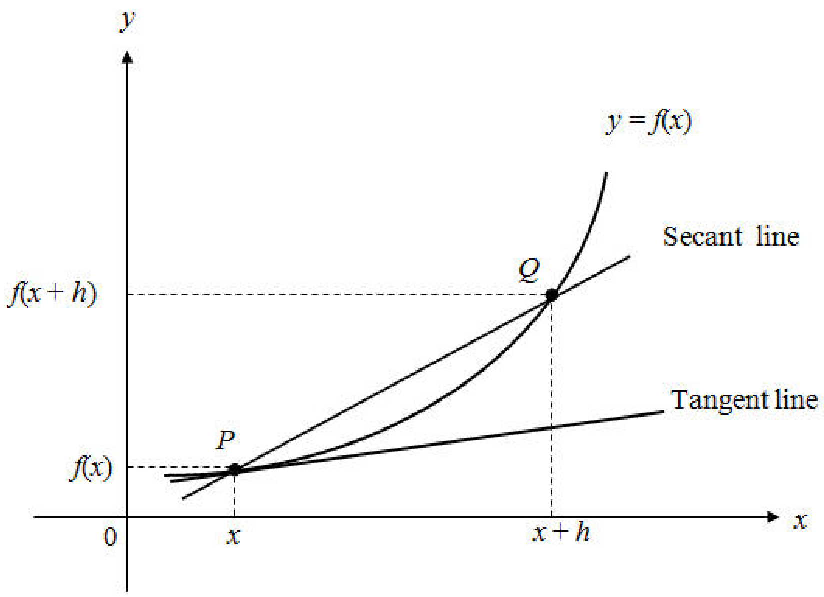
Chapter 3

Derivatives

3.1 Definition of Derivatives



The slope of the secant line through P and Q is ________________________________.

If Q approaches P or, equivalently, h tend to 0, then the secant line get progressively closer to the tangent line. Therefore the slope of the tangent line at the point $(x, f(x))$ is

__________________________________.

Definition 3.1. The derivative of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists.

- If $f'(a)$ exists, we say that ***f* is differentiable at $x = a$** or ***f* has a derivative at $x = a$**.
- If f is differentiable at all points in its domain, we say that ***f* is a differentiable function**.

- Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x).$$

- To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = y'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}.$$

From

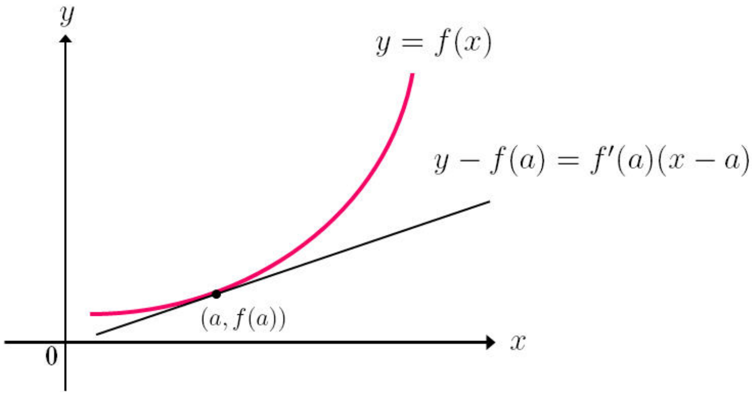
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if we write $x = a + h$, then $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore, an equivalent definition of $f'(a)$ is as follows:

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}.$$

From the definition of $f'(a)$, we see that $f'(a)$ is the slope of tangent line to the curve $y = f(x)$ at the point $x = a$ or $(a, f(a))$. Then

> the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is
> $$y - f(a) = f'(a)(x - a)$$



Example 3.1. Let $f(x) = x^2$.

(a) Find $f'(x)$.

(b) Find $f'(2)$.

(c) Find an equation of the tangent line to the curve $f(x) = x^2$ at the point $x = 2$.

Solution

Example 3.2. Define $f(x) = \begin{cases} x^2 & \text{if } x \leq 1, \\ \dfrac{x}{2} & \text{if } x > 1. \end{cases}$

Is f differentiable at $x = 1$?

