Linear Operators and Functional

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Seminar on Statistics 105c

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Definition

V_1, V_2 are two vector spaces. $L(V_1, V_2) = \{$all the linear maps $T : V_1 \to V_2\}$. $L(V_1, V_2)$ is a linear vector space.

$Dom(T)$: Domain of T
$Ker(T) = \{x \in V_1; Tx = 0\}$
$Im(T) = T(V_1)$
$Rank(T) = dim(Im(T))$

Definition

$(V_1, ||\cdot||_1)$, $(V_2, ||\cdot||_2)$ *are two vector spaces,* $T \in L(V_1, V_2)$, *define* $||T|| = sup_{x \in B[V_1]}\{||T(x)||_2\}$.

Property

$$||Tx||_2 \leqslant ||T||\ ||x||_1$$
$$||ST|| \leq ||S||\ ||T||,\ S \in L(V_1, V_2), T \in L(V_2, V_3)$$

Proof.

$$||Tx||_2/||x||_1 \leq ||T||$$
$$||(ST)x||_3 = ||S(Tx)||_3/||Tx||_2 \cdot ||Tx||_2/||x||_1 \leq ||S||\ ||T|| \qquad \square$$

Theorem

T is bounded $\Leftrightarrow$ T is uniformly continuous

Proof.

"$\Rightarrow$": $\forall x, y \in V_1$, $||T(x-y)||_2 \leqslant ||T||\ ||x-y||_1$
"$\Leftarrow$": T is continuous at 0 $\Rightarrow$ $||Tx||_2$ bounded, $x \in B[V_1]$. $\square$

Definition

$B(V_1, V_2) = \{T \in L(V_1, V_2);$ *T is uniformly continuous* $\}$, *then* $(B(V_1, V_2), || \cdot ||)$ *is a normed vector space.*

Property

$||T|| = \sup_{||x||_1 = 1} ||Tx||_2$, $T \in B(V_1, V_2)$

Proof.

If $||x||_1 < 1$, $||Tx||_2 \leq ||T||\ ||x||_1 < ||T||$ □

Example

$||x||_k = (\sum_n |x_n|^k)^{1/k}$, $x \in R^n$.

$B(R^q, R^p) \approx M_{pq}(R)$, $G \in B(R^q, R^p)$. $||G||_k = \max_{||x||_k=1} ||Gx||_k$ *is the k-norm of the matrix* G.

If $k = 1$, $||G||_1 = \max_j \sum_i |g_{ij}|$.

If $k = 2$, $||G||_2 = \max_{x^T x=1} \sqrt{x^T G^T G x} = \lambda$, λ^2 *is the largest eigenvalue of* $G^T G$.

If $k = \infty$, $||G||_1 = \max_i \sum_j |g_{ij}| = ||G^T||_1$.

Theorem

$(V_2, ||\cdot||_2)$ *Banach space* $\Rightarrow (B(V_1, V_2), ||\cdot||)$ *Banach space.*

Proof.

Take a Cauchy $\{T_n\}$, $||T_n x - T_m x||_2 \leq ||T_n - T_m||\ ||x||_1 \Rightarrow$ $\{T_n x\}$ Cauchy, then $\exists\, y_x \in V_2$, $T_n x \to y_x$.

Let $T : V_1 \to V_2$, $Tx = y_x$. $T \in B(V_1, V_2)$ since $\forall\, x, y \in V_1$, $||Tx - Ty||_2 \leq ||Tx - T_n x||_2 + ||T_n x - T_n y||_2 + ||T_n y - Ty||_2$.

$\forall\, \varepsilon > 0$, $\exists\, k$ s.t. $\forall\, x \in B[V_1]$, $||T_m x - T_n x||_2 \leq \varepsilon$, $n, m \geq k \Rightarrow$ $||Tx - T_m x||_2 \leq \varepsilon \Rightarrow T_n \to T$. □

V Banach space $\Rightarrow B(V)$ is a Banach algebra.

