Compact Operators

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Compact Operators

1. The Spectrum of Operators
    - Spectrum Decomposition
    - Singular Value Decomposition
2. Hilbert-Schmidt Operators
    - Eckart-Young Theorem
3. Trace Class Operators
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Definition

V_i normed vector space, $T \in L(V_1, V_2)$ is a compact operator: $\forall$ bound set $A \subset V_1$, $T(A) \subset\subset V_2$. We mark that $T \in K(V_1, V_2)$.

Theorem

T is compact $\Leftrightarrow T(B_1[V_1]) \subset\subset V_2$

Proof.

"$\Leftarrow$": Let bound set $A \subset B_1(0, r)$, then $T(A) \subset T(B_1(0; r))$ $= rT(B[V_1]) \subset\subset V_2$ $\square$

Note that $K(V_1, V_2) \subset B(V_1, V_2)$. If V_2 is banach space, we just need to show $T(B_1[V_1])$ is totally bounded.

Theorem

V_2 Banach space, then $K(V_1, V_2)$ is closed sub-space of $B(V_1, V_2)$.

Proof.

Let $T_n \to T$, $T_n \in K(V_1, V_2) \Leftrightarrow T_n(B[V_1]) \subset\subset V_2 \Leftrightarrow T_n(B[V_1])$ is totally bounded.

$\forall\ \varepsilon > 0$, $\exists\ T_n$ s.t. $||T - T_n|| < \varepsilon/3$. $T_n(B[V_1])$ is totally bounded $\Rightarrow \exists$ finite $\{T_n x_k\}$, $\forall\ x \in B[V_1]$, $\exists\ x_j \in B[V_1]$ s.t $||T_n x - T_n x_j||_2 < \varepsilon/3$.

$$||Tx - Tx_j||_2 \leq ||Tx - T_n x||_2 + ||T_n x - T_n x_j|| + ||T_n x_j - T x_j||$$

□

Property

$T \in B(V_1, V_2)$, $S \in B(V_2, V_3)$, $ST \in K(V_1, V_3)$ if S or T is compact.

Proof.

If T is compact, $\forall \{x_n\} \subset$ bound A, $\exists T(x_{n_k}) \to a \in V_2 \Rightarrow ST(x_{n_k}) \to Sa \in V_3$. □

If V_i infinite-dim normed vector space, let T be a bijection in $B(V_1, V_2)$, then T is not compact since $I = T^{-1}T$ is not compact.

Lemma

$T \in K(V_1, V_2)$, then $\overline{Im(T)}$ is separable.
$T \in B(V_1, V_2)$ s.t. $Rank(T) < \infty \Rightarrow T \in K(V_1, V_2)$.

Proof.

$Im(T) \subset \cup_n T(B(0;n))$ and $\overline{T(B(0;n))}$ is separable $\Rightarrow$ $\overline{Im(T)}$ separable.
$T(B[V_1]) \subset Im(T) \Rightarrow T(B[V_1]) \subset\subset Im(T) \subset V_2$. □

Theorem

$T \in K(H_1, H_2) \Leftrightarrow \exists\, T_n \to T,\ Rank(T_n) < \infty.$

Proof.

"$\Rightarrow$": $\overline{Im(T)}$ is separable, let $\{e_n\}$ is COB of $\overline{Im(T)}$.

Let $M_k = span\{e_n, n \leq k\}$, $T_n : H_1 \to M_k$, $T_k x = P_{M_k}(Tx)$, $Rank(T_k) < \infty$.

Note that $||Tx - P_{M_k}Tx||_2 \to 0$, $\forall x \in B[H_1]$ and $T(B[H_1])$ totally bounded, $\forall\ \varepsilon > 0$, $\exists$ finite x_m s.t. $||Tx - Tx_m||_2 < \varepsilon/3$.

$||Tx - T_kx||_2 \leq ||Tx - Tx_m||_2 + ||Tx_m - T_kx_m||_2 + ||T_kx_m - T_kx||_2 \leq 2||Tx - Tx_m||_2 + ||Tx_m - T_kx_m||_2 \to 0.$ □

$T \in K(H_1, H_2) \Leftrightarrow T^* \in K(H_2, H_1)$ since $||T - T_n|| = ||T^* - T_n^*||$ and $Rank(T) = Rank(T^*)$.

