ENG2005

Assignment 3

School of Mathematics
Monash University
Semester 1 2022

Due: **Friday, 20 May 2022, 23:55 (Clayton)/ 21:55 (Malaysia)**

Complete the following questions, scan, upload and submit them in Moodle in a pdf file. ***Late assignments will be penalised*** *at 10% of the maximum mark per day late. Justify all your answers.*

Learning outcomes:

- *Represent a periodic function with a Fourier series, determine their convergence, calculate even and odd series, and apply these to solving simple periodic systems*
- *Solve simple ordinary differential equations*
- *Use MATLAB and other appropriate software to assist in understanding these mathematical techniques*
- *Express and explain mathematical techniques and arguments clearly in words*

1. Consider [3+5+2+1+1+2=14 marks]

$$f(x) = \begin{cases} 3 & 0 \leq x < 1 \\ 0 & 1 \leq x < 2 \\ 1 & 2 \leq x < 3 \\ 0 & 3 \leq x < 4. \end{cases}$$

   (a) Sketch at least two periods of each of these periodic extensions of f....
   - as a periodic function of period 4
   - as the *even* periodic extension
   - as the *odd* periodic extension.

   (b) Compute the Fourier series for the even periodic extension.

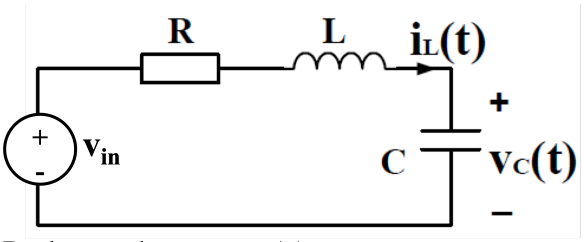
   (c) Write the partial sum of the first four non-zero terms of the Fourier series from part 1b (this is $S_3(x)$).

   (d) What does the limit of partial sums converge to at $x = 4$?

   (e) At which values of x would you expect to see the Gibbs phenomena?

   (f) Use MATLAB or other software to plot the partial sum S_3, and the partial sum S_{60}.

2. We consider the differential equation satisfied by the capacitor voltage $v_c(t)$ in the transient response of an RLC circuit: [2+2 +3 + 3 +1=11 marks]



$$\frac{d^2 v_c}{dt^2} + \frac{\omega_0}{Q}\frac{dv_c}{dt} + \omega_0^2 v_c = \omega_0 v_{\text{in}}(t), \tag{1}$$

where ω_0 is the undamped frequency, Q is the quality factor, v_{in} is the voltage source.

Background: `https://en.wikipedia.org/wiki/RLC_circuit`

(a) Consider a circuit with $\omega_0 = 10$ (rad/s) , $C = 1$F, $Q = 0.5$ and with the voltage source set to zero, so that $v_{\text{in}}(t) = 0$. Find the general solution for the capacitor voltage $v_c(t)$.

(b) A circuit is *underdamped* when the transient response includes oscillatory solutions. For the system given in (2a), how should we change Q in order for the circuit to be underdamped— should we increase, decrease, or keep Q the same?

