# ENG2005

Assignment 2

School of Mathematics
Monash University
**Semester 1 2022**

Due: **Friday, 29 April 2022, 23:55 (Clayton)/ 21:55 (Malaysia)**

*Complete the following questions, scan, upload and submit them in Moodle in a pdf file.* ***Late assignments will be penalised*** *at 10% of the maximum mark per day late. Justify all your answers.*

*Learning outcomes:*

- *Perform change of variables for multivariable functions with the chain rule, use polar coordinates, represent 2D and 3D curves parametrically and solve line integrals on these curves*
- *Manipulate and evaluate double and triple integrals in Cartesian, cylindrical and spherical coordinates*
- *Calculate the gradient, divergence and curl vector operations, and apply these in the evaluation of surface and volume integrals through the Gauss (divergence) and Stokes theorems*
- *Use MATLAB and other appropriate software to assist in understanding these mathematical techniques*
- *Express and explain mathematical techniques and arguments clearly in words*

**Marks**:

- Solutions must include clear justification as appropriate
- If solution is given but NO justification (e.g for partial derivatives calculated using wolfram alpha or similar), award no more than 1/4 of available marks.
- If justification is unclear then awards partial marks only.

1. Consider two charged particles, separated by distance $d > 0$, and located at points $(0, \frac{d}{2}, 0)$ and $(0, -\frac{d}{2}, 0)$ respectively. One has charge $q$ and **electric potential**

$$\varphi_1(x, y, z) = \frac{q}{\sqrt{x^2 + (y - \frac{d}{2})^2 + z^2}}$$

and the other has charge $(-q)$ and electric potential

$$\varphi_2(x, y, z) = \frac{-q}{\sqrt{x^2 + (y + \frac{d}{2})^2 + z^2}}.$$

Taken as a single system, the electric potential is $\varphi = \varphi_1 + \varphi_2$. The **electric field** is given by

$$\mathbf{E} = -\nabla\varphi.$$

(a) Find $\mathbf{E}$, and sketch the $\mathbf{i}, \mathbf{j}$ component of the vector field $\mathbf{E}$ in the plane $z = 0$. [1 mark]

**Solution:** *First for $\varphi_1$:*

$$\nabla\varphi_1 = \frac{\partial}{\partial x}\varphi_1\mathbf{i} + \frac{\partial}{\partial y}\varphi_1\mathbf{j} + \frac{\partial}{\partial z}\varphi_1\mathbf{k}$$

*Here*

$$\begin{aligned}\frac{\partial}{\partial x}\varphi_1 &= \frac{\partial}{\partial x}\left[q\left(x^2 + (y - \frac{d}{2})^2 + z^2\right)^{-1/2}\right] \\ &= q\left(-\frac{1}{2}\right)\left(x^2 + (y - \frac{d}{2})^2 + z^2\right)^{-3/2}(2x) \\ &= \frac{-qx}{(x^2 + (y - \frac{d}{2})^2 + z^2)^{3/2}}\end{aligned}$$

*and similarly*

$$\frac{\partial}{\partial y}\varphi_1 = \frac{-q(y-d/2)}{(x^2+(y-\frac{d}{2})^2+z^2)^{3/2}}, \qquad \frac{\partial}{\partial z}\varphi_1 = \frac{-qz}{(x^2+(y-\frac{d}{2})^2+z^2)^{3/2}},$$

*so*

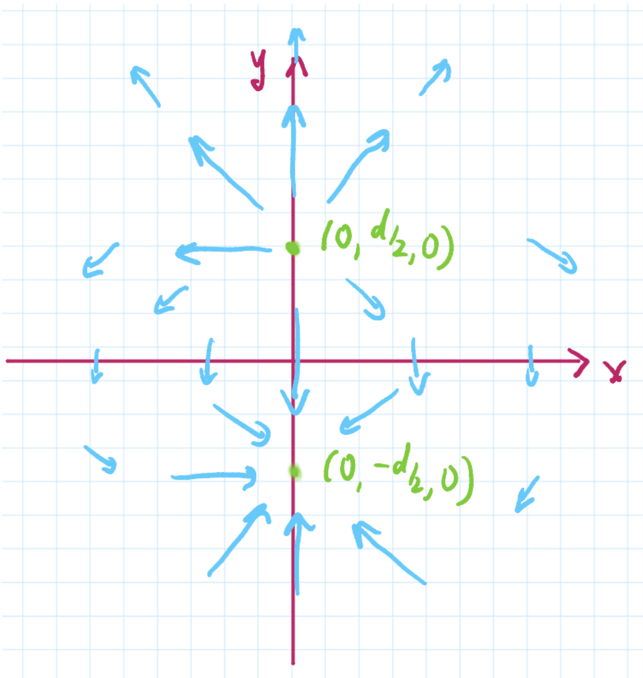
$$\nabla\varphi_1 = \frac{-q}{(x^2+(y-d/2)^2+z^2)^{3/2}}\left[x\mathbf{i}+(y-d/2)\mathbf{j}+z\mathbf{k}\right];$$