Solution

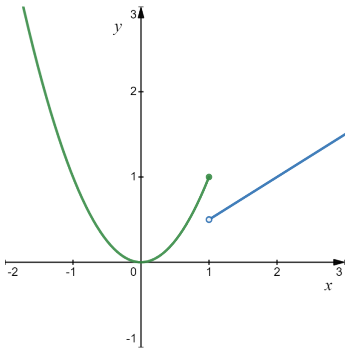
Theorem 3.1. Differentiability Implies Continuity

If f is differentiable at $x = a$, then f is continuous at $x = a$.

The statement in Theorem 3.1 is equivalent to

$$\boxed{\text{If } f \text{ is not continuous at } x = a, \text{ then } f \text{ is not differntiable at } x = a.}$$

Example 3.3. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 1, \\ \dfrac{x}{2} & \text{if } x > 1. \end{cases}$



We see that

$$\lim_{x \to 1^-} f(x) = \underline{\hspace{8cm}}$$

and $$\lim_{x \to 1^+} f(x) = \underline{\hspace{8cm}},$$

which implies that $\lim_{x \to 1} f(x)$ ____________________________

Then f is not continuous at $x = 1$.

By Theorem 3.1, we conclude that f is not differentiable at $x = 1$.

CAUTION The converse of Theorem 3.1 is not generally true. A function need not have a derivative at a point where it is continuous. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$. Let's explore further. For $f(x) = |x|$,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

Exercise 3.1

1. Let
$$f(x) = \begin{cases} \sqrt{x} & \text{if } 0 < x \leq 1, \\ \dfrac{x+1}{2} & \text{if } x > 1. \end{cases}$$
Is f differentiable at $x = 1$?

2. Let
$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1, \\ 2x - 1 & \text{if } 1 < x \leq 2, \\ \sqrt{x+4} & \text{if } x > 2. \end{cases}$$
Find $f'(1)$ and $f'(2)$.

3.2 Differentiation Rules and Differentiation Formulas

Basic Differentiation Rules

Let u and v be differentiable functions of x and let c be a constant. Then

1. **Constant Rule:** $\dfrac{d}{dx}(c) = 0$
2. **Constant Multiple Rule:** $\dfrac{d}{dx}(cu) = c\,\dfrac{du}{dx}$
3. **Sum/Difference Rule:** $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{du}{dx}$
4. **Product Rule:** $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
5. **Quotient Rule:** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Example 3.4. Let f and g be differentiable at the point $x = 1$ where

$$f(1) = 2, f'(1) = 1, g(1) = 1 \text{ and } g'(1) = 4.$$

Evaluate the following derivative.

1. $\dfrac{d}{dx}(6f(x) - g(x))\Big|_{x=1}$

2. $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=1}$

3. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\Big|_{x=1}$

Differentiation Formulas

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}, n$ is a real number

2. $\dfrac{d}{dx}(a^x) = a^x \ln a$

3. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

4. $\dfrac{d}{dx}(e^x) = e^x$

5. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

6. $\dfrac{d}{dx}(\sin x) = \cos x$

7. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

8. $\dfrac{d}{dx}(\cos x) = -\sin x$

9. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$

10. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

11. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

12. $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

13. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$

14. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

15. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

16. $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

17. $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$

Example 3.5. Find the derivatives of the following functions.

1. $y = 2\sqrt{x} - \dfrac{1}{x^2} + x^\pi + 3^\pi$

   Solution

2. $y = \sin x \cos x$

   Solution

3. $y = \dfrac{x^2}{1 + e^x}$

   Solution

The Chain Rule

The Chain rule is used to differentiate composite functions such as $y = \cos(x^2)$ and $y = \sqrt{x^3 + 1}$.

Recall the composite of f and g, denoted $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x)).$$

We refer to g as the inside function and f as the outside function.

We regard the composite as $f(u)$, where $u = g(x)$.

For example,

- $y = \cos(x^2)$ is the function $y = \cos u$ where $u = x^2$.
- $y = \sqrt{x^3 + 1}$ is the function $y = \sqrt{u}$ where $u = x^3 + 1$.

