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NUMERICAL ANALYSIS
(PHƯƠNG PHÁP TÍNH)

Chapter 04: Numerical Differentiation and Integration

Outline

- Differentiation of continuous functions
- Differentiation of discrete functions
- Trapezoid rule of Integration
- Simpson’s rule of Integration
- Romberg Integration
- Richardsons Extrapolation
- Gaussian Quadrature Rule of Integration

Differentiation of Continuous Functions

Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx'

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

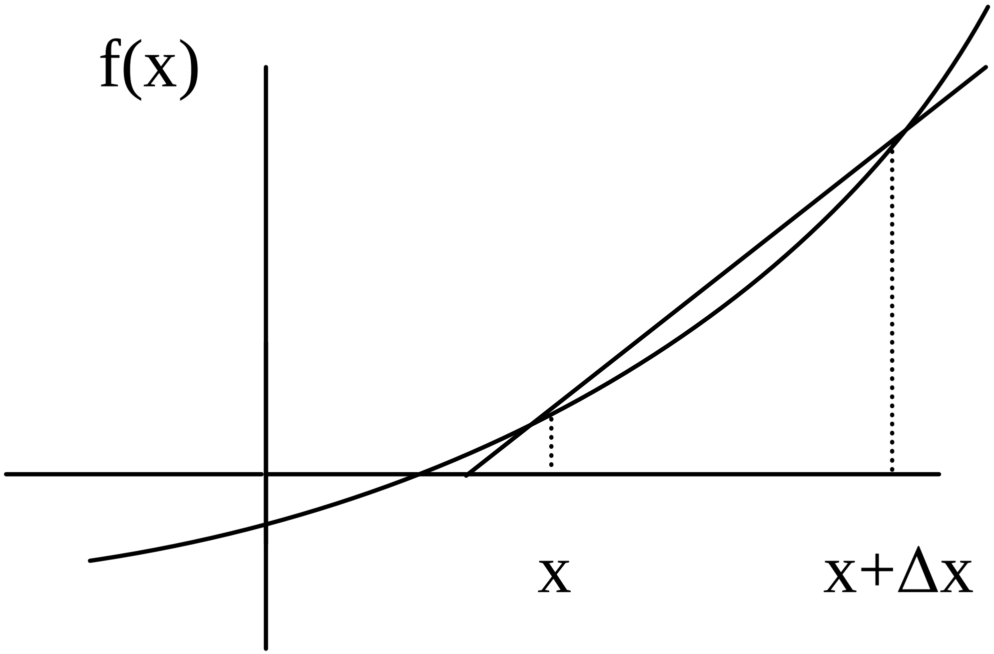


Figure 1 Graphical Representation of forward difference approximation of first derivative.

Example 1

The velocity of a rocket is given by

$$v(t) = 2000 \ln\left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

where 'v' is given in m/s and 't' is given in seconds.

a) Use forward difference approximation of the first derivative of $v(t)$ to calculate the acceleration at $t = 16s$. Use a step size of $\Delta t = 2s$.
b) Find the exact value of the acceleration of the rocket.
c) Calculate the absolute relative true error for part (b).

Example 1 Cont.

Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$a(16) \approx \frac{v(18) - v(16)}{2}$$

Example 1 Cont.

$$v(18) = 2000 \ln\left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(18)}\right] - 9.8(18)$$

$$= 453.02 \text{m/s}$$

$$v(16) = 2000 \ln\left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(16)}\right] - 9.8(16)$$

$$= 392.07 \text{m/s}$$

Hence

$$a(16) \approx \frac{v(18) - v(16)}{2}$$

Example 1 Cont.

$$\approx \frac{453.02-392.07}{2}$$

$$\approx 30.474\text{m/s}^2$$

b) The exact value of $a(16)$ can be calculated by differentiating

$$v(t)=2000\ln\left[\frac{14\times 10^4}{14\times 10^4-2100t}\right]-9.8t$$

as

$$a(t)=\frac{d}{dt}[v(t)]$$

Example 1 Cont.

Knowing that

$$\frac{d}{dt}[\ln(t)] = \frac{1}{t} \quad \text{and} \quad \frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}$$

$$a(t) = 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)\frac{d}{dt}\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)(-1)\left(\frac{14\times10^4}{(14\times10^4 - 2100t)^2}\right)(-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

Example 1 Cont.

$$a(16) = \frac{-4040 - 29.4(16)}{-200 + 3(16)}$$

$$= 29.674 \text{m/s}^2$$

The absolute relative true error is

$$|\in_t| = \left| \frac{\text{True Value - Approximate Value}}{\text{True Value}} \right| x100$$

$$= \left| \frac{29.674 - 30.474}{29.674} \right| x100$$

$$= 2.6967\%$$

Backward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite $'\Delta x'$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $'\Delta x'$ is chosen as a negative number,

$$f'(x) \approx \frac{f(x-\Delta x) - f(x)}{-\Delta x}$$

$$= \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation of the First Derivative Cont.

This is a backward difference approximation as you are taking a point backward from x. To find the value of $f'(x)$ at $x = x_i$, we may choose another point 'Δx' behind as $x = x_{i-1}$. This gives

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

where

$$\Delta x = x_i - x_{i-1}$$

Backward Difference Approximation of the First Derivative Cont.

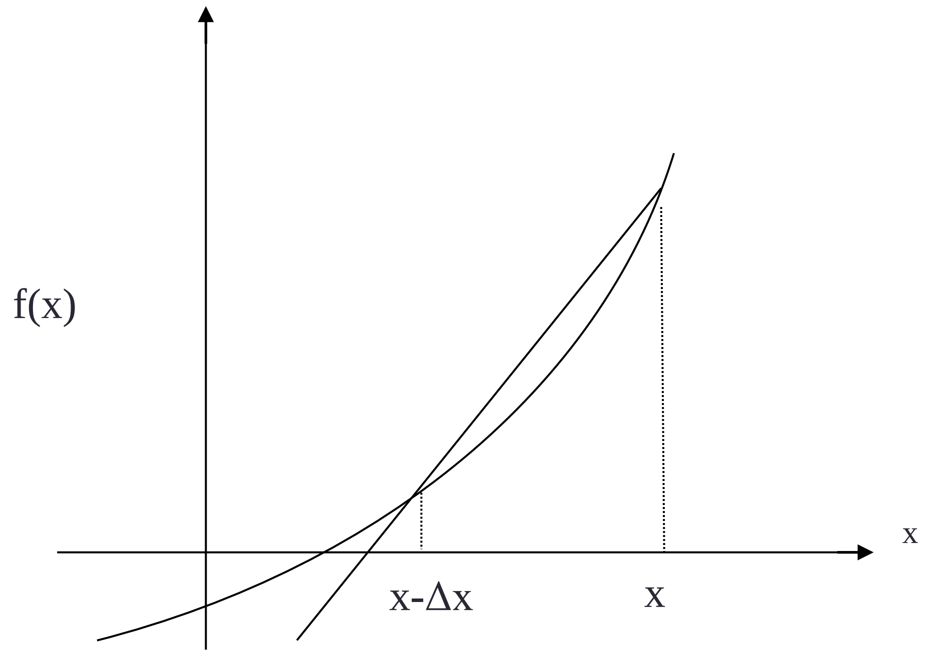


Figure 2 Graphical Representation of backward difference approximation of first derivative

Example 2

The velocity of a rocket is given by

$$v(t) = 2000 \ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

where $'v'$ is given in m/s and $'t'$ is given in seconds.

a) Use backward difference approximation of the first derivative of $v(t)$ to calculate the acceleration at $t = 16\,\text{s}$. Use a step size of $\Delta t = 2s$.
b) Find the absolute relative true error for part (a).

