Tutorial 11

$(a,b) = (2b,b)$

① reflexive ② antisym. ③ transitive

1. Determine whether the relation R is a partial order on the set A.

i) $A = \mathbb{Z}$, and $a\,R\,b$ if and only if $\boxed{a = 2b.}$ *

ii) $A = \mathbb{R}$, and $a\,R\,b$ if and only if $a \le b$.

iii) $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

i) $A = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$R = \{\ldots, (-4,-2), (-2,-1), (0,0), (2,1), (4,2), \ldots\}$

① R is not reflexive

$\therefore$ R is not a partial order on set A.

Refer to next slide

1) (i) $A = \mathbb{Z}$, aRb if and only if $a = 2b$

$R = \{(2,1), (4,2), (6,3), \dots\}$ ✓

- Reflexive (x) ✓
- Antisymmetric (v) ✓
- Transitive (x) ✓

$$\frac{(4,2)\quad(2,1)}{R \notin (4,1)}$$

$1 \to 2$

∴ R is not a partial order ✓

real numbers

(ii) $A = \mathbb{R}$, aRb if and only if $a \le b$

(1,2), (2,3)
$1 \le 2$ ✓, $2 \le 3$
$1 \le 3$ ✓

$R = \{(1,1), (1,2), \dots (2,2), (2,3), \dots\}$

- reflexive (v) ✓
- Antisymmetric (v) ✓
- Transitive (v) ✓

$$\frac{(1,2)\ ✓\quad(2,3)\ ✓}{R \in (1,3)\ ✓}$$

∴ R is a partial order ✓

(iii)

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \end{array}$$

$R = \{(1,1), (1,3), (2,2), (2,3), (3,3), (3,4), (4,1), (4,2), (4,4)\}$

- Reflexive (v) ✓
- Antisymmetric (v) ✓
- Transitive (x) ✓

$$\frac{(4,1)\quad(1,3)}{R \notin (4,3)}$$

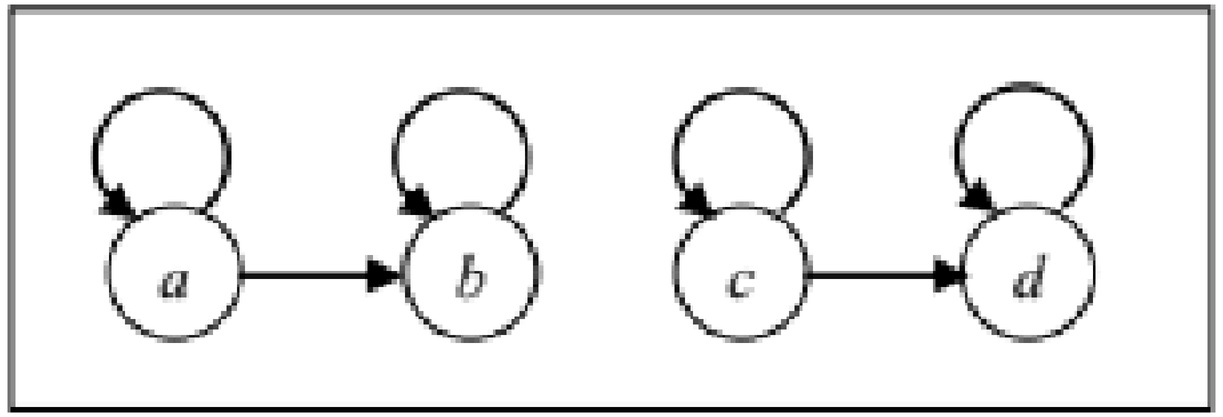
∴ R is not a partial order ✓

Q1

iv) $$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M_{ij} = 1$
$M_{ji} = 0$

$M_{12} = 1$
$M_{21} = 0$

v) 

(a,a) (a,b) (c,c), (c,d)

iv. $$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), (3,3), (3,5), (4,4), (5,5)\}$

- Reflexive (✓) ✓
- Ant. sym (✓) ✓
- transitive (✓) ✓

$(1,2)$
$(2,4)$
$R \in (1,4)$

R is partial order ✓

v. $$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R = \{(a,a), (a,b), (b,b), (c,c), (c,d), (d,d)\}$

- Reflexive (✓) ✓
- Anti sym (✓) ✓
- transitive (✓)

