Tutorial 6

$\frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

1. Determine each of the following real numbers is rational or irrational.

   i) $3.9602 = \frac{39602}{10000} = \frac{19801}{5000}$ ii) $\frac{2}{5} - \frac{5}{6}$

   iii) 0.30303030 . . . (where the digits 30 are assumed to repeat forever)

Q1) ✓

I)rational ✓can be written in fraction format

Ii)rational , -13/30

iii)rational✓number , 10/33

geometric progression

$a + ar + ar^2 + ar^3 + \ldots$

$0.30303030\ldots$

$= 0.3 + 0.003 + 0.00003 + 0.0000003 + \ldots = S_\infty$

$a = 0.3$, $r = \frac{0.003}{0.3} = \frac{1}{100}$

$= \frac{a}{1-r}$

$= \frac{0.3}{1 - \frac{1}{100}} = \frac{0.3}{\frac{99}{100}} = \frac{10}{33}$ #

2. Justify your answer to each of the following questions, for the integers m and n.

i) Is $2m(2m + 2)$ divisible by 4?

ii) If $m = (4n + 3)$ does 8 divide $m^2 - 1$?

iii) Does 24|6?

iv) Is $(3m + 1)(3m + 2)(3m + 3)$ divisible by 3?

i) yes, (4m^2 +4m) /4 = m^2 +1m $\Rightarrow (m^2+m) \in \mathbb{Z}$

ii) yes, $m^2-1 = (4n+3)^2-1 = 16n^2+24n+9-1 = 16n^2+24n+8 = 8(2n^2+3n+1)$

$m^2-1 = \underline{\underline{8k}}$, Let $k = 2n^2+3n+1$, $k \in \mathbb{Z}$

iii) no $24|6 = \frac{6}{24} = \frac{1}{4} \notin \mathbb{Z}$

iv) ~~no~~ yes.

$(3m+1)(3m+2)(3m+3) = 3\underbrace{(m+1)(3m+1)(3m+2)}$

$= \underline{\underline{3k}}$, Let $k = (m+1)(3m+1)(3m+2)$

$k \in \mathbb{Z}$

3. Find integers q and r such that $n = dq + r$, $0 \leq r < d$.

i) $n = 100$, $d = 10$ ii) $n = 9$, $d = 10$ iii) $n = -57$, $d = 5$

③	(i) $n=100$, $d=10$	(ii) $n=9$, $d=10$	(iii) $n=-57$, $d=5$
	$100 = 10(10)+0$ ✓ ✓	$9 = 10(0)+9$ ✓	$-57 = 5(-12)+3$ ✓
	$q=10$ ✓, $r=0$ ✓	$q=0$ ✓, $r=9$ ✓	$q=-12$ ✓, $r=3$ ✓

4. Evaluate

i) 59 div 3, 59 mod 3

$$
\begin{array}{r}
19 \rightarrow q \\
3\overline{)59} \\
\underline{3} \\
29 \\
\underline{27} \\
2 \rightarrow r
\end{array}
$$

$n = d \cdot q + r$

$59 = 3(19) + 2$

$\therefore$ 59 div 3 = 19

$\therefore$ 59 mod 3 = 2

ii) 45 div 6, 45 mod 6

$$
\begin{array}{r}
7 \\
6\overline{)45} \\
\underline{42} \\
3
\end{array}
$$

$n = d \cdot q + r$

$45 = 6(7) + 3$

$\therefore$ 45 div 6 = 7

$\therefore$ 45 mod 6 = 3

6. Use the floor notation to express
i) 589 *div* 12 and 589 *mod* 12.
ii) 123 *div* 5 and 123 *mod* 5.

i)

$589 = 12(49) + 1$

$\therefore$ 589 div 12 = 49 (q) ✓

589 mod 12 = 1 (r) ✓

$\left\lfloor \frac{589}{12} \right\rfloor = \left\lfloor 49\frac{1}{12} \right\rfloor = 49$

ii)

$123 = 5(24) + 3$ ✓

$\therefore$ 123 div 5 = 24 (q) ✓

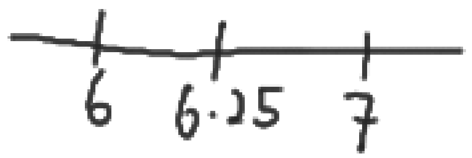
123 mod 5 = 3 (r) ✓

$\left\lfloor \frac{123}{5} \right\rfloor = \left\lfloor 24\frac{3}{5} \right\rfloor = 24$