Example 2 Cont.

Solution

$$a(t) \approx \frac{v(t_i) - v(t_{i-1})}{\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i-1} &= t_i - \Delta t \\ &= 16 - 2 \\ &= 14 \end{aligned}$$

$$a(16) \approx \frac{v(16) - v(14)}{2}$$

Example 2 Cont.

$$v(16) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$

$$= 392.07\text{m/s}$$

$$v(14) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(14)}\right] - 9.8(14)$$

$$= 334.24\text{m/s}$$

$$a(16) \approx \frac{v(16) - v(14)}{2}$$

$$= \frac{392.07 - 334.24}{2}$$

$$\approx 28.915\text{m/s}^2$$

Example 2 Cont.

The exact value of the acceleration at $t = 16\,\text{s}$ from Example 1 is

$$a(16) = 29.674\,\text{m/s}^2$$

The absolute relative true error is

$$|\in_t| = \left|\frac{29.674 - 28.915}{29.674}\right| x100$$

$$= 2.5584\%$$

Derive the forward difference approximation from Taylor series

Taylor's theorem says that if you know the value of a function $'f'$ at a point x_i and all its derivatives at that point, provided the derivatives are continuous between x_i and x_{i+1}, then

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + 0(\Delta x)$$

Derive the forward difference approximation from Taylor series Cont.

The $(0\Delta x)$ term shows that the error in the approximation is of the order of (Δx) Can you now derive from Taylor series the formula for backward divided difference approximation of the first derivative?

As shown above, both forward and backward divided difference approximation of the first derivative are accurate on the order of $(0\Delta x)$

Can we get better approximations? Yes, another method to approximate the first derivative is called the **Central difference approximation of the first derivative**.

Derive the forward difference approximation from Taylor series Cont.

From Taylor series

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

Subtracting equation (2) from equation (1)

$$f(x_{i+1}) - f(x_{i-1}) = f'(x_i)(2\Delta x) + \frac{2f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} - \frac{f'''(x_i)}{3!}(\Delta x)^2 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + 0(\Delta x)^2$$

Central Divided Difference

Hence showing that we have obtained a more accurate formula as the error is of the order of $0(\Delta x)^2$.

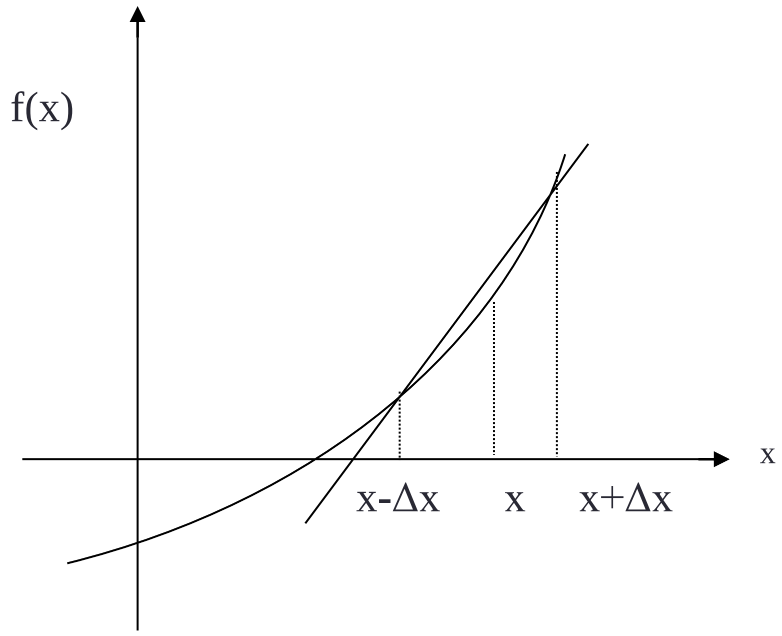


Figure 3 Graphical Representation of central difference approximation of first derivative

Example 3

The velocity of a rocket is given by

$$v(t) = 2000 \ln\left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

where $'v'$ is given in m/s and $'t'$ is given in seconds.

(a) Use central divided difference approximation of the first derivative of $v(t)$ to calculate the acceleration at $t = 16s$. Use a step size of $\Delta t = 2s$.
(b) Find the absolute relative true error for part (a).

Example 3 cont.

Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - v(t_{i-1})}{2\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} t_{i-1} &= t_i - \Delta t \\ &= 16 - 2 \\ &= 14 \end{aligned}$$

$$\begin{aligned} a(16) &\approx \frac{v(18) - v(14)}{2(2)} \\ &\approx \frac{v(18) - v(14)}{4} \end{aligned}$$

Example 3 cont.

$$\nu(18) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$

$$= 453.02\text{m/s}$$

$$\nu(14) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(14)}\right] - 9.8(14)$$

$$= 334.24\text{m/s}$$

$$a(16) \approx \frac{\nu(18) - \nu(14)}{4}$$

$$\approx \frac{453.02 - 334.24}{4}$$

$$\approx 29.694\text{m/s}^2$$

Example 3 cont.

The exact value of the acceleration at $t = 16\,\mathrm{s}$ from Example 1 is

$$a(16) = 29.674\,\mathrm{m/s}^2$$

The absolute relative true error is

$$|\in_t| = \left|\frac{29.674 - 29.694}{29.674}\right| \times 100$$

$$= 0.069157\%$$

Comparision of FDD, BDD, CDD

The results from the three difference approximations are given in Table 1.

Table 1 Summary of *a* (16) using different divided difference approximations

Type of Difference Approximation	$a(16)$ (m/s^2)	$\lvert \in_t \rvert \%$
Forward	30.475	2.6967
Backward	28.915	2.5584
Central	29.695	0.069157

Finding the value of the derivative within a prespecified tolerance

In real life, one would not know the exact value of the derivative – so how would one know how accurately they have found the value of the derivative.

A simple way would be to start with a step size and keep on halving the step size and keep on halving the step size until the absolute relative approximate error is within a pre-specified tolerance.

Take the example of finding $v'(t)$ for

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t$$

at $t = 16$ using the backward divided difference scheme.

Finding the value of the derivative within a prespecified tolerance Cont.

Given in Table 2 are the values obtained using the backward difference approximation method and the corresponding absolute relative approximate errors.

Table 2 First derivative approximations and relative errors for different Δt values of backward difference scheme

Δt	$v'(t)$	$\lvert \in_a \rvert \%$
2	28.915	
1	29.289	1.2792
0.5	29.480	0.64787
0.25	29.577	0.32604
0.125	29.625	0.16355

Finding the value of the derivative within a prespecified tolerance Cont.

From the above table, one can see that the absolute relative approximate error decreases as the step size is reduced. At $\Delta t = 0.125$ the absolute relative approximate error is 0.16355%, meaning that at least 2 significant digits are correct in the answer.

Finite Difference Approximation of Higher Derivatives

One can use Taylor series to approximate a higher order derivative.

For example, to approximate $f''(x)$, the Taylor series for

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2\Delta x) + \frac{f''(x_i)}{2!}(2\Delta x)^2 + \frac{f'''(x_i)}{3!}(2\Delta x)^3 + \ldots$$

where

$$x_{i+2} = x_i + 2\Delta x$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 \ldots$$

where

$$x_{i-1} = x_i - \Delta x$$

Finite Difference Approximation of Higher Derivatives Cont.