Theorem 3.2. The Chain Rule

If f and g are differentiable, then $(f \circ g)(x) = f(g(x))$ is differentiable and

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Suppose that $u = g(x)$ and

$$y = f(u) = f(g(x)).$$

Then, by the Chain Rule,

$$\frac{dy}{dx} = f'(u)g'(x) = \frac{df}{du} \cdot \frac{du}{dx}$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 3.6. Calculate $\dfrac{dy}{dx}$ for

1. $y = \sqrt{x^3 + 1}$

   Solution

2. $y = e^{\sin x}$

   Solution

3. $y = \ln(\ln x)$

   Solution

4. $y = \cos(\ln x)$

   Solution

5. $y = \tan(\sin x)$

   Solution

Example 3.7. Calculate $\dfrac{dy}{dx}$ for the following functions.

1. $y = \sin(\cos(2x - 5))$

   Solution

2. $y = \tan^2(4x^3 - 1)$

   Solution

Exercise 3.2

Find the derivative, $\dfrac{dy}{dx}$, of the following functions.

1. $y = x^{58} + \dfrac{58}{x} + (58)^x + \ln 58 + e^{58}$
2. $y = 3x^4 - \dfrac{4x^3}{3} + \dfrac{x^2}{2} + 3x - 9$
3. $y = 2\sqrt{x} - \dfrac{2}{\sqrt{x}} + \dfrac{1}{3x^2} - \dfrac{5}{2x} + 2^\pi$
4. $y = \dfrac{\operatorname{cosec} x + \pi^2}{5}$
5. $y = e^x \sin x$
6. $y = \dfrac{\cos x}{x^2 + 5}$
7. $y = \dfrac{2x - 1}{x^2 - 1}$
8. $y = \dfrac{x \ln x}{1 + \ln x}$
9. $y = (x + \sec x)^5$
10. $y = \dfrac{1}{(2x + 3)^5}$
11. $y = \tan^2 x$
12. $y = \sin(x^2)$
13. $y = \tan^{-1}(\ln x)$
14. $y = \ln(\sqrt{x}) + \sqrt{\ln x}$
15. $y = \log_3(x^2 \sin x + x)$
16. $y = (\ln 3)^x + x^{\ln 3}$
17. $y = \cos(\sin(x^2))$
18. $y = \tan(\cos(\sin(x^2)))$
19. $y = \sin^2\left(\sqrt{\log_2 x}\right)$
20. $y = x\big(\sin(\ln x) + \cos(\ln x)\big)$
21. $y = \cos(x^4) + \cos^4 x$
22. $y = x \sin^{-1} x + \sqrt{1 - x^2}$
23. $y = \dfrac{1}{\ln x} + \ln\left(\dfrac{1}{x}\right)$
24. $y = \dfrac{\sin(\sqrt{x})}{1 + \cos(\sqrt{x})}$
25. $y = \tan(x \sin x)$
26. $y = \ln(\ln(\ln x))$
27. $y = \tan^2(\sin^{-1} \sqrt{x})$
28. $y = \sin(\cos(\sin x))$
29. $y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$

3.3 Higher Order Derivatives

If f is differentiable, then f' can be found. And if f' is differentiable, then the derivative of f' is called **second order derivative of** f and denote by f''. The 3rd, 4th, ..., nth order derivative of f are defined similarly.

$$
\begin{aligned}
y' &= f'(x) = \frac{dy}{dx} \\
y'' &= f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
y''' &= f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \\
y^{(4)} &= f^{(4)}(x) = \frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) \\
&\vdots \\
y^{(n)} &= f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)
\end{aligned}
$$

Example 3.8. Given $f(x) = x^4 + 4x^3 + 2x^2 + 12x + 1$.

Find $f'(x), f''(x), f'''(x), f^{(4)}(x)$ and $f^{(5)}(x)$.

Solution

Example 3.9. Given $f(x) = \ln(2x + 3)$. Find $f^{(n)}(x)$.

Solution

Example 3.10. Find $\dfrac{d^n y}{dx^n}$ if $y = \sin x$.

