Functional Inference for Intraday Correlation Patterns*

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Abstract

We establish a functional central limit theorem for estimating the diurnal pattern of instantaneous correlations between two financial assets, using high frequency data over a long span of time. This is, to our knowledge, the first functional central limit theory ever built for such high frequency characteristics of asset returns correlations. An estimator of the asymptotic covariance operator is proposed, rendering the limit theorem feasible in a two-sample test for the equivalence of diurnal patterns of two non-overlapped time periods. Simulation evidence supports our theoretical findings, while empirical results on ??? show that ...

Keywords: instantaneous return correlation; intraday volatility curves; functional time series; high frequency data; Itô semimartingales.
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1 Introduction

Understanding the correlation between asset prices is of paramount importance in finance. Accurate predictions of intraday asset price correlations are critical for real-time portfolio management, risk management, and multi-asset derivatives pricing. Reliably measuring or predicting these correlations inevitably requires the assumption that intraday correlations remain stable over a certain period of time. Thus, formal functional inference procedures for intraday correlations are warranted so that the stability assumption can be statistically tested. This paper aims to devise a functional inference theory for the ubiquitous intraday pattern in asset price correlations, as documented in Section ???. As far as we are aware, this marks the inaugural development of a functional central limit theorem designed explicitly for studying intraday correlation patterns. The theorem naturally paves the way for devising a test to evaluate the equivalence between diurnal correlation patterns across two non-overlapping time periods.

Existing literature on dynamic conditional correlation models and inferences is mostly centered on discrete time models. These include early works such as the Constant Conditional Correlation (CCC) model by Bollerslev (1990), the Dynamic Conditional Correlation (DCC) models by Engle (2002) and Tse and Tsui (2002), the semi-generalized DCC (SGDCC) model by Hafner and Franses (2003)!!, the asymmetric DCC (ADCC) model by Cappiello et al. (2006), the Regime-Switching Conditional Correlation model by Pelletier (2006), the semi-parametric modelling approach by Hafner et al. (2006), the semi-parametric/nonparametric correlation models by Aslanidis and Casas (2013), and the smooth transition conditional correlation (STCC) model by Silvennoinen and Teräsvirta (2015). More recent progress is found in Saart and Xia (2021), where a functional time series approach is adopted to analyze conditional return correlations as functions of certain exogenous economic variables.

Despite the flexibility of discrete-time correlation models, continuous-time models are more appealing in capturing high-frequency characteristics of asset price correlations, such as intraday correlation patterns. There is a vast literature on the periodicity of intraday (or spot) volatilities, reflecting the critical role that understanding high-frequency characteristics of volatilities plays in financial econometrics. Wood et al. (1985), Dacorogna et al. (1993), Andersen and Bollerslev (1997), among others, have documented that intraday volatility in financial markets exhibits a distinct U-shaped pattern, peaking at the market open and close. Early studies on the estimation and tests pertaining to intraday volatility patterns include, e.g., Taylor and Xu (1997), Andersen et al. (2001), Boudt et al. (2011). More recent advancements in statistical inference of intraday volatility patterns can be found in, e.g., Christensen et al. (2018), Andersen et al. (2019) and Andersen et al. (2023).

In contrast to the extensive research on intraday volatility patterns, much less attention has been given to the intraday periodicity of asset price correlations. While studies on volatility have identified clear intraday patterns and established robust inference methods, the econometric theory concerning the fluctuation of asset price correlations throughout the trading day remains relatively underdeveloped. As documented in Section ???, pronounced intraday correlation patterns have been observed in most of the studied equities/equity indices from the Chinese financial market. Furthermore, these intraday correlation patterns visually demonstrate a certain degree of time variation. However, formally testing the invariance of intraday correlation patterns requires a functional inference theory. In this paper, we adopt a functional time series perspective to model correlation curve series while adhering to the continuous-time modeling framework widely used in high-frequency financial econometrics. A functional central limit theorem is formally established for the estimation of intraday correlation patterns. Estimating these patterns is challenging due to the latent nature of instantaneous volatility and correlation processes. Consequently, we begin by approximating these processes using high frequency asset returns. It is crucial to meticulously handle the errors introduced by these approximations in the subsequent development of our functional limit theory. To this end, we use a local time window with a shrinking size, encompassing a diverging number of returns, to approximate the spot volatilities and correlations. In fact, one could have used only a single pair of high frequency returns to construct a consistent estimator of the diurnal correlation pattern at a specific time of day. However, by allowing the number of observations in the local time window to diverge, our estimation strategy enhances efficiency. Importantly, this facilitates the development of a functional central limit theorem for correlation pattern estimation. Through the use of infill and long-span asymptotics, coupled with appropriate choices regarding the length of the local window, we demonstrate that the errors induced by the approximation of instantaneous volatilities and correlations are asymptotically negligible. As a result, our limit theorem arises as a consequence of a multivariate functional central limit theorem associated with scaled partial sums of centered individual latent volatility and covariation curve series, along with the functional delta method. The covariance function of the limiting Gaussian process is consistently estimated, rendering the developed functional central limit theorem feasible. A test for the equivalence of diurnal correlation patterns over two non-overlapping time periods is proposed.

We further elucidate our contribution by positioning the current paper within the context of existing literature on modeling correlation curves from a functional time series perspective. In this regard, the recent work by Saart and Xia (2021) stands out as particularly relevant to our study and, to our knowledge, remains the sole paper addressing asset return correlations from this perspective. However, Saart and Xia (2021) essentially model (conditional) correlations for the returns sampled at a *fixed* sampling frequency, such as five-minute intervals, as a *smooth deterministic* function of specific economic variables, including market return or volatility. Hence, their model is discrete-time in nature. The nonparametric Local Linear Regression Smoother, as proposed by Fan???@make sure that this IS the original work on LL regression@, is employed for estimating the smooth deterministic correlation functions of economic variables on a daily basis. The consistency of correlation curve estimators naturally hinges on infill asymptotics, where the number of observations within a trading day diverges. Unfortunately, this leads to a contradiction unless it is assumed that the correlation function is independent of the return sampling frequency. Such an assumption clearly deviates from the conventional modeling framework of high frequency financial econometrics. In contrast to Saart and Xia (2021), we adopt continuous-time diffusion models for the dynamics of asset prices, instantaneous volatility, and covariation processes. This choice ensures a well-defined target of interest and provides a clear interpretation of genuine *instantaneous* correlation between the logarithmic price processes of two assets. One might also consider whether the semi-parametric/nonparametric method proposed by Aslanidis and Casas (2013) can be applied to estimate intraday correlation curves, treating these curves as functions of time-of-day rather than functions of economic variables. However, the Local Linear Regression Smoothers employed by Aslanidis and Casas (2013) in estimating correlation curves would yield *smooth deterministic* functions in time of day. In contrast, our approach deals with random and nonsmooth correlation curves, which are time series of random functions in time of day. Importantly, the functional central limit theorem developed in this paper enables us to (at least partially) test for the stationarity of the series of correlation curves. This step is crucial before applying any stationarity-based inference procedure, such as the method proposed by Saart and Xia (2021).

The remainder of the paper is organized as follows. We introduce the model setup and assumptions in Section 2. Section 3 presents our generic estimation scheme and the associated functional central limit theorem. A consistent estimator for the limiting covariance function as well as the implementation details of a two-sample test are also provided. Sections 4 and 5 contain Monte Carlo simulations and an empirical study on a number of stocks and equity indices from the Chinese financial market, respectively. Section 6 concludes. All proofs are relegated to an Appendix.

2 Setup and Assumptions

All stochastic processes in this paper are defined on a common filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}(t)\}_{t\geq 0}, P)$. We are interested in the correlation between returns of two financial assets. Their log prices at time $t \geq 0$ are denoted by $X_1(t)$ and $X_2(t)$, and are governed by the following Itô processes,

$$\begin{cases} X_1(t) = X_1(0) + \int_0^t \mu_1(s)ds + \int_0^t \sigma_1(s)dW_1(s), \\ X_2(t) = X_2(0) + \int_0^t \mu_2(s)ds + \int_0^t \sigma_2(s)dW_2(s), \end{cases} \tag{1}$$

where $\mu_1(t)$, $\mu_2(t)$, $\sigma_1(t)$ and $\sigma_2(t)$ are adapted càdlàg processes; $W_1(t)$ and $W_2(t)$ are two standard Brownian motions with quadratic covariation $d[W_1, W_2](t) = \rho(t)dt$; $\rho(t)$ is a stochastic process that captures the time-varying instantaneous correlation between the two assets' return processes. In fact, for any $t \geq 0$ and small $\epsilon > 0$, we denote returns on the two assets over the short time interval $[t, t+\epsilon]$ as

$$r_{1,t}(\epsilon) = X_1(t+\epsilon) - X_1(t) \quad \text{and} \quad r_{2,t}(\epsilon) = X_2(t+\epsilon) - X_2(t).$$

It follows from simple calculations of, e.g., Shreve (2004), that, $\rho(t)$ has the following interpretation of instantaneous correlation between two log price processes:

$$\rho(t) = \lim_{\epsilon \downarrow 0} \frac{\mathrm{Cov}(r_{1,t}(\epsilon), r_{2,t}(\epsilon)|\mathcal{F}(t))}{\sqrt{\mathrm{Var}(r_{1,t}(\epsilon)|\mathcal{F}(t))\mathrm{Var}(r_{2,t}(\epsilon)|\mathcal{F}(t))}},$$

where $\mathrm{Cov}(\cdot,\cdot|\mathcal{F}(t))$ and $\mathrm{Var}(\cdot|\mathcal{F}(t))$ are the covariance and variance operators conditional on the information available up to time t.

We shall adopt multiplicative decompositions for volatility and correlation processes. That is, for $t \in [0, \infty)$ and $m = 1, 2$,

$$\sigma_m(t) := g_m(t - \lfloor t \rfloor)\tilde{\sigma}_m(t), \quad \text{and} \quad \rho(t) := g_\rho(t - \lfloor t \rfloor)\tilde{\rho}(t), \tag{2}$$

where $g_m(\kappa)$ and $g_\rho(\kappa)$ are bounded deterministic functions defined on $[0, 1]$, capturing the possible calendar effects present in volatility and correlation processes; $\tilde{\sigma}_m(t)$ and $\tilde{\rho}(t)$ are positive stationary processes. Additional comments are warranted regarding the structural form of volatility and correlation processes as outlined in (2). On the one hand, multiplicative decomposition models for volatility processes are widely used in financial econometrics literature due to the inherent positivity of volatilities. Similarly, the correlation process $\rho(t)$ always takes values in $(-1, 1)$ for all $t \geq 0$, this makes a multiplicative decomposition model more appealing than an alternative additive decomposition model in depicting intraday periodicity in instantaneous correlation curves. On the other hand, in line with the widely documented leverage effect, which highlights a negative correlation between an asset's return and its volatility (see, e.g., Bollerslev et al., 2006), one may anticipate that correlations between returns on two assets will demonstrate consistent signs over time as well. This is confirmed by the empirical results of @cite the asmb paper!@. Therefore, it is natural to impose the constraint that $\tilde{\rho}(t) \in (0,1)$ and $g_\rho(\kappa) \in (-1,1)$, i.e., the (random) stationary component $\tilde{\rho}(t)$ is a positive process and the (deterministic) periodic component $g_\rho(\kappa)$ determines the sign of correlations.

For notational convenience, we further define

$$\begin{cases} B(t) := \sigma_1(t)\sigma_2(t)\rho(t), \\ \tilde{B}(t) := \tilde{\sigma}_1(t)\tilde{\sigma}_2(t)\tilde{\rho}(t), \\ g_B(\kappa) := g_1(\kappa)g_2(\kappa)g_\rho(\kappa). \end{cases} \tag{3}$$

Since the quadratic covariation between $X_1(t)$ and $X_2(t)$ is given by $d[X_1, X_2](t) = B(t)dt$, we refer to $B(t)$ as the spot covariation process between the two asset price processes. $\tilde{B}(t)$ and $g_B(\kappa)$ are, respectively, the stationary stochastic and deterministic components of the spot covariation process $B(t)$. Furthermore, we assume that the stochastic components of volatility and covariation processes admit the following representations:

$$\begin{cases} \tilde{\sigma}_1^2(t) = \tilde{\sigma}_1^2(0) + \int_0^t \breve{\mu}_1(s)ds + \int_0^t \breve{\sigma}_1(s)d\breve{W}_1(s), \\ \tilde{\sigma}_2^2(t) = \tilde{\sigma}_2^2(0) + \int_0^t \breve{\mu}_2(s)ds + \int_0^t \breve{\sigma}_2(s)d\breve{W}_2(s), \\ \tilde{B}(t) = \tilde{B}(0) + \int_0^t \breve{\mu}_3(s)ds + \int_0^t \breve{\sigma}_3(s)d\breve{W}_3(s), \end{cases} \tag{4}$$

where for $m = 1,2,3$, $\breve{W}_m(s)$'s are standard Brownian motions, $\breve{\mu}_m$'s and $\breve{\sigma}_m$'s are adapted càdlàg processes. We do not impose any constraints on the relationship between $\breve{W}_1(t)$ and $\breve{W}_2(t)$, nor on the relationship between $(\breve{W}_1(t), \breve{W}_2(t))$ and $(W_1(t), W_2(t))$. Therefore, our modeling framework is highly flexible, allowing for leverage effect. Alternatively, one could specify an integral equation model for $\tilde{\rho}(t)$ directly instead of the one on the stochastic component of the spot covariation process $\tilde{B}(t) = \tilde{\sigma}_1(t)\tilde{\sigma}_2(t)\tilde{\rho}(t)$ in (4). The integral equation specification for $\tilde{B}(t)$ is then a straightforward consequence of Itô's formula and the integral equation specifications of the three individual processes. This latter modeling approach would result in an integral equation specification of the same type as that in (4) for $\tilde{B}(t)$. However, the former approach significantly simplifies the notation in the derivations of our limit theorems.

In this paper, we focus on establishing a functional central limit theorem for the estimation of *intraday correlation pattern* $f_\rho(\kappa)$ defined as

$$f_\rho(\kappa) := \frac{g_\rho(\kappa)}{\left(\int_0^1 g_\rho(\kappa)^2 d\kappa\right)^{1/2}}. \tag{5}$$

To this end, a series of technical assumptions are needed. We begin with the one concerning the existence of various moments of the drift and diffusion coefficients of price, volatility and correlation processes.

Assumption I (Moments of Various Drift and Diffusion Coefficients). *For* $m = 1, 2$ *and* $q = 1, 2, 3$, $\sup_{t\geq 0} Ee^{|\mu_m(t)|} + \sup_{t\geq 0} Ee^{|\sigma_m(t)|} + \sup_{t\geq 0} Ee^{|\rho(t)|} + \sup_{t\geq 0} Ee^{|\breve{\mu}_q(t)|} + \sup_{t\geq 0} Ee^{|\breve{\sigma}_q(t)|} < \infty$.

Our second assumption pertains to the stationarity and ergodicity of the volatility and correlation processes. Similar assumptions are made by Andersen, Su, Todorov, and Zhang (2023) in their analysis of volatility calendar effects.

Assumption II (Stationarity and Ergodicity). *The trivariate process* $Y(t)$ *with entries* $\tilde{\sigma}_1(t), \tilde{\sigma}_2(t)$, *and* $\tilde{\rho}(t)$ *given in* (2), *i.e.,*

$$Y(t) := (\tilde{\sigma}_1(t), \tilde{\sigma}_2(t), \tilde{\rho}(t)),$$

is stationary, ergodic and α*-mixing with coefficient* $\alpha_s = O(s^{-q-\iota})$ *for some* $q > 0$, *positive* *constant ι (which can be arbitrarily close to zero), where for $\mathscr{G}_t = \sigma(Y(u), u \le t)$, $\mathscr{G}^t = \sigma(Y(u), u \ge t)$ and $s > 0$, we denote,*

$$\alpha_s = \sup_{t \in \mathcal{T}} \sup \left\{ |P(A \cap B) - P(A)P(B)| : A \in \mathscr{G}_t, B \in \mathscr{G}^{t+s} \right\}.$$

The following technical condition on the drift coefficient of the stochastic component of the spot covariation process $\tilde{B}(t)$ is useful in establishing the tightness of $\zeta_{7,1}^{(3)}(\kappa)$???.

Condition DP. *The drift coefficient $\tilde{\mu}_3(t)$ of the stochastic component of the spot covariation process $\tilde{B}(t)$ satisfies*

$$\left| \mathrm{Cov} \left(\int_{i-1+r_1}^{i-1+r_2} \tilde{\mu}_3(t)dt, \int_{i+h-1+r_3}^{i+h-1+r_4} \tilde{\mu}_3(t)dt \right) \right| \le |r_1 - r_2| \, |r_3 - r_4| \, G(h),$$

for any integers $i \ge 1$ and $h \ge 0$, any real numbers $r_1, r_2, r_3, r_4 \in (0,1)$, and some deterministic function $G(h)$ that satisfies $\sum_{h=0}^{\infty} G(h) < \infty$.

If $\tilde{B}(t)$ follows the Cox-Ingersoll-Ross (CIR) model Cox et al. (1985a,b) @cite!!!@, then Condition DP is satisfied with $G(h) = \mathcal{C}_1 e^{-\mathcal{C}_2 h}$ for some constants $\mathcal{C}_1$ and $\mathcal{C}_2$ that depend on model parameters. Moreover, the following conditions on the deterministic functions that capture intraday patterns of volatility and correlation processes are needed.

Condition IP. *The deterministic periodic components of the volatility and correlation processes $\{g_m(\cdot)\}_{m=1,2}$ and $g_\rho(\cdot)$ are continuous on $[0,1]$, satisfying $g_1(0) = g_1(1)$, $g_2(0) = g_2(1)$, $g_\rho(0) = g_\rho(1)$ and, for any $t, s \in [0,1]$,*

$$|g_1(t) - g_1(s)| + |g_2(t) - g_2(s)| + |g_\rho(t) - g_\rho(s)| \le C|t - s|,$$

where C is a constant independent of t and s. Furthermore, for $m = 1, 2$, the deterministic *periodic components of the volatility and correlation processes satisfy* $\inf_{\kappa\in[0,1]} g_m(\kappa) \geq \underline{g_m} > 0$ *and* $\int_0^1 g_\rho^2(\kappa)d\kappa \geq \underline{g_\rho} > 0$ *for some constants* $\underline{g_m}$ *and* $\underline{g_\rho}$.