Definition

V normed vector space, dual space V^*: $B(V, R)$.
If V is a function space, then the elements of V^* is called linear functional.

We can easily conclude that the dual space V: $(V^*, ||\cdot||)$ is a Banach space.

Sometime when we meet some tough questions in V, we can transfer our attention to V^*, which can make the problem easier. So we want to ask a key question: $V \approxeq V^*$?

Theorem

H Hilbert space, $\forall T \in H^*$, $\exists!$ $e_T \in H$, called the representer of T s.t. $Tx = \langle x, e_T \rangle$ and $||T|| = ||e_T||$.

Proof.

If e_T exist, $e_T \in Ker(T)^\perp$, take $z \in Ker(T)^\perp$ s.t. $Tz = 1$, then $\langle x, z \rangle = \langle x - zTx, z \rangle + Tx\langle z, z \rangle = Tx||z||^2$, let $e_T = z/||z||^2$.

$||T|| = ||e_T||$ since that $||e_T|| = ||Te_T||/||e_T|| \leq ||T||$ and $|Tx| \leq ||x|| \, ||e_T|| \Rightarrow ||T|| \leq ||e_T||$.

e_T is unique since $\langle x, a - b \rangle = 0, \forall x \in H \Rightarrow a = b$. $\square$

Corollary

$H \approxeq H^*$, define $\langle T, G \rangle = \langle e_T, e_G \rangle$.

Definition

$T \in L(V_1, V_2)$, we say $\hat{T} \in L(V_1, V_2)$ is an extension of T if $Dom(T) \subset Dom(\hat{T})$ and $Tx = \hat{T}x$, $\forall x \in Dom(T)$.

Theorem

V_1, V_2 Banach space, $T \in B(V_1, V_2)$, then $\exists$ a unique extension $\hat{T}$ s.t. $Dom(\hat{T}) = \overline{Dom(T)}$ and $||\hat{T}|| = ||T||$.

Proof.

Let $x_n \to x$, since $||Tx_n - Tx_m||_2 \leq ||T||\ ||x_n - x_m||_1$, define $\hat{T}x = \lim_n Tx_n$, then $\hat{T} \in L(V_1, V_2)$ and $||T|| = ||\hat{T}||$.

If T_1, T_2 are extensions of T and $Dom(T_i) = \overline{Dom(T)}$, then $T_1 x = T_1(\lim_n x_n) = \lim_n T_1 x_n = \lim_n T_2 x_n = T_2 x \Rightarrow T_1 = T_2$. □

Definition

V normed vector space, $f : V \to R$ is called sub-linear function if $\forall x, y \in V$, $a \in R$, $f(x+y) \le f(x) + f(y)$, $f(ax) = af(x)$.

Let $T \in B(M, R)$, M is sub-space of V. Define a sub-linear function $f(x) = ||T||\ ||x||$, we know that $|Tx| \le f(x)$, $x \in M$. We want to find an extension of T s.t. $Dom(\hat{T}) = V$ and $|\hat{T}x| \le f(x)$, $x \in V$. If we achieve this, we can find a norm-preserved extension of T since that $||\hat{T}|| \le ||T||$.

This is easy in Hilbert space, without loss of generality, M is closed then M is Hilbert space. Then $Tx = \langle x, e_T \rangle$, $\forall x \in M$, we can define $\hat{T}$: $\langle x, e_T \rangle$, $\forall\ x \in V \Rightarrow \hat{T}(x) = 0$, $\forall x \in M^{\perp}$.

Lemma

$T \in M^*$, M *is a subspace of* V, $x \notin M$, $M_x = span\{x, M\}$. $f : V \to R$ *is a sub-linear function. If* $T(x) \leq f(x)$, $\forall x \in M$, *then* $\exists$ $\hat{T}$, $Dom(\hat{T}) = M_x$, $\hat{T}(x) \leq f(x)$, $\forall x \in M_x$.

