Chapter 6

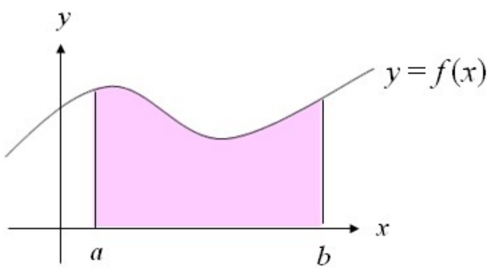
Applications of Integration

6.1 Area between Curves

6.1.1 Area between a curve and the $x-$axis

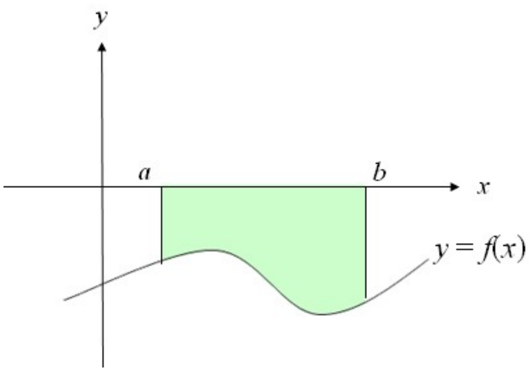
Let $y = f(x)$ be a continuous function on $[a, b]$ and let A be the area of the region that lies between the graph of $y = f(x)$ and the $x-$axis from $x = a$ to $x = b$.

$f(x) \geq 0$ for all $x \in [a, b]$



$$A = \int_a^b f(x)\, dx$$

$f(x) \leq 0$ for all $x \in [a, b]$

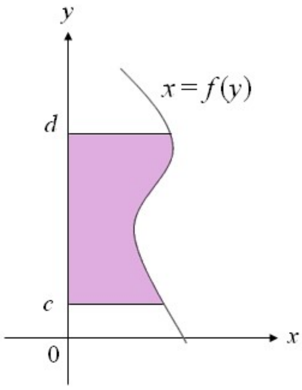


$$A = -\int_a^b f(x)\, dx$$

6.1.2 Area between a curve and the $y-$axis

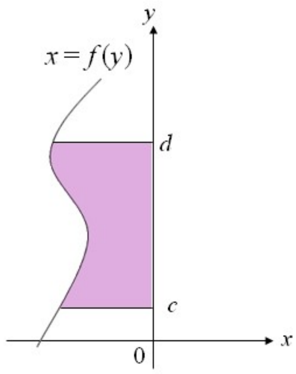
Let $x = f(y)$ be a continuous function on $[c, d]$ and let A be the area of the region that lies between the graph of $x = f(y)$ and the $y-$axis from $y = c$ to $y = d$.

$f(y) \geq 0$ for all $y \in [c, d]$



$$A = \int_c^d f(y)\,dy$$

$f(y) \leq 0$ for all $y \in [c, d]$

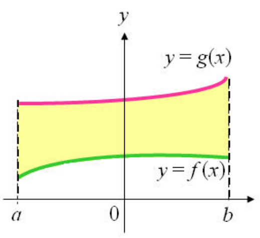


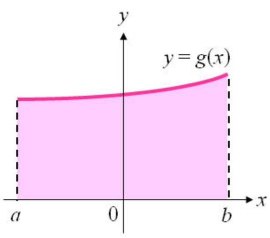
$$A = -\int_c^d f(y)\,dy$$

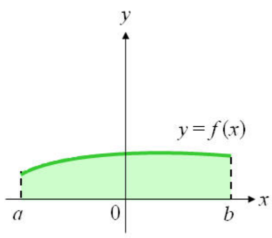
6.1.3 Area between two curves

Let $y = f(x)$ and $y = g(x)$ be continuous functions on $[a, b]$. Assume that $f(x) \leq g(x)$ for all $x \in [a, b]$. Let A be the area of the region between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$. We can find A as follows:

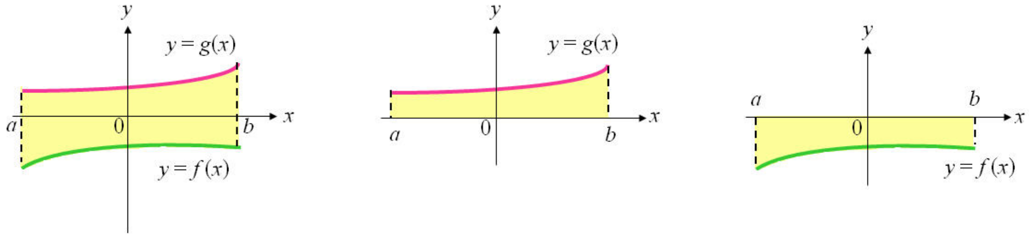
Case 1 : $0 \leq f(x) \leq g(x)$



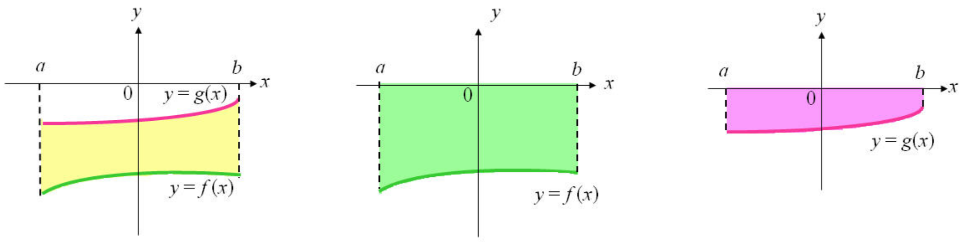




Case 2 : $f(x) \leq 0 \leq g(x)$



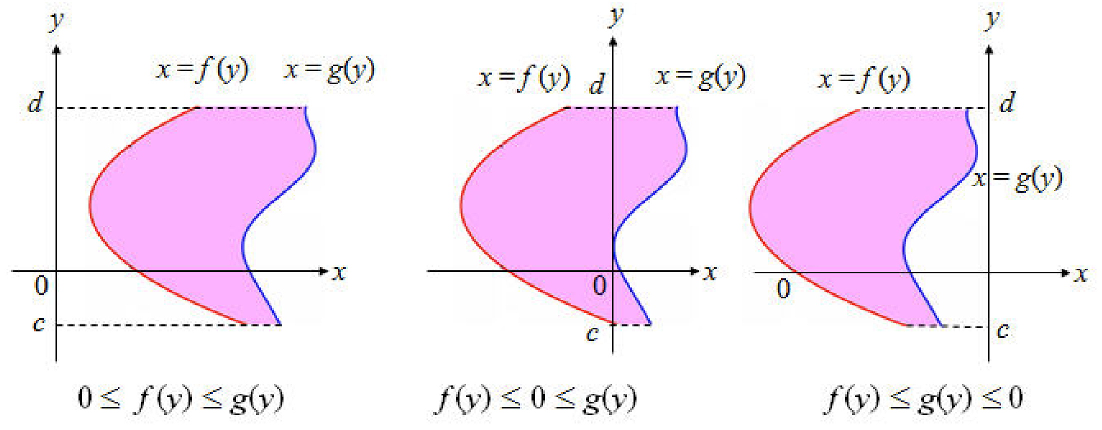
Case 3 : $f(x) \leq g(x) \leq 0$



Let A be the area of the region between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$. Then

$$\boxed{A = \int_a^b y_{\mathrm{U}} - y_{\mathrm{L}}\, dx}$$

Let $x = f(y)$ and $x = g(y)$ be continuous functions on $[c, d]$ and let A be the area of the region between the curves $x = f(y)$ and $x = g(y)$ from $y = c$ to $y = d$.