Definition

$T \in B(H)$, *if* $\lambda \in R$ *s.t.* $Ker(T - \lambda I) \neq \{0\}$, *then* λ *is the eigenvalue of* T *and* $Ker(T - \lambda I)$ *is the eigenspace of* λ.

$\sigma_p(T) = \{\lambda \in R, \lambda$ *is an eigenvalue* $T\}$.

Property

$e_i \in Ker(T - \lambda_i I)/\{0\}$, λ_i *is distinct and non-zero, then* e_i *is mutually linear independent. Moreover, if* T *is self-adjoint, then* e_i *is mutually orthonormal.*

Proof.

$$\langle e_i, e_j \rangle = 1/\lambda_i \langle \lambda_i e_i, e_j \rangle = 1/\lambda_i \langle e_i, \lambda_j e_j \rangle = \lambda_j/\lambda_i \langle e_i, e_j \rangle \qquad \square$$

Theorem

$T \in K(H)$, $Ker(T - \lambda I)$ is finite-dimensional for $\lambda \neq 0$.
$T \in K(H)$, $\sigma_p(T)/\{0\}$ is countable.

Proof.

$\forall x \in B[Ker(T - \lambda I)]$, $\lambda x = Tx \in T(B[H]) \Rightarrow$
$\lambda B[Ker(T - \lambda I)] \subset T(B[H]) \Rightarrow \lambda B[Ker(T - \lambda I)] \subset\subset H$.

We firstly proof $\forall \varepsilon > 0$, $A_\varepsilon = \sigma_p(T)/(-\varepsilon, \varepsilon)$ is finite. If not, we choose $x_n \in Ker(T - \lambda_n I)/\{0\}$, $\lambda_n \in A_\varepsilon$, use GS to $\{x_n\}$ and get a mutually orthonormal sequence $\{e_n\} \subset B[H]$.

$\{Te_n\}$ has no convergent subsequence since $||Te_n - Te_m||^2$ $= \lambda_n^2 + \lambda_m^2 \geq 2\varepsilon^2$. Contradiction. (Let ε be rational) $\square$

Lemma

If $T \in B(H)$ is self-adjoint, then $||T||$ or $-||T|| \in \sigma_p(T)$.

Proof.

Let $||Tx_n|| \to ||T||$, $||x_n|| = 1$, $||T^2x_n - ||T||^2x_n||^2 \leq 2||T||^4 - 2\langle T^2x_n, ||T||^2x_n\rangle = 2||T||^4 - 2||T||^2||Tx_n||^2 \to 0$. Then $(T - ||T||I)(T + ||T||I)x_n \to 0$.

If $||T||$ and $-||T||$ are both not in $\sigma_p(T) \Rightarrow Ker(T - ||T||I)$ $= Ker(T + ||T||I) = \{0\} \Rightarrow x_n = (T + ||T||I)^{-1}(T - ||T||I)^{-1}$ $(T - ||T||I)(T + ||T||I)x_n \to 0$, contradiction. □

T self-adjoint, let $m = \inf R_T(x)$, $M = \sup R_T(x)$. It's easy to show that $\sigma_p(T) \subset [m, M] \subset [-||T||, ||T||]$ and $\sigma_p(T) \neq \emptyset$.

Moreover, T is compact, then $\sigma_p(T)$ is countable. If $\sigma_p(T)$ is infinite set, we can easily show that $0 \in \overline{\sigma_p(T)}$.

Let $T \neq 0$, let $\{a_i\} = \sigma_p(T)/\{0\}$ s.t. $|a_i| > |a_{i+1}|$, and $\{x_{ik_i}\}$ is a finite basis of $Ker(T - a_i I)$.

Define mutually orthonormal sequence $\{e_n\}$ from $\cup_i \{x_{ik_i}\}$, λ_n is the eigenvalue of e_n then $|\lambda_n| \geq |\lambda_{n+1}| \to 0$. We call (λ_n, e_n) is a spectrum decomposition of T.

Theorem

$\overline{Im(T)} = \overline{span\{e_n\}}$ *and* $T = \sum_n \lambda_n e_n \otimes e_n$.

