

The summation over all  $x, x'$  can be represented in a new matrix.

$$H = \sum_k \check{c}_k^\dagger H_k \check{c}_k$$

involving the  $4N_x \times 4N_x$  matrix  $H_k$  and the  $4N_x$ -dimensional vector  $\check{c}_k$ .

$$H_k = \begin{pmatrix} H_{11k} & \cdots & H_{1N_x k} \\ \vdots & \ddots & \\ H_{N_x 1k} & & H_{N_x N_x k} \end{pmatrix}$$



Please note that [16] uses the notation  $D_k$  for  $\check{c}_k$ , but we don't want to confuse the reader with the already defined eigenvalues diagonal matrix of Eq.58. As before the  $y$  information is stored in the  $k$ -index, which is unique for each lattice  $y$ -slice. This said, we can diagonalize  $N_y$  times a  $4N_x \times 4N_x$  matrix. Each  $H_k$  represents the interaction of a  $y$ -line with itself. The eigenvalues are the same for each  $y$ -slices and the physical quantities are going to be expressed with this summation over  $k$  and the eigenvalues, -vectors of each  $k$  ( $y$ -slice).

On the other hand the vector we use to carry the creation and annihilation operators is given as

$$\check{c}_k^\dagger = \begin{pmatrix} c_{1k\uparrow}^\dagger & c_{1,-k,\downarrow} & \cdots & c_{N_x k\uparrow}^\dagger & c_{N_x,-k,\downarrow} \end{pmatrix} \in \mathbb{H}^{2N_x}$$

### A.2.1 BdG-transformation for a vertical Fourier transform

The eigenvalues equation is similar to Eq.56

$$H_k \mathfrak{X}_{nk} = E_{nk} \mathfrak{X}_{nk}. \quad (80)$$

The eigenvectors and -values are given as

$$\mathfrak{X}_{nk} = \begin{pmatrix} \mathfrak{x}_{n1k} \\ \vdots \\ \mathfrak{x}_{nN_x k} \end{pmatrix}, \quad \mathfrak{x}_{nk} = \begin{pmatrix} u_{nxk} \\ v_{nxk} \end{pmatrix}.$$

If we stick to the formalism we already derived in the earlier Sec.4.4 we obtain similar eigenvectors where  $u_{nxk}$  corresponds to  $c_{xk\uparrow}$  and  $v_{nxk}$  to  $c_{x,-k,\downarrow}$  (see Eq.81). We are now going to transform the  $c$  operators. First we need to define  $\mathfrak{X}_k = [\mathfrak{x}_{1k}, \dots, \mathfrak{x}_{2N_x k}] \in \mathbb{H}^{2N_x \times 2N_x}$  storing the number of lattice sites times the number of spins (2) in the first dimension, and the number of eigenvectors in the second dimension. Beside we have  $\mathfrak{g}_k = (\mathfrak{g}_{1k}, \dots, \mathfrak{g}_{2N_x k})^T \in \mathbb{H}^{2N_x}$  along with  $D_k = \mathfrak{X}_k \mathfrak{g}_k$ , delivering  $\mathfrak{g}_k = \mathfrak{X}_k^\dagger D_k$  which is equivalent to:

$$\mathfrak{g}_{nk} = \sum_{x \in N_x} u_{nxk} c_{xk\uparrow} + v_{nxk} c_{x,-k,\downarrow}^\dagger. \quad (81)$$

Looking at  $D_k = \mathfrak{X}_k \mathfrak{g}_k$  we can derive two very useful expressions:

$$c_{xk\uparrow} = \sum_{n \in [2N_x]} u_{nxk} \mathfrak{g}_{nk} \quad (82) \quad \left| \quad c_{x,-k,\downarrow}^\dagger = \sum_{n \in [2N_x]} v_{nxk} \mathfrak{g}_{nk}. \quad (83) \right.$$

that are the equivalent of Eqs.?? and ?? for the Fourier transformed  $c$ -operators along the  $y$ -axis.

