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NUMERICAL ANALYSIS
(PHƯƠNG PHÁP TÍNH)

Chapter 06: Linear System of Equations

Outline

- Gaussian elimination method
- Gauss-Seidel method
- LU Decomposition
- Cholesky and LDL^T Decomposition

Gaussian Elimination Method

Naïve Gaussian Elimination

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Forward Elimination

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}.

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$- \quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or $$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{22}^{'}x_2 + a_{23}^{'}x_3 + \ldots + a_{2n}^{'}x_n = b_2^{'}$$

$$a_{32}^{'}x_2 + a_{33}^{'}x_3 + \ldots + a_{3n}^{'}x_n = b_3^{'}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{n2}^{'}x_2 + a_{n3}^{'}x_3 + \ldots + a_{nn}^{'}x_n = b_n^{'}$$

End of Step 1

Forward Elimination

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a''_{n3}x_3 + \dots + a''_{nn}x_n = b''_n$$

End of Step 2

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad & \quad \vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1,\ldots,1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1,\ldots,1$$

Naïve Gauss Elimination Example

Example 1

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time, $t\,(\mathrm{s})$	**Velocity,** $v\,(\mathrm{m/s})$
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3\ , \qquad 5 \le \mathrm{t} \le 12.$$

Find the velocity at t=6 seconds .

Example 1 Cont.

Assume

$$v(t) = a_1 t^2 + a_2 t + a_3 \ , \quad 5 \le t \le 12.$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 1 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 144 & 12 & 1 & 279.2 \end{array}\right]$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is

$(n-1)=(3-1)=2$

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide Equation 2 by -4.8 and multiply it by -16.8, $\frac{-16.8}{-4.8} = 3.5$.

$$[0 \quad -4.8 \quad -1.56 \quad \vdots \quad -96.208] \times 3.5 = [0 \quad -16.8 \quad -5.46 \quad \vdots \quad -336.728]$$

Subtract the result from Equation 3

$$\begin{array}{r} [0 \quad -16.8 \quad -4.76 \quad \vdots \quad 335.968] \\ -[0 \quad -16.8 \quad -5.46 \quad \vdots \quad -336.728] \\ \hline [0 \quad 0 \quad 0.7 \quad \vdots \quad 0.76] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25}$$

$$= 0.290472$$

Naïve Gaussian Elimination Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

Example 1 Cont.

Solution

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$\begin{aligned} v(t) &= a_1 t^2 + a_2 t + a_3 \\ &= 0.290472t^2 + 19.6905t + 1.08571, \quad 5 \le t \le 12 \end{aligned}$$

$$\begin{aligned} v(6) &= 0.290472(6)^2 + 19.6905(6) + 1.08571 \\ &= 129.686 \text{ m/s}. \end{aligned}$$

Naïve Gauss Elimination Pitfalls

Pitfall#1. Division by zero

$$10x_2 - 7x_3 = 3$$

$$6x_1 + 2x_2 + 3x_3 = 11$$

$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 0 & 10 & -7 \\ 6 & 2 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?

$$12x_1 + 10x_2 - 7x_3 = 15$$
$$6x_1 + 5x_2 + 3x_3 = 14$$
$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}$$

Is division by zero an issue here? YES

$$
\begin{aligned}
12x_1 + 10x_2 - 7x_3 &= 15 \\
6x_1 + 5x_2 + 3x_3 &= 14 \\
24x_1 - x_2 + 5x_3 &= 28
\end{aligned}
$$

$$
\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 24 & -1 & 5 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =
\begin{bmatrix} 15 \\ 14 \\ 28 \end{bmatrix}
\longrightarrow
\begin{bmatrix} 12 & 10 & -7 \\ 0 & 0 & 6.5 \\ 12 & -21 & 19 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =
\begin{bmatrix} 15 \\ 6.5 \\ -2 \end{bmatrix}
$$

Division by zero is a possibility at any step of forward elimination

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Exact Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 6 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 5 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Is there a way to reduce the round off error?

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

Gauss Elimination with Partial Pivoting

Pitfalls of Naïve Gauss Elimination

- Possible division by zero
- Large round-off errors

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

What is Different About Partial Pivoting?

At the beginning of the k^{th} step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots\ldots, |a_{nk}|$$

If the maximum of the values is $|a_{pk}|$

in the p^{th} row, $k \le p \le n,$ then switch rows p and k.

Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Example (2nd step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -7 & 6 & 1 & 2 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -17 & 12 & 11 & 43 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 8 \\ 9 \\ 3 \end{bmatrix}$$

Which two rows would you switch?

Example (2^{nd} step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -17 & 12 & 11 & 43 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -7 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \\ 9 \\ -6 \end{bmatrix}$$

Switched Rows

Gaussian Elimination with Partial Pivoting

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

Same as naïve Gauss elimination method except that we switch rows before **each** of the (n-1) steps of forward elimination.

Example: Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{22}^{'}x_2 + a_{23}^{'}x_3 + \ldots + a_{2n}^{'}x_n = b_2^{'}$$

$$a_{33}^{''}x_3 + \ldots + a_{n}^{''}x_n = b_3^{''}$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \dots, 1$$

Gauss Elimination with Partial Pivoting Example

Example 2

Solve the following set of equations by Gaussian elimination with partial pivoting

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 2 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is $(n-1)=(3-1)=2$

Forward Elimination: Step 1

- Examine absolute values of first column, first row and below.

$$|25|, |64|, |144|$$

- Largest absolute value is 144 and exists in row 3.
- Switch row 1 and row 3.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 64, $\frac{64}{144} = 0.4444.$

$$[144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \times 0.4444 = [63.99 \quad 5.333 \quad 0.4444 \quad \vdots \quad 124.1]$$

Subtract the result from Equation 2

$$\begin{array}{r} [64 \quad 8 \quad 1 \quad \vdots \quad 177.2] \\ -[63.99 \quad 5.333 \quad 0.4444 \quad \vdots \quad 124.1] \\ \hline [0 \quad 2.667 \quad 0.5556 \quad \vdots \quad 53.10] \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 25, $\frac{25}{144} = 0.1736.$