*and similarly for $\varphi_2$:*

$$\nabla\varphi_2 = \frac{q}{(x^2+(y+d/2)^2+z^2)^{3/2}}\left[x\mathbf{i}+(y+d/2)\mathbf{j}+z\mathbf{k}\right].$$

*So we have*

$$\begin{aligned}\mathbf{E} &= -\nabla\varphi_1 - \nabla\varphi_2\\ &= \frac{q}{(x^2+(y-d/2)^2+z^2)^{3/2}}\left[x\mathbf{i}+(y-d/2)\mathbf{j}+z\mathbf{k}\right] - \frac{q}{(x^2+(y+d/2)^2+z^2)^{3/2}}\left[x\mathbf{i}+(y+d/2)\mathbf{j}+z\mathbf{k}\right]\end{aligned}$$



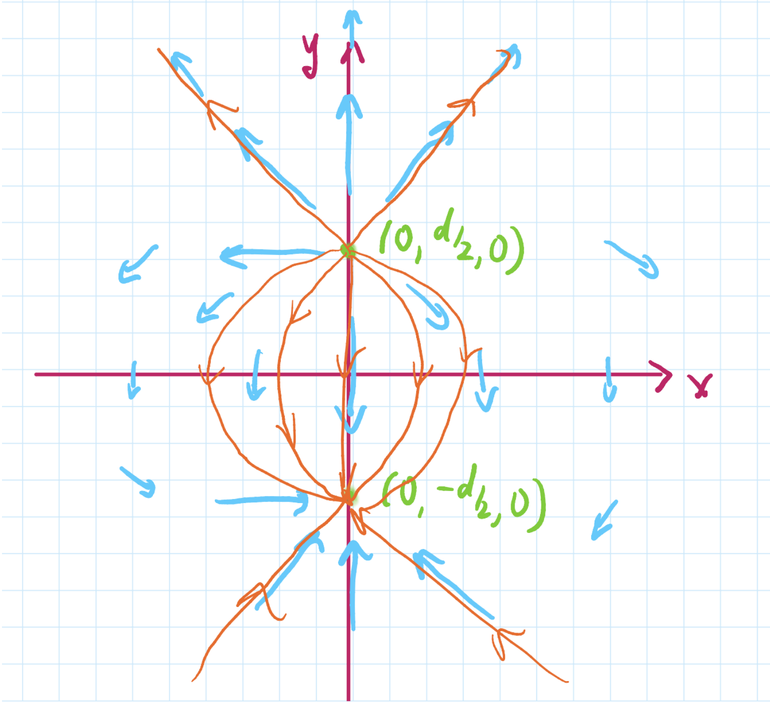

***Marks:***

*Calculate E* $\frac{1}{2}$ *mark*

*Sketch: vectors point outwards from first charge, inwards to second charge, smaller at greater distance from charges* $\frac{1}{2}$ *mark*

*Vector field may be drawn using software package, but should show above features.*

(b) Sketch some illustrative field lines (at least 6) for $\mathbf{E}$. *(Note: you can draw the field lines in the $z = 0$ plane, as in the previous question.)* [1 mark]




**Solution:**

***Marks:***

*Sketch: may be drawn by hand or software.* *1 mark*

(c) Find the field line for $\mathbf{E}$ passing through the point $(0, 0, 0)$ (either as a parameterised curve or as an equation). [1 mark]

**Solution:** *Per the sketch above, it looks like the straight line between the first and second charges is a field line, that is,*

$$\mathbf{r}(t) = (1-t)(d/2)\mathbf{j} + t(-d/2)\mathbf{j} = d\left(\frac{1}{2} - t\right)\mathbf{j}, \quad 0 \le t \le 1. \tag{1}$$