$$A = \int_c^d g(y)\,dy - \int_c^d f(y)\,dy \qquad A = \int_c^d g(y)\,dy + \left(-\int_c^d f(y)\,dy\right) \qquad A = -\int_c^d f(y)\,dy - \left(-\int_c^d g(y)\,dy\right)$$

Then

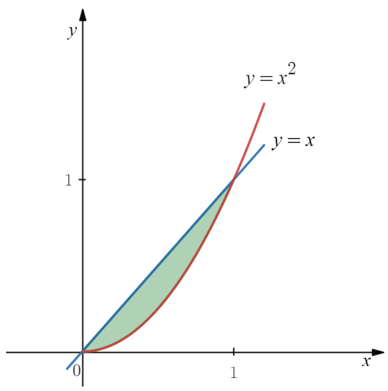
$$A = \int_c^d g(y) - f(y)\,dx.$$

Let A be the area of the region between the curves $x = f(y)$ and $x = g(y)$ from $y = c$ to $y = d$. Then

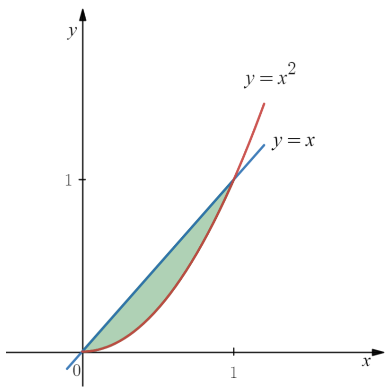
$$\boxed{A = \int_c^d x_{\mathrm{R}} - x_{\mathrm{L}}\,dy}$$

Example 6.1. Write the definite integrals for finding the area of the following regions.

1. The region enclosed by the curves $y = x^2$ and $y = x$.

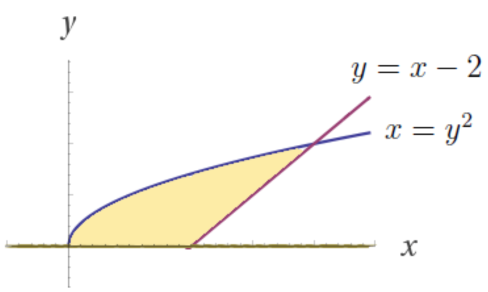


(a) Integrating with respect to x.

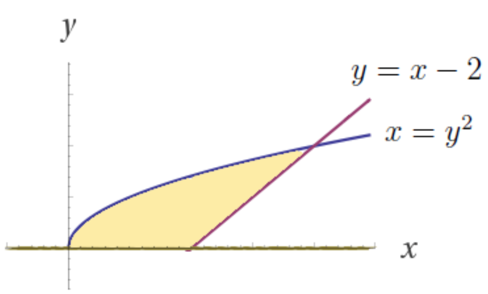


(b) Integrating with respect to y.

2. The region enclosed by the curves $x = y^2$, $y = x - 2$ and the $x-$ axis.

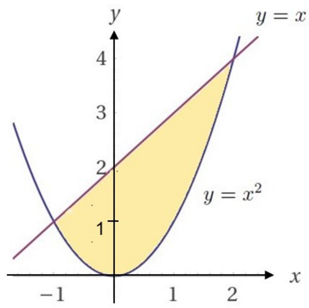


(a) Integrating with respect to x.

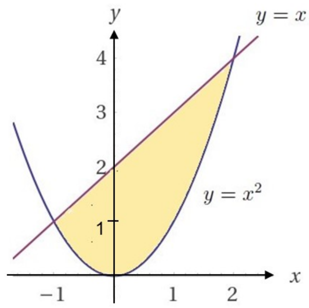


(b) Integrating with respect to y.

3. The region enclosed by the curves $y = x^2$ and $y = x + 2$.

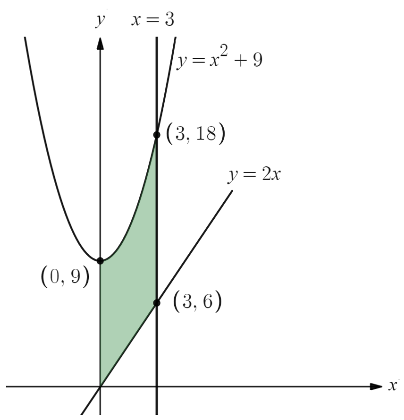


(a) Integrating with respect to x.

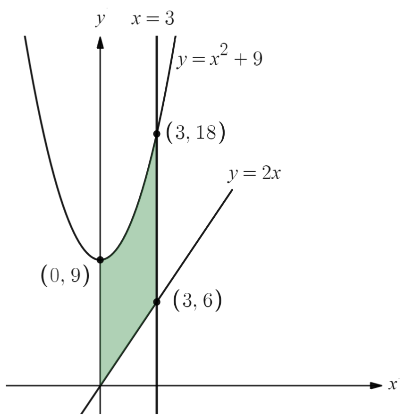


(b) Integrating with respect to y.

4. The region enclosed by the curves $y = x^2 + 9$, the lines $y = 2x, x = 3$ and the $y-$axis.



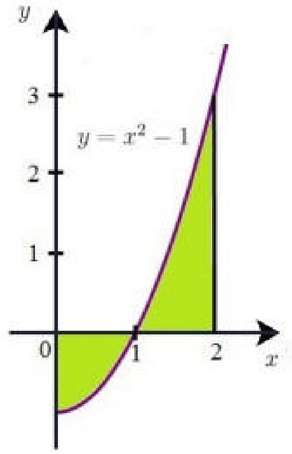
(a) Integrating with respect to x.



(b) Integrating with respect to y.

Exercise 6.1

Write the definite integrals for finding the area of the shaded region in the following pictures.

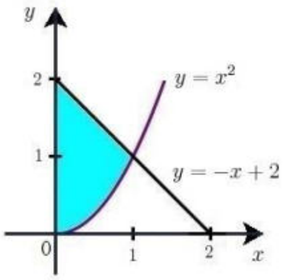
1. 

   (a) Integrating with respect to x.

   $A =$ ______________________________

   (b) Integrating with respect to y.

   $A =$ ______________________________

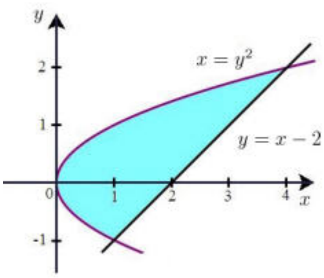
2. 

   (a) Integrating with respect to x.

   $A =$ ______________________________

   (b) Integrating with respect to y.

   $A =$ ______________________________

3. 

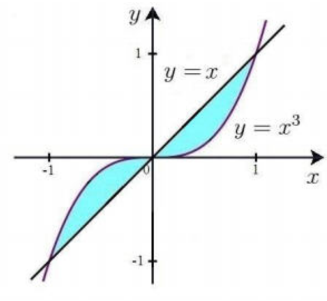
   (a) Integrating with respect to x.