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \times 0.1736 = \begin{bmatrix} 25.00 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \\ -\begin{bmatrix} 25 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$

Forward Elimination: Step 2

- Examine absolute values of second column, second row and below.

$$|2.667|, |2.917|$$

- Largest absolute value is 2.917 and exists in row 3.
- Switch row 2 and row 3.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Forward Elimination: Step 2 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Divide Equation 2 by 2.917 and multiply it by 2.667,

$$\frac{2.667}{2.917} = 0.9143.$$

$$[0 \quad 2.917 \quad 0.8264 \quad \vdots \quad 58.33] \times 0.9143 = [0 \quad 2.667 \quad 0.7556 \quad \vdots \quad 53.33]$$

Subtract the result from Equation 3

$$\begin{array}{r} [0 \quad 2.667 \quad 0.5556 \quad \vdots \quad 53.10] \\ -[0 \quad 2.667 \quad 0.7556 \quad \vdots \quad 53.33] \\ \hline [0 \quad 0 \quad -0.2 \quad \vdots \quad -0.23] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_3

$$-0.2a_3 = -0.23$$

$$a_3 = \frac{-0.23}{-0.2}$$

$$= 1.15$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_2

$$2.917a_2 + 0.8264a_3 = 58.33$$

$$a_2 = \frac{58.33 - 0.8264a_3}{2.917}$$

$$= \frac{58.33 - 0.8264 \times 1.15}{2.917}$$

$$= 19.67$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_1

$$144a_1 + 12a_2 + a_3 = 279.2$$

$$a_1 = \frac{279.2 - 12a_2 - a_3}{144}$$

$$= \frac{279.2 - 12 \times 19.67 - 1.15}{144}$$

$$= 0.2917$$

Gaussian Elimination with Partial Pivoting Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \\ 1.15 \end{bmatrix}$$

Gauss Elimination with Partial Pivoting Another Example

Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Determinant of a Square Matrix Using Naïve Gauss Elimination

Example

Theorem of Determinants

If a multiple of one row of $[A]_{nxn}$ is added or subtracted to another row of $[A]_{nxn}$ to result in $[B]_{nxn}$ then det(A)=det(B)

Theorem of Determinants

The determinant of an upper triangular matrix $[A]_{nxn}$ is given by

$$\det(\mathrm{A}) = a_{11} \times a_{22} \times \ldots \times a_{ii} \times \ldots \times a_{nn}$$

$$= \prod_{i=1}^{n} a_{ii}$$

Forward Elimination of a Square Matrix

Using forward elimination to transform $[A]_{nxn}$ to an upper triangular matrix, $[U]_{nxn}$.

$$[A]_{n\times n} \rightarrow [U]_{n\times n}$$

$$\det(A) = \det(U)$$

Example

Using naïve Gaussian elimination find the determinant of the following square matrix.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$.

$$[25 \quad 5 \quad 1] \times 2.56 = [64 \quad 12.8 \quad 2.56]$$

Subtract the result from Equation 2

$$\begin{array}{r} [64 \quad 8 \quad 1] \\ -[64 \quad 12.8 \quad 2.56] \\ \hline [0 \quad -4.8 \quad -1.56] \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$.

$$[25 \quad 5 \quad 1] \times 5.76 = [144 \quad 28.8 \quad 5.76]$$

Subtract the result from Equation 3

$$\begin{array}{r} [144 \quad 12 \quad 1] \\ -[144 \quad 28.8 \quad 5.76] \\ \hline [0 \quad -16.8 \quad -4.76] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Divide Equation 2 by -4.8 and multiply it by -16.8, $\frac{-16.8}{-4.8} = 3.5$.

$$([0 \quad -4.8 \quad -1.56]) \times 3.5 = [0 \quad -16.8 \quad -5.46]$$

Subtract the result from Equation 3

$$\begin{array}{r} [0 \quad -16.8 \quad -4.76] \\ -[0 \quad -16.8 \quad -5.46] \\ \hline [0 \qquad 0 \qquad 0.7] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the Determinant

After forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \to \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{aligned} \det(\mathrm{A}) &= u_{11} \times u_{22} \times u_{33} \\ &= 25 \times (-4.8) \times 0.7 \\ &= -84.00 \end{aligned}$$

Summary

-Forward Elimination

-Back Substitution

-Pitfalls

-Improvements

-Partial Pivoting

-Determinant of a Matrix

Gauss-Seidel Method

Gauss-Seidel Method

An ***iterative*** method.

Basic Procedure:

-Algebraically solve each linear equation for x_i

-Assume an initial guess solution array

-Solve for each x_i and repeat

-Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

Gauss-Seidel Method

Why?

The Gauss-Seidel Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and LU Decomposition are prone to prone to round-off error.

Also: If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss-Seidel Method

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss-Seidel Method

Algorithm

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ ← From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ ← From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$ ← From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ ← From equation n

Gauss-Seidel Method

Algorithm

General Form of each equation

$$x_1 = \frac{c_1 - \sum\limits_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum\limits_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_2 = \frac{c_2 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_n = \frac{c_n - \sum\limits_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

Gauss-Seidel Method

Algorithm

General Form for any row ‘i’

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

How or where can this equation be used?

Gauss-Seidel Method

Solve for the unknowns

Assume an initial guess for [X]

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Use rewritten equations to solve for each value of x_i.

Important: Remember to use the most recent value of x_i. Which means to apply values calculated to the calculations remaining in the **current** iteration.

Gauss-Seidel Method

Calculate the Absolute Relative Approximate Error

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a prespecified tolerance for all unknowns.

Gauss-Seidel Method: Example 1

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. Time data.

Time, t (s)	**Velocity** v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \, , 5 \le \mathrm{t} \le 12.$$

Gauss-Seidel Method: Example 1

Using a Matrix template of the form

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

The system of equations becomes

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Initial Guess: Assume an initial guess of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Gauss-Seidel Method: Example 1

Rewriting each equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Gauss-Seidel Method: Example 1

Applying the initial guess and solving for a_i

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Initial Guess

$$a_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510$$

$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36$$

When solving for a_2, how many of the initial guess values were used?

