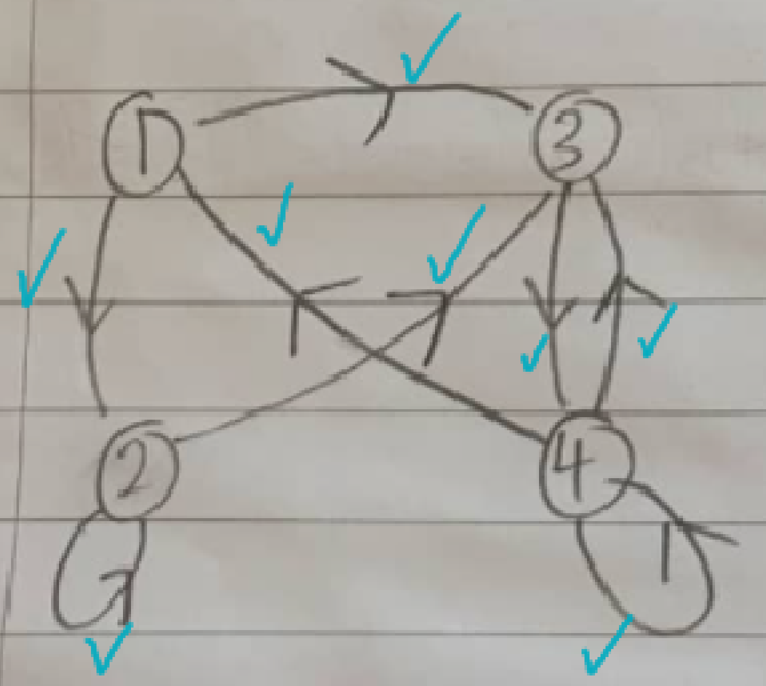
Tutorial 9

1. The following arrays describe a relation R on a set $A = \{1, 2, 3, 4\}$:
VERT = [1, 2, 6, 4]
TAIL = [1, 2, 2, 4, 4, 3, 4, 1]
HEAD = [2, 2, 3, 3, 4, 4, 1, 3]
NEXT = [8, 3, 0, 5, 7, 0, 0, 0]
Compute both the digraph of R and the matrix $\mathbf{M}_R$.

1.

VERT		TAIL		HEAD	NEXT
1	1	1	→	2	8
2	2	2	→	2	3
6	3	2	→	3	0
4	4	4	→	3	5
	5	4	→	4	7
	6	3	→	4	0
	7	4	→	1	0
	8	1	→	3	0



$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

2. Let $A = B = \{1, 2, 3\}$ and let $R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ and let $S = \{(2, 1), (3,1), (3, 2), (3, 3)\}$. Let R and S be the relations from A to B. Compute

i) $\overline{R}$

ii) $R \cap S$

iii) $R \cup S$

iv) S^{-1}

2. i) $\bar{R} = \{(1,3), (2,1), (2,2), (3,2), (3,3)\}$

2. ii) R ^ S = { (3, 1) }

optional:

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

interchange "0" and "1"

$$M_{\bar{R}} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

2. Let $A = B = \{1, 2, 3\}$ and let $R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ and let $S = \{(2, 1), (3,1), (3, 2), (3, 3)\}$. Let R and S be the relations from A to B. Compute

i) $\overline{R}$

ii) $R \cap S$

iii) $R \cup S$

iv) S^{-1}

Q2. $A = B = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,3), (3,1)\}$

$S = \{(2,1), (3,1), (3,2), (3,3)\}$

iii) $R \cup S = \{(1,1), (1,2), (2,1), (2,3), (3,1), (3,2), (3,3)\}$

2. Let $A = B = \{1, 2, 3\}$ and let $R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ and let $S = \{(2, 1), (3,1), (3, 2), (3, 3)\}$. Let R and S be the relations from A to B. Compute

iv) S^{-1}

$$S^{-1} = \{(1, 2), (1, 3), (2, 3), (3, 3)\}$$

Optional:

$$M_S = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} & \\ 3 & & & \end{array}$$

$$M_{S^{-1}} = (M_S)^T$$

$$= \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} & \\ 3 & & & \end{array}$$

3. Let $A = \{2, 4, 5, 7\}$ and let R and S be the relations on A described by $x\,R\,y$ if and only if $x + y$ is even and

$$M_S = \begin{array}{c} \\ 2 \\ 4 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

. List the ordered pairs belonging to the following relations.

i) S^{-1}

$S = \{(2,4), (4,5), (4,7), (5,4), (5,5), (5,7)\}$

s = {(1,2), (2,3), (2,4), (3,2), (3,3), (3,4)}

s^{-1} = {(2,1), (3,2), (4,2), (2,3), (3,3), (4,3)}

$S^{-1} = \{(4,2), (5,4), (7,4), (4,5), (5,5), (7,5)\}$

optional:

$$M_{S^{-1}} = (M_S)^T = \begin{array}{c} \\ 2 \\ 4 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

ii) $S^{-1} \cap R$

$= \{(4,2),(5,5),(7,5)\}$

$R = \{(2,2),(2,4),$
$(4,2),(4,4),(5,5),$
$(5,7), (7,5),(7,7)\}$

iii) $(S^{-1} \circ R)^{-1}$

$A = \{2, 4, 5, 7\}$

$$M_S = \begin{array}{c} \\ 2 \\ 4 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$S = \{(2,4), (4,5), (4,7), (5,4), (5,5),$
$(5,7)\}$

$$M_R = \begin{array}{c} \\ 2 \\ 4 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 5 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R = \{(2,2), (2,4), (4,2), (4,4),$
$(5,5), (5,7), (7,5), (7,7)\}$