Proof.

Let $\overline{Im(T)} = \overline{span\{e_n\}} \oplus N$. $\forall x \in N$, $\langle Tx, y\rangle = \langle x, Ty\rangle = 0$, $\forall y \in \overline{span\{e_n\}} \Rightarrow T(N) \subset N \Rightarrow T|_N$ is self-adjoint $\Rightarrow \pm||T|_N|| \in \sigma_p(T|_N) = \{0\} \Rightarrow T|_N = 0 \Rightarrow N \subset Ker(T)$.

Note that $H = Ker(T) \oplus \overline{Im(T)} \Rightarrow N = 0$.

$Tx = \sum_n \langle Tx, e_n\rangle e_n = \sum_n \langle x, \lambda_n e_n\rangle e_n = \sum_n \lambda_n e_n \otimes e_n x$.

□

Corollary

If T self-adjoint and compact, then T^k is compact and corresponding eigenvalue-eigenvector is (e_n, λ_n^k).

Moreover, $T \gg 0 \Leftrightarrow \sigma_p(T) \subset [0, \infty]$, and $T^{1/2}$ is compact and corresponding eigenvalue-eigenvector is $(e_n, \lambda_n^{1/2})$.

Theorem

$T \in K(H)$ is self-adjoint. $M_{k-1} = span\{e_n, n \leq k-1\}$, then $|\lambda_k| = ||T|_{M_{k-1}^\perp}|| = \max_{x \in M_{k-1}^\perp} |R_T(x)|$.

Moreover, if $T \gg 0$, sub-space $N_k \subset \overline{Im(T)}$, $dim(N_k) = k$, $\lambda_k = \min_{N_{k-1}} \max_{x \in N_{k-1}^\perp} R_T(x) = \max_{N_k} \min_{x \in N_k} R_T(x)$.

Proof.

If $N_{k-1}^\perp \cap M_{k-1} \neq \{0\}$, take $x \in N_{k-1}^\perp \cap M_{k-1}$ and $||x|| = 1$, $R_T(x) = \langle \sum_{i=1}^{k-1} e_n \langle e_n, x\rangle, T(\sum_{i=1}^{k-1} e_n \langle e_n, x\rangle)\rangle = \sum_{i=1}^{k-1} \lambda_i \langle x, e_n\rangle^2 \geq \lambda_k$.

$N_k \cap M_k^\perp \neq \{0\} \Rightarrow R_T(x) = \sum_{i=k+1}^{\infty} \lambda_i a_i^2 \leq \lambda_k$. $\square$

Corollary

$T, G \in K(H)$ *and* $T, G \gg 0$, $\{\lambda_n\}$ *and* $\{\phi_n\}$ *corresponding eigenvalue, then* $\sup_n |\lambda_n - \phi_n| \leq ||T - G||$.

Proof.

$||x|| = 1$, $R_T(x) = \langle Tx, x\rangle = \langle Gx, x\rangle + \langle (T - G)x, x\rangle$
$= R_G(x) + R_{T-G}(x) \Rightarrow \lambda_n \leq \phi_n + ||T - G||$. □

Theorem

$T \in K(H_1, H_2)$, then $\sigma_p(T^*T) = \sigma_p(TT^*)$.

Proof.

We know that $T^*T \in K(H_2)$ and $TT^* \in K(H_1)$ and they are both $\gg 0$. Let (λ_n^2, e_n) be the eigenvalue-eigenvector of T^*T, $TT^*(Te_n) = \lambda_n^2 Te_n \Rightarrow \sigma(T^*T) \subset \sigma(TT^*) \Rightarrow \sigma(T^*T) = \sigma(TT^*)$. and $||Te_n|| = \sqrt{\langle Te_n, Te_n \rangle} = \sqrt{\langle e_n, T^*Te_n \rangle} = \lambda_n$ $(\lambda_n > 0)$. $\square$

$(\lambda_n^2, Te_n/\lambda_n)$ is the eigenvalue-eigenvector of TT^*.

Theorem

$T \in K(H_1, H_2)$, *let* $e_{1n} = e_n$, $e_{2n} = Te_n/\lambda_n$, *then* $T = \sum_n \lambda_n\ e_{1n} \otimes_1 e_{2n}$ *and* $||T|| = \lambda_1$.