   $A =$ ______________________________

   (b) Integrating with respect to y.

   $A =$ ______________________________

4.



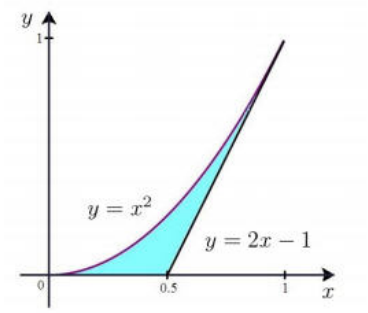
(a) Integrating with respect to x.

$A =$ ________________________________

(b) Integrating with respect to y.

$A =$ ________________________________

5.



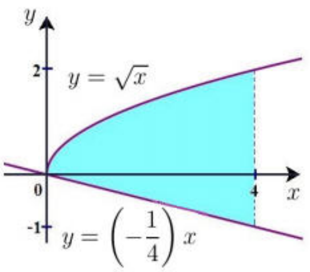
(a) Integrating with respect to x.

$A =$ ________________________________

(b) Integrating with respect to y.

$A =$ ________________________________

6.



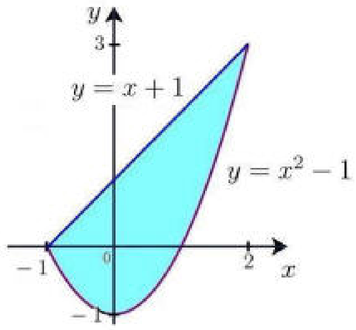
(a) Integrating with respect to x.

$A =$ ________________________________

(b) Integrating with respect to y.

$A =$ ________________________________

7.



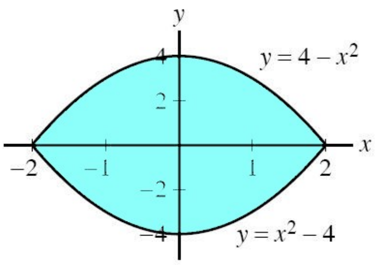
(a) Integrating with respect to x.

$A =$ ______________________________

(b) Integrating with respect to y.

$A =$ ______________________________

8.



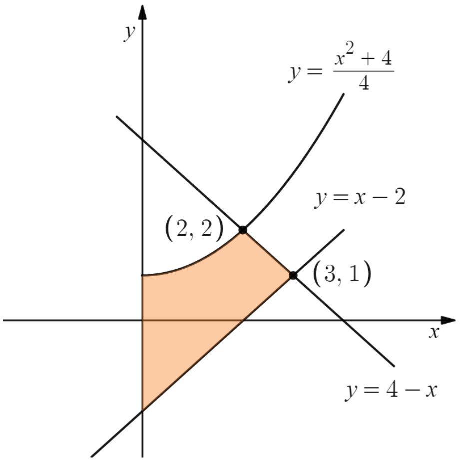
(a) Integrating with respect to x.

$A =$ ______________________________

(b) Integrating with respect to y.

$A =$ ______________________________

9.



(a) Integrating with respect to x.

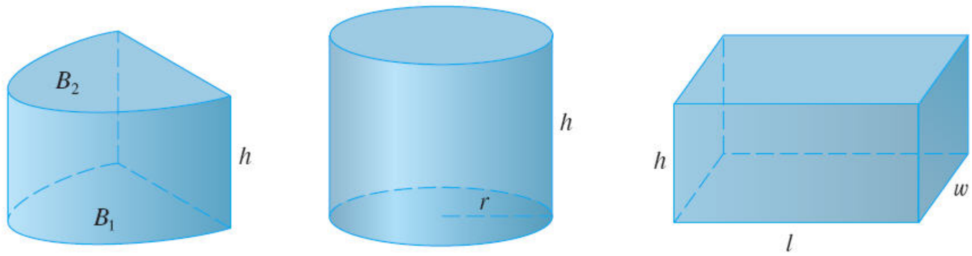
$A =$ ______________________________

(b) Integrating with respect to y.

$A =$ ______________________________

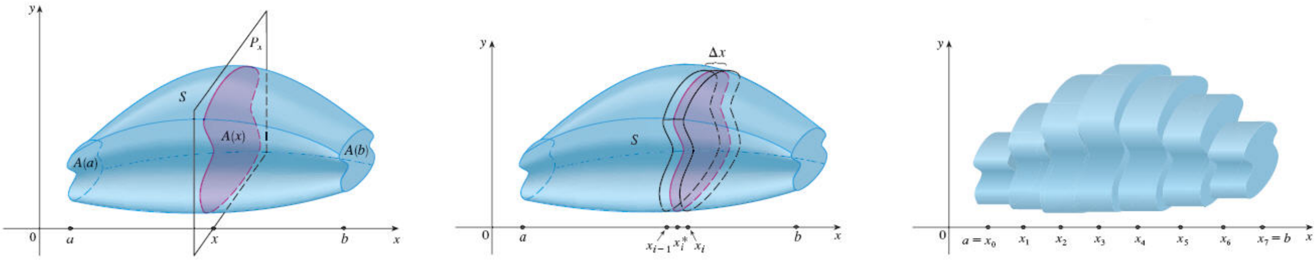
6.2 Volumes of solids

A cylinder is a solid where all cross sections are the same. The volume of a cylinder is Ah where A is the area of a cross section and h is the height of the cylinder.



(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

For a solid S that is not a cylinder we first cut S into n pieces and approximate each piece by a cylinder.



The areas of the cross sections of the solid shown on the left above vary as x varies. The areas of these cross sections are thus a function of x, $A(x)$, defined on the interval $[a, b]$. The volume V_i of the i^{th} slice of the solid above shown in the middle picture, is approximately the volume of a cylinder with height Δx and cross sectional area $A(x_i^*)$ where x_i^* is arbitrary point in $[x_{i-1}, x_i]$, that is,

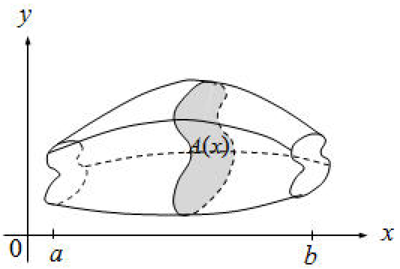
$$V_i \approx A(x_i^*)\Delta x.$$

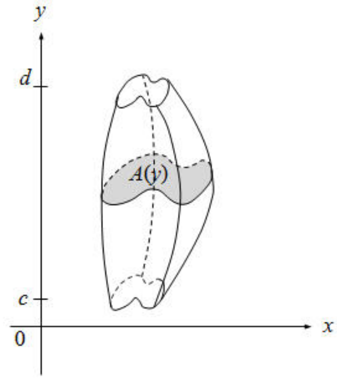
Adding the n slices, we find an approximation to the volume of the solid given by

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

This approximation appears to become better and better as $n \to \infty$. Therefore,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx.$$





DEFINITION OF VOLUME

The volume of a solid of known integrable cross-section area $A(x)$ from $x = a$ to $x = b$ is