Gauss-Seidel Method: Example 1

Finding the absolute relative approximate error

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\in_{\mathrm{a}}\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| x100 = 72.76\%$$

$$\left|\in_{\mathrm{a}}\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| x100 = 125.47\%$$

$$\left|\in_{\mathrm{a}}\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| x100 = 103.22\%$$

At the end of the first iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

The maximum absolute relative approximate error is 125.47%

Gauss-Seidel Method: Example 1

Using

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

from iteration #1

Iteration #2

the values of a_i are found:

$$a_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$

$$a_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$

$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

Gauss-Seidel Method: Example 1

Finding the absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| x100 = 69.543\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| x100 = 85.695\%$$

$$\left|\in_a\right|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| x100 = 80.540\%$$

At the end of the second iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

The maximum absolute relative approximate error is 85.695%

Gauss-Seidel Method: Example 1

Repeating more iterations, the following values are obtained

Iteration	a_1	$\lvert\in_a\rvert_1 \%$	a_2	$\lvert\in_a\rvert_2 \%$	a_3	$\lvert\in_a\rvert_3 \%$
1	3.6720	72.767	−7.8510	125.47	−155.36	103.22
2	12.056	69.543	−54.882	85.695	−798.34	80.540
3	47.182	74.447	−255.51	78.521	−3448.9	76.852
4	193.33	75.595	−1093.4	76.632	−14440	76.116
5	800.53	75.850	−4577.2	76.112	−60072	75.963
6	3322.6	75.906	−19049	75.972	−249580	75.931

Notice – The relative errors are not decreasing at any significant rate

Also, the solution is not converging to the true solution of $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.690 \\ 1.0857 \end{bmatrix}$

Gauss-Seidel Method: Pitfall

What went wrong?

Even though done correctly, the answer is not converging to the correct answer

This example illustrates a pitfall of the Gauss-Siedel method: not all systems of equations will converge.

Is there a fix?

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for all 'i'} \qquad \text{and} \quad |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for at least one 'i'}$$

Gauss-Seidel Method: Pitfall

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Which coefficient matrix is diagonally dominant?

$$[A]=\begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix}$$

$$[B]=\begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.

Gauss-Seidel Method: Example 2

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

The coefficient matrix is:

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Will the solution converge using the Gauss-Siedel method?

Gauss-Seidel Method: Example 2

Checking if the coefficient matrix is diagonally dominant

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true and at least one row is *strictly* greater than:

Therefore: The solution should converge using the Gauss-Siedel Method

Gauss-Seidel Method: Example 2

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Gauss-Seidel Method: Example 2

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000 - 1.0000}{0.50000}\right| \times 100 = 100.00\%$$

$$\left|\in_\mathrm{a}\right|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_\mathrm{a}\right|_3 = \left|\frac{3.0923 - 1.0000}{3.0923}\right| \times 100 = 67.662\%$$

The maximum absolute relative error after the first iteration is 100%

Gauss-Seidel Method: Example 2

After Iteration #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

Substituting the x values into the equations

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76 - 3(0.14679) - 7(4.900)}{13} = 3.8118$$

After Iteration #2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Gauss-Seidel Method: Example 2

Iteration #2 absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.61\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.889\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.874\%$$

The maximum absolute relative error after the first iteration is 240.61%

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?

Gauss-Seidel Method: Example 2

Repeating more iterations, the following values are obtained

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.876
3	0.74275	80.236	3.1644	17.408	3.9708	4.0042
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$ is close to the exact solution of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

Gauss-Seidel Method: Example 3

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Rewriting the equations

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Gauss-Seidel Method: Example 3

Conducting six iterations, the following values are obtained

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	A_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	−196.15	110.71	14.421	94.453	−462.30	110.96
3	−1995.0	109.83	−116.02	112.43	4718.1	109.80
4	−20149	109.90	1204.6	109.63	−47636	109.90
5	2.0364×10^5	109.89	−12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^5	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

The values are not converging.

Does this mean that the Gauss-Seidel method cannot be used?

Gauss-Seidel Method

The Gauss-Seidel Method can still be used

The coefficient matrix is not diagonally dominant

$$[A] = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

But this is the same set of equations used in example #2, which did converge.

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

If a system of linear equations is not diagonally dominant, check to see if rearranging the equations can form a diagonally dominant matrix.

Gauss-Seidel Method

Not every system of equations can be rearranged to have a diagonally dominant coefficient matrix.

Observe the set of equations

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 3x_2 + 4x_3 = 9$$

$$x_1 + 7x_2 + x_3 = 9$$

Which equation(s) prevents this set of equation from having a diagonally dominant coefficient matrix?

Gauss-Seidel Method

Summary

-Advantages of the Gauss-Seidel Method

-Algorithm for the Gauss-Seidel Method

-Pitfalls of the Gauss-Seidel Method

LU Decomposition Method

LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

LU Decomposition

Method

For most non-singular matrix [A] that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

How does LU Decomposition work?

If solving a set of linear equations	$[A][X] = [C]$
If $[A] = [L][U]$ then	$[L][U][X] = [C]$
Multiply by	$[L]^{-1}$
Which gives	$[L]^{-1}[L][U][X] = [L]^{-1}[C]$
Remember $[L]^{-1}[L] = [I]$ which leads to	$[I][U][X] = [L]^{-1}[C]$
Now, if $[I][U] = [U]$ then	$[U][X] = [L]^{-1}[C]$
Now, let	$[L]^{-1}[C]=[Z]$
Which ends with	$[L][Z] = [C] \quad (1)$
and	$[U][X] = [Z] \quad (2)$

LU Decomposition

How can this be used?

Given $[A][X] = [C]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [C]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

Is LU Decomposition better than Gaussian Elimination?

Solve $[A][X] = [B]$

T = clock cycle time and nxn = size of the matrix

Forward Elimination

$$CT|_{FE} = T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right)$$

Decomposition to LU

$$CT|_{DE} = T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right)$$

Back Substitution

$$CT|_{BS} = T\left(4n^2 + 12n\right)$$

Forward Substitution

$$CT|_{FS} = T\left(4n^2 - 4n\right)$$

Back Substitution

$$CT|_{BS} = T\left(4n^2 + 12n\right)$$

Is LU Decomposition better than Gaussian Elimination?

