



Proximity Effects in Altermagnetic Systems

~ o ~

a bachelor thesis.

Written by Arto Steffan,
under the supervision of prof. dr. Jacob Wüsthoff Linder,
and corrected by prof. dr. Wolfgang Belzig.

Universität
Konstanz



NTNU – Trondheim
Norwegian University of
Science and Technology

Acknowledgment

— ~ —

...

_____ ~ _____

Contents

1	Theoretical Background	1
1.1	Bosons and fermions	2
1.2	The second quantisation	3
1.2.1	Second quantisation for fermions	4
1.2.2	Second quantisation for bosons	5
1.3	Basis transformation	5
1.3.1	Field operators	6
1.4	Interactive electron gas	7
2	Superconductivity	9
2.1	Theoretical framework and BCS theory	10
2.1.1	The non-interacting electron gas	10
2.1.2	Fermi-Liquid - the interacting case	11
2.1.3	Instability due to attractive Interactions	13
2.1.4	Phonon-mediated attractive interactions reformulate	14
2.1.5	Contraction of the effective Hamiltonian	15
2.1.6	On our way to the BCS-theory	17
2.1.7	Generalized gap equation, s-wave and d-wave superconductivity	20
2.1.8	Transition temperature and energy gap	23
3	Bogoliubov-de Gennes Formalism	24
3.1	Tigh Binding Model	25
3.1.1	Non-interacting electrons	26
3.1.2	Superconductivity	26
3.2	A more symmertic Hamiltonian	27
3.3	Eigenvalues	28
3.4	Diagonalization	28
3.4.1	Superconducting Gap	30
4	Altermagnetism	31
4.1	Introduction and overview	31
4.2	Symmetries	32
4.3	Implementation of an altermagnet	33
5	Simulations and ..	33
5.1	The non-trivial choice of the parameters	34
5.2	Currents	35
6	An advanced superconductivity	39
6.1	Site dependend potential	39
6.2	In the vertical periodic boundary setup	40
6.2.1	BdG-transformation for a vertical foruier transform	42
6.2.2	Pairing amplitudes	42
6.2.3	In the open boundary setup	43
6.3	Advanced order parameters	44
6.4	Implementation of the advanced superconductivity	45
7	Literature	46

1 Theoretical Background

In order to describe the superconductors we are going the introduce the second quantisation formalism. It allows us to describe the wavefunction of a system using creation and anihinaltion operators over energystates of the system and simplify a lot the notation. The mathematical fundation of this formalism lays in the Hilbert space, its dual space and furthermore we are going to introduce the Foch space.

It's also relevant for our study that we are going to work on fermions. The formalism stays the same for bosons but the results are fundamentally different. One can mention the Pauli principle as an exemple which only applies on fermion is can be derived with the help of the second quantisation.

1.1 Bosons and fermions

We consider without loss of generality the following hamiltonian.

$$\hat{H} = \hat{H}_0 + \hat{H}_I \quad (1)$$

with the single particle operator \hat{H}_0 and the interaction operator \hat{H}_I :

$$\hat{H}_0 = \sum_{i \in \llbracket N \rrbracket} \hat{h}_i(x_i), \quad \hat{h}_i(x_i) = -\frac{\hbar^2}{2m} \nabla_i^2 + \hat{U}(x_i)$$

Where we introduce the notation $\llbracket N \rrbracket = \{n \in \mathbb{N} : n \leq N\}$. We call it single particle operator because the operator only applies on a particle. It may depend from the particle's position \mathbf{r} or spin s : $x_i := (\mathbf{r}, s) \in \mathcal{X} \subseteq \mathbb{R}^3 \times \mathbb{S}$. For exemple we have for an electron $\mathbb{S} = -\frac{1}{2}, \frac{1}{2}$. A single particle operator is in this case the sum of the kinetic- and potential energy operators.

Further we describe a quantum state that a particle can occupy with a wavefunction $\phi_\nu(x)$, which own a certain energy $\epsilon_\nu \in \mathbb{R}$. This energy depends on the wavevector and the spin of the particle: $\nu = (\mathbf{k}, \sigma)$. The fundamental equation of quantum mechanics relates the wave function with the hamiltonian using the energy of the state:

$$\hat{h}\phi_\nu(x) = \epsilon_\nu \phi_\nu(x)$$

The wave function lay in the Hilbert space [more details?]. Therefor $\phi_\nu(x)$ are eigenfunction or -states of the Hamiltonian with eigenvalues ϵ_ν . Further the wavefunction should build an orthonormal basis:

$$\int_{\mathcal{X}} \phi_{\nu'}(x) \phi_\nu(x) dx = \delta_{\nu'\nu}.$$

ν and ν' are two different states. We introduced here the korenker delta $\delta_{\nu'\nu}$ which is one when the two indicies are equal and zero otherwise. Because the spin s is not continuous one can understand the integral in the following way:

$$\int_{\mathcal{X}} dx = \sum_{s \in \mathbb{S}} \int_{\mathbb{R}^3} d^3r$$

where $\int_{\mathbb{R}^3} d^3r = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} dr_1 dr_2 dr_3$. We integrate over all possible states.

Now that we can describe one particle we want to describe a system containing many instance of that particle. The wavefunction sums up all possible combination of wavefunction in the system and should stay normalised. A combination is discribed as the product of the wavefunction of the particle in a certain state. These particle can be swaped and therefore we need to consider all combinations. We restrict ourselves to fermions and bosons. We admit having $N \in \mathbb{N}$ paricles in the system.

Bosons The many-particle wavefunction of the bosons is a symetric (exponent S) under swap of two particles.

$$\Phi^{(S)}(x_1, \dots, x_N) = \left(N! \prod_N (n_\nu)! \right)^{-\frac{1}{2}} \sum_{P \in S_n} P \phi_{\nu_1}(x_1) \cdot \dots \cdot \phi_{\nu_N}(x_N)$$

This represents the an eigenfunction of the non interacting bosonic-Hamiltonian. We used n_ν , which represents the number of particle in the state ν . Therefor we usaly call it the occupation number of the state ν . For fermion this integer has no constrain in general. The permutation set S_n contains all the possbile combinations of x_i in the state ν_j for $i, j \in \llbracket N \rrbracket$.

Fermions Fermions are a bit different, their many-particle fermion wavefunction is antisymmetric under swap of two particles. We denote it as

$$\Phi^{(A)}(x_1, \dots, x_N) = (N!)^{-\frac{1}{2}} \sum_{P \in S_n} \text{sgn}(P) \cdot P\phi_{\nu_1}(x_1) \cdot \dots \cdot \phi_{\nu_N}(x_N).$$

which is an eigenfunction of the non interacting fermionic-Hamiltonian. sgn represents the signum function. Applied on a permutation P it is one if P is even and minus one if P is even. We already know that the Pauli principle implies that it can be up to one particle in each energy state. We therefore have $n_\nu \in \{0, 1\}$. The normalisation factor is the same but the product over the ones vanishes.

At this point one might have recognised the formula of the determinant

$$\Phi^{(A)}(x_1, \dots, x_N) = (N!)^{-\frac{1}{2}} \det \begin{pmatrix} \varphi_{\nu_1}(x_1) & \cdots & \varphi_{\nu_1}(x_N) \\ \vdots & & \vdots \\ \varphi_{\nu_N}(x_1) & \cdots & \varphi_{\nu_N}(x_N) \end{pmatrix},$$

which vanishes if two rows or columns are identic. We usually describe this expression as the Slater determinant. This means that the probability of finding two fermions in the same state is zero. This is the Pauli principle. Only one or no particle may occupy each state.

Further we encounter a major problem. The many-particle wave function of fermions is defined up to a sign. For instance if we consider two particles “having” x_1 and x_2 , we have two possible state ν_1 and ν_2 . Two possible solutions are

$$\begin{aligned} \Phi^{(A_1)} &= \frac{1}{\sqrt{2}} (\varphi_{\nu_1}(x_1)\varphi_{\nu_2}(x_2) - \varphi_{\nu_1}(x_2)\varphi_{\nu_2}(x_1)) \\ \text{or } \Phi^{(A_2)} &= \frac{1}{\sqrt{2}} (\varphi_{\nu_1}(x_2)\varphi_{\nu_2}(x_1) - \varphi_{\nu_1}(x_1)\varphi_{\nu_2}(x_2)) \\ &= -\Phi^{(A_1)}. \end{aligned}$$

This sign difference may lead to computation errors. We aim to give a labeling to our states when we count them and keep it when it comes to build the Slater determinant.

These bosonic and fermionic wavefunctions are eigenstate of the Hamiltonian \hat{H}_0 and the corresponding eigenvalue E_ν is given by summing the energy of each state times its occupation number: $E_\nu = \sum_\nu \epsilon_\nu n_\nu$. For this reason it's important that they build an orthonormal basis:

$$\int_{\mathcal{X}^N} \Phi_a^*(x_1, \dots, x_N) \Phi_b(x_1, \dots, x_N) d^N x = \delta_{ab}.$$

Therefore we can expand any many-particle wavefunction Ψ as the linear combination of these:

$$\Psi = \sum_a f_a \Phi_a(x_1, \dots, x_N)$$

where f_a is a coefficient and a a labeling.

What we just discussed is the so called first quantisation- or wave function formalism. Now we intend to introduce a better way of describing our system.

1.2 The second quantisation

For a better description of the many-particle system we introduce a simpler notation. The second quantisation lays on three important objects. States described as “ket”. We put any relevant information between the ket e.g. $|\mathbf{k}, \sigma, \dots\rangle$. Then we need operators that act on these states to allow interactions in the system. We need an operator that creates a state and another that annihilates a state.

States In this section we describe a state as the number of particle that occupies each single-particle state. Therefore we order the state $1 < 2 < \dots < N$. We then can describe the wave function as follow $|n_1, \dots, n_N\rangle$.

Further the state where no particle are present is called the vacuum state and we denote it as $|0_{\nu_1}, \dots, 0_{\nu_N}\rangle = |0\rangle$.

1.2.1 Second quantisation for fermions

Creation operator c_ν^\dagger The creation operator adds a particle in the state that is concerned and rephase the state:

$$c_\nu^\dagger |n_1, \dots, n_\nu, \dots\rangle = (-1)^{\sum_{\mu < \nu} n_\mu} (1 - n_\nu) |n_1, \dots, n_\nu + 1, \dots\rangle$$

We notice the $(1 - n_\nu)$ term which avoid to create a particle at the state, if one already exist. This is the expression of the Pauli-principle. and we can then construct a state by applying this operator after another on the vacuum state. To avoid the minus one to add a negative sign, we start from the end and add the particle backwards in the order of the state:

$$|n_1, \dots, n_N\rangle = (c_1^\dagger)^{n_1} \dots (c_N^\dagger)^{n_N} |0\rangle$$

Annihilation operator c_ν Likewise the annihilation operator destroys a particle in the corresponding state. The operator reads

$$c_\nu |n_1, \dots, n_\nu, \dots\rangle = (-1)^{\sum_{\mu < \nu} n_\mu} (n_\nu) |n_1, \dots, n_\nu - 1, \dots\rangle.$$

We can easily recognise that due to the n_ν -term, destroying a particle that doesn't exist gives zero, so it's only possible to destroy particle that exist. Further we intend to introduce some few computation rules that are going to help us later.

The anticommutator of two operator reads $[A, B]_+$ or $\{A, B\} := AB + BA$ and is an operator as well. We're going to stick with $[A, B]_+$ since it's more consistent with the commutator notation $[A, B]_-$ (or simply $[A, B]$).

The following results are obtain by separating the $\nu = \mu$ from the $\nu \neq \mu$. We must also say that the dagger \dagger should be understand as the complex transpose of the operator and $(AB)^\dagger = B^\dagger A^\dagger$.

$$[c_\nu, c_\mu]_+ = 0 \quad (\text{fer}_1)$$

$$[c_\nu^\dagger, c_\mu^\dagger]_+ = 0 \quad (\text{fer}_2)$$

$$[c_\nu^\dagger, c_\mu]_+ = \delta_{\mu, \nu} \quad (\text{fer}_3)$$

We can then combine the creation and annihilation operator to count the number of particles in a state:

$$c_\nu^\dagger c_\nu |n_1, \dots, n_\nu, \dots\rangle = n_\nu |n_1, \dots, n_\nu, \dots\rangle.$$

From this we can define the number operator $\hat{n}_\nu := c_\nu^\dagger c_\nu$ which we can combine in the total number operator

$$\hat{N} = \sum_\nu \hat{n}_\nu, \quad \text{where logically } N = \sum_\nu n_\nu$$

if we apply the operator on a state.

Second quantisation description of the single- and two- particle operators We first need to make an important observation between the Slater determinant and the single particle state to understand the following. First we introduce two basis element $|\Phi_\alpha\rangle$ and $|\Phi_\beta\rangle$, which can be many-particles eigenstate of the system. We can also call them Slater determinant. Further we introduce the probability of the configuration $|\Phi_\alpha\rangle$ to scatter into the $|\Phi_\beta\rangle$ due to the action of an operator A (momentum, potential, interactions,...). This is described by the matrix element $\langle \Phi_\alpha | A | \Phi_\beta \rangle$ which involves the single particle states $|\alpha_1\rangle, \dots, |\alpha_N\rangle$ of $|\Phi_\alpha\rangle$ and $|\beta_1\rangle, \dots, |\beta_N\rangle$ of $|\Phi_\beta\rangle$.

$$\langle \Phi_\alpha | A | \Phi_\beta \rangle = \sum_{i,j} C_{ij} \langle \alpha_i | A | \beta_j \rangle$$

involving some constants C_{ij} . This describes the overlap of the two configurations, after that we modified $|\Phi_\beta\rangle$ with A . On the right hand side (r.h.s) we introduced the bracket scalar product notation. The bra $\langle\alpha|$ lives in the dual space of the Hilbert space. One reads it as the complex transpose of $|\alpha\rangle$.

We recall the single particle Hamiltonian we introduced earlier. Its second quantisation representation reads

$$\hat{H}_0 = \sum_{i \in \llbracket N \rrbracket} \hat{h}(x_i) \rightsquigarrow \sum_{\alpha, \beta} \langle \alpha | \hat{h} | \beta \rangle c_\alpha^\dagger c_\beta \quad (2)$$

where α and β are single-particle states of the system. $c_\alpha^\dagger c_\beta$ tries to transfert a fermion from the state $|\beta\rangle$ to the state $|\alpha\rangle$. We have

$$\langle \alpha | \hat{h} | \beta \rangle = \int \varphi_\alpha^*(x) \hat{h}(x) \varphi_\beta(x) dx \quad (3)$$

where x still represents the position and the spin of the particle.

\hat{h} is a single particle operator, it means it acts on one particle at a time. Two states are going to be changed. $|\alpha\rangle$ loses a particle and $|\beta\rangle$ gains one. We say for instance, that the configuration before the scattering is $|\Phi\rangle$ and after the scattering is $|\Phi'\rangle$. This means, if our two slatter determinant $|\Phi\rangle$ and $|\Phi'\rangle$ differs in more than two state, there are some scattering that we can't describe, so the overlap must be zero. We allow only two states to be modified. Otherwise the single-particle states differ and due to their orthonal properties, we get a zero.

Similarly for the two-particle operator we have

$$\hat{H}_I = \frac{1}{2} \sum_{i \neq j \in \llbracket N \rrbracket} \hat{v}(x_i, x_j) \rightsquigarrow \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \beta | \hat{v} | \gamma \delta \rangle c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta \quad (4)$$

involving a more nested overlap of the four states:

$$\langle \alpha \beta | \hat{v} | \gamma \delta \rangle = \int \int \varphi_\alpha^*(x) \varphi_\beta^*(x') \hat{v}(x, x') \varphi_\gamma(x) \varphi_\delta(x') dx dx'. \quad (5)$$

which modifies four states so the overlap of two slatter determinant vanishes if the determinant differ in at least four states.

The l.h.s of the equation is the matrix element $\langle \Phi_\alpha | \hat{v} | \Phi_\beta \rangle$ of the the operator \hat{v} , which involves two basis state $|\Phi_\alpha\rangle$ and $|\Phi_\beta\rangle$. On the r.h.s we have a description with wavefunctions, which own to the first quantisation. What we have here is the bridge between the first and the second quantisation. One could compute each side separately and notice that both formalism lead to the same result.

1.2.2 Second quantisation for bosons

1.3 Basis transformation

Until now considered the wavefunction in a restricted basis $\{\varphi_\alpha(x)\}$. A wave function is defined as the overlap between the basis and the state:

$$\phi(x) = \langle x | \alpha \rangle.$$

Let us now define a more general new operator that create a states $|\alpha\rangle = a_\alpha |0\rangle$ for fermion and bosons. We asume that we have another basis $\{|\tilde{\alpha}\rangle\}$. We now want to show, that the new operator a_α can be expressed as a linear combination of the other operator $a_{\tilde{\alpha}}$. This will be a usefull too to jump from a basis to another. We first notice the following identity relations:

$$\mathbb{I} = \sum_{\alpha} |\alpha\rangle \langle \alpha| = \sum_{\tilde{\alpha}} |\tilde{\alpha}\rangle \langle \tilde{\alpha}|$$

this allow us to write

$$a_\alpha^\dagger |0\rangle = |\alpha\rangle = \sum_{\tilde{\alpha}} |\tilde{\alpha}\rangle \underbrace{\langle \tilde{\alpha} | \alpha \rangle}_{\text{scalar}} = \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle |\tilde{\alpha}\rangle$$

inverting the indices leads to the same result, which leads to the transformation rules

$$\begin{aligned} a_\alpha^\dagger &= \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle a_{\tilde{\alpha}}^\dagger \\ a_\alpha &= \sum_{\tilde{\alpha}} \langle \alpha | \tilde{\alpha} \rangle a_{\tilde{\alpha}}. \end{aligned}$$

Further we can use these relations to express a wavefunction in the basis of another wavefunction. We have

$$\phi_\alpha(x) = \langle x | \alpha \rangle = \langle x | \left(\sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle | \tilde{\alpha} \rangle \right) = \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle \langle x | \tilde{\alpha} \rangle = \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle \varphi_{\tilde{\alpha}}(x).$$

Inverting α and $\tilde{\alpha}$ leads as well $\varphi_{\tilde{\alpha}}(x) = \sum_{\alpha} \langle \alpha | \tilde{\alpha} \rangle \phi_\alpha(x)$.

Moreover we can show that the basis transformation is unitary. This is an important feature because we can simplify the calculation by changing the basis but the result we seek for would remain the same after the unitary transformation. Such transformations plays a major role later in the thesis. We can save the $\langle \tilde{\alpha} | \alpha \rangle$ in a matrix D and prove that this matrix is unitary. We have

$$\begin{aligned} \langle \tilde{\alpha} | \tilde{\beta} \rangle &= \sum_{\gamma} \langle \tilde{\alpha} | \gamma \rangle \langle \gamma | \beta \rangle = \sum_{\gamma} \langle \tilde{\alpha} | \gamma \rangle \langle \beta | \gamma \rangle^* \\ &= \sum_{\gamma} D_{\alpha\gamma} D_{\beta\gamma}^* = \sum_{\gamma} D_{\alpha\gamma} D_{\gamma\beta}^\dagger = (DD^\dagger)_{\alpha\beta}. \end{aligned}$$

meanwhile $\langle \tilde{\alpha} | \tilde{\beta} \rangle = \delta_{\tilde{\alpha}\tilde{\beta}}$ so that $DD^\dagger = \mathbb{I}$. The matrix D is therefore unitary. [but do we also have the orthonormality of if we take two different basis?](#)

The last important step is to show that the basis transformation keeps the anti-, commutation relations. Let us for the seek of readability use the notation $[A, B]_\xi = AB + \xi BA$ where $\xi = -$ for bosons and $\xi = +$ for fermions. We have for exemple using $[a_\alpha, a_{\alpha'}^\dagger]_\xi = \delta_{\alpha\alpha'}$

$$[a_{\tilde{\alpha}}, a_{\tilde{\alpha}'}^\dagger]_\xi = \sum_{\alpha\alpha'} \langle \tilde{\alpha} | \alpha \rangle \langle \alpha' | \tilde{\alpha}' \rangle [a_\alpha, a_{\alpha'}^\dagger]_\xi = \langle \tilde{\alpha} | \tilde{\alpha}' \rangle = \delta_{\tilde{\alpha}\tilde{\alpha}'}.$$

The first step follows after splitting the commutator in two parts, insert the transformation and recombine the new commutator. The last step involves the orthonormality of the basis. On the same way one can prove $[a_{\tilde{\alpha}}, a_{\tilde{\alpha}'}] = [a_{\tilde{\alpha}}^\dagger, a_{\tilde{\alpha}'}^\dagger] = 0$.

1.3.1 Field operators

Later in this thesis we are going to use field operators to describe an order parameter of a superconductive system. These are creation and annihilation operators that are defined in the $|x\rangle$ -space-spin basis regarding another basis $\{|\alpha\rangle\}$. We here give the state as an argument and not as an index anymore. Despite the notation the following operators musn't be confused with a wavefunction.

$$\hat{\psi}^\dagger(x) = \sum_{\alpha} \langle \alpha | x \rangle a_\alpha^\dagger = \sum_{\alpha} \varphi_\alpha^*(x) a_\alpha^\dagger \quad (6)$$

$$\hat{\psi}(x) = \sum_{\alpha} \langle \alpha | x \rangle a_\alpha = \sum_{\alpha} \varphi_\alpha(x) a_\alpha \quad (7)$$

involving fermionic or bosonic operators a . We can then annihilate and create a particle at a spin-space location x . Because these operators are involved, the commutation property respects which particle we are describing. Using the result we had for $[a_\alpha, a_{\alpha'}]_\xi, [a_\alpha, a_{\alpha'}^\dagger]_\xi$ and $[a_\alpha^\dagger, a_{\alpha'}^\dagger]_\xi$ where $\xi = -$ for the bosons and $+$ for the fermions. We find the following commutation relations:

$$\begin{aligned} [\hat{\psi}(x), \hat{\psi}(x')]_\xi &= [\hat{\psi}(x)^\dagger, \hat{\psi}^\dagger(x)]_\xi = 0 \\ [\hat{\psi}(x), \hat{\psi}^\dagger(x')]_\xi &= \delta(x - x') \end{aligned}$$

In the last expression we obtain a $\langle x|x' \rangle$ which can't normalize the $\{|x\rangle\}$ -basis so instead of a Kronecker delta $\delta_{xx'}$ we get a delta distribution $\delta(x - x')$. The goal is now to describe the Hamiltonian using these field operators. Therefore we rebuild the field operator in the Hamiltonian using a $\{|x\rangle\}$ basis

$$\begin{aligned}\hat{H}_0 &= \int \hat{\psi}^\dagger(x) \hat{h}(x) \hat{\psi}(x) dx \\ \hat{H}_I &= \frac{1}{2} \int \int \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') \hat{v}(x, x') \hat{\psi}(x') \hat{\psi}(x) dx' dx\end{aligned}$$

We already know that the first and second quantisation are equivalent. We could now insert the definition of the field operators 6 and 7 in the Hamiltonian and obtain a wave function description. Then we can just use the first to second quantisation translation 3 and 5 to the second quantised representation. The result is a generalisation of the expression with the operators a .