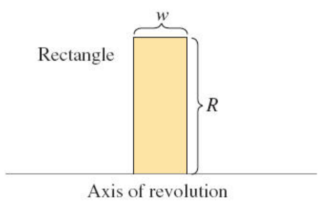
$$V = \int_a^b A(x)\,dx.$$

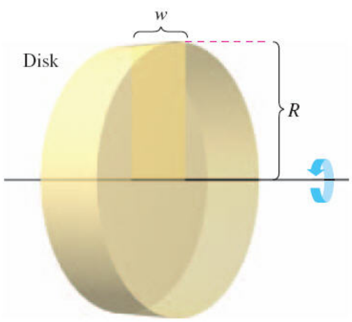
The volume of a solid of known integrable cross-section area $A(y)$ from $y = c$ to $x = d$ is

$$V = \int_c^d A(y)\,dy.$$

Volume of a solid of revolution

If a region in the plane is revolved about a line, the resulting solid is a **solid of revolution**, and the line is called **the axis of revolution**. The simplest such solid is a disk, which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, as shown in following figure

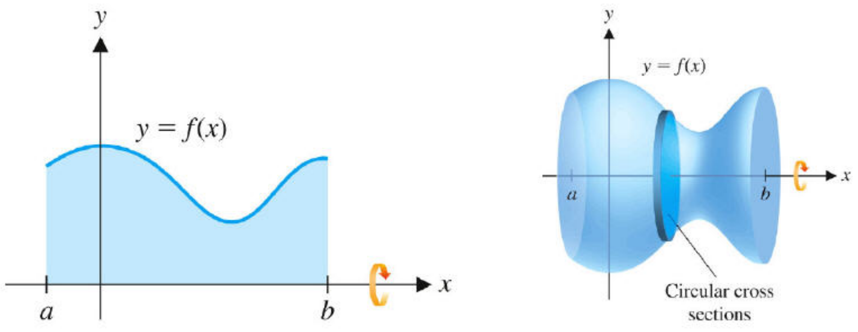




We can find the volume of a solid that is obtained by revolving a region in the plane about the axis of revolution by using the following method.

Disk Method

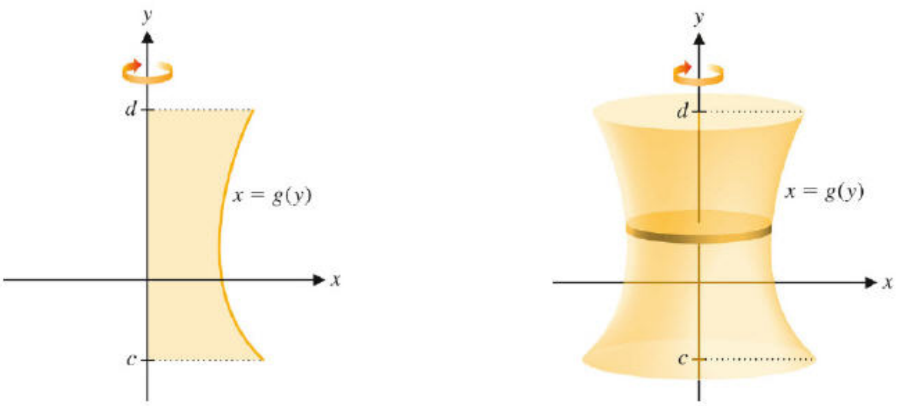
• **The $x-$ axis is the axis of revolution**



Disk Method : Rotating about the $x-$axis

$$V=\int_a^b A(x)\,dx=\int_a^b \pi R^2\,dx=\int_a^b \pi[f(x)]^2\,dx.$$

• **The $y-$axis is the axis of revolution**

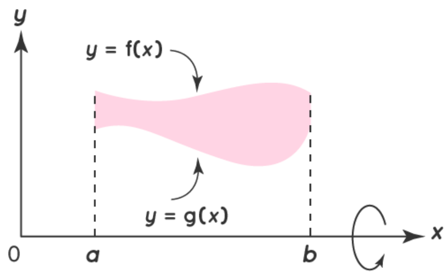


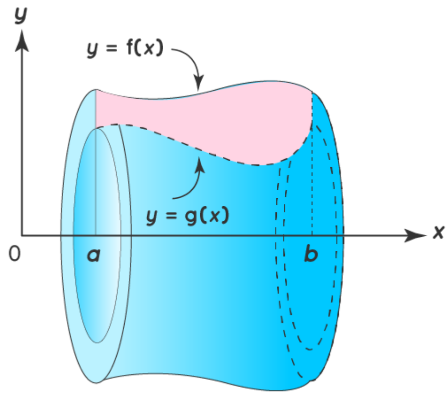
Disk Method : Rotating about the $y-$axis

$$V=\int_c^d A(y)\,dy=\int_c^d \pi R^2\,dy=\int_c^d \pi[g(y)]^2\,dy.$$

Washer Method

- **The $x-$ axis is the axis of revolution**

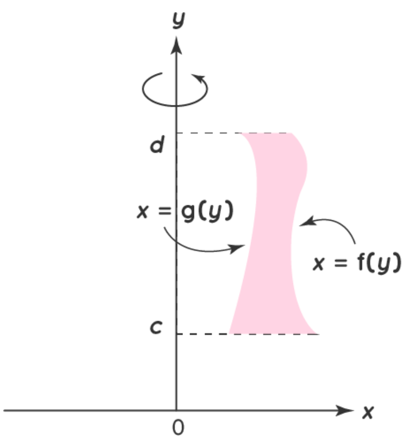


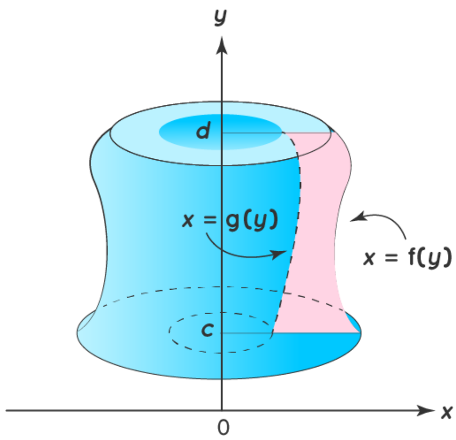


Washer Method : Rotating about the $x-$axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi R_{out}^2 - \pi R_{in}^2\,dx = \int_a^b \pi(f(x))^2 - \pi(g(x))^2\,dx = \int_a^b \pi y_{\mathrm{U}}^2 - \pi y_{\mathrm{L}}^2\,dx$$

- **The $y-$ axis is the axis of revolution**

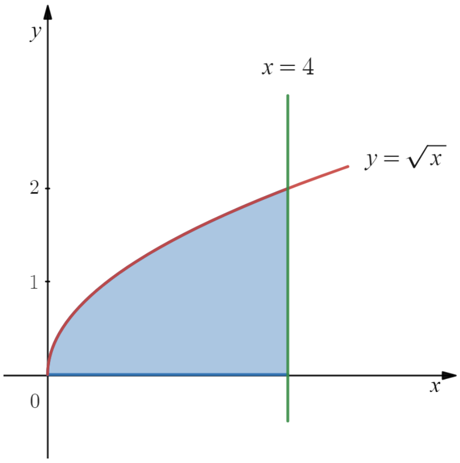




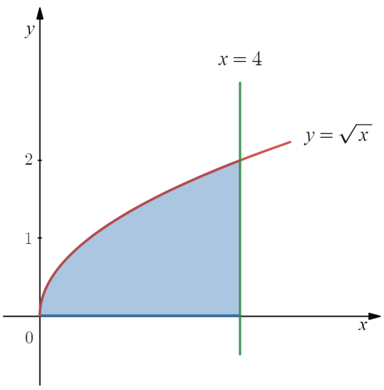
Washer Method : Rotating about the $y-$axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi R_{out}^2 - \pi R_{in}^2\,dy = \int_c^d \pi(f(y))^2 - \pi(g(y))^2\,dy = \int_d^d \pi x_{\mathrm{R}}^2 - \pi x_{\mathrm{L}}^2\,dy$$