Subtracting 2 times equation (4) from equation (3) gives

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)(\Delta x)^2 + f'''(x_i)(\Delta x)^3 \ldots$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} - f'''(x_i)(\Delta x) + \ldots$$

$$f''(x_i) \cong \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} + 0(\Delta x) \qquad (5)$$

Example 4

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times 10^4}{14\times 10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

Use forward difference approximation of the second derivative $v(t)$ of to calculate the jerk at $t = 16s$. Use a step size of $\Delta t = 2s$.

Example 4 Cont.

Solution

$$j(t_i) \approx \frac{\nu(t_{i+2}) - 2\nu(t_{i+1}) + \nu(t_i)}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} t_{i+2} &= t_i + 2(\Delta t) \\ &= 16 + 2(2) \\ &= 20 \end{aligned}$$

$$j(16) \approx \frac{\nu(20) - 2\nu(18) + \nu(16)}{(2)^2}$$

Example 4 Cont.

$$v(20) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(20)}\right] - 9.8(20)$$

$$= 517.35\text{m/s}$$

$$v(18) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$

$$= 453.02m/s$$

$$v(16) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$

$$= 392.07\text{m/s}$$

Example 4 Cont.

$$j(16) \approx \frac{517.35 - 2(453.02) + 392.07}{4}$$

$$\approx 0.84515 \text{m/s}^3$$

The exact value of $j(16)$ can be calculated by differentiating

$$v(t) = 2000 \ln\left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t}\right] - 9.8t$$

twice as

$$a(t) = \frac{d}{dt}[v(t)] \quad \text{and} \quad j(t) = \frac{d}{dt}[a(t)]$$

Example 4 Cont.

Knowing that

$$\frac{d}{dt}[\ln(t)] = \frac{1}{t} \quad \text{and} \quad \frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}$$

$$a(t) = 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)\frac{d}{dt}\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)(-1)\left(\frac{14\times10^4}{(14\times10^4 - 2100t)^2}\right)(-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

Example 4 Cont.

Similarly it can be shown that

$$j(t) = \frac{d}{dt}[a(t)]$$
$$= \frac{18000}{(-200+3t)^2}$$

$$j(16) = \frac{18000}{[-200+3(16)]^2}$$
$$= 0.77909 \text{m/s}^3$$

The absolute relative true error is

$$|\in_t| = \left|\frac{0.77909 - 0.84515}{0.77909}\right| \times 100$$

$$= 8.4797\,\%$$

Higher order accuracy of higher order derivatives

The formula given by equation (5) is a forward difference approximation of the second derivative and has the error of the order of (Δx) . Can we get a formula that has a better accuracy? We can get the central difference approximation of the second derivative.

The Taylor series for

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f''''(x_i)}{4!}(\Delta x)^4 \ldots \qquad (6)$$

where

$$x_{i+1} = x_i + \Delta x$$

Higher order accuracy of higher order derivatives Cont.

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f''''(x_i)}{4!}(\Delta x)^4 \ldots \qquad (7)$$

where

$$x_{i-1} = x_i - \Delta x$$

Adding equations (6) and (7), gives

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)(\Delta x)^2 + f'''(x_i)\frac{(\Delta x)^4}{12}$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} - \frac{f''''(x_i)(\Delta x)^2}{12}$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} + 0(\Delta x)^2$$

Example 5

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

Use central difference approximation of second derivative of $v(t)$ to calculate the jerk at $t = 16s$. Use a step size of $\Delta t = 2s$.

Example 5 Cont.

Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - 2v(t_i) + v(t_{i-1})}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} t_{i-1} &= t_i - \Delta t \\ &= 16 - 2 \\ &= 14 \end{aligned}$$

$$j(16) \approx \frac{v(18) - 2v(16) + v(14)}{(2)^2}$$

Example 5 Cont.

$$v(18)=2000\ln\left[\frac{14\times10^4}{14\times10^4-2100(18)}\right]-9.8(18)$$

$$=453.02\text{m/s}$$

$$v(16)=2000\ln\left[\frac{14\times10^4}{14\times10^4-2100(16)}\right]-9.8(16)$$

$$=392.07\text{m/s}$$

$$v(14)=2000\ln\left[\frac{14\times10^4}{14\times10^4-2100(14)}\right]-9.8(14)$$

$$=334.24\text{m/s}$$

Example 5 Cont.

$$j(16) \approx \frac{\nu(18) - 2\nu(16) + \nu(14)}{(2)^2}$$

$$\approx \frac{453.02 - 2(392.07) + 334.24}{4}$$

$$\approx 0.77969 \text{m/s}^3$$

The absolute relative true error is

$$|\in_t| = \left| \frac{0.77908 - 0.78}{0.77908} \right| \times 100$$

$$= 0.077992\%$$

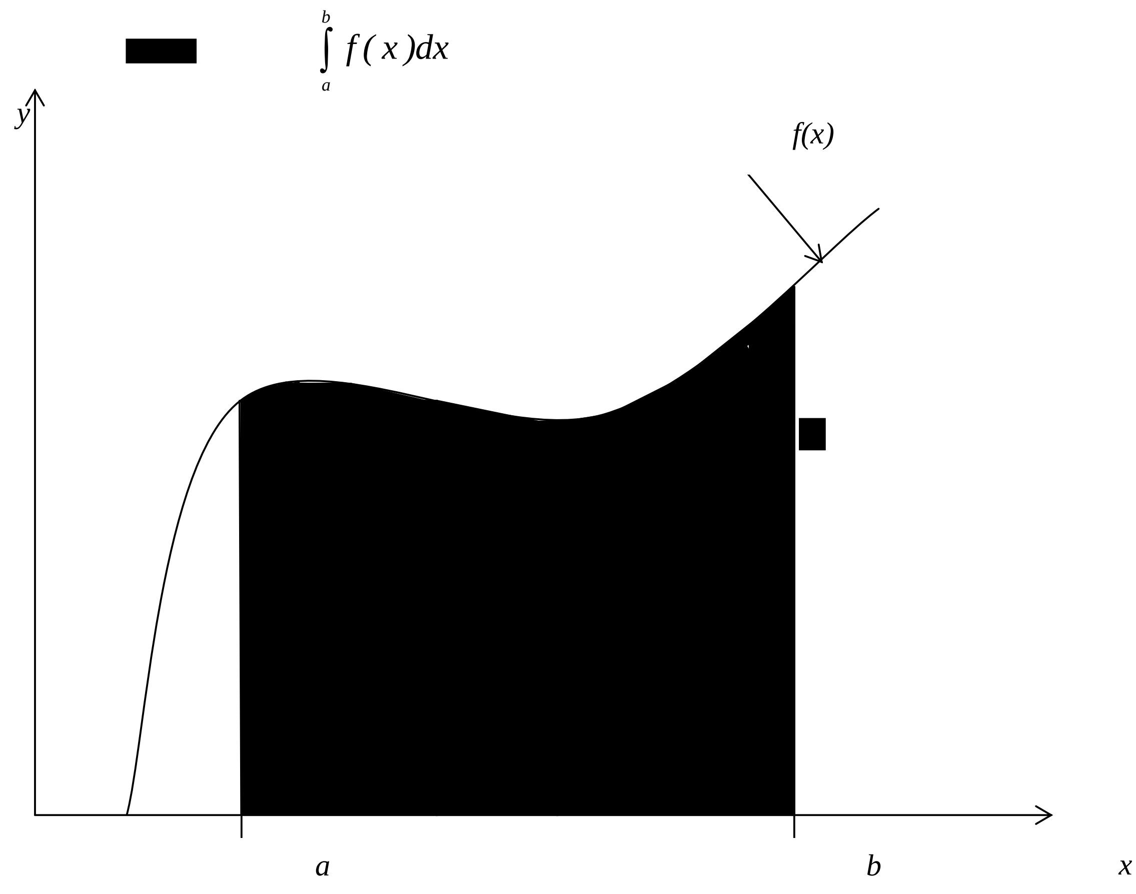
Trapezoidal Rule of Integration

What is Integration

Integration:

The process of measuring the area under a function plotted on a graph.

$$I = \int_a^b f(x)dx$$



Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration

Basis of Trapezoidal Rule

Trapezoidal Rule is based on the Newton-Cotes Formula that states if one can approximate the integrand as an n^{th} order polynomial…

$$I = \int_a^b f(x)dx \qquad \text{where} \qquad f(x) \approx f_n(x)$$

and $$f_n(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} + a_n x^n$$

Basis of Trapezoidal Rule

Then the integral of that function is approximated by the integral of that n^{th} order polynomial.

$$\int_a^b f(x) \approx \int_a^b f_n(x)$$

Trapezoidal Rule assumes n=1, that is, the area under the linear polynomial,

$$\int_a^b f(x)dx \quad = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

Derivation of the Trapezoidal Rule

Method Derived From Geometry

The area under the curve is a trapezoid. The integral

$$\int_a^b f(x)dx \approx Area\ of\ trapezoid$$

$$= \frac{1}{2}(Sum\ of\ parallel\ sides)(height)$$

$$= \frac{1}{2}\left(f(b)+f(a)\right)(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

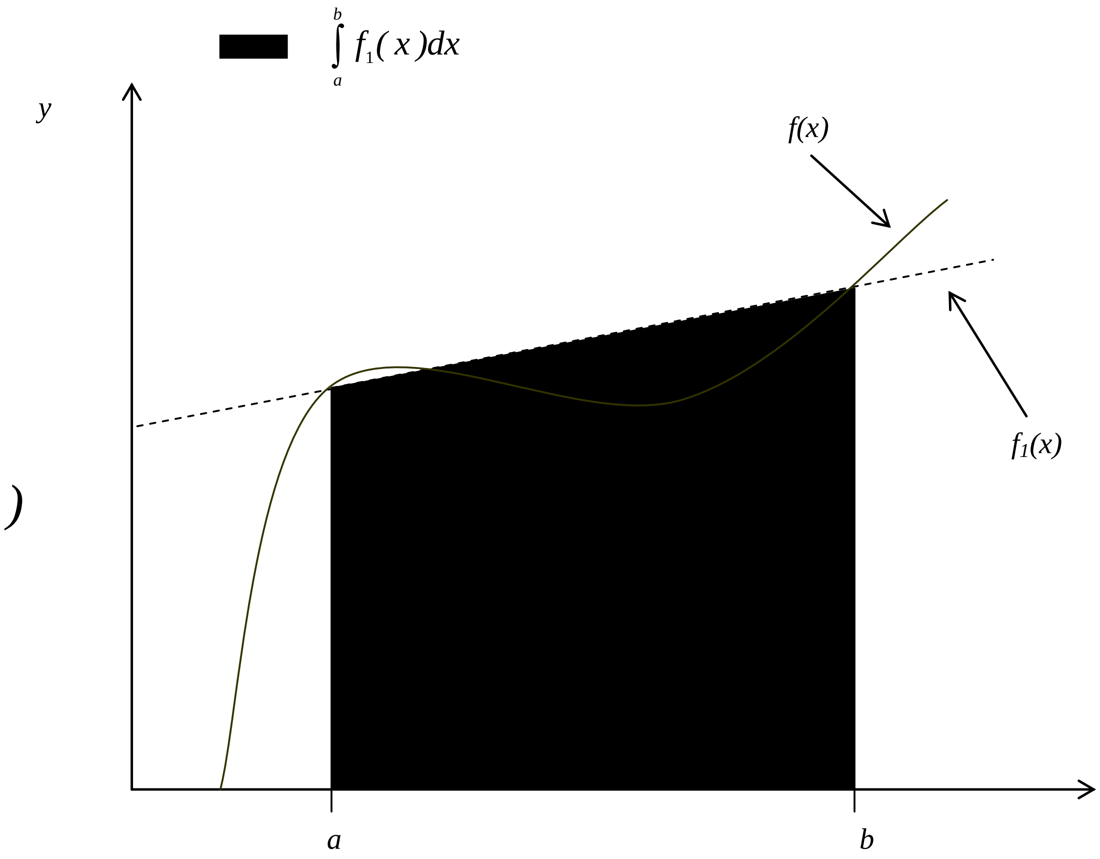


Figure 2: Geometric Representation

Example 1

The vertical distance covered by a rocket from t=8 to t=30 seconds is given by:

$$x = \int_{8}^{30} \left(2000 ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use single segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) $$I \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$a = 8 \qquad b = 30$$

$$f(t) = 2000 ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$f(8) = 2000 ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) \qquad = 177.27\ m/s$$

$$f(30) = 2000 ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) \quad = 901.67\ m/s$$

Solution (cont)

a) $$I = (30-8)\left[\frac{177.27+901.67}{2}\right]$$

$$= 11868\, m$$

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000 ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt \quad = 11061\, m$$

Solution (cont)

b) $$E_t = True\ Value - Approximate\ Value$$

$$= 11061 - 11868$$

$$= -807\ m$$

c) The absolute relative true error, $\left|\in_t\right|$, would be

$$\left|\in_t\right| = \left|\frac{11061 - 11868}{11061}\right| \times 100 = 7.2959\%$$

Multiple Segment Trapezoidal Rule

In Example 1, the true error using single segment trapezoidal rule was large. We can divide the interval [8,30] into [8,19] and [19,30] intervals and apply Trapezoidal rule over each segment.

$$f(t) = 2000\,ln\left(\frac{140000}{140000 - 2100t}\right) - 9.8t$$

$$\int_{8}^{30} f(t)dt = \int_{8}^{19} f(t)dt + \int_{19}^{30} f(t)dt$$

$$= (19-8)\left[\frac{f(8)+f(19)}{2}\right] + (30-19)\left[\frac{f(19)+f(30)}{2}\right]$$

Multiple Segment Trapezoidal Rule

With

$$f(8) = 177.27\ m/s$$

$$f(30) = 901.67\ m/s$$

$$f(19) = 484.75\ m/s$$

Hence:

$$\int_{8}^{30} f(t)dt = (19-8)\left[\frac{177.27+484.75}{2}\right] + (30-19)\left[\frac{484.75+901.67}{2}\right]$$

$$= 11266\ m$$

Multiple Segment Trapezoidal Rule

The true error is:

$$E_t = 11061 - 11266$$

$$= -205\ m$$

The true error now is reduced from -807 m to -205 m.

Extending this procedure to divide the interval into equal segments to apply the Trapezoidal rule; the sum of the results obtained for each segment is the approximate value of the integral.