1.4 Interactive electron gas

As we are later going to study, the electrons are allowed to interact with each other. During such processes a photon is usually carrying the momentum transfer of the scattering. We're going to use the formalism we introduced to find a second quantisation representation of the interacting Hamiltonian.

The system we're studying is a cube of side length L with volume $\Omega = L^3$ with N electrons. Further we consider a periodic boundary condition for an arbitrary position vector $\mathbf{r} = (x, y, z)$:

$$(L, y, z) = (0, y, z), \quad (x, L, z) = (x, 0, z), \quad (x, y, L) = (x, y, 0).$$

We use the general form of the Hamiltonian 1 introduced in the beginning. We have a Coulomb potential and a kinetic energy term.

$$H_0 = T + U = - \sum_{i \in [N]} \frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i \in [N]} u(x_i) \quad (8)$$

$$H_I = V = \sum_{i \neq j \in [N]} v(x_i, x_j) \quad (9)$$

Where the pairwise potential is just a Coulomb potential $v(x_i, x_j) = \frac{e^2}{4\pi\epsilon_0 |x_i - x_j|} = v(|x_i - x_j|)$. [drop of the hat?](#) We aim to describe the Hamiltonian in the momentum space, which is more convenient for the second quantisation. Therefore we first need to find an expression for the wavevector \mathbf{k} . We describe a state $\alpha = (\mathbf{k}, \sigma)$ at a spin-space coordinate $x = (\mathbf{r}, s)$. First we take a plane wave solution of the Schrödinger equation:

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k} \cdot \mathbf{r}} \chi_{\sigma}(s)$$

The periodicity of the system allows us to write $\psi(x=0) = \psi(x=L)$ using the boundary condition and obtain $1 = e^{ik_x L}$ as well as for the two other coordinates. The results reflect in the wavevector \mathbf{k} :

$$\mathbf{k} = \frac{2\pi}{L} (n_x, n_y, n_z), \quad n_i \in \mathbb{Z}.$$

As we saw earlier the eigenfunctions build a complete basis:

$$\begin{aligned}\int \psi_{\alpha'}^*(x) \psi_{\alpha}(x) dx &= \int \sum_s \psi_{\alpha'}^*(x) \psi_{\alpha}(x) dx \\ &= \frac{1}{\Omega} \int e^{i\mathbf{r} \cdot (\mathbf{k} - \mathbf{k}')} \underbrace{\sum_s \delta_{s\sigma} \delta_{s\sigma'}}_{\delta_{\sigma\sigma'}} dx \\ &= \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'} = \delta_{\alpha\alpha'}.\end{aligned}$$

where the integral over the exponential function diverges if $\mathbf{k} \neq \mathbf{k}'$ so we use the case $\mathbf{k} = \mathbf{k}'$ and the integral is zero otherwise.

The kinetic energy operator is a single particle operator so we have according to 2 we have:

$$T = \sum_{\alpha, \alpha'} \langle \alpha | \frac{\hat{\mathbf{p}}^2}{2m} | \alpha' \rangle c_{\alpha}^{\dagger} c_{\alpha'}.$$

Now we can just use $\hat{\mathbf{p}} = i\hbar\nabla$ and to compute this expression we use its first quantised form (see 3) involving the $\delta_{\sigma\sigma'}$ trick we introduced in the derivation of the complete basis. We also use $\mathbf{k}' = \mathbf{k}$ and hide the $\frac{1}{\Omega}$ is the $\delta_{\mathbf{k}\mathbf{k}'}$. The result is

$$T = \sum_{\alpha, \alpha'} \delta_{\alpha\alpha'} \frac{\hbar \mathbf{k}^2}{2m} c_{\alpha'}^{\dagger} c_{\alpha} = \sum_{\alpha} \underbrace{\frac{\hbar \mathbf{k}^2}{2m}}_{=: \epsilon_{\mathbf{k}}} c_{\alpha}^{\dagger} c_{\alpha} \quad (10)$$

we recognise the occupation number operator $\hat{n}_{\alpha} = c_{\alpha}^{\dagger} c_{\alpha}$ and the energy of the state $\epsilon_{\mathbf{k}}$. This variable plays a central role later. We obtain a quite meaningful result, the non-interacting energy part of the system is the product of the energy of a state with the number of particle within that state, summed over all states.

We don't consider the single particle external potential for now.

For the interaction potential, we have a two-particle operator. This is described by equation 4 and requires 5 to be solved.

$$V = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha\beta | v | \gamma\delta \rangle c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta}$$

Introducing these in V using $v(\mathbf{r}, \mathbf{r}') = v(\mathbf{r} - \mathbf{r}')$ leads to:

$$\langle \alpha\beta | v | \gamma\delta \rangle = \frac{1}{\Omega^2} \delta_{\sigma_{\alpha}\sigma_{\gamma}} \delta_{\sigma_{\beta}\sigma_{\delta}} \int \int e^{i\mathbf{r}(\mathbf{k}_{\gamma} - \mathbf{k}_{\alpha})} e^{i\mathbf{r}'(\mathbf{k}_{\delta} - \mathbf{k}_{\beta})} v(\mathbf{r} - \mathbf{r}') d\mathbf{r}' d\mathbf{r}$$

we can make a substitution $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, add and subtract a $(\mathbf{k}_{\gamma} - \mathbf{k}_{\alpha})\mathbf{r}'$ and obtain

$$\langle \alpha\beta | v | \gamma\delta \rangle = \frac{1}{\Omega^2} \delta_{\sigma_{\alpha}\sigma_{\gamma}} \delta_{\sigma_{\beta}\sigma_{\delta}} \underbrace{\int e^{-i\mathbf{R}(\mathbf{k}_{\alpha} - \mathbf{k}_{\delta})} v(\mathbf{R}) d\mathbf{R}}_{v_{\mathbf{k}_{\delta} - \mathbf{k}_{\alpha}}} \underbrace{\int e^{i\mathbf{r}'(\mathbf{k}_{\gamma} - \mathbf{k}_{\beta} + \mathbf{k}_{\delta} - \mathbf{k}_{\alpha})} d\mathbf{r}'}_{=\delta_{\mathbf{k}_{\gamma} - \mathbf{k}_{\beta} + \mathbf{k}_{\delta}, \mathbf{k}_{\alpha}}}.$$

Almost finished, we derive a combuermosom equation

$$V = \frac{1}{2\Omega} \sum_{\substack{\mathbf{k}_{\alpha}\mathbf{k}_{\beta}\mathbf{k}_{\gamma}\mathbf{k}_{\delta} \\ \sigma_{\alpha}\sigma_{\beta}\sigma_{\gamma}\sigma_{\delta}}} \delta_{\sigma_{\alpha}\sigma_{\gamma}} \delta_{\sigma_{\beta}\sigma_{\delta}} \delta_{\sigma_{\alpha}\sigma_{\gamma}} \delta_{\sigma_{\beta}\sigma_{\delta}} \delta_{\mathbf{k}_{\alpha}, \mathbf{k}_{\gamma} - \mathbf{k}_{\beta} + \mathbf{k}_{\delta}} v_{\mathbf{k}_{\alpha} - \mathbf{k}_{\delta}} c_{\mathbf{k}_{\alpha}\sigma_{\alpha}}^{\dagger} c_{\mathbf{k}_{\beta}\sigma_{\beta}}^{\dagger} c_{\mathbf{k}_{\gamma}\sigma_{\gamma}} c_{\mathbf{k}_{\delta}\sigma_{\delta}}.$$

we can sum over the \mathbf{k}_{α} , rename $\sigma_{\alpha} \rightarrow \sigma$ and $\sigma_{\beta} \rightarrow \sigma'$ and sum up over σ_{γ} and σ_{δ} to simplify the Kroneker deltas.

$$V = \frac{1}{2\Omega} \sum \sigma \sigma' \sum_{\mathbf{k}_{\beta}\mathbf{k}_{\gamma}\mathbf{k}_{\delta}} v_{\mathbf{k}_{\gamma} - \mathbf{k}_{\beta}} c_{\mathbf{k}_{\delta} + \mathbf{k}_{\gamma} - \mathbf{k}_{\beta}, \sigma}^{\dagger} c_{\mathbf{k}_{\beta}\sigma'}^{\dagger} c_{\mathbf{k}_{\gamma}\sigma'} c_{\mathbf{k}_{\delta}\sigma'}$$

where using $\mathbf{k}_{\alpha} = \mathbf{k}_{\gamma} - \mathbf{k}_{\beta} + \mathbf{k}_{\delta}$ from the Kroneker delta we get $v_{\mathbf{k}_{\alpha} - \mathbf{k}_{\delta}} = v_{\mathbf{k}_{\gamma} - \mathbf{k}_{\beta} + \mathbf{k}_{\delta} - \mathbf{k}_{\delta}}$. Then we can introduce the following variable transformations:

$$\mathbf{k}_{\delta} \rightarrow \mathbf{k}, \quad \mathbf{k}_{\gamma} \rightarrow \mathbf{k}', \quad \mathbf{k}_{\beta} \rightarrow \mathbf{k}' - \mathbf{q}$$

which yields

$$\begin{aligned} \mathbf{k}_{\delta} + \mathbf{k}_{\gamma} - \mathbf{k}_{\beta} &= \mathbf{k} + \mathbf{q} \\ \mathbf{k}_{\gamma} - \mathbf{k}_{\beta} &= -\mathbf{q} \end{aligned}$$

and we finally get our second quantised interaction operator

$$V = \frac{1}{2\Omega} \sum_{\mathbf{q}} v_{\mathbf{q}} \sum_{\substack{\mathbf{k}\sigma \\ \mathbf{k}'\sigma'}} c_{\mathbf{k} + \mathbf{q}, \sigma}^{\dagger} c_{\mathbf{k}' - \mathbf{q}, \sigma'}^{\dagger} c_{\mathbf{k}'\sigma'} c_{\mathbf{k}\sigma}.$$

This describes an electron transferring a momentum \mathbf{q} to another electron. The formula tells that we kill the electron we had before the interaction and create two in states where one loosed some momentum, that has been transferd the other electron. As we saw earlier this depends on the distance $\mathbf{r} - \mathbf{r}'$ between the electrons, but not \mathbf{r} and \mathbf{r}' respectively, so this process is translation invariant. The following diagramm is a good illustration of this effect.

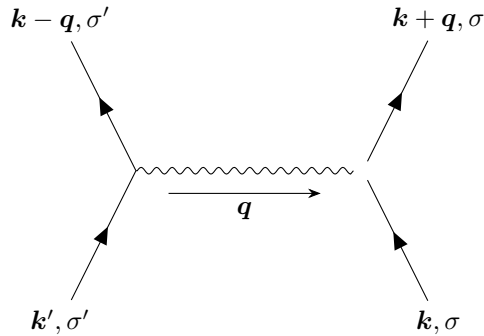


Figure 2: The interaction of two electrons modulated by a photon of momentum \mathbf{q} .

This closes the introduction theorie on the second quantisation. We saw how we can express integral over wavefunctions as bra-kets scalarproducts. This allowed us to introduce some operators that creates and annihilates particles in a certain state. Using this formalism we were able to compute the Hamiltonian of the system in the momentum sapce. We found that the non interacting-part relies on the energy of each state times there occupation number and that an interaction between two electrons can be described as a momentum transfert between them wich is modulated by the fourier transform of the Coulomb potential.

Now we can move on to the descirption of a superconductive state.

2 Superconductivity

Superconductivity can be illustrated as a phase transition of a meterial under a crital temper- ature. In the superconductive state the material become a perfect diamaget and its resistivity vanishes. We then observe some shielding currents that arise on it's surface and we can let flew a current for a very long time without loosing energy. The superconductive state is also described as Meissner state.

Suppose that we heat the material to the critical temperature T_c , some fluctuation effects arise and break the superconductive state. The shielding effects reacts different on the material. We usely distinguish type I and type II superconductors. The type I superconductor loose abruplty their magnetisation over T_c . Type II have a mixed state where the magnetisation slowly decreases until we can't measure it anymore.

The break of the superconductive state can be described as letting more and more filed flew inside of the material. Asuming that some particle are responsible for the superconductivity, the field achieve to penetrate where wo observe a lower density of these particles. The penetration is described as some magnetic field vortecies reaching a certain depth in the material.

The Meissner state is a thermodynamical state. We can show that the free energy of the superconductive state is higher than the normal state. This results in a lower entropy compered to the normal state.

Along the derivation of the superconductivity formalism we are going to follow closely the work of Fossheim and Sudbø [1] from chapter 2 to 4. For readability reason we set the reducted Planck constant $\hbar = 1$ in the following. Furhter they added the chemical potential to the energy of the state such that now $\epsilon_k + \mu \rightarrow \epsilon_k$ from eq. 10.

2.1 Theoretical framework and BCS theory

The Hamiltonian of the system is described by the solid state physics. We consider the energy of the electrons and the ions in a lattice.

$$H = H_{e^-e^-} + H_{e^-ion} + H_{ion ion}$$

Each term consists of a kinetic and potential energy term. For a more mathematical approach we consider a system of N electrons and L ions.

$$\begin{aligned} H_{e^-e^-} &= \sum_{i \in [N]} \frac{p_i^2}{2m} + \sum_{i,j \in [N]} V_{\text{Coulomb}}^{e^-e^-}(\mathbf{r}_i - \mathbf{r}_j) \\ H_{ion ion} &= \sum_{i \in [M]} \frac{p_i^2}{2M} + \sum_{i,j \in [L]} V_{\text{Coulomb}}^{\text{ion-ion}}(\mathbf{R}_i - \mathbf{R}_j) \\ H_{e^-ion} &= \sum_{i \in [N], j \in [L]} V_{\text{Coulomb}}^{e^-ion}(\mathbf{r}_i - \mathbf{R}_j) \end{aligned}$$

We have m and M as the mass of the electron and the ion. \mathbf{r} and \mathbf{R} are the position of the electron and the ion. The ion-ion potential freezes the ions into the lattice. We first go to introduce some concepts by describing a non-interacting electron and then improve it to include the interactions. Usually the potential of a lattice is periodic. If so the wavefunction of a particle moving in the system is a plane wave modulated by a periodic function. Eigenstates of the corresponding Hamiltonian are called Bloch states.

2.1.1 The non-interacting electron gas

In this case of study the Hamiltonian only includes a kinetic term

$$H = \sum_{\mathbf{k}, \sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma}. \quad (11)$$

We assume that it exists a ground state $|0\rangle$, where the system is filled up with a certain amount of electron until the Fermi-energy ϵ_F is reached. Associated with this energy we find a wave vector \mathbf{k}_F , the Fermi-momentum. The set of energy up to ϵ_F is called the Fermi-sea, as an analogy to the level zero of the topographic maps.

$$\hat{n}_{\mathbf{k}, \sigma} |0\rangle = \Theta(\epsilon_F - \epsilon_{\mathbf{k}}) |0\rangle.$$

We introduced here a very useful tool called the Heaviside-step function which is defined as:

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}, \quad \Theta(-x) = 1 - \Theta(x). \quad (12)$$

This means that if we count the particles that have an energy higher than the Fermi-energy ($\mathbf{k} > \mathbf{k}_F$) then we get zero.

We now want to study how the electron can scatter in different states. The function that we're using is called the propagator and gives the probability to find the particle at $|\mathbf{k}', \sigma\rangle$ at t' knowing it at $|\mathbf{k}, \sigma\rangle$, t . An important fact is that without interaction, the particle shouldn't scatter in another state due to energy conservation. Therefore

$$G_0(\mathbf{k}, \mathbf{k}', t' - t) = G_0(\mathbf{k}, t' - t) \delta_{\mathbf{k}, \mathbf{k}}.$$

which is zero if the wave-vectors between the two timepoints differ. We observe that only the past time $t' - t$ is relevant. This is due to the time independent property of the Hamiltonian. We are going to use the representation in the frequency space, using a Fourier-transformation.

$$G_0(\mathbf{k}, \omega) = \int_{\mathbb{R}} e^{i\omega t} G_0(\mathbf{k}, t) dt = \frac{1}{\omega - \epsilon_{\mathbf{k}} + i\delta_{\mathbf{k}}} \quad (13)$$

where $\delta_{\mathbf{k}} = \delta \cdot \text{sgn}(\epsilon_{\mathbf{k}} - \epsilon_F)$ involving a very small non zero number δ . We observe that this analytical function has a pole given by

$$\begin{aligned} \omega - \epsilon_{\mathbf{k}} + i\delta_{\mathbf{k}} &= 0 \\ \iff \omega &= \epsilon_{\mathbf{k}} - i\delta_{\mathbf{k}} \end{aligned}$$

where we denote i as the imaginary unit to avoid confusion with the index i . The frequency ω gives the so called spectrum of the excitation from the unique-particle system. The imaginary part serves as a damping term and is inversly proportional to the lifetime of the particle. δ is a small number due to the infinitely long lifetime. This is a direct result of the absence of scattering.

Further the propagator yields important informations on the system when considering the integration over its different arguments. First we take the imaginary part of the propagator called the single particle spectral weight.

$$A(\mathbf{k}, \omega) = -\frac{1}{\pi} \mathcal{I}m [G_0(\mathbf{k}, \omega)] = \frac{1}{\pi} \frac{\delta_{\mathbf{k}}}{(\omega - \epsilon_{\mathbf{k}})^2 + \delta_{\mathbf{k}}} = \delta(\omega - \epsilon_{\mathbf{k}}) \quad (14)$$

which informs us about the occupation of a state $|\mathbf{k}\rangle$ with energy ω . We can find a form for the momentum distribution $n(\mathbf{k})$

$$n(\mathbf{k}) = \int A(\mathbf{k}, \omega) d\omega$$

and for the density of state

$$D(\omega) = \int A(\mathbf{k}, \omega) d^3k, \text{ or for discontinuous state } \sum_{\mathbf{k}} A(\mathbf{k}, \omega). \quad (15)$$

2.1.2 Fermi-Liquid - the interacting case

Now that we've described the non interacting system, let us complexify the model by introducing the interactions. In an earlier section we saw how

$$H = \sum_{\mathbf{k}, \sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma} + \sum_{\mathbf{k}, \sigma, \mathbf{k}', \sigma'} V_{\mathbf{k}\mathbf{k}', \mathbf{q}} c_{\mathbf{k}-\mathbf{q}, \sigma}^\dagger c_{\mathbf{k}+\mathbf{q}, \sigma'}^\dagger c_{\mathbf{k}, \sigma} c_{\mathbf{k}', \sigma'} \quad (16)$$

represent the pairwise interaction of multiple electrons and their respective energy.

To extend the model we now want to introduce two new quantities, the propagator G and the one-particle irreducible self-energy Σ . The propagator maps in the complex space and sgives the probability amplitude of finding a the particle in the state $|\mathbf{k}, \sigma\rangle$ at a time t . On the other hand $\Sigma = \Sigma_R + i\Sigma_I$ contains the lifetime of the particle in this state and shift of energy of the particle due to the interaction with the surroundings. The frameworks defines the non-interacting energy of the particle as $\epsilon_{\mathbf{k}}$. Whene put in an interacting system the spectrum shifts and becomes $\tilde{\epsilon}_{\mathbf{k}} = \epsilon_{\mathbf{k}} + \Sigma_R$. Due to the interactions, the particle then has a much smaller lifetime. Σ_I is antiproportional to the particle's lifetime $\tau_{\mathbf{k}}$. We therfore expect Σ_I to be realy small in the non interacting case. These two quantities are linked trough the Dyson equation, which reads

$$(G(\mathbf{k}, \omega))^{-1} = (G_0(\mathbf{k}, \omega))^{-1} - \Sigma(\mathbf{k}, \omega).$$

One can use a Fourier-transformation to switch from the time representation to the frequency representation ω . Reordering the equation and using the result from 13 we obtain

$$G(\mathbf{k}, \omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} - \Sigma}.$$

This function has a pole at $\omega = \epsilon_{\mathbf{k}} + \Sigma_R + i\Sigma_I$, where in the none interacting case $\omega = \epsilon_{\mathbf{k}} + i\Sigma_I$. It makes makes sense, the particle spectrum is now shifted due to the finite lifetime of the particle as a result of the interaction.

The pole yields to an expression for the spectrum

$$\omega - \epsilon_{\mathbf{k}} - (\Sigma_R(\mathbf{k}, \omega) + i\Sigma_I(\mathbf{k}, \omega)) = 0 \quad (17)$$

In complexe analysis the order of a pole is given as n if $f(z)$ is meromorphic and has a pole at z_0 where

$$(z - z_0)^n f(z)$$

is also meromorphic in the neighbourhood of z_0 . In our case we're interested in the 0.th order of the pole *why?*. We therefore ignore the imaginary part of the pole and we get

$$\omega = \epsilon_{\mathbf{k}} + \Sigma_R(\mathbf{k}, \epsilon_{\mathbf{k}}) = \tilde{\epsilon}_{\mathbf{k}}.$$

If we take into account a tiny imaginary part of Σ we obtain a shifted pole. Performing a Taylor expansion of Σ_R in the neighbourhood of $\omega = \tilde{\epsilon}_{\mathbf{k}}$ will help us.

$$\Sigma_R(\mathbf{k}, \omega) = \Sigma_R(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}) + (\omega - \tilde{\epsilon}_{\mathbf{k}}) \left. \frac{\partial \Sigma_R}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}} + \mathcal{O}(\omega^2)$$

We aim to compute to the first order in Σ_I such that we evaluate it at $\omega = \tilde{\epsilon}_{\mathbf{k}}$. This leads to starting from 17 and, after inserting the Taylor expansion for Σ_R

$$\begin{aligned} \omega - \underbrace{(\epsilon_{\mathbf{k}} + \Sigma_R(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}))}_{=\tilde{\epsilon}_{\mathbf{k}}} - (\omega - \tilde{\epsilon}_{\mathbf{k}}) \left. \frac{\partial \Sigma_R}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}} - i\Sigma_I(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}) &= 0 \\ \iff (\omega - \tilde{\epsilon}_{\mathbf{k}}) \left(1 - \left. \frac{\partial \Sigma_R}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}} \right) - i\Sigma_I(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}) &= 0 \end{aligned} \quad (18)$$

We define the residue of the propagator as


$$z_{\mathbf{k}} = \frac{1}{1 - \left. \frac{\partial \Sigma}{\partial \omega} \right|_{\omega=\tilde{\epsilon}_{\mathbf{k}}}} \quad (19)$$

which we can insert in the inverse lifetime of the electron occupying the state $|\mathbf{k}, \sigma\rangle$

$$\frac{1}{\tau_{\mathbf{k}}} = -z_{\mathbf{k}} \Sigma_I(\mathbf{k}, \tilde{\epsilon}_{\mathbf{k}}). \quad (20)$$

This justifies the statement $\Sigma_I \propto 1/\tau_{\mathbf{k}}$. This residue is a decreasing function of the energy, which means that its influence is more important for low energies. An interpretation could be that the slow moving electrons have less time to interact with their fast homologues, which results in a longer lifetime.

