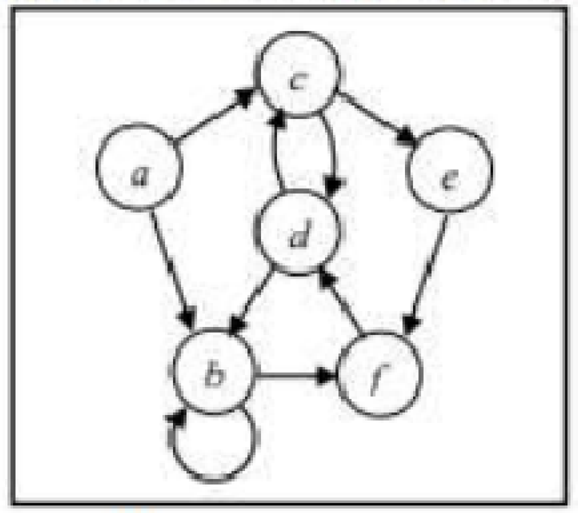
# Tutorial 8

A path of length $n$ involves $n + 1$ elements of $A$, although they are not necessarily distinct.

1. Let $R$ be the relation whose digraph is given as follow:




i) List all paths of length 1.
ii) List all paths of length 3 starting from vertex $a$.
iii) Find a cycle starting at vertex $d$.

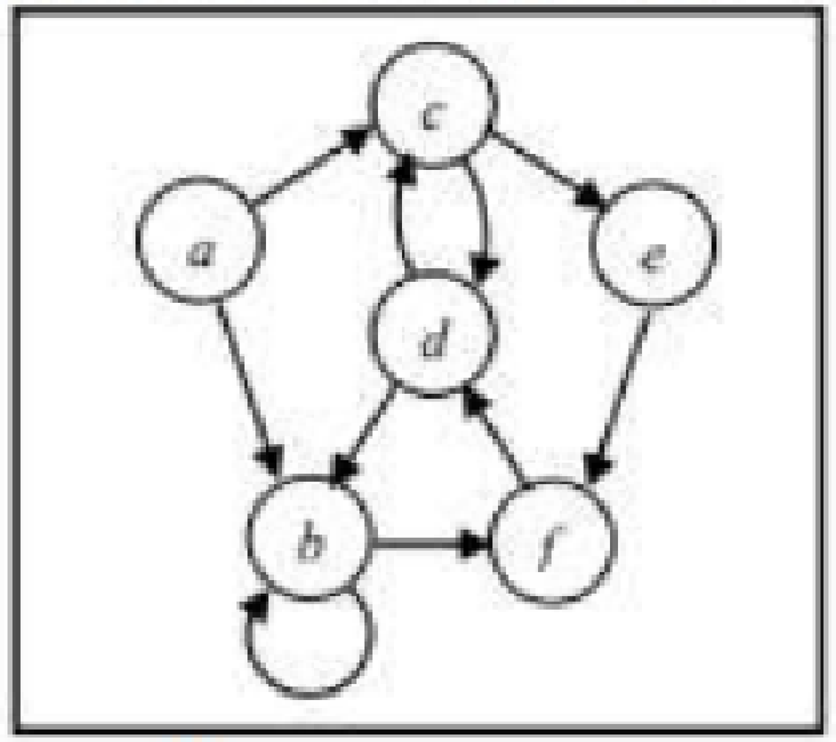
( i ) Path of length 1: a,c or a,b or b,b or b,f or c,d or c,e or d,c or d,b or e,f or f,d

{(a,c), (a,b), (b,b), (b,f), (c,d), (c,e), (d,c), (d,b), (e,f), (f,d)}

ii.
a,b,b,f or
a,b,f,d or
a,c,e,f or
a,c,d,b or
a,c,d,c or a,b,b,b

Path of length 3
= {(a,b,b,f), (a,b,f,d), (a,c,e,f), (a,c,d,b), (a,c,d,c), (a,b,b,b)}