R ~~not~~ is a partial order on set A

2. Find the lexicographic ordering of the following strings of lowercase English letters:

i) quack, quick, quicksilver, quicksand, quacking

ii) zoo, zero, zoom, zoology, zoological

i) quack, quick, quicksilver, quicksand, quacking

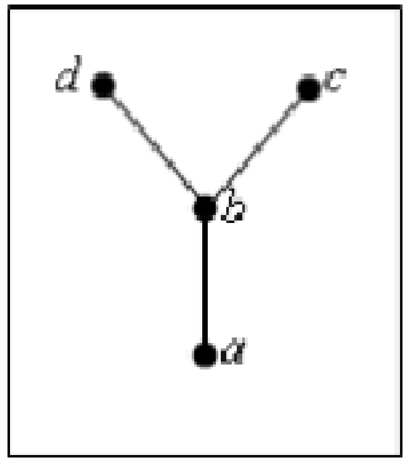
quack $\prec$ quacking $\prec$ quick $\prec$ quicksand $\prec$ quicksilver ✓

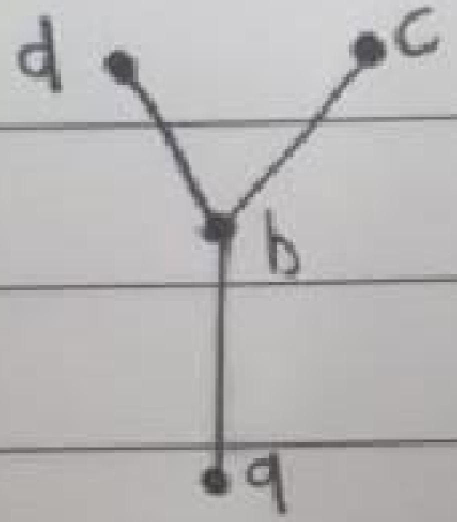
ii) zoo, zero, zoom, zoology, zoological

zero $\prec$ zoo $\prec$ zoological $\prec$ zoology $\prec$ zoom ✓

ref. ✓ antisym. transitive

3. List all ordered pairs in the partial order whose Hasse diagram is shown as below.





R = {(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (d, d)}

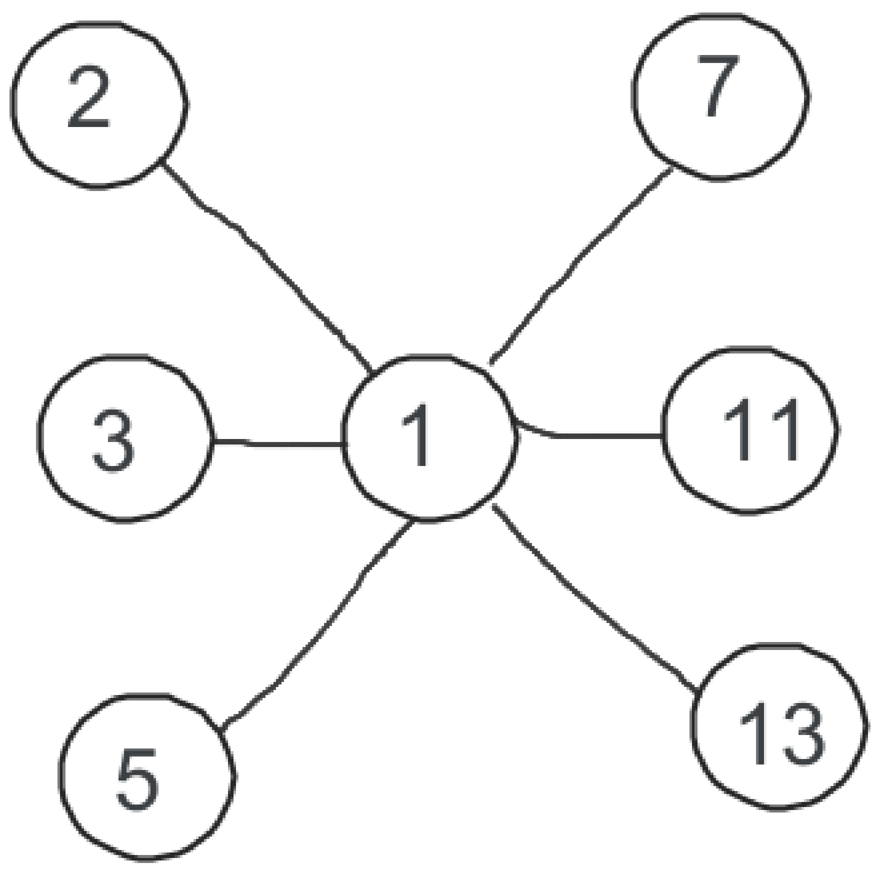
4. Draw the Hasse diagram for each of the following posets.