5. What are the floor and ceiling for the following values?
i) 8 ii) 25/4 iii) 19.67 iv) -76/12

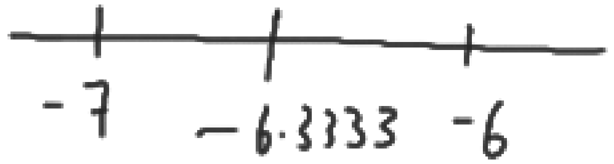
(i) Floor: 8; Ceiling: 8

(ii)
25/4 = 6.25
Floor: 6; Ceiling: 7



(iii) Floor: 19; Ceiling: 20

(iv)
-76/12 = -6.3333
Floor: -7; Ceiling: -6



$20 = 4\times5 = 2^2\times5$, $18 = 2\times \underline{9}^{3^2}$

7. Write the following integers in standard factored form.

i) 702 ii) 112385 iii) $(20!)^2$

(iii) $(20!)^2$

$2^2\times2\times2^4\times2\times2^2\times2\times2^3\times2\times2^4\times2 = 2^{2+1+4+1+2+1+3+1+4+1}$

$20! = 20\times19\times18\times17\times16\times15\times14\times13\times12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1$

$= (2^2\times5)\times19\times(2\times3^2)\times17\times2^4\times(3\times5)\times(2\times7)\times13\times(2^2\times3)\times11\times$
$(2\times5)\times3^2\times2^3\times7\times(2\times3)\times5\times2^2\times3\times2$

$= 2^{18}\times3^8\times5^4\times7^2\times11\times13\times17\times19$

$(20!)^2 = (2^{18}\times3^8\times5^4\times7^2\times11\times13\times17\times19)^2$

$= 19^2\times17^2\times13^2\times11^2\times7^4\times5^8\times3^{16}\times2^{36}$ #

Q7. i. $702 = 2\cdot3^3\cdot13$

$$\begin{array}{r|l} 2 & 702 \\ 3 & 351 \\ 3 & 117 \\ 3 & 39 \\ & 13 \end{array}$$

ii. $112385 = 5\cdot7\cdot13^2\cdot19$

$$\begin{array}{r|l} 5 & 112385 \\ 7 & 22477 \\ 13 & 3211 \\ 13 & 247 \\ & 19 \end{array}$$

iii. $(20!)^2 = [(2^2\cdot5)!]^2$

$$\begin{array}{r|l} 2 & 20 \\ 2 & 10 \\ & 5 \end{array}$$

8. Prove the following statements by the method of direct proof.
   i) If n is any odd integers, then $(-1)^n = -1$.
   ii) If r is any rational number, then $2r^2 - r + 1$ is rational.
   iii) For integers a, b and c. If $a|b$, then $a|bc$.

$$\boxed{a^{m+n} = a^m \times a^n}$$

Question 8

(i) Proof:

Suppose n is a particular but arbitarily chosen odd integer.

by the defination of odd integer, $n = 2k+1$, $k \in \mathbb{Z}$

LHS: $(-1)^n$

$= (-1)^{2k+1} = (-1)^{2k} \cdot (-1)^1 = (1)(-1) = -1$ (RHS)

Let k be any number, 1, 2, 3, 4, ...

$= (-1)^{2(4)+1}$

$= (-1)^9$

$= -1$ (proven)

Since the sum of products of integers is an integer, $2k+1$ is an integer and hence by defination of odd integer, $(-1)^{2k+1}$ still will be (-1)

Therefore, if n is any odd integer, then $(-1)^n = -1$

ii proof

Suppose r ~~are~~ is particular but arbitrarily chosen rational numbers

By definition of rational numbers

$r = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

Then $2r^2 - r + 1 = 2(\frac{a}{b})^2 - \frac{a}{b} + 1$ ~~= 2~~ $= 2(\frac{a^2}{b^2}) - \frac{a}{b} + 1$ $= \frac{2a^2 - ab + b^2}{b^2}$, $a, b \in \mathbb{Z}$, ~~$a^2 \in \mathbb{Z}$, $b^2 \in \mathbb{Z}$~~ $b^2 \neq 0$

Since the sum of product of integers is an integers, ~~a, b, a^2~~ $(2a^2 - ab + b^2)$ and b^2 are integers and b, $b^2 \neq 0$. Hence by the definition of rational number, r is rational. Therefore $2r^2 - r + 1$ is rational.