Proof.

$$\overline{span\{e_n\}} = \overline{Im(T^*T)} = Ker(T^*T)^\perp = Ker(T)^\perp.$$

$$\forall x \in H_1,\ y = P_{Ker(T)^\perp}x,\ Tx = Ty = T\sum_n \langle e_n, y\rangle_1 e_n$$
$$= \sum_n \langle e_n, x\rangle_1 Te_n = \sum_n (e_n \otimes_1 Te_n)x.$$

$$||T||^2 = ||T^*T|| = \lambda_1^2.$$

$\square$

Example

$A \in M_{p\times q}(R)$, $\exists$ *orthonormal matrices* $U_{p\times k}, V_{k\times q}$ *and* $\Lambda = diag(\lambda_1, ..., \lambda_k)$, $\lambda_i > 0$ *s.t.* $A = U\Lambda V^T$.

Theorem

$T \in B(H_1, H_2)$, *if* $\exists\ \lambda_n \downarrow 0$ *and MOS* $\{e_{1n}\} \subset H_1$ *and* $\{e_{2n}\} \subset H_2$, $\overline{span\{e_{1n}\}} = Ker(T)^{\perp}$, $\overline{span\{e_{2n}\}} = \overline{Im(T)}$ *s.t.* $T = \sum_n \lambda_n\ e_{1n} \otimes_1 e_{2n} \Leftrightarrow T \in K(H_1, H_2)$.

Proof.

"$\Rightarrow$": Let $T_k = \sum_{n=1}^{k} \lambda_n\ e_{1n} \otimes_1 e_{2n} \Rightarrow T_k \in K(H_1, H_2)$, then $\forall x \in H_1$, $||(T_k - T)x|| = ||\sum_{n=k+1}^{\infty} \lambda_n\ \langle e_{1n}, x\rangle_1 e_{2n}|| = \sqrt{\sum_{n=k+1}^{\infty} \lambda_n^2 \langle e_{1n}, x\rangle_1^2} \leq \lambda_{k+1}||x||_1 \Rightarrow T \in K(H_1, H_2)$.

"$\Leftarrow$": $T = \sum_n \lambda_n\ e_{1n} \otimes_1 e_{2n}$, $\lambda_n^2 \in \sigma_p(T^*T) \Rightarrow \lambda_n \to 0$. And $\overline{span\{e_{2n}\}} = \overline{Im(TT^*)} = Ker(TT^*)^{\perp} = Ker(T^*)^{\perp} = \overline{Im(T)}$. □

Theorem

$T \in K(H_1, H_2)$ *and the svd of* T *is* $\sum_n \lambda_n \, e_{1n} \otimes_1 e_{2n}$*, then* $T^\dagger = \sum_n e_{2n} \otimes_2 e_{1n}/\lambda_n$.

Proof.

$T^\dagger : Im(T) + Im(T)^\perp \to Ker(T)^\perp$, $\forall y \in Im(T) + Im(T)^\perp$, let y_1 is the part of y in $Im(T)$, $T^\dagger y = T^\dagger y_1 = \sum_n \langle T^\dagger y_1, e_{1n} \rangle_1 e_{1n}$ $= \sum_n \langle T^\dagger y_1, T^* T e_{1n}/\lambda_n \rangle_1 e_{1n}/\lambda_n = \sum_n \langle T T^\dagger y_1, e_{2n} \rangle_2 e_{1n}/\lambda_n$ $= \sum_n \langle P_{\overline{Im(T)}} y_1, e_{2n} \rangle_2 e_{1n}/\lambda_n = \sum_n \langle y_1, e_{2n} \rangle_2 e_{1n}/\lambda_n =$ $\sum_n \langle y, e_{2n} \rangle_2 e_{1n}/\lambda_n = (\sum_n e_{2n} \otimes_2 e_{1n}/\lambda_n) y$. $\square$

Corollary

$T \in K(H_1, H_2)$, if H_i is infinite-dim, then $Im(T)$ is not closed.

Proof.