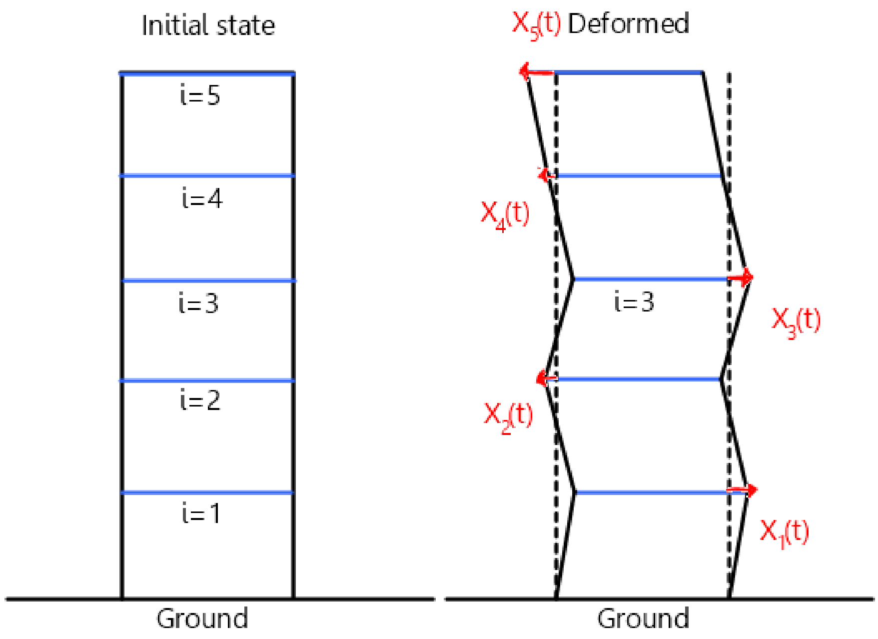
(c) For the circuit in (2a), find $v_c(t)$ for initial conditions $v_c(0) = 2$, and $i_c(0) = 5$A. Here i_c is the current through the capacitor, and it is related to v_c by $i_c = C\frac{dv_c}{dt}$.

(d) A voltage source is added to the circuit, such that $v_{\text{in}}(t) = \frac{1}{10}\sin(5t)$. Find a particular solution to (1), with the values of the constants C, ω_0, Q as in (2a).

(e) Find the general solution to (1) with C, ω_0, Q as in (2a), and v_{in} as in (2d).

3. [5+2+1+4+3=15 marks]

In this question you are going to numerically solve a system of ODEs describing the forced oscillations, floor by floor, in a simple model of a 5-storey building experiencing an earthquake ($N = 5$ is the number of 'floors'). The model is linear. It is assumed that the building can be modelled as a set of connected damped oscillators. A mass m_i, for each $i = 1, \ldots, 5$, is the mass associated with the floor i. Each floor i is connected to the floor beneath it by a connection (the walls and columns in the building) that, although rigid, can flex under extreme conditions. We will consider that they only move in the transverse (horizontal) direction as the building sways under the force of the earthquake. The transverse displacement is denoted $X_i(t)$ (see diagram). The walls of the building pull on each floor if there is a displacement between it and the floor above and below according to Hooke's law such that there is no elastic force if the the wall is vertical (not deformed). That is, we define $X_i - X_{i-1}$ as the displacement of floor i compared to floor $i - 1$ (or the ground if $i = 1$). In this case, the force experienced by floor i is $-k_i(X_i - X_{i-1})$. Each floor (except floor 5) also experiences a force due to the relative displacement $X_{i-1} - X_i$ of the floor above it which is equal to $k_{i+1}(X_{i+1} - X_i)$. The total elastic force for each floor is given by the sum of these forces. Each floor also experiences a drag force which is proportional to the relative velocity of the floor to the floor above and below; from below $-c_i(X_i' - X_{i-1}')$ and (except for floor 5, again) from above $c_{i+1}(X_{i+1}' - X_i')$.