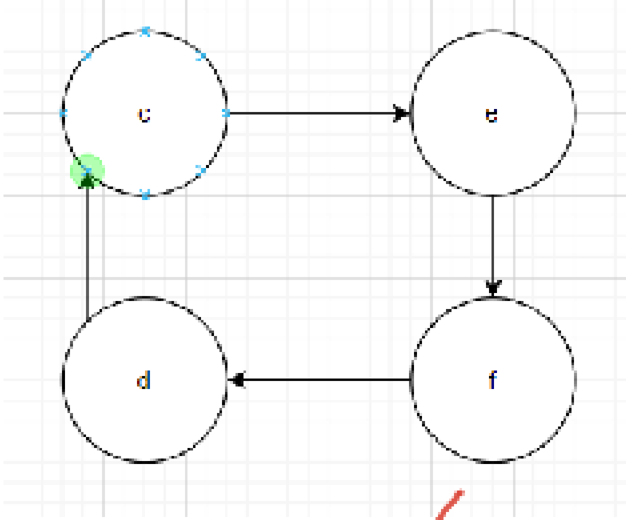
Q1

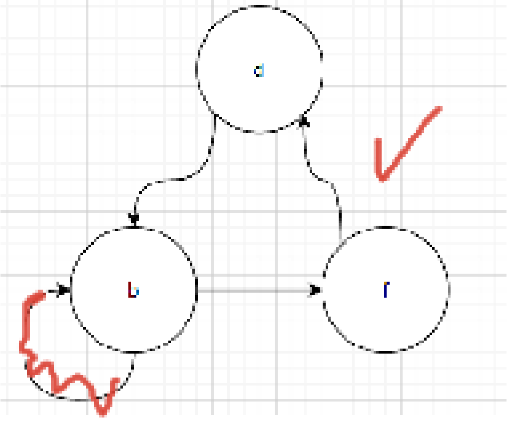



starting and ending point

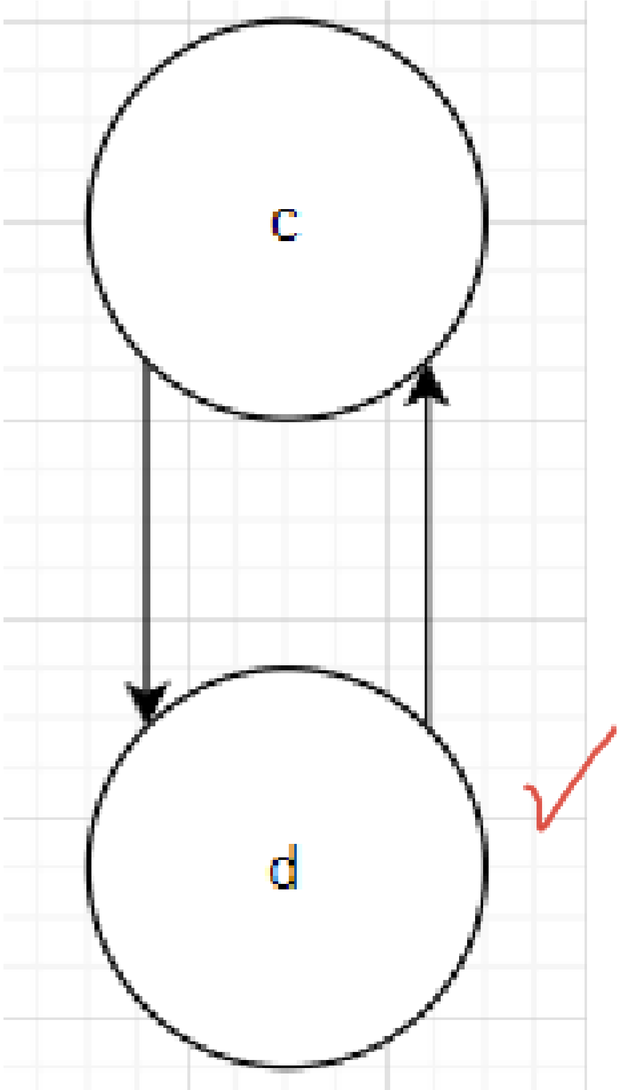
iii) Find a cycle starting at vertex $d$.