*We can check this: it certainly passes through $(0, 0, 0)$ (at $t = \frac{1}{2}$), and the tangent to this curve is*

$$\frac{d\mathbf{r}}{dt} = -d\mathbf{j}.$$

*Along this curve, we have*

$$\begin{aligned}
\mathbf{E}(\mathbf{r}(t)) &= \frac{q\,[x\mathbf{i} + (y - d/2)\mathbf{j} + z\mathbf{k}]}{(x^2 + (y - d/2)^2 + z^2)^{3/2}} - \frac{q\,[x\mathbf{i} + (y + d/2)\mathbf{j} + z\mathbf{k}]}{(x^2 + (y + d/2)^2 + z^2)^{3/2}}\Bigg|_{(x,y,z)=(0,y(t),0)} \\
&= \frac{q}{((y - d/2)^2)^{3/2}}\,[(y - d/2)\mathbf{j}] - \frac{q}{((y + d/2)^2)^{3/2}}\,[(y + d/2)\mathbf{j}] \\
&= \mathbf{j}q\left[\frac{(y - d/2)}{|y - d/2|^3} - \frac{(y + d/2)}{|y + d/2|^3}\right]
\end{aligned}$$

*— the important point here is that this is purely in the $-\mathbf{j}$ direction, and so tangent to the curve $\mathbf{r}(t)$ above. Hence $\mathbf{r}(t)$ is a field line.*

*Also acceptable: the curve*

$$\{(x, y, z) : x = 0, -d/2 \le y \le d/2, z = 0\}. \tag{2}$$

***Marks:***

*Identify an appropriate line, either as (**??**) or (**??**).* $\frac{1}{2}$ *mark*

*Justification that it is tangent to the field* $\frac{1}{2}$ *mark*

(d) A screen is placed between the two charges, occupying the region $\{(x, y, z) : y = 0\}$. What is the (pointwise) flux through the screen at a point $(x_0, 0, z_0)$? (You may take the normal to the screen to point in the direction of increasing $y$.) [2 marks]

**Solution:** *The unit normal to the screen is* $\mathbf{j}$*, and so*

$$\begin{aligned}\mathbf{E}\cdot\mathbf{n} &= \left(\frac{q\,[x\mathbf{i} + (y-d/2)\mathbf{j} + z\mathbf{k}]}{(x^2 + (y-d/2)^2 + z^2)^{3/2}} - \frac{q\,[x\mathbf{i} + (y+d/2)\mathbf{j} + z\mathbf{k}]}{(x^2 + (y+d/2)^2 + z^2)^{3/2}}\right)\cdot\mathbf{j} \\ &= \left(\frac{q\,[(y-d/2)\mathbf{j}]}{(x^2 + (y-d/2)^2 + z^2)^{3/2}} - \frac{q\,[(y+d/2)\mathbf{j}]}{(x^2 + (y+d/2)^2 + z^2)^{3/2}}\right) \qquad (3)\end{aligned}$$

*evaluated at a point on the screen* $(x, y, z) = (x_0, 0, z_0)$

$$\begin{aligned} &= \left(\frac{q\,[-d/2]}{(x_0^2 + (d/2)^2 + z_0^2)^{3/2}} - \frac{q\,[(d/2)]}{(x_0^2 + (d/2)^2 + z_0^2)^{3/2}}\right) \\ &= \frac{-qd}{(x_0^2 + (d/2)^2 + z_0^2)^{3/2}}. \qquad (4)\end{aligned}$$

***Marks:***

| | |
|---|---|
| *Identification of the normal* | $\frac{1}{2}$ *mark* |
| *Calculation of* $\mathbf{E}\cdot\mathbf{n}$ *(i.e* (**??**) *or similar)* | *1 mark* |
| *Simplification at* $(x, y, z) = (x_0, 0, z_0)$ *(i.e* (**??**) *or similar)* | $\frac{1}{2}$ *mark* |

(e) What is the net flux through the portion of the screen $\{(x, y, z) : y = 0, x^2 + z^2 \le 4\}$? [2 marks]

**Solution:** *Here we use a double integral to integrate the flux (calculated in the previous question) over the portion of the screen (which we label as* $S$*)*

$$\iint_S \mathbf{E}\cdot\mathbf{n}\,dS = \iint_S \frac{-qd}{(x^2 + (d/2)^2 + z^2)^{3/2}}\,dS. \qquad (5)$$