To solve $[A][X] = [B]$

Time taken by methods

Gaussian Elimination	LU Decomposition
$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$	$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$

T = clock cycle time and nxn = size of the matrix

So both methods are equally efficient.

To find inverse of [A]

Time taken by Gaussian Elimination

$$= n\left(CT|_{FE} + CT|_{BS}\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{DE} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right)$$

To find inverse of [A]

Time taken by Gaussian Elimination

$$T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right)$$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT\|_{\text{inverse GE}} / CT\|_{\text{inverse LU}}$	3.288	25.84	250.8	2501

For large n, $CT|_{\text{inverse GE}} / CT|_{\text{inverse LU}} \approx n/4$

Method: [A] Decomposes to [L] and [U]

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step.

[*L*] is obtained using the *multipliers* that were used in the forward elimination process

Finding the [U] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

Matrix after Step 1: $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Step 2: $\dfrac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$\ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = \ ?$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the [*L*] and [*U*] matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 10$$

$$2.56z_1 + z_2 = 177.2$$

$$5.76z_1 + 3.5z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for $[Z]$

$$
\begin{aligned}
z_1 &= 106.8 \\
z_2 &= 177.2 - 2.56 z_1 \\
&= 177.2 - 2.56(106.8) \\
&= -96.2 \\
z_3 &= 279.2 - 5.76 z_1 - 3.5 z_2 \\
&= 279.2 - 5.76(106.8) - 3.5(-96.21) \\
&= 0.735
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}
$$

Example

Set $[U][X] = [Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$0.7a_3 = 0.735$$

Example

From the 3rd equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example

Substituting in a_3 and a_2 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$\begin{aligned} a_1 &= \frac{106.8 - 5a_2 - a_3}{25} \\ &= \frac{106.8 - 5(19.70) - 1.050}{25} \\ &= 0.2900 \end{aligned}$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Finding the inverse of a square matrix

The inverse [B] of a square matrix [*A*] is defined as

$$[A][B] = [I] = [B][A]$$

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of [B] to be $[b_{11}\ \ b_{12}\ \ \ldots\ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of [B]

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of [B]

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in [B] can be found in the same manner

Example: Inverse of a Matrix

Find the inverse of a square matrix [*A*]

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the [*L*] and [*U*] matrices are found to be

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving for the each column of [*B*] requires two steps

1) Solve [*L*] [*Z*] = [*C*] for [*Z*]
2) Solve [*U*] [*X*] = [*Z*] for [*X*]

Step 1: $[L][Z]=[C]\rightarrow\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$

This generates the equations:

$$z_1=1$$

$$2.56z_1+z_2=0$$

$$5.76z_1+3.5z_2+z_3=0$$

Example: Inverse of a Matrix

Solving for [Z]

$$
\begin{aligned}
z_1 &= 1 \\
z_2 &= 0 - 2.56 z_1 \\
&= 0 - 2.56(1) \\
&= -2.56 \\
z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\
&= 0 - 5.76(1) - 3.5(-2.56) \\
&= 3.2
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}
$$

Example: Inverse of a Matrix

Solving [*U*][*X*] = [*Z*] for [*X*]

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$

Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of [*A*] is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix

The inverse of [*A*] is

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Cholesky and LDL^T Decomposition Method

Introduction

$$[A][x]=[b] \qquad (1)$$

where

$[A]$ = known coefficient matrix, with dimension $n \times n$

$[b]$ = known right-hand-side (RHS) $n \times 1$ vector

$[x]$ = unknown $n \times 1$ vector.

Symmetrical Positive Definite (SPD) SLE

A matrix $[A]_{n\times n}$ can be considered as SPD if either of the following conditions is satisfied:

(a) If each and every determinant of sub-matrix $A_{ii}(i=1,2,\ldots,n)$ is positive, or..

(b) If $y^T Ay > 0$, for any given vector $[y]_{n\times 1} \neq \vec{0}$

As a quick example, let us make a test a test to see if the given matrix

$$[A]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$ is SPD?

Symmetrical Positive Definite (SPD) SLE

Based on criteria (a):

The given 3×3 matrix is symmetrical, because

$$a_{ij} = a_{ji}$$

Furthermore,

$$\det\left|[A]_{1\times 1}\right| = |2| = 2 > 0$$

$$\det\left|[A]_{2\times 2}\right| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 3 > 0$$

$$\det\left|[A]_{3\times 3}\right| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 1 > 0$$

Hence is $[A]$ SPD.

<u>Based on criteria (b):</u> For any given vector

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \neq \vec{0}$$, one computes

$$\begin{aligned}
scalar &= y^T A y \\
&= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \left(2y_1^2 - 2y_1y_2 + 2y_2^2\right) + \left\{y_3^2 - 2y_2y_3\right\} \\
&= \left(y_1 - y_2\right)^2 + y_1^2 + y_2^2 + \left\{y_3^2 - 2y_2y_3\right\}
\end{aligned}$$

$$scalar = (y_1 - y_2)^2 + y_1^2 + (y_2 - y_3)^2 > 0$$

hence matrix is $[A]$ SPD

Step 1: Matrix Factorization phase

$$[A] = [U]^T [U] \tag{2}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{3}$$

Multiplying two matrices on the right-hand-side (RHS) of Equation (3), one gets the following 6 equations

$$u_{11} = \sqrt{a_{11}} \qquad u_{12} = \frac{a_{12}}{u_{11}} \qquad u_{13} = \frac{a_{13}}{u_{11}} \tag{4}$$

$$u_{22} = \left(a_{22} - u_{12}^2\right)^{\frac{1}{2}} \quad u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} \quad u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{\frac{1}{2}} \tag{5}$$

$$u_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} (u_{ki})^2 \right)^{\frac{1}{2}} \qquad (6)$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}} \qquad (7)$$

Step 1.1: Compute the numerator of Equation (7), such as

$$Sum = a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}$$

Step 1.2 If u_{ij} is an off-diagonal term (say $i < j$) then $u_{ij} = \frac{Sum}{u_{ii}}$ (See Equation (7)). Else, if u_{ij} is a diagonal term (that is, $i = j$), then $u_{ii} = \sqrt{Sum}$ (See Equation (6))

As a quick example, one computes:

$$u_{57} = \frac{a_{57} - u_{15}u_{17} - u_{25}u_{27} - u_{35}u_{37} - u_{45}u_{47}}{u_{55}} \quad (8)$$

Thus, for computing $u(i = 5, j = 7)$, one only needs to use the (already factorized) data in columns $\#i(= 5)$ and $\# j(= 7)$ of $[U]$ respectively.