In this paper, we treat instantaneous volatility and correlation processes from a functional time series perspective, splitting volatility and correlation processes into consecutive segments. To be specific, for $m = 1, 2$ and $i \in \mathbb{N}$, we define sequences of (random) volatility and correlation curves, i.e., $\sigma_{m,i} = \{\sigma_{m,i}(\kappa)\}_{\kappa\in[0,1]}$ and $\rho_i = \{\rho_i(\kappa)\}_{\kappa\in[0,1]}$, as follows:

$$\sigma_{m,i} := \{\sigma_m(i-1+\kappa)\}_{\kappa\in[0,1]} \quad \text{and} \quad \rho_i := \{\rho(i-1+\kappa)\}_{\kappa\in[0,1]},$$

where the time interval $[i-1, i]$ refers to the ith trading day. For $i \in \mathbb{N}$, functional time series $\tilde{\sigma}_{1,i} = \{\tilde{\sigma}_{1,i}(\kappa)\}_{\kappa\in[0,1]}$, $\tilde{\sigma}_{2,i} = \{\tilde{\sigma}_{2,i}(\kappa)\}_{\kappa\in[0,1]}$, and $\tilde{\rho}_i = \{\tilde{\rho}_i(\kappa)\}_{\kappa\in[0,1]}$ can be defined similarly. Moreover, for $i \in \mathbb{N}$, we define the time series of random covariation curves $B_i = \{B_i(\kappa)\}_{\kappa\in[0,1]}$ between two asset price processes as follows:

$$B_i(\kappa) := B(i-1+\kappa) = \sigma_{1,i}(\kappa)\,\sigma_{2,i}(\kappa)\,\rho_i(\kappa) \quad \text{for} \quad \kappa \in [0,1].$$

The time series of the stochastic components of the covariation curves $(\tilde{B}_i)_{i\in\mathbb{N}}$ is generated by $\tilde{B}(t)$ in a similar manner to $(B_i)_{i\in\mathbb{N}}$. We treat these random curves as random elements in the Skorokhod space $D[0,1]$ of càdlàg functions defined on $[0,1]$. Under the condition specified in IP, $g_m(\kappa)$ for $m = 1, 2$, $g_\rho(\kappa)$, and $f_\rho(\kappa)$ are all continuous functions. Therefore, they are elements of $C[0,1] \subset D[0,1]$, where $C[0,1]$ denotes the space of all continuous functions on $[0,1]$. Throughout, $a_n \asymp b_n$ refers to $1/C \leq a_n/b_n \leq C$ for some universal constant C.

3 Main Results

We introduce our estimation scheme in Section 3.1. Functional central limit theorem for the intraday-correlation-pattern estimator is formally established under a joint infill and long-span asymptotics in Section 3.2. While Section 3.3 consistently estimates the limiting covariance function rendering the developed central limit theorem feasible, Section 3.4 provides a test for the equivalence of intraday correlation patterns over two non-overlapping time periods.

3.1 Estimation Scheme

We provide a nonparametric method for estimating the correlation calendar effect $f_\rho(\kappa)$ defined in (5). We shall see in what follows that the method adopted in this paper is a generalization of Tan et al. (2024) The estimation problem is challenging because the intraday volatility and correlation curves are not observable and have to be approximated first. Suppose that the bivariate log price process $(X_1(t), X_2(t))$ is discretely observed and observation times are equally spaced over $[0, N]$. To be specific, during each trading day $[i-1, i]$, for $i = 1, 2, ..., N$, we observe price processes at $n+1$ observation times $(t_{i,j})_{0\leq j\leq n}$ with $i - 1 \equiv t_{i,0} < t_{i,1} < \ldots < t_{i,j} < \ldots < t_{i,n} \equiv i$, and,

$$t_{i,j} \;=\; i - 1 + j/n \quad \text{ with } \Delta = t_{i,j} - t_{i,j-1} \;=\; 1/n, \qquad \text{for} \quad j = 1, 2, ..., n.$$

The high-frequency log returns on the two assets are defined as follows,

$$\Delta_{i,j}^n X_m \;:=\; X_m\left(t_{i,j}\right) - X_m\left(t_{i,j-1}\right), \quad \text{for } i = 1, 2, ..., N, \quad j = 1, 2, ..., n, \quad \text{and} \quad m = 1, 2.$$

The spot volatilities and correlations and hence intraday volatility and correlation curves are approximated using observations within a *shrinking* local window of some time-of-period $\kappa \in (0,1]$. Let ℓ be an integer-valued function that diverges with respect to n. We consider intervals of the form $[t_{i,j_\kappa-\ell}, t_{i,j_\kappa}]$ within $[i-1,i]$, where,

$$j_\kappa := \lfloor \kappa n \rfloor .$$

For $m = 1$,2, we introduce local spot covariation and volatility estimators as follows,

$$\widehat{B}_i(\kappa) := \frac{1}{\ell\Delta} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \Delta_{i,j}^n X_1 \Delta_{i,j}^n X_2, \quad \text{and} \quad \hat{\sigma}_{m,i}^2(\kappa) := \frac{1}{\ell\Delta} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \left(\Delta_{i,j}^n X_m\right)^2 .$$

Note that the above spot covariation and volatility estimators $\widehat{B}_i(\kappa)$ and $\hat{\sigma}_{m,i}^2(\kappa)$ are not readily well defined when $\kappa \in [0, \ell\Delta)$ for fixed n (i.e., when $j_\kappa < \ell$). This issue can be addressed automatically by adopting the following convention. That is, we let the difference operator $\Delta_{i,j}^n$ apply to $-\ell+1 \leq j \leq 0$, with $t_{i,j} = t_{i-1,n+j}$ for $-\ell \leq j \leq 0$ and $i \in \{2, \cdots, N\}$. When $i = 1$ and $\kappa \in [0, \ell\Delta)$, we simply set $\widehat{B}_1(\kappa) = \widehat{B}_1(\ell\Delta)$ and $\hat{\sigma}_{m,1}^2(\kappa) = \hat{\sigma}_{m,1}^2(\ell\Delta)$ for $m = 1, 2$.

Built on $\widehat{B}_i(\kappa)$ and $\hat{\sigma}_{m,i}^2(\kappa)$, the following statistic

$$\hat{\mathfrak{g}}_\rho(\kappa) := \frac{\frac{1}{N}\sum_{i=1}^N \widehat{B}_i(\kappa)}{\left(\frac{1}{N}\sum_{i=1}^N \hat{\sigma}_{1,i}^2(\kappa)\right)^{\frac{1}{2}} \left(\frac{1}{N}\sum_{i=1}^N \hat{\sigma}_{2,i}^2(\kappa)\right)^{\frac{1}{2}}}, \tag{6}$$

up to a constant, estimates the deterministic intraday periodic component $g_\rho(\kappa)$. The constant is given by

$$c_0 := \frac{E\left(\tilde{\sigma}_1(\kappa)\tilde{\sigma}_2(\kappa)\tilde{\rho}(\kappa)\right)}{\sqrt{E\tilde{\sigma}_1^2(\kappa)E\tilde{\sigma}_2^2(\kappa)}}. \tag{7}$$

Note that under Assumption II, c_0 is independent of κ. Our estimator for the correlation diurnal pattern function $f_\rho(\kappa)$ reads as follows,

$$\hat{f}_\rho(\kappa) := \frac{\hat{\mathfrak{g}}_\rho(\kappa)}{\hat{\eta}}, \tag{8}$$

where

$$\hat{\eta} := \sqrt{\Delta \sum_{j=1}^{n} \hat{\mathfrak{g}}_\rho(j\Delta)^2}. \tag{9}$$

The ratio-type estimator (8) disposes of the unknown constant c_0 in (7) automatically. This paper aims at developing a functional central limit theorem for the intraday correlation pattern estimator $\hat{f}_\rho(\kappa)$.

One readily sees that the local window $[t_{i,j_\kappa-\ell}, t_{i,j_\kappa}]$ used to approximate spot covariations and volatilities at the time-of-day $\kappa \in (0,1]$ has a shrinking size, while the number of observations within it is diverging. In contrast, Tan et al. (2024) only use a single observation immediately preceding a time-of-day in the construction of their estimator. The estimation scheme employed here nests that of Tan et al. (2024) as a special case. This generalization reduces the magnitudes of errors in the approximation of spot covariations and volatilities and hence facilitates the development of a functional central limit theorem.

3.2 Functional Central Limit Theorem

To present our functional central limit theorem for the intraday correlation pattern estimator $\hat{f}_\rho(\kappa)$, we need some additional notations. For $i = 1, 2, \cdots, N$ and $\kappa \in [0,1]$, we define

$$
\begin{cases}
A_i^{(m)}(\kappa) = \sigma_{m,i}^2(\kappa) - E\sigma_{m,i}^2(\kappa) \quad \text{for} \quad m = 1 \text{ or } 2; \\
A_i^{(3)}(\kappa) = B_i(\kappa) - EB_i(\kappa),
\end{cases} \tag{10}
$$

where the covariation curve series $B_i(\kappa)$ is defined at the end of Section 2. To describe the limiting distribution, for $\kappa, \kappa' \in [0, 1]$, we introduce the following matrix-valued covariance function $\boldsymbol{C}(\kappa, \kappa')$:

$$
\boldsymbol{C}(\kappa, \kappa') := (C_{i,j}(\kappa, \kappa'))_{1 \le i,j \le 3} \quad \text{with} \quad C_{i,j}(\kappa, \kappa') := \sum_{h=-\infty}^{\infty} \phi_h^{(i,j)}(\kappa, \kappa'), \tag{11}
$$

where, for $h \ge 0$,

$$
\phi_h^{(i,j)}(\kappa, \kappa') = \mathrm{Cov}\left(A_1^{(i)}(\kappa), A_{1+h}^{(j)}(\kappa')\right),
$$

and for $h < 0$, $\phi_h^{(i,j)}(\kappa, \kappa') = \phi_{-h}^{(j,i)}(\kappa', \kappa)$. We further define a Gaussian process as follows,

$$
\widetilde{\mathcal{T}}(\kappa) := \frac{1}{\sqrt{E\sigma_1^2(\kappa)E\sigma_2^2(\kappa)}}\mathcal{T}_3(\kappa) - \frac{EB(\kappa)}{2\sqrt{\left(E\sigma_1^2(\kappa)\right)^3 E\sigma_2^2(\kappa)}}\mathcal{T}_1(\kappa) - \frac{EB(\kappa)}{2\sqrt{E\sigma_1^2(\kappa)\left(E\sigma_2^2(\kappa)\right)^3}}\mathcal{T}_2(\kappa), \tag{12}
$$

where

$$
\boldsymbol{\mathcal{T}}(\kappa) := (\mathcal{T}_1(\kappa), \mathcal{T}_2(\kappa), \mathcal{T}_3(\kappa))' \tag{13}
$$

is a three-dimensional Gaussian process with mean zero and matrix-valued covariance function $\boldsymbol{C}(\kappa, \kappa')$. The limiting process is, up to a multiplicative random variable, given by the following Gaussian process:

$$\breve{\mathcal{T}}(\kappa) := \widetilde{\mathcal{T}}(\kappa) - \frac{\int_0^1 g_\rho(s)\widetilde{\mathcal{T}}(s)ds}{\sqrt{\int_0^1 g_\rho^2(s)ds}} f_\rho(\kappa). \tag{14}$$

We note that the covariance function of $\widetilde{\mathcal{T}}(\kappa)$ is

$$\begin{aligned}
C_{\widetilde{\mathcal{T}}}(\kappa,\kappa') :=& \frac{EB(\kappa)EB(\kappa')}{4\sqrt{\left(E\sigma_1^2(\kappa)\right)^3\left(E\sigma_1^2(\kappa')\right)^3 E\sigma_2^2(\kappa)E\sigma_2^2(\kappa')}} C_{1,1}(\kappa,\kappa') \\
&+ \frac{EB(\kappa)EB(\kappa')}{4\sqrt{\left(E\sigma_1^2(\kappa)\right)^3 E\sigma_1^2(\kappa')E\sigma_2^2(\kappa)\left(E\sigma_2^2(\kappa')\right)^3}} C_{1,2}(\kappa,\kappa') \\
&- \frac{EB(\kappa)}{2\sqrt{\left(E\sigma_1^2(\kappa)\right)^3 E\sigma_1^2(\kappa')E\sigma_2^2(\kappa)E\sigma_2^2(\kappa')}} C_{1,3}(\kappa,\kappa') \\
&+ \frac{EB(\kappa)EB(\kappa')}{4\sqrt{E\sigma_1^2(\kappa)\left(E\sigma_1^2(\kappa')\right)^3\left(E\sigma_2^2(\kappa)\right)^3 E\sigma_2^2(\kappa')}} C_{2,1}(\kappa,\kappa') \\
&+ \frac{EB(\kappa)EB(\kappa')}{4\sqrt{E\sigma_1^2(\kappa)E\sigma_1^2(\kappa')\left(E\sigma_2^2(\kappa)\right)^3\left(E\sigma_2^2(\kappa')\right)^3}} C_{2,2}(\kappa,\kappa') \\
&- \frac{EB(\kappa)}{2\sqrt{E\sigma_1^2(\kappa)E\sigma_1^2(\kappa')\left(E\sigma_2^2(\kappa)\right)^3 E\sigma_2^2(\kappa')}} C_{2,3}(\kappa,\kappa') \\
&- \frac{EB(\kappa)}{2\sqrt{E\sigma_1^2(\kappa)\left(E\sigma_1^2(\kappa')\right)^3 E\sigma_2^2(\kappa)E\sigma_2^2(\kappa')}} C_{3,1}(\kappa,\kappa') \\
&- \frac{EB(\kappa)}{2\sqrt{E\sigma_1^2(\kappa)E\sigma_1^2(\kappa')E\sigma_2^2(\kappa)\left(E\sigma_2^2(\kappa')\right)^3}} C_{3,2}(\kappa,\kappa') \\
&+ \frac{1}{\sqrt{E\sigma_1^2(\kappa)E\sigma_1^2(\kappa')E\sigma_2^2(\kappa)E\sigma_2^2(\kappa')}} C_{3,3}(\kappa,\kappa').
\end{aligned}$$

Accordingly, the covariance function of $\breve{\mathcal{T}}(\kappa)$ is defined as

$C_{\breve{\mathcal{T}}}(\kappa,\kappa')$

$$:= C_{\widetilde{\mathcal{T}}}(\kappa,\kappa') - \frac{f_\rho(\kappa)}{\sqrt{\int_0^1 g_\rho^2(s)ds}} \int_0^1 g_\rho(s) C_{\widetilde{\mathcal{T}}}(\kappa',s)ds$$
$$- \frac{f_\rho(\kappa')}{\sqrt{\int_0^1 g_\rho^2(s)ds}} \int_0^1 g_\rho(s) C_{\widetilde{\mathcal{T}}}(\kappa,s)ds + \frac{f_\rho(\kappa) f_\rho(\kappa')}{\int_0^1 g_\rho^2(s)ds} \int_0^1 \int_0^1 g_\rho(t) g_\rho(s) C_{\widetilde{\mathcal{T}}}(t,s)dtds. \quad (15)$$

Our main limit theorem is formally stated as follows.

Theorem 1. *Suppose that Assumptions I, II with $q=3$ and Conditions DP and IP hold. Moreover, assume that $N \asymp n^b$ and $\ell \asymp n^c$, for some nonnegative exponents b and c, satisfying the following conditions,*

$$0 < b < 1 - \zeta_1 \quad \text{and} \quad \zeta_1 < c < \min\left\{1-\zeta_2, 1 - \frac{b}{2} - \frac{\zeta_2}{2}\right\} \quad (16)$$

for some small $\zeta_1, \zeta_2 > 0$ that can be arbitrarily close to 0. Then,

$$\sqrt{N}\left(\widehat{f}_\rho(\kappa) - f_\rho(\kappa)\right) \xrightarrow{d} \frac{\breve{\mathcal{T}}(\kappa)}{\eta} \quad \text{in} \quad D[0,1],$$

where $\eta := \sqrt{c_0^2 \int_0^1 g_\rho(t)^2 dt}$.

As a direct consequence of Theorem 1 and the continuous mapping theorem, the following corollary on the asymptotic behavior of Cramér-von Mises type statistics turns out to be very useful in making statistical inference about the intraday correlation pattern.

Corollary 2. *Suppose that all the assumptions and conditions of Theorem 1 hold. Then,*

$$N \int_0^1 \left(\widehat{f}_\rho(\kappa) - f_\rho(\kappa)\right)^2 d\kappa \xrightarrow{d} \mathcal{Z},$$

where $\mathcal{Z} = \left\|\breve{\mathcal{T}}(\kappa)/\eta\right\|^2$ is a weighted sum of independent $\chi^2(1)$ variables, defined on an *extension of the original probability space and independent from $\mathscr{F}$. The weights are given by the eigenvalues $(\pi_i)_{i\geq 1}$ of the covariance operator with kernel $C_{\breve{\mathcal{T}}}(\kappa,\kappa')/\eta^2$.*

3.3 Approximation of the Limiting Distribution

In this section, we present a consistent estimator for the matrix-valued limiting covariance function $\boldsymbol{C}(\kappa,\kappa')$ as defined in (11). This estimator plays a crucial role in approximating the distribution of the limiting Gaussian process $\breve{\mathcal{T}}(\kappa)/\eta$ as stated in Theorem 1, thereby enabling the feasibility of Theorem 1 and, consequently, Corollary 2.