Proof.

$\forall z \in M_x$, $\exists y \in M$, $a \in R$, $z = y + ax$, let $\hat{T}z = Ty + ah(x)$, the trick is in establishing the existence of h.

If $a > 0$, $\forall m_1, m_2 \in M$, $T(m_1 + m_2) \leq f(m_1 + m_2) \leq f(m_1 - x)$ $+f(m_2 + x) \Rightarrow Tm_1 - f(m_1 - x) \leq f(m_2 + x) - Tm_2$. Take $h(x) \in [\sup_{m \in M}(Tm - f(m - x))\ , \inf_{m \in M}(f(m + x) - Tm)]$, then $\hat{T}z = Ty + ah(x) = a(T(y/a) + h(x)) \leq af(y/a + x) = f(z)$. □

Theorem

$T \in M^*$, $f : V \to R$ is a sub-linear function. If $T(x) \leq f(x)$, $\forall x \in M$, then $\exists\ \hat{T}$, $Dom(\hat{T}) = V$, $\hat{T}(x) \leq f(x)$, $\forall x \in V$.

Proof.

Define (A, T_A): $A \subset V$, and T_A is an extension of T which domain is A and $T_A \leq f$. $\Theta = \{\text{All } (A, T_A)\}$ and define a partial order on Θ: $A_1 \leq A_2$ if $A_1 \subset A_2$ and T_{A_2} is an extension of T_{A_1}. Let $\{(A_\beta, T_{A_\beta})\}_{\beta \in B}$ be the collection of comparable sets.

Let $G = \cup_{\beta \in B} A_\beta$. $\forall x \in G$, $\exists A_\beta$ s.t. $x \in A_\beta$: $T_G(x) = T_{A_\beta}(x)$. Then $(G, T_G) \in \Theta$ is an upper bound on $\{(A_\beta, T_{A_\beta})\}_{\beta \in B}$. We apply Zorn's Lemma to conclude that $\{(A_\beta, T_{A_\beta})\}_{\beta \in B}$ has a maximal element $(V', T_{V'})$. It's easy to show that $V = V'$. □

Corollary

$T \in B(M, R)$, $\exists\ \hat{T} \in V^*$ s.t. $||T|| = ||\hat{T}||$.
$\forall x \in V$, $\exists\ T \in V^*$ s.t. $Tx = ||x||$ and $||T|| = 1$.

Proof.

Let $f(x) = ||T||\ ||x||$, then $|Tx| \leq f(x) \Rightarrow |\hat{T}x| \leq ||T||\ ||x||$ $\Rightarrow ||\hat{T}|| \leq ||T|| \Rightarrow ||\hat{T}|| = ||T||$

Define $T \in span\{x\}^*$, $T(ax) = a||x||$, then $Tx = ||x||$ and $|T(ax)| = ||ax|| \Rightarrow ||T|| = 1 \Rightarrow \hat{T}x = ||x||$, $||\hat{T}|| = 1$. $\square$

Definition

$\forall x \in V$, evaluation functional J_x: $J_x(T) = Tx$, $\forall T \in V^*$. $J_x \in V^{**}$. Define $J : V \to V^{**}$, $J(x) = J_x$.

Property

J is an injection.

J is a norm-preserved map and $J_x \in V^{**}$.

Proof.

$J_x = J_y \Rightarrow T(x-y) = 0$, $\forall T \in V^*$. And $I \in V^* \Rightarrow x = y$.

$\forall x \in V$, $||J(x)|| = \sup_{T \in B[V^*]}\{|T(x)|\} \leq ||x||$. $\exists\, T \in B[V^*]$ s.t. $Tx = ||x|| \Rightarrow ||J(x)|| = ||x|| \Rightarrow J_x \in V^{**}$. □

Definition

$(V, ||\cdot||)$ *is reflexive:* J *is surjection, then* $V \approx V^{**}$.

