# ENG2005

Assignment 1

School of Mathematics
Monash University
**Semester 1 2022**

Due: **Friday, 8 April 2022, 23:55 (Clayton)/ 21:55 (Malaysia)**

*Complete the following questions, scan, upload and submit them in Moodle in a pdf file.* ***Late assignments will be penalised*** *at 10% of the maximum mark per day late. Justify all your answers.*

*Learning outcomes:*

- *Use essential concepts related to $m \times n$ linear systems, including linear independence and basis*
- *Manipulate and evaluate double and triple integrals in Cartesian, cylindrical and spherical coordinates*
- *Express and explain mathematical techniques and arguments clearly in words*

**Marks**: If poor explanations are given, subtract up to 25% of marks.

**Question 1** (6 marks)
Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Find the eigenvalues and associated eigenvectors of $A$.

**Solution:** *To find the* ***eigenvalues*** *we find the characteristic equation:*

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} 2-\lambda & -1 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \\ &= (2-\lambda)\left[(1-\lambda)^2 + 1\right] = (2-\lambda)(2 - 2\lambda + \lambda^2). \end{aligned}$$

*So one eigenvalue is $\lambda = 2$. Solving $(2 - 2\lambda^2_\lambda) = 0$ gives $\lambda = 1 + i$, $\lambda = 1 - i$.*
*To find the* ***eigenvectors****, we solve $(A - \lambda I)\mathbf{v} = 0$ for $\mathbf{v}$, for each $\lambda$ in turn.*
*For $\lambda = 2$, we have*

$$A - 2I = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

*and so we have the augmented matrix*

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \sim \begin{array}{c} \\ \\ \textit{R3-R1} \end{array} \left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

*and so we can set the free variable $v_1 = t$, and $v_2 = 0 = v_3$. The eigenvector is*

$$\mathbf{v} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \text{ for any } t \neq 0, \text{ so choosing } t = 1 \text{ gives } \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

*For $\lambda = 1 + i$,*

$$A - (1+i)I = \begin{bmatrix} 1-i & -1 & -1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix}$$

*and so we have the augmented matrix*

$$\left[\begin{array}{ccc|c} 1-i & -1 & -1 & 0 \\ 0 & -i & 1 & 0 \\ 0 & -1 & -i & 0 \end{array}\right] \sim \textit{R2-iR3} \left[\begin{array}{ccc|c} 1-i & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -i & 0 \end{array}\right].$$

*Setting $v_3 = t$ implies (via the third row) that $v_2 = -it$, then (via the second row) that $v_1 = t$. Hence*

$$v = t\begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} \text{ and so if we set } t = 1 \text{ we have eigenvector } v = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

*(other values of t also valid).*
*For the third eigenvalue $\lambda = 1 - i$, this is the conjugate of the second, and so the eigenvector is the conjugate of the second eigenvector,*

$$v = \begin{bmatrix} 1 \\ +i \\ 1 \end{bmatrix}.$$

*(One can also find this in same way as the second.)*

| | |
|---|---|
| ***Marks**: Characteristic equation* | *1 mark* |
| *Real eigenvalue and eigenvector* | *2 marks* |
| *First complex eigenvalue and eigenvector* | *2 marks* |
| *Seond complex eigenvalue and eigenvector* | *1 mark* |

**Question 2** (20 marks)
The crew of a roughly cylindrical spaceship is loading a section of the ship, with radius 2m. At times, the spaceship will spin along the cylindrical axis.
A bulkhead extends across the ship, perpendicular to the cylindrical axis. The load must be secured to the bulkhead, and there are operational constraints on the distribution of mass.
The load consists of

- a circular plate of radius 1. It is in a fixed position, flat against the bulkhead and just touching the outer wall of the ship. It has variable density given by $\rho(r) = \rho_0 r\text{kg}/\text{m}^2$, where $r$ is the distance to the cylindrical axis, and $\rho_0$ is a dimensionless constant.
- two small, extremely dense pieces of metal, each of which has mass $3\rho_0$kg. Because they are so dense, we will treat them as point masses. These may be secured in any position on the bulkhead.