Estimating $\boldsymbol{C}(\kappa,\kappa')$ involves estimating $C_{i,j}(\kappa,\kappa')$ for $i,j=1,2,3$, as defined in (11). Hence, the problem further reduces to estimating (cross-)covariances $\phi_h^{(i,j)}(\kappa,\kappa')$'s. If, for $m=1,2,3$, $A_i^{(m)}(\kappa)$ and $A_i^{(m)}(\kappa')$ are observable, then $\phi_h^{(i,j)}(\kappa,\kappa')$'s can be naturally estimated via their sample counterparts. However, this is not the case in the problem at hand. We proceed by substituting estimators of $A_i^{(m)}(\kappa)$ and $A_i^{(m)}(\kappa')$ in the sample (cross-)covariance estimator of $\phi_h^{(i,j)}(\kappa,\kappa')$. This yields the following estimator of $C_{i,j}(\kappa,\kappa')$ for $i,j=1,2,3$:

$$\widehat{C}_{i,j}(\kappa,\kappa')=\frac{1}{N}\sum_{d=1}^{N}\widehat{A}_d^{(i)}(\kappa)\widehat{A}_d^{(j)}(\kappa')+\sum_{h=1}^{L_n}\frac{1}{N-h}\sum_{d=1}^{N}\left[\widehat{A}_d^{(i)}(\kappa)\left(\widehat{A}_{d+h}^{(j)}(\kappa')+\widehat{A}_{d-h}^{(j)}(\kappa')\right)\right], \quad (17)$$

where, for $m=1,2$, estimators of $A_i^{(m)}(\kappa)$ and $A_i^{(3)}(\kappa)$ are given by

$$\widehat{A}_i^{(m)}(\kappa)=\hat{\sigma}_{m,i}^2(\kappa)-\widehat{E\sigma_m^2}(\kappa) \quad \text{and} \quad \widehat{A}_i^{(3)}(\kappa)=\widehat{B}_i(\kappa)-\widehat{EB}(\kappa),$$

with

$$\widehat{E\sigma_m^2}(\kappa):=\frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}_{m,i}^2(\kappa) \quad \text{and} \quad \widehat{EB}(\kappa):=\frac{1}{N}\sum_{i=1}^{N}\widehat{B}_i(\kappa); \quad (18)$$

for $m = 1,2,3$, $\widehat{A}_i^{(m)}(\kappa) = \widehat{A}_i^{(m)}(\kappa') = 0$ if $i \leq 0$ or $i > N$; and the integer sequence L_n diverges with the rate being specified in Theorem 3.

The following theorem establishes the consistency of the estimator $\widehat{C}_{i,j}(\kappa,\kappa')$ for $C_{i,j}(\kappa,\kappa')$. This, in turn, ensures consistent estimation of $\boldsymbol{C}(\kappa,\kappa')$. We denote the estimator of $\boldsymbol{C}(\kappa,\kappa')$ as $\widehat{\boldsymbol{C}}(\kappa,\kappa') := \left(\widehat{C}_{i,j}(\kappa,\kappa')\right)_{1\leq i,j\leq 3}$.

Theorem 3. *Suppose that all the Assumptions in Theorem 1 hold. Moreover, $L_n \asymp n^{\varrho}$ for some strictly positive exponent ϱ which satisfies the following condition,*

$$\varrho < \min\left\{\frac{b}{2}, \frac{c}{2}, \frac{1-c}{2}\right\}. \tag{19}$$

Then, for any $i,j \in \{1,2,3\}$, we have

$$\int_0^1 \int_0^1 \left(\widehat{C}_{i,j}(\kappa,\kappa') - C_{i,j}(\kappa,\kappa')\right)^2 d\kappa d\kappa' \xrightarrow{p} 0.$$

Armed with Theorem 3, we are now ready to introduce our scheme for generating approximate sample paths from vector-valued Gaussian process $\boldsymbol{\mathcal{T}}(\kappa)$, as defined in (13), with matrix-valued covariance function $\boldsymbol{C}(\kappa,\kappa')$. We first split the $[0,1]$ interval into 100 equally spaced subintervals and denote $\kappa_i = i/100,\ i = 1,2,\ldots,100$. An approximate sample path of $\boldsymbol{\mathcal{T}}(\kappa)$ is generated over this grid of times. To be specific, we generate a sample from the 300-dimensional Gaussian distribution with covariance matrix $(\widehat{\boldsymbol{C}}(\kappa_k,\kappa_l))_{1\leq k,l\leq 100}$. The resulted sample is then an approximation to a sample path of $\boldsymbol{\mathcal{T}}(\kappa)$ being evaluated over the grid of time points $\{\kappa_i = i/100,\ i = 1,2,\ldots,100\}$. We denote this approximation as $(\widehat{\boldsymbol{\mathcal{T}}}(\kappa_1)', \ldots, \widehat{\boldsymbol{\mathcal{T}}}(\kappa_{100})')'$.

We next turn to the generation of approximate sample paths from the Gaussian process $\widetilde{\mathcal{T}}(\kappa)$ as defined in (12). To this end, it remains to generate approximations of $E\sigma_1^2(\kappa)$,

$E\sigma_2^2(\kappa)$ and $EB(\kappa)$ over the grid of time points $\{\kappa_i = i/100,\ i = 1, 2, \dots, 100\}$ by $\widehat{E\sigma_1^2}(\kappa)$, $\widehat{E\sigma_2^2}(\kappa)$ and $\widehat{EB}(\kappa)$ as defined in (18). Then, the approximation of a sample path of $\widetilde{\mathcal{T}}(\kappa)$ over the grid of time points $\{\kappa_i = i/100,\ i = 1, 2, \dots, 100\}$ is a direct consequence of equation (12) and the approximated sample path of $\boldsymbol{\mathcal{T}}(\kappa)$, i.e., $(\widehat{\boldsymbol{\mathcal{T}}}(\kappa_1)', \dots, \widehat{\boldsymbol{\mathcal{T}}}(\kappa_{100})')'$. We denote this approximation by $\left(\widehat{\widetilde{\mathcal{T}}}(\kappa_1), \dots, \widehat{\widetilde{\mathcal{T}}}(\kappa_{100})\right)$.

Finally, we consider the generation of approximate sample paths of $\breve{\mathcal{T}}(\kappa)$ as defined in (14). This involves approximating $g_\rho(\kappa)$ and $f_\rho(\kappa)$ over the grid of time points $\{\kappa_i = i/100,\ i = 1, 2, \dots, 100\}$. The approximation of $f_\rho(\kappa)$ is given by $\widehat{f}_\rho(\kappa)$ as defined in (8). Similarly, in the spirit of estimating $f_\rho(\kappa)$, we estimate $g_\rho(\kappa)$ by $\hat{\mathfrak{g}}_\rho(\kappa)$, as given in (6). This approach is justified because $g_\rho(\kappa)$ appears in both the numerator and the denominator of (14). The involved constants c_0, as defined in (7), cancel out in both the numerator and denominator of (14). Therefore, the approximation $(\widehat{\breve{\mathcal{T}}}(\kappa_1), \dots, \widehat{\breve{\mathcal{T}}}(\kappa_{100}))$ of a sample path of $\breve{\mathcal{T}}(\kappa)$ being evaluated over the grid of time points $\{\kappa_i = i/100,\ i = 1, 2, \dots, 100\}$ is given below: for $i = 1, \dots, 100$,

$$\widehat{\breve{\mathcal{T}}}(\kappa_i) = \widehat{\widetilde{\mathcal{T}}}(\kappa_i) - \frac{\frac{1}{100}\sum_{j=1}^{100} \hat{\mathfrak{g}}_\rho(\kappa_j)\widehat{\widetilde{\mathcal{T}}}(\kappa_i)}{\sqrt{\frac{1}{100}\sum_{j=1}^{100} \hat{\mathfrak{g}}_\rho(\kappa_j)^2}} \widehat{f}_\rho(\kappa_i). \tag{20}$$

A sample path of the limiting Gaussian process $\breve{\mathcal{T}}(\kappa)/\eta$ over the grid of time points $\{\kappa_i = i/100,\ i = 1, 2, \dots, 100\}$ is then approximated by $(\widehat{\breve{\mathcal{T}}}(\kappa_1)/\widehat{\eta}, \dots, \widehat{\breve{\mathcal{T}}}(\kappa_{100})/\widehat{\eta})$, where $\widehat{\eta}$ is given by (9). In our numerical implementations, a large number of i.i.d. copies of $(\widehat{\breve{\mathcal{T}}}(\kappa_1)/\widehat{\eta}, \dots, \widehat{\breve{\mathcal{T}}}(\kappa_{100})/\widehat{\eta})$ are generated to obtain an approximation of the limiting distribution of the Gaussian process $\breve{\mathcal{T}}(\kappa)/\eta$. In simulations, each of the involved estimates $(\widehat{\boldsymbol{C}}(\kappa_k, \kappa_l))_{1\le k,l\le 100}$, $\widehat{E\sigma_1^2}(\kappa)$, $\widehat{E\sigma_2^2}(\kappa)$, $\widehat{EB}(\kappa)$, $\widehat{f}_\rho(\kappa)$, $\hat{\mathfrak{g}}_\rho(\kappa)$ and $\widehat{\eta}$ is replaced with the average of the corresponding estimates produced using a large number of realizations of the pair of asset price processes. However, in empirical studies, these estimates are preserved in their original form and are calculated based on a single realization of the pair of asset price processes.

3.4 Feasible Implementation of a Two-Sample Test

In this section, we apply the feasible central limit theorem developed in Section 3.2 and the limiting distribution approximation scheme proposed in Section 3.3 to test for the equivalence of correlation diurnal patterns over two non-overlapping time periods. Let $f_{\rho,\mathrm{P}_1}(\kappa)$ and $f_{\rho,\mathrm{P}_2}(\kappa)$ be the intraday correlation patterns over two non-overlapping periods P_1 and P_2. The two periods consist of N_1 and N_2 trading days, respectively. We assume that $N_1/N_2 \to r$ for some $r \in (0, \infty)$. The hypotheses we shall test are as follows:

$$H_0: \int_0^1 \left(f_{\rho,\mathrm{P}_1}(\kappa) - f_{\rho,\mathrm{P}_2}(\kappa)\right)^2 d\kappa = 0 \quad \text{versus} \quad H_1: \int_0^1 \left(f_{\rho,\mathrm{P}_1}(\kappa) - f_{\rho,\mathrm{P}_2}(\kappa)\right)^2 d\kappa > 0. \tag{21}$$

Based on Corollary 2, we propose the test statistic

$$T_n := N_1 \int_0^1 \left(\hat{f}_{\rho,\mathrm{P}_1}(\kappa) - \hat{f}_{\rho,\mathrm{P}_2}(\kappa)\right)^2 d\kappa,$$

where $\hat{f}_{\rho,\mathrm{P}_1}(\kappa)$ and $\hat{f}_{\rho,\mathrm{P}_2}(\kappa)$ are the intraday-correlation-pattern estimators for the periods P and P$'$, respectively. Because the two periods are non-overlapping, it follows easily that $\sqrt{N_1}(\hat{f}_{\rho,\mathrm{P}_1}(\kappa) - f_{\rho,\mathrm{P}_1}(\kappa))$ and $\sqrt{N_2}(\hat{f}_{\rho,\mathrm{P}_2}(\kappa) - f_{\rho,\mathrm{P}_2}(\kappa))$ are asymptotically independent. Therefore, under the null hypothesis H_0, we have

$$T_n \xrightarrow{d} \int_0^1 \left(\frac{\breve{\mathcal{T}}_{\mathrm{P}_1}(\kappa)}{\eta} - \sqrt{r}\frac{\breve{\mathcal{T}}_{\mathrm{P}_2}(\kappa)}{\eta}\right)^2 d\kappa, \tag{22}$$

where $\breve{\mathcal{T}}_{\mathrm{P}_1}(\kappa)$ and $\breve{\mathcal{T}}_{\mathrm{P}_2}(\kappa)$ are independent Gaussian processes with the same distribution as that of $\breve{\mathcal{T}}(\kappa)$ in Theorem 1. Therefore, the same approximation scheme as that described in Section 3.3 can be employed to approximate the limiting distribution in (22). That is, for a grid of equally spaced partition points $\kappa_i = i/100,\ i = 1, 2, \ldots, 100$ of the $[0, 1]$ interval, the limiting distribution in (22) can be approximated by that of

$$\widehat{\mathcal{M}} = \frac{1}{100} \sum_{i=1}^{100} \left(\frac{\widehat{\breve{\mathcal{T}}}_{\mathrm{P}_1}(\kappa_i)}{\widehat{\eta}} - \sqrt{N_1/N_2} \frac{\widehat{\breve{\mathcal{T}}}_{\mathrm{P}_2}(\kappa_i)}{\widehat{\eta}} \right)^2,$$

where $\widehat{\breve{\mathcal{T}}}_{\mathrm{P}_1}(\kappa_i)$ and $\widehat{\breve{\mathcal{T}}}_{\mathrm{P}_2}(\kappa_i)$ are independent and identically distributed copies of $\widehat{\breve{\mathcal{T}}}(\kappa_i)$ as defined in (20), and $\widehat{\eta}$ is again given by (9). In our numerical implementations, the distribution of $\widehat{\mathcal{M}}$ is further approximated through Monte Carlo by the empirical distribution of a large number of i.i.d. copies of $\widehat{\mathcal{M}}$. Let $c_{1-\alpha}$ denote the $(1-\alpha)$th quantile of this empirical distribution. The rejection region is then naturally defined as follows,

$$\mathcal{R} := \{T_n \geq c_{1-\alpha}\}. \tag{23}$$

We shall implement this test in our numerical studies.

4 Monte Carlo Simulations

In this section, we investigate the finite-sample performance of our correlation diurnal pattern estimator (8) through Monte Carlo simulations under realistic settings. Following Tan et al. (2024), our simulation setting is as follows: The data generating processes considered in this simulation study are as follows. For $m = 1, 2$, the log price and volatility processes, i.e., X_m and σ_m^2, are given, respectively, by,

$$\begin{cases} dX_m(t) = \mu_m dt + \sigma_m(t) dW_m(t), \\ \sigma_m^2(t) = \breve{g}_m(t - \lfloor t \rfloor)\tilde{\sigma}_m^2(t), \\ d\tilde{\sigma}_m^2(t) = \lambda_m \left(\eta_m - \tilde{\sigma}_m^2(t)\right) dt + \xi_m \tilde{\sigma}_m(t) d\widetilde{W}_m(t), \end{cases} \tag{24}$$

where the volatility process $\sigma_m^2(t)$ admits a multiplicative decomposition with the stationary component $\tilde{\sigma}_m^2(t)$ being given by a Cox-Ingersoll-Ross process (see, e.g., Cox et al., 1985a) and the seasonality component being given by a deterministic periodic function $\breve{g}_m(t - \lfloor t \rfloor)$. $\breve{g}_m(t - \lfloor t \rfloor)$'s determine the diurnal effects in volatility processes and are calibrated to the empirical data used in Section 5. Here, the quadratic covariation between W_m and $\widetilde{W}_m$ is $d[W_m, \widetilde{W}_m](t) = \tilde{\rho}_m dt$, where the parameter $\tilde{\rho}_m$ captures the leverage effect, i.e., the widely documented negative correlations between returns and volatility changes.

Turning to the correlation between two stocks, we assume that

$$\begin{cases} \rho(t) = \breve{g}_\rho(t - \lfloor t \rfloor)\tilde{\rho}(t), \\ \tilde{\rho}(t) = \frac{1}{1+\exp\{-Z(t)\}}, \\ Z(t) = \tilde{\mu} + \exp(-\tilde{\lambda} t)(Z(0) - \tilde{\mu}) + \breve{\sigma} \int_0^t \exp\{-\tilde{\lambda}(t-s)\} d\breve{W}(s), \end{cases} \tag{25}$$

where $Z(t)$ is a Gaussian Ornstein-Uhlenbeck process driven by a standard Brownian motion $\breve{W}(t)$, and $\breve{g}_\rho(t - \lfloor t \rfloor)$ is the deterministic intraday periodic component that captures the correlation diurnal pattern. As is the case with volatility processes, we also calibrate $\breve{g}_\rho(t - \lfloor t \rfloor)$ to the empirical data studied in Section 5. In simulations, the relation between $W_1(t)$ and $W_2(t)$ and that between $W_m(t)$ and $\widetilde{W}_m(t)$ are given, respectively, by

$$W_2(t) = \rho(t) W_1(t) + \sqrt{1 - \rho(t)^2} W^{(p)}(t) \quad \text{and} \quad \widetilde{W}_m(t) = \tilde{\rho}_m W_m(t) + \sqrt{1 - \tilde{\rho}_m^2} W_m^{(v)}(t),$$

where standard Brownian motions $\breve{W}(t)$, $W_1(t)$, $W^{(p)}(t)$, $W_1^{(v)}(t)$ and $W_2^{(v)}(t)$ are independent of one another.

We next introduce the specific settings for the model parameters and the deterministic periodic components, i.e., $\breve{g}_m(t - \lfloor t \rfloor)$ in (24) and $\breve{g}_\rho(t - \lfloor t \rfloor)$ in (25), of volatility and correlation processes. The model parameters are fixed at

$$\begin{cases} X_1(0) = X_2(0) = 3.6, \ \mu_1 = \mu_2 = 0.0005, \ \lambda_1 = \lambda_2 = 0.5, \ \eta_1 = \eta_2 = 1, \\ \xi_1 = \xi_2 = 1, \ \tilde{\rho}_1 = \tilde{\rho}_2 = -0.5, \ \tilde{\mu} = 1, \ \tilde{\lambda} = 0.5, \ \breve{\sigma} = 0.3. \end{cases}$$

To ensure that our analysis is within an empirically relevant setting, we follow the method of, e.g., Andersen, Su, Todorov, and Zhang (2023) and Andersen, Tan, Todorov, and Zhang (2023), calibrating the deterministic periodic components $\breve{g}_m$ and $\breve{g}_\rho$ of volatility and correlation processes to the high frequency trade data of SSE Composite Index and SSE Financial Index used in our empirical study in Section 5. To be specific, we use the data sample spanning the period January 3, 2017–December 30, 2022 to compute $\hat{\mathfrak{g}}_{\sigma_m}$ and $\hat{\mathfrak{g}}_\rho$, respectively. The deterministic periodic component $\breve{g}_m$ of volatility process in (24) is set as $\hat{\mathfrak{g}}_{\sigma_m}$ divided by the expectation of $\breve{\sigma}_m^2(t)$, which equals η_m under the setting of (24). The mean of the spot volatility curves induced by model (24) with this empirically calibrated $\breve{g}_m$ is thus ensured to be comparable with that of the real data. The deterministic periodic component $\breve{g}_\rho$ of correlation process is set as $\hat{\mathfrak{g}}_\rho$. With this calibrated $\breve{g}_\rho$, we ensure that estimates $\hat{\mathfrak{g}}_\rho$ produced based on the real data and our simulated data are comparable with each other.

We set $n = 1,440$, which corresponds to a 1-minute sampling frequency across 24 trading hours, to mimic the length of the trading day in our empirical analysis on stocks and indices from exchange market. The reported results are based on 1000 simulation trials with each simulation trial consists of a 100-day long two-dimensional trade price series sampled at the 1-minute frequency. If one takes $b \approx 7/10$ and $c \approx 1/2$ for $N = 1,000$, $\ell = 20$ and $n = 1,440$, conditions (19) reduces to $\varrho <$. We take $L_n = 5$ in all our numerical illustrations. The condition $L_n \asymp n^{\varrho}$ could thus be deemed satisfied.

The following Figure ??? plots the histogram so as to verify Corollary 2.

