MONASH University

MONASH
ENGINEERING
ENG1060

ORDINARY DIFFERENTIAL EQUATIONS

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Slides by Tony Vo

Assisted by Tham Lai Kuan & Christopher Ng

GROUP OF EIGHT AUSTRALIA

HOUSEKEEPING

- Weekly Moodle post
  - Week 10 Moodle announcement

- Lab-related items
  - Lab 7 marks and feedback available on Moodle Grade Book
  - Lab 8 solutions available on Gdrive > Labs

- PASS Sessions

  1) Monday (3:30-5:30pm MYT , 6:30-8:30pm AEDT):
  https://monash.zoom.us/j/89128532133?pwd=VVVOenhDbW5xZ3h6ZFRZR1dieVhldz09
  2) Tuesday (12-2pm MYT , 3-6pm AEDT):
  https://monash.zoom.us/j/85226581851?pwd=d0YxeWVHd0tudnpIanFRYWU2ZGJRUT09

HOUSEKEEPING

- Assignment due next Friday (22 Jan 2021, 8pm MYT / 11pm AEDT)
  - Remember that it is an individual assessment
  - Use the support avenue available (e.g. discussion board, etc.)
  - Assignment-marking schedule release next week

	Group 01 (Tuesday 9am MYT / 12 Noon AEDT)		
	Christopher Ng		
Zoom link			
Zoom ID			
Time	Student ID	First Name	Last Name
9.00am - 9.30am	1234567	abc	def
9.30am - 10.00am			
10.00am - 10.30am			
10.30am - 11.00am			
11.00am - 11.30am			
11.30am - 12.00noon			

HOUSEKEEPING

- SETU questionnaire is now open for a limited time
  - Please spend 5-10 minutes to complete this during the workshop
  - Always seeking feedback and striving for continuous improvement

IN THIS WORKSHOP

1. Understanding methods for solving ordinary differential equations (ODEs)
    a. Euler's
    b. Heun's
    c. Midpoint
2. Creating function files for ODE-solving methods
3. Solving ODEs
4. Using ode45()

RECAP: ODEs

- The generic 1st-order ODE is given as

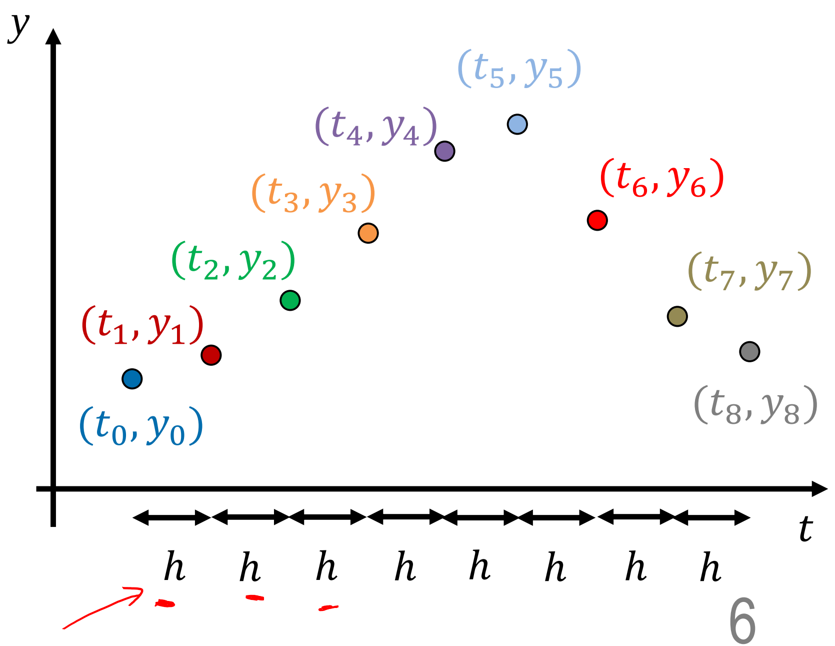
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$$

- Starting with initial condition (t_0, y_0)
  - Determine the next point (t_1, y_1) using slope ϕ information

$$\boxed{y_{i+1} \cong y_i + h\phi}$$

step

- Then use (t_1, y_1) and slope ϕ information to determine (t_2, y_2)
  - Repeat until you get to your desired t value