Noticed that T isn't invertible, svd of T: $\sum_n \lambda_n \; e_{1n} \otimes_1 e_{2n}$.

If $Im(T)$ is closed, let $y = \sum_k \lambda_{n_k} e_{2n_k}$, $\lambda_{n_k} \leq k^{-4}$, then $y \in Im(T)$.

$||T^\dagger y||_1^2 = ||\sum_n \langle y, e_{2n}\rangle_2 e_{1n}/\lambda_n||_1^2 = ||\sum_k e_{1n_k}||_1^2 = \infty \Rightarrow T^\dagger y$ doesn't exist, contradiction. □

Lemma

$T_1, T_2 \in B(H_1, H_2)$, H_i is separable Hilbert space, which COB is $\{e_{in}\}$, then $\sum_n \langle T_1 e_{1n}, T_2 e_{1n} \rangle_2 = \sum_m \langle T_1^* e_{2m}, T_2^* e_{2m} \rangle_1$ if one of them exists.

Proof.

$$\sum_n \langle T_1 e_{1n}, T_2 e_{1n} \rangle_2 = \sum_n \langle T_1 e_{1n}, \sum_m \langle T_2 e_{1n}, e_{2m} \rangle_2 e_{2m} \rangle_2$$
$$= \sum_n \sum_m \langle T_1 e_{1n}, e_{2m} \rangle_2 \langle T_2 e_{1n}, e_{2m} \rangle_2$$
$$= \sum_n \sum_m \langle e_{1n}, T_1^* e_{2m} \rangle_1 \langle e_{1n}, T_2^* e_{2m} \rangle_1 = \sum_m \langle T_1^* e_{2m}, T_2^* e_{2m} \rangle_1$$

□

If $\sum_n (||T_1 e_{1n}||_2^2 + ||T_2 e_{1n}||_2^2) < \infty$, then $\sum_n \langle T_1 e_{1n}, T_2 e_{1n} \rangle_2$ exist.

Definition

$T \in B(H_1, H_2)$, if $\sum_n ||Te_{1n}||_2^2 := ||T||_{HS}^2 < \infty$, we call T a Hilbert-Schmidt operator. We mark that $T \in B_{HS}(H_1, H_2)$.

Property

$B_{HS}(H_1, H_2)$ normed vector space. If $T \in B_{HS}(H_1, H_2)$, $\sum_n ||Te_{1n}||_2^2 = \sum_n ||T^* e_{2n}||_1^2$, then $T^* \in B_{HS}(H_1, H_2)$.

It also means that $||T||_{HS}$ doesn't depend on the COB of H_i.

Theorem

$$B_{HS}(H_1, H_2) \subset K(H_1, H_2).$$

Proof.

$\forall T \in B_{HS}(H_1, H_2)$, define $T_n x = \sum_{k \leq n} \langle Tx, e_{2k} \rangle_2 e_{2k}$,
$\forall x \in B[H_1]$, $||Tx - T_n x||_2^2 = \sum_{k>n} \langle Tx, e_{2k} \rangle_2^2 \leq \sum_{k>n} \langle x, T^* e_{2k} \rangle_1^2$
$\leq \sum_{k>n} ||T^* e_{2k}||_1^2$. □

Definition

Define $\langle \cdot, \cdot \rangle_{HS}$*:* $\langle T_1, T_2 \rangle_{HS} = \sum_n \langle T_1 e_{1n}, T_2 e_{1n} \rangle_2$

Without loss of generation, let $\{e_{1n}\}$ be the eigenvector of T^*T, and make a complement to get a COB of H_1, then

$$||T||_{HS}^2 = \sum_n ||Te_{1n}||_2^2 = \sum_n \lambda_n^2$$

Theorem

$B_{HS}(H_1, H_2)$ is separable Hilbert space and $\{e_{1n} \otimes_1 e_{2m}\}$ is a COB of $B_{HS}(H_1, H_2)$.

Proof.