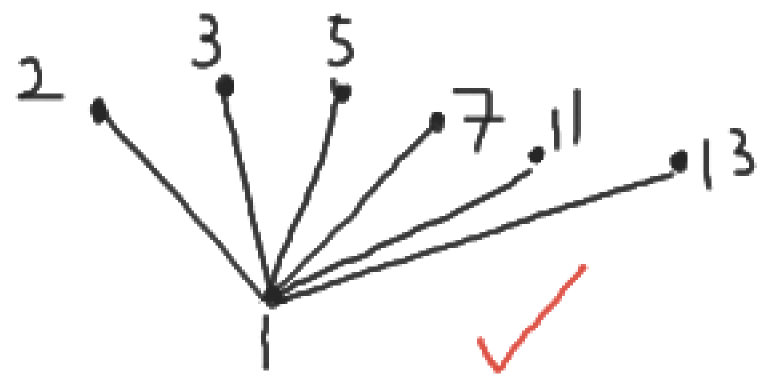
i) a is a divisor of b on the set $\{1, 2, 3, 5, 7, 11, 13\}$.

ii) X is a subset of Y on the set of all subsets of $\{1, 2, 3\}$.

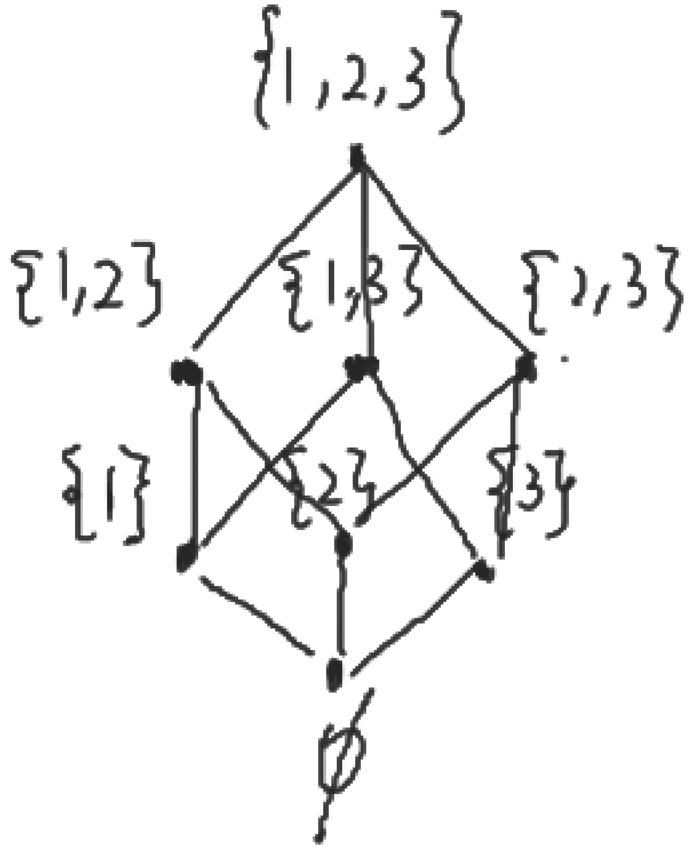
$\cancel{(1,1)}, \cancel{(2,2)}, \cancel{(3,3)}, \cancel{(5,5)}, \cancel{(7,7)}, \cancel{(11,11)}, \cancel{(13,13)}$

i) $R = \{(1,2),(1,3),(1,5),(1,7),(1,11),(1,13)\}$





ii) subsets$=\{\emptyset, \{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$



4. Draw the Hasse diagram for each of the following posets.

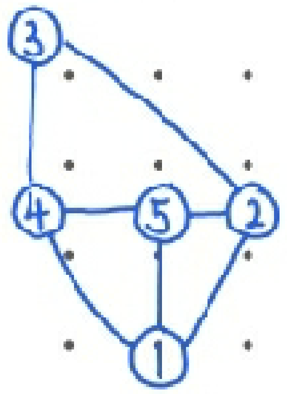
iii) $A = \{1, 2, 3, 4, 5\}$,

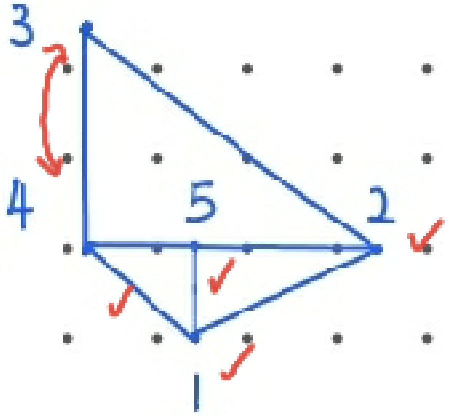
$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \left[\cancel{1}\right. & 1 & 1 & 1 & \left.1\right] \\ 2 & 0 & \cancel{1} & 1 & 1 & 1 \\ 3 & 0 & 0 & \cancel{1} & 1 & 1 \\ 4 & 0 & 0 & 0 & \cancel{1} & 1 \\ 5 & \left[0\right. & 0 & 0 & 0 & \left.\cancel{1}\right] \end{array}$$