We recall once again the difference of the propagators to conclude this section:

Free electron		Interacting electron
$G_0(\mathbf{k}, \omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} + i\delta_{\mathbf{k}}}$		$G(\mathbf{k}, \omega) = \frac{1}{\omega - \tilde{\epsilon}_{\mathbf{k}} + i\frac{1}{\tau_{\mathbf{k}}}}$

Interacting electrons are degraded version of the non-interacting case. There remains a $z_{\mathbf{k}} < 1$.

Quasi-particles The main question we have now is how does the residue look like on the fermi surface? We set us in the context of a low energy electron, close to the Fermi-surface and once we had a interaction. It turns out that if there is a $z_{\mathbf{k}_F} > 0$ we find a precise low energy single particle excitation. This excitation is very close to the exact eigenfunction of an non-interacting Hamiltonian. We call this state akin to the free electron a quasi-particle.

This is it, a Fermi-liquid is a system of interacting electrons and quasi-particles. These quasi-particles are not eigenstates of the interacting Hamiltonian anymore. We can't consider them as an electron like excitation in the interaction context. The interactions allows to scatter some states in and out of the Bloch-State.

The above expression for the momentum distribution $n(\mathbf{k})$ can be plotted and we can recognise a gap of $z_{\mathbf{k}_F} < 1$. This wich is an important feature of the Fermi-liquid. Whereas for the non interacting system the momentum distribution has a discontinuity from one to zero when crossing the Fermi surface.

An important feature to mention that is the key understanding of the Fermi-liquid and revealed itself to be very correct for metals, is the one-to-one correspondance of quantum numbers between the interacting and non-interacting electrons. This means we can start with simple Hamiltonian like Eq. 11 and add new particles by introducing some perturbations. The obtained quasi-particles are a result of the excitation of the non-interacting system. We then observe this one-to-one correspondance.

The last question one can ask is why does this approximation work so well in the most case? The answer lays in the reference system we choose to perturb around. Keeping in mind the Pauli-principle, we notice that the added particle are restricted in their scattering. This is so strong that it almost cancels out the “switching-on” of the interactions. It’s a good decision to perturb around the free system. Due to the fact that we are limited in the number of ground states, thermodynamic quantities moves in a smooth manner when perturbing.

We want to emphasize one last time that the Fermi-liquid formalism work because of the momentum restrictions due to the Pauli exclusion principle.

Repulsive interactions In condensed matter physics the dominant effect is the repulsive interactions between the electrons due to the coulomb potential. We already described it in the Hamiltonian H_{e-e} using some pairwise interactions in 16. Therefore the goal is now to find an expression for the potential $V_{\mathbf{k}\mathbf{k}',\mathbf{q}}$.

As we can see in eq. 16 the system is describe in the momentum-space. For this reason we need to consider the Fourier-transform of the real-space potential. We start with the coulomb potential which is predominant in the solid state physics. However the integral is going to diverge for $r \rightarrow 0$. To solve this problem We introduce the Yukawa potential which exponentially modulates the coulomb potential:

$$V_{\lambda}(r) = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} e^{-r\lambda} \quad (21)$$

this is a solid approximation since we can the fourier transform and let $\lambda \rightarrow 0$ in the result. [Result in appendix?](#). The result we get is

$$V_{\mathbf{k},\mathbf{k}',\mathbf{q}} = V_{\text{el}}(\mathbf{q}) := \frac{1}{4\pi\epsilon_0} \frac{2\pi e^2}{q^2} \quad (22)$$

with \mathbf{q} the momentum transfert during the interaction. The Fermi-liquid remains stable to the repulsive processes.

However as we’re going to see in the next section, attractive interactions also takes place due to an exchange of phonons between two electrons. This will be the ground stone to our description of the Meissner state.

2.1.3 Instability due to attractive Interactions

Hier we are going to show how the attractive interactions destabilise the Fermi-liquid. In other words we are going to show that new ground states open.

Leon Cooper introduced a very specific context of attraction that has a huge influence on the stability. Taking this exemple we are going to make clear that attractive interaction can in some cases exceed the Coulomb force resulting in unexpected new eigenstates.

Let’s assume we have an impotent Fermi sea where the electrons are considered non interacting. Adding to electrons requires to place them above the sea. The exotic context lays into the interactions. They can only interact if they are within a small cover ω_0 over the Fermi surface, one on each side, facing them through the complete surface. If this is not the case the interaction vanishes.

Fossheim and Sudbø derived a method in [1] from p.67 to 69. They conclude with the fact that allowing such interaction leads to a total energy of the interacting system E smaller than $2\epsilon_F$. This means in the attraction of the electrons shift them in a state that lays under the Fermi sea. Further they showed that if the Fermi sea vanishes (one could take electrons out of the system) then this attractive pairing disappears. The same is observed as we approach

the classical limit. By forming a pair the electron share their fermionic properties and act as bosons. The Pauli principle doesn't rule their energies anymore.

We now understand why attractive processes create instabilities. An energy gap opens next to the surface, reflecting the energy needed to break the new formed Cooper pair. The shell ω_0 is the maximum frequency delivering the “adesive” that allows the electron to pair. Now that we showed the influence of attractive interactions, we seek some candidate process that are attractive.

2.1.4 Phonon-mediated attractive interactions [reformulate](#)

As known from condensed matter physics, the lattice can have some intern oscillations called phonons resulting from the spring coupling between the ions. Now we can imagine that due to the Coulomb interactions, an electron can shift an ions producing a phonon. If this phonon travels and influences another electron on it way, we result in an effective electron-electron interaction thanks to the phonon. A similar case would be the exchange of a photon between two electrons. We are going to show how this exchange can lead to an attractive interaction.

In a dense lattice the ions moves much slowly around their equilibrium positions than the lights electrons who pass by. The electron moves the charges of the ions resulting in a small dipol moment. A second electron that also passes in the surrounding is going to feel the dipol moment and will be attracted. Then the ion shifts back in its new equilibrium position and the dipol moment vanishes long after the first electron passed.

Moreover due to the Coulomb interaction, the electrons aim to put as most distance as they can between them in a minimal amount of time. Therefore we can say that the \mathbf{k} -quantum number should be opposite between the two electron. If we target to put these concepts in a mathematical form, we use our previous Hamiltonian and add an electron-phonon interaction term.

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k},\sigma} + \sum_{\mathbf{k},\sigma,\mathbf{k}',\sigma'} V_{\mathbf{k}\mathbf{k}',\mathbf{q}} c_{\mathbf{k}-\mathbf{q},\sigma}^\dagger c_{\mathbf{k}+\mathbf{q},\sigma'}^\dagger c_{\mathbf{k},\sigma} c_{\mathbf{k}',\sigma'} + V_{e\text{-phonon}}$$

where $V_{e\text{-phonon}}$ usely depends on the sum of the phonons-modes λ . These modes are similar to the oscillations modes we have in a CO_2 -molecule. The expression form the phonon-depend potential in momentum space reads

$$V_{e\text{-phonon}} = \sum_{\mathbf{k},\mathbf{q},\sigma} M_{\mathbf{q}} (a_{-\mathbf{q}}^\dagger + a_{\mathbf{q}}) c_{\mathbf{k}+\mathbf{q},\sigma}^\dagger c_{\mathbf{k},\sigma}. \quad (23)$$

$M_{\mathbf{q}} (a_{-\mathbf{q}}^\dagger + a_{\mathbf{q}})$ is a matrix element of the coupling between the electron and the phonon. The $a_{\mathbf{q}}$ and $a_{\mathbf{q}}^\dagger$ are annihilation and creation operators of the phonon with wavevector \mathbf{q} . Further researches have shown that [source](#)

$$[a_{\mathbf{q}}, a_{\mathbf{q}'}^\dagger] = \delta_{\mathbf{q},\mathbf{q}'}$$

and therefor phonons act like bosons. Their number is however not conserved in a solid. The matrix element M is a function of the eigenfrequency [or energy?](#) of the phonon $\omega_{\mathbf{q},\lambda}$ and the fourier transform \tilde{V} of the electrostatic potential $V_\lambda(\mathbf{q})$ between the electron and the phonon of mode λ , if included. [more details?](#)

$$M_{\mathbf{q},\lambda} = i(\mathbf{q} \cdot \boldsymbol{\xi}_\lambda) \sqrt{\frac{\hbar}{2M\omega_{\mathbf{q},\lambda}}} \tilde{V}_\lambda(\mathbf{q}).$$

M can't be really computed due to its complexity, we hold it as a parameter here. This pairing is much waker than the electron-photon interaction. An other important fact ist than for $\mathbf{q} \rightarrow 0$ the matrix element M vanishes. M is proportional to \mathbf{q} which illustrate the the electron-phonon interaction happens between a point charge and a dipol.

The goal is now the implicitly express the phonon exchange with an effective electron-electron process. If we consider two diagrams, one aiming to describe the absorption of a phonon and one the emission of a phonon, we can combine them to get a new effective interaction like the photon exchange case.



Figure 3: The emission (left) and absorption (right) digramm of a phonon of wavevector \mathbf{q} by an electron.

If we represent this interaction by linking both \mathbf{q} -edges. The energy of phonon is simply defined as

$$H_{\text{phonon}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}$$

and we can write a propagator which describes the dashed line in a connected context similar to photon case described in the figure ??.

$$D_0(\mathbf{q}, \omega) = \frac{2\omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2 + i\eta}$$

involving a very small quantity η . From this we obtain an expression for the phonon-mediated interaction of two electrons. This is performed using $V_{\text{eff}}^{(ph)}(\mathbf{q}, \omega) = \text{Re}(|M_{\mathbf{q}}|^2 D_0(\mathbf{q}, \omega))$ and discarding the second order η term which is very small.

$$V_{\text{eff}}^{(ph)}(\mathbf{q}, \omega) = \frac{2|M_{\mathbf{q}}|^2 \omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2} \quad (24)$$

with \mathbf{q} the momentum transfert. We can now use a more complete potential in the Hamiltonian involving both the electrostatic 22 and effective phonon-mediated interactions 24.

$$V_{\text{eff}}(\mathbf{k}, \mathbf{k}', \mathbf{q}) = V_{\text{el}}(\mathbf{q}) + V_{\text{eff}}^{(ph)}(\mathbf{q}, \omega) = \frac{1}{4\pi\epsilon_0} \frac{2\pi e^2}{q^2} + \frac{2|M_{\mathbf{q}}|^2 \omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2} \quad (25)$$

Here we have reached a very important point. This new potential can be in some case negative, which means we result in an attractive interaction between the electrons. With other word, in some cases the phonon exchange can be attractive and even overcome the strong repulsive coulomb potential. [Graph of V and the simplified version of it, which is also a very good predication.](#)

2.1.5 Contraction of the effective Hamiltonian

We want to restrict ourselves in the case where the effective Hamiltonian is attractive. This happens in a small shell around the Fermi-surface. If we want to maximise the phase space for the scattering, the state before and after the scattering have to be in this shell. A good idea is to consider that the two electrons have opposite wavevectors. The folowing figure illustrates this process.



Figure 4: The scattering process of two electrons with opposite wavevectors \mathbf{k} and \mathbf{k}' . We have a momentum transfer \mathbf{q} between them. The thick line illustrates the Fermi-surface and the dotted one the thin shell. As we see if the electrons have opposite wavevectors, and the \mathbf{k} electron scatters into the shell, then the \mathbf{k}' electron scatters in the shell as well. This is not always the case if the wavevectors don't agree as we can see on the left figure. The right figure points out the maximisation process. With opposite wavevectors, we get the largest possibility spectrum for the scattering.

Further the attractivity is a short range effect, so if we want to consider it, we must think that the electrons are very close to each other. This requires the electrons to have opposite spins due to the Pauli principle. This is the same as a lattice site. This is the only possibility for them to cohabit the same neighbourhood. [check](#) The approximation turns out to be a good model.

Now we allow us to rename some variables:

$$\mathbf{k} + \mathbf{q} \longrightarrow \mathbf{k}, \quad \mathbf{k} \longrightarrow \mathbf{k}'$$

The Hamiltonian that follows from these transformations is called the BCS-reduced Hamiltonian

$$H = \sum_{\mathbf{k}, \sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^{\dagger} c_{\mathbf{k}, \sigma} + \sum_{\mathbf{k}, \sigma, \mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}, \sigma}^{\dagger} c_{-\mathbf{k}, -\sigma}^{\dagger} c_{-\mathbf{k}', -\sigma} c_{\mathbf{k}', \sigma} \quad (26)$$

$V_{\mathbf{k}\mathbf{k}'}$ is now a matrix element that acts if the wavevectors are close to the Fermi-surface. The electrons have to move in opposite directions with opposite spins. Due to the retardation processes we introduced earlier, there remains a distortion in the lattice long after the electrons passed. Due to the inducting dipole moment, the other electron is attracted towards the distortion with $M_{\mathbf{q}}$ [?](#). As we also saw, the coulomb repulsion causes a colinear displacement, close to the distortion of the homologue. This phenomenon is called the Cooper-pairing and is a coupling that happens in momentum space.

The reader might want to see a picture of what is happening in the real space. To achieve a representation two simple statements are enough. First, the Coulomb force aims to maximize the distance between the electrons in minimal time. This is achieved by moving them collinearly in opposite directions. Second, due to the retardation process, it's energetically more favourable for the electron to move along the distortion of the lattice. The result is illustrated in the figure [??](#).



Here the moved ions in blue are shifting back to their equilibrium position but it is important to remember that when the other electron passes, it experiences almost the same distortion everywhere. An interesting fact is that these interactions are the source of the superconductivity but they are also the main origin of resistivity in clean materials.

2.1.6 On our way to the BCS-theory

After this introduction on the phonon coupling between the electrons in the momentum space, or Cooper-pairing, we aim to describe the energy of the superconductor in a mean-field approach. The goal of it is to reduce the description with the neighbours to the description of a site, in the mean field of the other sites. Therefore we are going to describe a one-body problem which is easier to compute. As known mean-field approaches require self-consistent equations that will as well follow.

The first step is to introduce the following expectation values:

$$b_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \quad (27)$$

$$b_{\mathbf{k}}^{\dagger} = \langle c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger} \rangle \quad (28)$$

which lead to a new expression for the c operators:

$$c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} = b_{\mathbf{k}} + \underbrace{c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - b_{\mathbf{k}}}_{\delta b_{\mathbf{k}}} \quad (29)$$

where we can see the $\delta b_{\mathbf{k}}$ as a deviation, or fluctuation term. If we introduce it back into the Hamiltonian, we can write

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}, \mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} (b_{\mathbf{k}}^{\dagger} + \delta b_{\mathbf{k}}^{\dagger}) (b_{\mathbf{k}} + \delta b_{\mathbf{k}}).$$

We can compute the product of the two terms in parenthesis and forget the $\mathcal{O}(\delta b_{\mathbf{k}}^2)$ because the fluctuations are small. We then obtain the following expression

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}, \mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left(b_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + b_{\mathbf{k}'}^{\dagger} c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger} - b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}'} \right).$$

The next step is to define the superconducting gap parameter Δ :

$$\Delta_{\mathbf{k}} := - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}}^{\dagger} \quad (30)$$

$$\Delta_{\mathbf{k}}^{\dagger} := - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}'} \quad (31)$$

which brings our Hamiltonian in another form:

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + \Delta_{\mathbf{k}} c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger} - b_{\mathbf{k}}^{\dagger} \Delta_{\mathbf{k}} \right). \quad (32)$$

We took the liberty to spilt the sum, rename the \mathbf{k}' to \mathbf{k} and recombine the sum. We notice that this form involves a lot of creation and annihilation terms that are not common for an effective non-interacting electron gas. We remember that we aim at a one particle description in a mean field of its neighbours. This complexity will lead to some difficulties to express the quasi-particle spectrum. A good solution is to rotate the basis of the c operators to land in a basis that diagonalises the Hamiltonian and therefore minimises the number of operators.

The transformation involves two new fermionic operators η and γ that therefore respect $\mathfrak{F}\mathfrak{e}\mathfrak{r}_1$ to $\mathfrak{F}\mathfrak{e}\mathfrak{r}_3$ and reads

$$\begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \eta_{\mathbf{k}} \\ \gamma_{\mathbf{k}} \end{pmatrix} \quad (33)$$

along with the conjugate transpose of each component of the l.h.s vector rebuild into a matrix-vector equation

$$\begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger \\ c_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \eta_{\mathbf{k}}^\dagger \\ \gamma_{\mathbf{k}}^\dagger \end{pmatrix}. \quad (34)$$

We can reintroduce these into the Hamiltonian 32. Some of the multiplication involves $\cos(\theta)^2 - \sin(\theta)^2 = \cos(2\theta)$ and $\cos(\theta)^2 + \sin(\theta)^2 = 1$. Further due to the anticommutations we have $\eta\gamma^\dagger = -\gamma^\dagger\eta$ and so on for $\gamma\eta^\dagger = -\eta^\dagger\gamma$. We obtain the following expression

$$\begin{aligned} H = & \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \cdot 1 + \Delta_{\mathbf{k}} b_{\mathbf{k}}^\dagger \\ & + \sum_{\mathbf{k}} \left[\epsilon_{\mathbf{k}} \cos(2\theta) - \cos(\theta) \sin(\theta) \left(\Delta_{\mathbf{k}}^\dagger + \Delta_{\mathbf{k}} \right) \right] \eta_{\mathbf{k}}^\dagger \eta_{\mathbf{k}} \\ & - \sum_{\mathbf{k}} \left[\epsilon_{\mathbf{k}} \cos(2\theta) - \sin(\theta) \cos(\theta) \left(\Delta_{\mathbf{k}}^\dagger + \Delta_{\mathbf{k}} \right) \right] \gamma_{\mathbf{k}}^\dagger \eta_{\mathbf{k}} \\ & - \sum_{\mathbf{k}} \left[\Delta_{\mathbf{k}} \cos(\theta)^2 - \Delta_{\mathbf{k}}^\dagger \sin(\theta)^2 + 2\epsilon_{\mathbf{k}} \cos(\theta) \sin(\theta) \right] \eta_{\mathbf{k}}^\dagger \gamma_{\mathbf{k}} \\ & - \sum_{\mathbf{k}} \left[\Delta_{\mathbf{k}} \cos(\theta)^2 - \Delta_{\mathbf{k}}^\dagger \sin(\theta)^2 + 2\epsilon_{\mathbf{k}} \cos(\theta) \sin(\theta) \right] \gamma_{\mathbf{k}}^\dagger \eta_{\mathbf{k}}. \end{aligned} \quad (35)$$

The goal is to diagonalise the Hamiltonian in the (γ, η) basis. Therefore have to rotate with θ such that the terms with $\gamma^\dagger\eta$ and $\eta^\dagger\gamma$ vanish. A difficulty that we may encounter along with this idea is that Δ is an order parameter and own a complex phase fluctuation. With other words one can write $\Delta = |\Delta|e^{i\tau}$ where τ has some fluctuations. We are going to ignore them.

We can set

$$\Delta_{\mathbf{k}} = \Delta_{\mathbf{k}}^\dagger \quad \text{and} \quad \tan(2\theta) = -\frac{\Delta_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}$$

and introduce two new variables called the coherence factors

$$v_{\mathbf{k}}^2 := \sin(\theta)^2 = \frac{1}{2} \left(1 - \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}} \right), \quad (36)$$

$$u_{\mathbf{k}}^2 := \cos(\theta)^2 = \frac{1}{2} \left(1 + \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}} \right) \quad (37)$$

along with a new energy $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$. [Historical context?](#) These factors play an important role in NMR as well as in the ultra sound propagation in superconductors. Cooper and Schrieffer made some correct predictions in an advanced many-body system. This is labeled as one of the greatest achievements in condensed matter physics in the 20th century.

[source](#)

We obtain a Hamiltonian that looks very similar to a free fermion quasiparticle gas [source?](#)

$$H = \underbrace{\sum_{\mathbf{k}} \left(\epsilon_{\mathbf{k}} + \Delta_{\mathbf{k}} b_{\mathbf{k}}^\dagger \right)}_{=: H_0 \text{ constant mean-field term}} + \underbrace{\sum_{\mathbf{k}} E_{\mathbf{k}} \left(\eta_{\mathbf{k}}^\dagger \eta_{\mathbf{k}} - \gamma_{\mathbf{k}}^\dagger \gamma_{\mathbf{k}} \right)}_{\text{Spinless fermion system with two types of fermions of energies } E_{\mathbf{k}} \text{ and } -E_{\mathbf{k}}}. \quad (38)$$

where we don't describe any interaction, but two types of particles in a mean field. As [1] described it in chapter 3, p.83-84, one can note that we don't have any spins involved. This is

due to the fact that we describe the quasiparticles as a linear combination of electrons and holes with opposite spins. [source](#) Therefore talking about degrees of freedom the two electron-hole singlets replace the \uparrow, \downarrow degrees of freedom and they are preserved.

The energy $E_{\mathbf{k}}$ shows up a gap in the energy of the quasiparticle when looking at the Fermi-surface $\epsilon_{\mathbf{k}} = 0$. [We count relative to it? But the way it is defined in a non-interacting Hamiltonian is not relative to it right?](#) This is due to the fact that the expectation values we introduced in 27 and 28 are not zero. The gap is a result of the Cooper-pairing. These expectation values are the order parameters of the Meissner state and should be confused with the superconducting gap Δ . They are however zero at the same time.

