Definition

$\mathcal{M}$ *is a set and* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}$. $(\mathcal{M}, d)$ *is said to be a metric space* $(\mathcal{M}, d)$*:*

(a) $d(x,x) \geq 0$
(b) $d(x,y) = 0 \Leftrightarrow x = y$
(c) $d(x,y) = d(y,x)$
(d) $d(x,y) \leq d(x,z) + d(y,z)$

Notation: If $\mathcal{A} \subset \mathcal{M}$,

$B(a;\delta) = \{x \in \mathcal{M};\ d(x,a) < \delta\}$
int$(\mathcal{A}) = \{a \in \mathcal{A}; \exists\, \delta > 0, B(a;\delta) \subset \mathcal{A}\}$
$x_n \to x : d(x_n, x) \to 0$
$\mathcal{A}' = \{a \notin A; \exists\, \{a_n\} \subset \mathcal{A}, a_n \to a\}, \overline{A} = \mathcal{A} \cup \mathcal{A}'$

Definition

$\mathcal{O} \subset \mathcal{M}$ *is an open set* : $\mathcal{O} \subset$ *int*$(\mathcal{O})$.
$\mathcal{C} \subseteq \mathcal{M}$ *is a closed set* : $\mathcal{C}^c$ *is open.*

Theorem

$\mathcal{C} \subset \mathcal{M}$ *is closed* $\Leftrightarrow \mathcal{C}' \subset \mathcal{C}$

Property

$\emptyset$ *and* $\mathcal{M}$ *are both closed and open.*
$\overline{\mathcal{A}} = \cap_{\{\mathcal{A} \subset \mathcal{C}\}} \mathcal{C}$, *int*$(\mathcal{A}) = \cup_{\{\mathcal{O} \subset \mathcal{A}\}} \mathcal{O}$.

Definition

$\{x_n\} \subset \mathcal{M}$ is a Cauchy sequence if $\sup_{n,m>N}\{d(x_n, x_m) \to 0$. If every Cauchy $\{x_n\} \subset \mathcal{M}$, $\exists x \in \mathcal{M}$ s.t. $x_n \to x$, we call $(\mathcal{M}, d)$ is complete.

Property

If $x_n \to x$, then $\{x_n\}$ is Cauchy
$\{x_n\}$ is Cauchy and $x_{n_k} \to x$, then $x_n \to x$

Theorem

$(\mathcal{M}, d)$ is a metric space, then $\exists$ a metric space $(\mathcal{M}', d')$ s.t. $M \subset \mathcal{M}'$ and $(\mathcal{M}', d)$ is complete.

Proof.

$\mathcal{M}' = \{$ all the Cauchy sequences in $\mathcal{M}$ $\}$, we can extend the metric d to $\mathcal{M}'$: $d'(\{x_n\}, \{y_n\}) = \lim_n d(x_n, y_n)$ and define the equivalence relation of $\mathcal{M}'$: $\{x_n\} = \{y_n\}$: $\lim_n d(x_n, y_n) = 0$. □

We should carefully check the existence, uniqueness and ambiguity of the definition of metric d'.

Definition

If $\mathcal{B} \subset \mathcal{A}$ s.t. $\mathcal{A} \subset \overline{\mathcal{B}}$, we call $\mathcal{B}$ is dense in $\mathcal{A}$.
$\mathcal{A}$ is separable if $\exists$ a countable dense set $\mathcal{B}$.

Property

If a metric space $(\mathcal{M}, d)$ is separable, then the sub-metric space $(\mathcal{A}, d^{'})$ is separable.

Property

$\mathcal{A} \subset \mathcal{M}$ is complete in $\mathcal{A} \Rightarrow \mathcal{A}$ is closed. If $\mathcal{M}$ is complete, then $\mathcal{A}$ complete in $\mathcal{A} \Leftrightarrow \mathcal{A}$ is closed.

Proof.