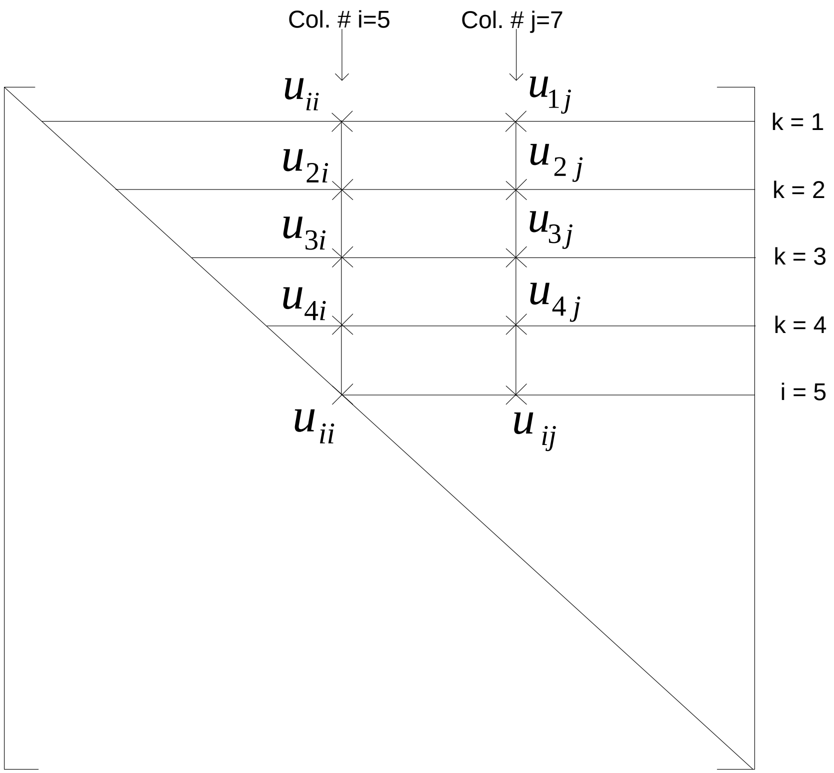

Figure 1: Cholesky Factorization for the term u_{ij}

Step 2: Forward Solution phase

Substituting Equation (2) into Equation (1), one gets:

$$[U]^T[U][x]=[b] \tag{9}$$

Let us define:

$$[U][x]\equiv[y] \tag{10}$$

Then, Equation (9) becomes:

$$[U]^T\,[y]=[b] \tag{11}$$

$$\begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \tag{12}$$

$$u_{11}y_1 = b_1$$

$$y_1 = \frac{b_1}{u_{11}} \tag{13}$$

From the 2nd row of Equation (12), one gets

$$u_{12}y_1 + u_{22}y_2 = b_2$$

$$y_2 = b_2 - \frac{u_{12}y_1}{u_{22}} \tag{14}$$

Similarly

$$y_3 = \frac{b_3 - u_{13}y_1 - u_{23}y_2}{u_{33}} \tag{15}$$

In general, from the j^{th} row of Equation (12), one has

$$y_j = \frac{b_j - \sum_{i=1}^{j-1} u_{ij} y_i}{u_{jj}} \qquad (16)$$

Step 3: Backward Solution phase

As a quick example, one has (See Equation (10)):

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{17}$$

From the last (or $n^{th} = 3^{rd}$) row of Equation (17),

one has $$u_{33}x_3 = y_3$$

hence
$$x_3 = \frac{y_3}{u_{33}} \tag{18}$$

Similarly:
$$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}} \tag{19}$$

and
$$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}} \tag{20}$$

In general, one has:

$$x_j = \frac{y_j - \sum_{i=j+1}^{n} u_{ji} x_i}{u_{jj}} \tag{21}$$

$$[A]=[L][D][L]^T \tag{22}$$

For example,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Multiplying the three matrices on the RHS of Equation (23), one obtains the following formulas for the “diagonal” $[D]$, and “lower-triangular” $[L]$ matrices:

$$d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 d_{kk} \tag{24}$$

$$l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk} \right) \times \left(\frac{1}{d_{jj}} \right) \tag{25}$$

Step1: Factorization phase

$$[A]=[L][D][L]^T \qquad \text{(22, repeated)}$$

Step 2: Forward solution and diagonal scaling phase

Substituting Equation (22) into Equation (1), one gets:

$$[L][D][L]^T[x]=[b] \qquad (26)$$

Let us define:

$$[L]^T[x]=[y]$$

Also, define: $[D][y]=[z]$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad (29)$$

$$y_i=\frac{z_i}{d_{ii}},\ for\ i=1,2,3,\ldots\ldots,n \qquad (30)$$

Then Equation (26) becomes:

$$[L][z]=[b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{31}$$

$$z_i = b_i - \sum_{k=1}^{i-1} L_{ik} z_k \quad for\ i = 1,2,3,\ldots\ldots,n \tag{32}$$

Step 3: Backward solution phase

$$\begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_i = y_i - \sum_{k=i+1}^{n} l_{ki} x_k ; \; for \; i = n, n-1, \ldots\ldots, 1$$

Numerical Example 1
(Cholesky algorithms)

Solve the following SLE system for the unknown vector $[x]$?

$$[A][x]=[b]$$

where

$$[A]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[b]=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