RECAP: ODE-SOLVING METHODS

MONASH University

$$y_{i+1} \cong y_i + h\phi$$

Method	Evaluate derivative at …	Local error	Global error
Euler	Point i derivative $\phi = \frac{\mathrm{d}y_i}{\mathrm{d}t_i} = f(t_i, y_i)$ current pt.	$O(h^2)$	$O(h)$
Heun's	Point i and predicted $i+1$ – then averaged ypred $\phi = \frac{f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)}{2}$	$O(h^3)$	$O(h^2)$
Midpoint	Half way between point i and $i+1$ $\phi = f(t_{i+1/2}, y_{i+1/2})$	$O(h^3)$	$O(h^2)$

RECAP: EULER'S METHOD

$$y_{i+1} \cong y_i + h\phi$$

Steps for Euler's method:

$$\boxed{y_{i+1} = y_i + hf(t_i, y_i)}$$

1. Starting condition $(i = 0)$

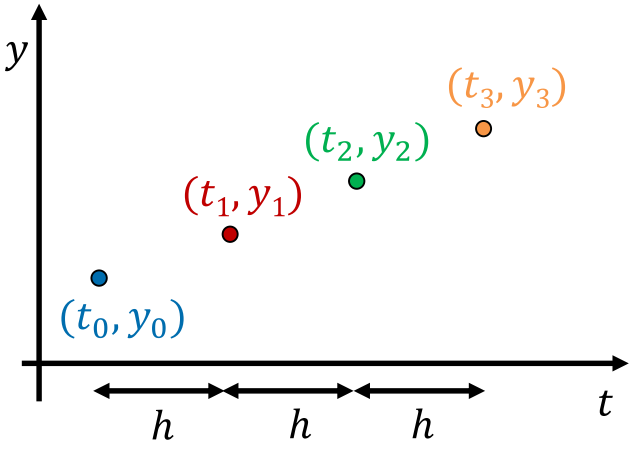
$$y_1 = y_0 + hf(t_0, y_0)$$

2. Euler's method for $i = 1$

$$y_2 = y_1 + hf(t_1, y_1)$$

3. Euler's method for $i = 2$

$$y_3 = y_2 + hf(t_2, y_2)$$



[20 MINS]

ACTIVITY: STEEPNESS

EULER.M, STEEPNESS.M

The gradient of a terrain is is described by $\frac{\mathrm{d}y}{\mathrm{d}x}$, where x is the horizontal distance and y is the vertical distance

Process:

1. Understand Euler's method by hand
2. Write a function file for Euler's method
3. Solve the ordinary differential equation

Activity involves:

1. Hand calculations
2. Writing a function file

Equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$$

$$y(0) = 2$$

$$h = 0.5$$

$$y_{i+1} \cong y_i + hf(t_i, y_i)$$

MATLAB commands:

```
for i = …
y = ones(…)
error(…)
f = @(x,y) …
```

[20 MINS]

ACTIVITY: STEEPNESS

EULER.M, STEEPNESS.M

MATLAB commands:

```
f = @(x,y) …
```

Equations:

derivative

$$\frac{dy}{dx} = x - y^2$$

$x_i = 0$

$$y(0) = 2$$

$$h = 0.5$$

$$y_{i+1} \cong y_i + hf(t_i, y_i)$$

The gradient of a terrain is is described by $\frac{dy}{dx}$, where x is the horizontal distance and y is the vertical distance

→ y? when x = 1 = 0.25

1. Solve for $y(1)$ by hand

2. Write a function with the following header: [t,y] = euler(dydt,tspan,y0,h)

3. Use euler() to verify $y(1)$ in step 1

4. Modify the code so that it can solve for $y(1.25)$ using $h = 0.5$

$x - y^2$

i	x_i	y_i	dy_i/dx_i
0	0	2	-4
1	0.5 $x_{i+1} = x_i + h = 0 + 0.5$	0 $= 2 + 0.5(-4) = 0$	0.5
2	1 $= 0.5 + 0.5 = 1$	0.25 $= 0 + 0.5(0.5) = 0.25$	0.9375

RECAP: MIDPOINT METHOD

$$y_{i+1} \cong y_i + h\phi$$

Steps for the midpoint method:

= t(i) + h/2

thalf yhalf.

$$\boxed{y_{i+1/2} = y_i + \frac{h}{2} f(t_i, y_i)} \qquad \boxed{y_{i+1} = y_i + hf(t_{i+1/2}, y_{i+1/2})}$$

1. Starting condition $(i = 0)$

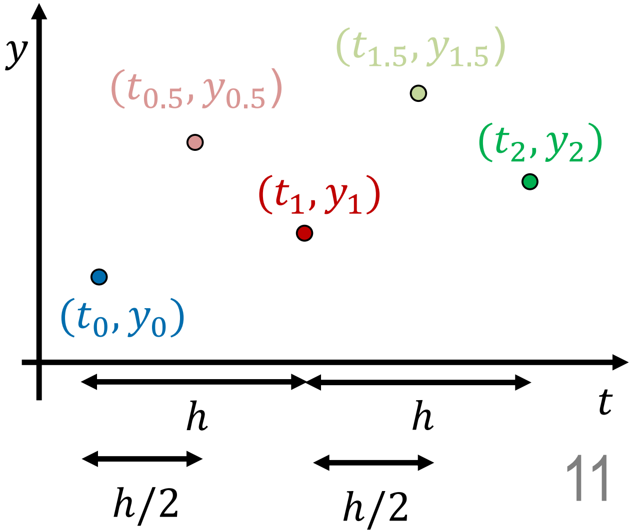
$$y_{0.5} = y_0 + \frac{h}{2} f(t_0, y_0)$$

$$y_1 = y_0 + hf(t_{0.5}, y_{0.5})$$

2. Euler's method for $i = 1$

$$y_{1.5} = y_1 + \frac{h}{2} f(t_1, y_1)$$

$$y_2 = y_1 + hf(t_{1.5}, y_{1.5})$$



[15 MINS]

ACTIVITY: OBJECT

MIDPOINT.M, OBJECT.M

Equations:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = t - v$$

$$v(0) = 1$$

$$h = 0.5$$

$$y_{i+1/2} = y_i + \frac{h}{2} f(t_i, y_i)$$

$$y_{i+1} = y_i + hf(t_{i+1/2}, y_{i+1/2})$$

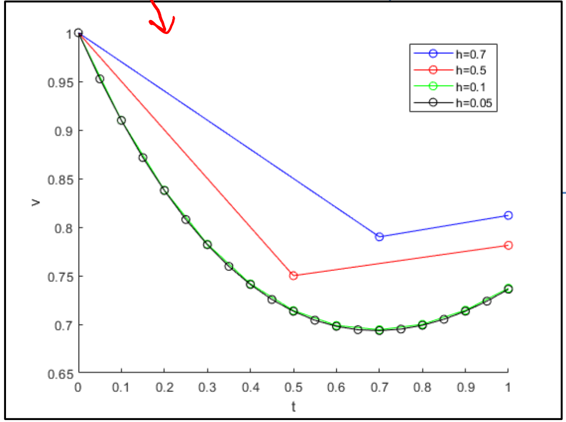
An accelerating object is heavily resisted by an unknown fluid, which is described by $\frac{\mathrm{d}v}{\mathrm{d}t}$

Process:

1. Understand the midpoint method by hand
2. Write the midpoint method function file
3. Solve the ordinary differential equation

MATLAB commands:

```
for i = …
error(…)
y = ones(…)
f = @(x,y) …
```



Activity involves:

1. Hand calculations
2. Writing a function file

[15 MINS]

ACTIVITY: OBJECT

MIDPOINT.M, OBJECT.M

MATLAB commands:

```
f = @(x,y) …
```

Equations:

$$\frac{dv}{dt} = t - v$$

$$v(0) = 1$$

$$h = 0.5$$

$$y_{i+1/2} = y_i + \frac{h}{2} f(t_i, y_i)$$

$$y_{i+1} = y_i + h\boxed{f(t_{i+1/2}, y_{i+1/2})}$$

y half

$t_{half} = t(i) + \frac{h}{2}$

v? when t = 1

1. Solve for $v(1)$ by hand

2. Write a function with the following header: [t,y] = midpoint(dydt,tspan,y0,h)

3. Plot v against t for $h = 0.7, 0.5, 0.1$ and 0.05?

$v(0) = 1$

$\frac{dv}{dt} = t - v$

thalf

vhalf

$v_i + \frac{h}{2} f(t_i, y_i)$

$= 1 + \frac{0.5}{2}(-1)$

i	t_i	v_i	$f(t_i, v_i)$	$t_{i+1/2}$	$v_{i+1/2}$	$f(t_{i+1/2}, v_{i+1/2})$
0	0	1	-1	0+0.5/2 0.25	0.75	-0.5 $= t_{half} - v_{half} =$ -0.5
1	0.5	0.75	-0.25	0.75	0.6875	0.0625
2	1	?				

$1 + 0.5(-0.5)$

RECAP: HEUN'S METHOD

$$y_{i+1} \cong y_i + h\phi$$

Steps for Heun's method:

$$y_{i+1}^0 = y_i + hf(t_i, y_i)$$

avg grad

y pred.

$$y_{i+1} = y_i + \frac{h}{2}\left(f(t_i, y_i) + f(t_{i+1}, \boxed{y_{i+1}^0})\right)$$

1. Starting condition $(i = 0)$

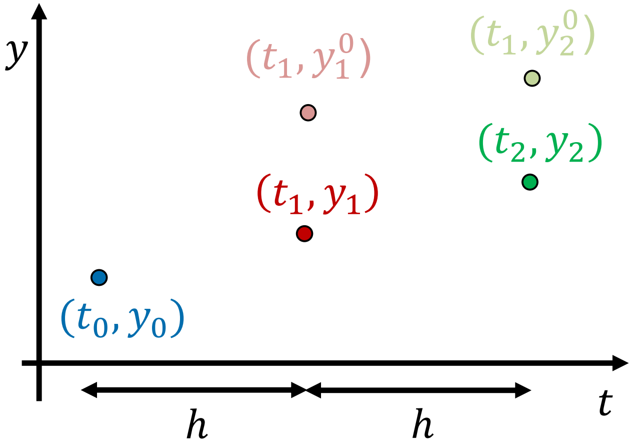
$$y_1^0 = y_0 + hf(t_0, y_0)$$

$$y_1 = y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, y_1^0)\right)$$

2. Euler's method for $i = 1$

$$y_2^0 = y_1 + hf(t_1, y_1)$$

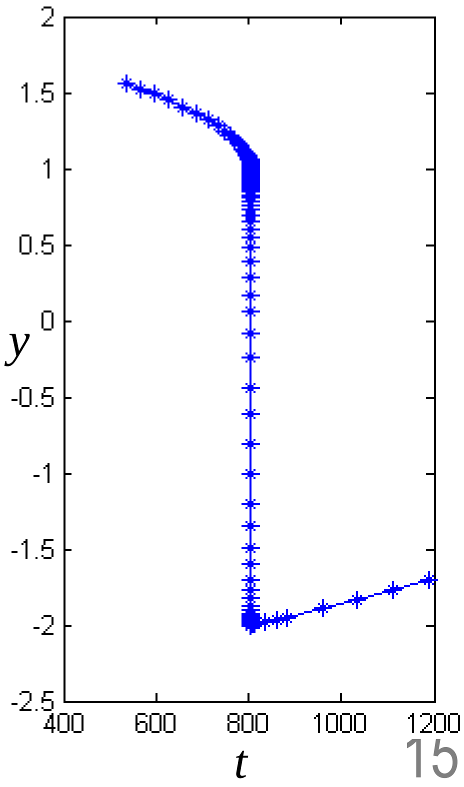
$$y_2 = y_1 + \frac{h}{2}\left(f(t_1, y_1) + f(t_2, y_2^0)\right)$$



ADAPTIVE STEP-SIZE METHODS

- Function gradients can change rapidly
  - For most of the range of t, y changes gradually, so a large step size can be used
  - In regions where the solution undergoes an abrupt change, a much smaller step size is required for accuracy

- Adaptive step-size methods dynamically adjust their step size based on an estimate of the local gradient of the solution