"$\Rightarrow$": $x_n \to x$, $x_n \in \mathcal{A} \Rightarrow \{x_n\}$ Cauchy in $\mathcal{A} \Rightarrow x \in \mathcal{A}$.
"$\Leftarrow$": $\{x_n\}$ Cauchy in $\mathcal{A} \Rightarrow \exists\, x \in \mathcal{M}$, $x_n \to x \Rightarrow x \in \mathcal{A}$. $\square$

Definition

$\mathcal{A} \subset \mathcal{M}$ is compact: $\forall$ $\{x_n\} \subset \mathcal{A}$, $\exists\{x_{n_k}\}$ and $x \in \mathcal{A}$ s.t. $x_{n_k} \to x$.

Compact set $\mathcal{A}$ is complete since that $\forall$ Cauchy $\{x_n\}$, $\exists$ $\{x_{n_k}\}$ and $x \in \mathcal{M}$ s.t. $x_{n_k} \to x \Rightarrow x_n \to x$.

Definition

$\mathcal{A} \subset\subset M$: $\forall$ $\{x_n\} \subset \mathcal{A}$, $\exists$ a convergent subsequence $\{x_{n_k}\}$, which limit is in $\mathcal{M}$.

$\mathcal{A}$ is totally bounded: for $\forall \varepsilon > 0$, $\exists$ finite $\{a_k\} \subset \mathcal{A}$ s.t. $A \subset \cup_k B(a_k; \varepsilon)$.

Property

The closure of totally bound set $\mathcal{A}$ is separable.

Proof.

Take $b_{jk} \in B(a_{jj_k}; 1/j)$, then $\{b_{jk_j}\}$ is countable and dense in $\overline{\mathcal{A}}$, since $\forall a \in A$, $d(a, b_{jk}) < d(a, a_{jj_k}) + d(a_{jj_k}, b_{jk})$. $\square$

Lemma

$\mathcal{A}$ is totally bounded $\Leftrightarrow \forall\{x_n\} \subset \mathcal{A}$, $\exists$ a Cauchy $\{x_{n_k}\}$

Proof.

"$\Rightarrow$" : $\forall\{x_n\} \subset \mathcal{A}$, $\exists$ finite $\{a_{1k}\}$, $\{x_n\} \subset \cup_k B(a_{1k}; 1)$
$\Rightarrow \exists g_1 \in \{a_{1k}\}$, infinite term in $\{x_n\} \in B(a_{1_{g_1}}; 1)$
$\Rightarrow$ infinite term in $\{x_n\} \in B(a_{1_{g_1}}; 1) \cap (\cup_k B(a_{2k}; 1/2))$
$\Rightarrow$ infinite term in $\{x_n\} \in B(a_{1_{g_1}}; 1) \cap B(a_{2_{g_2}}; 1/2)$
$\Rightarrow$ infinite term in $\{x_n\} \in \cap_j B(a_{jg_j}; 1/j)$.
Take $x_{n_k} \in \cap_{j=1}^{k} B(a_{jg_j}; 1/j)$, then $\{x_{n_k}\}$ Cauchy
"$\Leftarrow$" : $\forall\varepsilon > 0$ and a fixed x_1, if $A\mathcal{A} \subset B(x_1; \varepsilon)$, claim holds.
If not, take $x_2 \in A - B(x_1; \varepsilon)$, if $A \in \cup_{j=1}^{2} B(x_j; \varepsilon)$, claim holds.
If it never ends, then we get a $\{x_n\}$ and it has no Cauchy. $\square$

Theorem

$A \subset\subset M \Rightarrow A$ is totally bounded. If M is complete, then $A \subset\subset M \Leftrightarrow A$ is totally bounded.

M is compact $\Leftrightarrow M$ is complete and totally bounded.

Theorem

A is compact $\Leftrightarrow$ $\forall$ open covering of A, $\exists$ finite sub-covering.

Proof.

"$\Rightarrow$" : G is an open covering of A, $\forall x \in A$, define $\delta_x = \sup\{d; B(x; d) \subset \Omega, \Omega \in G\}$, $\delta = inf_{x \in A}\{\delta_x\}$.