This context of a free case is a good opportunity to introduce the grand canonical ensemble of the Hamiltonian. Even if we could give the direct result, deriving this ensemble involves a lot of important steps, so for the sake of completeness we are going to derive it.

We define the particle number operator $N = \sum_{\mathbf{k}} \mu (n_{\eta\mathbf{k}} + n_{\gamma\mathbf{k}})$ where the index \mathbf{k} is proper to this equation. Further the possible states \mathbf{k} are in a “continuous” set $\mathfrak{K} = \{\mathbf{k}_1, \mathbf{k}_2, \dots\}$. $\{n_{\mathbf{k}}\}$ is the set of the different occupation numbers of all the states \mathbf{k} . Further we consider fermions so the occupation numbers of the particle type η or γ in state \mathbf{k} are labeled $n_{\eta\mathbf{k}}$ and $n_{\gamma\mathbf{k}}$ and equals either 0 or 1.

$$\begin{aligned}
Z_G &= \text{Tr} \left(e^{-\beta(H+\mu N)} \right) \\
&= \sum_{\{n_{\mathbf{k}}\}} e^{-\beta H_0} \langle \{n_{\mathbf{k}}\} | \exp \left(\sum_{\mathbf{k}'} -\beta E_{\mathbf{k}'} \eta_{\mathbf{k}'}^\dagger \eta_{\mathbf{k}'} + \beta E_{\mathbf{k}'} \gamma_{\mathbf{k}'}^\dagger \gamma_{\mathbf{k}'} \right) \exp(-\beta \mu N) | \{n_{\mathbf{k}}\} \rangle \\
&= \sum_{\{n_{\eta\mathbf{k}}\}} \sum_{\{n_{\gamma\mathbf{k}}\}} e^{-\beta H_0} \exp \left(\sum_{\mathbf{k}} -\beta \left[E_{\mathbf{k}} \eta_{\mathbf{k}}^\dagger \eta_{\mathbf{k}} + \mu \right] + \beta \left[E_{\mathbf{k}} \gamma_{\mathbf{k}}^\dagger \gamma_{\mathbf{k}} - \mu \right] \right) \\
&= e^{-\beta H_0} \sum_{\substack{n_{\eta\mathbf{k}_1}, \\ n_{\eta\mathbf{k}_2}, \dots}} \sum_{\substack{n_{\gamma\mathbf{k}_1}, \\ n_{\gamma\mathbf{k}_2}, \dots}} \exp \left(\sum_{\mathbf{k}} -\beta \left[E_{\mathbf{k}} n_{\eta\mathbf{k}} + \mu \right] + \beta \left[E_{\mathbf{k}} n_{\gamma\mathbf{k}} - \mu \right] \right) \\
&= e^{-\beta H_0} \sum_{\substack{n_{\eta\mathbf{k}_1}, \\ n_{\eta\mathbf{k}_2}, \dots}} \prod_{\mathbf{k}} \exp(-\beta [E_{\mathbf{k}} n_{\eta\mathbf{k}} + \mu]) \sum_{\substack{n_{\gamma\mathbf{k}_1}, \\ n_{\gamma\mathbf{k}_2}, \dots}} \prod_{\mathbf{k}} \exp(\beta [E_{\mathbf{k}} n_{\gamma\mathbf{k}} - \mu]) \\
&= e^{-\beta H_0} \sum_{n_{\eta\mathbf{k}_1}} \exp(-\beta [E_{\mathbf{k}_1} n_{\eta\mathbf{k}_1} + \mu]) \sum_{n_{\eta\mathbf{k}_2}} \exp(-\beta [E_{\mathbf{k}_2} n_{\eta\mathbf{k}_2} + \mu]) \dots \\
&\quad \sum_{n_{\gamma\mathbf{k}_1}} \exp(\beta [E_{\mathbf{k}_1} n_{\gamma\mathbf{k}_1} - \mu]) \sum_{n_{\gamma\mathbf{k}_2}} \exp(\beta [E_{\mathbf{k}_2} n_{\gamma\mathbf{k}_2} - \mu]) \dots \\
&= e^{-\beta H_0} \prod_{\mathbf{k}} \left(1 + e^{-\beta(E_{\mathbf{k}} + \mu)} \right) \left(1 + e^{\beta(E_{\mathbf{k}} - \mu)} \right)
\end{aligned}$$

This partition function isn't very useful like this in our case. But we can derive the free energy which will help us to solve the self-consistency equation for the gap.

According to [3] p.99 the free energy can be derived from the partition function as ([Appendix for more details?](#))

$$F = -\frac{1}{\beta} \ln(Z_G) = H_0 - \frac{1}{\beta} \sum_{\mathbf{k}} + \ln(1 + e^{-\beta E_{\mathbf{k}}}) \ln(1 + e^{\beta E_{\mathbf{k}}})$$

According to [1] we obtain in the limit of low temperatures ($\beta \rightarrow \infty$)

$$\begin{aligned}
F &= H_0 + \sum_{\mathbf{k}} E_{\mathbf{k}} \theta(-E_{\mathbf{k}}) + E_{\mathbf{k}} \theta(E_{\mathbf{k}}) \\
&= H_0 + \sum_{\mathbf{k}} E_{\mathbf{k}} - E_{\mathbf{k}} \theta(E_{\mathbf{k}}) + E_{\mathbf{k}} \theta(E_{\mathbf{k}}) \\
&= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} + \Delta_{\mathbf{k}} b_{\mathbf{k}}^\dagger E_{\mathbf{k}}.
\end{aligned}$$

where we used the properties from the Heaviside step-function described in 12. The minimization of the free energy F should self-consistently determine the gap $\Delta_{\mathbf{k}}$. This is a statement that depends neither on the momentum space structure nor on the attractiveness of the potential [1]. $E_{\mathbf{k}}$ has an implicit $\Delta_{\mathbf{k}}$ -dependence. We compute the derivation $\frac{\partial F}{\partial \Delta_{\mathbf{k}}}$ and search the argument of the zero-position. Derivating one \mathbf{k} from the sum is enough. We demand the following to be satisfied:

$$\frac{\partial F}{\partial \Delta_{\mathbf{k}}} = 0, \quad \frac{\partial F}{\partial \Delta_{\mathbf{k}}^\dagger} = 0 \quad (39)$$

This leads to [Appendix?](#)

$$b_{\mathbf{k}}^\dagger = \Delta_{\mathbf{k}} \underbrace{\frac{\tanh(\beta E_{\mathbf{k}}/2)}{E_{\mathbf{k}}}}_{\chi(\mathbf{k})}$$

$\chi(\mathbf{k})$ is the pair-susceptibility and gives how capable the system is to create Cooper-pairs [1]. If we allow us to relabel $\mathbf{k} \rightarrow \mathbf{k}'$ then multiply on both sides with $-\sum_{\mathbf{k}} V_{\mathbf{k}\mathbf{k}'}$ and introduce 31 we obtain the self-consistency equation for the gap. This is usually denominated as the BCS gap equation.

$$\Delta_{\mathbf{k}} = - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \Delta_{\mathbf{k}'} \frac{\tanh(\beta E_{\mathbf{k}'}/2)}{E_{\mathbf{k}'}}. \quad (40)$$

Fossheim and Sudbø [1] emphasize that this equation is not limited to phonon mediated interaction, as long we don't specify the potential. Further Cooper-pairs can condensate because in the pair the electron involved have opposite spins and this adds up to zero. The pair is not ruled anymore by the Pauli principle and can take part into the superconducting condensate. Even if the formation temperature of Cooper-pairs is higher than the critical temperature T_C , this mean field approach makes this temperatures agree.

As we introduced earlier the gap has a complex phase fluctuation. This phase is hard to vary in good metals because the number of carrier electrons is high. On the other side in poor metals the fluctuations that break the pair are more easily reached, i.e. at a lower temperature than in good metal. Superconductivity is therefore more stable in metals of good value.

In this section we saw how to reduce the many body Hamiltonian to a single-body problem in the mean field of its neighbours. The system that follows from this is a non interacting gas of a type electrons and holes. This involves the introduction of the superconducting gap parameter that we can thanks to the statistical mechanics formalism express in a self-consistent way.

2.1.7 Generalized gap equation, s-wave and d-wave superconductivity

As we introduced in an earlier section, the formalism follows from the attractive phonon exchange but can be generalized. The superconductivity is the result of the pairing of the electrons into Cooper-pairs. These pairs shift into a condensate of take part of a coherent matter-wave. Making the analogy with the coherent light wave, superconductivity can be viewed as the *analogos* of lasers like Fossheim and Sudbø highlight it in [1].

To achieve such generalisation we let the formalism be open to other kind of interactions and don't restrict it to the thin shell around the Fermi-surface. The potential matrix-element can therefore take a complex form than introduced in 25.

We recall that the electrons move into a lattice. We can assume that this crystal owns some symmetries that are reflected in its crystallographic (complete) basis $\{g_\eta(\mathbf{k})\}$. Similar to the Bloch state, we could imagine that the symmetries emphasize some physical quantities. Assuming this we propose a form for the potential:

$$V_{\mathbf{k}\mathbf{k}'} = \sum_{\eta} \lambda_{\eta} g_{\eta}(\mathbf{k}) g_{\eta}(\mathbf{k}')$$

η is also often labeled as the channel [1]. Assuming we have a square lattice,



with interaction strength $U/2$ on site, for nearest neighbours $2V$ and $4W$ for second-nearest neighbours. [need of more details from Jacob](#). Taking the Fourier transform of this leads:

$$f(k, k') = \cos(k) \cos(k') + \sin(k) \sin(k')$$

$$V_{\mathbf{k}\mathbf{k}'} = U + 2V (f(k_x, k'_x) + f(k_y, k'_y)) + 4W (f(k_x, k'_x) \cdot f(k_y, k'_y))$$

We can then introduce the basis-functions $\{g_\eta(\mathbf{k})\}$ we talked about a few paragraphs ago.

$$g_1(\mathbf{k}) = \frac{1}{2\pi}$$

$$g_2(\mathbf{k}) = \frac{1}{2\pi} (\cos(k_x) + \cos(k_y)) \quad (\text{s-wave})$$

$$g_3(\mathbf{k}) = \frac{1}{2\pi} \cos(k_x) \cos(k_y)$$

$$g_4(\mathbf{k}) = \frac{1}{2\pi} (\cos(k_x) - \cos(k_y)) \quad (\text{d-wave})$$

$$g_5(\mathbf{k}) = \frac{1}{2\pi} \sin(k_x) \sin(k_y)$$

from which involves the following lambdas $\lambda_1 = 2U\pi^2$, $\lambda_2 = \lambda_4 = 4V\pi^2$, $\lambda_3 = \lambda_5 = 4W\pi^2$ and for greater η ge set $\lambda_{\eta \geq 6} = 0$.

For $\eta \in \{1, 2\}$, the corresponding g is the identity under all symmetries of C_{v4} and for $\eta \in \{3, 4, 5\}$ we observe a change of sign under $\pi/2$ rotations. The same behaviour is found in the spherical harmonics for resp. the quantum numbers $l = 0$ and $l = 2$. From this we can deduct the use of the terms s-wave and d-wave. [Does this need a source?](#)

Furhter we can reintroduce the BCS gap equation 40 and obtain a lattice-dependent form for the gap:

$$\begin{aligned} \Delta_{\mathbf{k}} &= - \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \Delta_{\mathbf{k}'} \chi(E_{\mathbf{k}}) \\ &= - \sum_{\eta \in \llbracket 5 \rrbracket} \lambda_\eta g_\eta(\mathbf{k}) \underbrace{\sum_{\mathbf{k}'} g_\eta(\mathbf{k}') \Delta_{\mathbf{k}'} \chi(E_{\mathbf{k}'})}_{=: \Delta_\eta / \lambda_\eta} \\ &= \sum_{\eta \in \llbracket 5 \rrbracket} \Delta_\eta g_\eta(\mathbf{k}) \end{aligned} \quad (41)$$

This is a nice form to have as Fossheim and Sudbø [1] point out p.92. The gap is a linear combination of the physical quantities $g_\eta(\mathbf{k})$ which gives a physical quantities as well. The newly introduced Δ_η are independent from the wavevector \mathbf{k} but are function if the temperature. We can express them in the basis of antoher η' :

$$\Delta_\eta = \sum_{\eta' \in \llbracket 5 \rrbracket} \Delta_{\eta'} \mathcal{M}_{\eta, \eta'}$$

$$\mathcal{M}_{\eta, \eta'} = - \lambda_\eta \sum_{\mathbf{k}} g_\eta(\mathbf{k}) g_{\eta'}(\mathbf{k}) \chi(E_{\mathbf{k}}).$$

These numerically combuersom to compiute but taking $U > 0$, $V < 0$ and $W = 0$ in the square lattice we obtain no attraction $V_{\mathbf{k}\mathbf{k}'} \eta=1 > 0$? in the λ_1 -channel but attraction in the s- and d-wave channels. This tells us a lot like [1] analysed it. First the gap is highly associated to the Fourier transform of the wavefunction of the Cooper-pairss [Which variable describes their](#)

[wavefunction?](#) Further the gap accomodate itself to be zero in the channel where the wavefunction is non-zero “[for zero separation between the electrons](#)”. The s- and d-wave channels are preferred which as consequence avoid the on site coulomb repulsion (i.e the Coulomb force between the two members of the pair). This can be put in comparison with the special case of the phonon pairing. In the latest the electrons used retardation processes to cancel out the repulsion. They avoided themselves in time. Here the avoidance takes place in space. ([How to recognise the angular momentum coupling here?](#))

The manifestation of the retardation effects are actually a direct consequence of restricting ourselves to a thin shell around the Fermi-surface. [But members of a Cooper pairs must have opposite spin and momenta. The opposite momentum is a result of the shell so is this statement valid in general?](#). This means without restrictions, the whole Brillouin zone is taken into account to compute the gap.

Further we can express the Fourier transform of the gap as $\Delta(\mathbf{r}) = \sum_{\mathbf{k}} \Delta_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}}$ which leads $\Delta(0) = \sum_{\mathbf{k}} \Delta_{\mathbf{k}}$. However a repulsive interaction is obtained for $\Delta(0) \neq 0$ so we need to find $\Delta_{\mathbf{k}}$ s whose sum satisfy this. This can be done using the s-wave and d-wave channels, i.e $\Delta_{\mathbf{k}} = \Delta_0(T)g_2(\mathbf{k})$ and $\Delta_{\mathbf{k}} = \Delta_0(T)g_4(\mathbf{k})$. We already motivated superconductivity and its difficulty to maintain due to the phase fluctuations above the (freezing) critical temperature T_c . However superconductivity involving d-wave channels is believed [1] p.92 to be found in materials with high T_c . $\Delta_{\mathbf{k}} = \Delta_0(T)g_4(\mathbf{k})$ is therefore a good case of study.

Density of Meissner states

We imagine some fluctuation [in?](#) $g(\mathbf{k})$ that only depend on $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$. Further we assume Δ being \mathbf{k} - independent for now. We can introduce the normal density of states.

$$D_n(\epsilon) = \frac{1}{N} \sum_{\mathbf{k}} \delta(\epsilon - \epsilon_{\mathbf{k}})$$

with N the number of Fourier modes. This is the same as the number of lattice sites for us [1] p.93.

$$\sum_{\mathbf{k}} g(\mathbf{k}) = \int D_n(\epsilon) g(\sqrt{\epsilon^2 - \Delta^2}) d\epsilon$$

as well as a variable transformation $E = \sqrt{\epsilon^2 + \Delta^2}$ which leads to

$$dE = \frac{\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} d\epsilon \iff d\epsilon = \frac{E}{\epsilon} dE,$$

$$\sum_{\mathbf{k}} g(\mathbf{k}) = \int D_n(\epsilon) g(E) \frac{E}{\epsilon} dE.$$

Further we have $\frac{\partial \epsilon}{\partial E} = \frac{E}{\sqrt{E^2 - \Delta^2}} = \frac{E}{\epsilon}$ so that

$$\sum_{\mathbf{k}} g(\mathbf{k}) = \int g(E) \underbrace{D_n(\epsilon) \frac{\partial \epsilon}{\partial E}}_{D_s(E)} dE. \quad (\text{take total or partial derivative?})$$

Then we use a method we already used ([sure?](#)) to simplify the expression. We assume that $D_n(\epsilon)$ is slowly varying so that $D_n(\epsilon) \approx D_n(0)$ where ϵ is measured relative to the Fermi-surface. Hopefully we don't have any sharpness around the Fermi-surface [1] p.93. We can therefore exclude the Van Hove singularities close to the Fermi-surface.

$$\frac{D_s(E)}{D_n(0)} \frac{E}{\sqrt{E^2 - \Delta^2}} = \frac{E}{\sqrt{E^2 - \Delta^2}} \underbrace{\frac{D_n(\epsilon)}{D_n(0)}}_{=\Theta(E-\Delta)}.$$

For the \mathbf{k} dependence of the gap we can use the spectral weight introduced in 14. In analogy to the propagator or Green function introduced in ?? for the electrons, we can define one for the superconducting condensate that involves time-dependent fermionic operators.

$$G(\mathbf{k}, t; \sigma) = -i \langle 0 | c_{\mathbf{k}\sigma}(t) c_{\mathbf{k}\sigma}^\dagger(0) | 0 \rangle$$

involving the BCS superconducting ground state $|0\rangle$ which is an eigenstate of 38. As before we can consider the Fourier transform of the propagator $G(\mathbf{k}, \omega; \sigma)$

$$G(\mathbf{k}, \omega; \sigma) = \frac{1}{2\pi} \int e^{i\omega t} G(\mathbf{k}, \omega; \sigma) dt$$

We recall that Cooper pairs are two made of two electrons with opposite spin. We can first take a look at the density of state of the pair-member with $\sigma = \uparrow$. However are only considering spin singlet pairing so proving that the spectral weight A is spin independent should show that computing it for $\sigma = \uparrow$ is enough as Fossheim and Sudbø argued in [1] p.94.

So without loss of generality we set $\sigma = \uparrow$ and reintroduce the rotation of the basis of the c operators we performed in 33 and 34.

$$G(\mathbf{k}, t; \uparrow) = -i\langle 0 | \left(\cos(\theta)\eta_{\mathbf{k}}^\dagger(t) - \sin(\theta)\gamma_{\mathbf{k}}^\dagger(t) \right) \cdot \left(\cos(\theta)\eta_{\mathbf{k}}(0) - \sin(\theta)\gamma_{\mathbf{k}}(0) \right) | 0 \rangle$$

The goal of these rotated operators was to diagonalise the hamiltonian. It follows

$$\langle 0 | \eta_{\mathbf{k}}^\dagger \gamma_{\mathbf{k}}^\dagger | 0 \rangle = 0 = \langle 0 | \eta_{\mathbf{k}} \gamma_{\mathbf{k}} | 0 \rangle$$

and therefore

$$G(\mathbf{k}, t; \uparrow) = -i \cos(\theta)^2 \langle 0 | \eta_{\mathbf{k}}^\dagger(t) \eta_{\mathbf{k}}(0) | 0 \rangle - i \sin(\theta)^2 \langle 0 | \eta_{\mathbf{k}}^\dagger(t) \eta_{\mathbf{k}}(0) | 0 \rangle. \quad (42)$$

We can nicely split the propagator in the sum of free η - and γ -particles in the superconducting state. Employing to coherence factors 37 and 36 in the Fourier transform of we obtain

$$G(\mathbf{k}, \omega; \uparrow) = \frac{u_{\mathbf{k}}^2}{\omega - E_{\mathbf{k}} + i\delta_{\mathbf{k}}} + \frac{v_{\mathbf{k}}^2}{\omega + E_{\mathbf{k}} - i\delta_{\mathbf{k}}}.$$

which gives the spectral weight

$$A(\mathbf{k}, \omega; \uparrow) = u_{\mathbf{k}}^2 \delta(\omega - E_{\mathbf{k}}) + v_{\mathbf{k}}^2 \delta(\omega + E_{\mathbf{k}}).$$

This value is spin independent and using the arguments we've just given in such a case, we find

$$\begin{aligned} D_s(\omega) &= \frac{1}{N} \sum_{\mathbf{k}} A(\mathbf{k}, \omega) \\ &= \frac{1}{N} \sum_{\mathbf{k}} (u_{\mathbf{k}}^2 \delta(\omega - E_{\mathbf{k}}) + v_{\mathbf{k}}^2 \delta(\omega + E_{\mathbf{k}})). \end{aligned}$$

The density of state of the superconductive condensate D_s is also spin independent since we're considering spin-singlet pairings.

2.1.8 Transition temperature and energy gap

The goal of this discussion will to derive a universal ratio between Δ and the critical temperature. In the last section we already introduced some expressions for $\Delta_{\mathbf{k}}(T)$ and $V_{\mathbf{k}\mathbf{k}}$. Let us consider the simplest case where $V_{\mathbf{k}\mathbf{k}} = V$.

The phonon modulated interaction has a cover $\omega_0 = \omega_D$ the Debye-frequency. Inserting it back to the BCS gap equation 40 we see that the gap loses its \mathbf{k} -dependence and results as the identity when applying the symmetries ruling the crystal:

$$1 = V \sum_{\mathbf{k}'} \frac{\tanh(\beta E_{\mathbf{k}'})}{2E_{\mathbf{k}'}}.$$

This equation can be easily solved for $T = T_C$ or $T = 0$.

Considering T approaching T_C from below, we can assume that the gap vanishes. We replace the \mathbf{k} -sum by an integral over the normal density of state $D_n(\epsilon)$. Due to the shell the sum occurs in a tiny volume around the Fermi-Surface so that $D_n(\epsilon)$ is evaluated close to the surface. We assume that in this neighbourhood D_n varies slowly such that avoid some van Hove singularities

we simply approximate $D_n(\epsilon) \rightarrow D_n(0)$ because ϵ is counted relative to the surface in our early thoughts. We introduce $\lambda = VD_n(0)$. We get

$$\begin{aligned} 1 &= \lambda \int_{[0, \omega_D]} \frac{\tanh(\beta\epsilon/2)}{\epsilon} d\epsilon \\ &= \lambda \ln \left(\frac{2e^\gamma \beta \omega_D}{\pi} \right) \end{aligned}$$

using $\gamma := \lim_{m \rightarrow \infty} \left(\sum_{l \in \llbracket m \rrbracket} 1/l - \ln(m) \right)$ the Euler-Mascheroni constant. More details are provided in [1] p.88-89. We obtain

$$k_B T_C \approx 1.13 \cdot \omega_D e^{-1/\lambda}$$

For $T \rightarrow 0$ the gap equation takes a simpler form to solve:

$$1 = V \sum_{\mathbf{k}'} \frac{1}{2E_{\mathbf{k}'}} = \lambda \int_{[0, \omega_D]} \frac{1}{\sqrt{\epsilon^2 + \Delta^2}} d\epsilon$$

leads

$$\Delta(T=0) = 2\omega_D e^{-1/\lambda}$$

according to the same source. We see how these expressions are closely dependent on λ . Moreover we can interpret the essential singularity at $\lambda \rightarrow 0$ as following: The attractive processes are singular perturbations of the non interacting electron gas. m is very demending to solve even for simple metals and is a function of multiple small details of the system. We aim here to aquire a qualitative knowledge. Let us now bring the ratio

$$\frac{2\Delta(T=0)}{k_B T_C} = \frac{2\pi}{e^\gamma} \approx 3.52$$

which is a universal ratio and does not depend anymore on the properties of the material. Knowing the critical temperature one can know the gap at 0K.