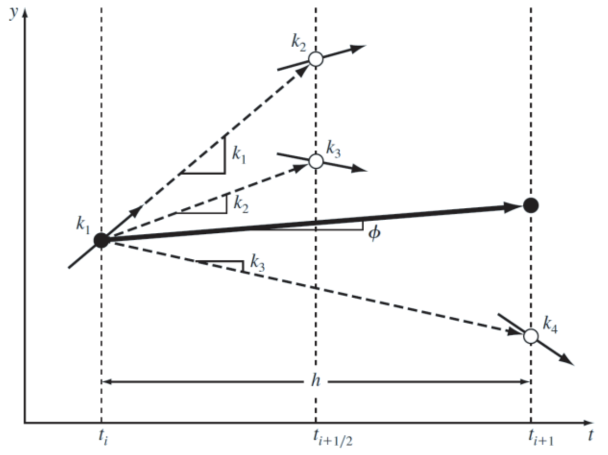


IN-BUILT MATLAB ODE SOLVERS

- MATLAB provides several built-in functions for adaptive methods
  – Most common are ode23, ode45, ode113 (there are others)

- ode45() simultaneously uses 4th and 5th-order Runge-Kutta methods
  – Algorithm developed by Dormand and Prince (1980)
  – Use ode45 first if the characteristics of the system are not well known

4th-order RK



ODE45

column

ind dep fn hd h

[T, Y] = ode45(odefun, tspan, Y0)

`odefun`	A function handle that evaluates the RHS of the differential equation
`tspan`	A vector specifying the interval in ascending order $[t_0 \;\; t_f]$ – displays solution at the adaptive independent values $[t_0 \;\; t_1 \;\; t_2 \ldots t_f]$ – displays solution at specified independent values
`Y0`	Initial condition
`T`	Column vector of the independent variable
`Y`	Solution array. Each row in Y corresponds to the solution at a time returned in the corresponding row of T

[20 MINS]

ACTIVITY: OBJECT II

HEUN.M, OBJECT2.M

An accelerating object heavily resisted by an unknown fluid is described by $\frac{\mathrm{d}v}{\mathrm{d}t}$

Process:

1. Understand the Heun's method by hand
2. Write the Heun's function file
3. Solve the ordinary differential equation

Activity involves:

1. Writing a function file
2. Using ode45()

Equations:

$$\frac{\mathbf{d}v}{\mathbf{d}t} = t - v^2$$

$$v(5) = 1$$

$$h = 0.5$$

$$y_{i+1}^0 = y_i + hf(t_i, y_i)$$

$$y_{i+1} = y_i + \frac{h}{2}\left(f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)\right)$$

MATLAB commands:

```
for i = …
error(…)
y = ones(…)
f = @(x) …
[t, y] = ode45(…)
```

[20 MINS]

ACTIVITY: OBJECT II

HEUN.M, OBJECT2.M

1. Write a function with the following header: [t,y] = heun(dydt,tspan,y0,h)

2. Plot v for $t = 5$ to 10 using
   a. Euler's, Heun's and midpoint methods
   b. ode45

Equations:

$$\frac{dv}{dt} = t - v^2$$

$$v(5) = 1$$

$$h = 0.5$$

ypred

$$y_{i+1}^0 = y_i + hf(t_i, y_i)$$

$$y_{i+1} = y_i + \frac{h}{2}\left(f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)\right)$$

ypred

MATLAB commands:

```
f = @(x,y) …
[t, y] = ode45(…)
```

h=0 v(5)=1 t(i)+h=5 to 5

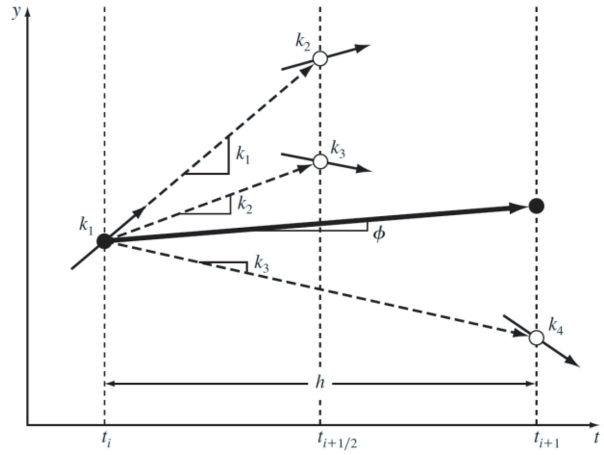
i	t_i	v_i	dv_i/dt_i	t_{i+1}	v_{i+1}^0	dv_{i+1}^0/dt_{i+1}
0	5	1	4	5.5	3	-3.5
1	5.5	1.125	4.2344	6	3.2422	-4.5118
2	6	1.0556				