Claim that $\delta > 0$. Take $\delta_{x_n} \to \delta$, $\exists\, x \in A$, $x_{n_k} \to x$ s.t. $B(x_{n_k}; \delta_x/2) \subset B(x; \delta_x) \Rightarrow \delta_{x_{n_k}} \geq \delta_x/2$. Claim hold. A totally bounded $\Rightarrow$ $\exists$ finite $\{a_n\}$ $A \subset \cup_n B(a_n; \delta/2) \subset \cup_n \Omega_n$, and $B(a_n; \delta/2) \subset \Omega_n \in G$.

"$\Leftarrow$" : Take a mutually different $\{x_n\}$, if A is not compact $\Rightarrow$ $\overline{\{x_n\}} \cap A = \{x_n\}$. Let $\Omega_i = (\overline{\{x_n\}})^c \cup \{x_i\}$, $(\Omega_i)^c = \overline{\{x_n\}} - \{x_i\}$ (closed), $\cup_i \Omega_i = (\overline{\{x_n\}})^c \cup A \supset A \Rightarrow \cup_i \Omega_i$ is an open covering of A, but it have no finite covering. □

Definition

$(M_1, d_1), (M_2, d_2)$ *are two metric spaces.* $f : M_1 \to M_2$ *is continuous at x, if* $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.* $f(B_1(x;\delta)) \subset B_2(f(x);\varepsilon)$

Theorem

f *continuous* $\Leftrightarrow f^{-1}(\Omega)$ *is open, for* $\forall$ *open set* Ω

Proof.

"$\Rightarrow$" : $\forall \varepsilon > 0, \forall x \in \Omega$, $\exists \delta > 0, B_1(x;\delta) \subset f^{-1}(B_2(f(x);\varepsilon))$, If ε is small enough, then $f^{-1}(B_2(f(x);\varepsilon)) \subset f^{-1}(\Omega)$

"$\Leftarrow$" : $f^{-1}(B_2(f(x);\varepsilon))$ is an open set contained x □

Theorem

$(M_1, d_1), (M_2, d_2)$ are two metric spaces. And A is a compact set in M_1, $f : M_1 \to M_2$ is continuous in A. So $f(A)$ is a compact set in M_2.

Proof.

$\cup_{\alpha \in F} \Omega_\alpha$ is an open covering of $f(A) \Rightarrow \cup_{\alpha \in F} f^{-1}(\Omega_\alpha)$ is an open covering of $A \Rightarrow \exists$ a finite sub-covering $\cup_n f^{-1}(\Omega_n)$ of $A \Rightarrow \cup_n \Omega_n$ is a finite sub-covering of $f(A)$ $\square$

Definition

Finite-dim vector space: $\exists$ *finitely linear independent* $\{v_n\} \subset V$ *s.t.* $V = span\{v_n\}$.

Definition

V *is a vector space over field* F, *and a norm* $||\cdot|| : V \to R$, *we say* $(V, ||\cdot||)$ *is said to be a normed vector space if:*

(a) $||x|| \geq 0$ *and* $||x|| = 0 \Rightarrow x = 0$

(b) $||ax|| = |a|\ ||x||$ *and (c)* $||x+y|| \leq ||x|| + ||y||$.

Noticed that $\forall x, y \in V$, we can define $d(x,y) = ||x-y||$, so (V,d) is a metric space.

Theorem

Any p-dim vector space V over filed R is isomorphic to R^p

Proof.

Take basis $\{v_n\}$. $\forall x \in V$, $\exists! \{a_n\} \subset R$, $x = \sum_n a_n v_n$ □

So V and R^p are the same thing, and a norm of V can be regarded as the norm of R^p. R^p equipped with an Euclidean norm is complete and separable.

Definition

$||\cdot||_i$ norms of V, we say $||\cdot||_1, ||\cdot||_2$ are equivalent if $\exists c, C > 0, \forall x \in V, c||x||_1 \leq ||x||_2 \leq C||x||_1$. We mark that: $||\cdot||_1 \backsim ||\cdot||_2$. "$\backsim$" is an equivalence relation.