$= \{ (d,c,d), (d,c,e,f,d), (d,b,f,d) \}$




iii)



2. Determine whether the given relation on $A = \{1, 2, 3, 4\}$ is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers.

i) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$

R is reflexive

R is not irreflexive since $(1,1) \in R$

2 i) R is reflexive

R is not irreflexive since $(1,1), (2,2), (3,3), (4,4) \in R$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

R is symmetric

$(1,2), (2,1) \in R$

R is not asymmetric, since there exists two-way street and loop $(1,1), (2,2), (1,2), (2,1) \in R$

$(1,2), (2,1) \in R$ and $1 \neq 2$

R is not antisymmetric since there exists two-way street

R is transitive since $M_R = (M_R)^2_\odot$.

$$(M_R)^2_\odot = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = M_R$$

$A = \{1, 2, 3, 4\}$

ii) $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$

$$R = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

R is not reflexive since $(2,2) \notin R$

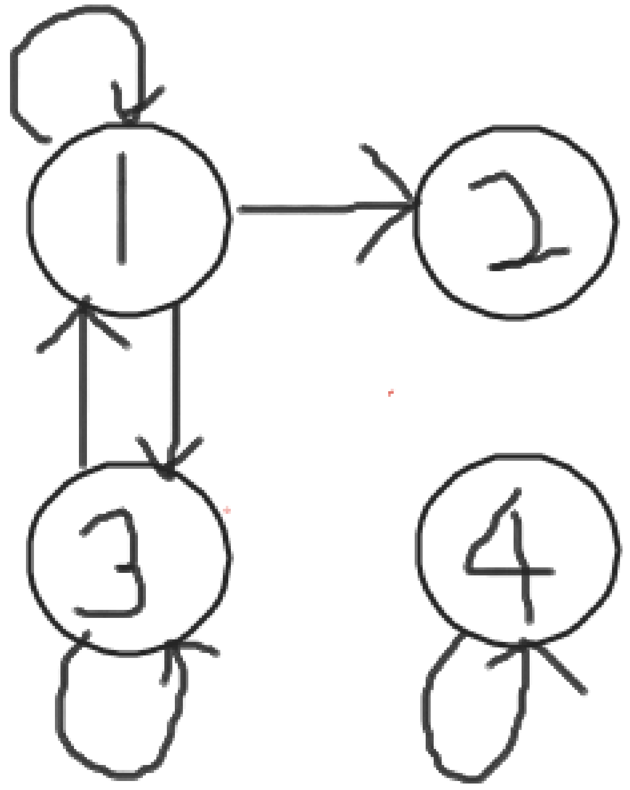
R is not irreflexive since $(1,1) \in R$

R is not symmetric since $(1,2) \in R$ but $(2,1)$ does not occur belong to R.

R is not asymmetric since (1,3) and (3,1) $\in R$

R is not antisymmetric since (1,3) and (3,1) $\in R$ and $\neq 3$

R is not transitive since it have (3,1) and (1,2) but do not have (3,2).



iii) $R = \varnothing$

(iii) R is not reflexive since $R = \phi \subseteq A \times A$, then $(1,1) \notin R$

R is irreflexive since $R = \phi$

R is symmetric since $R \neq \phi$, $M_R = (M_R)^T$

R is asymmetric since $\phi \notin R$

R is antisymmetric $\phi \in R$

R is transitive since $\phi \in R$

This is vacuously true because we cannot find any counterexample.

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

1
1
1
1

ref. ✓

iv) $R = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 0 & 1 & 0 & 0 \end{array}$$