We can find the moment of inertia for a flat body $R$ of density $\rho$, perpendicular to the $z$ axis, rotating around the $z$ axis using

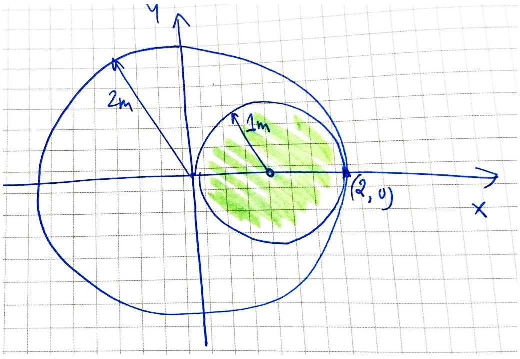
$$I_z = \iint_R (x^2 + y^2)\rho(x, y)\, dx\, dy.$$

a) What is the mass, centre of mass, and moment of inertia (relative to the cylindrical axis) of the plate alone?

**Solution:** ***The mass**: we need only to integrate the density over the region occupied by the object. Denoting the plate by $P$,*

$$\textit{mass} = \iint_P \rho(x, y)\, dA. \tag{1}$$

*We can arrange this like so–*




*We can either use Cartesian coordinates or polar coordinates.*

*In Cartesian coordinates, the plate is given by a disc bounded by the circle $(x-1)^2 + y^2 = 1$, so the region becomes*

$$-\sqrt{1-(x-1)^2} \leq y \leq \sqrt{1-(x-1)^2}, \quad 0 \leq x \leq 2. \tag{2}$$

*The density is $\rho(x,y) = \rho_0\sqrt{x^2+y^2}$. The integral is*

$$\text{mass} = \iint_P \rho(x,y)\,dA = \int_{x=0}^{2}\int_{y=-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} \rho_0\sqrt{x^2+y^2}\,dy\,dx. \tag{3}$$

*In polar coordinates, the circle bounding the plate $(x-1)^2+y^2 = 1$ becomes $(r\cos\theta-1)^2+r^2\sin^2\theta = 1$, which we can rearrange to $r = 2\cos\theta$. The region is given by*

$$0 \leq r \leq 2\cos\theta, \quad -\pi/2 \leq \theta \leq \pi/2. \tag{4}$$

*The area element is $dA = r\,d\theta\,dr$. The integral becomes*

$$\text{mass} = \iint_P \rho(x,y)\,dA = \int_{\theta=\pi/2}^{\pi/2}\int_{r=0}^{2\cos\theta} \rho_0 r\,r\,dr\,d\theta = \rho_0\int_{\theta=\pi/2}^{\pi/2}\int_{r=0}^{2\cos\theta} r^2\,dr\,d\theta. \tag{5}$$

*Integrating in polar coordinates:*

$$\text{mass} = \rho_0\int_{\theta=\pi/2}^{\pi/2}\int_{r=0}^{2\cos\theta} r^2\,dr\,d\theta$$

*integrate with respect to $r$*

$$\begin{aligned}
&= \rho_0\int_{\theta=-\pi/2}^{\pi/2}\left[\frac{1}{3}r^3\right]_{r=0}^{2\cos\theta} d\theta \\
&= \rho_0\int_{\theta=-\pi/2}^{\pi/2}\frac{1}{3}(2\cos\theta)^3\,d\theta \\
&= \rho_0\frac{2^3}{3}\int_{\theta=-\pi/2}^{\pi/2}(\cos\theta)^3\,d\theta
\end{aligned}$$

*substitute $u = \sin\theta$, $du = \cos\theta\,d\theta$, $\cos^2\theta = 1-\sin^2\theta = 1-u^2$*

$$\begin{aligned}
&= \rho_0\frac{2^3}{3}\int_{\theta=-\pi/2}^{\pi/2}(1-u^2)\,du \\
&= \rho_0\frac{2^3}{3}[u-\frac{1}{3}u^3]_{\theta=-\pi/2}^{\pi/2} \\
&= \rho_0\frac{2^3}{3}[\sin\theta-\frac{1}{3}(\sin\theta)^3]_{\theta=-\pi/2}^{\pi/2} \\
&= \rho_0\frac{2^3}{3}\left[1-\frac{1}{3}-(-1+\frac{1}{3})\right] = \rho_0\frac{2^3}{3}\frac{4}{3} = \rho_0\frac{32}{9}.
\end{aligned}$$