*we parameterise* $S$ *by* $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{k}$, $0 \le v \le 2\pi$, $0 \le u \le 2$, *and hence with the polar coordinates Jacobian* $|J(u, v)| = u$

$$\begin{aligned} &= \int_{u=0}^{2}\int_{v=0}^{2\pi} \frac{-qd}{(x^2 + (d/2)^2 + z^2)^{3/2}}\,u\,dv\,du. \qquad (6) \\ &= -qd\int_{u=0}^{2}\int_{v=0}^{2\pi} \frac{1}{(u^2 + (d/2)^2)^{3/2}}\,u\,dv\,du.\end{aligned}$$

*integrate with respect to* $v$

$$= -qd(2\pi)\int_{u=0}^{2} \frac{1}{(u^2 + (d/2)^2)^{3/2}}\,u\,du$$

*substitute* $u^2 + (d/2)^2 = w$*, so that* $dw = 2u\,du$

$$\begin{aligned} &= -qd\pi\int_{u=0}^{2} w^{-3/2}\,dw \\ &= -qd\pi\left[-2w^{-1/2}\right]_{u=0}^{2} = -qd\pi\left[-2(u^2 + (d/2)^2)^{-1/2}\right]_{u=0}^{2} \\ &= 2q\pi\left[\frac{d}{(4 + (d/2)^2)^{1/2}} - 2\right].\end{aligned}$$

***Marks:***

| | |
|---|---|
| *Correct set up for integral (ie RHS of* (**??**)*, or* (**??**) *)* | *1 mark* |
| *Evaluation of integral* | *1 mark* |

(f) Let $\mathcal{C}$ be the closed curve around the edge of the portion of the screen in question (**??**), with a positive orientation with respect to the normal vector. Find

$$\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r}.$$

[2 marks]

**Solution:** *Here we can use that* $\mathbf{E}$ *is conservative (since it is the gradient of a potential function), so* $\nabla \times \mathbf{E} = 0$ *(or we can calculate the curl directly). Stokes' theorem applies, since on* $S$, $\mathbf{E}$ *has continuous first partial derivatives (the only place where it doesn't is at the locations of the two charges, which can be excluded from the region of interest) and gives that*

$$\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{E}\, dS = 0.$$

***Alternative approach:*** *Notice that in the* $y = 0$ *plane,*

$$\begin{aligned}\mathbf{E}|_{y=0} &= \frac{q}{(x^2 + (d/2)^2 + z^2)^{3/2}} [x\mathbf{i} + (d/2)\mathbf{j} + z\mathbf{k}] - \frac{q}{(x^2 + (d/2)^2 + z^2)^{3/2}} [x\mathbf{i} + (d/2)\mathbf{j} + z\mathbf{k}] \\ &= \frac{qd}{(x^2 + (d/2)^2 + z^2)^{3/2}}\mathbf{j},\end{aligned}$$

*that is, purely in a* $\mathbf{j}$ *direction.*
*Parameterising the curve around the piece of screen by*

$$\mathbf{r}(t) = 2\cos(t)\mathbf{i} - 2\sin(t)\mathbf{k}$$

*we can see that the tangent vector*

$$\frac{\partial}{\partial t}\mathbf{r}(t) = -2\sin(t)\mathbf{i} - 2\cos(t)\mathbf{k}$$

*is purely in* $\mathbf{i}, \mathbf{k}$ *directions, so*

$$\mathbf{E} \cdot \frac{\partial}{\partial t}\mathbf{r}(t) = 0.$$

*Consequently*

$$\int_C \mathbf{E} \cdot d\mathbf{r} = 0.$$

***Marks:***

| | |
|---|---|
| *Either* ***Stokes approach:*** *Justify that* $\nabla \times \mathbf{E} = 0$ | *1 mark* |
| *Apply Stokes* | *1 mark* |
| *Or,* ***line integral:*** *Find* $\mathbf{E}$ *and* $\frac{\partial \mathbf{r}}{\partial r}$ | *1 mark* |
| *Evaluate* $\int \mathbf{E} \cdot d\mathbf{r}$ | *1 mark.* |

(g) Let $B$ be a ball of radius $R$ centred at $(2R, 2R, 2R)$, where $R >> d$ ($R$ is much larger than $d$). Let $T$ be the boundary of the ball. Calculate the net electric flux through the boundary of the ball,

$$\iint_T \mathbf{E} \cdot \mathbf{n}\, dS.$$

Hint: you can use an integral theorem here. [4 marks]

