Week 1
CSIT 2024
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1. Probabilty

7. Prove that $P(A \cap B) \geq P(A) + P(B) - 1$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \geq 1 \quad (1)$$
$$1 \geq P(A) + P(B) - P(A \cap B) \quad (2)$$
$$P(A \cap B) \geq P(A) + P(B) - 1 \quad (3)$$

8. Prove $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

Through sheer force of will,

Consider the marginal $B_j = A_j - \bigcup_{i=1}^{i-1} A_i$

$$\begin{aligned} P(\bigcup_{i=1}^{n} A_i) &= P(\bigcup_{j=1}^{n} B_j) \\ &= \sum_{j=1}^{n} P(B_j) \\ &\leq \sum_{i=1}^{n} P(A_i) \end{aligned}$$

15. How many different meals can be made from 4 kinds of meat, 6 vegetables, and 3 starches if a meal consists of one selection from each group?

$4 * 6 * 3 = 72$

35. Prove the following 2 identities both algebraically and combinatorially

$\binom{n}{r} = \binom{n}{n-r}$

$$\begin{aligned} \binom{n}{r} &= \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(n-r)!r!} \\ &= \binom{n}{n-r} \end{aligned}$$

$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Consider an objeect that is either in the set or not in the set. If it is in the set, then there are $r-1$ objects left to choose from $n-1$ objects. If it is not in the set, then there are r objects left to choose from $n-1$ objects.

2. Random Variables

21. If X is a geometric RV, show that

$P(X > n + k - 1 | X > n - 1) = P(X > k)$

$$P(X = k) = (1-p)^{k-1}p$$

$$\begin{aligned} P(X > n+k-1 | X > n-1) &= \frac{\sum_{i=n+k}^{\infty} P(X=i)}{\sum_{i=n}^{\infty} P(X=i)} \\ &= 1 - \frac{\sum_{i=n}^{n+k-1}(1-p)^{i-1}p}{\sum_{i=n}^{\infty}(1-p)^{i-1}p} \\ &= 1 - \frac{\sum_{i=n}^{n+k-1}(1-p)^{i-1}}{\sum_{i=n}^{\infty}(1-p)^{i-1}} \\ &= 1 - \frac{(1-p)^{n-1}\sum_{i=0}^{k-1}(1-p)^{i}}{(1-p)^{n-1}\sum_{i=0}^{\infty}(1-p)^{i}} \\ &= 1 - \frac{\sum_{i=0}^{k-1}(1-p)^{i}}{\sum_{i=0}^{\infty}(1-p)^{i}} \\ &= 1 - \frac{\sum_{i=0}^{k-1}(1-p)^{i}}{\frac{1}{p}} \\ &= 1 - \sum_{i=0}^{k-1}(1-p)^{i}p \\ &= P(X > k) \end{aligned}$$

In a sequence of independent trials with probability p of success, what is the probability that there are r successes before the kth failure?

Consider the combination of k-1 failures and r successes. The last trial wil be the kth failure.

$$(1-p) * \binom{r+k-1}{k-1} * (1-p)^{k-1} * p^r = (1-p)^k * p^r * \binom{r+k-1}{k-1}$$