The factorized, $[U]$ upper triangular matrix can be computed by either referring to Equations (6-7), or looking at Figure 1, as following:

$$\left.\begin{aligned}
u_{11} &= \sqrt{a_{11}} \\
&= \sqrt{2} \\
&= 1.414 \\
u_{12} &= \frac{a_{12}}{u_{11}} \\
&= \frac{-1}{1.414} \\
&= -0.7071 \\
u_{13} &= \frac{a_{13}}{u_{11}} \\
&= \frac{0}{1.414} \\
&= 0
\end{aligned}\right\} row\,1\,of\ [U]$$

$$\left.\begin{aligned}
u_{22} &= \left\{ a_{22} - \sum_{k=1}^{i-1=1} (u_{ki})^2 \right\}^{\frac{1}{2}} \\
&= \left\{ 2 - (u_{12})^2 \right\}^{\frac{1}{2}} \\
&= \sqrt{2 - (-0.7071)^2} \\
&= 1.225 \\
u_{23} &= \frac{a_{23} - \sum_{k=1}^{i-1=1} u_{ki} u_{kj}}{U_{22}} \\
&= \frac{-1 - u_{12} \times u_{13}}{1.225} \\
&= \frac{-1 - (-0.7071)(0)}{1.225} \\
&= -0.8165
\end{aligned}\right\} row\ 2\ of\ [U]$$

$$\left.\begin{aligned}
u_{33} &= \left\{a_{33} - \sum_{k=1}^{i-1=2} \left(u_{ki}\right)^2\right\}^{\frac{1}{2}} \\
&= \left\{a_{33} - u_{13}^2 - u_{23}^2\right\}^{\frac{1}{2}} \\
&= \sqrt{1 - (0)^2 - (-0.8165)^2} \\
&= 0.5774
\end{aligned}\right\} row\,3\,of\;[U]$$

Thus, the factorized matrix

$$[U] = \begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

The forward solution phase, shown in Equation (11), becomes:

$$[U]^T[y]=[b]$$

$$\begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y_1 = \frac{b_1}{u_{11}}$$

$$= \frac{1}{1.414}$$

$$= 0.7071$$

$$y_2 = \frac{b_2 - \sum_{i=1}^{j-1=1} u_{ij} y_i}{u_{jj}}$$

$$= \frac{0 - (u_{12} = -0.7071)(y_1 = 0.7071)}{(u_{22} = 1.225)}$$

$$= 0.4082$$

$$y_3 = \frac{b_3 - \sum_{i=1}^{j-1=2} u_{ij} y_i}{u_{jj}}$$

$$= \frac{0 - (u_{13} = 0)(y_1 = 0.7071) - (u_{23} = -0.8165)(y_2 = 0.4082)}{(u_{33} = 0.5774)}$$

$$= 0.5774$$

The backward solution phase, shown in Equation (10), becomes:

$$[U][x]=[y]$$

$$\begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

$$x_3 = \frac{y_j}{u_{jj}}$$

$$= \frac{y_3}{u_{33}}$$

$$= \frac{0.5774}{0.5774}$$

$$= 1$$

$$x_2 = \frac{y_j - \sum_{i=j+1=3}^{N=3} u_{ji} x_i}{u_{jj}}$$

$$= \frac{y_2 - u_{23} x_3}{u_{22}}$$

$$= \frac{0.4082 - (-0.8165)(1)}{1.225}$$

$$= 1$$

$$x_1 = \frac{y_j - \sum\limits_{i=j+1=2}^{N=3} u_{ji} x_i}{u_{jj}}$$

$$= \frac{y_1 - u_{12} x_2 - u_{13} x_3}{u_{11}}$$

$$= \frac{0.7071 - (-0.7071)(1) - (0)(1)}{1.414}$$

$$= 1$$

Hence

$$[x] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Numerical Example 2

LDL^T (Algorithms)

Using the same data given in Numerical Example 1, find the unknown vector $[x]$ by LDL^T algorithms?

Solution:

The factorized matrices $[D]$ and $[L]$ can be computed from Equation (24), and Equation (25), respectively.

$$\left.\begin{aligned}
d_{11} &= a_{11} - \sum_{k=1}^{j-1=0} l_{jk}^{2} d_{kk} \\
&= a_{11} \\
&= 2 \\
l_{11} &= 1\,(always!) \\
l_{21} &= \frac{a_{21} - \sum_{k=1}^{j-1=0} l_{ik} d_{kk} l_{jk}}{d_{jj}} \\
&= \frac{a_{21}}{d_{11}} \\
&= \frac{-1}{2} \\
&= -0.5 \\
l_{31} &= \frac{a_{31}}{d_{11}} \\
&= \frac{0}{2} \\
&= 0
\end{aligned}\right\} Column\,1\,of\ matrices\,of\,[D]\,and\,[L]$$

$$\left.\begin{aligned}
d_{22} &= a_{22} - \sum_{k=1}^{j-1=1} l_{jk}^2 d_{kk} \\
&= 2 - l_{21}^2 d_{11} \\
&= 2 - (-0.5)^2(2) \\
&= 1.5 \\
l_{22} &= 1\,(always!) \\
l_{32} &= \frac{a_{32} - \sum_{k=1}^{j-1=1} l_{31} d_{11} l_{21}}{d_{22}} \\
&= \frac{-1-(0)(2)(-0.5)}{1.5} \\
&= -0.6667
\end{aligned}\right\} Column\,2\,of\;\;matrices\,[D]\,and\,[L]$$

$$\left.\begin{aligned} d_{33} &= a_{33} - \sum_{k=1}^{j-1=2} l_{jk}^2 d_{kk} \\ &= 1 - l_{31}^2 d_{11} - l_{32}^2 d_{22} \\ &= 1 - (0)^2(2) - (-0.6667)^2(1.5) \\ &= 0.3333 \end{aligned}\right\} Column\, 3\, of\;\; matrices\, [D]\, and\, [L]$$