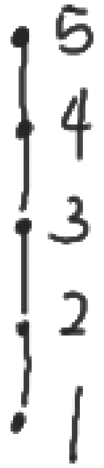
(2,3)(3,4)
(1,2) (2,3)

A = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (4,5), (5,5)}

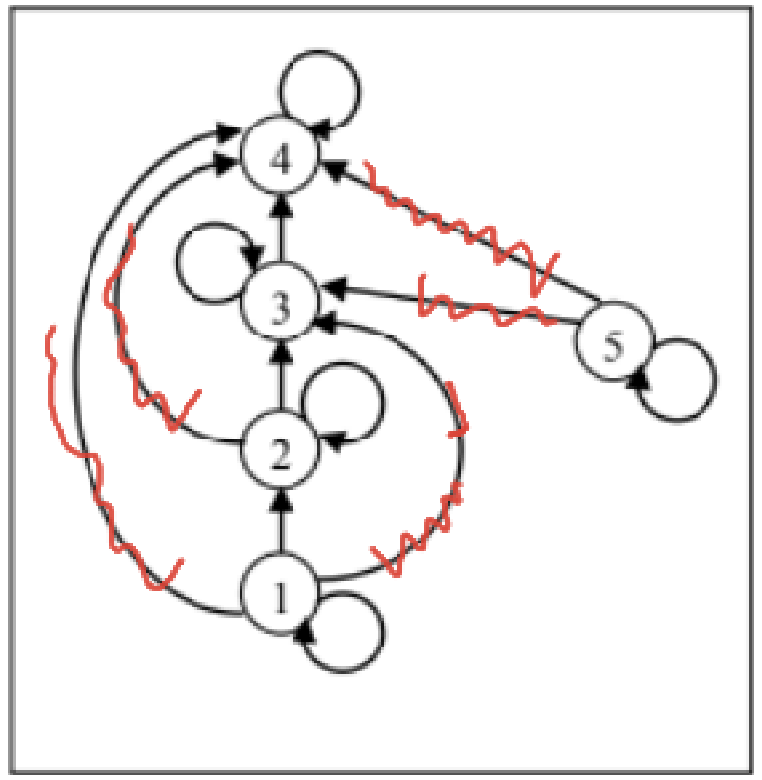
transitive
(1,2) (2,4)
(1,2) (2,5)
(2,3) (3,5)



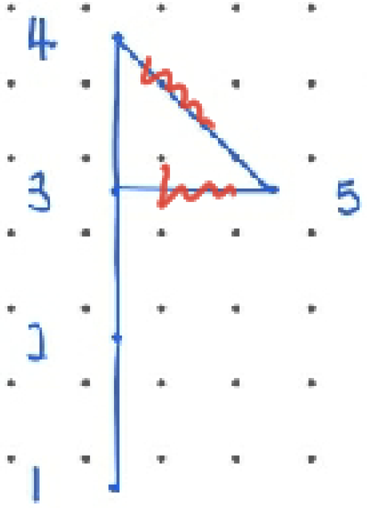


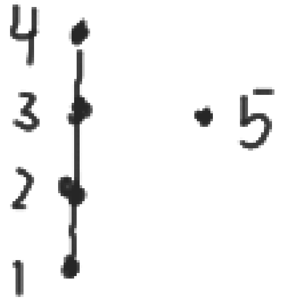


iv)



(1,2), (2,3) (3,4)
(2,3) (3,4)
(1,2) (2,3)