Proof:

Suppose: $\frac{a}{b}$ is true, $a \neq 0$

Since a is not 0, b and c are integers. Therefore, $a|bc$ is true.

→ next slide

Q8 (iii)

$a|b = \frac{b}{a} = k \,, \quad k \in \mathbb{Z}$

$b = ka$

Proof:
Suppose a, b and c are particular but arbitrarily chosen integers such that a divides b.
By definition of divisibility, b=ka, k$\in$ Z.
Then, a | bc $= \frac{bc}{a}$

$= \frac{k\cancel{a}c}{\cancel{a}}$

$= kc \in \mathbb{Z}$

Since the product of integers is an integer, kc is an integer and hence by definition of divisibility, a divides bc.
Therefore for all integers a, b and c. If a | b, then a | bc.

using counter example

9. Disprove the following statements.
   i) For all integers n, if n is prime then $(-1)^n = -1$. = not prime, e.g. 4, 6, 8, 100, .
   ii) For all integers m, if $m > 2$, then $m^2 - 4$ is composite.
   iii) If m and n are positive integers and mn is a perfect square, then m and n are perfect square.

9. i) n is prime LHS: $(-1)^n = -1$

Counterexample: 2 is prime (valid) if n = 2

$= (-1)^2 = -1$

$= 1 \neq -1$ (invalid) RHS

$\therefore$ n = 2 is prime but $(-1)^n \neq -1$

9ii) ① m > 2

② $m^2 - 4$ is composite

Counterexample:

if m = 3

$m^2 - 4$

① 3 > 2 (valid)

② $= 3^2 - 4$ is composite

= 5 is composite (invalid)

m > 2 but $m^2 - 4$ is not composite

(iii) m, n is positive integers

Counter example:

if m = 2, n = 8   m = 2, n = 8 is not perfect square

mn = 2(8)

= 16

(~~invalid~~)

$\therefore$ m and n ~~is~~ are positive integers but mn is ~~not~~ a perfect square

~~and~~ but m and n ~~is~~ are not perfect squares

⑩ Prove the following statements by contradiction. $p \to q \equiv (\sim p \vee q)$ $\downarrow$ negate $(p \wedge \sim q)$

i) For all integers n, if (n is even) [p], then (n^2+n+1 is odd) [q]

Negation: For some integers n, n is even, and n^2+n+1 is even ✓

Proof: ✓

Suppose not. ✓

Suppose for some integers n, n is even and n^2+n+1 is even ✓

By definition of even, $n = 2k$, $k \in \mathbb{Z}$ ✓

$$
\begin{aligned}
\text{Then } n^2+n+1 &= (2k)^2 + 2k + 1 \\
&= 4k^2 + 2k + 1 \\
&= 2(2k^2+k)+1 \\
&= 2l + 1, \quad l = 2k^2+k, \; l \in \mathbb{Z} ✓
\end{aligned}
$$

Since the sum of product of integers is an integer so $(2k^2+k)$ is an integer ~~and n is even, but~~ $n^2+n+1 = 2l+1$ is odd ~~and~~ by definition of odd, ~~$n = 2k+1$, $k \in \mathbb{Z}$~~. Thus, n^2+n+1 is an odd integer. ✓

This contradicts with the supposition and so the supposition is false and the theorem is true. ✓

10 ii)

ii) The product of any nonzero rational number and any irrational number is irrational.

Negation: The product of ~~any~~ at least one nonzero ~~i~~rational number and ~~any~~ at least one irrational number is rational.