Hence

$$[D]=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

and

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.6667 & 1 \end{bmatrix}$$

The forward solution shown in Equation (31), becomes:

$$[L][z]=[b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.667 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

or,

$$z_i = b_i - \sum_{k=1}^{i-1} l_{ik} z_k \qquad (32, \text{ repeated})$$

Hence

$$
\begin{aligned}
z_1 &= b_1 = 1 \\
z_2 &= b_2 - L_{21} z_1 \\
&= 0 - (-0.5)(1) \\
&= 0.5 \\
z_3 &= b_3 - L_{31} z_1 - L_{32} z_2 \\
&= 0 - (0)(1) - (-0.6667)(0.5) \\
&= 0.3333
\end{aligned}
$$

The diagonal scaling phase, shown in Equation (29), becomes

$$[D][y]=[z]$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0.5 \\ 0.3333 \end{bmatrix}$$

or

$$y_i = \frac{z_i}{d_{ii}}$$

Hence

$$y_1 = \frac{z_1}{d_{11}} = \frac{1}{2} = 0.5$$

$$y_2 = \frac{z_2}{d_{22}} = \frac{0.5}{1.5} = 0.3333$$

$$y_3 = \frac{z_3}{d_{33}} = \frac{0.3333}{0.3333} = 1$$

The backward solution phase can be found by referring to Equation (27)

$$[L]^T[x] = [y]$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.667 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.333 \\ 1 \end{bmatrix}$$

$$x_i = y_i - \sum_{k=i+1}^{N} l_{ki} x_k \qquad \text{(28, repeated)}$$

Hence

$$x_3 = y_3$$
$$= 1$$

$$x_2 = y_2 - l_{32} x_3$$
$$= 0.3333 - (-0.6667) \times 1$$

$$x_2 = 1$$

$$x_1 = y_1 - l_{21} x_2 - l_{31} x_3$$
$$x_1 = 0.5 - (-0.5)(1) - (0)(1)$$
$$= 1$$

Hence

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Re-ordering Algorithms For Minimizing Fill-in Terms [1,2].

During the factorization phase (of Cholesky, or LDL^T Algorithms), many "zero" terms in the original/given $[A]$ matrix will become "non-zero" terms in the factored matrix $[U]$. These new non-zero terms are often called as "fill-in" terms (indicated by the symbol F) It is, therefore, highly desirable to minimize these fill-in terms , so that both computational time/effort and computer memory requirements can be substantially reduced.

For example, the following matrix $[A]$ and vector $[b]$ are given:

$$[A]=\begin{bmatrix} 112 & 7 & 0 & 0 & 0 & 2 \\ 7 & 110 & 5 & 4 & 3 & 0 \\ 0 & 5 & 88 & 0 & 0 & 1 \\ 0 & 4 & 0 & 66 & 0 & 0 \\ 0 & 3 & 0 & 0 & 44 & 0 \\ 2 & 0 & 1 & 0 & 0 & 11 \end{bmatrix} \quad (33)$$

$$[b]=\begin{bmatrix} 121 \\ 129 \\ 94 \\ 70 \\ 47 \\ 14 \end{bmatrix} \quad (34)$$

The Cholesky factorization matrix $[U]$, based on the original matrix $[A]$ (see Equation 33) and Equations (6-7), or Figure 1, can be symbolically computed as:

$$[U]=\begin{bmatrix} \times & \times & 0 & 0 & 0 & \times \\ 0 & \times & \times & \times & \times & F \\ 0 & 0 & \times & F & F & \times \\ 0 & 0 & 0 & \times & F & F \\ 0 & 0 & 0 & 0 & \times & F \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \tag{35}$$

IPERM (<u>new</u> equation #) = {<u>old</u> equation #} (36)

such as, for this particular example:

$$IPERM\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}=\begin{bmatrix}6\\5\\4\\3\\2\\1\end{bmatrix} \tag{37}$$

Using the above results (see Equation 37), one will be able to construct the following re-arranged matrices:

$$[A^*] = \begin{bmatrix} 11 & 0 & 0 & 1 & 0 & 2 \\ 0 & 44 & 0 & 0 & 3 & 0 \\ 0 & 0 & 66 & 0 & 4 & 0 \\ 1 & 0 & 0 & 88 & 5 & 0 \\ 0 & 3 & 4 & 5 & 110 & 7 \\ 2 & 0 & 0 & 0 & 7 & 112 \end{bmatrix} \quad (38)$$

and $[b^*] = \begin{bmatrix} 14 \\ 47 \\ 70 \\ 94 \\ 129 \\ 121 \end{bmatrix}$ (39)

Remarks:

In the original matrix A (shown in Equation 33), the nonzero term A^* (old row 1, old column 2) = 7 will move to new location of the new matrix A^* (new row 6, new column 5) = 7, etc.

The non zero term A (old row 3, old column 3) = 88 will move to A^* (new row 4, new column 4) = 88, etc.

The value of b (old row 4) = 70 will be moved to (or located at) b^* (new row 3) = 70, etc

Now, one would like to solve the following modified system of linear equations (SLE) for $[x^*]$,

$$[A^*][x^*]=[b^*] \quad (40)$$

rather than to solve the original SLE (see Equation1). The original unknown vector $\{x\}$ can be easily recovered from $[x^*]$ and $[IPERM]$, shown in Equation (37).

The factorized matrix $[U^*]$ can be "symbolically" computed from $[A^*]$ as (by referring to either Figure 1, or Equations 6-7):

$$[U^*] = \begin{bmatrix} \times & 0 & 0 & \times & 0 & \times \\ 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 & \times & 0 \\ 0 & 0 & 0 & \times & \times & F \\ 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \tag{41}$$