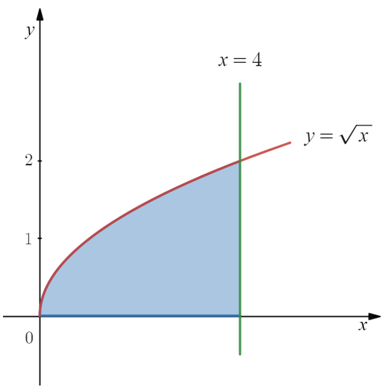
Example 6.2. Let R be the region enclosed by the curve $y = \sqrt{x}$, the $x-$axis and the line $x = 4$.



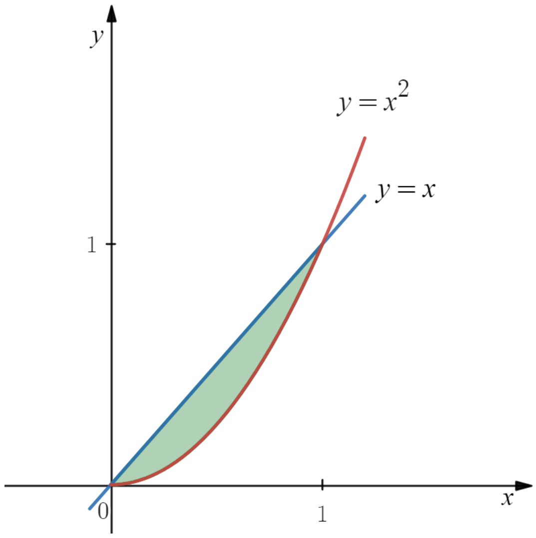
1. Write the definite integrals for finding the volume of the solid generated by revolving the region R <u>about the $x-$axis</u>.



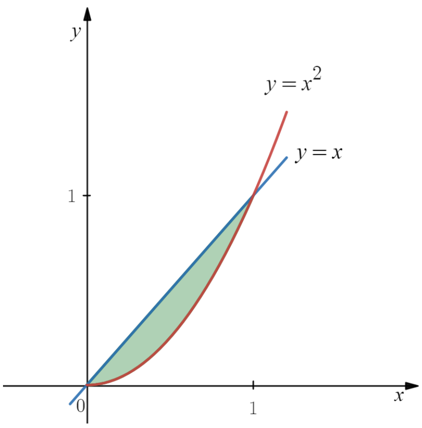
2. Write the definite integrals for finding the volume of the solid generated by revolving the region R <u>about the $y-$axis</u>.



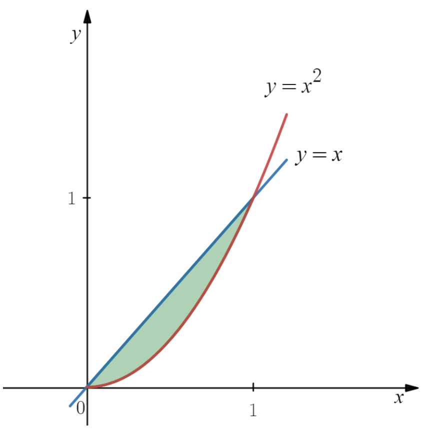
Example 6.3. Let R be the region enclosed by the curve $y = x$ and $y = x^2$.



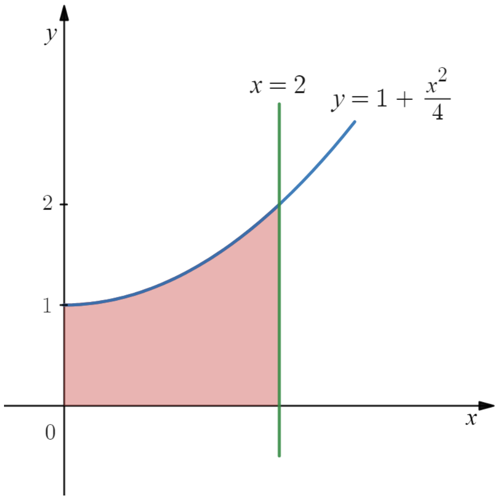
1. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $x-$axis.



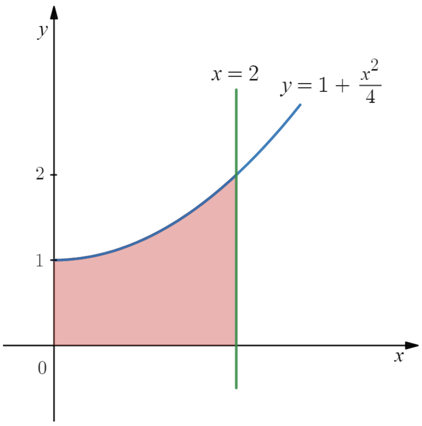
2. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $y-$axis.



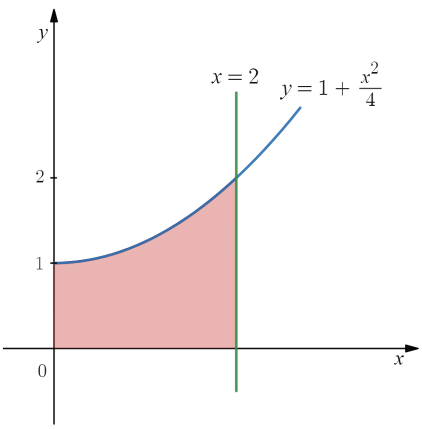
Example 6.4. Let R be the region bounded by the curve $y = 1 + \dfrac{x^2}{4}$, the $x-$axis, the $y-$axis and the line $x = 2$.



1. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $x-$axis.



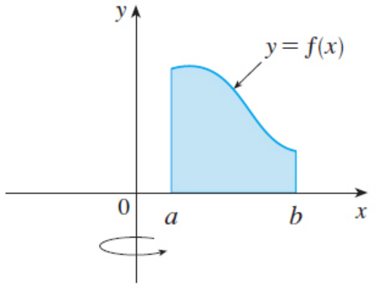
2. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $y-$axis.