NOT-EXAMINABLE: 4th-ORDER RUNGE-KUTTA

The 4th-order Runge-Kutta method uses a weighted average of four slopes.

$$y_{i+1} = y_i + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)$$

- $k_1 = hf(x_i, y_i)$
- $k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$
- $k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$
- $k_4 = hf(x_i + h, y_i + k_3)$



IN THIS WORKSHOP

1. Understanding methods for solving ordinary differential equations (ODEs)
   a. Euler's
   b. Heun's
   c. Midpoint
2. Creating function files for ODE-solving methods
3. Solving ODEs
4. Using ode45()

PART B: NUMERICAL METHODS

7. ~~Roots and optimisation~~
8. ~~Curve fitting~~
9. ~~Numerical integration~~
10. ~~Ordinary differential equations~~
11. Linear systems
12. Exam information

You can now complete lab 10!

SUPPLEMENTARY SLIDES

RECAP: EULER'S METHOD

Steps for Euler's method:

$$\boxed{y_{i+1} \cong y_i + hf(t_i, y_i)}$$

1. Starting condition (i = 0)

$$y_1 \cong y_0 + hf(t_0, y_0)$$

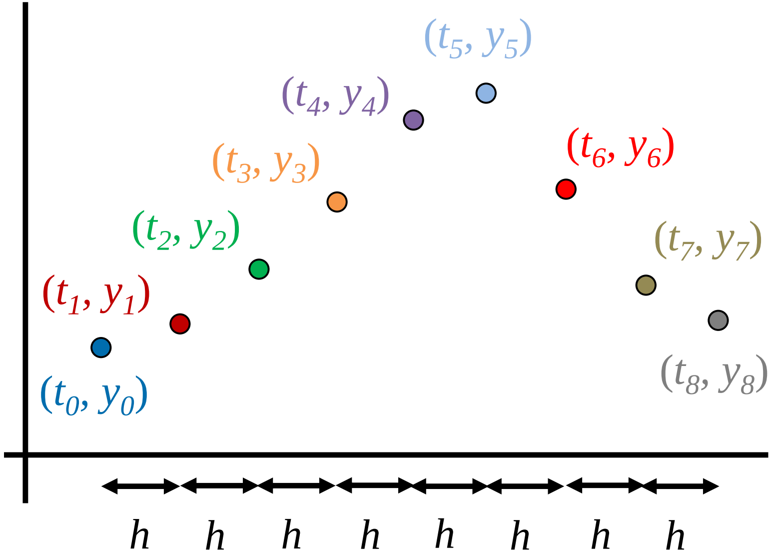
2. Euler's method for i = 1

$$y_2 \cong y_1 + hf(t_1, y_1)$$

3. Euler's method for i = 2

$$y_3 \cong y_2 + hf(t_2, y_2)$$

$$\boxed{y_{i+1} \cong y_i + h\phi}$$

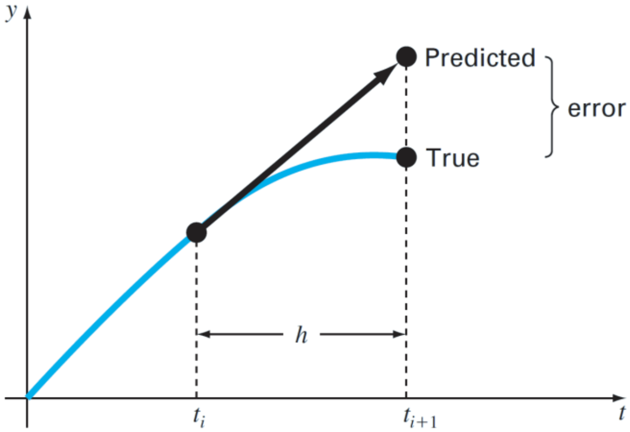


RECAP: ERROR IN EULER'S METHOD

$$y_{i+1} \cong y_i + hf(t_i, y_i)$$

- Local truncation error:
  - Arises from the application of Euler's method over a single step
  - A consequence of the method only being approximate
  - The error in a single step of Euler's method given by

$$\varepsilon_{\text{loc}} \cong \frac{h^2}{2} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\bigg|_{t=t_i}$$