Theorem

All the norm for R^p are equivalent.

Proof.

$||\cdot||_p$ is the Euclidean norm for R^p and $||\cdot||_a$ is the other norm for R^p, which can be view as a function $f : (R^p, ||\cdot||_p) \to (R, ||\cdot||_1)$.

$\forall x, y \in R^p$, $||f(x) - f(y)||_1 \leqslant f(x-y) = ||x-y||_a \leq \sum_i |x_i - y_i|\,||e_i||_a \leq ||x-y||_p\, (\sum_i ||e_i||_a^2)^{1/2} \Rightarrow f$ is continuous.

Then $f(\{x \in R^p ; ||x||_p = 1\})$ is a bound closed set. $\exists c_a, C_a$, $c_a \leqslant ||x||_a \leqslant C_a, \forall ||x||_p = 1 \Leftrightarrow \forall x \in R^p$, $c_a \leqslant ||x||_a / ||x||_p \leqslant C_a \Rightarrow ||\cdot||_a \backsim ||\cdot||_p$ □

Theorem

$dim(V) < \infty, A \subset V$. Then A is compact $\Leftrightarrow$ A is closed and bounded.

Proof.

"$\Rightarrow$" : A is closed and totally bounded.

"$\Leftarrow$" : $\exists\, M > 0,\ A \subset B(0; M)$. Finite-dim means $B(0; M)$ is totally bounded and complete $\Rightarrow B(0; M)$ is compact. Then a closed subset A of $B(0; M)$ is compact. □

Definition

Infinite-dim vector space: $\forall$*finite* $\{v_n\}$, $span\{v_n\} \subsetneq V$, $span\{v_n\}$ *is a closed subspace of* V.

Lemma

V *normed vector space,* $M \subsetneq V$ *is a closed vector subspace. Then* $\forall \varepsilon > 0$, $\exists v$ *s.t.* $||v|| = 1$, $||v - M|| \geqslant 1 - \varepsilon$

Proof.

Take $u \in M^c$, $d := ||u - M|| > 0$, and $\exists m \in M$ s.t. $d \leq ||u - m|| \leq d/(1 - \varepsilon)$. Let $v = (u - m)/||u - m||$, then $||v - M|| = ||u - m - M||/||u - m|| = ||u - M||/||u - m|| \geqslant 1 - \varepsilon$ $\square$

Theorem

V normed vector space, define $B[V] = \{x \in V; ||x|| \leqslant 1\}$, $B[V]$ is not compact $\Leftrightarrow dim(V) = \infty$.

Proof.

"$\Rightarrow$" : We consider $V = R^p$. Then $B[V]$ is compact.

"$\Leftarrow$" : If $dim(V) = \infty$. Take closed sub-space sequence $\{V_n\}$ s.t. $V_n \subsetneq V_{n+1} \subsetneq V$. $\exists\, x_n \in V_n$ s.t. $||x_n|| = 1$, $||x_n - V_{n-1}|| \geqslant 1/2 \Rightarrow \{x_n\} \subset B[V]$ but it has no convergent subsequence. $\square$

Definition

$(V, ||\cdot||)$ over F is a Banach space if $(V, ||\cdot||)$ is a complete normed vector space.

Theorem

V Banach space, $A \subset V$ is separable, which countable dense subset of A is $\{x_n\}$, then $\overline{span(A)} = \overline{\cup_{X\in G} span(X)}$, G is a collection of all the finite sub-sequences of $\{x_n\}$.

Proof.