3 Bogoliubov-de Gennes Formalism

The Bogoliubov-de Gennes transformation allows us to express the hamiltonian in a diagonal way and express our quantities by looking at the eigenvectors of the hamiltonian. The resulting matrix is expressed in a huge space and is very sparse.

To give a taste of it, it will allow us to rewrite our hamiltonian as following

$$H = E_0 - \frac{1}{2} \tilde{c}^\dagger \tilde{H} \tilde{c}, \quad (43)$$

involving $\tilde{c} = (\hat{c}_1, \dots, \hat{c}_N)$, where each \hat{c}_i is a vector containing the creation and annihilation operators of a lattice site i : $\hat{c}_i = (c_{i\uparrow}, c_{i\downarrow}, c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger)$.

As we see we just describe each site with the four possible c -operators. This means for each lattice site, we have a 4×4 -submatrix that reflects the possible combinations of creation and anihinaltion operators of both spins. For the readability we are going to drop the comma between the site and spin indices.

For exemple if one has (without loss of generality) a chemical potential at the site i , then the hamiltonian is discribed in the following way:

$$H_{\mu i} = - \sum_{\sigma} \mu_i c_{i,\sigma}^\dagger c_{i,\sigma}$$

If we want to discribe it in therm of \hat{c}_i we have:

$$H_{\mu,i} = \hat{c}_i^\dagger \cdot \mu_i \mathbb{I}_4 \cdot \hat{c}_i = \begin{pmatrix} c_{i\uparrow}^\dagger \\ c_{i\downarrow}^\dagger \\ c_{i\uparrow} \\ c_{i\downarrow} \end{pmatrix} \cdot \mu_i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{i\uparrow} \\ c_{i\downarrow} \\ c_{i\uparrow}^\dagger \\ c_{i\downarrow}^\dagger \end{pmatrix}$$

Please not that in later discussions we will discuss about how the matrix must exhibit a kind of symmetry to respect the fermionic relations. This was just an example.

Depending on the interaction we wish to describe, we can figure out what combination of operators we want and design the 4×4 matrix accordingly. To achieve a full description of the system we can consider the interaction between to site i, j as a 4×4 matrix involving the \hat{c}_i^\dagger and c_j operators. Then we can build a huge matrix \tilde{H} based on 4×4 matrices at \tilde{H}_{ij} and the vector we multiply it to is just the \hat{c}_i^\dagger and c_j operators stack above one and other forming the above-introduced \tilde{c} vector. As a result, one gets the first formula introduced in this section 43. We can then compute the eigenvalues and -vectors express the quantities we're interested in. This is what we call the Bogoliubov-de Gennes transformation.

Now that the motivation is clear, we need to bring our Hamiltonian in a form that involves the fermionic operators $c_{i\sigma}$ and $c_{i\sigma}^\dagger$.

3.1 Tigh Binding Model

Our goal is now to fix our particle on lattice sites and describe their interactions. We are therefore going to translate our wavefunction formalism in an on site plus nearest neighbour description.

For the generalities, assume we have the Hamiltonian in the second quantisation formalism:

$$H = \sum_{\sigma\sigma'} \int \phi_\sigma^\dagger(\mathbf{r}) H_{\sigma\sigma'}(\mathbf{r}) \psi_{\sigma'}(\mathbf{r}) d^3r \\ + \sum_{\sigma\sigma'} \int \int \phi_\sigma^\dagger(\mathbf{r}) \phi_{\sigma'}^\dagger(\mathbf{r}') V_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') \phi_{\sigma'}(\mathbf{r}') \phi_\sigma(\mathbf{r}) d^3r' d^3r$$

We introduce a basis of so called Wannier orbitals $w(\mathbf{r} - \mathbf{R}_i)$ with \mathbf{R}_i an atom location. They should be large in the neighbourhood of \mathbf{R}_i and vanishes when the distance tends to infinity. They are therefore called "localised". The basis is complete, the orbitals verify the orthonormality condition:

$$\int w^*(\mathbf{r} - \mathbf{R}_i) w(\mathbf{r} - \mathbf{R}_j) d^3r = \delta_{ij}.$$

therefore we can define some field operator in this basis, based on creation and annihilation operators that acts on a lattice site i :

$$\phi_\sigma(\mathbf{r}) := \sum_i w(\mathbf{r} - \mathbf{R}_i) c_{i\sigma} \quad \phi_\sigma^\dagger(\mathbf{r}) := \sum_i w^*(\mathbf{r} - \mathbf{R}_i) c_{i\sigma}^\dagger \quad (44)$$

which is not a continuous description anymore. Inserting these operator back into our above Hamiltonian and using the orthonormality allows us to have an on site/nearest neighbour Hamiltonian. Taking for instance the first part of the Hamiltonian:

$$H = \sum_{\sigma\sigma'} \int \psi_\sigma^\dagger(\mathbf{r}) H_{\sigma\sigma'}(\mathbf{r}) \psi_{\sigma'}(\mathbf{r}) d^3r \\ = \sum_{ij\sigma\sigma'} c_{i\sigma}^\dagger c_{j\sigma'} \int w^*(\mathbf{r} - \mathbf{R}_i) H_{\sigma\sigma'}(\mathbf{r}) w(\mathbf{r} - \mathbf{R}_j) d^3r \\ := \sum_{i\sigma\sigma'} \epsilon_i^{\sigma\sigma'} c_{i\sigma}^\dagger c_{i\sigma'} - \sum_{\langle ij \rangle \sigma\sigma'} t_{ij}^{\sigma\sigma'} c_{i\sigma}^\dagger c_{j\sigma'} + \dots$$

In the last line we include a local energy term ϵ and the so called hopping term t_{ij} , which is the interaction with the nearest neighbour sites j of i . For a more precise description one could consider more neighbour. The spin dependent term can be used to describe spin orbit coupling or spin-flip processes.

We now aim to define the useful process for this thesis using this formalism.

3.1.1 Non-interacting electrons

The two main components of the non-interacting system Hamiltonian H_N are the chemical potential μ_i which is specific to each site and the hopping term t_{ij} . The chemical potential is modulated by the number of particles on the site i and the hopping term gives the amplitudes of moving an electron from site i to j . We assume it as spin-independent here.

$$H_N = - \sum_{i\sigma} \mu_i c_{i\sigma}^\dagger c_{i\sigma} - \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} \quad (45)$$

where $\langle ij \rangle$ is a commonly-used notation to sum over i and its nearest neighbours j , skipping $i = j$. We label it the normal Hamiltonian.

The hopping amplitude can be computed from the overlap of the orbitals under a kinetic operator $-\nabla^2/(2m)$, which explains the meaning “hopping”:

$$\begin{aligned} t_{ij} &= - \int w^*(\mathbf{r} - \mathbf{R}_i) \frac{\nabla^2}{2m} w(\mathbf{r} - \mathbf{R}_j) d^3r \\ &= + \frac{1}{2m} \int (\nabla w(\mathbf{r} - \mathbf{R}_i))^* (\nabla w(\mathbf{r} - \mathbf{R}_j)) d^3r. \end{aligned}$$

We used a partial integration considering the boundary conditions of the Wannier orbitals $w(\pm\infty) = 0$. Therefore one part of the partial integration vanishes and we integrate/differentiate the integrands in the other integral, leading to two ∇ s. Further we see that $t_{ij} = t_{ji}^*$ by swapping the two integrands.

3.1.2 Superconductivity

Previous study of ours on the superconductivity have led us to the following Hamiltonian:

$$H_S = - \int U(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\uparrow^\dagger(\mathbf{r}) \psi_\uparrow(\mathbf{r}) \psi_\downarrow(\mathbf{r}) d^3r$$

on which we can apply a mean field approximation $\Delta(\mathbf{r}) = U(\mathbf{r}) \langle \psi_\uparrow(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \rangle$. This yields to a common BCS-Hamiltonian for regular superconductors.

$$H_S = - \int \left(\Delta(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\uparrow^\dagger(\mathbf{r}) + \Delta(\mathbf{r})^* \psi_\uparrow(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \right) d^3r.$$

we see that the second integrand is just the complex conjugate of the first one. To spare some place, we are going to focus ourselves on the first one and denote its homologue with *h.c.* “hermitian conjugate”.

We insert 44 and obtain:

$$\begin{aligned} H_S &= - \sum_{ij} c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger \int \Delta(\mathbf{r}) w^*(\mathbf{r} - \mathbf{R}_i) w(\mathbf{r} - \mathbf{R}_j) d^3r + \text{h.c.} \\ &:= - \sum_{ij} \Delta_{ij} c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger + \text{h.c.} \end{aligned}$$

$\Delta(\mathbf{r})$ is an order parameter and doesn't vary too much in the coherence length, which is much bigger than the atomic length. Therefore we can say that the orbitals vary faster than the gap. Moreover these orbitals are peaked in the neighbourhood of the atomic location \mathbf{R}_i and \mathbf{R}_j . Achieving the integral we get $\Delta_{ij} = \Delta_i \delta_{ij}$. We can then reintroduce the *h.c.* and we get

$$H_S = - \sum_i \Delta_i c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger + \Delta_i^* c_{i\uparrow} c_{i\downarrow}. \quad (46)$$

We however we're missing the mean field term E_0 :

$$E_0 = \int U \langle \psi_\downarrow^\dagger \psi_\uparrow^\dagger \rangle \langle \psi_\uparrow \psi_\downarrow \rangle d^3r = \int U \frac{\Delta^*}{U} \frac{\Delta}{U} d^3r = \int \frac{|\Delta|^2}{U} d^3r.$$

and after applying the tight binding formalism we get:

$$E_0 = \sum_i \frac{|\Delta_i|^2}{U},$$

wich is a term we can add to the Hamiltonian 46. Form these equations we have the final Hamiltonian for the superconducting system:

$$H = E_0 + H_N + H_S.$$

3.2 A more symmertic Hamiltonian

As we introduced it while motivating the Bogoliubov-de Gennes formalism, we aspire to describe each state as a vector-matrix-vector product of

$$\hat{c}_i = \left(c_{i\uparrow}, c_{i\downarrow}, c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger \right).$$

However using the form we have in the superconducting 46 and normal 45 Hamiltonian will later not act as a fermionic operator upon the transformation we're about to do. We need to rewrite the Hamiltonian in a more symmertic way to later respect the anticommutation relations.

The chemical potential term can be expressed using the anticommutation relations of the fermionic operators $[c_{i\sigma}^\dagger, c_{i\sigma}]_+ = 1$:

$$\sum_{i\sigma} \mu_i c_{i\sigma}^\dagger c_{i\sigma} = \frac{1}{2} \sum_{i\sigma} \mu_i \left(c_{i\sigma}^\dagger c_{i\sigma} - c_{i\sigma} c_{i\sigma}^\dagger + 1 \right) \quad (47)$$

The trick we used is quite straight forward but not obvious. Taking two different states α and γ we have:

$$c_{i\alpha}^\dagger c_{i\gamma} = \frac{1}{2} c_{i\alpha}^\dagger c_{i\gamma} + \frac{1}{2} c_{i\alpha}^\dagger c_{i\gamma} = \frac{1}{2} c_{i\alpha}^\dagger c_{i\gamma} + \underbrace{\frac{1}{2} c_{i\alpha}^\dagger c_{i\gamma} + \frac{1}{2} c_{i\gamma} c_{i\alpha}^\dagger}_{\frac{1}{2} [c_{i\alpha}^\dagger, c_{i\gamma}]_+ = \frac{1}{2} \delta_{\alpha\gamma}} - \frac{1}{2} c_{i\gamma} c_{i\alpha}^\dagger. \quad (\mathfrak{T}1)$$

The hopping term can in the same way be expressed as:

$$\sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} = \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} \left(c_{i\sigma}^\dagger c_{j\sigma} - c_{j\sigma} c_{i\sigma}^\dagger \right).$$

we can take the liberty to reorder the indicies in a term of a sum and use the fact that $t_{ij} = t_{ji}^*$:

$$\sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} = \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - t_{ij} c_{i\sigma} c_{j\sigma}^\dagger = \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - t_{ij}^* c_{i\sigma} c_{j\sigma}^\dagger. \quad (48)$$

The superconducting term for its part takes the form:

$$\sum_i \Delta_i c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger + \Delta_i^* c_{i\uparrow} c_{i\downarrow} = \frac{1}{2} \sum_i \Delta_i \left(c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger - c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \right) + \Delta_i^* (c_{i\uparrow} c_{i\downarrow} - c_{i\downarrow} c_{i\uparrow}). \quad (49)$$

We then finish this section by using Eq.47, 48 and 49 in the Hamiltonian and obtain the following form:

$$\begin{aligned} H = E_0 & - \frac{1}{2} \sum_{i\sigma} \mu_i \left(c_{i\sigma}^\dagger c_{i\sigma} - c_{i\sigma} c_{i\sigma}^\dagger \right) \\ & - \frac{1}{2} \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - t_{ij}^* c_{i\sigma} c_{j\sigma}^\dagger \\ & - \frac{1}{2} \sum_i \Delta_i \left(c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger - c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \right) + \Delta_i^* (c_{i\uparrow} c_{i\downarrow} - c_{i\downarrow} c_{i\uparrow}). \end{aligned} \quad (50)$$

The constant term $\frac{1}{2} \sum_{i\sigma} \mu_i$ of the normal Hamiltonian just vanished in the E_0 . [right?](#) We can now rewrite the Hamiltonian in a more compact way:

$$H = E_0 - \frac{1}{2} \sum_{i,j} \hat{c}_i^\dagger \hat{H}_{ij} \hat{c}_j \quad (51)$$

where the on site matrix reads

$$\hat{H}_{ij} = \begin{pmatrix} \mu_i \mathbb{I}_2 \delta_{ij} + t_{ij} & -i\sigma_2 \Delta_i \delta_{ij} \\ i\sigma_2 \Delta_i^* \delta_{ij} & -\mu_i \mathbb{I}_2 \delta_{ij} - t_{ij}^* \end{pmatrix} = \begin{pmatrix} H_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -H_{ij}^* \end{pmatrix} \quad (52)$$

where we use \mathbb{I}_n as an n -dimensional identity matrix. We haven't explicitly removed the t_{ij} if we're not considering nearest neighbours. At this point it's interesting to note that if we wish to build some periodic boundary conditions, it might be the case that a site on side is neighbour with a site on the other side.

We can further compress our \hat{c}_i operator by introducing

$$\tilde{c} = (\hat{c}_1, \dots, \hat{c}_N)$$

along with the system Hamiltonian-matrix $\tilde{H}_{ij} := \hat{H}_{ij}$ which allows us to rewrite the Hamiltonian 51 as:

$$H = E_0 - \frac{1}{2} \tilde{c}^\dagger \tilde{H} \tilde{c}. \quad (53)$$

3.3 Eigenvalues

We now have a look at the following eigenvalue problem, which later helps from the diagonalization of the Hamiltonian:

$$\tilde{H} \tilde{\chi}_n = E_n \tilde{\chi}_n \quad (54)$$

n runs over the number of the eigenvalue and $\tilde{\chi}_n$ is the corresponding eigenvector. we can decompose the $\tilde{\chi}_n$ to reflect each lattice site: $\tilde{\chi}_n = (\hat{\chi}_{n1}, \dots, \hat{\chi}_{nN})$. This means $\chi_{n,i}$ refers to a 4×4 block, i.e. the on the submatrix we had earlier talked about. Therefore this $\chi_{n,i}$ contains four values, grouped in two vectors of length two, one for each spin: $\chi_{n,i} = (u_{ni}, v_{ni})$. Further $u_{ni} = (u_{ni\uparrow}, u_{ni\downarrow})$ couples to the two first components $(c_{i\uparrow}, c_{i\downarrow})$ we had in \hat{c} and similarly $v_{ni} = (v_{ni\uparrow}, v_{ni\downarrow})$ to the two last components $(c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger)$ of the four operator \hat{c} .

We can simplify the eigenvalue problem by taking a look only at a site i . We then only sum up over i .th row of \tilde{H}_{ij} with the components of $\tilde{\chi}_n$:

$$\sum_{j \in [N]} \hat{H}_{ij} \hat{\chi}_{nj} = E_n \hat{\chi}_{ni}.$$

We remember that \tilde{H}_{ij} represent a complex scalar and \hat{H}_{ij} is a 4×4 matrix with complex entries. So it follows by reintroducing 52 the following set of equations:

$$\begin{cases} \sum_{j \in [N]} H_{ij} u_{nj} + \Delta v_{nj} = E_n u_{nj} \\ \sum_{j \in [N]} \Delta^\dagger u_{nj} - H_{ij}^* v_{nj} = E_n v_{nj} \end{cases} \xrightarrow{(1)} \begin{cases} \sum_j H_{ij} u_{nj} + \Delta v_{nj} = E_n u_{nj} \\ \sum_j H_{ij} v_{nj}^* + \Delta^\dagger u_{nj}^* = -E_n v_{nj}^* \end{cases} \quad (55)$$

Where in (1) we took the conjugate of the second equation and used $\Delta^\dagger = -\Delta^*$. This is an important result, because it shows that if $\tilde{\chi}_n = (u_{n1}, v_{n1}, u_{n2}, v_{n2}, \dots)$ is an eigenvector with eigenvalue E_n , then so should be $(v_{n1}^*, u_{n1}^*, v_{n2}^*, u_{n2}^*, \dots)$ with the eigenvalue $-E_n$.

This leads to a symmetry in the energy spectrum of $H = E_0 \pm \frac{1}{2} \tilde{c}^\dagger \tilde{H} \tilde{c}$. This flexibility allows us to choose the version of H with the positive sign, which is more commonly used.

3.4 Diagonalization

Our goal is now to express the Hamiltonian relative to its energy eigenvalues, which is more practical to work with. As we have seen in the last section, eigenvectors χ_n allows us to compute the energies. Therefore we are going to diagonalize the Hamiltonian by using the eigenvectors χ_n to express the Hamiltonian according to its eigenvalues.

First we define a row-vector of our eigenstate $\check{X} = [\check{\chi}_{\pm 1}, \dots, \check{\chi}_{\pm 2N}]$ and introduce a diagonal matrix $\check{D} = \text{diag}(E_{\pm 1}, \dots, E_{\pm 2N})$ with the eigenvalues. Then we can write the Hamiltonian as:

$$\check{H} = \check{X} \check{D} \check{X}^{-1} = \check{X} \check{D} \check{X}^\dagger$$

we can then transform the Hamiltonian with $\check{c} := \check{X} \gamma$

$$\begin{aligned} H &= E_0 - \frac{1}{2} \check{c}^\dagger \check{H} \check{c} = E_0 - \frac{1}{2} \gamma^\dagger \check{X}^\dagger \check{H} \check{X} \gamma \\ &= E_0 - \frac{1}{2} \gamma^\dagger \underbrace{\check{X}^\dagger \check{X}}_{=\mathbb{I}} \underbrace{\check{D} \check{X}^{-1} \check{X}}_{=\mathbb{I}} \gamma \\ &= E_0 - \frac{1}{2} \gamma^\dagger \check{D} \gamma \\ &= E_0 - \frac{1}{2} \sum_{n \in \mathcal{N}} \end{aligned}$$

where $\mathcal{N} = \{\pm n : n \in \llbracket N \rrbracket\}$. Reagrranging the transformation of \check{c} we get $\gamma = \check{X}^\dagger \check{c}$. Now that we've made the structure of the involved variables clear in the last section, we find the expression of the γ wich is $2N$ -dimensional:

$$\begin{aligned} \gamma_n &= \sum_i \left(u_{ni\uparrow}^* c_{i\uparrow} + v_{ni\uparrow}^* c_{i\uparrow}^\dagger + u_{ni\downarrow}^* c_{i\downarrow} + v_{ni\downarrow}^* c_{i\downarrow}^\dagger \right) \\ &= \sum_{i\sigma} \left(u_{ni\sigma}^* c_{i\sigma} + v_{ni\sigma}^* c_{i\sigma}^\dagger \right) \end{aligned}$$

and due to the symmerty we saw erlier,

$$\gamma_{-n} = \sum_{i\sigma} \left(v_{ni\sigma} c_{i\sigma} + u_{ni\sigma} c_{i\sigma}^\dagger \right)$$

for $n \in \llbracket N \rrbracket$. We now take a look at the conjugate transpose of γ_{-n} . Because scalar are dimension 1×1 we have $(uc^\dagger)^\dagger = (c^\dagger)^\dagger u^\dagger = c^\dagger u^* = u^* c$ and it follows:

$$\gamma_{-n}^\dagger = \sum_{i\sigma} \left(v_{ni\sigma}^* c_{i\sigma}^\dagger + u_{ni\sigma}^* c_{i\sigma} \right) = \gamma_n.$$

Using this we can link each γ_i to the corresponding eigenvalue E_i : γ_n to the corresponding eigenvalue E_n and γ_{-n} to the corresponding eigenvalue $E_{-n} = -E_n$. We recall that we had $2N$ degrees of freedom $c_{i\sigma}$ due to the spins and after the transformation we get $4N$ degrees into \hat{c}_i . But because our energies E_n and E_{-n} are realted to eachother, we can keep the positive $2N$ eigenvalues and this maintain the total number of degree of freedom.