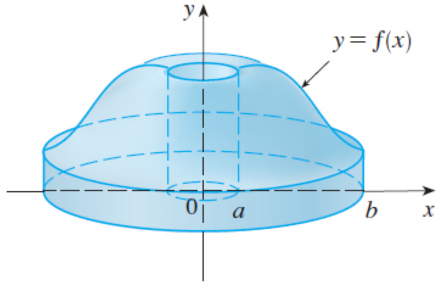


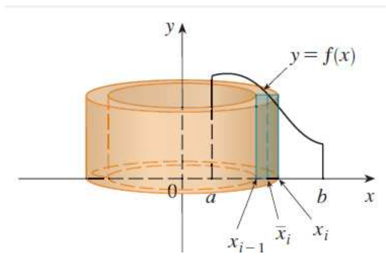
Cylindrical Shell Method

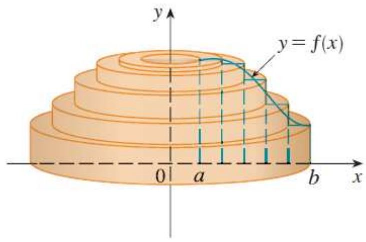
Consider the region that lies under the curve $y = f(x)$ from $x = a$ to $x = b$.

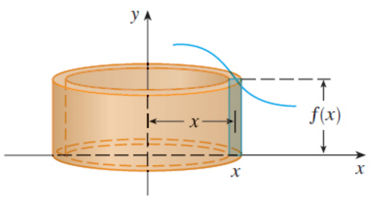
If we rotate this region about the $y-$axis, we obtain a solid. The volume of this solid can be computed by means of definite integral.

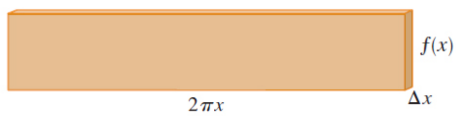






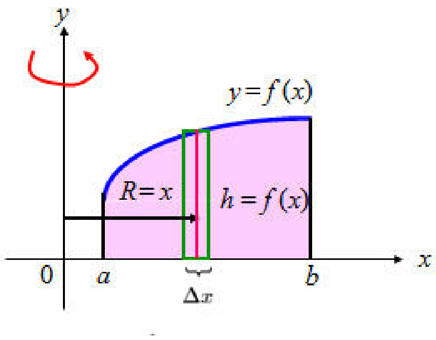


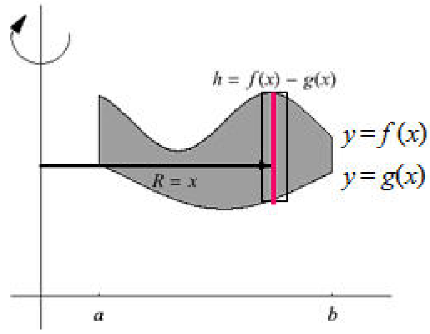




We can see that the volume $V \approx \sum_{i=1}^{n} 2\pi\overline{x}_i f(\overline{x}_i)\Delta x$ where Δx is the thickness of each shell. This approximation appears to become better and better as $n \to \infty$. Therefore,

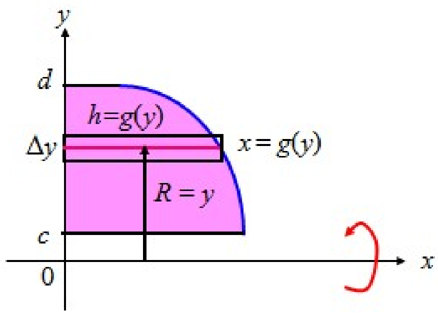
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\overline{x}_i f(\overline{x}_i)\Delta x = \int_a^b \underbrace{(2\pi x)}_{circumference} \underbrace{f(x)}_{height} \underbrace{dx}_{thickness} .$$

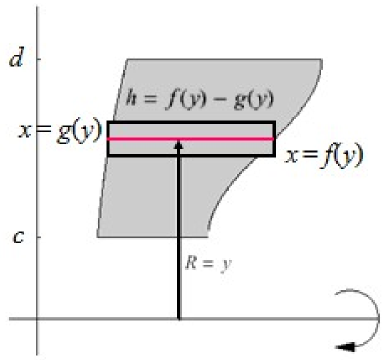




Cylindrical Shell Method : Rotating about the y−axis

$$V = \int_a^b 2\pi Rh\,dx = \int_a^b 2\pi x(y_{\mathrm{U}} - y_{\mathrm{L}})\,dx$$

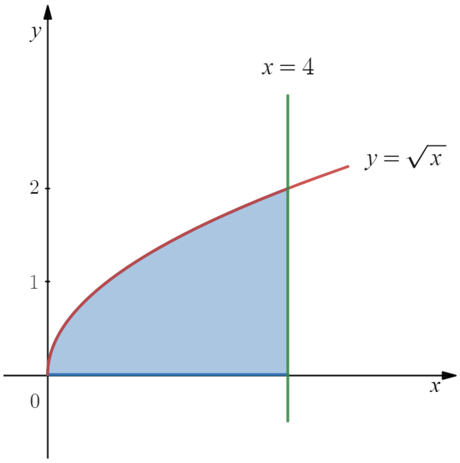




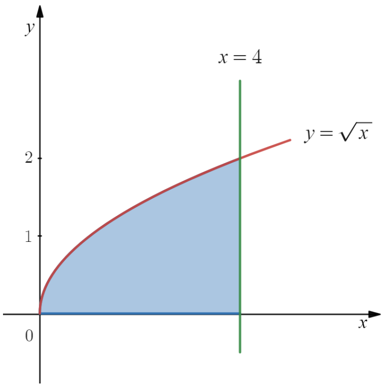
Cylindrical Shell Method : Rotating about the x−axis

$$V = \int_c^d 2\pi Rh\,dy = \int_c^d 2\pi y(x_{\mathrm{R}} - x_{\mathrm{L}})\,dy$$

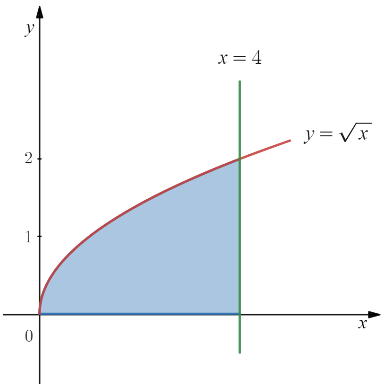
Example 6.5. Let R be the region enclosed by the curve $y = \sqrt{x}$, the $x-$axis and the line $x = 4$.



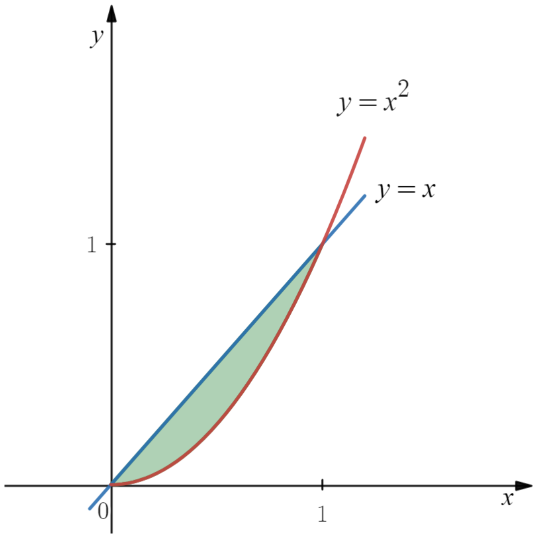
1. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $x-$axis using the Cylindrical Shell Method.