Theorem

H *Hilbert space, then* H *is reflexive.*

Proof.

$\forall J \in V^{**}, J(T) = \langle T, E_J \rangle = \langle e_T, e_{E_J} \rangle = T(e_{E_J})$. $\square$

Definition

$\mathcal{F}$ is a collection of A and $(A, \mathcal{F})$ is called a topological space if:
(a) $\emptyset, A \in \mathcal{F}$.
(b) $\forall B, C \in \mathcal{F}$, $B \cap C \in \mathcal{F}$.
(c) $\cup_{\alpha \in H} B_\alpha \in \mathcal{F}$, if $B_\alpha \in \mathcal{F}$.

We can similarly define $x_n \to x$: $\forall B \in \mathcal{F}$, $x \in B$, $\exists N$ s.t. $\forall n \geq N$, $x_n \in B$. In this setting, $B' \subset B$ is not equivalent to B is closed.

Definition

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two topological structures of A and $\mathcal{F}_1 \subset \mathcal{F}_2$, then we say $(A, \mathcal{F}_1)$ is weaker than $(A, \mathcal{F}_2)$.

We noticed that if $\{x_n\}$ converges in $(A, \mathcal{F}_2)$, then $\{x_n\}$ converges in $(A, \mathcal{F}_1)$. We say $\{x_n\}$ weakly converges in $(A, \mathcal{F}_1)$.

If we want to ensure convergence of $\{x_n\}$, we can consider the convergence in some weaker topological spaces.

But it may sacrifice the uniqueness of the limitation of $\{x_n\}$. Defining a weak convergence is a technical problem.

Definition

V Banach space, $x_n \in V$ converges weakly to x: $Tx_n \to Tx$ $\forall T \in V^*$.

If $\{x_n\}$ weakly converges, the limitation of $\{x_n\}$ is unique since $\exists\ T \in V^*$ s.t. $T(x-y) = ||x-y||$.

$x_n \to x$, then x_n converges weakly to x since that $|Tx - Tx_n|$ $\leq ||T||\ ||x_x - x||$, we mark that $x_n \xrightarrow{w} x$.

Theorem

If V Hilbert space, $x_n \xrightarrow{w} x \Leftrightarrow \forall y \in V$, $\langle x_n, y\rangle = \langle x, y\rangle$.

Proof.

$\langle x_n, y\rangle = T_y x_n \to T_y x = \langle x, y\rangle$ $\square$

If $||x_n|| \to ||x||$, then $x_n \to x$ since $||x - x_n||^2 = ||x||^2 - 2\langle x, x_n\rangle + ||x_n||^2 \to 0$.

Theorem

V Banach space, V reflexive $\Leftrightarrow B[V]$ is weakly compact.

Theorem

H Hilbert space, then $B[H]$ is weakly compact.

Proof.

$\forall \{x_n\} \subset B[H]$, take COB $\{e_m\}$ of $S := \overline{span\{x_n\}}$.

$\{\langle x_n, e_1\rangle\}$ bounded $\Rightarrow \exists \{x_{n_k}^{(1)}\}$ s.t. $\langle x_{n_k}^{(1)}, e_1\rangle \to a_1 \Rightarrow \exists \{x_{n_k}^{(m)}\} \subset \{x_{n_k}^{(m-1)}\}$ s.t. $\langle x_{n_k}^{(m)}, e_m\rangle \to a_m$.

Let $\{y_g\} = \cap_m \{x_{n_k}^{(m)}\}$, then for $\forall m$, $\langle y_g, e_m\rangle \to a_m \Rightarrow \forall z \in S$, $\langle y_g, z\rangle \to a_z \Rightarrow \forall z \in H$, $\langle y_g, z\rangle \to a_z$.