① Proof:

② Suppose not. ③ Supose there is a rational number r and an irrational number s such that rs is rational

~~Suppose the product of any nonzero irrational number, s and any rational number, r is rational.~~

By definition of ~~irrational and~~ rational integer,

$s = a\sqrt{b}$, $a,b \in \mathbb{Z}$, $a,b \neq 0$ $r = \frac{c}{d}$, $c,d \in \mathbb{Z}$, $d \neq 0$

Then, $sr = a\sqrt{b}\left(\frac{c}{d}\right)$

$= \frac{ac\sqrt{b}}{d}$

Then $sr = \frac{a}{b}$, $a,b \in \mathbb{Z}$, $b \neq 0$

$s\left(\frac{c}{d}\right) = \frac{a}{b}$

$s = \frac{ad}{bc}$, $a,b,c,d\ \mathbb{Z}$, $bc \neq 0$

Since the product of integers is an integer, ~~as irrational number while r is rational number and sr is not rational number~~ hence, ad and bc are integers, with $bc \neq 0$. Hence by definition of rational number ~~and irrational number~~, ~~$\frac{ac\sqrt{b}}{d}$~~ s is a ~~ir~~rational number.

Therefore this contradicts with the supposition and the supposition is false and the original theorem is true.

11. Prove the following statements by contraposition.

i) For all integers a, b and c, if (a does not divide the product of b and c) p then (a does not divide b) q

Statement: $p \to q$

contrapositive: $\sim q \to \sim p$

$2|10 = \frac{10}{2} = 5 \in \mathbb{Z}$

$10 = 2(5)$

$a|b = \frac{b}{a} = k \in \mathbb{Z}$

$b = ak$

11) i)

p: a does not divide the product of b and c ✓

q: a does not divide b ✓

contrapositive: $\sim q \to \sim p$ ✓

✓ $\forall$ integers a, b and c, If a divides b, then a divides the product of b and c

By definition of divisibility,

$b = a \cdot k$ ✓, $k \in \mathbb{Z}$ ✓

Then $bc = a \cdot l$

$(a \cdot k)(c) = a \cdot l$

$c = \frac{l}{k}$, $l, k \in \mathbb{Z}$

$a|bc = \frac{bc}{a} = \frac{akc}{a} = kc$

Since the product of integers is an integer, ✓ kc is an integer and hence by definition of divisibility, a divides the product of b and c ✓

Therefore for all integers a, b and c, if a divide b, then a divide the product of b and c. ✓

11 11)

Equivalent form

If x is an irrational number, then $\sqrt{x}$ is irrational

Contrapositive

If $\sqrt{x}$ is rational, then x is rational ✓

✓ Proof : by the definition of rational, $\sqrt{x} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

Then SBS : $x = \frac{a^2}{b^2}$, $b \neq 0$

Suppose that $\sqrt{x}$ is rational, and ~~$x \neq 0$, Then there exists~~

~~integers p such that $\sqrt{x} \neq \sqrt{p}$, This would mean that $p \neq 0$, Since $p \neq 0$, then $x = \sqrt{x} = \sqrt{p}$. Hence x can be written as a quotient of two integers with a nonzero denominator, Thus x is rational~~

Since the square of integer is an integer a^2 and b^2 are integers. Hence, by definition of rational, x is rational. Therefore for all x, if $\sqrt{x}$ is rational then x is rational. The statement is true.

12. Use the Euclidean algorithm to find the greatest common divisor of the following pair of integers.

i) 27 and 72 ii) 3510 and 672

12.(i) 27 and 72

$27 = 72(0) + 27$ ✓

$72 = 27(2) + 18$ ✓

$27 = 18(1) + 9$ ✓

$18 = 9(2) + 0$ ✓

$\therefore \gcd(9, 0) = 9$ ✓

12.(ii) 3510 and 672

$3510 = 672(5) + 150$ ✓

$672 = 150(4) + 72$ ✓

$150 = 72(2) + 6$ ✓

$72 = 6(12) + 0$ ✓

$\therefore \gcd(3510, 672) = 6$ ✓

3) i) 27 and 72   LCM = 216

27 : 27, 54, 81, 108, 135, 162, 189, 216, 243, 270

72 : 72, 144, 216

$$\text{LCM}(27,72) = \frac{27 \times 72}{\text{GCD}(27,72)}$$

$$= \frac{27 \times 72}{9}$$

$$= 216$$

ii) 3510 and 672   LCM = 393120

3510 : 3510, 7020, 10530

~~672 : 672, 1344, 2016, 2688, 3360, 4032, 4704, 5376, 6048, 6720, 7392, 8064, 8736, 9408, 10080, 10752~~

$$\text{LCM}(3510, 672) = \frac{3510 \times 672}{\text{GCD}(3510, 672)}$$

$$= \frac{3510 \times 672}{6}$$

$$= 393120$$