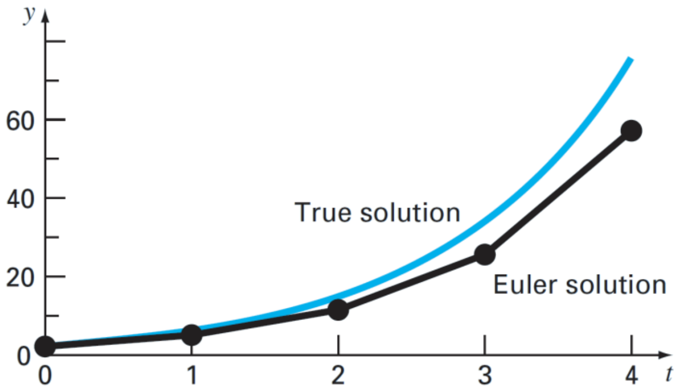
- Error decreases quadratically
  - Smaller step size = smaller error
  - E.g. ½ step size = ¼ error

RECAP: ERROR IN EULER'S METHOD

- Propagated truncation error:
  - Accumulation of local truncation errors from the previous steps

- Global truncation error
  - Arises from an accumulation of local errors PLUS propagation of error in the solution from previous steps

$$\boxed{y_{i+1} \cong y_i + hf(t_i, y_i)}$$



Errors: $\varepsilon_{\text{loc}} \sim O(h^2)$ and $\varepsilon_{\text{global}} \sim O(h)$

RECAP: HEUN'S METHOD

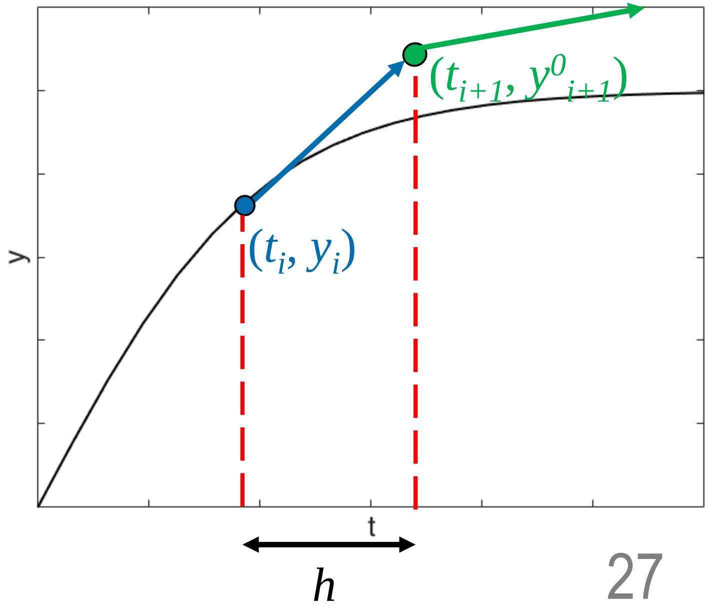
- Heun's method averages
  - The slope at the beginning of the step and
  - The slope at the end of the step
- Predictor step: y_{i+1}^{0}
  - Estimated using Euler's method