$$(M_R)^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

R is not reflexive since $(2,2) \notin R$ and $(4,4) \notin R$ ✓

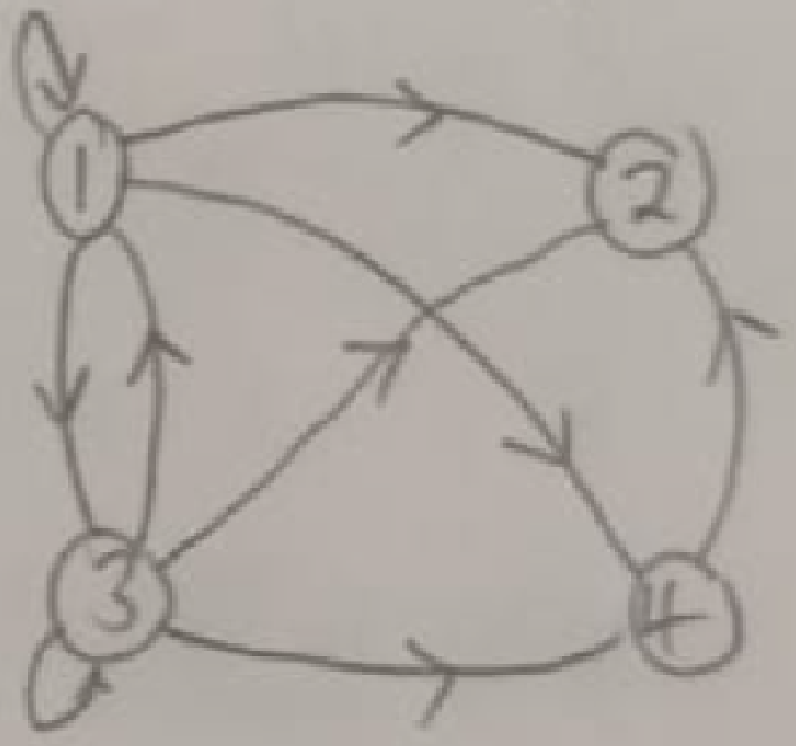
R is not irreflexive since $(1,1) \in R$ ✓

R is not symmetric since $M_R \neq (M_R)^T$ ✓ / $(4,2) \in R$ but $(2,4) \notin R$

R is not asymmetric since $(1,1) \in R$ / $(1,3)$ and $(3,1) \in R$ ✓

R is not antisymmetric since $(1,3)$ and $(3,1) \in R$, $1 \neq 3$ ✓

R is transitive since $(1,3)$, $(3,4)$ and $(1,4) \in R$ ✓




Since there is no elements $a, b, c$ such that $(a,b), (b,c) \in R$ but $(a,c) \notin R$ therefore R is transitive.

3 i)



3. i) $M_R = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\!\!\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$ $(M_R)^T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

- R is symmetric since $M_R = (M_R)^T$ ✓
- R is not reflexive ✓ since $(w,w) \notin R$ ✓, $(x,x) \notin R$, $(y,y) \notin R$, $(z,z) \notin R$
- R is irreflexive ✓ since there is $(w,w) \in R$, $(x,x) \in R$, $(y,y) \in R$, $(z,z) \in R$ ✓ ✓
- R is not asymmetric ✓ since $(x,y)$ and $(y,x) \in R$ ✓
- R is not antisymmetric ✓ since $(x,y)$ and $(y,x) \in R$ ✓ and $x \neq y$
- R is not transitive since $(M_R)^2_\odot \neq M_R$ / since $(y,x)$, $(x,z) \in R$ ~~…~~, but $(y,z) \notin R$.

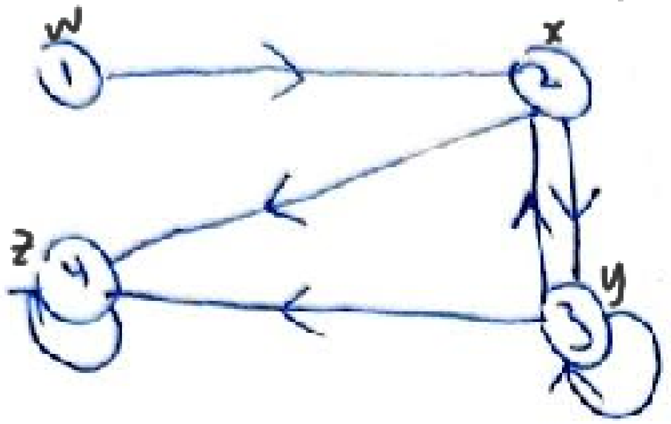
$(M_R)^2_\odot = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ $0+1+0+1 = 2$

$= \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = (M_R)^2_\odot \neq M_R$

Q3 ii)

3) ii)

$$\begin{array}{c|cccc} & w & x & y & z \\ \hline w & & & & \\ x & & \left[\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}\right] & & \\ y & & & & \\ z & & & & \end{array}$$




R is not reflexive since not all elements related to itself / since $(w,w) \notin R$.