*The units here are kg, since we integrate the density (kg/m$^2$) with respect to area (m$^2$).*

***Marks**: A mass integral in some form, e.g.* (1) *1 mark*
*Setting up the region (either* (2) *or* (4)*)* *1 mark*
*Setting up the integral (either* (3) *or* (5)*)* *1 mark*
*Integrating (either in Cartesian or polar)* *2 marks*

***The* centre of mass*:*** *$(\overline{x}, \overline{y})$ is given by*

$$\overline{x} = \frac{1}{m} \iint_P x\rho(x,y)\,dx\,dy, \quad \overline{y} = \frac{1}{m} \iint_P y\rho(x,y)\,dx\,dy, \tag{6}$$

*where $m$ is the mass. If we again use polar coordinates, then $x = r\cos\theta$ and we find*

$$\overline{x} = \frac{1}{m} \iint_P x\rho(x,y)\,dx\,dy = \frac{1}{m}\rho_0 \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} r\cos\theta r^2\,dr\,d\theta. \tag{7}$$

*integrate with respect to r*

$$\begin{aligned}
&= \frac{1}{m}\rho_0 \int_{\theta=-\pi/2}^{\pi/2} \cos\theta \left[\frac{1}{4}r^4\right]_{r=0}^{2\cos\theta} d\theta \\
&= \frac{1}{m}\rho_0 \int_{\theta=-\pi/2}^{\pi/2} \cos\theta \frac{1}{4}(2\cos\theta)^4\,d\theta \\
&= \frac{1}{m}2^2\rho_0 \int_{\theta=-\pi/2}^{\pi/2} (\cos\theta)^5\,d\theta
\end{aligned}$$

*substitute $u = \sin\theta$, $du = \cos\theta\,d\theta$, $\cos^4\theta = (1-\sin^2\theta)^2 = (1-u^2)^2$*

$$= \frac{1}{m}2^2\rho_0 \int_{\theta=-\pi/2}^{\pi/2} (1 - 2u^2 + u^4)\,du$$

*integrate with respect to $u$*

$$\begin{aligned}
&= \frac{1}{m}2^2\rho_0 \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{\theta=\pi/2}^{\pi/2} \\
&= \frac{1}{m}2^2\rho_0 \left[\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\right]_{\theta=\pi/2}^{\pi/2} \\
&= \frac{1}{m}2^2\rho_0 2\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{1}{m}2^2\rho_0\frac{16}{1}5 = \frac{1}{m}\rho_0\frac{64}{15}.
\end{aligned}$$

*Together with $m = \rho_0\frac{32}{9}$ from above, we find*

$$\overline{x} = \frac{9}{32\rho_0}\rho_0\frac{64}{15} = \frac{6}{5}.$$

*We could make a similar calculation to find $\overline{y}$, but notice that the region is symmetric across the line $y = 0$, while the integrand $y\rho$ is* antisymmetric *(the same size, but the opposite sign) across the line $y = 0$— hence the two halves of the integral cancel out, and $\overline{y} = 0$.*

*So the centre of mass of the plate is $(\frac{6}{5}, 0)$. Units here are metres, as a position.*

*Double check: this point is within the plate, and notice that it's a little to the right of the centre of the plate– because the density is greater on the outer edge of the disc (the part closer to the wall of the space ship).*

*Marks: Full marks for solutions using Cartesian or polar coordinates*

| | |
|---|---|
| *A centre of mass formula in some form, e.g. (6)* | *1 mark* |
| *Setting up the integral (e.g (7))* | *1 mark* |
| *Integrating for $\overline{x}$* | *2 marks* |
| *Finding $\overline{y}$* | *1 mark* |