(a) Setting up the problem

- Noting Newtons second law, $F_i = m_i X_i''$, where F_i is the total force experienced by floor i, write down the set of equations that govern the displacement of all of the floors in the matrix form
$$MX'' = KX + CX'$$
where $X = [X_1, X_2, X_3, X_4, X_5]^\intercal$ by explaining what the matricies M, K, and C are in terms of the masses m_i, spring constants k_i and dampening constants c_i of the building. Note that we shall be using the equation for the acceleration which is found by multiplying by M^{-1},
$$X'' = M^{-1}KX + M^{-1}CX'.$$
- You will find a skeleton code that we will use to solve this problem provided. You should begin by entering in a N by 1 vector ($N = 5$ floors) for the masses, spring constants and dampening constants of each floor. We shall take $k_i = 10,000$ for each floor and $c_i = 200$ for each floor. The masses of each floor should be 100 times the last 5 digits of your student ID number (that is, if your student number is 12345678 then your masses should me $m_1 = 4$, $m_2 = 5$,... $m_5 = 8$ – Note that if one of these numbers is a 0 then you should use a 1 instead).
- You should then write the code that determines the matrices M, K, and C.
- We shall be considering an earthquake that provides an acceleration of $A_{forced}(t) = a\cos(\omega t)$ in the first floor. The other floors are only affected by the forces placed on them by their neighbouring floors. The actual set of equations that therefore need to be solved is
$$X'' = M^{-1}KX + M^{-1}CX' + f(t),$$
where $f(t) = [A_{forced}(t), 0, 0, 0, 0]^\intercal$ with initial conditions $X(0) = 0$ and $X'(0) = 0$. You need to write this system of second order equations into the form $x' = Ax + F(t)$ where $x = [X_1, X_2, X_3, X_4, X_5, X_1', X_2', X_3', X_4', X_5']^\intercal$ by finding A and F in terms of M, C, K and $f(t)$. In the skeleton code, I have already provided you with $F(t)$, determine A and enter this into the skeleton code. The problem also needs an initial condition for $x(0)$ but don't worry, I have done that for you.

(b) Euler's method solution

- Now that the problem has been written in the form $x'(t) = g(x,t)$, we can apply Euler's method to solving it. Do this in the skeleton code. You are referred back to Module 2 reading material where this is explained. You should keep the same time step and other parameters in the skeleton code where not stated in the questions. For now, you should comment out the construction of Figure 2 in the skeleton code and run your code noting that the code for Figure 1 should animate for you a diagramatical representation of your building as it moves during the earthquake. Depending on your ID number you will see different things. Importantly, Eulers method is fairly inaccurate with the timestep that we are using (and it is very slow to use with sufficiently small timestep to get rid of these inaccuracies). The inaccuracies you may not notice but many of you will have a completely unstable system and this will be obvious from your animation.

(c) Plotting the displacement between floors

- The solution $X(t)$ (not $x(t)$) is the displacement of each floor relative to the ground. What is more important is the displacement of each floor relative to the floor below it (or the ground if talking about the first floor). This gives you an indication of the extent to which the building is warped between each floor. Define the displacement of each floor relative to the floor below it so that it may be plotted for each floor.

(d) Heun's method solution

- To fix the inaccuracies of the Euler method. Simply replace the code that updates the solution each timestep with the Heun's method. When you run the code now, you will see that the solution is a lot more reasonable. For the building materials used in this building, if a floor is displaced from the floor below it by more than 0.04 in magnitude it breaks. Using your plot, determine which floor of your building is likely to break and potentially should be reinforced.

(e) Building protection

- See https://www.mitsuifudosan.co.jp/english/corporate/news/2013/0729/index.html which explains methods for protecting buildings from shaking during an earthquakes and high wind. One of the devices used is essentially a giant pendulum that is placed at the top of the building to dampen vibrations. You can model this pendulum as though it were a 6th floor placed on top of the building. The mass of this new floor is the mass of the pendulum however the pendulum is not nearly as damped as the walls of the building so $c_6 = 20$ instead of 200. We shall also keep $k_6 = 10,000$. Modify your code so that it models the affect of the pendulum. Because all of your buildings are different, they are going to require a different mass on the pendulum to ensure that the building doesn't break and furthermore some of you may find that the pendulum does not help at all. Play around with the mass of the pendulum m_6 and try to find the smallest value of m_6 which will save the building from destruction during the earthquake and show the plot of the floor displacements where m_6 are just smaller than this cut-off and just larger than this cut-off respectively (try to be within 10% accuracy, the accuracy is not important, demonstrating you understand what is happening is important). If your building cannot be saved, show and describe what affect the pendulum has if any.

In your solution to this question, you should submit three codes. The first code is the code that you have at the time you reach the end of Part (c), the second is the code at the end of Part (d) and the final code at the end of Part (e). If you have changed the code in any way away from the comments prompting your changes for each part, then you should write a very clear comment to draw the attention of markers and justify what you are doing.