### A.2.2 Pairing amplitudes

For the derivation of the pairing amplitudes along the axis  $a \in \{\hat{x}, \hat{y}\}$ , we are going to abstractize. With  $F_i^{\pm a}$  we mean  $F_i^{\pm \hat{x}}$  or  $F_i^{\pm \hat{y}}$  which are equivalent to  $F_{i,i\pm x}$  and  $F_{i,i\pm y}$ . Doing so, we will use our BdG-transformation as we did earlier to be able to solve these parameters self-consistently using the eigenvectors and -values. We remember that the site  $i$  has the coordinate  $(x, y)$  and its neighbour  $j$  the coordinate  $(x', y')$ .

$$F_i^{\pm a} = \langle c_{i\uparrow} c_{i\pm a, \downarrow} \rangle = \frac{1}{N_y} \sum_{kk'} \langle c_{xk\uparrow} c_{x\pm \delta_x, k', \downarrow} \rangle e^{i(ky + k'y')} \\ \frac{1}{N_y} \sum_{kk'} \langle c_{xk\uparrow} c_{x\pm \delta_x, k', \downarrow} \rangle e^{i(k+k')y} e^{\pm i k' \delta_y}$$

using  $y \pm \delta_y = y'$  and  $x \pm \delta_x = x'$ . Naturally we know that for  $a = \hat{x}$  there is no hopping along the  $y$ -axis, such that  $\delta_y = 0$  and in the same way for  $a = \hat{y}$  we have  $\delta_x = 0$ . The expectation value is independent of the site vertical coordinate  $y$ . This means that we can achieve a site description by making the average of the value over a  $x$ -slice. In other words we can write:

$$F_i^{\pm a} = \frac{1}{N_y} \sum_{y \in [N_y]} \frac{1}{N_y} \sum_{kk'} \langle c_{xk\uparrow} c_{x\pm \delta_x, k', \downarrow} \rangle e^{i(k+k')r_i} e^{\pm i k' \delta_y}$$

which after the summation over  $i$  yields, as we covered earlier, to  $1/N_y \sum_y e^{i(k+k')y} = \delta_{k,-k'}$ :

$$F_i^{\pm x} = \frac{1}{N_y} \sum_k \langle c_{xk\uparrow} c_{x\pm\delta_x, k, \downarrow} \rangle e^{\mp i k \delta_y}.$$

Now that we have simplified the Fourier transform we can incorporate the BdG-transformation. The process is very similar to Eq.62 and after doing this, we obtain the following expression:

$$F_i^{\pm a} = \frac{1}{N_y} \sum_k \sum_{nn'} u_{nxk} v_{n', x\pm\delta_x, k}^* \langle \mathbf{g}_{nk} \mathbf{g}_{n'k}^\dagger \rangle e^{\mp i k \delta_y} \quad (84)$$

Recalling the  $a$  abstractization of the axis, we get:

$$F_i^{\pm x} = \frac{1}{N_y} \sum_{nk} u_{nxk} v_{n, x\pm\delta_x, k}^* (1 - f(E_{nk})) \quad (85)$$

$$F_i^{\pm y} = \frac{1}{N_y} \sum_{nk} u_{nxk} v_{nxk}^* (1 - f(E_{nk})) e^{\mp i k}. \quad (86)$$

where  $x \pm \delta_x$  means  $x \pm 1$ .

### A.2.3 In the open boundary setup

Here we need to reuse the equations of the Nambu-Spin formalism described in chapter 4. As we saw we need to first take care that the part of the Hamiltonian we want to introduce has [a kind of symmetry](#) in the creation and annihilation operators. If we transform the Hartree-Fock Hamiltonian as it is in Eq.74 we see that only the upper right quadrant of the matrix  $H_{V,ij}^{HF}$  of  $\check{c}_i^\dagger \cdot (H_V^{HF})_{ij} \cdot \check{c}_j$  is filled. We need to use the anticommutation relations to solve this. This delivers the following expression:

$$\begin{aligned} H_V^{HF} &= - \sum_{\langle ij \rangle} V_{ij} \left( F_{ij}^{\uparrow\downarrow} c_{j\downarrow}^\dagger c_{i\uparrow}^\dagger + F_{ij}^{\uparrow\downarrow*} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger \right) \\ &= - \sum_{\langle ij \rangle} V_{ij} \left( F_{ji}^{\uparrow\downarrow} c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger + F_{ij}^{\uparrow\downarrow*} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger \right) \\ &= - \frac{1}{2} \sum_{\langle ij \rangle} V_{ij} \left[ F_{ij}^{\uparrow\downarrow} (c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger - c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger) + F_{ij}^{\uparrow\downarrow*} (c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger) \right] \end{aligned} \quad (87)$$

which is achieved using Eq.31,  $V_{ij} = V_{ji}$ ,  $F_{ij}^{\uparrow\downarrow} = -F_{ji}^{\downarrow\uparrow} = F_{ji}^{\uparrow\downarrow}$  using the anticommutation relation and a spin exchange. We then use  $F_{ij}^{\uparrow\downarrow} = F_{ij}$  since all ambiguity has been removed. Therefore the matrix we obtain is given as

$$H_{V,ij}^{HF} = -\frac{1}{2} \begin{pmatrix} \mathcal{O} & & -F_{ij}^{\uparrow\downarrow} \\ & F_{ij}^{\uparrow\downarrow} & \\ F_{ij}^{\uparrow\downarrow*} & & \mathcal{O} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \mathcal{O} & -i\sigma_2 F_{ij} \\ i\sigma_2 F_{ij}^* & \mathcal{O} \end{pmatrix} \quad (88)$$

an takes only place between two neighbouring sites  $i$  and  $j$ . For the accuracy one could multiply it with  $(\delta_{i+1,j} + \delta_{i-1,j} + \delta_{i+N_x,j} + \delta_{i-N_x,j})$ . This matrix can then be added to Eq.54 replacing the on-site superconductive  $\Delta$ -term by a neighbouring superconductive  $F$ -term.

Now that we have expressed the Hartree-Fock Hamiltonian in the correct maner, we can use our BdG-transformation to diagonalize the Hamiltonian and find a self-consistent solution for the  $F$ -parameters. Here we use the transformation between the creation and annihilation operators, and the eigenvectors, -values of the Hamiltonian given by Eq.60 and Eq.61.

$$F_{ij} = \langle c_{i\uparrow} c_{j\downarrow} \rangle = \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{nj\downarrow}^* (1 - f(E_{nk})) + v_{ni\uparrow}^* u_{nj\downarrow} f(E_{nk}) \quad (89)$$

in a very similar way as Eq.62.

### A.3 Advanced order parameters

In the more simple description of the superconductivity we outlined how the superconducting order parameter  $\Delta$  depends on the pairing amplitude  $F$ . Because the potential was isotropic, we simply had  $\Delta = U\langle c_{i\uparrow}c_{i\downarrow} \rangle$ . Here however the potential is anisotropic and we obtain a linear combination of the pairing amplitudes in the different directions of the lattice. Achieving different combinations of the  $F$ s we obtain different superconducting states. Here is an exhaustive list. [replace V with U for consistency?](#)

$$\Delta_{s,i} = VF_{s,i} = \frac{V}{4} \left( F_i^{x+(S)} + F_i^{x-(S)} + F_i^{y+(S)} + F_i^{y-(S)} \right) \quad (90)$$

$$\Delta_{d,i} = VF_{d,i} = \frac{V}{4} \left( F_i^{x+(S)} + F_i^{x-(S)} - F_i^{y+(S)} - F_i^{y-(S)} \right) \quad (91)$$

$$\Delta_{p_x,i} = VF_{p_x,i} = \frac{V}{2} \left( F_i^{x+(T)} - F_i^{x-(T)} \right) \quad (92)$$

$$\Delta_{p_y,i} = VF_{p_y,i} = \frac{V}{2} \left( F_i^{y+(T)} - F_i^{y-(T)} \right) \quad (93)$$