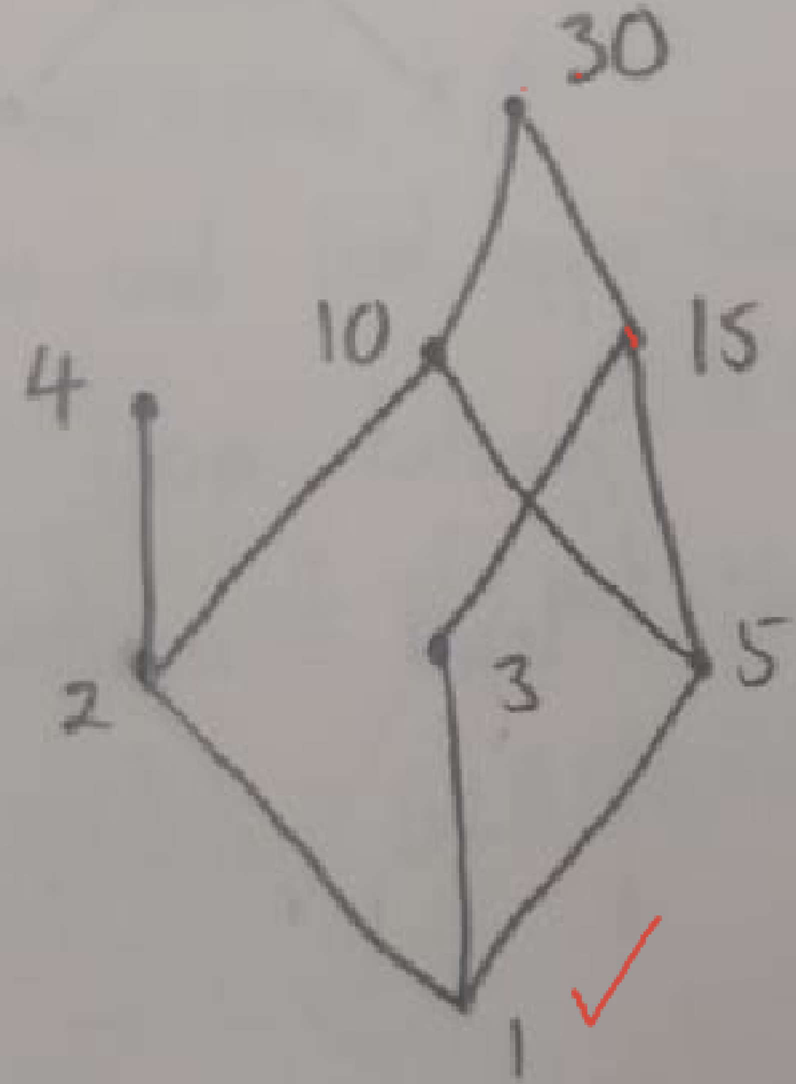


(1,2) (2,3)
(5,5) (5,4)
(5,5) (5,3)

⑤ Consider the partial order of divisibility on the set A. Draw the Hasse diagram of the poset and determine which posets are linearly ordered.

1) $A = \{1, 2, 3, 4, 5, 10, 15, 30\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,10), (1,15), (1,30), (2,2), (2,4), (2,10), (2,30), (3,3), (3,15), (3,30), (4,4), (5,5), (5,10), (5,15), (5,30), (10,10), (10,30), (15,15), (15,30), (30,30)\}$



$(1,2)(2,4)$

$\therefore$ Not linearly ordered.

i-, ii)