$$\boxed{y_{i+1}^{0} = y_i + hf(t_i, y_i)}$$

- Averages slopes at t_i and t_{i+1}

$$\frac{f(t_i, y_i) + f(t_{i+1}, y_{i+1}^{0})}{2}$$

$$\boxed{y_{i+1} \cong y_i + h\phi}$$



RECAP: HEUN'S METHOD

$$y_{i+1} \cong y_i + h\phi$$

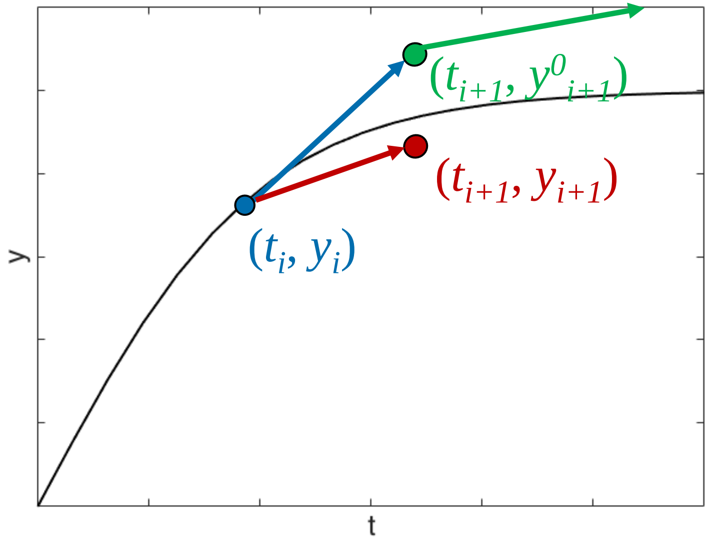
- Corrector step: y_{i+1}
  - Uses the averaged slope at t_i and t_{i+1}

$$y_{i+1} = y_i + \frac{h}{2}\left(f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)\right)$$

- That is, the slope is given by

$$\phi = \frac{h}{2}\left(\frac{\mathrm{d}y_i}{\mathrm{d}t_i} + \frac{\mathrm{d}y_{i+1}^0}{\mathrm{d}t_{i+1}}\right) = \frac{f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)}{2}$$

- Errors: $\varepsilon_{\text{loc}} \sim O(h^3)$ and $\varepsilon_{\text{global}} \sim O(h^2)$



RECAP: MIDPOINT METHOD

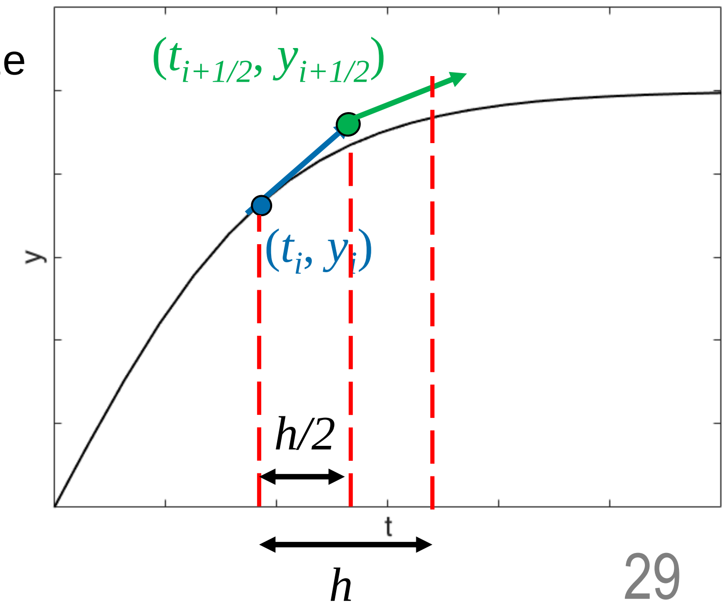
$$y_{i+1} \cong y_i + h\phi$$

- Midpoint method uses the slope at the midpoint
- Predictor step: $y_{i+1/2}$
  - Estimated using Euler's method with half step size

$$y_{i+1/2} = y_i + \frac{h}{2} f(t_i, y_i)$$

- Slope at midpoint is given by

$$f\left(t_{i+1/2}, y_{i+1/2}\right)$$



RECAP: MIDPOINT METHOD

- Corrector step: y_{i+1}
  - Uses the slope at $t_{i+1/2}$ for the full step h

$$y_{i+1} = y_i + hf(t_{i+1/2}, y_{i+1/2})$$

- Slope is given by

$$\phi = \frac{dy_{i+1/2}}{dt_{i+1/2}} = f(t_{i+1/2}, y_{i+1/2})$$

- Errors: $\varepsilon_{\text{loc}} \sim O(h^3)$ and $\varepsilon_{\text{global}} \sim O(h^2)$

$$y_{i+1} \cong y_i + h\phi$$