"$\overline{\cup_{X\in G} span(X)} \subset \overline{span(A)}$" is apparent. $\forall x \in \overline{span(A)}$, $\forall \varepsilon > 0$, $\exists$ finite $a_m \in A$ and $k_m \in F$, $||x - \sum_m a_m k_m|| \leq \varepsilon/2$. Take $x^{(m)} \in \{x_n\}$ s.t. $x^{(m)}$ and a_m are sufficiently closed to achieve $||x - \sum_m x^{(m)} k_m|| \leq \varepsilon$ □

Definition

V is a vector space over R, an inner product

$\langle \cdot, \cdot \rangle : V \times V \to R$

(a) $\langle x, x \rangle \in R$ *and* ≥ 0

(b) $\langle x, y \rangle = \langle y, x \rangle$

(c) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

(d) $x = 0 \Leftrightarrow \langle x, x \rangle = 0$

When "2,4" are not satisfied, we call $\langle \cdot, \cdot \rangle$ semi-inner-product.

Property

$\langle\cdot,\cdot\rangle$ is a semi-inner-product of vector space V, define $||x|| = \sqrt{\langle x,x\rangle}$, then $|\langle x,y\rangle| \leq ||x||\ ||y||$

Proof.

$\hat{y} = \langle x,y\rangle/||x||^2\ x$, $\hat{y}$ is the projection of y onto x. Let $r = y - \hat{y} \Rightarrow r \perp \hat{y}$, we call r and $\hat{y}$ are orthonormal.

$||y||^2 = ||\hat{y}+r||^2 = \langle \hat{y}+r, \hat{y}+r\rangle \geq ||\hat{y}||^2 = |\langle x,y\rangle|^2/||x||^2$. $\square$

$$||x+y||^2 \leq ||x||^2 + ||y||^2 + 2||x||\ ||y|| = (||x|| + ||y||)^2.$$

Corollary

$(V,\langle\cdot,\cdot\rangle)$ inner product space, then V is normed vector space.

Property

Parallelogram rule: $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$

$\langle \cdot, \cdot \rangle$ is a norm $|| \cdot ||^2$, but a norm $|| \cdot ||$ may not be induced by an inner product.

Theorem

If a norm $|| \cdot ||$ *satisfies parallelogram rule, we can define* $\langle x, y \rangle = (||x+y||^2 - ||x-y||^2)/4$*, then* $\langle \cdot, \cdot \rangle$ *is an inner product and* $|| \cdot ||$ *is induced by* $\langle \cdot, \cdot \rangle$*.*

Property

$(V, \langle\cdot,\cdot\rangle)$ inner product space, and $x_n \to x$, $y_m \to y$, then $\langle x_n, y_m\rangle \to \langle x, y\rangle$.

Proof.

$$|\langle x_n, y_m\rangle - \langle x, y\rangle| = |\langle x_n, y_m\rangle - \langle x, y_m\rangle + \langle x, y_m\rangle - \langle x, y\rangle|$$
$$\leq |\langle x_n - x, y_m\rangle| + |\langle x, y_m - y\rangle| \leq ||x_n - x||\ ||y_m|| + ||x||\ ||y_n - y||$$

□

Definition

$(V, \langle\cdot,\cdot\rangle)$ *inner product space,* $A \subset V$*, then orthogonal complement of* A*:* $A^{\perp} = \{x \in V; \langle x, y\rangle = 0, \forall y \in A\}$

Property

$x \perp V \Leftrightarrow x = 0$
$A \cap A^{\perp} \subset \{0\}$
$A \subset B \Rightarrow B^{\perp} \subset A^{\perp}$
$x \perp y$, $||x + y||^2 = ||x||^2 + ||y||^2$
$A^{\perp}$ *is a closed sub-space and* $A \subset (A^{\perp})^{\perp}$

Definition

$(V, \langle\cdot,\cdot\rangle)$ *inner product space,* A *is a sub-space. For* $x \in V$*, if* $\exists\, x_0 \in A, x_1 \in A^{\perp}$ *s.t.* $x = x_0 + x_1$*, then* x_0 *is the projection of* x *onto* A*.*

x_0 is unique since if $x = x_0 + x_1 = y_0 + y_1, x_0 - y_0 \in A \cap A^{\perp}$ $\Rightarrow x_0 - y_0 = 0$.