Define $Tz = a_z$, $|Tz| = |\lim_g \langle y_g, z\rangle| \leq \lim_g ||y_g||\ ||z|| \leq ||z|| \Rightarrow T \in B(H)$, then $a_z = Tz = \langle z, e_T\rangle \Rightarrow \langle y_g, z\rangle \to \langle e_T, z\rangle$. □

Theorem

H_1, H_2 Hilbert spaces, $\forall T \in B(H_1, H_2)$, $\exists!\ T^* \in B(H_2, H_1)$ s.t. $\langle Tx_1, x_2\rangle_2 = \langle x_1, T^*x_2\rangle_1$, $\forall x_i \in H_i$.

Proof.

Define $G_{x_2} : H_1 \to H_2,\ G_{x_2}(x_1) = \langle Tx_1, x_2\rangle_2$, $G_{x_2} \in H_1^*$ since $|G_{x_2}x_1| \leq ||T||\ ||x_1||_1\ ||x_2||_2 \Rightarrow ||G_{x_2}|| \leq ||T||\ ||x_2||_2$.

Then $\exists!\ y$ s.t. $\langle Tx_1, x_2\rangle_2 = \langle x_1, y\rangle_1$. Let $T^*x_2 = y$.

$||T^*x_2||_1 = ||y|| = ||G_{x_2}|| \leq ||T||\ ||x_2||_2 \Rightarrow T^* \in B(H_2, H_1)$. □

We say T^* is adjoint to T and $||T|| = ||T^*||$ since that $||T|| \leq ||T^*||$ and $||T^*|| \leq ||T||$.

Definition

$H_1 = H_2$, *if* $T = T^*$, *we call* T *self-adjoint.*

Theorem

If $T \in B(H)$ *self-adjoint,* $||T|| = \sup_{||x||=1} |\langle x, Tx\rangle|$.

Proof.

Let $M = \sup_{||x||=1} |\langle x, Tx\rangle|$. $|\langle x, Tx\rangle| \leq ||T||\ ||x||^2 \Rightarrow M \leq ||T||$.
$||x|| = ||y|| = 1$, $4\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle$
$\Rightarrow |\langle Tx, y\rangle| \leq M(||x+y||^2 + ||x-y||^2)/4 = M(||x||^2 + ||y||^2)/2 = M$
Let $y = Tx/||Tx||$, then $||Tx|| \leq M \Rightarrow ||T|| \leq M$. □

Let $m = \inf R_T(x)$, $M = \sup R_T(x)$, $[m, M] \subset [-||T||, ||T||]$.

Property

$T \in B(H_1, H_2)$, H_i Hilbert space, then $||T^*T|| = ||T||^2$.

Proof.

$$||T^*T|| \leq ||T^*||\ ||T|| = ||T||^2$$
$$||Tx||_2^2 = \langle Tx, Tx\rangle_2 = \langle x, T^*Tx\rangle_1 \leq ||x||_1\ ||T^*T||\ ||x||_1 \Rightarrow$$
$||T|| \leq ||T^*T||^{1/2}$. □

Property

$T \in B(H_1, H_2)$, $Ker(T) = Im(T^*)^{\perp}$.

Proof.

"$\subset$": $\forall\, x \in Ker(T)$, $\langle T^* y, x\rangle_1 = \langle y, Tx\rangle_2 = 0$

"$\supset$": $\forall\, x \in Im(T^*)^{\perp}$, $\langle T^* y, x\rangle_1 = \langle y, Tx\rangle_2 = 0 \Rightarrow Tx = 0$ $\square$

Property

$Ker(T^*T) = Ker(T)$ and $\overline{Im(T^*T)} = \overline{Im(T^*)}$
$H_1 = Ker(T) \oplus \overline{Im(T^*)} = Ker(T^*T) \oplus \overline{Im(T^*T)}$

Proof.