**Solution:**

*The divergence theorem gives that*

$$\iint_T \mathbf{E} \cdot \mathbf{n}\, dS = \iiint_B \nabla \cdot \mathbf{E}\, dV$$

*so calculating the divergence:*

$$\nabla \cdot \mathbf{E} = -\nabla \cdot \nabla\varphi_1 - \nabla \cdot \nabla\varphi_2$$

*where*

$$-\nabla\cdot\nabla\varphi_1 = \frac{\partial}{\partial x}\frac{-qx}{(x^2+(y-d/2)^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{-q(y-d/2)}{(x^2+(y-d/2)^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{-qz}{(x^2+(y-d/2)^2+z^2)^{3/2}}$$

*Each of these terms is very similar, with*

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{-qx}{(x^2+(y-d/2)^2+z^2)^{3/2}}\right) &= \frac{-q}{(x^2+(y-d/2)^2+z^2)^{3/2}} + \frac{qx(3x)}{(x^2+(y-d/2)^2+z^2)^{5/2}}\\ &= \frac{q}{(x^2+(y-d/2)^2+z^2)^{5/2}}\left[x^2+(y-d/2)^2+z^2-3x^2\right]\end{aligned}$$

$$\frac{\partial}{\partial y}\left(\frac{-q(y-d/2)}{(x^2+(y-d/2)^2+z^2)^{3/2}}\right) = \frac{q}{(x^2+(y-d/2)^2+z^2)^{5/2}}\left[x^2+(y-d/2)^2+z^2-3(y-d/2)^2\right]$$

$$\frac{\partial}{\partial z}\left(\frac{-qz}{(x^2+(y-d/2)^2+z^2)^{3/2}}\right) = \frac{q}{(x^2+(y-d/2)^2+z^2)^{5/2}}\left[x^2+(y-d/2)^2+z^2-3z^2\right]$$

*Notice that since the ball is well away from the charge at* $(0, d/2, 0)$*, the term in the denominator is never zero, and these are all well defined and continuous. Adding the three terms above we find*

$$-\nabla\cdot\nabla\varphi_1 = \frac{q}{(x^2+(y-d/2)^2+z^2)^{5/2}}\left[3(x^2+(y-d/2)^2+z^2)-3x^2-3(y-d/2)^2-3z^2\right] = 0.$$

*We can make a similar calculation (with* $+d/2$ *instead of* $-d/2$*) for* $\varphi_2$*, so that*

$$-\nabla\cdot\nabla\varphi_2 = \frac{q}{(x^2+(y+d/2)^2+z^2)^{5/2}}\left[3(x^2+(y+d/2)^2+z^2)-3x^2-3(y+d/2)^2-3z^2\right] = 0.$$

*Again, since the ball does not include the charge at* $(0, -d/2, 0)$ *the term in the denominator is never zero, and the partial derivatives are continuous. Hence the divergence theorem can be applied to* $\mathbf{E}$*. The divergence is*

$$\nabla\cdot\mathbf{E} = -\nabla\cdot\nabla\varphi_1 - \nabla\cdot\nabla\varphi_2 = 0.$$

*We conclude that*

$$\begin{aligned}\iint_T \mathbf{E}\cdot\mathbf{n}\,dS &= \iiint_B \nabla\cdot\mathbf{E}\,dV \quad \textit{by the divergence theorem}\\ &= 0 \quad \textit{since } \nabla\cdot\mathbf{E}=0.\end{aligned}$$

***Marks****:*

| | |
|---|---|
| *Calculation that* $\nabla\cdot\mathbf{E}=0$ | *2 marks* |
| *Check that conditions of div theorem are satisfied (eg that ball does not enclose a point where the partial derivatives are not continuous)* | *1 mark* |
| *Use of divergence theorem* | *1 mark* |
| *Alternatively: full marks for calculating the flux across the surface correctly* | |
| *Alternatively: if an approximate flux integral is calculated (e.g by assuming that far from the charges,* $(y\pm d/2)^2\sim y^2$*)* ***with appropriate error bounds*** | *max 2 marks* |
| *For a physics answer of the form "the net flux is the net enclosed charge and hence zero" or similar* | *max 1 mark* |