Figure **??** plots the true correlation diurnal pattern, and the mean together with the corresponding 95th and 5th percentiles of the sample of correlation diurnal pattern estimates produced based on 1000 simulation trials. The results demonstrate that the proposed estimator performs quite well, tracing the true correlation diurnal pattern closely.

5 Empirical Studies

6 Concluding Remarks

Appendix A Proofs

We start by introducing additional notations utilized in the decomposition of errors pertaining to the approximations of the mean levels of spot volatilities and covariations at a fixed time-of-day. For $\kappa \in [0, 1]$ and $m = 1, 2$, we define the components of an appropriate decomposition of the errors induced by the approximations of the mean levels of spot volatilities as follows:

$$
\begin{cases}
\zeta_1^{(m)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\int_{t_{i,j-1}}^{t_{i,j}} \mu_m(s)\right)^2, \\
\zeta_2^{(m)}(\kappa) := \frac{2}{N\ell\Delta}\sum_{i=1}^N \sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \mu_m(s)ds \int_{t_{i,j-1}}^{t_{i,j}} \sigma_m(s)dW_m(s), \\
\zeta_3^{(m)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\left(\int_{t_{i,j-1}}^{t_{i,j}} \sigma_m(s)dW_m(s)\right)^2 - \int_{t_{i,j-1}}^{t_{i,j}} \sigma_m^2(s)ds\right), \\
\zeta_4^{(m)}(\kappa) := \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{\ell\Delta}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \sigma_m^2(s)ds - \sigma_{m,i}^2(\kappa)\right), \\
\zeta_5^{(m)}(\kappa) := \frac{1}{N}\sum_{i=1}^N \sigma_{m,i}^2(\kappa) - E\sigma_m^2(\kappa) = \frac{1}{N}\sum_{i=1}^N A_i^{(m)}(\kappa).
\end{cases}
\tag{26}
$$

According to the convention we adopt in the definitions of covariation and volatility estimators in Section 3.1, for $i = 2, \ldots, N$ and $\kappa \in [0, \ell\Delta)$, the upper and lower integral limits involved in the first four terms of (26) should be interpreted as $t_{i,j} = t_{i-1,n+j}$ for $-\ell \leq j \leq 0$. Moreover, for $i = 1$ and $\kappa \in [0, \ell\Delta)$, the inner summation index variable k involved in the first four terms of (26) should always range from 1 to ℓ; that is, $j_\kappa \equiv \ell$ for $\kappa \in [0, \ell\Delta)$ and $i = 1$. The individual components of an appropriate decomposition of the approximation error for the average spot covariation $EB(\kappa) \equiv EB_1(\kappa)$ at the time-of-day $\kappa \in [0, 1]$ are provided as follows:

$$
\begin{cases}
\zeta_1^{(3)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \mu_1(s)ds \int_{t_{i,j-1}}^{t_{i,j}} \mu_2(s)ds, \\
\zeta_2^{(3)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \mu_1(s)ds \int_{t_{i,j-1}}^{t_{i,j}} \sigma_2(s)dW_2(s), \\
\zeta_3^{(3)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \sigma_1(s)dW_1(s) \int_{t_{i,j-1}}^{t_{i,j}} \mu_2(s)ds, \\
\zeta_4^{(3)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \sigma_2(s) \int_{t_{i,j-1}}^{s} \sigma_1(u)dW_1(u)dW_2(s); \\
\zeta_5^{(3)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \sigma_1(s) \int_{t_{i,j-1}}^{s} \sigma_2(u)dW_2(u)dW_1(s); \\
\zeta_6^{(3)}(\kappa) := \frac{1}{N\ell\Delta}\sum_{i=1}^N \sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} \left(B(s) - B_i(\kappa)\right) ds; \\
\zeta_7^{(3)}(\kappa) := \frac{1}{N}\sum_{i=1}^N B_i(\kappa) - EB(\kappa) = \frac{1}{N}\sum_{i=1}^N A_i^{(3)}(\kappa) = \frac{g_B(\kappa)}{N}\sum_{i=1}^N \tilde{B}_i^c(\kappa),
\end{cases}
\tag{27}
$$

where $\tilde{B}_i^c(\kappa) := \tilde{B}_i(\kappa) - E\tilde{B}_i(\kappa)$. According to the convention we adopt in the definitions of covariation and volatility estimators in Section 3.1 again, for $i = 2, \ldots, N$ and $\kappa \in [0, \ell\Delta)$, the upper and lower integral limits involved in the first six terms of (27) should be interpreted as $t_{i,j} = t_{i-1,n+j}$ for $-\ell \leq j \leq 0$. Moreover, for $i = 1$ and $\kappa \in [0, \ell\Delta)$, the inner summation index variable k involved in the first six terms of (27) should always range from 1 to ℓ; that is, $j_\kappa \equiv \ell$ for $\kappa \in [0, \ell\Delta)$ and $i = 1$. It then follows easily from Itô's formula that, for $m = 1, 2$,@Pls really check carefully if the following decomposition is still true with the adjustments of definitions of various estimators!@

$$\frac{1}{N}\sum_{i=1}^{N} \hat{\sigma}_{m,i}^2(\kappa) - E\sigma_m^2(\kappa) = \sum_{k=1}^{5} \zeta_k^{(m)}(\kappa),$$

and

$$\frac{1}{N}\sum_{i=1}^{N} \widehat{B}_i(\kappa) - EB(\kappa) = \sum_{k=1}^{7} \zeta_k^{(3)}(\kappa).$$

The following lemma provides a bound for the average (pointwise) differences between the true daily volatility and covariation curves, i.e., $\sigma_{m,i}^2$ for $m = 1, 2$ and B_i, and their estimates, i.e., $\hat{\sigma}_{m,i}^2$ for $m = 1, 2$ and $\hat{B}_i$.

Lemma 4. *Suppose that Assumptions I, II with $q = 3$ hold. Then, for any $\kappa \in [0, 1]$, $m = 1, 2$, and $d \geq 2$, when $\Delta \to 0$ and $\ell \to \infty$,*

$$E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\sigma}_{m,i}^2(\kappa) - \sigma_{m,i}^2(\kappa)\right)\right|^d \leq C\left(\Delta^{\frac{d}{2}} \vee \frac{1}{(N\ell)^{\frac{d}{2}}} \vee (\ell\Delta)^d \vee \left(\frac{\ell\Delta}{N}\right)^{\frac{d}{2}}\right), \tag{28}$$

and

$$E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{B}_i(\kappa) - B_i(\kappa)\right)\right|^d \leq C\left(\Delta^{\frac{d}{2}} \vee \frac{1}{(N\ell)^{\frac{d}{2}}} \vee (\ell\Delta)^d \vee \left(\frac{\ell\Delta}{N}\right)^{\frac{d}{2}}\right). \tag{29}$$

Proof. We focus on proving (29). Same arguments can be applied to derive (28). The remainder of the proof is divided into two parts.

Part 1. The case $\kappa \in [\ell\Delta, 1]$.

We first consider the scenario where $\kappa \in [\ell\Delta, 1]$. Note first that, by Itô's formula, we have the following decomposition:

$$\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{B}_i(\kappa) - B_i(\kappa)\right) = \sum_{k=1}^{6}\zeta_k^{(3)}(\kappa).$$

Hence, by C_r-inequality, we have

$$E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{B}_i(\kappa) - B_i(\kappa)\right)\right|^d \leq C\sum_{k=1}^{6}E\left|\zeta_k^{(3)}(\kappa)\right|^d. \tag{30}$$

In what follows, we deal with the six summands on the right hand side of the above inequality one by one.

For the first summand on the right hand side of the inequality (30), we have

$$\begin{aligned}
E\left|\zeta_1^{(3)}(\kappa)\right|^d &\leq \frac{C}{N\ell\Delta^d}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}E\left|\int_{t_{i,j-1}}^{t_{i,j}}\mu_1(s)ds\int_{t_{i,j-1}}^{t_{i,j}}\mu_2(s)ds\right|^d \\
&\leq \frac{C}{N\ell\Delta^d}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\left(E\left|\int_{t_{i,j-1}}^{t_{i,j}}\mu_1(s)ds\right|^{2d}\right)^{\frac{1}{2}}\left(E\left|\int_{t_{i,j-1}}^{t_{i,j}}\mu_2(s)ds\right|^{2d}\right)^{\frac{1}{2}} \\
&\leq \frac{C\Delta^{d-1}}{N\ell}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\left(\int_{t_{i,j-1}}^{t_{i,j}}E\left|\mu_1(s)\right|^{2d}ds\right)^{\frac{1}{2}}\left(\int_{t_{i,j-1}}^{t_{i,j}}E\left|\mu_2(s)\right|^{2d}ds\right)^{\frac{1}{2}} \leq C\Delta^d,
\end{aligned}$$

where the first and third inequalities follow from Jensen's inequality, the second inequality follows from Cauchy–Schwarz inequality, and the last inequality follows from Assumption

I.

As to the second summand on the right hand side of the inequality (30), we have

$$
\begin{aligned}
E\left|\zeta_2^{(3)}(\kappa)\right|^d &\leq \frac{C}{N\ell\Delta^d}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa} E\left|\int_{t_{i,j-1}}^{t_{i,j}}\mu_1(s)ds\int_{t_{i,j-1}}^{t_{i,j}}\sigma_2(s)dW_2(s)\right|^d \\
&\leq \frac{C}{N\ell\Delta^d}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\left(E\left|\int_{t_{i,j-1}}^{t_{i,j}}\mu_1(s)ds\right|^{2d}\right)^{\frac{1}{2}}\left(E\left|\int_{t_{i,j-1}}^{t_{i,j}}\sigma_2(s)dW_2(s)\right|^{2d}\right)^{\frac{1}{2}} \\
&\leq \frac{C}{N\ell\Delta^d}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\left(E\left|\int_{t_{i,j-1}}^{t_{i,j}}\mu_1(s)ds\right|^{2d}\right)^{\frac{1}{2}}\left(E\left|\int_{t_{i,j-1}}^{t_{i,j}}\sigma_2^2(s)d(s)\right|^{d}\right)^{\frac{1}{2}} \\
&\leq \frac{C\Delta^{\frac{d}{2}-1}}{N\ell}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\left(\int_{t_{i,j-1}}^{t_{i,j}}E\left|\mu_1(s)\right|^{2d}ds\right)^{\frac{1}{2}}\left(\int_{t_{i,j-1}}^{t_{i,j}}E\left|\sigma_2^2(s)\right|^{d}d(s)\right)^{\frac{1}{2}} \leq C\Delta^{\frac{d}{2}},
\end{aligned}
$$

where the first and fourth inequalities follow from Jensen's inequality, the second inequality follows from Cauchy–Schwarz inequality, the third inequality follows from Burkholder–Davis–Gundy inequality and the last inequality follows from Assumption I. By the same arguments as those employed in dealing with $E\left|\zeta_2^{(3)}(\kappa)\right|^d$, we also have

$$
E\left|\zeta_3^{(3)}(\kappa)\right|^d \leq C\Delta^{\frac{d}{2}}.
$$

We now turn to the fourth summand on the right hand side of the inequality (30). It can be bounded as follows,

$$
\begin{aligned}
E\left|\zeta_4^{(3)}(\kappa)\right|^d &\leq CE\left|\frac{1}{N^2\ell^2\Delta^2}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}}\sigma_2^2(s)\left(\int_{t_{i,j-1}}^{s}\sigma_1(u)dW_1(u)\right)^2 ds\right|^{\frac{d}{2}} \\
&\leq \frac{C}{(N\ell\Delta)^{\frac{d}{2}+1}}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}}E\left|\sigma_2^d(s)\left(\int_{t_{i,j-1}}^{s}\sigma_1(u)dW_1(u)\right)^d\right|ds
\end{aligned}
$$

$$\leq \frac{C}{(N\ell\Delta)^{\frac{d}{2}+1}} \sum_{i=1}^{N} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,j-1}}^{t_{i,j}} \left(E\left|\sigma_2(s)\right|^{2d}\right)^{\frac{1}{2}} \left(E\left|\int_{t_{i,j-1}}^{s} \sigma_1(u)dW_1(u)\right|^{2d}\right)^{\frac{1}{2}} ds$$

$$\leq \frac{C}{(N\ell\Delta)^{\frac{d}{2}+1}} \sum_{i=1}^{N} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,j-1}}^{t_{i,j}} \left(E\left|\int_{t_{i,j-1}}^{s} \sigma_1^2(u)du\right|^{d}\right)^{\frac{1}{2}} ds$$

$$\leq \frac{C}{(N\ell\Delta)^{\frac{d}{2}+1}} \sum_{i=1}^{N} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,j-1}}^{t_{i,j}} \left(\Delta^{d-1}\int_{t_{i,j-1}}^{s} E\left|\sigma_1(u)\right|^{2d} du\right)^{\frac{1}{2}} ds \leq \frac{C}{(N\ell)^{\frac{d}{2}}},$$

where the first and fourth inequalities follows from Burkholder–Davis–Gundy inequality, second and fifth inequalities follow from Jensen's inequality, the third inequality follows from Cauchy–Schwarz inequality and the last inequality follows from Assumption I. By the same arguments as those used in bounding $E\left|\zeta_4^{(3)}(\kappa)\right|^d$, we also obtain

$$E\left|\zeta_5^{(3)}(\kappa)\right|^d \leq \frac{C}{(N\ell)^{\frac{d}{2}}}.$$

We finally turn to the sixth summand $E\left|\zeta_6^{(3)}(\kappa)\right|^d$ on the right hand side of the inequality (30). By (4), we can further decompose $\zeta_6^{(3)}(\kappa)$ as follows

$$\zeta_6^{(3)}(\kappa) = \sum_{i=1}^{4} \zeta_{6,i}^{(3)}(\kappa),$$

where

$$\begin{cases} \zeta_{6,1}^{(3)}(\kappa) := \frac{g_B(\kappa)}{N\ell\Delta} \sum_{i=1}^{N} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,j-1}}^{t_{i,j}} \int_{s}^{i-1+\kappa} \breve{\mu}_3(u)duds; \\ \zeta_{6,2}^{(3)}(\kappa) := \frac{g_B(\kappa)}{N\ell} \sum_{i=1}^{N} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,j-1}}^{i-1+\kappa} \breve{\sigma}_3(u)d\breve{W}_3(u); \\ \zeta_{6,3}^{(3)}(\kappa) := -\frac{g_B(\kappa)}{N\ell\Delta} \sum_{i=1}^{N} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,j-1}}^{t_{i,j}} \int_{t_{i,j-1}}^{s} \breve{\sigma}_3(u)d\breve{W}_3(u)ds; \\ \zeta_{6,4}^{(3)}(\kappa) := \frac{1}{N\ell\Delta} \sum_{i=1}^{N} \sum_{j=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,j-1}}^{t_{i,j}} \left(g_B(s-i+1) - g_B(\kappa)\right) \tilde{B}(s)ds; \end{cases} \tag{31}$$

By C_r-inequality, we have

$$E\left|\zeta_6^{(3)}(\kappa)\right|^d \le C\sum_{i=1}^4 E\left|\zeta_{6,i}^{(3)}(\kappa)\right|^d.$$

Hence, it suffices to deal with the four terms in (31) one by one. For the first term in (31), we have

$$E\left|\zeta_{6,1}^{(3)}(\kappa)\right|^d \le \frac{C}{N\ell\Delta}\sum_{i=1}^N\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}} E\left|\int_s^{i-1+\kappa}\breve{\mu}_3(u)du\right|^d ds \le C\left(\ell\Delta\right)^d,$$

where the first inequality follows from the boundedness of deterministic functions $g_1(\kappa)$, $g_2(\kappa)$ and $g_\rho(\kappa)$ and Jensen's inequality, and the second inequality follows from Assumption I.

As to the second term $\zeta_{6,2}^{(3)}(\kappa)$, we define, for $s \in [0, N]$,

$$M_2(s) := \frac{1}{N\ell}\sum_{i=1}^N\left[\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}\wedge s}^{t_{i,j}\wedge s}(j+\ell-j_\kappa)\breve{\sigma}_3(u)d\breve{W}_3(u) + \int_{t_{i,j_\kappa}\wedge s}^{(i-1+\kappa)\wedge s}\ell\breve{\sigma}_3(u)d\breve{W}_3(u)\right],$$

which is a continuous martingale over the interval $[0, N]$. One readily sees that $E\left|\zeta_{6,2}^{(3)}(\kappa)\right|^d = E|M_2(N)|^d$. Moreover, the quadratic variation of $M_2(s)$ is

$$[M_2, M_2](s) := \frac{1}{(N\ell)^2}\sum_{i=1}^N\left[\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}\wedge s}^{t_{i,j}\wedge s}(j+\ell-j_\kappa)^2\breve{\sigma}_3^2(u)du + \int_{t_{i,j_\kappa}\wedge s}^{(i-1+\kappa)\wedge s}\ell^2\breve{\sigma}_3^2(u)du\right].$$

Now by Jensen's inequality, Burkholder-Davis-Gundy inequality and Assumption I, we obtain that,

$$E\left|\zeta_{6,2}^{(3)}(\kappa)\right|^d = E\left|M_2(N)\right|^d \le CE\left|[M_2, M_2](N)\right|^{\frac{d}{2}}$$

$$= CE\left|\frac{1}{(N\ell)^2}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}}(j+\ell-j_\kappa)^2\breve{\sigma}_3^2(u)du+\frac{1}{N^2}\sum_{i=1}^{N}\int_{t_{i,j_\kappa}}^{(i-1+\kappa)}\breve{\sigma}_3^2(u)du\right|^{\frac{d}{2}}$$
$$\leq \frac{C\ell^{\frac{d}{2}-1}}{N^{\frac{d}{2}+1}}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}E\left|\int_{t_{i,j-1}}^{t_{i,j}}\breve{\sigma}_3^2(u)du\right|^{\frac{d}{2}}+\frac{C}{N^{\frac{d}{2}+1}}\sum_{i=1}^{N}E\left|\int_{t_{i,j_\kappa}}^{(i-1+\kappa)}\breve{\sigma}_3^2(u)du\right|^{\frac{d}{2}}$$
$$\leq C\left(\left(\frac{\ell\Delta}{N}\right)^{\frac{d}{2}}\vee\left(\frac{\Delta}{N}\right)^{\frac{d}{2}}\right)=C\left(\frac{\ell\Delta}{N}\right)^{\frac{d}{2}}.$$

Turning to the third term $\zeta_{6,3}^{(3)}(\kappa)$ in (31), by the arguments employed in the proof of Lemma 2.22 on page 144 of Mykland and Zhang (2012), Burkholder-Davis-Gundy inequality and Assumption I, we have that,

$$E\left|\zeta_{6,3}^{(3)}(\kappa)\right|^d \leq CE\left|\frac{1}{N\ell\Delta}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}}(t_{i,j}-s)\breve{\sigma}_3(s)dW(s)\right|^d$$
$$\leq CE\left|\frac{1}{(N\ell\Delta)^2}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}}(t_{i,j}-s)^2\breve{\sigma}_3^2(s)ds\right|^{\frac{d}{2}}\leq C\left(\frac{\Delta}{N\ell}\right)^{\frac{d}{2}}.$$

For the last term $\zeta_{6,4}^{(3)}(\kappa)$ in (31), under condition IP and Assumption I, by Jensen's inequality, we have

$$E\left|\zeta_{6,4}^{(3)}(\kappa)\right|^d\leq\frac{1}{N\ell\Delta}\sum_{i=1}^{N}\sum_{j=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,j-1}}^{t_{i,j}}\left|g_B(s-i+1)-g_B(\kappa)\right|^d E\left|\tilde{B}(s)\right|^d ds\leq C\left(\ell\Delta\right)^d.$$

Therefore, we have obtained that,

$$E\left|\zeta_6^{(3)}(\kappa)\right|^d\leq C\left((\ell\Delta)^d\vee\left(\frac{\ell\Delta}{N}\right)^{\frac{d}{2}}\right). \tag{32}$$

By the derived upper bound of term $E\left|\zeta_k^{(3)}(\kappa)\right|^d$ for $k=1,\ldots,6$ and (30), we obtain the desired result in (29) in the case where $\kappa \in [\ell\Delta, 1]$.