"$\subset$": $\forall x \in Ker(T^*T)$, $\langle T^*Tx, x\rangle_1 = ||Tx||_2 = 0$.
$Ker(T)^\perp = (Im(T^*)^\perp)^\perp = \overline{Im(T^*)} \Rightarrow \overline{Im(T^*T)} = \overline{Im(T^*)}$.
$H_1 = Ker(T) \oplus Ker(T)^\perp = Ker(T) \oplus \overline{Im(T^*)}$. □

Property

$$Rank(T) = Rank(T^*)$$

Proof.

$\forall x \in H_1$, $\exists\, x_0 \in Ker(T)$, $x_1 \in Ker(T)^{\perp}$ s.t. $x = x_0 + x_1$, then $Tx = Tx_1 \Rightarrow Im(T) \subset T(\overline{Im(T^*)}) \Rightarrow dim(Im(T)) \leq dim(\overline{Im(T^*)})$.

If $dim(Im(T^*)) < \infty$, then $dim(Im(T)) \leq dim(Im(T^*))$ and $dim(Im(T)) < \infty \Rightarrow Rank(T) = Rank(T^*)$.

If $dim(Im(T^*)) = \infty$, $dim(Im(T^*)) \leq dim(\overline{Im(T)}) \Rightarrow dim(\overline{Im(T)}) = \infty \Rightarrow dim(Im(T)) = \infty$. $\square$

Definition

$T \in B(H)$ is non-negative: T is self-adjoint and $\langle Tx, x\rangle \geq 0$, $\forall x \in H$.

$T_1 \geq T_2$: $T_1 - T_2$ is non-negative.

T^*T is non-negative, since that $\langle T^*Tx, x\rangle = ||Tx||^2 \geq 0$.

Let $\sqrt{1-x} := 1 + \sum_n c_n x^n$, $|x| \leq 1$, $c_n < 0$. Let T be non-negative, we use this to ensure the existence of $\sqrt{T}$.

If $||T|| \leq 1$, $||I-T|| = \sup_{||x||=1} |\langle x, x\rangle - \langle x, Tx\rangle| \leq 1$. Then we can define $\sqrt{T} := \sqrt{I-(I-T)} = I + \sum_n c_n (I-T)^n$.

Definition

$\sqrt{T} = ||T||^{1/2} I + ||T||^{1/2} \sum_n c_n (I - T/||T||)^n$.

$\sqrt{T}$ also non-negative since:

$$\langle \sqrt{T}x, x\rangle = ||x||^2 + \sum_n c_n \langle (I-T)^n x, x\rangle \geqslant ||x||^2 (1 + \sum_n c_n) \geq 0$$

Definition

M closed sub-space of H, then $\forall x \in H$, $\exists\, x_1 \in M$ as a projection of x onto M, let $P_M : H \to H$, $P_M x = x_1$.

Property

P_M is self-adjoint, $P_M = P_M^2$ and $||P_M|| = 1$.

Proof.

$\langle P_M x, y\rangle = \langle x_1, y\rangle = \langle x_1, y_1\rangle = \langle x, y_1\rangle = \langle x, P_M y\rangle$
$P_M^2 x = P_M x \Rightarrow ||P_M|| = ||P_M^2|| \leqslant ||P_M||^2 \Rightarrow ||P_M|| \geqslant 1$, and $||P_M x|| \leqslant ||x|| \Rightarrow ||P_M|| \leqslant 1$ □

Definition

H_i Hilbert spaces, $x_i \in H_i$. The tensor product operator $x_1 \otimes_1 x_2 \in L(H_1, H_2)$, $(x_1 \otimes_1 x_2)y = \langle x_1, y \rangle_1 x_2$, $y \in H_1$.

Theorem

$||x_1 \otimes x_2|| = ||x_1||_1 \; ||x_2||_2$ and $x_1 \otimes x_2 \in B(H_1, H_2)$

Proof.