R is not irreflexive since not all elements not related to itself / since $(y,y) \in R$

R is not symmetric. since $(w,x) \in R$ but $(x,w) \notin R$

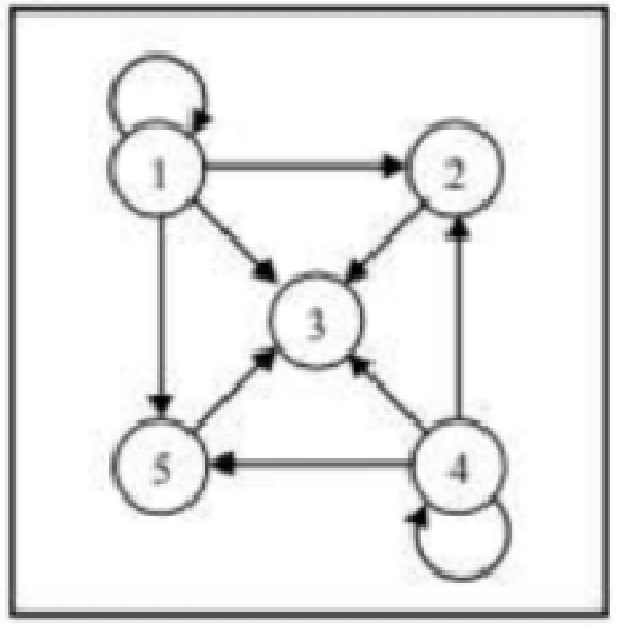
R is not asymmetric - since $(x,y)$ and $(y,x) \in R$

R is not antisymmetric since $(x,y)$, $(y,x) \in R$ and $x \neq y$

R is not transitive since $(w,x)$, $(x,y) \in R$ ~~( , ) ∈ R~~ but $(w,y) \notin R$

4. Let $A = \{1, 2, 3, 4, 5\}$. Determine whether the relation $R$ whose digraph is given below is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers.

i)




R is not reflixive since not all elements have loop/ since $(2,2) \notin$ R

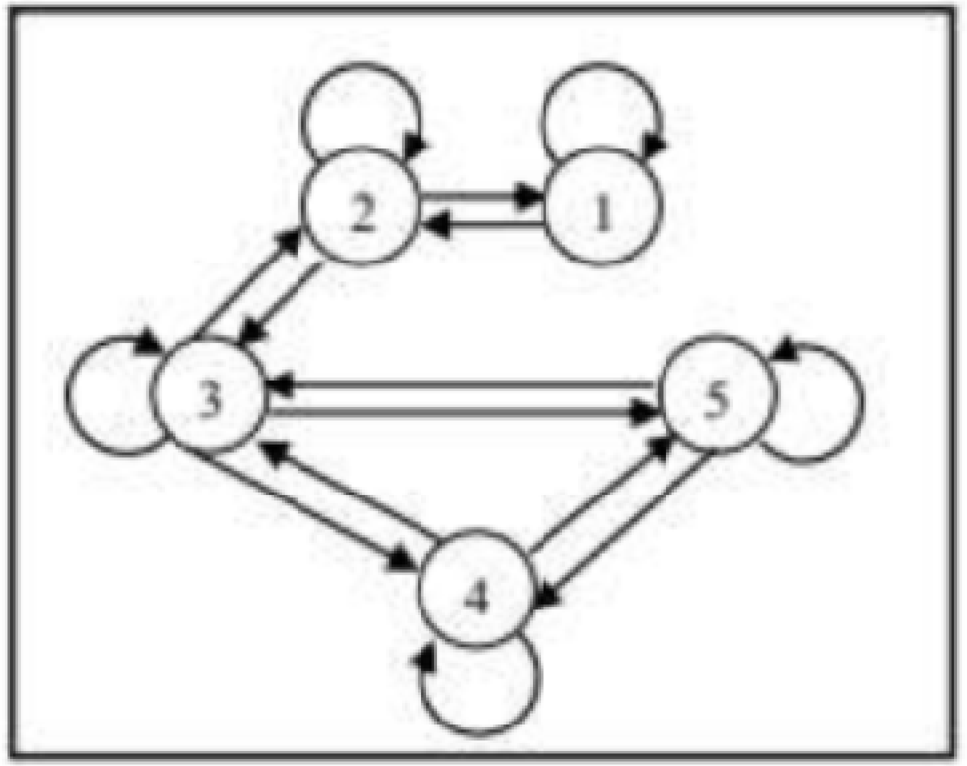
R is not irreflexive since not all elements do not have loop/
since $(1,1) \in$ R