Part 2. The case $\kappa \in [0, \ell\Delta)$.

We now turn to the scenario where $\kappa \in [0, \ell\Delta)$. To this end, by setting $N = 1$ in (29), we first obtain that

$$E\left|\widehat{B}_1(\kappa) - B_1(\ell\Delta)\right|^d \le C\left(\frac{1}{\ell^{\frac{d}{2}}} \vee (\ell\Delta)^{\frac{d}{2}}\right).$$

Second, by the same argument as that used in dealing with term $\zeta_6^{(3)}(\kappa)$, we have

$$\begin{aligned}
& E\left|B_1(\ell\Delta) - B_1(\kappa)\right|^d \\
& \le C\left|g_B(\ell\Delta) - g_B(\kappa)\right|^d E\left|\tilde{B}_1(\ell\Delta)\right|^d + CE\left|\int_\kappa^{\ell\Delta} \breve{\mu}_3(u)du\right|^d + CE\left|\int_\kappa^{\ell\Delta} \breve{\sigma}_3(u)d\breve{W}_3(u)\right|^d \\
& \le C\left(\ell\Delta - \kappa\right)^{\frac{d}{2}} \le C\left(\ell\Delta\right)^{\frac{d}{2}}.
\end{aligned}$$

Third, by using exactly the same arguments as those employed in dealing with the scenario $\kappa \in [\ell\Delta, 1]$ in Part 1 of the proof, we have that, for $\kappa \in [0, \ell\Delta)$,@Need to be double checked!@

$$E\left|\frac{1}{N}\sum_{i=2}^N\left(\widehat{B}_i(\kappa) - B_i(\kappa)\right)\right|^d \le C\left(\Delta^{\frac{d}{2}} \vee \frac{1}{(N\ell)^{\frac{d}{2}}} \vee (\ell\Delta)^d \vee \left(\frac{\ell\Delta}{N}\right)^{\frac{d}{2}}\right).$$

It then follows straightforwardly that

$$E\left|\frac{1}{N}\sum_{i=1}^N\left(\widehat{B}_i(\kappa) - B_i(\kappa)\right)\right|^d$$

$$\leq CE\left|\frac{1}{N}\sum_{i=2}^{N}\left(\widehat{B}_i(\kappa)-B_i(\kappa)\right)\right|^d+\frac{C}{N^d}E\left|\widehat{B}_1(\kappa)-B_1(\ell\Delta)\right|^d+\frac{C}{N^d}E\left|B_1(\ell\Delta)-B_1(\kappa)\right|^d$$
$$\leq C\left(\Delta^{\frac{d}{2}}\vee\frac{1}{(N\ell)^{\frac{d}{2}}}\vee(\ell\Delta)^d\vee\left(\frac{\ell\Delta}{N}\right)^{\frac{d}{2}}\right).$$

Therefore, (29) holds uniformly for $\kappa\in[0,\ell\Delta)$ as well. This completes the proof.

□

We now proceed to establish functional central limit theorems in the Skorohod space for the following sequence of three-dimensional vector processes

$$\boldsymbol{A}_i(\kappa):=\left(A_i^{(1)}(\kappa),A_i^{(2)}(\kappa),A_i^{(3)}(\kappa)\right)',$$

where $A_i^{(m)}(\kappa)$, for $m=1,2,3$, are defined in (10). We begin with finite-dimensional convergence.

Lemma 5. *Suppose that Assumptions I and II with $q=3$ hold. Then, for any fixed integer d and $\kappa_1<\kappa_2<\ldots<\kappa_d$ with $\kappa_1,\kappa_2,\ldots,\kappa_d\in[0,1]$, as $N\to\infty$,*

$$\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{A}_i(\kappa_1)',\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{A}_i(\kappa_2)',\cdots,\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{A}_i(\kappa_d)'\right)'\xrightarrow{d}N_{3d}(\mathbf{0},\boldsymbol{\Lambda}),$$

where $N_{3d}(\mathbf{0},\boldsymbol{\Lambda})$ refers to the $3d$-dimensional normal distribution with mean zero and covariance matrix $\boldsymbol{\Lambda}$ whose entries are given by $\Lambda_{3m_1+q_1,3m_2+q_2}=C_{q_1,q_2}(\kappa_{m_1+1},\kappa_{m_2+1})$ for $m_1,m_2\in\{0,1,\cdots,d-1\}$ and $q_1,q_2\in\{1,2,3\}$; and $C_{q_1,q_2}(\cdot,\cdot)$ is defined in (11).

Proof. For any fixed $\kappa\in[0,1]$, $m=1,2,3$, and any positive integer l, define

$$\tilde{A}_{i,l}^{(m)}(\kappa):=\sum_{k=0}^{l-1}\left[E_i\left(A_{i+k}^{(m)}(\kappa)\right)-E_{i-1}\left(A_{i+k}^{(m)}(\kappa)\right)\right],\text{ and}$$

$$R_{N,l}^{(m)}\left(\kappa\right) := \frac{1}{N}\sum_{i=1}^{N}\left(A_i^{(m)}\left(\kappa\right) - \tilde{A}_{i,l}^{(m)}\left(\kappa\right)\right),$$

where $E_i(\cdot)$ refers to expectation conditional on the sigma-field $\mathscr{G}_i$ defined in Assumption II. For $m = 1$, 2 or 3, we further define

$$\tilde{\boldsymbol{A}}_{i,\infty}\left(\kappa\right) := \left(\tilde{A}_{i,\infty}^{(1)}\left(\kappa\right), \tilde{A}_{i,\infty}^{(2)}\left(\kappa\right), \tilde{A}_{i,\infty}^{(3)}\left(\kappa\right)\right)', \quad \text{where} \quad \tilde{A}_{i,\infty}^{(m)}\left(\kappa\right) := \lim_{l\to\infty}\tilde{A}_{i,l}^{(m)}\left(\kappa\right),$$

and

$$\boldsymbol{R}_{N,\infty}\left(\kappa\right) := \left(R_{N,\infty}^{(1)}\left(\kappa\right), R_{N,\infty}^{(2)}\left(\kappa\right), R_{N,\infty}^{(3)}\left(\kappa\right)\right)', \quad \text{where} \quad R_{N,\infty}^{(m)}\left(\kappa\right) := \lim_{l\to\infty} R_{N,l}^{(m)}\left(\kappa\right).$$

By the same argument as that used in the proof of Lemma 14 in Andersen et al. (2023), we have that for any $\kappa \in [0,1]$ and positive integer N, $\tilde{\boldsymbol{A}}_{i,\infty}\left(\kappa\right)$ and $\boldsymbol{R}_{N,\infty}\left(\kappa\right)$ exist almost surely. Our proof is based on approximating $\boldsymbol{A}_i(\kappa)$ by $\tilde{\boldsymbol{A}}_{i,\infty}\left(\kappa\right)$. By the similar argument as that used in the proof of Lemma 6 in Andersen et al. (2023), we have that, for any fixed $\kappa \in [0,1]$,

$$\sqrt{N}\left\|\boldsymbol{R}_{N,\infty}(\kappa)\right\|_2 \leq C\sum_{m=1}^{3}\sqrt{N}R_{N,\infty}^{(m)}(\kappa)^2 \xrightarrow{p} 0.$$

Therefore, in the following, it suffices to prove that

$$\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{\boldsymbol{A}}_{i,\infty}\left(\kappa_1\right)', \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{\boldsymbol{A}}_{i,\infty}\left(\kappa_2\right)', \cdots, \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{\boldsymbol{A}}_{i,\infty}\left(\kappa_d\right)'\right)' \xrightarrow{d} N_{3d}(\mathbf{0}, \boldsymbol{\Lambda}),$$

It follows from Part II of the proof of Lemma 10 in the Supplementary Appendix of Andersen et al. (2023) that for $m = 1, 2, 3$, $\{1/\sqrt{N}\tilde{A}_{i,\infty}^{(m)}\left(\kappa\right)\}_{i\in\mathbb{N}}$ are martingale differences with respect to filtration $\{\mathscr{G}_i\}_{i\in\mathbb{N}}$. On the one hand, by the same argument as that used in the proof of Lemma 14 in Andersen et al. (2023), for any $\kappa \in [0,1]$ and any integer $i \geq 1$, we have

$$\sum_{m=1}^{3} E\left|\tilde{A}_{i,\infty}^{(m)}(\kappa)\right|^{3} \leq C < \infty.$$

The above fact together with the ergodic theory implies that, as $N \to \infty$,

$$\frac{1}{N^{3/2}} \sum_{m=1}^{3} \sum_{i=1}^{N} E_{i-1}\left(\left|\tilde{A}_{i,\infty}^{(m)}(\kappa)\right|^{3}\right) \xrightarrow{p} 0.$$

On the other hand, for any fixed $\kappa, \kappa' \in [0,1]$ and $m_1, m_2 \in \{1,2,3\}$, by the same argument as that used in the Step 1 of the proof of Lemma 14 of Andersen et al. (2023), we readily obtain that

$$\begin{aligned} E\left[\tilde{A}_{1,\infty}^{(m_1)}(\kappa)\,\tilde{A}_{1,\infty}^{(m_2)}(\kappa')\right] &= \sum_{k=0}^{\infty} E\left[A_{1+k}^{(m_1)}(\kappa)\,A_{1}^{(m_2)}(\kappa')\right] + \sum_{p=1}^{\infty} E\left[A_{1}^{(m_1)}(\kappa)\,A_{1+p}^{(m_2)}(\kappa')\right] \\ &= C_{m_1,m_2}(\kappa,\kappa'). \end{aligned}$$

Now by Assumption II and the ergodic theory, we have the following convergence for the conditional (co)variance processes,

$$\frac{1}{N} \sum_{i=1}^{N} E_{i-1}\left(\tilde{A}_{i,\infty}^{(m_1)}(\kappa)\,\tilde{A}_{i,\infty}^{(m_2)}(\kappa')\right) \xrightarrow{p} E\left(\tilde{A}_{1,\infty}^{(m_1)}(\kappa)\,\tilde{A}_{1,\infty}^{(m_2)}(\kappa')\right) = C_{m_1,m_2}(\kappa,\kappa').$$

The proof is therefore completed by an application of the martingale central limit theorem (see, e.g., Corollary 3.1 on page 58 of Hall and Heyde, 1980 and Theorem A.1 of Zhang et al., 2005).

$\square$

Lemma 6. *Suppose that Assumptions I, II with $q = 3$ and Condition DP hold. Moreover, assume that $N \asymp n^b$ and $\ell \asymp n^c$ with b and c satisfying condition (16).@pls double check!@*

Then,

$$\sqrt{N}\left(\begin{pmatrix}\frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}_{1,i}^{2}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}_{2,i}^{2}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}\widehat{B}_{i}(\kappa)\end{pmatrix}-\begin{pmatrix}E\sigma_{1}^{2}(\kappa)\\ E\sigma_{2}^{2}(\kappa)\\ EB(\kappa)\end{pmatrix}\right)\xrightarrow{d}\boldsymbol{\mathcal{T}}(\kappa)\quad in\quad D^{3}[0,1].$$

Proof. Since the finite dimensional convergence together with tightness implies convergence in Skorokhod space, see, e.g., Theorem 15.6 on page 128 of Billingsley (1968), we divide the proof into two steps. In the first step, we establish finite-dimensional convergence, followed by verification of the tightness condition in the second step.

Step 1. Finite dimensional convergence

Note that we have the following decomposition

$$\begin{aligned}&\sqrt{N}\left(\begin{pmatrix}\frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}_{1,i}^{2}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}_{2,i}^{2}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}\widehat{B}_{i}(\kappa)\end{pmatrix}-\begin{pmatrix}E\sigma_{1}^{2}(\kappa)\\ E\sigma_{2}^{2}(\kappa)\\ EB(\kappa)\end{pmatrix}\right)\\
&=\sqrt{N}\left(\begin{pmatrix}\frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}_{1,i}^{2}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}_{2,i}^{2}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}\widehat{B}_{i}(\kappa)\end{pmatrix}-\begin{pmatrix}E\sigma_{1}^{2}(\kappa)\\ E\sigma_{2}^{2}(\kappa)\\ EB(\kappa)\end{pmatrix}-\begin{pmatrix}\frac{1}{N}\sum_{i=1}^{N}A_{i}^{(1)}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}A_{i}^{(2)}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}A_{i}^{(3)}(\kappa)\end{pmatrix}\right)+\sqrt{N}\begin{pmatrix}\frac{1}{N}\sum_{i=1}^{N}A_{i}^{(1)}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}A_{i}^{(2)}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}A_{i}^{(3)}(\kappa)\end{pmatrix}\\
&=\begin{pmatrix}\sqrt{N}\sum_{k=1}^{4}\zeta_{k}^{(1)}(\kappa)\\ \sqrt{N}\sum_{k=1}^{4}\zeta_{k}^{(2)}(\kappa)\\ \sqrt{N}\sum_{k=1}^{6}\zeta_{k}^{(3)}(\kappa)\end{pmatrix}+\sqrt{N}\begin{pmatrix}\frac{1}{N}\sum_{i=1}^{N}A_{i}^{(1)}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}A_{i}^{(2)}(\kappa)\\ \frac{1}{N}\sum_{i=1}^{N}A_{i}^{(3)}(\kappa)\end{pmatrix}.\end{aligned}$$

The finite dimensional convergence in distribution is then a direct consequence of the finite dimensional convergence of $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{A}_{i}(\kappa)$ in Lemma 5 and the asymptotic negligibility, under condition (16) as implied by Lemma 4, of $\sqrt{N}\sum_{k=1}^{4}\zeta_{k}^{(1)}(\kappa)$, $\sqrt{N}\sum_{k=1}^{4}\zeta_{k}^{(2)}(\kappa)$ and $\sqrt{N}\sum_{k=1}^{6}\zeta_{k}^{(3)}(\kappa)$.

Step 2. Tightness

Note that coordinate-wise tightness in $D[0,1]$ implies tightness in $D^3[0,1]$. Hence, without loss of generality, we focus on proving that $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\widehat{B}_i(\kappa)-EB(\kappa)\right)=\sqrt{N}\sum_{k=1}^{7}\zeta_k^{(3)}(\kappa)$ is tight in $D[0,1]$. To this end, it suffices to prove that $\sqrt{N}\zeta_k^{(3)}(\kappa)$ is tight for each $k\in\{1,2,\cdots,7\}$.

We commence with terms $\sqrt{N}\zeta_k^{(3)}(\kappa)$ for $k\in\{1,2,\cdots,5\}$. Fix any $\kappa_1,\kappa_2,\kappa_3\in[0,1]$ such that $\kappa_1<\kappa_2<\kappa_3$. On the one hand, when $\kappa_3-\kappa_1\leq 1/n$, for any $\epsilon>0$, we have

$$\left|\sqrt{N}\zeta_k^{(3)}(\kappa_3)-\sqrt{N}\zeta_k^{(3)}(\kappa_2)\right|^{\epsilon}\left|\sqrt{N}\zeta_k^{(3)}(\kappa_2)-\sqrt{N}\zeta_k^{(3)}(\kappa_1)\right|^{\epsilon}\equiv 0.$$

On the other hand, when $\kappa_3-\kappa_1>1/n$, we have that, for any $\varpi_1>1$,

$$\begin{aligned}
&E\left(\left|\sqrt{N}\zeta_k^{(3)}(\kappa_3)-\sqrt{N}\zeta_k^{(3)}(\kappa_2)\right|^{\varpi_1}\left|\sqrt{N}\zeta_k^{(3)}(\kappa_2)-\sqrt{N}\zeta_k^{(3)}(\kappa_1)\right|^{\varpi_1}\right)\\
&\leq\left(E\left|\sqrt{N}\zeta_k^{(3)}(\kappa_3)-\sqrt{N}\zeta_k^{(3)}(\kappa_2)\right|^{2\varpi_1}\right)^{\frac{1}{2}}\left(E\left|\sqrt{N}\zeta_k^{(3)}(\kappa_2)-\sqrt{N}\zeta_k^{(3)}(\kappa_1)\right|^{2\varpi_1}\right)^{\frac{1}{2}}\\
&\leq C\left(E\left|\sqrt{N}\zeta_k^{(3)}(\kappa_3)\right|^{2\varpi_1}+E\left|\sqrt{N}\zeta_k^{(3)}(\kappa_2)\right|^{2\varpi_1}\right)^{\frac{1}{2}}\left(E\left|\sqrt{N}\zeta_k^{(3)}(\kappa_2)\right|^{2\varpi_1}+E\left|\sqrt{N}\zeta_k^{(3)}(\kappa_1)\right|^{2\varpi_1}\right)^{\frac{1}{2}}\\
&\leq C\left((N\Delta)^{\varpi_1}\vee\frac{1}{\ell^{\varpi_1}}\right)\leq C\Delta^{\alpha}\leq C\left|\kappa_3-\kappa_1\right|^{\alpha},
\end{aligned}$$

for some $\alpha>1$, where the first inequality follows from Cauchy–Schwarz inequality, the third inequality follows from the proof of Lemma 4, and the fourth inequality follows from condition (16) and by letting $\varpi_1>\alpha/\zeta_1$. Hence, when $k\in\{1,2,\cdots,5\}$, the tightness condition (see, e.g., Theorem 15.6 on page 128 of Billingsley, 1968) is obviously satisfied.