$F_s$ ,  $F_d$ ,  $F_{p_x}$  and  $F_{p_y}$  are the pairing amplitudes for the  $s$ ,  $d$  (also called  $d_{x^2-y^2}$  because of its expression),  $p_x$  and  $p_y$ -wave superconductivity. The  $S$  and  $T$  are the singlet and triplet expressions of the pairing amplitudes. More precisely we define them as

$$F_{ij}^{(S)} = \frac{F_{ij} - F_{ji}}{2} = F_{ij} \quad (94)$$

$$F_{ij}^{(T)} = \frac{F_{ij} + F_{ji}}{2} \quad (95)$$

Where we shorten the expression using  $F_{ij}^{\uparrow\downarrow(S)} = F_{ij}^{(S)}$ .

**Symmetry discussion** We see that these parameter depends on the spin and the momentum. Therefore it's a good idea to look at their respective behaviour under exchange of these variables.

The discussion for the spin exchange is quite straight forward.

$$F_{ij}^{\uparrow\downarrow(S)} = \frac{F_{ij}^{\uparrow\downarrow} + F_{ji}^{\uparrow\downarrow}}{2} = \frac{\langle c_{i\uparrow}c_{j\downarrow} \rangle + \langle c_{j\uparrow}c_{i\downarrow} \rangle}{2}$$

$$F_{ij}^{\uparrow\downarrow(T)} = \frac{F_{ij}^{\uparrow\downarrow} - F_{ji}^{\uparrow\downarrow}}{2} = \frac{\langle c_{i\uparrow}c_{j\downarrow} \rangle - \langle c_{j\uparrow}c_{i\downarrow} \rangle}{2}$$

[What's happening when we invert both spin?](#) using  $\langle c_{i\uparrow}c_{j\downarrow} \rangle = -\langle c_{i\downarrow}c_{j\uparrow} \rangle$ . Using the linearity of the expression we obtain

$$F_{ij}^{\uparrow\downarrow(S)} = -F_{ij}^{\downarrow\uparrow(S)}$$

$$F_{ij}^{\uparrow\downarrow(T)} = F_{ij}^{\downarrow\uparrow(T)}.$$

This means that the singlet wave-pairing amplitude is antisymmetric under spin exchange and the triplet wave pairing is symmetric under spin exchange. Their names find place in the analogy of the wavefunction formalism.

For the momentum exchange we are going to take a look at the Fourier transformation of the pairing amplitudes. Using the transformation we made to reach Eq.84 with any translation  $x \rightarrow \mathbf{r}$ ,  $r \in \{\mathbf{e}_x, \mathbf{e}_y\}$  we obtain

$$F_{i,i+\mathbf{r}} = \frac{1}{N} \sum_{\mathbf{k}} \langle c_{\mathbf{k}\uparrow}c_{-\mathbf{k},\downarrow} \rangle e^{-i\mathbf{k}\mathbf{r}}.$$

Changing the sign of the momentum we obtain  $\frac{1}{N} \sum_{\mathbf{k}} \langle c_{-\mathbf{k},\uparrow}c_{\mathbf{k},\downarrow} \rangle e^{i\mathbf{k}\mathbf{r}} = F_{i,i-\mathbf{r}}$  because of the  $\delta_{\mathbf{k},\mathbf{k}'}$  trick. For this reason  $F_{i,i+\mathbf{r}} + F_{i,i-\mathbf{r}}$  is symmetric under momentum exchange while  $F_{i,i+\mathbf{r}} - F_{i,i-\mathbf{r}}$  is antisymmetric.

Referring back to the order parameter definition we see that the  $s$  and  $d$ -wave are symmetric under momentum exchange, where the  $p_x$  and  $p_y$ -wave are antisymmetric in such exchange.

## A.4 Implementation of the advanced superconductivity

The individual  $F$  always change but the end result in  $F_d$  can converge so we compute  $F_{d,i}$  on each step to see if it converges.