***The moment of inertia of the plate:*** *This is given by integrating the distance from the axis* ***squared****, multiplied by the density, so again using polar coordinates*

$$I_z = \iint_P (x^2+y^2)\rho(x,y)\,dx\,dy. = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{2\cos\theta} r^2(\rho_0 r)r\,dr\,d\theta \tag{8}$$

$$= \rho_0 \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{1}{5}r^5\right]_{r=0}^{2\cos\theta} d\theta \qquad \textit{integrating w.r.t } r$$

$$= \rho_0 \int_{\theta=-\pi/2}^{\pi/2} \frac{1}{5}(2\cos\theta)^5 - 0\,d\theta$$

$$= \rho_0 \frac{2^5}{5}\int_{\theta=-\pi/2}^{\pi/2} (\cos\theta)^5\,d\theta$$

*substitute $u=\sin\theta$, $du=\cos\theta d\theta$, $(\cos\theta)^4=(1-\sin^2\theta)^2=(1-u^2)^2$*

$$= \rho_0\frac{2^5}{5}\int_{u=-1}^{1} 1-2u^2+u^4\,du \qquad \textit{integrate w.r.t } u$$

$$= \rho_0\frac{2^5}{5}\left[u-\frac{2}{3}u^3+\frac{1}{5}u^5\right]_{u=-1}^{1}$$

$$= \rho_0\frac{2^5}{5}\left(2-\frac{4}{3}+\frac{2}{5}\right) = \rho_0\frac{2^5}{5}\frac{16}{15} = \rho_0\frac{512}{75} = 6.83\rho_0.$$

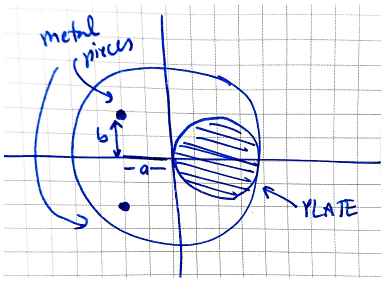
*Marks: Full marks for solutions using Cartesian or polar coordinates*

| | |
|---|---|
| *Setting up the integral (e.g (8))* | *1 mark* |
| *Integrating* | *2 marks* |

b) For this question, let $\rho_0 = 1$. Where should the two pieces of metal be attached so that the load (plate and pieces of metal) as a whole has centre of mass located on the cylindrical axis, and moment of inertia 15kg? (Answer to the nearest centimeter.)

**Solution:** *To ensure the centre of mass of the load is on the cyclindrical axis (that is, so that $\overline{x}=0=\overline{y}$) we observe that the centre of mass of the ensemble is given by adding that of the component parts, then dividing by the sum of masses.*

*For the $y$-component of the centre of mass of the pieces to be zero, we begin by* ***assuming*** *that the pieces of metal are located at $(-a,b)$ and $(-a,-b)$ (in $xy$-coordinates, with the $x$-axis running through the centre of the plate.). The $y$-components of the two pieces cancel with each other.*



*Then the $x$-coordinate of the centre of mass of the ensemble is zero if*

$$0 = \overline{x} = \frac{1}{m_{\textit{ensemble}}}\left(\iint_P x\rho\,dA\right.$$

$$\left. + m(\textit{piece 1}) \times (\ x\textit{-coordinate of piece 1}) + m(\textit{piece 2}) \times (\ x\textit{-coordinate of piece 2})\right)$$

$$= \frac{1}{m_{\textit{ensemble}}}\left(\rho_0\frac{64}{15} \quad + 3\rho_0(-a) + 3\rho_0(-a)\right)$$

*Solving for $a$ gives $a = 64/90 = 32/45$.*

*For the piece of metal at $(-a, b)$, this has moment of inertia given by m(piece 1) $\times$ (distance to axis)$^2$, that is*

$$I_z(\textit{piece 1}) = m(\textit{piece }1)(a^2 + b^2) = 3\rho_0((32/45)^2 + b^2).$$