To deal with term $\sqrt{N}\zeta_6^{(3)}(\kappa)$, we need a preliminary result. For any $k>2$ and $s,t\in$

$[0,1]$ such that $s < t$, we have

$$
\begin{aligned}
& E\left|\frac{1}{N}\sum_{i=1}^{N}\left(B_i(t)-B_i(s)\right)\right|^k \\
\leq\; & CE\left|\frac{g_B(t)-g_B(s)}{N}\sum_{i=1}^{N}\tilde{B}_i(t)\right|^k + CE\left|\frac{g_B(s)}{N}\sum_{i=1}^{N}\left(\tilde{B}_i(t)-\tilde{B}_i(s)\right)\right|^k \\
\leq\; & C\left|g_B(t)-g_B(s)\right|^k \frac{1}{N}\sum_{i=1}^{N} E\left|\tilde{B}_i(t)\right|^k + CE\left|\frac{1}{N}\sum_{i=1}^{N}\int_{i-1+s}^{i-1+t}\breve{\mu}_3(u)du\right|^k \\
& + CE\left|\frac{1}{N}\sum_{i=1}^{N}\int_{i-1+s}^{i-1+t}\breve{\sigma}_3(u)d\breve{W}_3(u)\right|^k \leq C\left(|t-s|^k \vee \frac{|t-s|^{\frac{k}{2}}}{N^{\frac{k}{2}}}\right), \qquad (33)
\end{aligned}
$$

where the last inequality follows from Jensen's inequality, Burkholder-Davis-Gundy inequality, Assumptions I and the Lipschitz continuity of $g_B(\cdot)$. We are now ready to deal with term $\sqrt{N}\zeta_6^{(3)}(\kappa)$. We again fix any $\kappa_1, \kappa_2, \kappa_3 \in [0,1]$ such that $\kappa_1 < \kappa_2 < \kappa_3$. Three different situations emerge. The first situation refers to the case where $|\kappa_3 - \kappa_1| \leq \Delta$ and $\kappa_1, \kappa_2, \kappa_3$ locate in the same time interval between two successive prices at the highest frequency. In this case, $j_{\kappa_1} = j_{\kappa_2} = j_{\kappa_3}$. Hence, we have

$$
\begin{aligned}
& E\left(\left|\sqrt{N}\zeta_6^{(3)}(\kappa_3)-\sqrt{N}\zeta_6^{(3)}(\kappa_2)\right|^2\left|\sqrt{N}\zeta_6^{(3)}(\kappa_2)-\sqrt{N}\zeta_6^{(3)}\left(\kappa_1\right)\right|^2\right) \\
\leq\; & \left(E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(B_i(\kappa_3)-B_i(\kappa_2)\right)\right|^4\right)^{1/2}\left(E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(B_i(\kappa_2)-B_i(\kappa_1)\right)\right|^4\right)^{1/2} \\
\leq\; & C\left(N^2\left|\kappa_3-\kappa_1\right|^4 \vee \left|\kappa_3-\kappa_1\right|^2\right) \leq C\left|\kappa_3-\kappa_1\right|^2,
\end{aligned}
$$

where the first inequality follows from Cauchy–Schwarz inequality, the second inequality follows from (33), and the last inequality is a consequence of condition (16). The second situation refers to the case where $|\kappa_3 - \kappa_1| \leq \Delta$ and only two of κ_1, κ_2 and κ_3 locate in the same interval between two successive observations. Without loss of generality, we assume that κ_3 and κ_2 locate in the same interval. In this case, we have

$$
\begin{aligned}
& E\left(\left|\sqrt{N}\zeta_6^{(3)}(\kappa_3)-\sqrt{N}\zeta_6^{(3)}(\kappa_2)\right|^4\left|\sqrt{N}\zeta_6^{(3)}(\kappa_2)-\sqrt{N}\zeta_6^{(3)}(\kappa_1)\right|^4\right) \\
\leq\; & C\left(E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(B_i(\kappa_3)-B_i(\kappa_2)\right)\right|^8\right)^{1/2}\left(E\left|\sqrt{N}\zeta_6^{(3)}(\kappa_2)\right|^8+E\left|\sqrt{N}\zeta_6^{(3)}(\kappa_1)\right|^8\right)^{1/2} \\
\leq\; & C\left|\kappa_3-\kappa_1\right|^2,
\end{aligned}
$$

where the first inequality follows from Cauchy–Schwarz inequality and the second inequality follows from (32), (33), and condition (16). We now turn to the third situation where $|\kappa_3-\kappa_1|>\Delta$. By result (32) for the term $\zeta_6^{(3)}(\kappa)$ in the proof of Lemma 4, Cauchy-Schwarz inequality and condition (16), we have that, for some $\alpha>1$, as long as $\varpi_2>\alpha/\zeta_2$,

$$
\begin{aligned}
& E\left(\left|\sqrt{N}\zeta_6^{(3)}(\kappa_3)-\sqrt{N}\zeta_6^{(3)}(\kappa_2)\right|^{\varpi_2}\left|\sqrt{N}\zeta_6^{(3)}(\kappa_2)-\sqrt{N}\zeta_6^{(3)}(\kappa_1)\right|^{\varpi_2}\right) \\
\leq\; & N^{\varpi_2}\left(E\left|\zeta_6^{(3)}(\kappa_3)-\zeta_6^{(3)}(\kappa_2)\right|^{2\varpi_2}\right)^{\frac{1}{2}}\left(E\left|\zeta_6^{(3)}(\kappa_2)-\zeta_6^{(3)}(\kappa_1)\right|^{2\varpi_2}\right)^{\frac{1}{2}} \\
\leq\; & CN^{\varpi_2}\left(E\left|\zeta_6^{(3)}(\kappa_3)\right|^{2\varpi_2}+E\left|\zeta_6^{(3)}(\kappa_2)\right|^{2\varpi_2}\right)^{\frac{1}{2}}\left(E\left|\zeta_6^{(3)}(\kappa_2)\right|^{2\varpi_2}+E\left|\zeta_6^{(3)}(\kappa_1)\right|^{2\varpi_2}\right)^{\frac{1}{2}} \\
\leq\; & C\left(N^{\varpi_2}(\ell\Delta)^{2\varpi_2}\vee(\ell\Delta)^{\varpi_2}\right)\leq C\Delta^{\alpha}\leq C\left|\kappa_3-\kappa_1\right|^{\alpha}.
\end{aligned}
$$

We finally turn to term $\sqrt{N}\zeta_7^{(3)}(\kappa)$. Recalling the definitions in (3), (4) and (27), we further define

$$
\begin{cases}
\zeta_{7,1}^{(3)}(\kappa) := \frac{g_B(\kappa)}{N}\sum_{i=1}^{N}\int_{i-1}^{i-1+\kappa}\breve{\mu}_3(u)du, \\
\zeta_{7,2}^{(3)}(\kappa) := \frac{g_B(\kappa)}{N}\sum_{i=1}^{N}\int_{i-1}^{i-1+\kappa}\breve{\sigma}_3(u)d\breve{W}_3(u); \\
\zeta_{7,3}^{(3)}(\kappa) := -\frac{g_B(\kappa)}{N}\sum_{i=1}^{N}\tilde{B}_i^c(0).
\end{cases}
$$

It is easy to see that $\zeta_7^{(3)}(\kappa) = \sum_{i=1}^{3} \zeta_{7,i}^{(3)}(\kappa)$. Hence, it suffices to show that each of the three terms $\sqrt{N}\zeta_{7,1}^{(3)}(\kappa)$, $\sqrt{N}\zeta_{7,2}^{(3)}(\kappa)$ and $\sqrt{N}\zeta_{7,3}^{(3)}(\kappa)$ is tight. To this end, we again consider any fixed $\kappa_1, \kappa_2, \kappa_3 \in [0,1]$ such that $\kappa_1 < \kappa_2 < \kappa_3$. For term $\sqrt{N}\zeta_{7,1}^{(3)}(\kappa)$, we have

$$
\begin{aligned}
&E\left|\sqrt{N}\zeta_{7,1}^{(3)}(\kappa_3) - \sqrt{N}\zeta_{7,1}^{(3)}(\kappa_2)\right|^2 \\
&\leq CE\left|\frac{g_B(\kappa_3) - g_B(\kappa_2)}{\sqrt{N}} \sum_{i=1}^{N} \int_{i-1}^{i-1+\kappa_3} \breve{\mu}_3(u)du\right|^2 + CE\left|\frac{g_B(\kappa_2)}{\sqrt{N}} \sum_{i=1}^{N} \int_{i-1+\kappa_2}^{i-1+\kappa_3} \breve{\mu}_3(u)du\right|^2 \\
&\leq C|g_B(\kappa_3) - g_B(\kappa_2)|^2 \sum_{h=0}^{N-1} \frac{N-h}{N} G(h) + C|\kappa_3 - \kappa_2|^2 \sum_{h=0}^{N-1} \frac{N-h}{N} G(h) \leq C|\kappa_3 - \kappa_2|^2,
\end{aligned}
$$

where the second inequality follows from Condition DP and the last inequality follows from the Lipschitz continuity of $g_B(\cdot)$. Therefore, by Cauchy-Schwarz inequality and the above result, we obtain

$$
\begin{aligned}
&E\left(\left|\sqrt{N}\zeta_{7,1}^{(3)}(\kappa_3) - \sqrt{N}\zeta_{7,1}^{(3)}(\kappa_2)\right| \left|\sqrt{N}\zeta_{7,1}^{(3)}(\kappa_2) - \sqrt{N}\zeta_{7,1}^{(3)}(\kappa_1)\right|\right) \\
\leq &\left(E\left|\sqrt{N}\zeta_{7,1}^{(3)}(\kappa_3) - \sqrt{N}\zeta_{7,1}^{(3)}(\kappa_2)\right|^2\right)^{\frac{1}{2}} \left(E\left|\sqrt{N}\zeta_{7,1}^{(3)}(\kappa_2) - \sqrt{N}\zeta_{7,1}^{(3)}(\kappa_1)\right|^2\right)^{\frac{1}{2}} \leq C\left|\kappa_3 - \kappa_1\right|^2.
\end{aligned}
$$

To deal with term $\sqrt{N}\zeta_{7,2}^{(3)}(\kappa)$, we need the following notations:

$$
M_5(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{t\wedge i-1}^{t\wedge i-1+\kappa_3} \breve{\sigma}_3(u)d\breve{W}_3(u), \quad \text{and} \quad M_6(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{t\wedge i-1+\kappa_2}^{t\wedge i-1+\kappa_3} \breve{\sigma}_3(u)d\breve{W}_3(u).
$$

which are continuous martingales on the interval $[0, N]$. Their quadratic variation processes are, respectively, given by

$$
\left[M_5, M_5\right](t) = \frac{1}{N} \sum_{i=1}^{N} \int_{t\wedge i-1}^{t\wedge i-1+\kappa_3} \breve{\sigma}_3^2(u)du \quad \text{and} \quad \left[M_6, M_6\right](t) = \frac{1}{N} \sum_{i=1}^{N} \int_{t\wedge i-1+\kappa_2}^{t\wedge i-1+\kappa_3} \breve{\sigma}_3^2(u)du.
$$

It is easy to see that

$$\sqrt{N}\zeta_{7,2}^{(3)}(\kappa_3) - \sqrt{N}\zeta_{7,2}^{(3)}(\kappa_2) = (g_B(\kappa_3) - g_B(\kappa_2))\, M_5(N) + g_B(\kappa_2)M_6(N).$$

Hence, by Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned}
&E\left|\sqrt{N}\zeta_{7,2}^{(3)}(\kappa_3) - \sqrt{N}\zeta_{7,2}^{(3)}(\kappa_2)\right|^4 \\
&\leq C\left|g_B(\kappa_3) - g_B(\kappa_2)\right|^4 E\left|[M_5, M_5](N)\right|^2 + CE\left|[M_6, M_6](N)\right|^2 \\
&\leq C|\kappa_3 - \kappa_2|^4 + C|\kappa_3 - \kappa_2|^2 \leq C|\kappa_3 - \kappa_2|^2.
\end{aligned}$$

Similarly, we also obtain

$$E\left|\sqrt{N}\zeta_{7,2}^{(3)}(\kappa_2) - \sqrt{N}\zeta_{7,2}^{(3)}(\kappa_1)\right|^4 \leq C|\kappa_2 - \kappa_1|^2.$$

Therefore, by Cauchy-Schwarz inequality, we have

$$E\left(\left|\sqrt{N}\zeta_{7,2}^{(3)}(\kappa_3) - \sqrt{N}\zeta_{7,2}^{(3)}(\kappa_2)\right|^2 \left|\sqrt{N}\zeta_{7,2}^{(3)}(\kappa_2) - \sqrt{N}\zeta_{7,2}^{(3)}(\kappa_1)\right|^2\right) \leq C\left|\kappa_3 - \kappa_1\right|^2.$$

Turning to term $\zeta_{7,3}^{(3)}(\kappa)$, we have

$$\begin{aligned}
&E\left(\left|\sqrt{N}\zeta_{7,3}^{(3)}(\kappa_3) - \sqrt{N}\zeta_{7,3}^{(3)}(\kappa_2)\right| \left|\sqrt{N}\zeta_{7,3}^{(3)}(\kappa_2) - \sqrt{N}\zeta_{7,3}^{(3)}(\kappa_1)\right|\right) \\
&\leq C\left|g_B(\kappa_3) - g_B(\kappa_1)\right|^2 E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{B}_i^c(0)\right|^2 \leq C\left|\kappa_3 - \kappa_1\right|^2,
\end{aligned}$$

where the last inequality follows from Assumption II and the same arguments as that used in the proof immediately following Eq.(C.4) in the online appendix of Andersen et al. (2023). Therefore, the tightness condition on page 128 of Billingsley (1968) is satisfied.

We are done.

$\square$

We first clarify the norms used for different function spaces. We equip the space $D[0,1]$ with supremum norm, i.e., $\|x\| = \sup_{t\in[0,1]} |x(t)|$ for any $x \in D[0,1]$. The norm for the product space $D^3[0,1] := D[0,1] \times D[0,1] \times D[0,1]$ is defined as follows: for any $z = (z_1, z_2, z_3)' \in D^3[0,1]$,

$$\|z\| = \|z_1\| \vee \|z_2\| \vee \|z_3\|.$$

Hence, the convergence of sequences in $D^3[0,1]$ is equivalent to the coordinatewise convergence. Now we introduce two maps $\phi_1 : D^3[0,1] \mapsto D[0,1]$ and $\phi_2 : D[0,1] \mapsto D[0,1]$. They are defined, respectively, as follows: for some $\gamma \in C[0,1]$ and any $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) \in D^3[0,1]$ and $\theta \in D[0,1]$,

$$\begin{cases} \phi_1(\boldsymbol{\theta}) := \frac{\theta_3}{\sqrt{\theta_1\theta_2}}, \\ \phi_2(\theta) := \theta - \gamma\sqrt{\int_0^1 \theta^2(s)ds}. \end{cases}$$

The following lemma shows that, under appropriate conditions, the two maps $\phi_1 : D^3[0,1] \mapsto D[0,1]$ and $\phi_2 : D[0,1] \mapsto D[0,1]$ are Hadamard differentiable.

Lemma 7. *(i) Suppose that $\boldsymbol{\theta}_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in C^3[0,1]$ such that $\inf_{\kappa\in[0,1]} \theta_{0,m}(\kappa) \geq \underline{c} > 0$ for $m = 1, 2$ and some constant $\underline{c}$. Then, $\phi_1(\cdot)$ is Hadamard differentiable at $\boldsymbol{\theta}_0$ tangentially to $C^3[0,1]$;*

(ii) Suppose that $\gamma, \theta_0 \in C[0,1]$ such that $\int_0^1 \theta_0^2(\kappa)d\kappa \geq \underline{c} > 0$ for some constant $\underline{c}$. Then, $\phi_2(\cdot)$ is Hadamard differentiable at θ_0 tangentially to $C[0,1]$.

Proof. We prove (i) first. For any $\boldsymbol{h}_t \in D^3[0,1]$ such that

$$\boldsymbol{h}_t := (h_{1,t}, h_{2,t}, h_{3,t})' \to \boldsymbol{h} := (h_1, h_2, h_3)' \in C^3[0,1]$$

as $t \to 0$. Note that $\|\boldsymbol{h}_t - \boldsymbol{h}\| \to 0$ readily implies the coordinatewise convergence, i.e.,

$$\|h_{i,t} - h_i\| \to 0$$

for $i = 1, 2, 3$. For the fixed $\boldsymbol{\theta}_0$, we further introduce a map $\varphi^{(1)}_{\boldsymbol{\theta}_0} : C^3[0,1] \mapsto C[0,1]$ defined as follows: for any $\boldsymbol{x} = (x_1, x_2, x_3) \in C^3[0,1]$,

$$\varphi^{(1)}_{\boldsymbol{\theta}_0}(\boldsymbol{x}) := \frac{x_3}{\sqrt{\theta_{0,1}\theta_{0,2}}} - \frac{\theta_{0,3}x_1}{2\theta_{0,1}^{\frac{3}{2}}\theta_{0,2}^{\frac{1}{2}}} - \frac{\theta_{0,3}x_2}{2\theta_{0,1}^{\frac{1}{2}}\theta_{0,2}^{\frac{3}{2}}}.$$

We next prove that $\varphi^{(1)}_{\boldsymbol{\theta}_0}(\boldsymbol{h})$ is the derivative of ϕ_1 at $\boldsymbol{\theta}_0$ along the direction $\boldsymbol{h}$. To this end, we bound the difference between $\frac{\phi_1(\boldsymbol{\theta}_0 + t\boldsymbol{h}_t) - \phi_1(\boldsymbol{\theta}_0)}{t}$ and $\varphi^{(1)}_{\boldsymbol{\theta}_0}(\boldsymbol{h})$ as follows:

$$\begin{aligned}
&\left\| \frac{\phi_1(\boldsymbol{\theta}_0 + t\boldsymbol{h}_t) - \phi_1(\boldsymbol{\theta}_0)}{t} - \varphi^{(1)}_{\boldsymbol{\theta}_0}(\boldsymbol{h}) \right\| \\
\leq &\left\| \frac{h_{3,t}}{\sqrt{(\theta_{0,1} + th_{1,t})(\theta_{0,2} + th_{2,t})}} - \frac{h_3}{\sqrt{\theta_{0,1}\theta_{0,2}}} \right\| \\
&+ \left\| \left(\frac{h_{2,t}}{\sqrt{\theta_{0,1}\theta_{0,2}} + \sqrt{(\theta_{0,1} + th_{1,t})(\theta_{0,2} + th_{2,t})}} \right.\right. \\
&\qquad \left.\left. \times \frac{1}{\sqrt{(\theta_{0,1} + th_{1,t})(\theta_{0,2} + th_{2,t})}} - \frac{h_2}{2\theta_{0,1}\theta_{0,2}} \right) \frac{\theta_{0,3}\sqrt{\theta_{0,1}}}{\sqrt{\theta_{0,2}}} \right\| \\
&+ \left\| \left(\frac{h_{1,t}}{\sqrt{\theta_{0,1}\theta_{0,2}} + \sqrt{(\theta_{0,1} + th_{1,t})(\theta_{0,2} + th_{2,t})}} \right.\right.
\end{aligned}$$

$$\times \frac{1}{\sqrt{(\theta_{0,1}+th_{1,t})(\theta_{0,2}+th_{2,t})}} - \frac{h_1}{2\theta_{0,1}\theta_{0,2}}\Bigg) \frac{\theta_{0,3}\sqrt{\theta_{0,2}}}{\sqrt{\theta_{0,1}}}\Bigg\|$$

$$+ \left\| \frac{t\theta_{0,3}h_{1,t}h_{2,t}}{\sqrt{\theta_{0,1}\theta_{0,2}(\theta_{0,1}+th_{1,t})(\theta_{0,2}+th_{2,t})}} \frac{1}{\sqrt{\theta_{0,1}\theta_{0,2}}+\sqrt{(\theta_{0,1}+th_{1,t})(\theta_{0,2}+th_{2,t})}} \right\|$$

$$=: I + II + III + IV.$$

We next deal with terms I, II, III, and IV one by one.