$\forall T \in B_{HS}(H_1, H_2)$, $Tx = \sum_m \langle T \sum_n \langle x, e_{1n}\rangle_1 e_{1n}, e_{2m}\rangle_2 e_{2m}$ $= \sum_{n,m} \langle x, e_{1n}\rangle_1 \langle Te_{1n}, e_{2m}\rangle_2 e_{2m} = \sum_{n,m} a_{nm}(e_{1n} \otimes_1 e_{2m})x$, $a_{nm} = \langle Te_{1n}, e_{2m}\rangle_2 \Rightarrow T = \sum_{n,m} a_{nm}(e_{1n} \otimes_1 e_{2m})$

Noticed that $\langle e_{1n} \otimes_1 e_{2m}, e_{1j} \otimes_1 e_{2k}\rangle_{HS} = \delta_{nj}\langle e_{2m}, e_{2k}\rangle_2 \Rightarrow$ $||T||_{HS} = \sum_{n,m} a_{nm}^2 < \infty \Rightarrow B_{HS}(H_1, H_2) \cong l^2$.

Let $\langle T, e_{1n} \otimes_1 e_{2m}\rangle_{HS} = 0 \ \forall n, m$, then $\langle Te_{1n}, e_{2m}\rangle = 0 \Rightarrow$ $Te_{1n} = 0 \Rightarrow T = 0$, then $\{e_{1n} \otimes_1 e_{2m}\}$ is a COB of $B_{HS}(H_1, H_2)$. $\square$

Theorem

$T \in B_{HS}(H_1, H_2)$ *with* $(\lambda_n, e_{1n}, e_{2n})$. $\forall G \in B_{HS}(H_1, H_2)$, $Rank(G) \leq n$, $||T - G||_{HS} \geq ||T - \sum_{k=1}^{n} \lambda_k e_{1k} \otimes_1 e_{2k}||_{HS}$.

Note that $||T - \sum_{k=1}^{n} \lambda_k e_{1k} \otimes_1 e_{2k}||_{HS}^2 = \sum_{k>n} \lambda_k^2$.
$\exists \ \{x_k\}, \ \{y_k\}$ MOS s.t. $G = \sum_{k=1}^{n} \mu_k (x_k \otimes_1 y_k)$.
$||G||_{HS} = \sum_{k \leq n} \sum_j ||\mu_k \langle x_k, e_{1j} \rangle_1 y_k||_2^2 = \sum_{k \leq n} \mu_k^2$.
$\langle T, \mu_k (x_k \otimes_1 y_k) \rangle_{HS} = \mu_k \sum_j \langle T e_{1j}, \langle x_k, e_{1j} \rangle_1 y_k \rangle_2 = \mu_k \langle x_k, T^* y_k \rangle_1$
$\mu_k^2 - 2\mu_k \langle x_k, T^* y_k \rangle_2 = ||\mu_k x_k - T^* y_k||_1^2 - ||T^* y_k||_1^2 \geq -||T^* y_k||_1^2$
$||T - G||_{HS}^2 \geq ||T||_{HS}^2 - \sum_{k=1}^{n} ||T^* y_k||_1^2$, then we just need to prove that $\sum_{k \leq n} ||T^* y_k||_1^2 \leq \sum_{k \leq n} \lambda_k^2$.

Lemma

$\sum_{k\leq n} ||T^* y_k||_1^2 \leq \sum_{k\leq n} \lambda_k^2.$

Proof.

$||T^* y_k||_1^2 = ||\sum_m \lambda_m \langle e_{2m}, y_k\rangle_2 e_{1m}||_1^2 = \sum_m \lambda_m^2 \langle e_{2m}, y_k\rangle_2^2$
$= \sum_{m=1}^n \lambda_m^2 \langle e_{2m}, y_k\rangle_2^2 + \sum_{m>n} \lambda_m^2 \langle e_{2m}, y_k\rangle_2^2 - \lambda_n^2 \sum_m \langle e_{2m}, y_k\rangle_2^2$
$+\lambda_n^2 \leq \sum_{m=1}^n (\lambda_m^2 - \lambda_n^2)\langle e_{2m}, y_k\rangle_2^2 + \lambda_n^2$
$\sum_{m,k\leq n}((\lambda_m^2 - \lambda_n^2)\langle e_{2m}, y_k\rangle_2^2 + n\lambda_n^2 \leq \sum_{m\leq n}(\lambda_m^2 - \lambda_n^2)$
$\sum_{k\leq n}\langle e_{2m}, y_k\rangle_2^2 + n\lambda_n^2 \leq \sum_{m\leq n} \lambda_m^2.$ $\square$

Definition

$T \in B(H_1, H_2)$, H_i separable Hilbert space, then T is a trace class operator if COB $\{e_n\}$ of H_1, $tr(T) := \sum_n \langle (T^*T)^{1/2} e_n, e_n \rangle_1$ $= ||(T^*T)^{1/4}||_{HS}^2 < \infty$.