R is not symmetric since there are no two way street

R is not asymmetric since there exists element that has loop /
since $(1,1) \in R$

R is antisymmetric since all edges are one way street

R is transitive since (1,2) , (2,3) and (1,3) E R




4. ii. $R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4),$
$(3,5), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5)\}$

R is reflexive, $(2,2) \in R$

R not irreflexive, $(1,1) \in R$

R is symmetric, $(2,3) \in R$, ~~but~~ and $(3,2) \in R$

R not asymmetric, $(1,2)$ and $(2,1) \in R$ ~~$(4,4), (3,3), (2,2) \in R$~~

R not antisymmetric, $(1,2), (2,1) \in R$, but $2 \neq 1$

R not transitive, $(1,2), (2,3) \in R$, but ~~$(1,5)$~~ $(1,3)$ $\notin R$

① reflexive ② symmetric ③ transitive

(5) Let $A = \{a, b, c\}$. Determine whether the relation R whose matrix $M_R$ is given is an equivalence relation. If yes, find $A|R$.

i) $\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 1 \\ c & 0 & 1 & 1 \end{array} = M_R$ R is reflexive since matrix of R have all 1's in the main diagonal

$(M_R)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M_R$ R is symmetric since $M_R = (M_R)^T$

$(M_R)^2_{\odot} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M_R$ R is transitive since $M_R = (M_R)^2_{\odot}$

∴ R is equivalence relation.

$R = \{(a,a), (b,b), (b,c), (c,b), (c,c)\}$

$A|R = \{\underbrace{\{a\}}_{A_1}, \underbrace{\{b,c\}}_{A_2}\}$

5ii) R={ (a,a), (a,c), (b,b), (b,c), (c,a), (c,c) }

∴ relation R is not a equivalence relation because it is reflexive but not symmetric and transitive

R is reflexive: (a,a), (b,b), (c,c)
R is not symmetric since (b,c) ∈ R but (c,b) is not belongs to R
R is not transitive: since (b,c) and (c,a) belong to R but (a,b) does not belong to R.

1) $A_1 \cap A_2 = \emptyset$
2) $A_1 \cup A_2 = A$

(b,c), (c,b), (b,b) ∈ R

(c,b), (b,c), (c,c) ∈ R

transitive:
(a,b) (b,c) and (a,c) ∈ R

c. (i) $R = \{ (1,1), (1,2), (1,5), (2,2), (2,1), (2,5),$
$(5,2), (5,1), (5,5), (3,3), (3,4), (3,6),$
$(4,4), (4,3), (4,6), (6,3), (6,6), (6,4) \}$

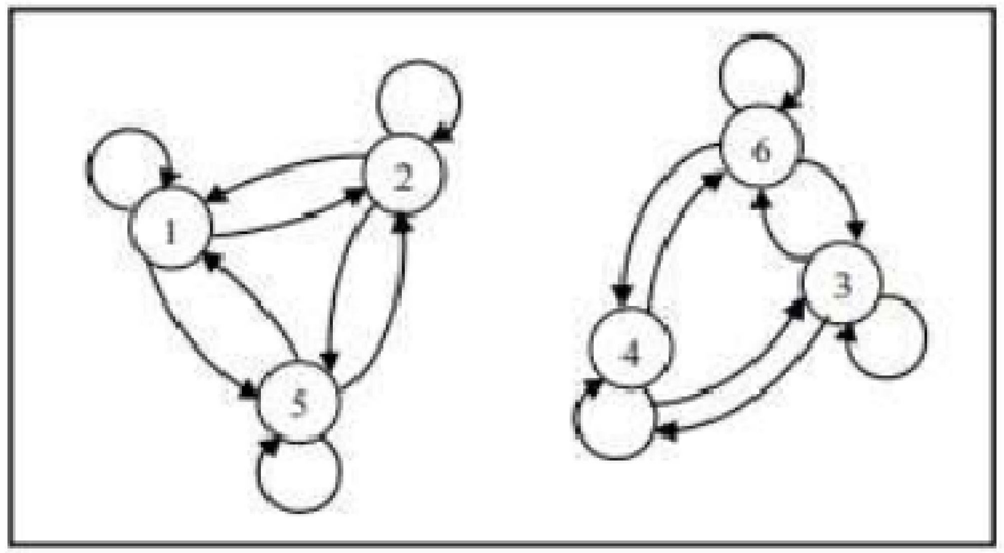
$\therefore$ R is an equivalence relation