Multiple Segment Trapezoidal Rule

Divide into equal segments as shown in Figure 4. Then the width of each segment is:

$$h = \frac{b-a}{n}$$

The integral I is:

$$I = \int_a^b f(x)dx$$

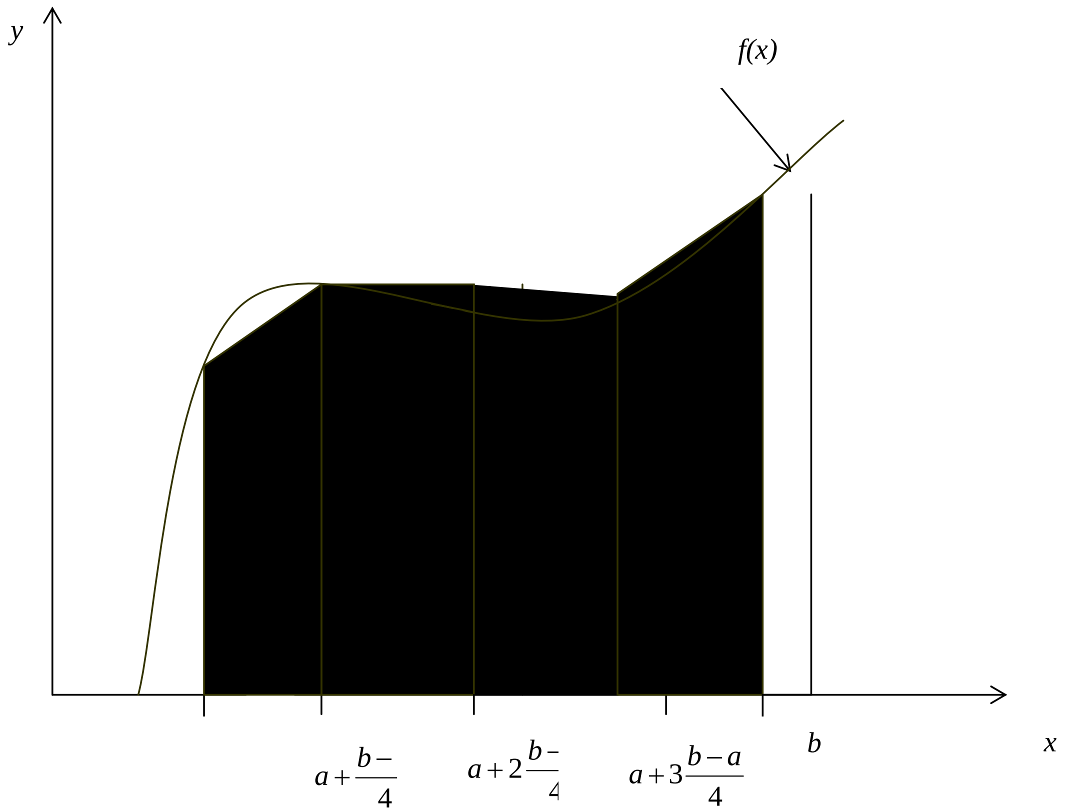


Figure 4: Multiple (n=4) Segment Trapezoidal Rule

Multiple Segment Trapezoidal Rule

The integral *I* can be broken into *h* integrals as:

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx$$

Applying Trapezoidal rule on each segment gives:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

Example 2

The vertical distance covered by a rocket from to seconds is given by:

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

a) Use two-segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) The solution using 2-segment Trapezoidal rule is

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

$$n = 2 \qquad a = 8 \qquad b = 30$$

$$h = \frac{b-a}{n} = \frac{30-8}{2} = 11$$

Solution (cont)

Then:

$$I = \frac{30-8}{2(2)}\left[f(8) + 2\left\{ \sum_{i=1}^{2-1} f(a+ih) \right\} + f(30) \right]$$

$$= \frac{22}{4}\left[f(8) + 2f(19) + f(30) \right]$$

$$= \frac{22}{4}\left[177.27 + 2(484.75) + 901.67 \right]$$

$$= 11266\ m$$

Solution (cont)

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt \quad = 11061\,m$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061 - 11266$$

Solution (cont)

The absolute relative true error, $|\in_t|$, would be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{11061 - 11266}{11061}\right| \times 100$$

$$= 1.8534\%$$

Solution (cont)

Table 1 gives the values obtained using multiple segment Trapezoidal rule for

$$x=\int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

Exact Value=11061 m

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Example 3

Use Multiple Segment Trapezoidal Rule to find the area under the curve

$$f(x)=\frac{300x}{1+e^{x}} \quad \text{from} \quad x=0 \quad \text{to} \quad x=10$$

Using two segments, we get $h=\frac{10-0}{2}=5$ and

$$f(0)=\frac{300(0)}{1+e^{0}}=0 \qquad f(5)=\frac{300(5)}{1+e^{5}}=10.039 \qquad f(10)=\frac{300(10)}{1+e^{10}}=0.136$$

Solution

Then:

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

$$= \frac{10-0}{2(2)}\left[f(0) + 2\left\{ \sum_{i=1}^{2-1} f(0+5) \right\} + f(10) \right]$$

$$= \frac{10}{4}\left[f(0) + 2f(5) + f(10) \right] = \frac{10}{4}\left[0 + 2(10.039) + 0.136 \right]$$

$$= 50.535$$

Solution (cont)

So what is the true value of this integral?

$$\int_0^{10} \frac{300x}{1+e^x}\,dx = 246.59$$

Making the absolute relative true error:

$$|\in_t| = \left|\frac{246.59 - 50.535}{246.59}\right| \times 100\%$$

$$= 79.506\%$$

Solution (cont)

Table 2: Values obtained using Multiple Segment Trapezoidal Rule for:

$$\int_0^{10} \frac{300x}{1+e^x} dx$$

n	Approximate Value	E_t	$\lvert \in_t \rvert$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

Error in Multiple Segment Trapezoidal Rule

The true error for a single segment Trapezoidal rule is given by:

$$E_t = \frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$ where ζ is some point in $[a,b]$

What is the error, then in the multiple segment Trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment Trapezoidal rule.

The error in each segment is

$$E_1 = \frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= \frac{h^3}{12} f''(\zeta_1)$$

Error in Multiple Segment Trapezoidal Rule

Similarly:

$$E_i = \frac{[(a+ih)-(a+(i-1)h)]^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih$$

$$= \frac{h^3}{12} f''(\zeta_i)$$

It then follows that:

$$E_n = \frac{[b-\{a+(n-1)h\}]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= \frac{h^3}{12} f''(\zeta_n)$$

Error in Multiple Segment Trapezoidal Rule

Hence the total error in multiple segment Trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i = \frac{h^3}{12}\sum_{i=1}^{n} f''(\zeta_i) = \frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

The term $\dfrac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the $f''(x), a < x < b$

Hence:

$$E_t = \frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

Error in Multiple Segment Trapezoidal Rule

Below is the table for the integral $\int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$

as a function of the number of segments. You can visualize that as the number of segments are doubled, the true error gets approximately quartered.

n	**Value**	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
2	11266	-205	1.854	5.343
4	11113	-51.5	0.4655	0.3594
8	11074	-12.9	0.1165	0.03560
16	11065	-3.22	0.02913	0.00401

Simpson's 1/3rd Rule of Integration

Basis of Simpson's 1/3rd Rule

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson's 1/3rd rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

Where $f_2(x)$ is a second order polynomial.

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Basis of Simpson's 1/3rd Rule

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \quad \text{and} \quad (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Basis of Simpson's 1/3rd Rule

Solving the previous equations for a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Basis of Simpson's 1/3rd Rule

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Basis of Simpson's 1/3rd Rule

Substituting values of a_0, a_1, a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since for Simpson's 1/3rd Rule, the interval [a, b] is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Basis of Simpson's 1/3rd Rule

Hence

$$\int_a^b f_2(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Because the above form has 1/3 in its formula, it is called Simpson's 1/3rd Rule.