For the first term I, by the condition that $\|\boldsymbol{h}_t - \boldsymbol{h}\| \to 0$ and $\inf_{\kappa\in[0,1]} \theta_{0,m}(\kappa) \geq \underline{c} > 0$ for $m = 1, 2$ and some constant $\underline{c}$, for sufficiently small $t > 0$, we have

$$\left\| \frac{h_{3,t} - h_3}{\sqrt{(\theta_{0,1}+th_{1,t})(\theta_{0,2}+th_{2,t})}} \right\| \leq C \left\| h_{3,t} - h_3 \right\|.$$

Similarly, for sufficiently small $t > 0$, we also have

$$\begin{aligned}
&\left\| \frac{h_3}{\sqrt{(\theta_{0,2}+th_{2,t})}} \left(\frac{1}{\sqrt{\theta_{0,1}+th_{1,t}}} - \frac{1}{\sqrt{\theta_{0,1}}} \right) \right\| \\
&\leq \left\| \frac{h_3}{\sqrt{\theta_{0,2}+th_{2,t}}} \right\| \left\| \frac{1}{\sqrt{\theta_{0,1}+th_{1,t}}} - \frac{1}{\sqrt{\theta_{0,1}}} \right\| \\
&= t \left\| \frac{h_3}{\sqrt{\theta_{0,2}+th_{2,t}}} \right\| \left\| \frac{h_{1,t}}{\sqrt{\theta_{0,1}\left(\theta_{0,1}+th_{1,t}\right)}\left(\sqrt{\theta_{0,1}+th_{1,t}}+\sqrt{\theta_{0,1}}\right)} \right\| \leq Ct \left\| h_3 \right\| \left\| h_{1,t} \right\|,
\end{aligned}$$

and

$$\left\| \frac{h_3}{\sqrt{\theta_{0,1}}} \left(\frac{1}{\sqrt{\theta_{0,2}+th_{2,t}}} - \frac{1}{\sqrt{\theta_{0,2}}} \right) \right\| \leq Ct \left\| h_3 \right\| \left\| h_{2,t} \right\|.$$

Therefore, as $t \to 0$, we obtain that

$$
\begin{aligned}
I \leq & \left\| \frac{h_{3,t} - h_3}{\sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}} \right\| + \left\| \frac{h_3}{\sqrt{(\theta_{0,2} + t h_{2,t})}} \left(\frac{1}{\sqrt{\theta_{0,1} + t h_{1,t}}} - \frac{1}{\sqrt{\theta_{0,1}}} \right) \right\| \\
& + \left\| \frac{h_3}{\sqrt{\theta_{0,1}}} \left(\frac{1}{\sqrt{\theta_{0,2} + t h_{2,t}}} - \frac{1}{\sqrt{\theta_{0,2}}} \right) \right\| \to 0.
\end{aligned}
$$

As to the second term II, we bound it as follows:

$$
\begin{aligned}
II \leq & \left\| \frac{\theta_{0,3}\sqrt{\theta_{0,1}}}{\sqrt{\theta_{0,2}}} \right\| \\
& \times \left\| \frac{h_{2,t}}{\sqrt{\theta_{0,1}\theta_{0,2}} + \sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}} \frac{1}{\sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}} - \frac{h_2}{2\theta_{0,1}\theta_{0,2}} \right\| \\
\leq & \, C \left\| \frac{h_{2,t}}{\sqrt{\theta_{0,1}\theta_{0,2}} + \sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}} \left(\frac{1}{\sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}} - \frac{1}{\sqrt{\theta_{0,1}\theta_{0,2}}} \right) \right\| \\
& + C \left\| \frac{h_{2,t}}{\sqrt{\theta_{0,1}\theta_{0,2}}} \left(\frac{1}{\sqrt{\theta_{0,1}\theta_{0,2}} + \sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}} - \frac{1}{2\sqrt{\theta_{0,1}\theta_{0,2}}} \right) \right\| \\
& + C \left\| \frac{h_{2,t} - h_2}{2\theta_{0,1}\theta_{0,2}} \right\| := II_1 + II_2 + II_3.
\end{aligned}
$$

We take II_1 for example. The other two terms can be treated using the same arguments as those employed for dealing with terms I and II_1. For sufficiently small $t > 0$, we have

$$
\begin{aligned}
II_1 = & \, C \left\| \frac{h_{2,t}\left(t\theta_{0,1} h_{2,t} + t\theta_{0,2} h_{1,t} + t^2 h_{1,t} h_{2,t}\right)}{\left(\sqrt{\theta_{0,1}\theta_{0,2}} + \sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}\right)^2} \frac{1}{\sqrt{\theta_{0,1}\theta_{0,2}}\sqrt{(\theta_{0,1} + t h_{1,t})(\theta_{0,2} + t h_{2,t})}} \right\| \\
\leq & \, Ct \left\| \theta_{0,1} h_{2,t}^2 \right\| + Ct \left\| \theta_{0,2} h_{1,t} h_{2,t} \right\| + Ct^2 \left\| h_{1,t} h_{2,t}^2 \right\|.
\end{aligned}
$$

Hence, as $t \to 0$, $II_1 \to 0$. Therefore, we obtain

$$\lim_{t\to 0} II = 0.$$

By the same arguments as those used in dealing with terms I and II, we also have

$$\lim_{t\to 0} III + \lim_{t\to 0} IV = 0.$$

To sum up, we have obtained that

$$\lim_{t\to 0} \left\| \frac{\phi_1(\boldsymbol{\theta}_0 + t\boldsymbol{h}_t) - \phi_1(\boldsymbol{\theta}_0)}{t} - \varphi^{(1)}_{\boldsymbol{\theta}_0}(\boldsymbol{h}) \right\| = 0,$$

completing the proof of claim (i).

We now turn to prove (ii). For any $h_t \in D[0,1]$ such that

$$h_t \to h \in C[0,1],$$

as $t \to 0$. For the fixed $\theta_0 \in C[0,1]$, we further introduce a map $\varphi^{(2)}_{\theta_0} : C[0,1] \mapsto C[0,1]$ defined as follows: for any $x \in C[0,1]$,

$$\varphi^{(2)}_{\theta_0}(x) := x - \gamma \frac{\int_0^1 \theta_0(s)x(s)ds}{\sqrt{\int_0^1 \theta_0^2(s)ds}}.$$

We next prove that $\varphi^{(2)}_{\theta_0}(h)$ is the derivative of ϕ_2 at θ_0 along the direction h. To this end, we bound the difference between $\frac{\phi_2(\theta_0+th_t)-\phi_2(\theta_0)}{t}$ and $\varphi^{(2)}_{\theta_0}(h)$ as follows:

$$\begin{aligned}
&\left\|\frac{\phi_2(\theta_0+th_t)-\phi_2(\theta_0)}{t}-\varphi^{(2)}_{\theta_0}(h)\right\| \\
&\leq \|h_t-h\|+\|\gamma\|\left|\frac{2\int_0^1\theta_0(s)h_t(s)ds}{\sqrt{\int_0^1\left(\theta_0(s)+th_t(s)\right)^2ds}+\sqrt{\int_0^1\theta_0^2(s)ds}}-\frac{\int_0^1\theta_0(s)h(s)ds}{\sqrt{\int_0^1\theta_0^2(s)ds}}\right| \\
&\quad +t\|\gamma\|\left|\frac{\int_0^1 h_t^2(s)ds}{\sqrt{\int_0^1\left(\theta_0(s)+th_t(s)\right)^2ds}+\sqrt{\int_0^1\theta_0^2(s)ds}}\right| =: (I)+(II)+(III).
\end{aligned}$$

Obviously, $(I)\to 0$ as $t\to 0$. We next deal with term (II). Under the condition that $\int_0^1\theta_0^2(\kappa)d\kappa\geq \underline{c}>0$ for some constant $\underline{c}$, by using the same arguments as those used in the proof of claim (i), we have that

$$\begin{aligned}
(II)&\leq C\left|\frac{\int_0^1\theta_0(s)\left(h_t(s)-h(s)\right)ds}{\sqrt{\int_0^1\left(\theta_0(s)+th_t(s)\right)^2ds}+\sqrt{\int_0^1\theta_0^2(s)ds}}\right| \\
&\quad +C\left|\frac{\int_0^1\theta_0(s)h(s)ds\int_0^1\left(2t\theta_0(s)h_t(s)+t^2h_t^2(s)\right)ds}{\left(\sqrt{\int_0^1\left(\theta_0(s)+th_t(s)\right)^2ds}+\sqrt{\int_0^1\theta_0^2(s)ds}\right)^2\sqrt{\int_0^1\theta_0^2(s)ds}}\right|\to 0
\end{aligned}$$

as $t\to 0$. It is obvious that

$$\lim_{t\to 0}(III)=0.$$

To sum up, we have obtained that

$$\lim_{t\to 0}\left\|\frac{\phi_2(\theta_0+th_t)-\phi_2(\theta_0)}{t}-\varphi^{(2)}_{\theta_0}(h)\right\|=0,$$

completing the proof of (ii).

We are done.

$\square$

Proof of Theorem 1. Note first that we have the following decomposition:

$$\begin{aligned}\sqrt{N}\left(\hat{f}_\rho(\kappa)-f_\rho(\kappa)\right) &= \frac{\sqrt{N}}{\hat{\eta}}\left(\hat{\mathfrak{g}}_\rho(\kappa)-f_\rho(\kappa)\hat{\eta}\right)\\ &= \frac{\sqrt{N}}{\hat{\eta}}\left(\left(\hat{\mathfrak{g}}_\rho(\kappa)-f_\rho(\kappa)\hat{\eta}\right)-\left(c_0 g_\rho(\kappa)-f_\rho(\kappa)\eta\right)\right),\end{aligned}$$

where

$$\eta = c_0\left(\int_0^1 g_\rho(\kappa)^2 d\kappa\right)^{1/2}.$$

Now define $\boldsymbol{\theta}_0 \in D^3[0,1]$ as follows: for any $\kappa \in [0,1]$,

$$\boldsymbol{\theta}_0(\kappa) = (\theta_{0,1}(\kappa), \theta_{0,2}(\kappa), \theta_{0,3}(\kappa))' := \left(E\sigma_1^2(\kappa), E\sigma_2^2(\kappa), EB(\kappa)\right)'.$$

Under Condition IP, we obviously have that $\boldsymbol{\theta}_0 \in C^3[0,1]$ and $\inf_{\kappa\in[0,1]}\theta_{0,m}(\kappa) \geq \underline{c} > 0$ for $m = 1, 2$ and some constant $\underline{c}$. It then follows from (i) of Lemma 7 that $\phi_1(\cdot)$ is Hadamard differentiable at $\boldsymbol{\theta}_0$ tangentially to $C^3[0,1]$. Now by Theorem 20.8 of Van der Vaart (2000) and Lemma 6, we obtain that, under the assumptions and conditions of the theorem,

$$\sqrt{N}\left(\hat{\mathfrak{g}}_\rho(\kappa)-c_0 g_\rho(\kappa)\right) \xrightarrow{d} \widetilde{\mathcal{T}}(\kappa) \quad \text{in} \quad D[0,1], \tag{34}$$

where the limiting process $\widetilde{\mathcal{T}}(\kappa)$ is defined by (12) in conjunction with (11).

Convergence result in (34) readily implies that,

$$\hat{\mathfrak{g}}_\rho(\kappa)-c_0 g_\rho(\kappa) = O_p\left(\frac{1}{\sqrt{N}}\right) = o_p(1) \quad \text{in} \quad D[0,1].$$

Note that @make sure the numerical implementation follows this strictly???@

$$\hat{\eta} = \sqrt{\Delta \sum_{j=1}^{n} \hat{\mathfrak{g}}_\rho(j\Delta)^2} = \sqrt{\int_0^1 \hat{\mathfrak{g}}_\rho(\kappa)^2 d\kappa}.$$

We define a map $\phi_3 : D[0,1] \mapsto \mathbb{R}$ as follows: for any $g \in D[0,1]$,

$$\phi_3(g) := \sqrt{\int_0^1 g(\kappa)^2 d\kappa}.$$

It follows easily that the map ϕ_3 is continuous at any point $x \in C_0[0,1] \subset D[0,1]$ such that $x \in C_0[0,1]$ is continuous and satisfies $\int_0^1 x(\kappa)^2 d\kappa > \underline{c} > 0$ for some constant $\underline{c}$. Under Condition IP, $c_0 g_\rho$ is continuous with $\int_0^1 (c_0 g_\rho(\kappa))^2 d\kappa$ being bounded away below from zero. It then follows from continuous mapping theorem that,

$$\hat{\eta} - \eta = o_p(1). \tag{35}$$

We finally let

$$\theta_0(\kappa) = c_0 g_\rho(\kappa) \quad \text{and} \quad \gamma = f_\rho(\kappa).$$

Under Condition IP, $\theta_0 \in C[0,1]$, $\int_0^1 \theta_0^2(\kappa) d\kappa$ is bounded away below from zero, and $\gamma \in C[0,1]$. It then follows from (ii) of Lemma 7 and Theorem 20.8 of Van der Vaart (2000) again that,

$$\sqrt{N}\Big(\big(\hat{\mathfrak{g}}_\rho(\kappa) - f_\rho(\kappa)\hat{\eta}\big) - \big(c_0 g_\rho(\kappa) - f_\rho(\kappa)\eta\big)\Big) \xrightarrow{d} \breve{\mathcal{T}}(\kappa) \quad \text{in} \quad D[0,1]. \tag{36}$$

The conclusion of the theorem is then a consequence of (35), (36) and an application of Slutsky's theorem.

$\square$

Proof of Theorem 3. Without loss of generality, we prove the theorem for the case where $i = 1$ and $j = 3$. Note that, for any $t, s \in [0, 1]$,

$$
\begin{aligned}
&(C_{1,3}(t,s) - \widehat{C}_{1,3}(t,s))^2 \\
&\leq C \Bigg[E\left(A_1^{(1)}(t)A_1^{(3)}(s)\right) + \sum_{h=1}^{\infty} E\left(A_1^{(1)}(t)A_{1+h}^{(3)}(s) + A_{1+h}^{(1)}(t)A_1^{(3)}(s)\right) \\
&\quad - \left(\frac{1}{N}\sum_{i=1}^{N} A_i^{(1)}(t)A_i^{(3)}(s) + \sum_{h=1}^{L_n} \frac{1}{N-h} \sum_{i=1}^{N} \left[A_i^{(1)}(t)\left(A_{i+h}^{(3)}(s) + A_{i-h}^{(3)}(s)\right)\right]\right)\Bigg]^2 \\
&\quad + C \Bigg[\frac{1}{N}\sum_{i=1}^{N} A_i^{(1)}(t)A_i^{(3)}(s) + \sum_{h=1}^{L_n} \frac{1}{N-h} \sum_{i=1}^{N} \left[A_i^{(1)}(t)\left(A_{i+h}^{(3)}(s) + A_{i-h}^{(3)}(s)\right)\right] \\
&\quad - \left(\frac{1}{N}\sum_{i=1}^{N} \widehat{A}_i^{(1)}(t)\widehat{A}_i^{(3)}(s) + \sum_{h=1}^{L_n} \frac{1}{N-h} \sum_{i=1}^{N} \left[\widehat{A}_i^{(1)}(t)\left(\widehat{A}_{i+h}^{(3)}(s) + \widehat{A}_{i-h}^{(3)}(s)\right)\right]\right)\Bigg]^2 =: I + II
\end{aligned}
$$

with the convention that $A_i^{(1)}(t) = \widehat{A}_i^{(1)}(t) \equiv 0$ for $i \leq 0$ and $i > N$.