Solution

Exercise 3.3

Find $f^{(n)}(x)$ of the following functions.

1. $f(x) = \dfrac{1}{2x+1}$

2. $f(x) = e^{-x}$

3. $f(x) = \ln(1-x)$

4. $f(x) = \cos x$

5. $f(x) = \sin(2x)$

3.4 Derivatives of Implicit Functions

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Suppose that y is determined instead by an equation such as

$$x^2 + y^2 - 25 = 0, \quad y^2 - x = 0, \quad x^3 + y^3 - 9xy = 0.$$

These equations define an implicit relation between the variables x and y in the form $F(x, y) = 0$. These kind of functions is called **implicit function**.

How to find $\boldsymbol{\frac{dy}{dx}}$ of implicit function $\boldsymbol{F(x, y) = 0}$?

1. Differentiate both sides of the equation $F(x, y) = 0$ with respect to x, treating y as a function of x and using the Chain Rule. We obtain the equation containing $\frac{dy}{dx}$.
2. Move the terms containing $\frac{dy}{dx}$ to the left-hand side and the remainder terms to the right-hand side of the equation.
3. Solve the equation to get $\frac{dy}{dx}$.

Example 3.11. Given $x^2 + y^2 = 1$. Compute $\frac{dy}{dx}$.

Solution

Example 3.12. Find $\dfrac{dy}{dx}$ if $\ln y = e^y \sin x$.

Solution

Example 3.13. Given $\ln\left(\dfrac{y}{x}\right) + e^{2x} = \tan(xy)$. Find $\dfrac{dy}{dx}$.

Solution

The equation of the tangent line to the curve $F(x, y) = 0$ at the point (x_0, y_0) is

$$y - y_0 = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} (x - x_0).$$

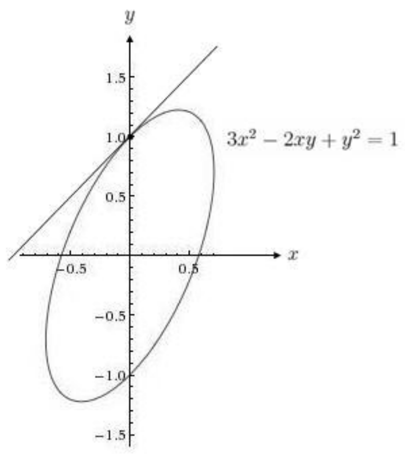
Example 3.14. Given $3x^2 - 2xy + y^2 = 1$.

1. Find $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{(0,1)}$.

2. Find an equation of the tangent line of the graph of $3x^2 - 2xy + y^2 = 1$ at the point $(0, 1)$.

Solution 1.

2.



The slope of the tangent line of the graph of $3x^2 - 2xy + y^2 = 1$ at the point $(0, 1)$ is

_______________.

Then the equation of the tangent line of the graph of $3x^2 - 2xy + y^2 = 1$ at the point $(0, 1)$ is

Logarithmic Differentiation

We now consider the derivative of explicit functions of the form

$$y = f(x)^{g(x)}$$

for $f(x) > 0$. It is not possible to use differentiation formulas to find the derivative of this kind of functions. However, it can be done by the following steps:

1. Take natural logarithm both sides of the equation to get implicit function of the form

$$\ln y = g(x)\ln(f(x)).$$

2. Use implicit differentiation to get $\dfrac{dy}{dx}$.

Example 3.15. Given $y = x^x, \;\; x > 0$. Find $\dfrac{dy}{dx}$.

Solution

Example 3.16. Given $y = (\sin x)^{\tan^{-1} x}$. Compute $\dfrac{dy}{dx}$.

Solution

Example 3.17. Given $x^y = y^x$ where $x, y > 1$. Find $\dfrac{dy}{dx}$.

Solution

Example 3.18. Given $y = \sqrt[3]{\dfrac{(x^2+1)^3(2x+3)}{(x^2+5)^2}}$. Compute $\dfrac{dy}{dx}$.

Solution

Exercise 3.4

Find $\dfrac{dy}{dx}$ of the following functions.