Theorem

If the projection of x *exists, then* $\exists!\, x_0 \in A$ *s.t.* $||x - x_0|| = \inf_{y \in A} ||x - y||$*, and* x_0 *is the projection of* x *onto* A*.*

Proof.

$$||x - y||^2 = ||x - x_0||^2 + ||x_0 - y||^2 \geq ||x - x_0||^2$$ □

If x_0 minimizes the distance between x and A, x_0 is the projection of x? The answer is no since that we can not ensure the existence of projection of x.

Lemma

A is a sub-space, if $\exists\ x_0 \in A$ minimizes the distance between x and A, then x_0 is the projection of x.

Proof.

$\forall z \in A$, $||x-z||^2 \geq ||x-x_0||$, let $z = x_0 + \lambda y$.

$||x - x_0 + \lambda y||^2 = ||x-x_0||^2 + \lambda^2||y||^2 + 2\lambda\langle x - x_0, y\rangle$
$\geq ||x-x_0||^2$, $\forall y \in A$, $\lambda \in R \Rightarrow (\lambda||y||^2 + 2\langle x - x_0, y\rangle)\lambda \geq 0 \Rightarrow$
$\langle x - x_0, y\rangle = 0$, $\forall y \in A$. $\square$

Lemma

V Banach space, which norm satisfied parallelogram rule, A is a closed convex subset, then $\forall x \in V$, $\exists!\ x_0 \in A$ s.t. $||x - x_0|| = \inf_{y \in A} ||x - y||$.

Proof.

Take $y_n \in A$ s.t. $\lim_n ||x - y_n|| = \inf_{y \in A} ||x - y||$, $\{y_n\}$ Cauchy since $||y_n - y_m||^2 = 2||x - y_n||^2 + 2||x - y_m||^2 - ||2x - y_n - y_m||^2 = 2||x - y_n||^2 + 2||x - y_m||^2 - 4||x - (y_n + y_m)/2||^2$.

$(y_n + y_m)/2 \in A$, then $||x - (y_n + y_m)/2|| \geq \inf_{y \in A} ||x - y||$ $\Rightarrow \{y_n\}$ Cauchy $\Rightarrow \exists! x_0 \in A$ s.t. $y_n \to x_0$. $\square$

Definition

A complete inner-product space H is called a Hilbert space.

Theorem

H Hilbert space, and A closed sub-space, then $H = A \oplus A^{\perp}$.

Proof.

A is convex $\Rightarrow \exists\, x_0 \in A$ minimizes the distance between x and A. Then x_0 is the projection of x and $x - x_0 \in A^{\perp}$. $\square$

Corollary

H Hilbert space, A is a sub-space, then $\overline{A} = (A^{\perp})^{\perp}$

Proof.

$$A \subset (A^{\perp})^{\perp} \Rightarrow \overline{A} \subset (A^{\perp})^{\perp},\ (A^{\perp})^{\perp} = \overline{A} \oplus (\overline{A}^{\perp} \cap (A^{\perp})^{\perp})$$

$$\forall x \in (A^{\perp})^{\perp},\ \exists! x_0 \in \overline{A},\ x_1 \in \overline{A}^{\perp} \cap (A^{\perp})^{\perp} \subset A^{\perp} \cap (A^{\perp})^{\perp} \text{ s.t.}$$

$x = x_0 + x_1 = x_0 \in \overline{A}$. □

Definition

A mutually orthogonal countable $\{e_n\}$ *s.t.* $||e_n|| = 1$ *in a inner-product space* V *is said to be an orthonormal sequence.*

Theorem

H *Hilbert space,* $\{x_n\}$ *is a linear independent. Define* $\{e_n\}$*:* $e_1 = x_1/||x_1||$, $v_n = x_n - \sum_{k=1}^{n-1}\langle x_n, e_k\rangle e_k$, $e_n = v_n/||v_n||$, *then* $\{e_n\}$ *is an orthonormal sequence and* $\overline{span\{x_n\}} = \overline{span\{e_n\}}$.