R is reflective as $(1,1), (2,2), (3,3), (4,4), (5,5), (6,6) \in R$

R is symmetric as $(1,2)$ and $(2,1) \in R$

R is transitive as $(1,2), (2,5), (5,1) \in R$

i)




$(1,2), (2,5), (1,5)$

$$A|R = \{\overset{A_1}{\{1,2,5\}}, \overset{A_2}{\{3,4,6\}}\}$$

1) $A_1 \cap A_2 = \phi$

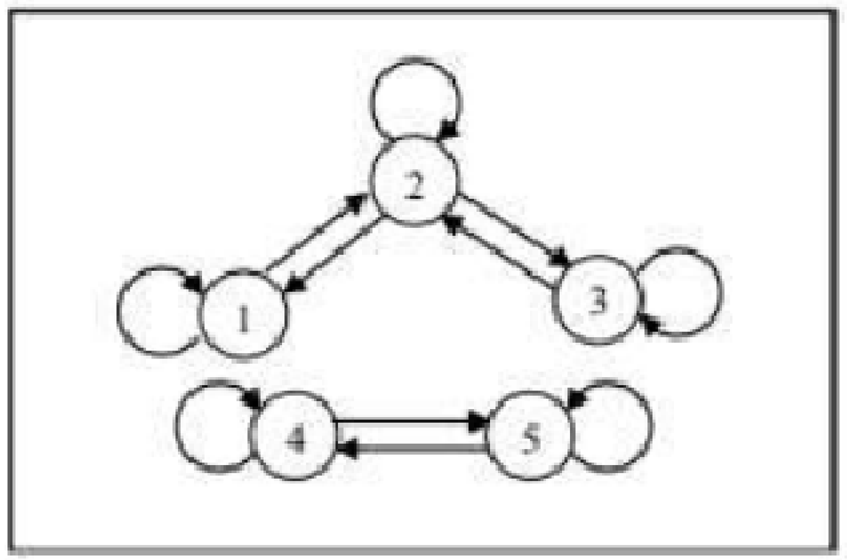
2) $A_1 \cup A_1 = A$

Q6(ii) $R = \{(1,1),(1,2),(2,2),(2,1),(2,3),(3,3),(3,2),$
$(4,4),(4,5),(5,5),(5,4)\}$

$$M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} = (M_R)^T$$

∴ R is not an equivalence relation because it is reflexive and symmetric but not Transition. ✓

ii)




R is Reflexive = (1,1), (2,2) ... ∈ R
R is Symmetric = (1,2), (2,1) ... ∈ R
R is not Transition since (1,2), (2,3) ∈ R but (1, 3) ∉ R

7. Determine whether the following relation $R$ on the set $A$ is an equivalence relation. If yes, find $A/R$.

i) $A = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$

ii) $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)\}$

$A = \{a, b, c, d\}$, $R = \{(a,a), (a,b), (b,a), (b,b), (c,c), (c,d), (d,c), (d,d)\}$

$R = \{(a,a), (b,b), (c,c), (d,d)\}, (a,b), (b,a), (c,d), (d,c)\}$

R is not an ~~sym~~ equivalence Relation.

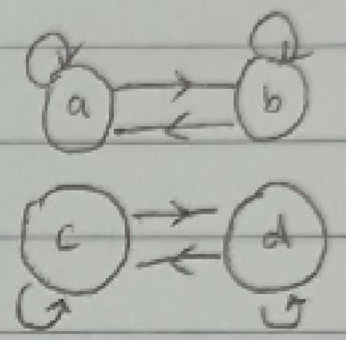
R is Reflexive ✓ $\{(a,a), (b,b), (c,c), (d,d)\} \in R$

R is Symmetric ✓ $\{(a,b), (b,a), (c,d), (d,c)\} \in R$

R is Transitive : ~~{ }~~

~~Since R is not a transitive~~, it is an

equivalence relation.