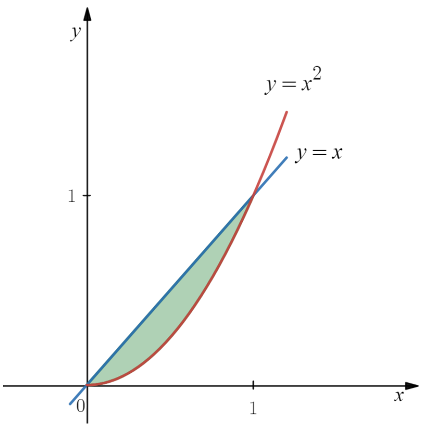
2. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $y-$axis using the Cylindrical Shell Method.



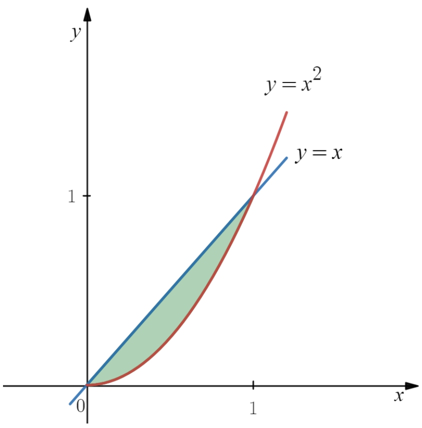
Example 6.6. Let R be the region enclosed by the curve $y = x$ and $y = x^2$.



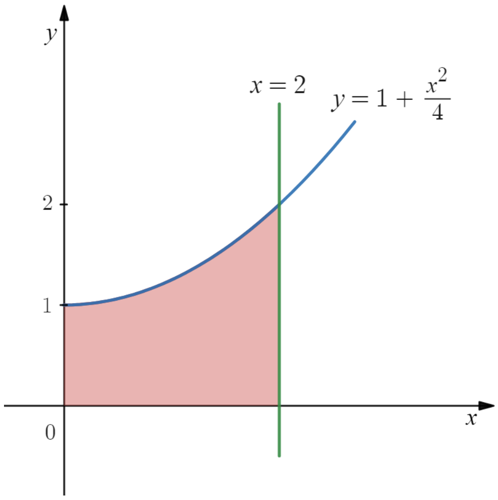
1. Write the definite integrals for finding the volume of the solid generated by revolving the region R <u>about the $x-$axis</u> using the Cylindrical Shell Method.



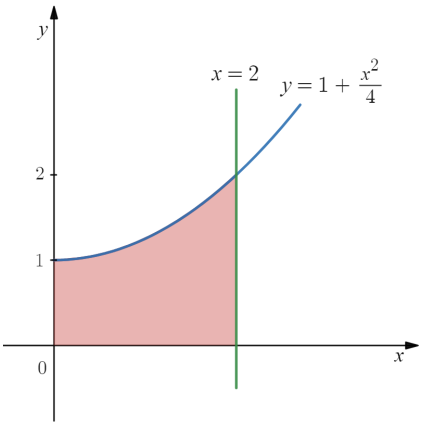
2. Write the definite integrals for finding the volume of the solid generated by revolving the region R <u>about the $y-$axis</u> using the Cylindrical Shell Method.



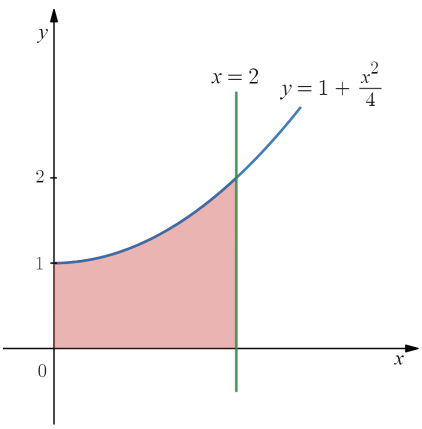
Example 6.7. Let R be the region bounded by the curve $y = 1+\dfrac{x^2}{4}$, the $x-$axis, the $y-$axis and the line $x = 2$.



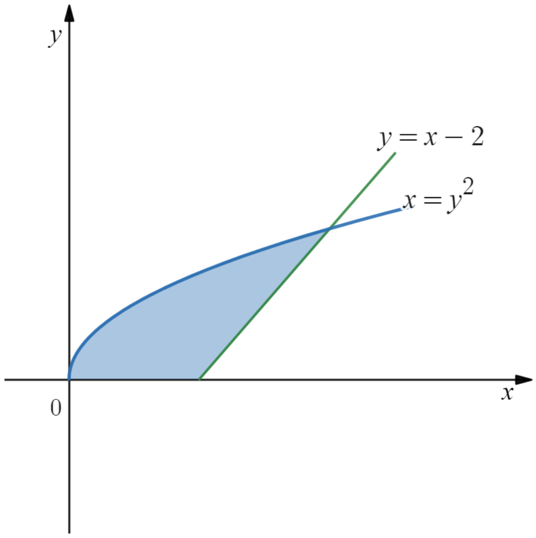
1. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $x-$axis using the Cylindrical Shell Method.



2. Write the definite integrals for finding the volume of the solid generated by revolving the region R about the $y-$axis using the Cylindrical Shell Method.



Example 6.8. Let R be the region bounded by the curve $x = y^2, y = x - 2$ and the $x-$axis.



1. Write the definite integral to find the volume of the solid generated by revolving the region R about the $x-$axis.

   a) By the Disk or Washer Method

   b) By the Cylindrical Shell Method

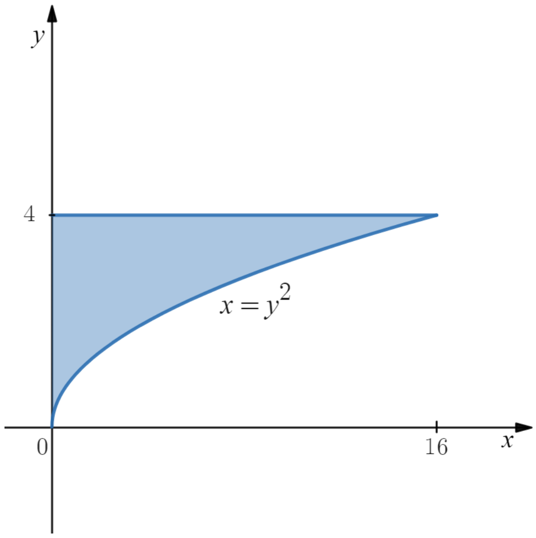
2. Write the definite integral to find the volume of the solid generated by revolving the region R about the $y-$axis.