2. **[10 marks]**

In this question we consider the distribution of smoke in an open-plan, single-storey house. The floor of the house is given by $z = 0$. The origin of the floor plan at $x = 0$ and $y = 0$ is centred in the kitchen where dinner is burning. The smoke disperses throughout the house in such a way that the density of smoke $u(x, y, z, t)$ changes with time $t$. The density of smoke is given by

$$u(x, y, z, t) = (4\pi Dt)^{-1} \exp\left(\frac{-(x^2 + y^2)}{4Dt} - \frac{\alpha z}{D}\right),$$

where $D = 3$ is the Einstein diffusion coefficient and $\alpha = mg/\gamma = 1$ where $mg$ is the force of gravity on a smoke particle and $\gamma$ is the coefficient of friction.

The only smoke detector in the house is in the living room. The living room is a rectangular prism $\Omega$ given by $x_{\text{min}} < x < x_{\text{min}} + 2$, $y_{\text{min}} < y < y_{\text{min}} + 3$ and $0 < z < 3$. The smoke detector will activate if total smoke mass in the living room $\rho$ is above 0.015. The value of $x_{\text{min}}$ is the final digit of your student ID number and the value of $y_{\text{min}}$ is the penultimate digit of your student ID number.

You have two numerical tasks as follows which you should do in a single .m file. You should submit this code by submitting the text (either as a screenshot or as text) in your submission (you do not need to submit the actual .m file).

(a) You need to numerically compute the total mass of smoke in the living room

$$\rho(t) = \iiint_{\Omega} u(x, y, z, t)\, dV.$$

To do this, you should first discretise the living room rectangular 'voxels'. For simplicity, you should use $N = 50$ intervals in each dimension of the room to discretise the room, each with width $h_x$, breadth $h_y$ and height $h_z$. Assuming that the value of $u$ in each voxel is uniform and equal to $u(x_i, y_j, z_k, t)$ where $(x_i, y_j, z_k)$ is the centre of each voxel, the total mass of smoke at any moment in time approximated by

$$\rho(t) \approx \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} u(x_i, y_j, z_k, t)\, h_x h_y h_z.$$

Plot $\rho(t)$ over a suitable interval of time in blue. Making reference to your plot, does the smoke detector in the living room ever activate and, if so, when?

**Solution:**

2a)

```
N = 50;
xmin = 4; %last number in ID
xmax = xmin+2;
x = linspace(xmin,xmax,N+1);
hx = x(2)-x(1);
ymin = 4; %second last number in ID
ymax = ymin+3;
```

← These need to be updated to the individual students ID

(1 mark)

(* 1 mark)

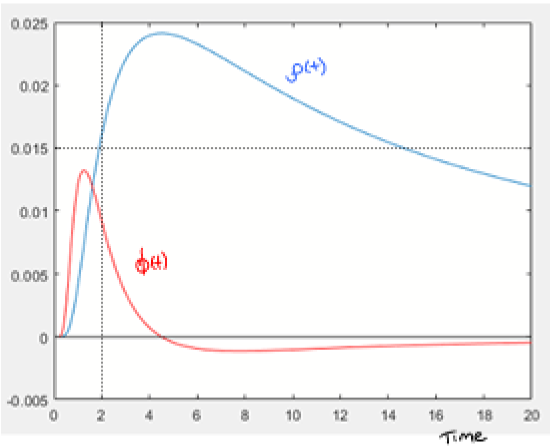
```
% ----- Volume Integral (Part a) -----
rho = zeros(Nt,1);

for n = 1:Nt
    rhoval = 0;
    for i = 1:N
        for j = 1:N
            for k = 1:N

                rhoval = rhoval + u(x(i),y(j),z(k),t(n),D,alpha)*hx*hy*hz;

            end
        end
    end
    rho(n) = rhoval;
end
```

*

\#

(# 2marks)

*




The blue curve is ρ(t) and since it reaches 0.015 at about t=2 this is when the alarm activates.

*Note this will not happen for all students if their ID numbers are large since the smoke is sufficiently diluted before reaching the room.