Example 1

The distance covered by a rocket from t=8 to t=30 is given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Simpson's 1/3rd Rule to find the approximate value of x

b) Find the true error, E_t

c) Find the absolute relative true error, $\left| \in_t \right|$

Solution

a)

$$x = \int_{8}^{30} f(t)dt$$

$$x = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \left(\frac{30-8}{6}\right)\left[f(8) + 4f(19) + f(30)\right]$$

$$= \left(\frac{22}{6}\right)\left[177.2667 + 4(484.7455) + 901.6740\right]$$

$$= 11065.72\ m$$

Solution (cont)

b) The exact value of the above integral is

$$x = \int_{8}^{30} \left(2000\, ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

$$= 11061.34\ m$$

True Error

$$E_t = 11061.34 - 11065.72$$

$$= -4.38\ m$$

Solution (cont)

a)c) Absolute relative true error,

$$|\in_t| = \left|\frac{11061.34 - 11065.72}{11061.34}\right| \times 100\%$$

$$= 0.0396\%$$

Multiple Segment Simpson's 1/3rd Rule

Multiple Segment Simpson's 1/3rd Rule

Just like in multiple segment Trapezoidal Rule, one can subdivide the interval [a, b] into n segments and apply Simpson's 1/3rd Rule repeatedly over every two segments. Note that n needs to be even. Divide interval [a, b] into equal segments, hence the segment width

$$h = \frac{b-a}{n} \qquad \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

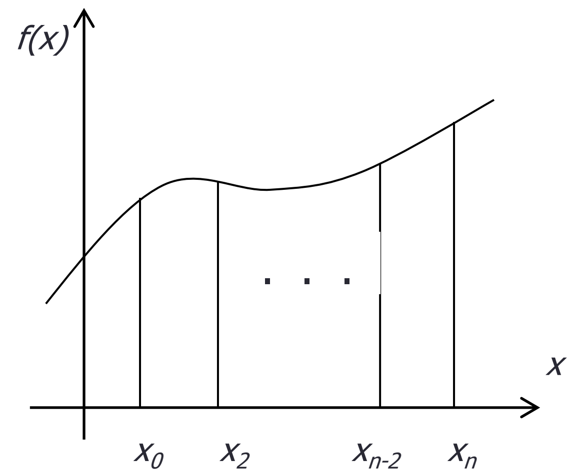
where

$$x_0 = a \qquad x_n = b$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots..$$

$$\ldots. + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$



Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + \ldots$$

$$+ (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

Multiple Segment Simpson's 1/3rd Rule

$$\ldots + (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + \ldots$$

$$+ (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h \qquad i = 2, 4, \ldots, n$$

Multiple Segment Simpson's 1/3rd Rule

Then

$$\int_a^b f(x)dx = 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right]+\dots$$

$$+2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right]+\dots$$

$$+2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right]+\dots$$

$$+2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\}+\ldots]$$

$$\ldots+2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\}+f(x_n)\}]$$

$$=\frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\i=odd}}^{n-1}f(x_i)+2\sum_{\substack{i=2\\i=even}}^{n-2}f(x_i)+f(x_n)\right]$$

$$=\frac{b-a}{3n}\left[f(x_0)+4\sum_{\substack{i=1\\i=odd}}^{n-1}f(x_i)+2\sum_{\substack{i=2\\i=even}}^{n-2}f(x_i)+f(x_n)\right]$$

Example 2

Use 4-segment Simpson's 1/3rd Rule to approximate the distance

covered by a rocket from t= 8 to t=30 as given by

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use four segment Simpson's 1/3rd Rule to find the approximate value of *x*.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $\left| \in_a \right|$ for part (a).

Solution

a) Using n segment Simpson's 1/3rd Rule,

$$h = \frac{30-8}{4} = 5.5$$

So

$$f(t_0) = f(8)$$

$$f(t_1) = f(8+5.5) = f(13.5)$$

$$f(t_2) = f(13.5+5.5) = f(19)$$

$$f(t_3) = f(19+5.5) = f(24.5)$$

$$f(t_4) = f(30)$$

Solution (cont.)

$$x = \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1 \\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{2} f(t_i) + f(30) \right]$$

$$= \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30) \right]$$

Solution (cont.)

cont.

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30) \right]$$

$$= \frac{11}{6}\left[177.2667 + 4(320.2469) + 4(676.0501) + 2(484.7455) + 901.6740 \right]$$

$$= 11061.64\ m$$

Solution (cont.)

b) In this case, the true error is

$$E_t = 11061.34 - 11061.64 = -0.30\ m$$

c) The absolute relative true error

$$|\in_t| = \left|\frac{11061.34 - 11061.64}{11061.34}\right| \times 100\%$$

$$= 0.0027\%$$

Solution (cont.)

Table 1: Values of Simpson's 1/3rd Rule for Example 2 with multiple segments

n	Approximate Value	E_t	$\lvert \in_t \rvert$
2	11065.72	4.38	0.0396%
4	11061.64	0.30	0.0027%
6	11061.40	0.06	0.0005%
8	11061.35	0.01	0.0001%
10	11061.34	0.00	0.0000%

Error in the Multiple Segment Simpson's 1/3rd Rule

The true error in a single application of Simpson's 1/3rd Rule is given as

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In Multiple Segment Simpson's 1/3rd Rule, the error is the sum of the errors in each application of Simpson's 1/3rd Rule. The error in n segment Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1) = -\frac{h^5}{90} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2) = -\frac{h^5}{90} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

Error in the Multiple Segment Simpson's 1/3rd Rule

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i) = -\frac{h^5}{90} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

.

.

.

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{4}\left(\zeta_{\frac{n}{2}}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Error in the Multiple Segment Simpson's 1/3rd Rule

Hence, the total error in Multiple Segment Simpson's 1/3rd Rule is

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i \quad = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \quad = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