We treat term I first as follows:

$$
\begin{aligned}
I &\leq C\left(E\left(A_1^{(1)}(t)A_1^{(3)}(s)\right) - \frac{1}{N}\sum_{i=1}^{N} A_i^{(1)}(t)A_i^{(3)}(s)\right)^2 \\
&\quad + C\left(\sum_{h=L_n+1}^{\infty} E\left(A_1^{(1)}(t)A_{1+h}^{(3)}(s) + A_{1+h}^{(1)}(t)A_1^{(3)}(s)\right)\right)^2 \\
&\quad + C\left(\sum_{h=1}^{L_n}\left\{E\left(A_1^{(1)}(t)A_{1+h}^{(3)}(s) + A_{1+h}^{(1)}(t)A_1^{(3)}(s)\right) - \frac{1}{N-h}\sum_{i=1}^{N}\left[A_i^{(1)}(t)\left(A_{i+h}^{(3)}(s) + A_{i-h}^{(3)}(s)\right)\right]\right\}\right)^2 \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

As to term I_1, because of the stationarity of $A_1^{(1)}(t)$, as implied by Assumption II,

$$E\left(A_i^{(1)}(t)A_i^{(3)}(s)\right) = E\left(A_1^{(1)}(t)A_1^{(3)}(s)\right).$$

By Corollary 14.3 on page 212 of Davidson (1994), we have that, for $j > i$,

$$\begin{aligned}\mathrm{Cov}\left(A_i^{(1)}(t)A_i^{(3)}(s), A_j^{(1)}(t)A_j^{(3)}(s)\right) &\leq C\alpha_{j-i}^{1-\frac{2}{\omega}}\left(E\left|A_i^{(1)}(t)A_i^{(3)}(s)\right|^{\omega}\right)^{\frac{1}{\omega}}\left(E\left|A_j^{(1)}(t)A_j^{(3)}(s)\right|^{\omega}\right)^{\frac{1}{\omega}} \\ &\leq C\alpha_{j-i}^{1-\frac{2}{\omega}}\end{aligned}$$

for any $\omega > 2$. Then, by taking $\omega > 2(3+\iota)/(2+\iota)$ where ι is given in Assumption II with $q = 3$, we obtain

$$\begin{aligned}EI_1 &= E\left|\frac{C}{N}\sum_{i=1}^{N}\left[A_i^{(1)}(t)A_i^{(3)}(s) - E\left(A_i^{(1)}(t)A_i^{(3)}(s)\right)\right]\right|^2 \\ &\leq \frac{C}{N^2}\sum_{i=1}^{N}\sum_{j=i}^{N}\mathrm{Cov}\left(A_i^{(1)}(t)A_i^{(3)}(s), A_j^{(1)}(t)A_j^{(3)}(s)\right) \leq \frac{C}{N}.\end{aligned}$$

As to term I_2, by taking $\omega > 2(3+\iota)/\iota$ where ι is given in Assumption II with $q = 3$, we have

$$\begin{aligned}&\left|\sum_{h=L_n+1}^{\infty} E\left(A_1^{(1)}(t)A_{1+h}^{(3)}(s) + A_{1+h}^{(1)}(t)A_1^{(3)}(s)\right)\right| \\ &\leq \sum_{h=L_n+1}^{\infty}\left[\left(E\left|A_1^{(1)}(t)\right|^{\omega}\right)^{\frac{1}{w}}\left(E\left|E_1A_{1+h}^{(3)}(s)\right|^{\frac{\omega}{w-1}}\right)^{1-\frac{1}{\omega}} + \left(E\left|A_1^{(3)}(s)\right|^{\omega}\right)^{\frac{1}{w}}\left(E\left|E_1A_{1+h}^{(1)}(t)\right|^{\frac{\omega}{w-1}}\right)^{1-\frac{1}{\omega}}\right] \\ &\leq C\sum_{h=L_n+1}^{\infty}\alpha_h^{1-\frac{2}{\omega}} \leq \frac{C}{L_n^2},\end{aligned}$$

where the first inequality follows from Hölder's inequality and the second inequality follows from Lemma 3.102 on page 497 of Jacod and Shiryaev (2003) and Assumption II with

$q = 3$. Therefore, we have

$$I_2 = C\left|\sum_{h=L_n+1}^{\infty} E\left(A_1^{(1)}(t)A_{1+h}^{(3)}(s) + A_{1+h}^{(1)}(t)A_1^{(3)}(s)\right)\right|^2 \le \frac{C}{L_n^4}.$$

We now turn to term I_3. On the one hand, by Hölder's inequality, Lemma 3.102 on page 497 of Jacod and Shiryaev (2003) and Assumption II with $q = 3$, for $k \le h$ and any $\omega > 2$, we have,

$$\begin{aligned}
&\left|EA_i^{(1)}(t)A_{i+k}^{(1)}(t)A_{i+h}^{(3)}(s)A_{i+k+h}^{(3)}(s)\right| \\
&= \left|EA_i^{(1)}(t)\left(E_iA_{i+k}^{(1)}(t)A_{i+h}^{(3)}(s)A_{i+k+h}^{(3)}(s) - EA_{i+k}^{(1)}(t)A_{i+h}^{(3)}(s)A_{i+k+h}^{(3)}(s)\right)\right| \\
&\le \left(E\left|A_i^{(1)}(t)\right|^{\omega}\right)^{\frac{1}{\omega}}\left(E\left|E_iA_{i+k}^{(1)}(t)A_{i+h}^{(3)}(s)A_{i+k+h}^{(3)}(s) - EA_{i+k}^{(1)}(t)A_{i+h}^{(3)}(s)A_{i+k+h}^{(3)}(s)\right|^{\frac{\omega}{\omega-1}}\right)^{1-\frac{1}{\omega}} \\
&\le C\alpha_k^{1-\frac{2}{\omega}}.
\end{aligned}$$

Similarly, for any $\omega > 2$, we have

$$\left|EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right| \le C\alpha_h^{1-\frac{2}{\omega}} \quad \text{and} \quad \left|EA_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s)\right| \le C\alpha_h^{1-\frac{2}{\omega}}.$$

Hence, by taking $\omega > 2(3+\iota)/(2+\iota)$ where ι is given in Assumption II with $q = 3$, we have

$$\begin{aligned}
&\frac{2}{(N-h)^2}\sum_{i=1}^{N-h}\sum_{k=1}^{h}\left|E\left[A_i^{(1)}(t)A_{i+h}^{(3)}(s) - EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right]\left[A_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s) - EA_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s)\right]\right| \\
&\le \frac{2}{(N-h)^2}\sum_{i=1}^{N-h}\sum_{k=1}^{h}\left[\left|EA_i^{(1)}(t)A_{i+k}^{(1)}(t)A_{i+h}^{(3)}(s)A_{i+k+h}^{(3)}(s)\right| + \left|EA_i^{(1)}(t)A_{i+h}^{(3)}(s)EA_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s)\right|\right] \\
&\le \frac{C}{(N-h)^2}\sum_{i=1}^{N-h}\sum_{k=1}^{h}\alpha_k^{1-\frac{2}{\omega}} + \frac{C}{(N-h)^2}\sum_{i=1}^{N-h}\sum_{k=1}^{h}\alpha_h^{2-\frac{4}{\omega}} \le \frac{C}{N}.
\end{aligned}$$

On the other hand, by the same arguments as that used in dealing with term I_1, we have, for $k > h$,

$$E\left(A_i^{(1)}(t)A_{i+h}^{(3)}(s) - EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right)\left(A_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s) - EA_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s)\right) \leq C\alpha_{k-h}^{1-\frac{2}{\omega}}$$

for any $\omega > 2$. Hence, we obtain that, for $\omega > 2(3+\iota)/(2+\iota)$ where ι is given in Assumption II with $q = 3$,

$$\begin{aligned}
&\frac{2}{(N-h)^2}\sum_{i=1}^{N-h}\sum_{k=h+1}^{N-h-i} E\left(A_i^{(1)}(t)A_{i+h}^{(3)}(s) - EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right)\left(A_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s) - EA_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s)\right) \\
&\leq \frac{C}{N}.
\end{aligned}$$

Combining the above results@be specific, cite the explicit equation numbers@, we obtain

$$\begin{aligned}
& E\left|\frac{1}{N-h}\sum_{i=1}^{N-h}\left[A_i^{(1)}(t)A_{i+h}^{(3)}(s) - EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right]\right|^2 \\
={}& \frac{1}{(N-h)^2}\sum_{i=1}^{N-h} E\left[A_i^{(1)}(t)A_{i+h}^{(3)}(s) - EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right]^2 \\
& + \frac{2}{(N-h)^2}\sum_{i=1}^{N-h}\sum_{k=1}^{h} E\left[A_i^{(1)}(t)A_{i+h}^{(3)}(s) - EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right] \\
& \times \left[A_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s) - EA_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s)\right] \\
& + \frac{2}{(N-h)^2}\sum_{i=1}^{N-h}\sum_{k=h+1}^{N-h-i} E\left[A_i^{(1)}(t)A_{i+h}^{(3)}(s) - EA_i^{(1)}(t)A_{i+h}^{(3)}(s)\right] \\
& \times \left[A_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s) - EA_{i+k}^{(1)}(t)A_{i+k+h}^{(3)}(s)\right] \leq \frac{C}{N}.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
EI_3 \le \sum_{h=1}^{L_n} CL_n E\left|\frac{1}{N-h}\sum_{i=1}^{N-h}\left[A_i^{(1)}(t)A_{i+h}^{(3)}(s) - E\left(A_i^{(1)}(t)A_{i+h}^{(3)}(s)\right)\right]\right|^2 \\
+ \sum_{h=1}^{L_n} CL_n E\left|\frac{1}{N-h}\sum_{i=h+1}^{N}\left[A_i^{(1)}(t)A_{i-h}^{(3)}(s) - E\left(A_i^{(1)}(t)A_{i-h}^{(3)}(s)\right)\right]\right|^2 \le \frac{CL_n^2}{N}.
\end{aligned}
$$

Combining the results for terms I_1, I_2 and I_3, we obtain

$$
EI \le C\left(\frac{1}{L_n^4} \vee \frac{L_n^2}{N}\right).
$$

We now turn to term II. Note first that

$$
\begin{aligned}
II &\le C\left[\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{A}_i^{(1)}(t)\widehat{A}_i^{(3)}(s) - A_i^{(1)}(t)A_i^{(3)}(s)\right)\right]^2 \\
&\quad + C\left\{\sum_{h=1}^{L_n}\frac{1}{N-h}\sum_{i=1}^{N}\left[\widehat{A}_i^{(1)}(t)\left(\widehat{A}_{i+h}^{(3)}(s) + \widehat{A}_{i-h}^{(3)}(s)\right) - A_i^{(1)}(t)\left(A_{i+h}^{(3)}(s) + A_{i-h}^{(3)}(s)\right)\right]\right\}^2 \\
&\le \frac{C}{N}\sum_{i=1}^{N}\left(\widehat{A}_i^{(1)}(t)\widehat{A}_i^{(3)}(s) - A_i^{(1)}(t)A_i^{(3)}(s)\right)^2 + \sum_{h=1}^{L_n}\frac{CL_n}{N-h}\sum_{i=1}^{N-h}\left(\widehat{A}_i^{(1)}(t)\widehat{A}_{i+h}^{(3)}(s) - A_i^{(1)}(t)A_{i+h}^{(3)}(s)\right)^2 \\
&\quad + \sum_{h=1}^{L_n}\frac{CL_n}{N-h}\sum_{i=h+1}^{N}\left(\widehat{A}_i^{(1)}(t)\widehat{A}_{i-h}^{(3)}(s) - A_i^{(1)}(t)A_{i-h}^{(3)}(s)\right)^2 =: II_1 + II_2 + II_3.
\end{aligned}
$$

The above three terms II_1, II_2 and II_3 can be treated by using the same arguments. Without loss of generality, we shall focus on term II_2. Term II_2 can be further bounded as follows,

$$
II_2 \le \sum_{h=1}^{L_n}\frac{CL_n}{N-h}\sum_{i=1}^{N-h}\left[\left(\widehat{A}_{i+h}^{(3)}(s) - A_{i+h}^{(3)}(s)\right)^2\left(\widehat{A}_i^{(1)}(t)\right)^2 + \left(\widehat{A}_i^{(1)}(t) - A_i^{(1)}(t)\right)^2\left(A_{i+h}^{(3)}(s)\right)^2\right]
$$

$$
\begin{aligned}
&\le \sum_{h=1}^{L_n} \frac{CL_n}{N-h} \sum_{i=1}^{N-h} \left[\left(\widehat{B}_{i+h}(s) - B_{i+h}(s) \right)^2 \left(\hat{\sigma}^2_{1,i}(t) \right)^2 \right] \\
&\quad + \left[\sum_{h=1}^{L_n} \frac{CL_n}{N-h} \sum_{i=1}^{N-h} \left(\widehat{B}_{i+h}(s) - B_{i+h}(s) \right)^2 \right] \left(\frac{1}{N} \sum_{i=1}^{N} \hat{\sigma}^2_{1,i}(t) \right)^2 \\
&\quad + \left[\sum_{h=1}^{L_n} \frac{CL_n}{N-h} \sum_{i=1}^{N-h} \left(\hat{\sigma}^2_{1,i}(t) \right)^2 \right] \left(\frac{1}{N} \sum_{i=1}^{N} \widehat{B}_i(s) - EB(s) \right)^2 \\
&\quad + CL_n^2 \left(\frac{1}{N} \sum_{i=1}^{N} \hat{\sigma}^2_{1,i}(t) \right)^2 \left(\frac{1}{N} \sum_{i=1}^{N} \widehat{B}_i(s) - EB(s) \right)^2 \\
&\quad + \sum_{h=1}^{L_n} \frac{CL_n}{N-h} \sum_{i=1}^{N-h} \left[\left(\hat{\sigma}^2_{1,i}(t) - \sigma^2_{1,i}(t) \right)^2 \left(A^{(3)}_{i+h}(s) \right)^2 \right] \\
&\quad + \left[\sum_{h=1}^{L_n} \frac{CL_n}{N-h} \sum_{i=1}^{N-h} \left(A^{(3)}_{i+h}(s) \right)^2 \right] \left(\frac{1}{N} \sum_{i=1}^{N} \hat{\sigma}^2_{1,i}(t) - E\sigma^2_1(t) \right)^2 =: \sum_{i=1}^{6} II_{2,i}.
\end{aligned}
$$

We treat terms $II_{2,1}$–$II_{2,6}$ one by one. Because many of these terms involve the fourth-order moment of $\hat{\sigma}^2_{1,i}(t)$, we provide a bound for $E\left(\hat{\sigma}^2_{1,i}(t)\right)^4$ first as follows. Note that by Assumption I and (28) with $N = 1$ and $d = 2$, under the condition that $\ell \to \infty$ with $\ell\Delta \to 0$, we have

$$
E\left|\hat{\sigma}^2_{1,i}(t)\right|^4 \le CE\left|\hat{\sigma}^2_{1,i}(t) - \sigma^2_{1,i}(t)\right|^4 + CE\left|\sigma^2_{1,i}(t)\right|^4 \le C. \tag{37}
$$

which further implies

$$
E\left(\hat{\sigma}^2_{1,i}(t)\right)^2 \le C \quad \text{and} \quad E\left(\frac{1}{N}\sum_{i=1}^{N}\hat{\sigma}^2_{1,i}(t)\right)^2 \le C. \tag{38}
$$

Additionally, note that by C_r-inequality and Assumption I, for any $\omega > 2$, we have

$$
E\left(A^{(1)}_i(t)\right)^\omega + E\left(A^{(3)}_i(t)\right)^\omega \le C. \tag{39}
$$

Based on this@be specific...cite equation numbers...@, by the same argument as that used in the proof immediately following Eq.(C.4) in the online appendix of Andersen et al. (2023), we readily obtain that

$$E\left|\frac{1}{N}\sum_{i=1}^{N}A_i^{(1)}(t)\right|^2+E\left|\frac{1}{N}\sum_{i=1}^{N}A_i^{(3)}(t)\right|^2\leq\frac{C}{N},\tag{40}$$

Therefore, by (28) with $d=2$, when $\ell\to\infty$ with $\ell\Delta\to 0$, we have

$$E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\sigma}_{1,i}^2(t)-E\sigma_1^2(t)\right)\right|^2+E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{B}_i(t)-EB(t)\right)\right|^2\leq C\left(\Delta\vee(\ell\Delta)^2\vee\frac{1}{N}\right).\tag{41}$$

We are now ready to treat terms $II_{2,1}$ to $II_{2,6}$. By Cauchy-Schwarz inequality, (29) with $N=1$ and $d=4$ and (37), we obtain

$$EII_{2,1}\leq\sum_{h=1}^{L_n}\frac{CL_n}{N-h}\sum_{i=1}^{N-h}\left[E\left(\widehat{B}_{i+h}(s)-B_{i+h}(s)\right)^4E\left(\hat{\sigma}_i^2(t)\right)^4\right]^{\frac{1}{2}}\leq C\left[L_n^2\left(\frac{1}{\ell}\vee\ell\Delta\right)\right].$$

Similarly, by Cauchy-Schwarz inequality, (28) with $N=1$ and $d=4$ and (39) we have

$$EII_{2,5}\leq\sum_{h=1}^{L_n}\frac{CL_n}{N-h}\sum_{i=1}^{N-h}\left[E\left(\hat{\sigma}_{1,i}^2(t)-\sigma_{1,i}^2(t)\right)^4E\left(A_{i+h}^{(3)}(s)\right)^4\right]^{\frac{1}{2}}\leq C\left[L_n^2\left(\frac{1}{\ell}\vee\ell\Delta\right)\right].$$

For term $II_{2,2}$, by (29) with $N=1$ and $d=2$ and (38), we have

$$II_{2,2}=O_p\left[L_n^2\left(\frac{1}{\ell}\vee\ell\Delta\right)\right].$$

Similarly, by (38), (39) and (41) we have

$$II_{2,3}+II_{2,4}+II_{2,6}=O_p\left[L_n^2\left(\Delta\vee(\ell\Delta)^2\vee\frac{1}{N}\right)\right].$$

To sum up, we have proved

$$II_2 = O_p\left[L_n^2\left(\frac{1}{\ell} \vee \ell\Delta \vee \frac{1}{N}\right)\right].$$

By the same arguments as above, we have

$$II_1 = O_p\left(\frac{1}{\ell} \vee \ell\Delta \vee \frac{1}{N}\right), \quad \text{and} \quad II_3 = O_p\left[L_n^2\left(\frac{1}{\ell} \vee \ell\Delta \vee \frac{1}{N}\right)\right].$$

Therefore,

$$II = O_p\left[L_n^2\left(\frac{1}{\ell} \vee \ell\Delta \vee \frac{1}{N}\right)\right].$$

Finally, combining the results for terms I and II, we have proved that

$$\left(C_{1,3}(t,s) - \widehat{C}_{1,3}(t,s)\right)^2 = O_p\left(\frac{1}{L_n^4} \vee \frac{L_n^2}{N} \vee \frac{L_n^2}{\ell} \vee L_n^2\ell\Delta\right),$$

which converges to 0 under the condition (19).

The same arguments as above apply to other configurations of (i,j) besides $(i,j) = (1,3)$ with $i,j \in \{1,2,3\}$. Thus, we complete the proof. $\square$

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