We can split the sum over the $n \in \mathcal{N}$ in two parts: $\mathcal{N}_+ = \{n \in \mathcal{N} : n > 0\}$, $\mathcal{N}_- = \{n \in \mathcal{N} : n < 0\}$

$$\begin{aligned} H &= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n + \frac{1}{2} \sum_{n \in \mathcal{N}_-} E_n \gamma_n^\dagger \gamma_n \\ &= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_{-n} \gamma_{-n}^\dagger \gamma_{-n} \\ &= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n - \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_{-n}^\dagger \gamma_{-n} \\ &= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n^\dagger \gamma_n - \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n \gamma_n \gamma_n^\dagger \\ &= E_0 + \frac{1}{2} \sum_{n \in \mathcal{N}_+} E_n (\gamma_n^\dagger \gamma_n - \gamma_n \gamma_n^\dagger) \end{aligned}$$

where with used the energy symmetry and $\gamma_{-n}^\dagger = \gamma_n$, $\gamma_{-n} = \gamma_n^\dagger$.

Using this knowledge, we can express a final formula for the Hamiltonian by using the anti-commutation properties of the fermionic γ -operators: $[\gamma_n^\dagger, \gamma_n]_+ = 1$, so using the trick $\Im \mathbf{r} 1$ and bringing the $\frac{1}{2}$ prefactor in the sum:

$$H = E_0 - \sum_{n \in \llbracket N \rrbracket} E_n \left(\gamma_n^\dagger \gamma_n - \frac{1}{2} \right). \quad (56)$$

This is the final form of the Hamiltonian in the Bogoliubov-de Gennes formalism. As a user one should build the Hamiltonian and computes its eigenvalues, -vector and transform them into the γ operators.

3.4.1 Superconducting Gap

We already covered how the superconducting gap Δ is a relevant property of the Meissner state. We now aim to use the mean field theorie in order to find the gap. This requires a self consistency equation, which we can be derived from the Hamiltonian.

The gap was defined as $\Delta(\mathbf{r}) := U(\mathbf{r}) \langle \psi_\uparrow(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \rangle$. Back to the tight binding formalism, the gap now depends on the lattice site i and reads $\Delta_i = U_i \langle c_{i\uparrow} c_{i\downarrow} \rangle$ and we can express $c_{i\sigma}$ in terms of the γ -operators:

$$\begin{aligned} c_{i\sigma} &= \sum_{n \in \mathcal{N}} u_{ni\sigma} \gamma_n \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma} \gamma_n + u_{-n,i\sigma} \gamma_{-n} \quad (57) \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma} \gamma_n + v_{ni\sigma}^\dagger \gamma_n^\dagger \end{aligned} \quad \begin{array}{c} \circ \\ | \\ \circ \end{array} \quad \begin{aligned} c_{i\sigma}^\dagger &= \sum_{n \in \mathcal{N}_+} (u_{ni\sigma} \gamma_n)^\dagger + (v_{ni\sigma}^\dagger \gamma_n^\dagger)^\dagger \\ &= \sum_{n \in \mathcal{N}_+} \gamma_n^\dagger u_{ni\sigma}^\dagger + \gamma_n (v_{ni\sigma}^\dagger)^\dagger \quad (58) \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma}^* \gamma_n^\dagger + v_{ni\sigma} \gamma_n \end{aligned}$$

where we used the symmetry of the eigenvectors. We can now compute expectation value involved in the gap:

$$\begin{aligned} \langle c_{i\uparrow} c_{i\downarrow} \rangle &= \sum_{n, m \in \mathcal{N}_+} \langle (u_{ni\uparrow} \gamma_n + v_{ni\uparrow}^\dagger \gamma_n^\dagger) (u_{mi\downarrow} \gamma_m + v_{mi\downarrow}^\dagger \gamma_m^\dagger) \rangle \\ &= \sum_{n, m \in \mathcal{N}_+} \langle (u_{ni\uparrow} u_{mi\downarrow} \gamma_n \gamma_m + u_{ni\uparrow} v_{mi\downarrow}^* \gamma_n \gamma_m^\dagger + v_{ni\uparrow}^* u_{mi\downarrow} \gamma_n^\dagger \gamma_m + v_{ni\uparrow}^* v_{mi\downarrow}^* \gamma_n^\dagger \gamma_m^\dagger) \rangle \\ &\stackrel{(*)}{=} \sum_{n \in \mathcal{N}_+} \langle u_{ni\uparrow} v_{ni\downarrow}^* \gamma_n \gamma_n^\dagger \rangle + \langle v_{ni\uparrow}^* u_{ni\downarrow} \gamma_n^\dagger \gamma_n \rangle \quad (59) \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{ni\downarrow}^* \langle \gamma_n \gamma_n^\dagger \rangle + v_{ni\uparrow}^* u_{ni\downarrow} \langle \gamma_n^\dagger \gamma_n \rangle \\ &= \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{ni\downarrow}^* (1 - f(E_n)) + v_{ni\uparrow}^* u_{ni\downarrow} f(E_n) \end{aligned}$$

where f is the Fermi-Dirac distribution. In $(*)$ we notice no $\gamma\gamma$ or $\gamma^\dagger\gamma^\dagger$ terms in the Hamiltonian, so their expectation value is zero ¹.

The expectation value $\langle a \hat{A} \rangle_\Phi$ of a scalar times an operator reads $\langle \Phi | a \hat{A} | \Phi \rangle_\Phi = a \langle \Phi | \hat{A} | \Phi \rangle_\Phi = a \langle \hat{A} \rangle_\Phi$. To convince ourselves, we just take a look at the first quantisation expression of this braket. This result leads to the self consistency equation:

$$\Delta_i = U_i \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{ni\downarrow}^* (1 - f(E_n)) + u_{ni\downarrow} v_{ni\uparrow}^* f(E_n) \quad (60)$$

We plan to solve this equation numerically, inserting some guess in the Hamiltonian, diagonalize it, update Δ_i and reinsert it into H and repeat until we reach a fixpoint.

□ _____ □

¹This is like the expectation value of killing twice a fermion in a state. It is not possible, because we can't annihilate a state that has a possession number of zero. And in the same way due to the Pauli-principle we can't have more than one particle in the same state, so $\langle \gamma^\dagger \gamma^\dagger \rangle = 0$. We are not finding these terms in the Hamiltonian. Here we additionally removed the indices, in fact the diagonalization also takes place on the indices so that we end just with n . The Hamiltonian is diagonal in $\gamma\gamma^\dagger$ and $\gamma^\dagger\gamma$ [right?](#)

4 Altermagnetism

4.1 Introduction and overview

The reader might already be familiar with ferromagnets and all the regular magnetic models. Taking into account the spin and wave vector of each particle in a site, one can derive some symmetries under transform operations. For example a ferromagnet is symmetric under spin-flip (also called time reversal) and a rotation. In the example of the antiferromagnetism, we have two sub-lattice with opposite spins. In such systems the spin compensate each other resulting in a null magnetism. The system is symmetric under spin-flip and translation and was theorised in 1948 by Louis Néel as the Néel antiferromagnets. Allowing more complexity one could imagine multiple ions in a unit cell linked by rotation, screw, etc symmetries. The half ions of the cell owns an opposite spin from the other half. These material are labeled as zero q antiferromagnets. Both of these material keep the same electron spectra under such transformations. The spectra compoente eachother resulting in a zero net magnetisation. This section summerises the historical work of [4].

Altermagnets implement two or more sub-lattice which are not related between eachother by translation nor inversion but rotation. These class of material was pinpointed in 2019. These material are similar to complexer structure where the sublattices aren't linked with the crystal symmetries. Therefore summing up the spin might not result in a trivial expression like the antiferromagnetic material. In fact the overall spin projection might almost be zero but not exactly.

However studying half metal and insulator leads to different result. Half metal can be seen as an insulator in one spin channel but not both [4]. The Quin Luttinger's theorem, showed that such substance must have an integer number of Bohr spin magnetic moment. Therefore structures with a small net magnetisation see these quantity be flooTamYellow to zero.

On the search of pratical applciation, reaseachers have found (an exaustive list is provided in [4]) that domain like tunneling magnetoresistance (TMR) are limited under the properties of the material. For instance the actual use of ferromagnets limits the frequency to the gigahertz range. This has to deal with the ferromagnetic resonance that connects the magnetisation with electromagnetic waves. The altermagnets could be used in the terahertz range. Mentioning that TMR is a key component of the magnetic random access memory (MRAM), the reader can easly estimate the performance improvment such upgrade could deliver in a computer.

In a more formal way, we can distinguish two types of altermagnet. In the first type, the altermagnet's lattice site have different distance to the neighbours depending on the linking axis and the spin of the particle in the site. This is illustrated in (a). On the other hand we can consider a unit cell beeing unsymmetric in its ion position. We consider a square lattice were each unit cell has a non magnetic ion and two magnetic ion with opposite spin. Please consider the second schema (b).

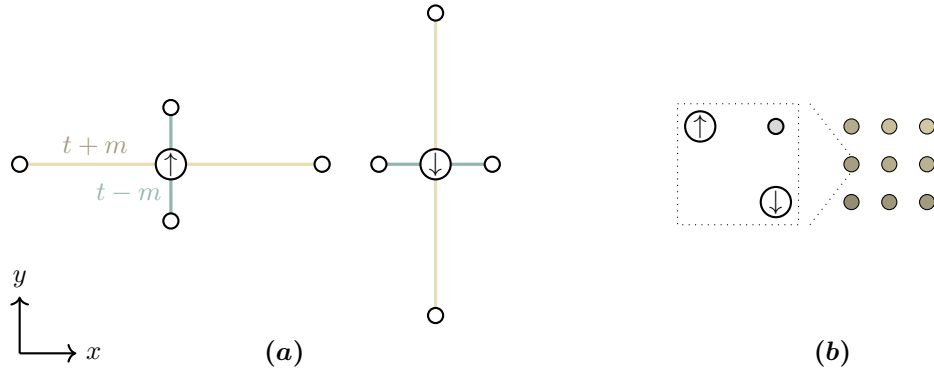


Figure 5: Altermagnet of type 1 and 2.

4.2 Symmetries

After giving an introduction on the geometric background of altermagnet we can now focus on the symmetries of the material. In this section we are going to focus on two simple transformation. The inversion P and the spin flip (time reversal) T .

$$\begin{aligned} P : \mathbf{r} &\mapsto -\mathbf{r} & \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ T : t &\mapsto -t \hat{=} \sigma \mapsto -\sigma & \mathbb{R} &\rightarrow \mathbb{R} \end{aligned}$$

We can first assume that the energy ϵ of the particle (or band structure) depends on the spin σ and the wave vector \mathbf{k} . Knowing $\mathbf{k} = d\mathbf{r}/dt$ leads to

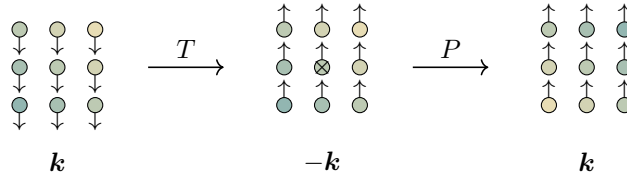
$$\begin{aligned} P(\epsilon(\mathbf{k}, \sigma)) &= \epsilon(-\mathbf{k}, \sigma) \\ T(\epsilon(\mathbf{k}, \sigma)) &= \epsilon(-\mathbf{k}, -\sigma) \end{aligned}$$

We observe the PT operation $P \circ T$. If the system is PT symmetric one could get $\epsilon(\mathbf{k}, \sigma) = P \circ T(\epsilon(\mathbf{k}, \sigma))$. Assuming it's the case we can write

$$\epsilon(\mathbf{k}, \sigma) = P \circ T(\epsilon(\mathbf{k}, \sigma)) = \epsilon(\mathbf{k}, \downarrow)$$

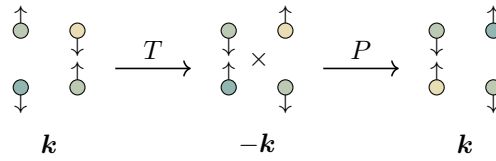
and therefore the PT -symmetric systems are spin-degenerated. We have the same energy for a given momentum at two opposite spins. Reciprocely this means that the existence of $\epsilon(\mathbf{k}, \sigma)$ is followed by an observation of $\epsilon(\mathbf{k}, -\sigma)$.

A few examples We can imagine having a small section for the seek of readability. One should imagine the point where we apply the transformation in a the center of a 6×6 lattice, where we only show the second quadrant. We consider a ferromagnetic lattice where we apply the space inversion in the middle where the black cross is.

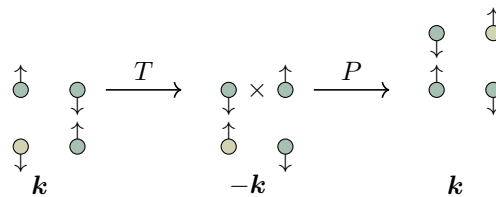


Letting the spin points upwards again will require to use another operation. Therefore the ferromagnet is not PT -symmetric and we observe no spin degeneracy as the equation shows it.

The antiferromagnet has an spin switch in every directions. If, as the ferromagnet, we pick a lattice point to apply the PT transformation to, we observe that this doesn't lead to a PT symmetry



However if we apply the inversion in the inbetween we observe something different.



This point we apply the P transformation around makes the antiferromagnets having a PT symmetry. We see that the lattice site have moved but only the spin and the wave vector takes

part to the band ϵ . We therefore have a degeneracy in the energy spectrum. Finding at least one point where the PT symmetry works makes the system PT symmetric.

If we now take a look at an altermagnet pictured in Fig.??, we have to different models to test. In the first one **(a)** the effect of the time reversal T makes the hopping change $m \rightarrow -m$. Then we find no point to invert the system around so that this first model doesn't have a PT symmetry. For the second one **(b)** we can first invert all spin but afterwards inverting the space doesn't bring back the system in the original configuration. In fact neither applying the P after T on the lattice point nor on the space inbetween leads to a the lattice we began with. In both case the degeneracy is lifted, we have different energies for the different spin-orientation.

Beyond the scope We refer as spin-orbit coupling (SOC) the interaction between the spin of the electron and the orbital motion. When this is disregarded we observe a rotation symmetry around the spins. A rotation R can turn the spin back after the application of T resulting in a RT symmetry. $RT(\epsilon(\mathbf{k}, \sigma)) = \epsilon(-\mathbf{k}, \sigma)$ must equal $\epsilon(\mathbf{k}, \sigma)$.

If we now consider a collinear magnet without SOC, we have a RT but no PT symmetry. Under such circumstances the spin-splitting must be symmetric under $\epsilon(-\mathbf{k}, \sigma) = \epsilon(\mathbf{k}, \sigma)$ which describes an even band structure. Such systems are called inversion symmetric altermagnets.

4.3 Implementation of an altermagnet

After introducing the basic properties of an altermagnet we now aim to describe this system with the already introduced formalism. We are going to consider the first model involving a spin-dependent hopping. The Hamiltonian is given by

$$H_{AM} = - \sum_{\langle i,j \rangle \sigma \sigma'} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma \sigma'} c_{i\sigma}^\dagger c_{j\sigma'}$$

involving $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ the Pauli matrices. \mathbf{m}_{ij} is the actual spin dependent hopping term. If the connection line between the two sites i and j lays on the \mathbf{e}_x axis we have $\mathbf{m}_{ij} = m(0, 0, 1)$ and $\mathbf{m}_{ji} = m(0, 0, -1)$ along \mathbf{e}_y with m the desired hopping amplitude. \mathbf{m}_{ij} scales and masks the pauli matrices.

Since the BdG formalism uses fermionic operators we again need to bring the matrix in a way that reflects the fermionic properties regarding to the \tilde{c} operator. We can remake use of Eq.??:

$$H_{AM} = - \sum_{\langle i,j \rangle \sigma \sigma'} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma \sigma'} \frac{1}{2} \left(c_{i\sigma}^\dagger c_{j\sigma'} - c_{i\sigma'} c_{j\sigma}^\dagger \right).$$

For the last summand we splitted the sum, exchanged the labeling of the states i and j and recombined the sums. However due to the matrix element in the front that we want to factorise with, we can't exchange the labelings of σ and σ' . Bringing this Hamiltonian in the Nambu formalism we obtain by defining $(\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma \sigma'} = \mathcal{M}_{\sigma \sigma'}$ the following Hamiltonian:

$$H_{ij}^{(AM)} = \frac{1}{2} \begin{pmatrix} -\mathcal{M} & \mathcal{O} \\ \mathcal{O} & \mathcal{M}^T \end{pmatrix}, \quad \text{where} \quad \mathcal{M} = \begin{pmatrix} \mathcal{M}_{\uparrow\uparrow} & \mathcal{M}_{\uparrow\downarrow} \\ \mathcal{M}_{\downarrow\uparrow} & \mathcal{M}_{\downarrow\downarrow} \end{pmatrix}.$$

Making use of this matrix in an altermagnetic lattice point, we can generate the according eigenvectors and -values. We can then study how the physical quantities behave in this material.

5 Simulations and ..

In the last chapters we introduced the theoretical background of superconductivity and altermagnets. The goal of this thesis is to use numerical methods to help us find some properties of the system. We remind here that all the code is available on a GitHub repository at

<https://github.com/Tamwyn001/Thesis>.

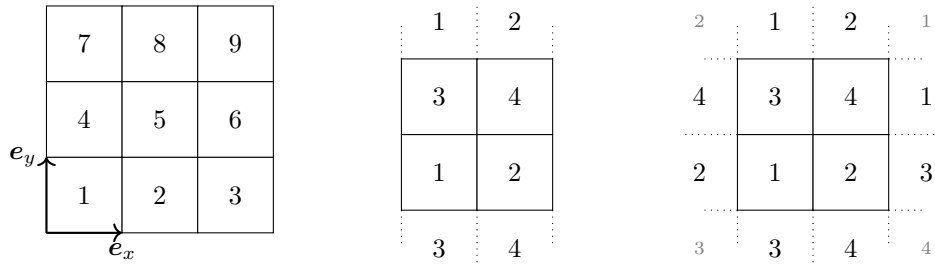
The repository contains the matlab code (.m) used to generate the data in the form of (coordinate, value) files (.dat). Gnuplot scripts (.gp) allow us to produce some plots that we can then directly link to the Latex source code of this thesis. Finally some powershell scripts (.ps1) automate the gnuplot-latex process.

The matlab code contains two important files. The system.m file is a class that describes which material we are using for the lattice, the physical context like the temperature and the chemical potential as well as the use or not of periodic boundary conditions. It contains also the Hamiltonian of the system and keeps track of all lattice sites.

The LatticeSites.m describes the position of a lattice site, stores its neighbours and monitors the lattice site specific physical quantities. These quantities can be the superconducting gap or the current, etc.

We introduced earlier the need of a self-consistent solution for the gap to correctly describe the Hamiltonian. This is achieved in the GapEquation.m file which is the main script we run. We start with a guess for Δ_i and introduce it in the Hamiltonian. This guess can depend on the material, or the site if we aim to introduce some phase shift at specific locations. We then derive a new site-dependent value using 60 for all sites and reinsert it into the Hamiltonian. This requires to get eigenvectors and values of the Hamiltonian. This loop will be performed until the difference of each gap with its previous value reaches a threshold. We then have a proper Hamiltonian and can start to derive some values like the current. Matlab is a program that works well with matrices so using it is a wise choice.

The site coordinate is stored as (x, y) -tuple as well as an index i that iterates over y -row and then starts again at the beginning of the next row. This means that with different boundary conditions we get the following shape for the system:



Left we have a 3×3 -system. The others are 2×2 with vertical periodic boundary conditions (PBC) and on the right with both vertical and horizontal PBC.

5.1 The non-trivial choice of the parameters

We want to describe a system including Cooper pairs. We already saw how the thermal fluctuations can break them. We pick a very low temperature of 10^{-3}K . For the chemical potential we need to pay closer attention. The band structure formula is given as

$$\epsilon_{\mathbf{k}} = -2t (\cos(k_x a) + \cos(k_y a)) - \mu$$

The Fermi surface is given by $\epsilon_{\mathbf{k}} = 0$ when the chemical potential is included which means

$$-\frac{\mu}{2t} = (\cos(k_x a) + \cos(k_y a))$$

We can find a solution to this equation if

$$-4t \leq \mu \leq 4t.$$

In order to have a properly filled system we are going to choose $\mu = 3.75t$.

5.2 Currents

Using the charge conservation and the Heisenberg picture we are going to derive an expression for the current in the lattice. The system we are taking into account is two dimensional.

The charge conservation reads

$$\partial_t \rho_i = -\nabla \cdot \mathbf{j}_i$$

and identifies the time variation of the charge density on the site i as the negative divergence of the current density. Now, performing some transformations, we bring this expression in a more useful form. The goal here is to integrate on both side over our two-dimensional surface Ω . For the charge density this yields to the charge at a site:

$$\int_{\Omega} \partial_t \rho_i d\mathbf{r} = \partial_t Q_i.$$

For the current density we can use the Gauß law to change the integration set:

$$\int_{\Omega} \nabla \cdot \mathbf{j}_i d\mathbf{r} = \int_{\partial\Omega} \mathbf{j}_i \cdot \mathbf{n} = \sum_n J_{i,n} a = \sum_n I_{i,n}$$

where $\partial\Omega$ is the boundary of Ω . The normal vector \mathbf{n} points in the 2D-plane, outward from the boundary. Assuming we have a square lattice, we can assign to each lattice site a square unit cell with side length a . The sum over n happens to be over all the side.

Now introducing the Heisenberg picture with $\hbar = 1$ we get

$$\partial_t Q_i = i[H, Q_i].$$

Finlay we can introduce the second quantisation in the charge:

$$Q_i = \sum_{\sigma} c_{i\sigma}^{\dagger} c_{i\sigma} = \sum_{\sigma} n_{i,\sigma}$$

which is quite trivial, summing over all the particle at a site leads the charge of the site. After putting all together, this yields

$$I_i^{+x} + I_i^{+y} + I_i^{-x} + I_i^{-y} = -i \left[H, \sum_{\sigma} n_{i,\sigma} \right] \quad (61)$$

This mean the last step to perform is to compute the commutator of the different terms of the Hamiltonian with the charge at a site i .

We remind here that our Hamiltonian contains a chemical potential, a hopping, a superconducting and an altermagnetic term.