$A = \{3, 6, 12, 36, 72\}$

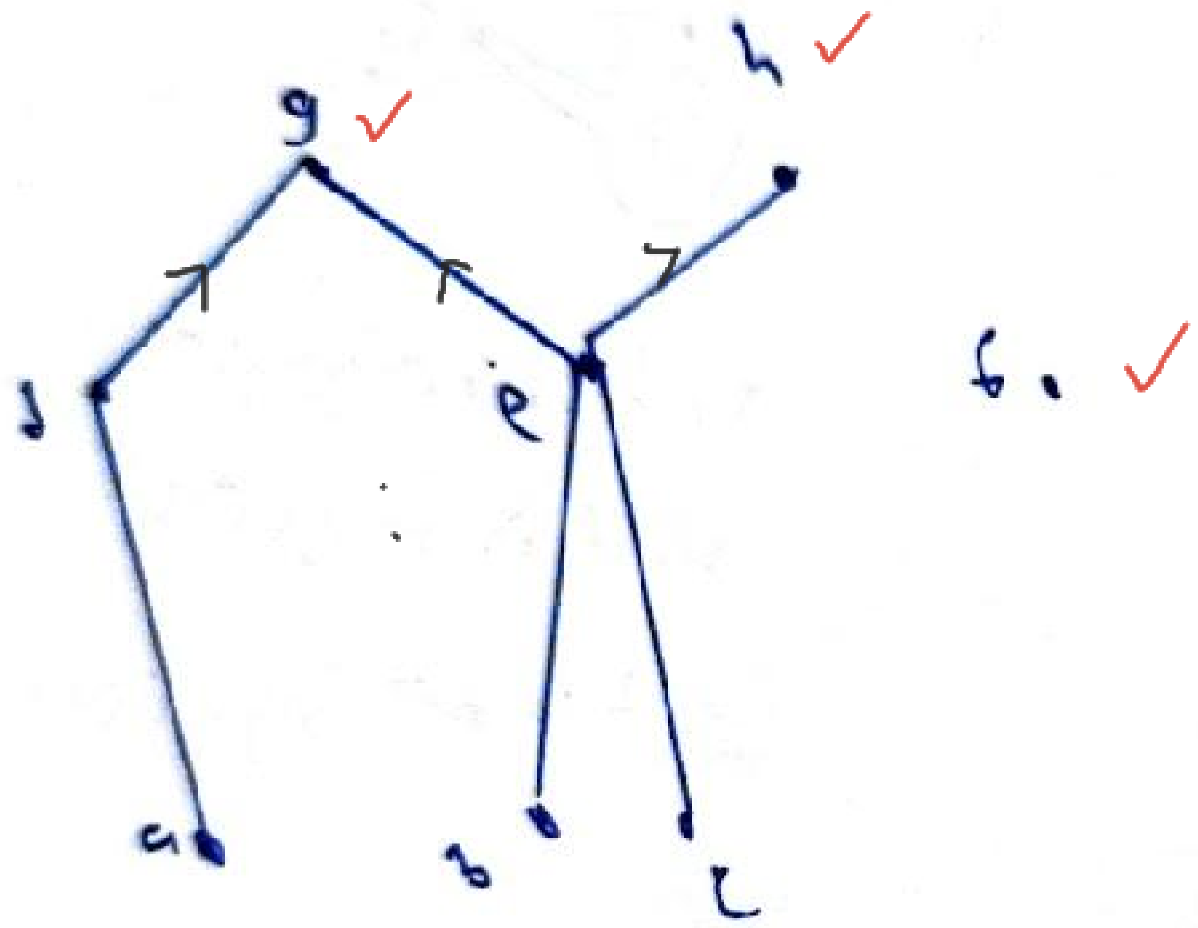
$R = \{(3,3), (3,6), (3,12), (3,36), (3,72), (6,6), (6,12), (6,36), (6,72),$
$(12,12), (12,36), (12,72), (36,36), (36,72), (72,72)\}$

$(3, 6)(6, 12)$

$(3, 6)\ (6, 36)$

linearly order

2b)



(a,d) (d,g)

(c,e) (e,g)

(c,e) (e,h)

R = {~~(a,a) b, c, d, e, h, g~~} = {(a,a), (a,d), (a,g), (b,b), (b,e), (b,g), (b,h), (c,c), (c,e), (c,g), (c,h), (d,g), (d,d), (e,e), (e,g), (e,h), (f,f), (g,g), (h,h)}

maximal elements = {g, h, f}

minimal elements = {a, b, c, f}

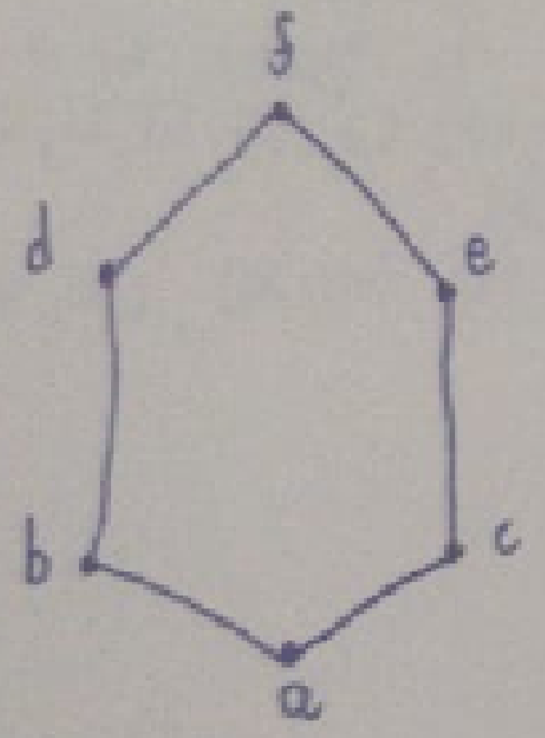
max = $\{g, h, f\}$

min = $\{a, b, c\}$

only one answer

(7) Determine the greatest and least elements, if exist, of the following posets.