For a matrix A, $tr(A) = tr((A^T A)^{1/2}) = tr(V^T (A^T A)^{1/2} V)$

Property

If $tr(T) < \infty$, *then* $T \in K(H_1, H_2)$.

Proof.

$\{e_{1n}\}$ is the eigen-vector of T^*T and $\{e_{2n}\}$ is TT^*. $\forall x \in H_1$:

$$
\begin{gathered}
Tx = \sum_n \langle T \sum_m \langle x, e_{1m}\rangle_1 e_{1m}, e_{2n}\rangle_2 e_{2n} \\
= \sum_{n,m} \langle Te_{1m}, e_{2n}\rangle_2 \langle x, e_{1m}\rangle_1 e_{2n} \\
= \sum_{n,m} \langle T^*Te_{1m}, e_{1n}\rangle_1 / \lambda_n (e_{1m} \otimes_1 e_{2n})x \\
= \sum_{n,m} \langle T^*Te_{1m}, e_{1n}\rangle_1 (e_{1m} \otimes_1 e_{2n})x \\
= \sum_n \langle (T^*T)^{1/2} e_{1n}, e_{1n}\rangle_1 (e_{1n} \otimes_1 e_{2n})x
\end{gathered}
$$

Then $T = \sum_n \lambda_n (e_{1n} \otimes_1 e_{2n})$. $\square$

Property

$tr(T) = \sum_n \lambda_n$, λ_n *is singular value of* T.

Proof.

$tr(T) = \langle (T^*T)^{1/2} e_n, e_n \rangle_1 = \sum_n \lambda_n$. □

If $T \in B_{HS}(H_1, H_2)$ since $||T||_{HS}^2 = \sum_n \lambda_n^2 \leq \lambda_1 tr(T)$.

Let $B_T(H_1, H_2)$ denote the operator which trace is finite. We now can conclude that:

$$B_T(H_1, H_2) \subset B_{HS}(H_1, H_2) \subset K(H_1, H_2) \subset B(H_1, H_2)$$

Theorem

$T, G \in B_{HS}(H_1, H_2)$, *and* $\{\lambda_n\}$ *and* $\{\phi_n\}$ *are corresponding singular values, then* $\sum_n (\lambda_n - \phi_n)^2 \leq tr((T-G)^*(T-G))$.

Proof.

$tr((T-G)^*(T-G)) = tr(T*T) + tr(G^*G) - 2tr(G^*T) = \sum_n(\lambda_n^2 + \phi_n^2) - 2\langle Te_{1n}^T, Ge_{1n}^T\rangle_2$

$$\langle Te_{1n}^T, Ge_{1n}^T\rangle_2 = \lambda_n \langle e_{2n}^T, Ge_{1n}^T\rangle_2 = \lambda_n \langle e_{2n}^T, \sum_m \langle Ge_{1n}^T, e_{2m}^G\rangle_2 e_{2m}^G\rangle_2$$
$$= \sum_m \lambda_n \langle e_{2n}^T, \langle e_{1n}^T, G^* e_{2m}^G\rangle_1 e_{2m}^G\rangle_2$$
$$= \sum_m \lambda_n \langle e_{2n}^T, \langle e_{1n}^T, \phi_m e_{1m}^G\rangle_1 e_{2m}^G\rangle_2 = \lambda_n \phi_m \langle e_{2n}^T, e_{2m}^G\rangle_2 \langle e_{1n}^T, e_{1m}^G\rangle_1$$

And $\sum_{n,m} \lambda_n \phi_m \langle e_{2n}^T, e_{2m}^G\rangle_2 \langle e_{1n}^T, e_{1m}^G\rangle_1 \leq \sum_n \lambda_n \phi_n$ $\square$