   a) By the Disk or Washer Method

   b) By the Cylindrical Shell Method

Exercise 6.2

1. Let R be the region bounded by the curve $x = y^2$, the line $y = 4$ and the $y-$axis.



1.1. Write the definite integral to find the volume of the solid generated by revolving the region R about the $x-$axis.

a) By the Disk or Washer Method

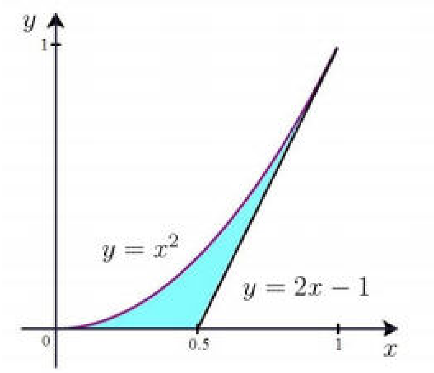
b) By the Cylindrical Shell Method

1.2. Write the definite integral to find the volume of the solid generated by revolving the region R about the $y-$axis.

a) By the Disk or Washer Method

b) By the Cylindrical Shell Method

2. Let R be the region bounded by the curve $y = x^2$, the line $y = 2x - 1$ and the $x-$ axis.



2.1. Write the definite integral to find the volume of the solid generated by revolving the region R about the $x-$axis.

a) By the Disk or Washer Method

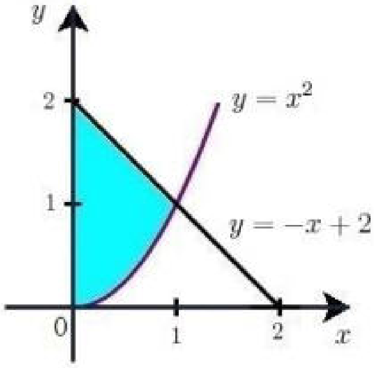
b) By the Cylindrical Shell Method

2.2. Write the definite integral to find the volume of the solid generated by revolving the region R about the $y-$axis.

a) By the Disk or Washer Method

b) By the Cylindrical Shell Method

3. Let R be the region bounded by the curve $y = x^2$, the line $y = -x + 2$ and the $y-$ axis.



3.1. Write the definite integral to find the volume of the solid generated by revolving the region R about the $x-$axis.

a) By the Disk or Washer Method

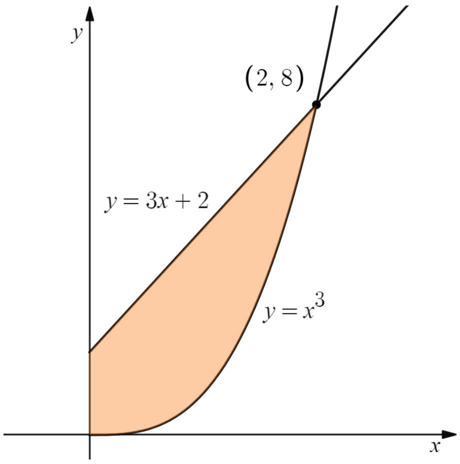
b) By the Cylindrical Shell Method

3.2. Write the definite integral to find the volume of the solid generated by revolving the region R about the $y-$axis.

a) By Disk or Washer Method

b) By the Cylindrical Shell Method

4. Let R be the region bounded by the curve $y = x^3$, the line $y = 3x + 2$ and the $y-$ axis as shown in the following figure.



4.1. Write the definite integral for finding the volume of the solid generated by revolving the region R about the $x-$axis.

a) Use the Disk or Washer Method

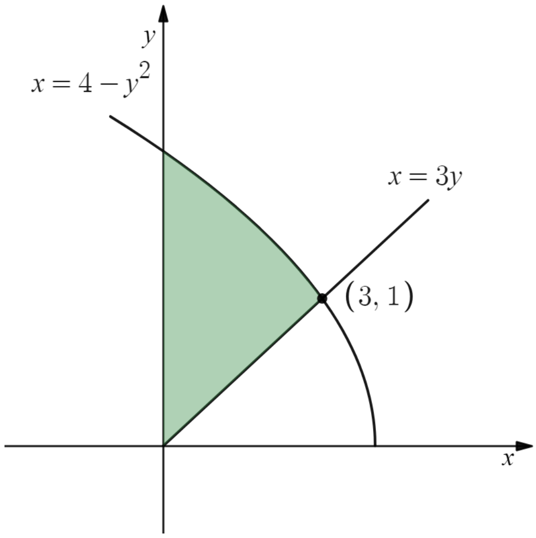
b) Use the Cylindrical Shell Method

4.2. Write the definite integral for finding the volume of the solid generated by revolving the region R about the $y-$axis.

a) Use the Disk or Washer Method

b) Use the Cylindrical Shell Method

5. Let R be the region between the curve $x = 4 - y^2$, the line $x = 3y$ and the $y-$ axis as shown in the following figure.



5.1. Write the definite integral for finding the volume of the solid generated by revolving the region R about the $x-$axis.

a) Use the Disk or Washer Method

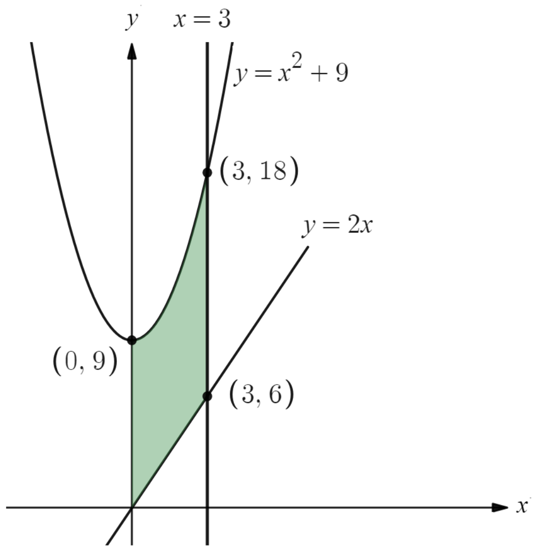
b) Use the Cylindrical Shell Method

5.2. Write the definite integral for finding the volume of the solid generated by revolving the region R about the $y-$axis.

a) Use the Disk or Washer Method

b) Use the Cylindrical Shell Method

6. Let R be the region between the curve $y = x^2 + 9$, the line $y = 2x, x = 3$ and the $y-$ axis. The region R is shown as the shaded region in the following figure.



6.1. Write the definite integral for finding the volume of the solid generated by revolving the region R about the $x-$axis.

a) Use the Disk or Washer Method

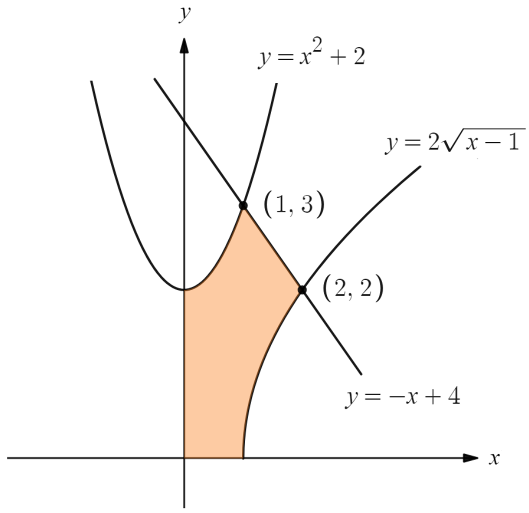
b) Use the Cylindrical Shell Method

6.2. Write the definite integral for finding the volume of the solid generated by revolving the region R about the $y-$axis.

a) Use the Disk or Washer Method

b) Use the Cylindrical Shell Method

7. Let R be the region enclosed by the curve $y = x^2 + 2, y = 2\sqrt{x-1}, y = -x + 4$, the $y-$ axis, and the $x-$ axis.



7.1. Write the definite integral to find the volume of the solid generated by revolving the region R about the $y-$axis using the Disk or Washer Method.

7.2. Write the definite integral to find the volume of the solid generated by revolving the region R about the $x-$axis using the Cylindrical Shell Method.