**Marks**:

i. The living room is located at the correct place (different for each student) by defining xmin and ymin using their ID number. 1 mark

ii. The loops for the volume integral are correct and the counter for the sum is initialised in the correct place between the time and spatial for loops and stored likewise in the correct place after the spatial loops. 1 mark.

iii. rhoval is updated correctly but maybe a small mistake in either u or the elemental volume ... 1 mark.

iv. ... and if it is completely correct 1 mark.

v. The graph is presented and the student correctly identifies if rho gets to 0.015 (some will and some wont) from the graph and correctly identifies the time that it does (if it does). Half a mark if the graph or if any conclusion is missing. 1 mark.

(b) Numerically you should find the total flux of smoke into the living room $\phi(t)$. Since the flux of smoke is 0 out of the top and bottom of the living room, you only need to compute

$$\nabla\phi(t) = \iint_S D\nabla u(x,y,z,t)\cdot\mathbf{n}\,dS,$$

where $S$ represents the four walls of the room and $\mathbf{n}$ is the outward facing normal and $\nabla$ is the gradient with respect to spatial coordinates. To calculate the flux numerically, you can do this by adding the flux over each of the four walls and you can compute the integral over a single wall using the same discretisation as the first part of this exercise. That is, for the wall at $x = x_{\min}$ the component of the total flux approximated by

$$\sum_{i=1}^{N}\sum_{j=1}^{N} D\nabla u(x_{\min}, y_i, z_j, t)\cdot\mathbf{n}\; h_y h_z.$$

I leave the calculation over the other three walls to you to sum together for $\phi(t)$. You should plot this flux in red on the same axes as $\rho(t)$ from the previous part of the exercise. What do you notice about the relationship between $\rho(t)$ and $\phi(t)$ just by looking at the curves? Can you explain this? You may like to plot a horizontal line in Matlab to visualise better the $t$-axis.

**Solution:**

2b)

We have $u(x,y,z,t) = \frac{1}{4\pi D t}\exp\left(-\frac{(x^2+y^2)}{4Dt} - \frac{\alpha z}{D}\right)$

To calculate $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial y}{\partial z}\right) = \left(\frac{-2x}{4Dt}\cdot\frac{1}{4\pi Dt}\exp\left(-\frac{(x^2+y^2)}{4Dt} - \frac{\alpha z}{D}\right), \frac{-2y}{4Dt}\cdot\frac{1}{4\pi Dt}\exp\left(-\frac{(x^2+y^2)}{4Dt} - \frac{\alpha z}{D}\right), -\frac{\alpha}{D}\cdot\frac{1}{4\pi Dt}\exp\left(-\frac{(x^2+y^2)}{4Dt} - \frac{\alpha z}{D}\right)\right)$

$$= \left(\frac{-2x}{4Dt}u, \frac{-2y}{4Dt}u, \frac{-\alpha}{D}u\right)$$ (simpler to input into code)

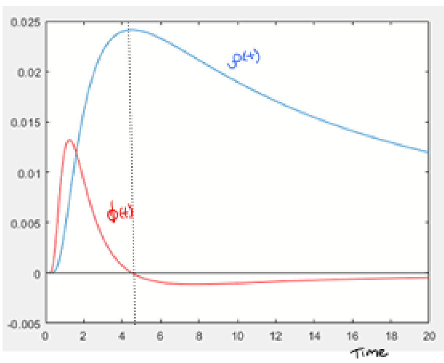
```
u = @(x,y,z,t,D,alpha) 1/(4*pi*D*t) * exp(-(x^2+y^2)/(4*D*t) - alpha*z/D); %alpha = mg/(gamma)
gradu = @(x,y,z,t,D,alpha) [-2*u(x,y,z,t,D,alpha)*x/(4*D*t); ...
    -2*u(x,y,z,t,D,alpha)*y/(4*D*t); ...
    -u(x,y,z,t,D,alpha)*alpha/D];
```

} 1 mark.

```
% ---- Flux -----

phi = zeros(Nt,1);

for n = 1:Nt
    phival = 0;
    for i = 1:N
        for j = 1:N

            %x = xmin surface
            phival = phival + D*([-1,0,0]*gradu(xmin,y(i),z(j),t(n),D,alpha))*hy*hz;
            %x = xmax surface
            phival = phival + D*([1,0,0]*gradu(xmax,y(i),z(j),t(n),D,alpha))*hy*hz;
            %y = ymin surface
            phival = phival + D*([0,-1,0]*gradu(x(i),ymin,z(j),t(n),D,alpha))*hx*hz;
            %y = ymax surface
            phival = phival + D*([0,1,0]*gradu(x(i),ymax,z(j),t(n),D,alpha))*hx*hz;

        end
    end
    phi(n) = phival;
end
```

* 1 mark

} 2 marks




It is clear to see that as $\phi(t)>0$ $\rho(t)$ increases, as $\phi(t)<0$ then $\rho(t)$ decreases. This is because $\rho$ is the total smoke in the room whilst $\phi(t)$ describes the rate of flow into and out of the room since there is no flow out top or bottom.
Indeed, the relationship is more precisely $\rho'(t) = \phi(t)$.