$(a,b), (b,a), (a,a) \in R$

$(b,a), (a,b), (b,b) \in R$

$(c,d), (d,c), (c,c) \in R$

$(d,c), (c,d), (d,d) \in R$

$A/R = \{\{a,b\}, \{c,d\}\}$

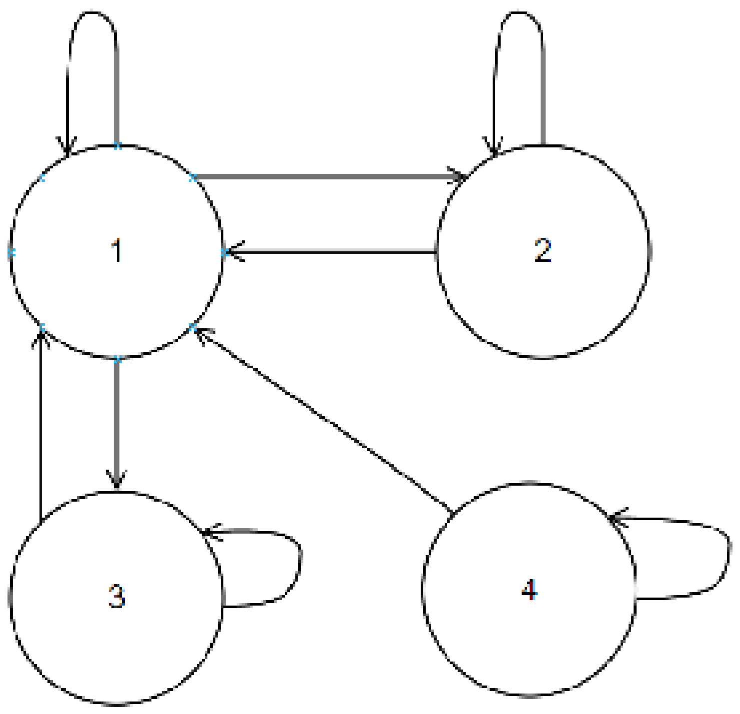
Q7 ii)

A= {1,2,3,4}

R={(1,1),(1,2),(2,1),(2,2),(3,1),(3,3),(1,3),(4,1),(4,4)}

R={(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(1,3),(3,1),(4,1)}

R is not an equivalence relation. Because R is reflexive and not transitive, ~~but~~ not symmetric.



R is Reflexive ✓ : a<->a : (1,1),(2,2),(3,3),(4,4) $\in R$

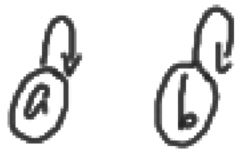
R is not Transitive : a R b, b R c,a R c : (~~1,2),(2,1),(1,1~~) $(2,1),(1,3) \in R$ but $(2,3) \notin R$

R is not Symmetric : a R b , b R a : (1,2),(2,1), $\in R$

(1,3),(3,1), $\in R$

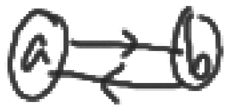
✓ Missing (1,4) $\notin R$,(4,1) $\in R$

① reflexive : loop

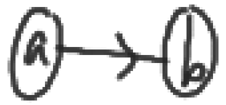



② irreflexive : no loop


③ symmetric : 2-way

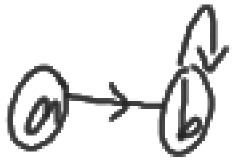



④ asymmetric : 1-way
no loop



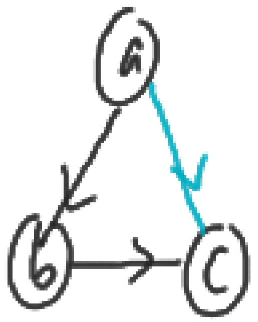

⑤ antisymmetric : 1-way
can have loop


or

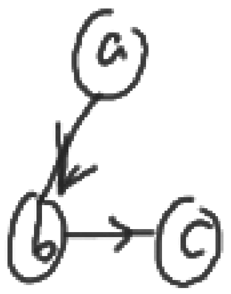



⑥ transitive : (a, b), (b, c),

(a, c)




✓ transitive




X transitive