(iii) $(S^{-1} \circ R)^{-1}$

$R^{-1} = \{(2,2), (4,2), (2,4), (4,4), (5,5), (7,5), (5,7),$
$(7,7)\}$

$(S^{-1} \circ R)^{-1} = (S \circ R^{-1}) = \{(2,2), (2,4), (4,5), (4,7), \cancel{(4,5)}, \cancel{(4,7)},$
$(5,4), (5,2), (5,5), (5,7), \cancel{(4,5)}, \cancel{(5,7)}\}$

$= (R^{-1} \circ S)$

start

$= \{(2,2), (2,4), (4,5), (4,7), (5,4), (5,2),$
$(5,5), (5,7)\}$

4. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3\}$. The matrices $\mathbf{M}_R$ and $\mathbf{M}_S$ of the relation R and S be the relations from A to B are given by $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Compute

i) $\mathbf{M}_{R \cup S}$ ii) $\mathbf{M}_{R \cap S}$

iii) $\mathbf{M}_{R^{-1}}$ iv) $\mathbf{M}_{\bar{S}}$

(i)

$$M_{R \cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

M(R and S) =

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

4. iii) $M_{R^{-1}} = (M_R)^T$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(iv)

$$M_{\bar{S}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Let $A = \{a, b, c, d, e\}$ and let the equivalence relations R and S on A be given by

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

i) Compute

a) $\mathbf{M}_{R \circ R}$ b) $\mathbf{M}_{S \circ R}$

c) $\mathbf{M}_{R \circ S}$ d) $\mathbf{M}_{S \circ S}$

i) a) $M_{R \circ R} = M_R \odot M_R$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $M_{S \circ R} = M_R \odot M_S$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

c) $M_{R \circ S} = M_S \odot M_R$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d) $M_{S \circ S} = M_S \odot M_S$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

5. Let $A = \{a, b, c, d, e\}$ and let the equivalence relations R and S on A be given by

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

ii) Compute the partition of A corresponding to $R \cap S$

$$R \cap S = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cap \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ✓$$

$R \cap S = \{(a,a), (b,b), (b,c), (c,b), (c,c), (d,d), (e,e)\}$ ✓✓✓✓✓✓✓

Partition of $A = \{\{a\}, \{b,c\}, \{d\}, \{e\}\}$ ✓✓✓✓

$A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$

$A_1 \cup A_2 \cup A_3 \cup A_4 = A$

diagonal relation, $\Delta = \{(1,1),(2,2)\ldots\}$ $\begin{bmatrix} x & & \\ & x & \\ & & x \end{bmatrix}$

6. $A = \{1,2,3,4\}$

$\{(2,1)\ \ (3,1)\ \ldots\} = R^{-1}$

$R = \{(1,2),(1,3),(2,3),(3,4)\}$

- reflexive closure : $R \cup \Delta$ ✓

$= \{(1,2),(1,3),(2,3),(3,4),\overbrace{(1,1),(2,2),(3,3),(4,4)}^{\Delta}\}$ ✓

- symmetric closure : $R \cup R^{-1}$ ✓

$= \{(1,2),(1,3),(2,3),(3,4),\overbrace{(2,1),(3,1),(3,2),(4,3)}^{R^{-1}}\}$ ✓

27) i)

$$M_R = W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_1 : 2$
$R_1 : 2$
Add $(1,2)$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 = 1,2$
$R_2 = 1,2,3$
Add : $(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)$

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 = 1,2$
$R_3 = 4$
Add = $(1,4), (2,4)$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = 1,2,3$
$R_4 = -$
Add : $-$

$$W_4 = W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_{R^\infty}$$

$$R^\infty = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$$

Tutorial 9

Question 7

7 (ii) let $A = \{1,2,3,4\}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_1 = -$

$R_1 = 2$

Add : $-$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 = 1, 3$

$R_2 : 3, 4$

Add : $(1,3), (1,4), (3,3), (3,4)$

$$W_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 = 1, 2, 3$

$R_3 = 2, 3, 4$

Add : $(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)$

$$W_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 = 1, 2, 3, 4$

$R_4 : 4$

Add : $(1,4), (2,4), (3,4), (4,4)$

$$W_4 = W_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M_{R^\infty}$$

8. Let $A = \{1, 2, 3, 4\}$ and let R and S be relations on A described by

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$(M_{R\cup S})^{\infty}$

Use Warshall's algorithm to compute the transitive closure of $R \cup S$.

8. $$M_{R\cup S} : \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$W_0 : \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$ ✓

$$(R\cup S)^{\infty} = \{(1,1), (1,2), (1,4), (2,2), (3,2), (3,3), (4,2), (4,3), (4,4)\}$$

Continued Q8.

$$W_0 = M_{R\cup S} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

C_1: 1

R_1: 1, 2, 4

Add: (1,1), (1,2), (1,4)

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

C_2: 1, 2, 3, 4

R_2: 2

Add: (1,2), (2,2), (3,2), (4,2)

$W_2 = W_1$

C_3: 3, 4

R_3: 2, 3

Add: (3,2), (3,3), (4,2), (4,3)

$W_3 = W_2$

C_4: 1, 4 , R_4: 2, 3, 4

Add: (1,2), (1,3), (1,4), (4,2), (4,3), (4,4)

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M_{(R\cup S)^\infty}$$