Theorem

V inner product space, $\{e_n\}$ orthonormal sequence. $\forall x \in V$, *the Fourier series of x: $\sum_n \langle x, e_n\rangle e_n$ converges in V and $\sum_n \langle x, e_n\rangle e_n$ is the projection of x onto $\overline{span\{e_n\}}$.*

Proof.

Bessel Inequality: $\sum_n \langle x, e_n\rangle^2 \leq ||x||^2$, it holds since that $||x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k||^2 = ||x||^2 - 2\sum_n \langle x, e_n\rangle^2 + \sum_n \langle x, e_n\rangle^2 \Rightarrow ||x||^2 - \sum_n \langle x, e_n\rangle^2 \geq 0$.

$|| \sum_n \langle x, e_n\rangle e_n||^2 = \sum_n \langle x, e_n\rangle^2 \Rightarrow \sum_n \langle x, e_n\rangle e_n$ exist. and it's the projection of $\overline{span\{e_n\}}$ since that $\forall i$, $\langle x - \sum_n \langle x, e_n\rangle e_n, e_i\rangle = \langle x, e_i\rangle - \langle x, e_i\rangle = 0$. $\square$

Theorem

H Hilbert space, $\{e_n\}$ orthonormal sequence. $\forall x \in H$,
(a) $||x||^2 = \sum_n \langle x, e_n\rangle^2 \Leftrightarrow$ (b) $x = \sum_n \langle x, e_n\rangle e_n$
$\Leftrightarrow$ (c) If $\langle x, e_n\rangle = 0, \ \forall n$, then $x = 0 \Leftrightarrow$ (d) $\overline{span\{e_n\}} = H$.

Proof.

$(a) \Leftrightarrow (b)$: $||x - \sum_n \langle x, e_n\rangle e_n||^2 = ||x||^2 - \sum_n \langle x, e_n\rangle^2$
$(b) \Leftrightarrow (c)$: $\langle x - \sum_n \langle x, e_n\rangle e_n, e_m\rangle = 0, \forall m$
$(b) \Rightarrow (d)$: $\overline{span\{e_n\}}$ closed sub-space, and $H \subset \overline{span\{e_n\}}$
$(b) \Leftarrow (d)$: $\sum_n \langle x, e_n\rangle e_n$ is the projection of x. □

Definition

$\{e_n\}$ is a complete orthonormal basis in Hilbert space H if $\{e_n\}$ is an orthonormal sequence and $\overline{span\{e_n\}} = H$

Theorem

H Hilbert space is separable $\Leftrightarrow$ H has a COB.

Proof.

"$\Rightarrow$": $\overline{\{e_n\}}$ is dense in H. $\overline{\forall x \in H}$, $\forall \varepsilon > 0$, $\exists\, e_k$ s.t. $||x - e_k|| \leq \varepsilon \Rightarrow x \in \overline{span\{e_n\}} \Rightarrow H \subset \overline{span\{e_n\}}$

"$\Leftarrow$": $\{e_n\}$ COB, let $A = \{x \in H; \langle x, e_k\rangle \in Q, \forall k\}$. Then A is countable and dense in H. $\square$

Definition

$l^2 = \{(a_1, a_2, ...); a_k \in R\}$ and $\langle a, b\rangle = \sum_n a_n b_n$, then l^2 is a separable Hilbert space.

Theorem

Any infinite-dim separable Hilbert space H isometrically isomorphic to l^2. We mark that $H \approx l^2$.

Proof.

$\{e_n\}$ COB, $\forall x \in H$, $x = \sum_n \langle x, e_n\rangle e_n$, define $f : H \to l^2$, $f(x) = ((\langle x, e_n\rangle))_n$. f is a bijection.

$||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y\rangle = \sum_n \langle x, e_n\rangle^2 + \sum_n \langle y, e_n\rangle^2$ $-2\sum_n \langle x, e_n\rangle\langle y, e_n\rangle = \sum_n (\langle x, e_n\rangle - \langle y, e_n\rangle)^2$. □