$||(x_1 \otimes_1 x_2)y||_2 = ||\langle x_1, y \rangle_1 x_2||_2 \leq ||x_1||_1 \; ||x_2||_2 \; ||y||_1$ and let $y = x_1/||x_1||_1$, $||(x_1 \otimes_1 x_2)y||_2 = ||x||_1 \; ||x||_2$. $\square$

Property

$x \otimes x \gg 0$ *and* $(x_1 \otimes_1 x_2)^* = x_2 \otimes_2 x_1$.

Proof.

$$\langle x \otimes x\, y, z \rangle = \langle x, y \rangle \langle x, z \rangle = \langle y, x \otimes x\, z \rangle$$
$$\langle (x_1 \otimes_1 x_2) y, z \rangle_2 = \langle \langle x_1, y \rangle_1 x_2, z \rangle_2 = \langle x_2, z \rangle_2 \langle x_1, y \rangle_1$$
$$\langle y, (x_2 \otimes_2 x_1) z \rangle_1 = \langle y, \langle x_2, z \rangle_2 x_1 \rangle_1 = \langle x_2, z \rangle_2 \langle x_1, y \rangle_1$$

□

Example

$H_i = R^{p_i}$, $(x_1 \otimes_1 x_2) y = x_2 \langle x_1, y \rangle_1 = x_2 x_1^T y \Rightarrow x_1 \otimes_1 x_2 = x_2 x_1^T$.

Definition

V_i *Banach space,* $T \in B(V_1, V_2)$, T *is invertible:* $\exists\, T^{-1}$ *s.t.* $TT^{-1} = T^{-1}T = I$.

Property

$T \in B(V)$, V *Banach space. If* $||T|| < 1$, *then* $I - T$ *invertible and* $(I - T)^{-1} = I + \sum_n T^n$. *If* $S, T \in B(V)$ *is invertible, then* $(T + US^{-1}V)^{-1} = T^{-1} - T^{-1}U(S + VT^{-1}U)^{-1}VT^{-1}$.

Theorem

V_i Banach space, $T \in B(V_1, V_2)$. The follow is equivalence:
(a) If T is surjection, then $T(\Omega)$ is open for all open set Ω.
(b) If T is invertible, then $T^{-1} \in B(V_2, V_1)$.

Proof.

$(a) \Rightarrow (b)$: $\exists\ r > 0$ s.t. $B_2(0; r) \subset T(B_1(0; 1))$, then for $\forall$ $y \in B[V_2]$, $||T^{-1}y||_1 = ||T^{-1}ry||_1/r \leq 1/r$. □

Corollary

If $||\cdot||_1$ *and* $||\cdot||_2$ *are two norms of Banach space* V*, and* $\exists$ $c > 0$ *s.t.* $||\cdot||_1 \leq c||\cdot||_2$*, then* $||\cdot||_1 \sim ||\cdot||_2$*.*

Proof.

$I : (V, ||\cdot||_1) \to (V, ||\cdot||_2)$, $I(x) = x$, then I is invertible. Then $||x||_1 = ||I^{-1}x||_1 \leq C||x||_2$. □

Theorem

V_i Banach space, $W \subset B(V_1, V_2)$ and $\sup_{T\in W} ||Tx||_2 \leq \infty$, $\forall$ $x \in V_1$, then W bounded.

Proof.

Define $||x||_1^{'} = max\{||x||_1, \sup_{T\in W} ||Tx||_2\}$, we can carefully check $||\cdot||_1^{'}$ is a norm of V_1 and $(V_1, ||\cdot||_1^{'})$ is also Banach space.

$||x||_1 \leq ||x||_1^{'} \Rightarrow \exists\ C > 0$, $||x||_1^{'} \leq C||x||_1 \Rightarrow ||Tx||_2 \leq C||x||_1 \Rightarrow ||T|| \leq C$. □

Theorem

H Hilbert space and $T \in B(H)$, if T self-adjoint and $\exists\, C > 0$ s.t. $||Tx|| \geq C||x||\ \forall x$, the T is invertible.