Error in the Multiple Segment Simpson's 1/3rd Rule

The term $\dfrac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of

$$f^{(4)}(x),\ a < x < b$$

Hence

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

Romberg Rule of Integration

Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

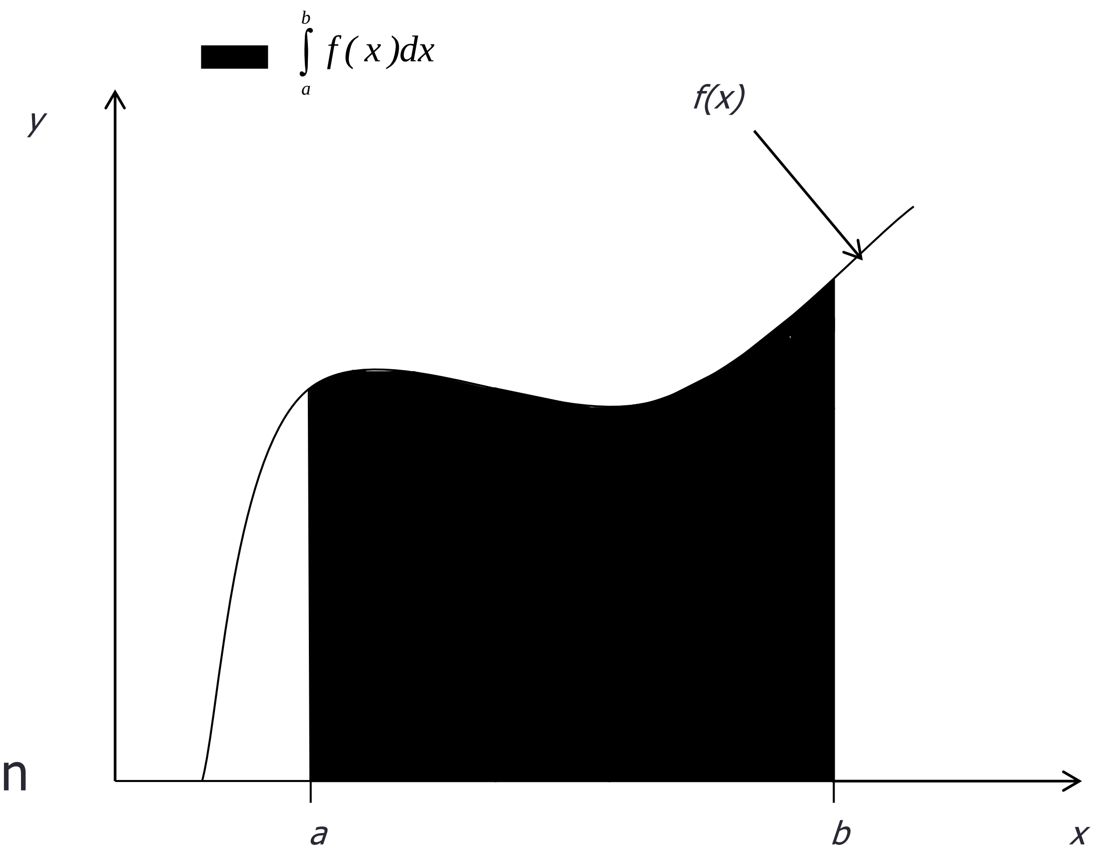
$$I = \int_a^b f(x)\,dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x)dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

where for each *i*, ξ_i is a point somewhere in the domain , $[a+(i-1)h, a+ih]$.

Error in Multiple Segment Trapezoidal Rule

The term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$x = \int_{8}^{30} \left(2000 \, ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Value	E_t	$\vert\in_t\vert\%$	$\vert\in_a\vert\%$
1	11868	807	7.296	---
2	11266	205	1.854	5.343
3	11153	91.4	0.8265	1.019
4	11113	51.5	0.4655	0.3594
5	11094	33.0	0.2981	0.1669
6	11084	22.9	0.2070	0.09082
7	11078	16.8	0.1521	0.05482
8	11074	12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the *n*-segment Trapezoidal rule is estimated as

$$E_t \approx \frac{C}{n^2}$$

where *C* is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value

Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \approx TV - I_{2n}$$

when the segment size is doubled and that

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

Example 1

The vertical distance covered by a rocket from 8 to 30 seconds is given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Richardson's rule to find the distance covered. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\epsilon_a|$ for part (a).

Solution

a) $I_2 = 11266m$ $I_4 = 11113m$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing *n=2,*

$$TV \approx I_4 + \frac{I_4 - I_2}{3} = 11113 + \frac{11113 - 11266}{3}$$

$$= 11062m$$

Solution (cont.)

b) The exact value of the above integral is

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

$$= 11061\ m$$

Hence

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061 - 11062$$

$$= -1\ m$$

Solution (cont.)

c) The absolute relative true error $\left|\in_t\right|$ would then be

$$\left|\in_t\right| = \left|\frac{11061 - 11062}{11061}\right| \times 100$$

$$= 0.00904\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Solution (cont.)

Table 2: The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$x = \int_{8}^{30} \left(2000\, ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Trapezoidal Rule	$\lvert \in_t \rvert$ for Trapezoidal Rule	Richardson's Extrapolation	$\lvert \in_t \rvert$ for Richardson's Extrapolation
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

Table 2: Richardson's Extrapolation Values

Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$\left(I_{2n}\right)_R = I_{2n} + \frac{I_{2n} - I_n}{3} \quad = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg Integration

Note that the variable TV is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\approx$ is replaced by = sign.

Hence the estimate of the true value now is

$$TV \approx (I_{2n})_R + Ch^4$$

Where Ch^4 is an approximation of the true error.

Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$TV \approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

$$= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}$$

Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index *k* represents the order of extrapolation. k=1 represents the values obtained from the regular Trapezoidal rule, k=2 represents values obtained using the true estimate as $O(h^2)$. The index *j* represents the more and less accurate estimate of the integral.

Example 2

The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = 11868 \qquad I_{1,2} = 11266$$

$$I_{1,3} = 11113 \qquad I_{1,4} = 11074$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

Solution (cont.)

To get the first order extrapolation values,

$$
\begin{aligned}
I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\
&= 11266 + \frac{11266 - 11868}{3} \\
&= 11065
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\
&= 11113 + \frac{11113 - 11266}{3} \\
&= 11062
\end{aligned}
$$

$$
\begin{aligned}
I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\
&= 11074 + \frac{11074 - 11113}{3} \\
&= 11061
\end{aligned}
$$

Solution (cont.)

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 11062 + \frac{11062 - 11065}{15}$$

$$= 11062$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= 11061 + \frac{11061 - 11062}{15}$$

$$= 11061$$

Solution (cont.)

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= 11061 + \frac{11061 - 11062}{63}$$

$$= 11061m$$

Table 3 shows these increased correct values in a tree graph.

Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		First Order	*Second Order*	*Third Order*
1-segment	11868			
		11065		
2-segment	1126		11062	
		11062		11061
4-segment	11113		11061	
		11061		
8-segment	11074			

Gauss Quadrature Rule of Integration

What is Integration?

Integration

The process of measuring the area under a curve.

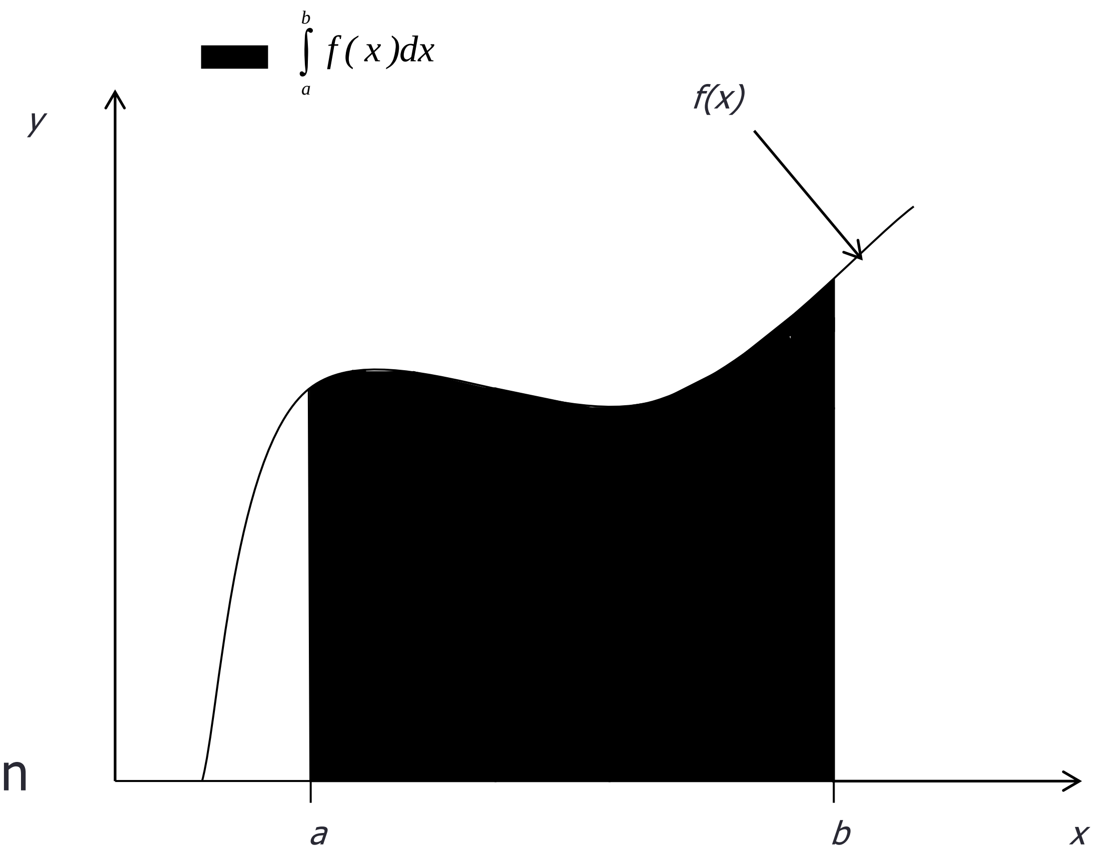
$$I = \int_a^b f(x)\,dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