**Marks**:

i. The gradient is calculated correctly. Full marks if working is shown leads to correct result. If simply transcribed incorrectly (but shown correct in working) deduct no marks. If the correct result is in the code but no communication about how this was obtained, half marks. 1 mark.

ii. Half a mark for each of the lines which adds elements of flux to the flux calculation. There are four walls so this is worth 2 marks. Note that each line is similar and contain four main parts 1. the normal vector, 2. the call to gradu, 3. the correct arguments of this gradient and 4. the correct element of area. If one of these is incorrect in all 4 expressions then you can consider this as 3/4 complete and therefore worth 1.5 marks, etc. 2 marks

iii. The flux is summed correctly using for loops and stored in phi(n). 1 mark

iv. There is a reasonable attempt at explaining, using the graph, that phi(t) is the derivative of rho(t) and that this is because the rate at which matter builds up is equal to the rate (or flux) at which it flows into the volume (as no material flows out the top or bottom and it is not created or destroyed). They can also make reference to the fact rho(t) changes from being increasing to decreasing when phi(t)=0 to make this point. There ought to be 1. some reference to the graph behavior and 2. some indication that they understand why this is in a practical sense. 1 mark.

I include with this assignment a skeleton code to help you complete this question.

**Solution:**

```
clear all
close all

D = 3;
alpha = 1;
u = @(x,y,z,t,D,alpha) 1/(4*pi*D*t) * exp(-(x^2+y^2)/(4*D*t) - alpha*z/D)
    ; %alpha = mg/(gamma)
gradu = @(x,y,z,t,D,alpha) [-2*u(x,y,z,t,D,alpha)*x/(4*D*t); ...
    -2*u(x,y,z,t,D,alpha)*y/(4*D*t); ...
    -u(x,y,z,t,D,alpha)*alpha/D];


N = 50;
xmin = 4; %last number in ID
xmax = xmin+2;
x = linspace(xmin,xmax,N+1);
hx = x(2)-x(1);
ymin = 4; %second last number in ID
ymax = ymin+3;
y = linspace(ymin,ymax,N+1);
hy = y(2)-y(1);
zmin = 0;
zmax = 3;
z = linspace(zmin,zmax,N+1);
hz = z(2)-z(1);

x = x(1:end-1)+hx/2;
y = y(1:end-1)+hy/2;
z = z(1:end-1)+hz/2;

tmin = 0;
tmax = 20;
Nt = 200;
t = linspace(tmin,tmax,Nt+1); t = t(2:end);

% ----- Volume Integral (Part a) -----
rho = zeros(Nt,1);

for n = 1:Nt
    rhoval = 0;
```

```
    for i = 1:N
        for j = 1:N
            for k = 1:N

                rhoval = rhoval + u(x(i),y(j),z(k),t(n),D,alpha)*hx*hy*hz
                    ;

            end
        end
    end
    rho(n) = rhoval;
end

figure(1)
cla
plot(t,rho)
hold on





% ---- Flux -----

phi = zeros(Nt,1);

for n = 1:Nt
    phival = 0;
    for i = 1:N
        for j = 1:N

            %x = xmin surface
            phival = phival + D*([-1,0,0]*gradu(xmin,y(i),z(j),t(n),D,
                alpha))*hy*hz;
            %x = xmax surface
            phival = phival + D*([1,0,0]*gradu(xmax,y(i),z(j),t(n),D,
                alpha))*hy*hz;
            %y = ymin surface
            phival = phival + D*([0,-1,0]*gradu(x(i),ymin,z(j),t(n),D,
                alpha))*hx*hz;
            %y = ymax surface
            phival = phival + D*([0,1,0]*gradu(x(i),ymax,z(j),t(n),D,
                alpha))*hx*hz;

        end
    end
    phi(n) = phival;
end

figure(1)
plot(t,phi,'r')
plot(t,zeros(length(t),1),'k')
```