*The other piece has the same moment of inertia. The moment of inertia of the ensemble is given by summing the various pieces, that is*

$$\begin{aligned} I_z &= I_z(\textit{plate}) + I_z(\textit{piece 1}) + I_z(\textit{piece 2}) \\ &= \rho_0\frac{512}{75} + 3\rho_0((32/45)^2 + b^2) + 3\rho_0((32/45)^2 + b^2) \end{aligned}$$

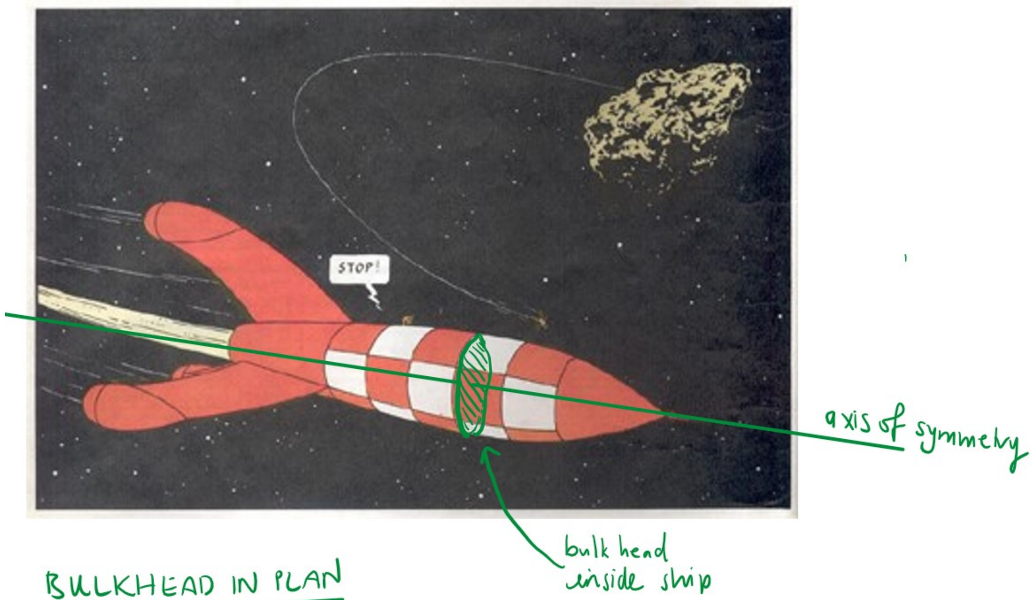
*Setting $I_z = 15$ and $\rho_0 = 1$ we find*

$$6b^2 = 15 - \frac{512}{75} - 6(32/45)^2,$$

*and so $b = 0.925$. Hence the pieces of metal should be located at $(-0.71, 0.93)$, $(-0.71, -0.93)$ respectively, given in metres.*

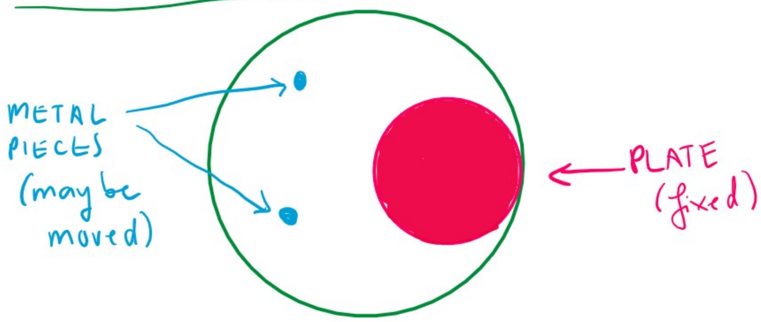
***Note: the assumption that the $x$-coordinate of the two pieces is the same is not necessary— other solutions are possible, and acceptable, as long as everything is explained.***

| | |
|---|---|
| ***Marks**: Explanation of implementation of the centre of mass condition* | *3 marks* |
| *Explanation of implementation of the inertia condition* | *3 marks* |
| *Correct answer* | *1 mark* |




[1]

BULKHEAD IN PLAN




[1]Image credit: *Explorers on the Moon* (1954), Hergé