Proof.

$||Tx|| \geq C||x|| \Rightarrow Ker(T) = \{0\}$, then T is an injection. And $H = Ker(T) \oplus \overline{Im(T)} \Rightarrow \overline{Im(T)} = H$.

Claim that $Im(T)$ closed. Let $Tx_n \to y$, $\{x_n\}$ is Cauchy since that $||T(x_n - x_m)|| \geq C||x_n - x_m|| \Rightarrow \exists x : \lim_n x_n = x \Rightarrow Tx = y \Rightarrow y \in Im(T) \Rightarrow T$ surjection. □

For $\forall T \in B(H_1, H_2)$, the inverse of T may not exist. The problem is that $Ker(T) \neq \{0\}$ or $Im(T) \neq H_2$.

We take $G = T|_{Ker(T)^\perp}$, then $Ker(G) = \{0\}$, $Im(G) = Im(T)$, then $G^{-1} \in B(Im(T), Ker(T)^\perp)$, which is the key to define a generalized inverse of T.

We can simply recognize that generalized inverse is just a norm-preserved extension of G^{-1}.

Definition

Define $T^{\dagger} : Im(T) + Im(T)^{\perp} \to Ker(T)^{\perp}$, $T^{\dagger}y = G^{-1}P_{\overline{Im(T)}}y$.

If $Im(T)$ closed, then $Im(T) \oplus Im(T)^{\perp} = H_2$, $T^{\dagger} = G^{-1}P_{Im(T)}$.

Property

$Ker(T^{\dagger}) = Im(T)^{\perp}$, $Im(T^{\dagger}) = Ker(T)^{\perp}$

If T invertible, $Ker(T^{\dagger}) = \{0\}$, $Im(T^{\dagger}) = H_1 \Rightarrow T^{\dagger} = T^{-1}$

Property

$$T^\dagger T = I - P_{Ker(T)},\ TT^\dagger = P_{\overline{Im(T)}}$$
$$TT^\dagger T = T,\ T^\dagger TT^\dagger = T^\dagger$$
If T_1, T_2 *bounded,* $(T_1T_2)^\dagger = T_2^\dagger T_1^\dagger$

Proof.

$\forall x \in H_1,\ T^\dagger Tx = G^{-1}Tx = G^{-1}TP_{Ker(T)^\perp}x = P_{Ker(T)^\perp}x \Rightarrow$
$T^\dagger T = P_{Ker(T)^\perp} = I - P_{Ker(T)}$

$\forall y \in Im(T) + Im(T)^\perp,\ TT^\dagger y = TG^{-1}P_{\overline{Im(T)}}y = P_{\overline{Im(T)}}y \Rightarrow$
$TT^\dagger = P_{\overline{Im(T)}}$

$$TT^\dagger T = P_{\overline{Im(T)}}T = T.$$
$$T^\dagger TT^\dagger = P_{Ker(T)^\perp}T^\dagger = T^\dagger.$$

□

Theorem

H_i Hilbert space, $T \in B(H_1, H_2)$. $\forall\ y \in Dom(T^\dagger)$, the solution x of $Tx = y$ which minimizes $||y - Tx||_2$ is $M = \{x \in H_1;\ Tx = P_{\overline{Im(T)}}y\}$.

$M = T^\dagger y + Ker(T)$ since that $T(T^\dagger y) = P_{\overline{Im(T)}}y$.

Property

$T^\dagger y = (T^*T)^\dagger T^* y, \forall y \in Dom(T^\dagger)$.

Proof.

$T^*T(T^\dagger y) = T^* y$ since $y - T(T^\dagger)y \in Im(T)^\perp = Ker(T^*)$, then $T^\dagger y \in (T^*T)^\dagger T^* y + Ker(T^*T) = (T^*T)^\dagger T^* y + Ker(T)$. □