Two-Point Gaussian Quadrature Rule

Basis of the Gaussian Quadrature Rule

Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

Basis of the Gaussian Quadrature Rule

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as a and b but as unknowns x_1 and x_2. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Basis of the Gaussian Quadrature Rule

The four unknowns x_1, x_2, c_1 and c_2 are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

Hence

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

Basis of the Gaussian Quadrature Rule

It follows that

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3\right)$$

Equating Equations the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

$$= c_1\left(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2\left(c_1x_1^2 + c_2x_2^2\right) + a_3\left(c_1x_1^3 + c_2x_2^3\right)$$

Basis of the Gaussian Quadrature Rule

Since the constants a_0, a_1, a_2, a_3 are arbitrary

$$b-a=c_1+c_2$$

$$\frac{b^2-a^2}{2}=c_1x_1+c_2x_2$$

$$\frac{b^3-a^3}{3}=c_1x_1{}^2+c_2x_2{}^2$$

$$\frac{b^4-a^4}{4}=c_1x_1{}^3+c_2x_2{}^3$$

Basis of Gauss Quadrature

The previous four simultaneous nonlinear Equations have only one acceptable solution,

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \qquad x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2} \qquad c_2 = \frac{b-a}{2}$$

Basis of Gauss Quadrature

Hence Two-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

Higher Point Gaussian Quadrature Formulas

Higher Point Gaussian Quadrature Formulas

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

is called the three-point Gauss Quadrature Rule.

The coefficients c_1, c_2, and c_3, and the functional arguments x_1, x_2, and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right)dx$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n)$$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for n-point Gauss Quadrature Rule are given for integrals

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

as shown in Table 1.

Table 1: Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
2	c_1 = 1.000000000 c_2 = 1.000000000	x_1 = -0.577350269 x_2 = 0.577350269
3	c_1 = 0.555555556 c_2 = 0.888888889 c_3 = 0.555555556	x_1 = -0.774596669 x_2 = 0.000000000 x_3 = 0.774596669
4	c_1 = 0.347854845 c_2 = 0.652145155 c_3 = 0.652145155 c_4 = 0.347854845	x_1 = -0.861136312 x_2 = -0.339981044 x_3 = 0.339981044 x_4 = 0.861136312

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Table 1 (cont.) : Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
5	c_1 = 0.236926885 c_2 = 0.478628670 c_3 = 0.568888889 c_4 = 0.478628670 c_5 = 0.236926885	x_1 = -0.906179846 x_2 = -0.538469310 x_3 = 0.000000000 x_4 = 0.538469310 x_5 = 0.906179846
6	c_1 = 0.171324492 c_2 = 0.360761573 c_3 = 0.467913935 c_4 = 0.467913935 c_5 = 0.360761573 c_6 = 0.171324492	x_1 = -0.932469514 x_2 = -0.661209386 x_3 = -0.2386191860 x_4 = 0.2386191860 x_5 = 0.661209386 x_6 = 0.932469514

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$? The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$ Let

$$x = mt + c$$

If $x = a,$ then $t = -1$

If $x = b,$ then $t = 1$

Such that:

$$m = \frac{b-a}{2}$$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Then $c = \frac{b+a}{2}$ Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \qquad dx = \frac{b-a}{2}dt$$

Substituting our values of x, and dx into the integral gives us

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}dt$$

Example 1

For an integral $\int_a^b f(x)dx$, derive the one-point Gaussian Quadrature Rule.

Solution

The one-point Gaussian Quadrature Rule is

$$\int_a^b f(x)dx \approx c_1 f\left(x_1\right)$$

Solution

The two unknowns x_1, and c_1 are found by assuming that the formula gives exact results for integrating a general first order polynomial,

$$f(x) = a_0 + a_1 x.$$

$$\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} \right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right)$$

Solution

It follows that

$$\int_a^b f(x)dx = c_1(a_0 + a_1x_1)$$

Equating Equations, the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) = c_1(a_0 + a_1x_1) = a_0(c_1) + a_1(c_1x_1)$$

Basis of the Gaussian Quadrature Rule

Since the constants a_0, and a_1 are arbitrary

$$b - a = c_1$$

$$\frac{b^2 - a^2}{2} = c_1 x_1$$

giving

$$c_1 = b - a$$

$$x_1 = \frac{b + a}{2}$$

Solution

Hence One-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f(x_1) = (b-a)\ f\left(\frac{b+a}{2}\right)$$

Example 2

a) Use two-point Gauss Quadrature Rule to approximate the distance covered by a rocket from t=8 to t=30 as given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

b) Find the true error, E_t for part (a).

c) Also, find the absolute relative true error, $|\in_a|$ for part (a).

Solution

First, change the limits of integration from [8,30] to [-1,1]

by previous relations as follows

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x+\frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f(11x+19)dx$$

Solution (cont)

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$$c_1 = \ \ 1.000000000$$

$$x_1 = -0.577350269$$

$$c_2 = \ \ 1.000000000$$

$$x_2 = \ \ 0.577350269$$

Solution (cont.)

Now we can use the Gauss Quadrature formula

$$11\int_{-1}^{1} f(11x+19)dx \approx 11c_1 f(11x_1+19)+11c_2 f(11x_2+19)$$

$$=11f(11(-0.5773503)+19)+11f(11(0.5773503)+19)$$

$$=11f(12.64915)+11f(25.35085)$$

$$=11(296.8317)+11(708.4811)$$

$$=11058.44\ m$$

Solution (cont)

since

$$f(12.64915) = 2000\,ln\left[\frac{140000}{140000 - 2100(12.64915)}\right] - 9.8(12.64915)$$

$$= 296.8317$$

$$f(25.35085) = 2000\,ln\left[\frac{140000}{140000 - 2100(25.35085)}\right] - 9.8(25.35085)$$

$$= 708.4811$$

Solution (cont)

b) The true error, E_t , is

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061.34 - 11058.44$$

$$= 2.9000\ m$$

c) The absolute relative true error, $|\in_t|$, is (Exact value = 11061.34m)

$$|\in_t| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100\%$$

$$= 0.0262\%$$

THE END