4. On-Line Chess-Like Game For Reordering/Factorized Phase [4].

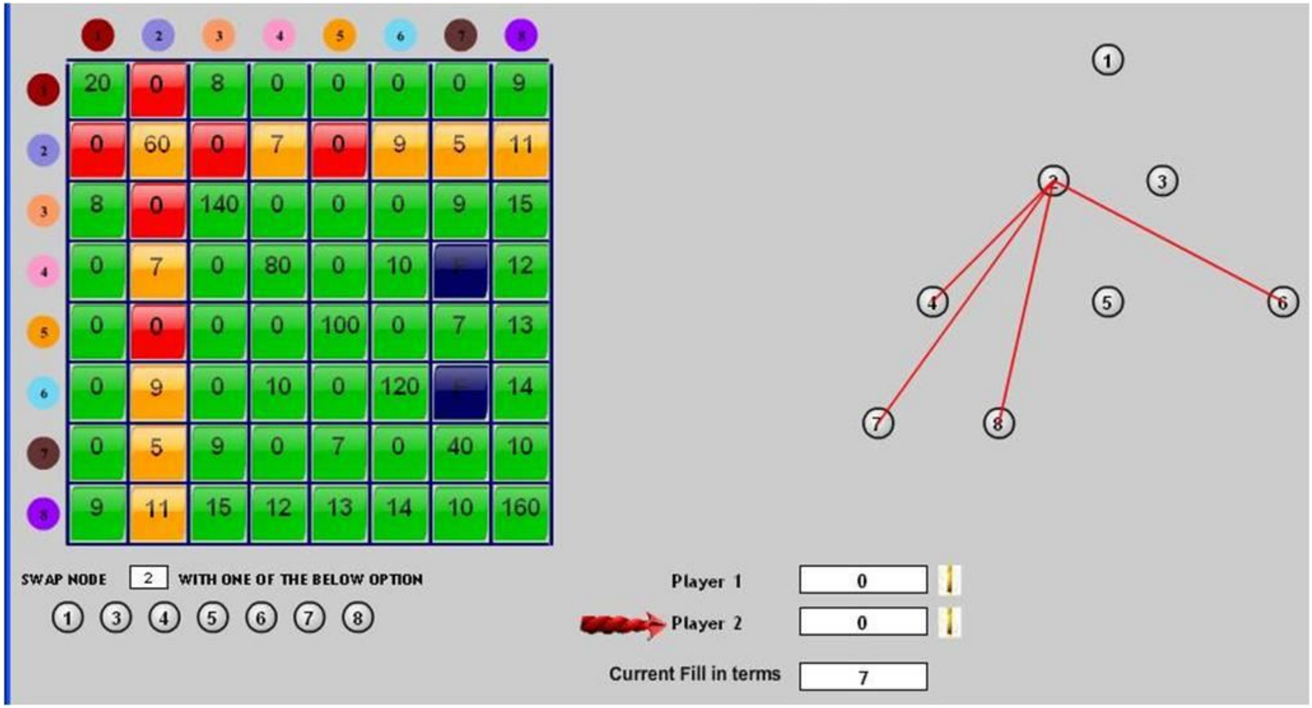

Figure 2 A Chess-Like Game For Learning to Solve SLE.

(A)Teaching undergraduate/HS students the process how to use the reordering output IPERM(-), see Equations (36-37) for converting the original/given matrix $[A]$, see Equation (33), into the new/modified matrix $[A^*]$, see Equation (38). This step is reflected in Figure 2, when the "Game Player" decides to swap node (or equation) $"i"$ (say $i = 2$) with another node (or equation $"j"$), and click the "CONFIRM" icon!

Since node $"i = 2"$ is currently connected to nodes $j = 4,6,7,8$; hence swapping node $i = 2$ with the above nodes $"j"$ will "NOT" change the number/pattern of "Fill-in" terms. However, if node $i = 2$ is swapped with node $j = 1, or\ 3, or\ 5,$ then the fill-in terms pattern may change (for better or worse)!

(B) Helping undergraduate/HS students to understand the “symbolic” factorization” phase, by symbolically utilizing the Cholesky factorized Equations (6-7). This step is illustrated in Figure 2, for which the “game player” will see (and also hear the computer animated sound, and human voice), the non-zero terms (including fill-in terms) of the original matrix $[A]$ to move to the new locations in the new/modified $[A^*]$ matrix .

(C) Helping undergraduate/HS students to understand the "numerical factorization" phase, by numerically utilizing the same Cholesky factorized Equations (6-7).

(D) Teaching undergraduate engineering/science students and even high-school (HS) students to "understand existing reordering concepts", or even to "discover new reordering algorithms"

5. Further Explanation On The Developed Game

1. In the above Chess-Like Game, which is available on-line [4], powerful features of FLASH computer environments [3], such as animated sound, human voice, motions, graphical colors etc… have all been incorporated and programmed into the developed game-software for more appealing to game players/learners.

2. In the developed “Chess-Like Game”, fictitious monetary (or any kind of ‘scoring system”) is rewarded (and broadcasted by computer animated human voice) to game players, based on how he/she swaps the node (or equation) numbers, and consequently based on how many fill-in terms occurred. In general, less fill-in $"F"$ terms introduced will result in more rewards!

3. Based on the original/given matrix $[A]$, and existing re-ordering algorithms (such as the Reverse Cuthill-Mckee, or RCM algorithms [1-2]) the number of fill-in terms $("F")$ can be computed (using RCM algorithms). This internally generated information will be used to judge how good the players/learners are, and/or broadcast “congratulations message” to a particular player who discovers new “chess-like move” (or, swapping node) strategies which are even better than RCM algorithms!

4. Initially, the player(s) will select the matrix size (8×8, or larger is recommended), and the percentage (50%, or larger is suggested) of zero-terms (or sparsity of the matrix). Then, “START Game” icon will be clicked by the player.

5. The player will then CLICK one of the selected node "i" (or equation) numbers appearing on the computer screen. The player will see those nodes "j" which ar connected to node "i" (based on the given/generated matrix $[A]$). The player then has to decide to swap node "i" with one of the possible node "j"

After confirming the player's decision, the outcomes/ results will be announced by the computer animated human voice, and the monetary-award will (or will NOT) be given to the players/learners, accordingly. In this software, a maximum of $1,000,000 can be earned by the player, and the "exact dollar amount" will be INVERSELY proportional to the number of fill-in terms occurred (as a consequence of the player's decision on how to swap node $"i"$ with another node $"j"$).

6. The next player will continue to play, with his/her move (meaning to swap the i^{th} node with the j^{th} node) based on the current best non-zero terms pattern of the matrix.

THE END