The hopping term We set remember the use of a constant hopping amplitude $t_{ij} = t$.

$$\left[\sum_{\langle ij \rangle \sigma} c_{i\sigma}^{\dagger} c_{j\sigma}, \sum_{\sigma} n_{l\sigma} \right] = \sum_{\langle ij \rangle \sigma \sigma'} c_{i\sigma}^{\dagger} c_{j\sigma} n_{l\sigma'} - n_{l\sigma'} c_{i\sigma}^{\dagger} c_{j\sigma}$$

We can then introduce a useful trick that involves the commutator $[n_{\mu}, c_{\nu}] = -\delta_{\mu\nu} c_{\mu}$

$$\begin{aligned} c_{i\sigma}^{\dagger} c_{j\sigma} n_{i,\sigma'} &= c_{i\sigma}^{\dagger} \left(\underbrace{c_{j\sigma} n_{l\sigma'} - n_{l\sigma'} c_{j\sigma}}_{-[n_{l\sigma'}, c_{j\sigma}]} + n_{l\sigma'} c_{j\sigma} \right) \\ &= c_{i\sigma}^{\dagger} (\delta_{\sigma'\sigma} \delta_{lj} c_{i\sigma'} + n_{l\sigma'} c_{j\sigma}). \end{aligned}$$

Following the same schema we derive the other part of the commutator. Here the expressions involves $[n_{\mu}, c_{\nu}^{\dagger}] = \delta_{\mu\nu} c_{\mu}^{\dagger}$:

$$\begin{aligned} n_{l\sigma'} c_{i\sigma}^{\dagger} c_{j\sigma} &= \left(\underbrace{n_{l\sigma'} c_{i\sigma}^{\dagger} - c_{i\sigma}^{\dagger} n_{l\sigma'}}_{[n_{l\sigma'}, c_{i\sigma}^{\dagger}]} + c_{i\sigma}^{\dagger} n_{l\sigma'} \right) c_{j\sigma} \\ &= (\delta_{\sigma'\sigma} \delta_{li} c_{j\sigma'}^{\dagger} + c_{i\sigma}^{\dagger} n_{l\sigma'}) c_{j\sigma} \end{aligned}$$

After subtracting the second term from the first one we are left with

$$\begin{aligned} \left[\sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma}, \sum_{\sigma} n_{i\sigma} \right] &= \sum_{\langle ij \rangle \sigma \sigma'} \delta_{\sigma' \sigma} \left(\delta_{lj} c_{i\sigma}^\dagger c_{j\sigma'} - \delta_{li} c_{i\sigma'}^\dagger c_{j\sigma} \right) \\ &= \frac{1}{2} \sum_{i \delta \sigma \sigma'} \delta_{\sigma' \sigma} \left(\delta_{l, i+\delta} c_{i\sigma}^\dagger c_{i+\delta, \sigma'} - \delta_{li} c_{i\sigma'}^\dagger c_{i+\delta, \sigma} \right) \end{aligned}$$

Because of the squared lattice we can summerise the neighbour set $\langle ij \rangle$ to $\{i + \delta_x, i - \delta_x, i + \delta_y, i - \delta_y\}$ involving δ_{axis} the displacmnt from the site to neighbour one along the given axis. This is abstracted in $i + \delta$. We obtain after summing up over the σ' and i , and writing explictely everything we obtain

$$\begin{aligned} &= \frac{1}{2} \sum_{\sigma} (c_{l-\delta_x, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l+\delta_x, \sigma}) + (c_{l+\delta_x, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l-\delta_x, \sigma}) \\ &\quad + (c_{l-\delta_y, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l+\delta_y, \sigma}) + (c_{l+\delta_y, \sigma}^\dagger c_{l, \sigma} - c_{l\sigma}^\dagger c_{l-\delta_y, \sigma}), \end{aligned} \tag{62}$$

where the one half factor avoids summing twice over the nearest neighbours. This proportional to the current at a site l . The current of one side of the square unit cell is represented by a pair of $c_{l\pm\delta_x, \sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l\pm\delta_x, \sigma}$ where the same displacement δ_x is involved. This will be an important consideration when coming to the altermagnet.

Chemical potentail term For the chemical potential term it is useful to introduce that the commutator between two number operator vanishes. Since the charge and the chenical potential operators involves only number operator, we find that this term dont't take part to the current.

Superconducting term The superconduting term has a particular behaviour. If one can sovl the gap (self consistently), this term doesn't contribute to the current as we are going to see. We first form the commutator between the Hubbard term and the charge operator:

$$\left[\sum_i \left(\Delta_i c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + \Delta_i^* c_{i\uparrow} c_{i\downarrow} \right), \sum_{\sigma} n_{l, \sigma} \right]$$

again make the already introded commutator appear, we obtain

$$\begin{aligned} &= \sum_{i\sigma} \delta_{il} \left(\Delta_i^\dagger (\delta_{\sigma\downarrow} + \delta_{\sigma\uparrow}) c_{i\downarrow} c_{i\uparrow} - \Delta_i (\delta_{\sigma\downarrow} + \delta_{\sigma\uparrow}) c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \right) \\ &= 2 \left(\Delta_i^\dagger c_{i\downarrow} c_{l\uparrow} - \Delta_i c_{l\uparrow}^\dagger c_{l\downarrow}^\dagger \right) \end{aligned}$$

This expression mixes creation and annihilation operators and makes it hard to recognise a current. One can view this term as a source [need citation?](#) meaning we havw a new term \mathcal{C}_i that appears in the equation

$$-\partial_t Q_i = \mathcal{C}_i + \sum_n I_{i,n}.$$

To know the contribution to the current we can investigate the rate of charge generation of this term. This is achieved by taking the quantum expectation and the thermal average of the system.

$$2 \left(\Delta_l^\dagger \langle c_{l\downarrow} c_{l\uparrow} \rangle - \Delta_l \langle c_{l\uparrow}^\dagger c_{l\downarrow}^\dagger \rangle \right) = \frac{2}{U_l} \left(-\Delta_l^\dagger \Delta_l + \Delta_l \Delta_l^\dagger \right)$$

This is achieved using the anticommutation relation of the fermionic operators Eq. [\S{eq1}](#) and Eq. [\S{eq2}](#). We have as well $\Delta_l^\dagger = U_i^* \langle (c_{l\uparrow} c_{l\downarrow})^\dagger \rangle = U_i \langle c_{l\downarrow}^\dagger c_{l\uparrow}^\dagger \rangle$. The dagger operator is asumed to be inserted into the expectation value using the linearity of the integral defining $\langle \cdot \rangle$.

Solving Δ self-consistently will lead to $\Delta \Delta^\dagger = \Delta^\dagger \Delta$ and therefore we find that the value vanishes.

Altermagnetic term As we introduced it in a previous discussion, the altermagnetic term is more complicated than the last one treated and need more work. In essence we describe an advanced hopping term, that changes regarding of the hopping axis. We can first bring the commutator where the matrix element $(\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'}$ is a scalar:

$$\begin{aligned} \left[\sum_{\langle ij \rangle \sigma \sigma'} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \cdot c_{i\sigma}^\dagger c_{j\sigma'}, \sum_{\tilde{\sigma}} n_{l,\tilde{\sigma}} \right] &= \sum_{\substack{\langle ij \rangle \\ \sigma \sigma' \tilde{\sigma}}} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left(c_{i\sigma}^\dagger c_{j\sigma'} n_{l\tilde{\sigma}} - n_{l\tilde{\sigma}} c_{i\sigma}^\dagger c_{j\sigma'} \right) \\ &= \sum_{\substack{\langle ij \rangle \\ \sigma \sigma' \tilde{\sigma}}} (\mathbf{m}_{ij} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \delta_{\sigma\tilde{\sigma}} \left(\delta_{lj} c_{i\sigma}^\dagger c_{j\tilde{\sigma}} - \delta_{li} c_{i\tilde{\sigma}}^\dagger c_{j\sigma'} \right) \\ &= \frac{1}{2} \sum_{i\delta\sigma\sigma'} (\mathbf{m}_{i,i+\delta} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left(\delta_{l,i+\delta} c_{i\sigma}^\dagger c_{i+\delta,\sigma} - \delta_{li} c_{i\sigma}^\dagger c_{i+\delta,\sigma'} \right) \end{aligned}$$

Using the same transformation we made earlier to introduce the commutator between n, c and n, c^\dagger we obtain after summing over the $\tilde{\sigma}$ introducing a new set made of $\boldsymbol{\delta}$ s as we did before. The summation over the $\boldsymbol{\delta}$ results in the following [combersom](#) expression

$$\begin{aligned} &= \sum_{\sigma} \left[\sum_{\sigma'} (\mathbf{m}_{l,l+\delta_x} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left(c_{l-\delta_x,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l+\delta_x,\sigma'} \right) \right. \\ &\quad + \sum_{\sigma'} (\mathbf{m}_{l,l-\delta_x} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left(c_{l+\delta_x,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l-\delta_x,\sigma'} \right) \\ &\quad + \sum_{\sigma'} (\mathbf{m}_{l,l+\delta_y} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left(c_{l-\delta_y,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l+\delta_y,\sigma'} \right) \\ &\quad \left. + \sum_{\sigma'} (\mathbf{m}_{l,l-\delta_y} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left(c_{l+\delta_y,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l-\delta_y,\sigma'} \right) \right] \end{aligned} \quad (63)$$

When it will come to identify the current on each side of the unit cell, we will have to take into account the displacement δ that is involved in the $c_{l\pm\delta_x,\sigma}^\dagger c_{l\sigma} - c_{l\sigma}^\dagger c_{l\pm\delta_x,\sigma'}$. This has to be consistent with what we already did for the hopping term. This means each current flowing through one side will be dependent on the spin hopping on both direction of the side's axis. This will become clear in the derivation.

Side currents Each unit cell were introduced as a square. We can from the last derived equation identify the contribution of the current on each side of the cell. We have a side on each axis direction $+x, +y, -x, -y$ as introduced in Eq.61. Due to the symmetric properties of the current, we can abstract the notation for $r \in \{x, y\}$. Using Eq.62 and Eq.63 we can write the current on each side r as follow. We as well reintroduce the minus of every summand of the Hamiltonian (due to their attractive nature) we let aside for the commutator relations. This multiplies with the minus one from the Heisenberg picture resulting in a plus sign. Further taking $i = -1/i$ we get:

$$(I_i^{\pm r})_{hop} = -\frac{t}{2i} \sum_{\sigma} c_{i\pm\delta_r,\sigma}^\dagger c_{i\sigma} - c_{i\sigma}^\dagger c_{i\pm\delta_r,\sigma} \quad (64)$$

$$(I_i^{\pm r})_{AM} = -\frac{1}{2i} \sum_{\sigma\sigma'} c_{i\pm\delta_r,\sigma}^\dagger c_{i\sigma} (\mathbf{m}_{i,i\mp\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} - c_{i\sigma}^\dagger c_{i\pm\delta_r,\sigma'} (\mathbf{m}_{i,i\pm\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \quad (65)$$

In this sense the hopping and altermagnetic term are very similar from nature. However the altermagnetic term is scaled by the hopping amplitude t on each summand, which shows a spin-dependent behaviour.

However we're not quite finished. The way we defined the matrix element, is to be isotropic on each axis. By doing so we can assume that $(\mathbf{m}_{i,i-\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} = (\mathbf{m}_{i,i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'}$ so that:

$$(I_i^{\pm r})_{AM} = -\frac{1}{2i} \sum_{\sigma\sigma'} (\mathbf{m}_{i,i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left(c_{i\pm\delta_r,\sigma}^\dagger c_{i\sigma} - c_{i\sigma}^\dagger c_{i\pm\delta_r,\sigma'} \right). \quad (66)$$

Total currents Until now we are able to describe how the current flows through each face of the unit cell. For this we can simply split the term we derived. From this we now aim to derive the current flows on each axis. This can easily be done assuming that the current flowing in the $-x$ direction subtracted from the current in the positive x direction forms the total current in \mathbf{e}_x . From this we get for $r \in \{x, y\}$:

$$I_i^r = I_i^{+r} - I_i^{-r}.$$

The real current that we can measure can be obtained by taking the quantum expectation value and the thermal average of the currents. Further we also introduce the BdG-transformed operators with the eigenvalues 57 and 58.

For the sake of readability we are going to stick with this r -notation. In fact the total currents takes a disproportionate size on the page so we abstract a bit. Further due to the linearity of the commutator one can split the current in different terms. We first start with the derivation of the physical current that comes from the hopping term.

$$\langle I_i^r \rangle_{\text{hop}} = \frac{t}{2i} \left[\sum_{\sigma} \langle c_{i-\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle - \langle c_{i\sigma}^{\dagger} c_{i-\delta_r, \sigma} \rangle - \langle c_{i+\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle + \langle c_{i\sigma}^{\dagger} c_{i+\delta_r, \sigma} \rangle \right]$$

and after introducing the BdG-transformed operators we obtain for the first term:

$$\begin{aligned} \langle c_{i-\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle &= \sum_{n, m \in \mathcal{N}_+} u_{n, i-\delta_r, \sigma}^* u_{ni\sigma} \underbrace{\langle \gamma_n^{\dagger} \gamma_m \rangle}_{\delta_{mn} f(E_n)} + u_{n, i-\delta_r, \sigma}^* v_{mi\sigma}^* \underbrace{\langle \gamma_n^{\dagger} \gamma_m^{\dagger} \rangle}_0 \\ &\quad + v_{n, i-\delta_r, \sigma} u_{mi\sigma} \underbrace{\langle \gamma_n \gamma_m \rangle}_0 + v_{n, i-\delta_r, \sigma} v_{mi\sigma}^* \underbrace{\langle \gamma_n \gamma_m^{\dagger} \rangle}_{\delta_{mn} (1-f(E_n))}. \end{aligned}$$

In the same way, we obtain for the other terms:

$$\begin{aligned} -\langle c_{i\sigma}^{\dagger} c_{i-\delta_r, \sigma} \rangle &= - \sum_{n \in \mathcal{N}_+} u_{ni\sigma}^* u_{n, i-\delta_r, \sigma} f(E_n) + v_{ni\sigma} v_{n, i-\delta_r, \sigma}^* (1 - f(E_n)), \\ -\langle c_{i+\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle &= - \sum_{n \in \mathcal{N}_+} u_{n, i+\delta_r, \sigma}^* u_{ni\sigma} f(E_n) + v_{n, i+\delta_r, \sigma} v_{ni\sigma}^* (1 - f(E_n)), \\ \langle c_{i\sigma}^{\dagger} c_{i+\delta_r, \sigma} \rangle &= \sum_{n \in \mathcal{N}_+} u_{ni\sigma}^* u_{n, i+\delta_r, \sigma} f(E_n) + v_{ni\sigma} v_{n, i+\delta_r, \sigma}^* (1 - f(E_n)). \end{aligned}$$

Here we can recognise a very useful relation that is going to simplify everything. We have $\langle c_{i\pm\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle = \langle c_{i\sigma}^{\dagger} c_{i\pm\delta_r, \sigma} \rangle^*$. We can then recall a useful relation that we can use in these expressions. Have z a complex we obtain $\text{Im}(z) = \frac{1}{2i}(z - z^*)$. Further $\text{Im}(z) + \text{Im}(z') = \text{Im}(z + z')$. After summing up the terms we obtain the following expression for current of the hopping term along the axis r :

$$\begin{aligned} \langle I_i^r \rangle_{\text{hop}} &= t \cdot \text{Im} \left[\sum_{\sigma} \sum_{n \in \mathcal{N}_+} f(E_n) u_{ni\sigma} (u_{n, i-\delta_r, \sigma}^* - u_{n, i+\delta_r, \sigma}^*) \right. \\ &\quad \left. + (1 - f(E_n)) v_{ni\sigma}^* (v_{n, i-\delta_r, \sigma} - v_{n, i+\delta_r, \sigma}) \right]. \end{aligned} \tag{67}$$

And inside the atom we add the following term using the same derivation method. The matrix element $(\mathbf{m}_{i, i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'}$ is a constant regarding the states-brackets. In this sense the quantum expectation and thermal average leave this quantity unchanged. Finally we obtain the exact same result than a t hopping but with a prefactor. This is consistent with the observation already made.

$$\langle I_i^r \rangle_{\text{AM}} = \frac{1}{2i} \sum_{\sigma\sigma'} (\mathbf{m}_{i, i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \left[\langle c_{i-\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle - \langle c_{i\sigma}^{\dagger} c_{i-\delta_r, \sigma'} \rangle - \langle c_{i+\delta_r, \sigma}^{\dagger} c_{i\sigma} \rangle + \langle c_{i\sigma}^{\dagger} c_{i+\delta_r, \sigma'} \rangle \right]$$

which after using the same derivation method as before yields:

$$\begin{aligned} \langle I_i^r \rangle_{\text{AM}} = & \frac{1}{2i} \left[\sum_{\sigma\sigma'} (\mathbf{m}_{i,i+\delta_r} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} \right. \\ & \sum_{n \in \mathcal{N}_+} f(E_n) \left[u_{ni\sigma} (u_{n,i-\delta_r,\sigma}^* - u_{n,i+\delta_r,\sigma}^*) + u_{ni\sigma}^* (u_{n,i+\delta_r,\sigma'} - u_{n,i-\delta_r,\sigma'}) \right] \\ & \left. + (1 - f(E_n)) \left[v_{ni\sigma}^* (v_{n,i-\delta_r,\sigma} - v_{n,i+\delta_r,\sigma}) + v_{ni\sigma} (v_{n,i+\delta_r,\sigma'}^* - v_{n,i-\delta_r,\sigma'}^*) \right] \right]. \end{aligned} \quad (68)$$

6 An advanced superconductivity

6.1 Site dependent potential

In this section we are going to highlight how the neighbour-depending potential can lead to a new kind of superconductivity. These new superconductive states are label s , p and extended d -wave superconductivity. The derivations we are going to make are closely based on the work of A.H. Mjøs and J. Linder in [2].

The Hamiltonian beeing the groundstone of this discussion, we're going to begin with it. It differs slightly from the symmetric one we derived in Eq.50. This extended version contains a neighbour-depending potential H_V that is going to produce the new superconductive Δ -part of the Hamiltonian. This potential is as well attractive such that $V > 0$, similar to the BCS theorie.

$$H = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} - \mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} - \frac{V}{2} \sum_{\langle ij \rangle \sigma} n_{i\sigma} n_{j\bar{\sigma}}. \quad (69)$$

Here the $\bar{\sigma}$ -notation means the opposite spin of σ . In other words the attraction finds only place between particles of opposite spin, mirroring the formation process of Cooper-pairs. We have a factor of one half to avoid counting twice the neighbours.

Trained eyes will recognise that H_V is not quadratic in the creation and annihilation operators which makes the Hamiltonian impossible to write in the BdG-formalism. For this reason we can use the so called Hartree-Fock mean field approximation $H_V \rightarrow H_V^{HF}$ defined as:

$$H_V^{HF} = -\frac{1}{2} \sum_{\langle ij \rangle \sigma} V_{ij} \left(F_{ij}^{\sigma\bar{\sigma}} c_{j\bar{\sigma}}^\dagger c_{i\sigma}^\dagger + \text{h.c.} + |F_{ij}^{\sigma\bar{\sigma}}|^2 \right) \quad (70)$$

Propagate the minus in the equations. involving the pairing amplitude $F_{ij}^{\sigma\bar{\sigma}} = \langle c_{i\sigma} c_{j\bar{\sigma}} \rangle$ that we already introduced. The V_{ij} are the neighbour-depending potential. The last term is a constant energy term which we can discard during the diagonalization process. If one wanted to compute the free energy this must be included there. From this point simplifications can be made to reach out final Hamiltonian.

Using the fermionic property $[c_{i\sigma}^\dagger, c_{j\sigma}^\dagger]_+ = 0$ we have $c_{i\sigma}^\dagger c_{j\sigma}^\dagger = -c_{j\sigma}^\dagger c_{i\sigma}^\dagger$ which leads to $\langle c_{i\sigma}^\dagger c_{j\sigma}^\dagger \rangle = -\langle c_{j\sigma}^\dagger c_{i\sigma}^\dagger \rangle$. We can use this in the last step of the following simplification:

$$\begin{aligned} H_V^{HF} &= \frac{1}{2} \sum_{\langle ij \rangle \sigma} V_{ij} \left(F_{ij}^{\sigma\bar{\sigma}} c_{j\bar{\sigma}}^\dagger c_{i\sigma}^\dagger + \text{h.c.} \right) \\ &= \frac{1}{2} \sum_{\langle ij \rangle} V_{ij} \left(F_{ij}^{\uparrow\downarrow} c_{j\downarrow}^\dagger c_{i\uparrow}^\dagger + F_{ij}^{\downarrow\uparrow} c_{j\uparrow}^\dagger c_{i\downarrow}^\dagger + \text{h.c.} \right) \\ &= \sum_{\langle ij \rangle} V_{ij} \left(F_{ij}^{\uparrow\downarrow} c_{j\downarrow}^\dagger c_{i\uparrow}^\dagger + \text{h.c.} \right) \end{aligned} \quad (71)$$

Using the relation $F_{ij}^{\uparrow\downarrow} = -F_{ji}^{\downarrow\uparrow}$ and the symmetry of the potential $V_{ij} = V_{ji}$ we can add the two terms up and remove the one half factor. The Hamiltonian is now ready to be diagonalized but first, we are going to discuss which advantages a Fourier transform of the Hamiltonian could bring us.

In an homogenous material we can consider a lattice and imagine some periodic boundary conditions in all directions. There is a translation invariance. In heterostructures however the material may vary, let's assume without loss of generality, in a direction. As we now, a periodic signal is a good candidate for a Fourier transform, which we can then express in a finite set of coefficients. This is very handfull. However the direction we want to transform on has to show a periodicity. For this reason an homogenous two-dimensional lattice involves a Fourier transformation in two space dimensions while a heterostructure we transform only in the direction where the material is the same. For instance a multilayer material in the x -direction can be described by combining a real space description in the x -axis combined to a Fourier transformation in the y -direction. In this thesis we are going to focus ourselves on a multilayer material in the x -direction.

6.2 In the vertical periodic boundary setup

The description of the creation and annihilation operators with a step in the momentum space for the y -direction can be expressed as follow:

$$c_{xy\sigma} = \frac{1}{\sqrt{N_y}} \sum_{k_y} c_{xk_y\sigma} e^{ik_y y} \quad (72)$$

$$c_{xy\sigma}^\dagger = \frac{1}{\sqrt{N_y}} \sum_{k_y} c_{xk_y\sigma}^\dagger e^{-ik_y y} \quad (73)$$

How do we find an expression for k_y ? Well the periodicity (for ex. in the c operator) yields $c_{xy\sigma} = c_{x,y+N_y,\sigma}$ this means the following condition must be fulfilled:

$$\begin{aligned} c_{xy\sigma} = c_{x,y+N_y,\sigma} &\Leftrightarrow e^{ik_y y} = e^{ik_y (y+N_y)} \Leftrightarrow e^{ik_y N_y} = 1 \\ &\Leftrightarrow k_y = \frac{2\pi n}{N_y}. \end{aligned}$$

We know that the momentum index should cover the entire first Brillouin zone. This covers the momentum from $-\pi/a$ to π/a where a is the lattice constant. Further due to the $2\pi/a$ periodicity we have the same k_y at $-\pi/a$ and π/a . For this reason we need to map k_y in $[-\pi/a; \pi/a)$. This means we have $n \in [-N_y/2; N_y/2 - 1]$ including 0. For $n = 4$ we then have $k_y \in \{-\pi, -\pi/2, 0, \pi/2\}$.