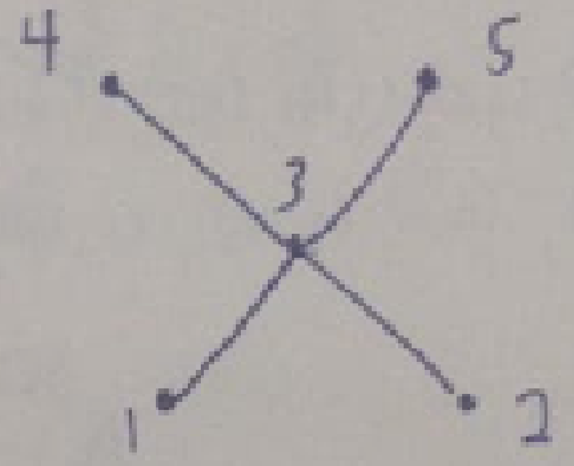
i)



greatest = f

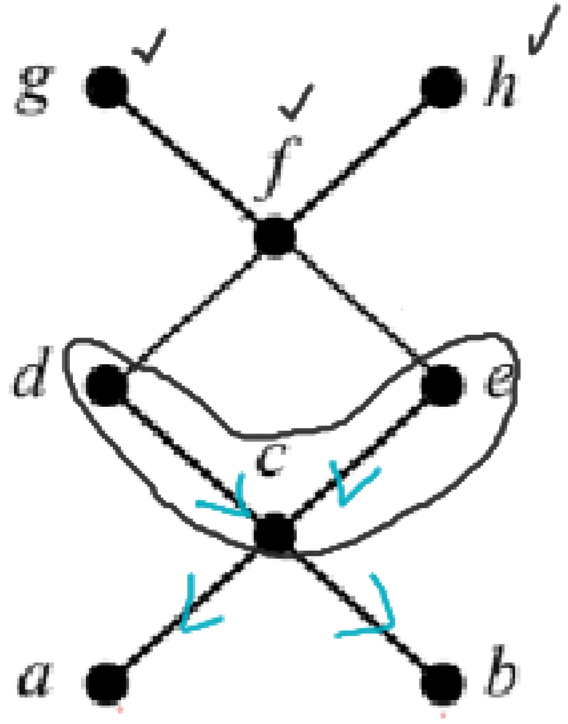
least = a

ii)



greatest = none // ϕ

least = none // ϕ



$B = \{c, d, e\}$

(c,c), (c,d), (c,e) ....

Q8 (a) Maximal : $\{g, h\}$

Minimal : $\{a, b\}$

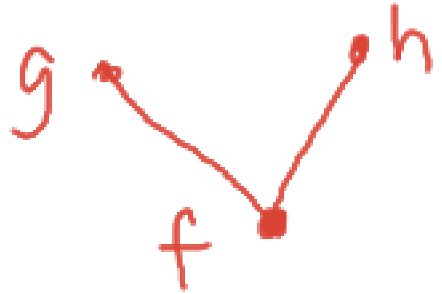
upper bound of B : $\{f, g, h\}$

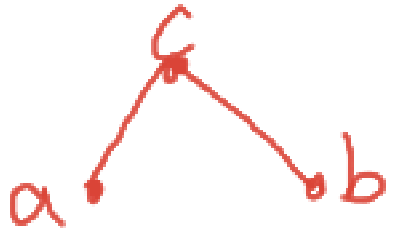
lower bound of B : $\{a, b, c\}$

Least upper bound of B : f

~~le lower up~~

greatest lower bound of B : c





b)

$B = \{b, c, d\}$

i) $\max = \{f\}$

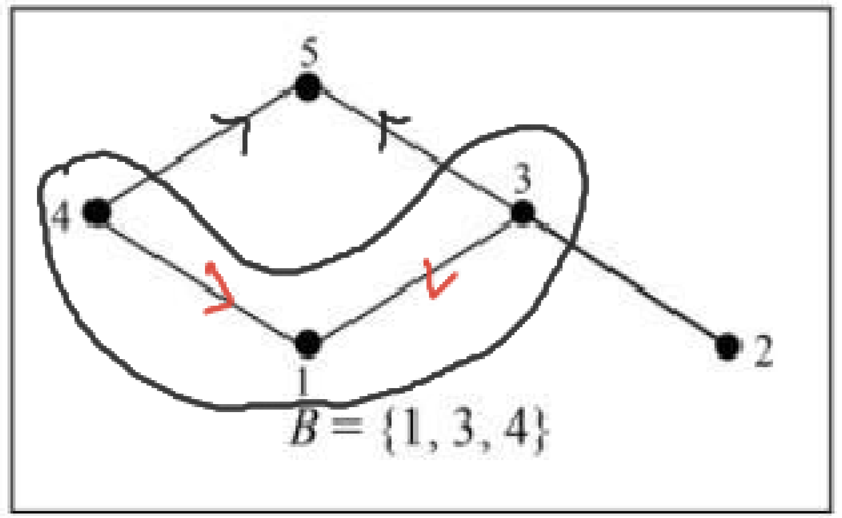
$\min = \{a\}$

ii) $UB(B) = \{d, e, f\}$

iii) $LB(B) = \{a, b\}$

iv) $LUB(B) = d$

v) $GLB(B) = b$



(1,1), (1,3), (1,4), ...