1. $xy + y^2 = x^2y^2 - 1$
2. $\dfrac{x}{y} + \dfrac{y}{x} = 3$
3. $\ln(xy) = e^{x+y}$
4. $x\ln(y^2) + y\ln x = 1$
5. $\cos y = \ln(x+y)$
6. $e^{xy} = \sqrt{x^2+y^2}$
7. $\tan y = 3x^2 + \tan(x+y)$
8. $e^x \cos y = xe^y$
9. $y^x = xy, \quad y > 0$
10. $x^{\tan(x-y)} = e^{x+y}$
11. $y = (4x+1)^{\cos(3x)}$
12. $y = (2+\sin x)^{\cos x}$
13. $y = (2\sin x + 3\cos x)^{x^3}$
14. $y = x^{x^x}, x > 1$
15. $y = \dfrac{(x^2+1)\sqrt{3x+4}}{\sqrt[5]{(2x-3)(x^2-4)}}$
16. $y = (x+1)(x+2)(x+3)(x+4)(x+5)$

3.5 Derivatives of Parametric Equations

Definition 3.2. Parametric Curve

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

Parametric Formula for $\dfrac{dy}{dx}$

Let $x = f(t)$ and $y = g(t)$ be differentiable at t and y be also a differentiable function of x. Then the derivatives $\dfrac{dy}{dt}$, $\dfrac{dx}{dt}$, and $\dfrac{dy}{dx}$ are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $\dfrac{dx}{dt} \neq 0$, then

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Parametric Formula for $\dfrac{d^2y}{dx^2}$

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $\dfrac{dx}{dt} \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}.$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Example 3.19. Given $x = 2t + 1$ and $y = \sqrt{t}$. Find $\dfrac{dy}{dx}$.

Solution

Example 3.20. Find $\dfrac{d^2y}{dx^2}$ where $x = t + \cos t$ and $y = \sin t$.

Solution

Example 3.21. Given

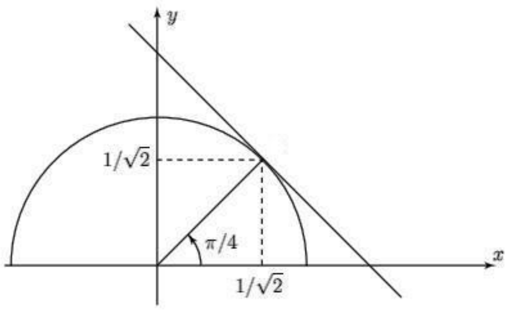
$$x = \cos t \ \text{ and } \ y = \sin t \quad \text{where } \ 0 \leq t \leq \pi.$$

1. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

2. Find an equation of the tangent line to the curve at the point $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.

Solution 1.

2.



Exercise 3.5

Find $\dfrac{dy}{dx}$ of the following parametric equations.

1. $x = t^2, \quad y = t^3 + 1$

2. $x = \ln(\cos t), \quad y = \sec t, \ t \in [0, 1]$

3. $x = 2\cos t, \quad y = 6\sin t, \ t \geq 0$

4. $x = 3t - 4\sin(\pi t), \quad y = t^2 + t\cos(\pi t), \ 0 \leq t \leq 4$

5. $x = te^{-t}, \quad y = 2t^2 + 1$

6. $x = 2t^3 + 1, \quad y = t^2 \cos t$

Find $\dfrac{d^2y}{dx^2}$ of the following parametric equations.

7. $x = t^2, \quad y = t^3$

8. $x = 2\cos t, \quad y = 2\sin t$

9. $x = e^{-t}, \quad y = t^3 + t + 1$

10. $x = 3t^2 + 4t, \quad y = \sin(2t)$

11. Given

$$x = \ln(\tan t) \text{ and } y = \sec^2 t \text{ where } 0 < t < 1.$$

    11.1) Find $\dfrac{dy}{dx}$.

    11.2) Find $\dfrac{d^2y}{dx^2}$.

    11.3) Find an equation of the tangent line to the curve at the point $t = \dfrac{\pi}{4}$.