For the readability we are going to use $k_y \rightarrow k$. Further to an index i can associate (x, y) and to j , (x', y') . We first want to transform the hopping term:

$$H_{\text{hop}} = - \sum_{\langle ij \rangle \sigma} t_{ij} \sum_{kk'} c_{xk\sigma}^\dagger c_{x'k'\sigma} e^{i(k'y' - ky)}$$

Here we can use the neighbour shift properties $y' = y + \delta_y$ where $\delta_y = \pm 1$. Doing so we have

$$e^{i(k'y' - ky)} = e^{i(k'(y + \delta_y) - ky)} = e^{i(k' - k)y} e^{ik'\delta_y}$$

No we need to express the neighbourhood sum. Precily we have

$$\begin{aligned} H_{\text{hop}} &= - \sum_{\langle ij \rangle \sigma} t_{ij} \sum_{kk'} c_{xk\sigma}^\dagger c_{x'k'\sigma} e^{i(k-k')y} e^{ik'\delta_y} \\ &= - \sum_{xy\sigma} \sum_{kk'} \left(\underbrace{t_{x,x+1} c_{xk\sigma}^\dagger c_{x+1k'\sigma} e^{ik' \cdot (0)}}_{+x \text{ hopping, no } \delta_y} + \underbrace{t_{x,x-1} c_{xk\sigma}^\dagger c_{x-1k'\sigma} e^{ik' \cdot (0)}}_{-x \text{ hopping, no } \delta_y} \right. \\ &\quad \left. + \underbrace{t_{x,y-1} c_{xk\sigma} c_{xk'\sigma}^\dagger e^{ik' \cdot (-1)}}_{\delta_y = -1} + \underbrace{t_{x,y+1} c_{xk\sigma}^\dagger c_{xk'\sigma} e^{ik' \cdot (1)}}_{\delta_y = 1} \right) e^{i(k-k')y} \end{aligned}$$

As we see the y direction is now expressed in the k -index, which is unique for each lattice y -slice. The information is then conserved. We know that system have different material on the x axis. This means $t_{x,y-1} = t_{x,x} = t_{x,y+1}$ because the material are isotropic but every material has

a different hopping term. Beside we can use the following relation $1/N_y \sum_y e^{i(k-k')y} = \delta_{kk'}$. Perfroming both expression leads after a summation over k' to

$$H_{\text{hop}} = - \sum_{xk\sigma} t_{x,x+1} c_{xk\sigma}^\dagger c_{x+1k\sigma} + t_{x,x-1} c_{xk\sigma}^\dagger c_{x-1k\sigma} + t_{x,x} c_{xk\sigma}^\dagger c_{xk\sigma} (e^{ik} + e^{-ik})$$

And now we can reintroduce an arbitray second coordinate x' to describe the neighbours.

$$H_{\text{hop}} = - \sum_{xx'k\sigma} t_{x,x'} c_{xk\sigma}^\dagger c_{x'k\sigma} (\delta_{x+1,x'} + \delta_{x-1,x'} + \delta_{x,x'} 2 \cos(k)) \quad (74)$$

The chemical potential term is more easily given. In fact the number opertor yields to use two operators $c^\dagger c$ at a same coordinate i .

$$H_\mu = -\mu \sum_{xk'k\sigma} c_{xk\sigma}^\dagger c_{xk'\sigma} e^{i(k-k')y} = - \sum_{xx'k\sigma} \mu c_{xk\sigma}^\dagger c_{xk'\sigma} \delta_{xx'} \quad (75)$$

We finally have for the terms involving $c_{xx'\sigma}^\dagger c_{xx'\sigma}$:

$$H_{\text{hop}} + H_\mu = \sum_{xx'k\sigma} \epsilon_{xx'k\sigma} c_{xk\sigma}^\dagger c_{x'k\sigma}$$

using

$$\epsilon_{xx'k\sigma} = -t_{xx'} (\delta_{x+1,x'} + \delta_{x-1,x'}) - (t_{xx'} 2 \cos(k) + \mu) \delta_{xx'}$$

Moving on to the potential term we have to introduce a new notation. $i \pm \hat{x} = i \pm 1$ and $i \pm \hat{y} = i \pm N_x$. [Write a single clear message at the beggining of the lattice introduction..](#) For the brevety we use $f(a) = V_{ia} F_{ia} c_{a\downarrow}^\dagger c_{i\uparrow}^\dagger$!! [sign check for V and in frotn oh Hv!](#)

$$\begin{aligned} H_V &= \sum_{\langle ij \rangle} V_{ij} F_{ij} c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger + \text{h.c.} \\ &= \sum_i f(i-1) + f(i+1) + f(i+N_x) + f(i-N_x) + \text{h.c.} \end{aligned}$$

We can now insert our Fourier transformation introduced in Eq.73 and Eq.72 to obtain

$$\begin{aligned} H_V &= \sum_{xy} \frac{1}{N_y} \sum_{kk'} \left(V_{x,x+1} F_x^{x+} c_{x+1,k,\downarrow}^\dagger c_{xk'\uparrow}^\dagger + V_{x,x-1} F_x^{x-} c_{x+1,k,\downarrow}^\dagger c_{xk'\uparrow}^\dagger \right. \\ &\quad \left. + V_{xx} (F_x^{y+} e^{-ik} - F_x^{y-} e^{ik}) c_{xk\downarrow}^\dagger c_{xk'\uparrow}^\dagger \right) e^{-i(k+k')y} + \text{h.c.} \end{aligned}$$

Defining a more general form for the summand involving two site i and j :

$$\begin{aligned} F_{xx'k} &= -V_{x,x'} (F_x^{x+} \delta_{x+1,x'} + F_x^{x-} \delta_{x-1,x'} \\ &\quad + (F_x^{y+} e^{-ik} - F_x^{y-} e^{ik}) \delta_{xx'}) , \end{aligned}$$

where we got $k' = -k$ from the sum over the y -direction. We can rewrite the expression of H_V as

$$H_V = \sum_x \sum_{kk'} F_{xx'k} c_{xk\uparrow}^\dagger c_{x',-k,\downarrow}^\dagger + F_{xx'k}^* c_{x',-k,\downarrow} c_{xk\uparrow}.$$

Now using H_{hop} , H_μ and H_V we can write the full Hamiltonian as

$$\begin{aligned} H &= \sum_{xx'k} D_{xk}^\dagger H_{xx'k} D_{x'k} \\ &= \sum_{xx'k} \begin{pmatrix} c_{xk\uparrow}^\dagger & c_{x,-k,\downarrow} \end{pmatrix} \begin{pmatrix} \epsilon_{xx'k\uparrow} & F_{xx'k} \\ F_{xx'k}^* & -\epsilon_{xx'k\downarrow} \end{pmatrix} \begin{pmatrix} c_{x'k\uparrow} \\ c_{x',-k,\downarrow}^\dagger \end{pmatrix}. \end{aligned} \quad (76)$$

The summation over all x, x' can be represented in a new matrix.

$$H = \sum_k D_k^\dagger H_k D_k$$

involving the $4N_x \times 4N_x$ matrix H_k and the $4N_x$ -dimensional vector D_k .

$$H_k = \begin{pmatrix} H_{11k} & \cdots & H_{1N_x k} \\ \vdots & \ddots & \\ H_{N_x 1k} & & H_{N_x N_x k} \end{pmatrix}$$

As before the y information is stored in the k -index, which is unique for each lattice y -slice. This said, we can diagonalize N_y times a $4N_x \times 4N_x$ matrix. H_k represent the interaction of a y -line with itself. The eigenvalues are the same for each y -slices and physical quantities are going to be expressed with this summation over k and the eigenvalues, -vectors of each k (y -slice).

On the other hand the vector we use to carry the creation and annihilation operators is given as

$$D_k^\dagger = \begin{pmatrix} c_{1k\uparrow}^\dagger & c_{1,-k,\downarrow} & \cdots & c_{N_x k\uparrow}^\dagger & c_{N_x,-k,\downarrow} \end{pmatrix} \in \mathbb{H}^{2N_x}$$

6.2.1 BdG-transformation for a vertical fourier transform

The eigenvalues equation is similar to Eq.54

$$H_k \mathfrak{X}_{nk} = E_{nk} \mathfrak{X}_{nk} \quad (77)$$

The eigenvectors and -values are given as

$$\mathfrak{X}_{nk} = \begin{pmatrix} \mathfrak{x}_{n1k} \\ \vdots \\ \mathfrak{x}_{nN_x k} \end{pmatrix}, \quad \mathfrak{x}_{n x k} = \begin{pmatrix} u_{n x k} \\ v_{n x k} \end{pmatrix}$$

if we stick to the formalism we already derived in the earlier Sec.3.4 we obtain similar eigenvectors where u_{nik} corresponds to c . We are now going to transform the c operators. First we need to define $\mathfrak{X}_k = [\mathfrak{X}_{1k}, \dots, \mathfrak{X}_{2N_x k}] \in \mathbb{H}^{2N_x \times 2N_x}$ storing the number of lattice sites times the number of spins (2) in the first dimension and the number of eigenvectors in the second dimension. Beside we have $\mathfrak{g}_k = (\mathfrak{g}_{1k}, \dots, \mathfrak{g}_{2N_x k})^T \in \mathbb{H}^{2N_x}$ along with $D_k = \mathfrak{X}_k \mathfrak{g}_k$, delivering $\mathfrak{g}_k = \mathfrak{X}_k^\dagger D_k$ which is equivalent to:

$$\mathfrak{g}_{nk} = \sum_{x \in N_x} u_{n x k} c_{x k \uparrow} + v_{n x k} c_{x, -k, \downarrow}^\dagger. \quad (78)$$

Looking at $D_k = \mathfrak{X}_k \mathfrak{g}_k$ we can derive two very usefull expressions:

$$c_{x k \uparrow} = \sum_{n \in \llbracket 2N_x \rrbracket} u_{n x k} \mathfrak{g}_{nk} \quad (79) \quad \begin{array}{c} \circ \\ \vdots \\ \circ \end{array} \quad c_{x, -k, \downarrow}^\dagger = \sum_{n \in \llbracket 2N_x \rrbracket} v_{n x k} \mathfrak{g}_{nk}. \quad (80)$$

6.2.2 Pairing amplitudes

For the derivation of the pairing amplitudes on the axis $a \in \{\hat{x}, \hat{y}\}$. Doing so, we will use our BdG-transformation as we did earlier to be able to solve these parameters self-consistently using the eigenvectors and -values.

$$F_i^{\pm a} = \langle c_{i\uparrow} c_{i \pm a, \downarrow} \rangle = \frac{1}{N_y} \sum_{k k'} \langle c_{i x k \uparrow} c_{i x \pm a, k', \downarrow} \rangle e^{i(k+k')i_y} e^{\pm i k' \delta_y}$$

using $y \pm a = y'$. Here for the brevety we have to sacrifice the notation. This is unformal and might be confusing so we clarify the point in first place.

So a describes a displacement in the x - or y -direction. On one hand $i + a$ refers to the lattice index when moving from i with a translation a . On the other hand, having " $i = (x, y)$ ", $x + a$ refers to the x -coordinate of the lattice site $i + a$. This is analogue for y .

This then means or $a = \hat{y}$ we have $x \pm a = x$ and $k'_a = 0$.

The expectaion value is independant of the site vertical coordinate y . This means that we can achieve a site description by making the average of the value over a x -slice. In other words we can write:

$$F_i^{\pm a} = \frac{1}{N_y} \sum_{y \in \llbracket N_y \rrbracket} \frac{1}{N_y} \sum_{k k'} \langle c_{i x k \uparrow} c_{i x \pm a, k', \downarrow} \rangle e^{i(k+k')r_i} e^{\pm i k' \delta_y}$$

which after the sumation over i results as we covered earlier as $1/N_y \sum_y e^{i(k+k')i_y} = \delta_{k,-k'}$

$$F_i^{\pm x} = \frac{1}{N_y} \sum_k \langle c_{i_x k \uparrow} c_{i_x \pm a, k, \downarrow} \rangle e^{\mp i k a}$$

Now that we have simplified the Fourier transform we can incorporate the BdG-transformation. The process is very similar to Eq.59 and yields

$$F_i^{\pm a} = \frac{1}{N_y} \sum_k \sum_{nn'} u_{ni_x k} v_{n', i_x \pm a, k}^* \langle g_{nk} g_{n'k}^\dagger \rangle e^{\mp i k \delta_y} \quad (81)$$

where we can as well write $i_x = x$. Recalling the use of a to abstractise the axis we get:

$$F_i^{\pm x} = \frac{1}{N_y} \sum_{nk} u_{ni_x k} v_{n, i_x \pm x, k}^* (1 - f(E_{nk})) \quad (82)$$

$$F_i^{\pm y} = \frac{1}{N_y} \sum_{nk} u_{ni_x k} v_{ni_x k}^* (1 - f(E_{nk})) e^{\mp i k}. \quad (83)$$

6.2.3 In the open boundary setup

Here we need to reuse the equations of the Nambu-Spin formalism described in chapter 3. As we saw we need to first take care that the part of the Hamiltonian we want to introduce has [a kind of symmerty](#) in the creation and annihilation operators. If we transform the Hartree-Fock Hamiltonian as it is in Eq.71 we see that only the upper right quadrant of the matrix $H_{V,ij}^{HF}$ of $\check{c}_i^\dagger \cdot (H_V^{HF})_{ij} \cdot \check{c}_j$ is filled. We need to use the anticommutation relations to solve this. This delivers the following expression:

$$\begin{aligned} H_V^{HF} &= - \sum_{\langle ij \rangle} V_{ij} \left(F_{ij}^{\uparrow\downarrow} c_{j\downarrow}^\dagger c_{i\uparrow}^\dagger + F_{ij}^{\uparrow\downarrow*} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger \right) \\ &= - \sum_{\langle ij \rangle} V_{ij} \left(F_{ji}^{\uparrow\downarrow} c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger + F_{ij}^{\uparrow\downarrow*} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger \right) \\ &= - \frac{1}{2} \sum_{\langle ij \rangle} V_{ij} \left[F_{ij}^{\uparrow\downarrow} (c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger - c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger) + F_{ij}^{\uparrow\downarrow*} (c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger) \right] \end{aligned} \quad (84)$$

which is achieved using Eq.3r1, $V_{ij} = V_{ji}$, $F_{ij}^{\uparrow\downarrow} = -F_{ji}^{\downarrow\uparrow} = F_{ji}^{\uparrow\downarrow}$ using the anticommutation realtion and a spin exchange. We then use $F_{ij}^{\uparrow\downarrow} = F_{ij}$ since all ambiguity has been removed. Therefore the matrix we obtain is given as

$$H_{V,ij}^{HF} = -\frac{1}{2} \begin{pmatrix} \mathcal{O} & & -F_{ij}^{\uparrow\downarrow} \\ & F_{ij}^{\uparrow\downarrow} & \\ -F_{ij}^{\uparrow\downarrow*} & F_{ij}^{\uparrow\downarrow*} & \mathcal{O} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \mathcal{O} & -i\sigma_2 F_{ij} \\ i\sigma_2 F_{ij}^* & \mathcal{O} \end{pmatrix} \quad (85)$$

an takes only place between two neighbouring sites i and j . For the accuracy one could multiply it with $(\delta_{i+1,j} + \delta_{i-1,j} + \delta_{i+N_x,j} + \delta_{i-N_x,j})$. This matrix can then be added to Eq.52 replacing the on-site superconductive Δ -term by a neighbouring superconductive F -term.

Now that we have expressed the Hartree-Fock Hamiltonian in the correct maner, we can use our BdG-transformation to diagonalize the Hamiltonian and find a self-consistent solution for the F -parameters. Here we use the transformation between the creation and annihilation operators, and the eigenvectors, -values of the Hamiltonian given by Eq.57 and Eq.58.

$$F_{ij} = \langle c_{i\uparrow} c_{j\downarrow} \rangle = \sum_{n \in \mathcal{N}_+} u_{ni\uparrow} v_{nj\downarrow}^* (1 - f(E_{nk})) + v_{ni\uparrow}^* u_{nj\downarrow} f(E_{nk}) \quad (86)$$

in a very similar way as Eq.59.

6.3 Advanced order parameters

In the more simple description of the superconductivity we outlined how the superconducting order parameter Δ depends on the pairing amplitude F . Because the potential was isotropic, we simply had $\Delta = U \langle c_{i\uparrow} c_{i\downarrow} \rangle$. Here however the potential is anisotropic and we obtain a linear combination of the pairing amplitudes in the different directions of the lattice. Achieving different combinations of the F s we obtain different superconducting states. Here is an exhaustive list. [replace V with U for consistency?](#)

$$\Delta_{s,i} = V F_{s,i} = \frac{V}{4} \left(F_i^{x+(S)} + F_i^{x-(S)} + F_i^{y+(S)} + F_i^{y-(S)} \right) \quad (87)$$

$$\Delta_{d,i} = V F_{d,i} = \frac{V}{4} \left(F_i^{x+(S)} + F_i^{x-(S)} - F_i^{y+(S)} - F_i^{y-(S)} \right) \quad (88)$$

$$\Delta_{p_x,i} = V F_{p_x,i} = \frac{V}{2} \left(F_i^{x+(T)} - F_i^{x-(T)} \right) \quad (89)$$

$$\Delta_{p_y,i} = V F_{p_y,i} = \frac{V}{2} \left(F_i^{y+(T)} - F_i^{y-(T)} \right) \quad (90)$$

F_s , F_d , F_{p_x} and F_{p_y} are the pairing amplitudes for the s , d (also called $d_{x^2-y^2}$ because of its expression), p_x and p_y -wave superconductivity. The S and T are the singlet and triplet expressions of the pairing amplitudes. More precily we define them as

$$F_{ij}^{(S)} = \frac{F_{ij} - F_{ji}}{2} = F_{ij} \quad (91)$$

$$F_{ij}^{(T)} = \frac{F_{ij} + F_{ji}}{2} \quad (92)$$

Where we shorten the expression using $F_{ij}^{\uparrow\downarrow(S)} = F_{ij}^{(S)}$.

Symmetry discussion We see that these parameter depends on the spin and the momentum. Therefore it's a good idea to look at their respective behaviour under exchange of these variables.

The discussion for the spin exchange is quite straight forward.

$$F_{ij}^{\uparrow\downarrow(S)} = \frac{F_{ij}^{\uparrow\downarrow} + F_{ji}^{\uparrow\downarrow}}{2} = \frac{\langle c_{i\uparrow} c_{j\downarrow} \rangle + \langle c_{j\uparrow} c_{i\downarrow} \rangle}{2}$$

$$F_{ij}^{\uparrow\downarrow(T)} = \frac{F_{ij}^{\uparrow\downarrow} - F_{ji}^{\uparrow\downarrow}}{2} = \frac{\langle c_{i\uparrow} c_{j\downarrow} \rangle - \langle c_{j\uparrow} c_{i\downarrow} \rangle}{2}$$

[What's happening when we invert both spin?](#) using $\langle c_{i\uparrow} c_{j\downarrow} \rangle = -\langle c_{i\downarrow} c_{j\uparrow} \rangle$. Using the linearity of the expression we obtain

$$F_{ij}^{\uparrow\downarrow(S)} = -F_{ij}^{\downarrow\uparrow(S)}$$

$$F_{ij}^{\uparrow\downarrow(T)} = F_{ij}^{\downarrow\uparrow(T)}.$$

This means that the singlet wave-pairing amplitude is antisymmetric under spin exchange and the triplet wave pairing is symmetric under spin exchange. Their names find place in the analogy of the wavefunction formalism.

For the momentum exchange we are going to take a look at the Fourier transformation of the pairing amplitudes. Using the transformation we made to reach Eq.81 with any translation $x \rightarrow \mathbf{r}$, $\mathbf{r} \in \{\mathbf{e}_x, \mathbf{e}_y\}$ we obtain

$$F_{i,i+\mathbf{r}} = \frac{1}{N} \sum_{\mathbf{k}} \langle c_{\mathbf{k}\uparrow} c_{-\mathbf{k},\downarrow} \rangle e^{-i\mathbf{k}\mathbf{r}}.$$

Changing the sign of the momentum we obtain $\frac{1}{N} \sum_{\mathbf{k}} \langle c_{-\mathbf{k},\uparrow} c_{\mathbf{k},\downarrow} \rangle e^{i\mathbf{k}\mathbf{r}} = F_{i,i-\mathbf{r}}$ because of the $\delta_{\mathbf{k},\mathbf{k}'}$ trick. For this reason $F_{i,i+\mathbf{r}} + F_{i,i-\mathbf{r}}$ is symmetric under momentum exchange while $F_{i,i+\mathbf{r}} - F_{i,i-\mathbf{r}}$ is antisymmetric.

Referring back to the order parameter definition we see that the s and d -wave are symmetric under momentum exchange, where the p_x and p_y -wave are antisymmetric in such exchange.

6.4 Implementation of the advanced superconductivity

The individual F always change but the end result in F_d can converge so we compute $F_{d,i}$ on each step to see if it converges.

7 Literature

Books

- [1] Kristian Fossheim and Asle Sudbø. John Wiley & Sons, Ltd, 2004. ISBN: 9780470020784. DOI: <https://doi.org/10.1002/0470020784>. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/0470020784>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/0470020784>.

Articles

- [2] Andreas Halkjelsvik Mjøs and Jacob Linder. “Magnetism and Superconductivity in the Hubbard Model”. In: *NTNU Norwegian University of Science and Technology, Department of Physics* (2019).

Online

- [3] Prof. Dr. Matthias Fuchs and Philipp Baumgärtel. Last public available version 2010. 2023. URL: http://theorie.physik.uni-konstanz.de/lmfuchs/lectures/statmech0910/Stat.Mech0910_Skript.pdf.
- [4] Igor Mazin. 2024. URL: <https://physics.aps.org/articles/v17/4>.