8)c) maximal elements: {5}
minimum elements: {1,2}
Upper Bounds of B: {5}
Lower Bounds of B: {1}
Least Upper Bound of B: {5}
Greatest Lower Bound of B: {1}

(3, 9) (9, 45)   (5, 15) (15, 45)

Let A= {3,5,9,15,24,45}

(A, |):

R ={(3,3),(3,9),(3,15),(3,24),(3,45),(5,5),(5,15),(5,45),(9,9),(9,45),
(15,15),(15,45),(24,24),(45,45)}

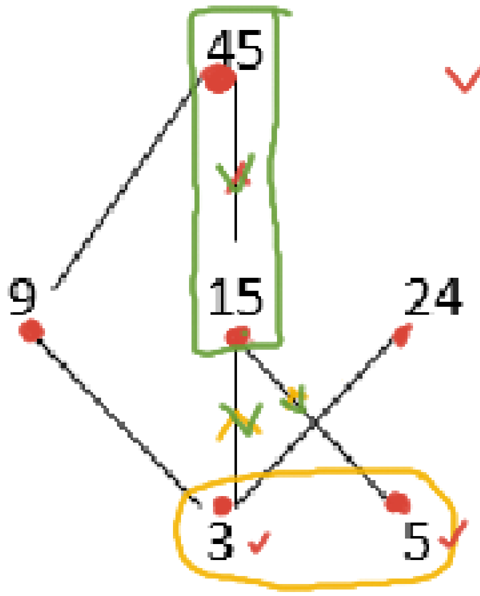
(3,9),(3,15),(3,24),(5,15),(9,45),(15,45)

9. Answer the following questions concerning the poset ({3, 5, 9, 15, 24, 45}, |).

i) Find the maximal and minimal elements.
ii) Determine the greatest element and least element, if exist.
iii) Find all upper bounds and least upper bounds of {3, 5}, if exist.
iv) Find all lower bounds of {15, 45}. Hence determine the greatest lower bound of {15, 45}, if exist.

Maximal elements: {24,45}

Minimal elements: {3,5}



Greatest element: none

Least element: none

Upper bounds of {3,5} = 15, 45

LUB of {3,5} = 15

(15, 15) (15, 45)   (5, 45) (3, 45)
(5, 15) (3, 15)

Lower bounds of {15,45} = 3, 5, 15

GLB of {15,45} = 15

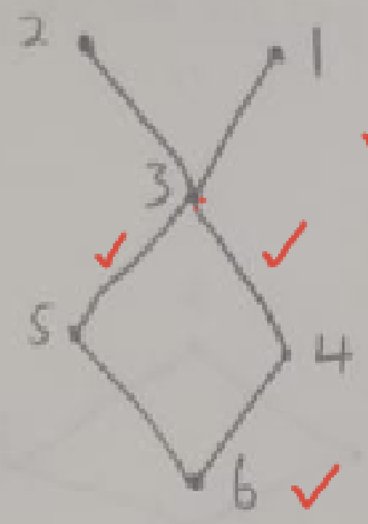
Q10

(10) Let $A = \{1,2,3,4,5,6\}$ and consider the partial order R on A as ~~$R = \{(6,6),(5,5),(6,4),(6,3),(6,2)$~~

$R = \{(6,6),(6,5),(6,4),(6,3),(6,2),(6,1),(5,5),(5,3),(5,2),(5,1),(4,4),(4,3),(4,2),(4,1),(3,3),(3,2),(3,1),(2,2),(1,1)\}$

2 is the upper B of 5
2 is " " " " 2

i) Draw a Hasse diagram of the poset $[A,R]$.



ii) Find the minimal and maximal elements of the poset $[A,R]$

minimal elements = $\{6\}$
maximal elements = $\{2,1\}$

iii) Find the least upper bound of $\{2,5\}$, if it exists.

$UB(\{2,5\}) =$ ~~none~~ 2 $\quad LUB(\{2,5\}) =$ ~~none~~ 2

iv) Find the greatest lower bound of $\{3,4\}$, if exist.

$LB(\{3,4\}) = \{6\}$ $\quad GLB(\{3,4\}) = 6$