Chapter 5

Integrations

5.1 Indefinite Integrals

$$\xrightarrow{\frac{d}{dx}} \quad \boxed{f(x) = x^2} \quad \xrightarrow{\frac{d}{dx}}$$

Antiderivatives Function Derivative

$\Downarrow$

The indefinite integral of x^2 (general antiderivative of x^2)

We use the symbol $\displaystyle\int x^2\,dx$ for the indefinite integral of x^2.

Therefore, $\displaystyle\int x^2\,dx =$

Definition 5.1. A function F is an **antiderivative** of f on an interval I if

$$\frac{d}{dx}F(x) = f(x) \quad \text{for all } x \in I.$$

If F is an antiderivative of f on an interval I, then the **general antiderivative** of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

The general antiderivative of f is denoted by $\int f(x)dx$ and is called an **indefinite integral** of f. That is,

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

For a function f, the derivative of f is unique but an antiderivative is not unique.

Example 5.1.

1. If $\int f(x)\,dx = h(x)$, then $\dfrac{d}{dx}h(x) =$ ________________________.
2. If $\int f(x)\,dx = 5\sin x + 2x + C$, then $f(x) =$ ________________________.
3. If $\dfrac{d}{dx}g(x) = 5x - 1$, then $\int 5x - 1\,dx =$ ________________________.
4. $\int \sin x\,dx =$ ________________________.

5.1.1 Basic Integrals

Basic Indefinite Integral Rules

Let f and g be functions of x. Then

1. $\displaystyle\int du = u + C$.
2. $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant.
3. $\displaystyle\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.
4. $\displaystyle\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$.

Indefinite Integral Formulas

1. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + C$
4. $\displaystyle\int e^x dx = e^x + C$
5. $\displaystyle\int \cos x\,dx = \sin x + C$
6. $\displaystyle\int \sin x\,dx = -\cos x + C$
7. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
8. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$
9. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
10. $\displaystyle\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$
11. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$
12. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$
13. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
14. $\displaystyle\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + C$
15. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
16. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ a > 0$
17. $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
18. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C,\ a > 0$

Example 5.2. Find the following indefinite integrals.

1. $\displaystyle\int (e^x + \cos x)\ dx$

   Solution

2. $\displaystyle\int \left(\frac{1}{4+x^2} - \frac{1}{\sqrt{9-x^2}} \right)\ dx$

   Solution

3. $\displaystyle\int (2^x + \sin x + \tan x - \sec x)\ dx$

   Solution

4. $\displaystyle\int \left(3x^2 + \frac{1}{x} + 3\sqrt{x} + 3 \right)\ dx$

   Solution

Example 5.3. Evaluate

1. $\displaystyle\int \tan^2 x \, dx$

   Solution

2. $\displaystyle\int \frac{\sin x}{\cos^2 x} \, dx$

   Solution

3. $\displaystyle\int \frac{\sin^2 x}{\cos x} \, dx$

   Solution

Exercise 5.1.1

Find the following indefinite integrals.

1. $\displaystyle\int\left(\frac{6}{x}+2\sqrt{x}+3\cos x-5^x\right)\,dx$
2. $\displaystyle\int\left(e^x+x^6-\sqrt[6]{x}-\frac{1}{6x^2}+6^x\ln 6\right)\,dx$
3. $\displaystyle\int(\pi^x-x^\pi+2^\pi)\,dx$
4. $\displaystyle\int(6x^2-2x^6+6^{2x})\,dx$
5. $\displaystyle\int\left(\frac{5}{25+x^2}-\frac{1}{\sqrt{4-x^2}}\right)\,dx$
6. $\displaystyle\int\left(\frac{1}{4\sec x}-\operatorname{cosec} x\cot x\right)\,dx$
7. $\displaystyle\int\left(7x^6+\frac{2}{\sqrt{x}}+\operatorname{cosec} x-\sin x\right)\,dx$
8. $\displaystyle\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{1+x^2}+\sec x\tan x+\sec^2 x\right)\,dx$
9. $\displaystyle\int\frac{1+\cos^2 x}{\cos^2 x}\,dx$
10. $\displaystyle\int\frac{\sin 2x}{\sin x}\,dx$
11. $\displaystyle\int\frac{\sin x}{1-\sin^2 x}\,dx$

5.1.2 Techniques of Integration

1. Integration by u-substitution
2. Integration by parts
3. Integration by partial fractions
4. Integration by trigonometric substitution

Definition 5.2. If $y = f(x)$ is differentiable, then the **differential** dx is an independent variable. The **differential** dy is then defined in terms of dx by the equation

$$dy = \frac{dy}{dx}dx = f'(x)dx.$$

Example 5.4.

1. $u = x^2 + 1$ $\qquad \frac{du}{dx} =$ ____________ $\qquad du =$ ____________
2. $u = \sec x$ $\qquad \frac{du}{dx} =$ ____________ $\qquad du =$ ____________
3. $u = \ln x$ $\qquad \frac{du}{dx} =$ ____________ $\qquad du =$ ____________
4. $u = 1 + \tan x$ $\qquad \frac{du}{dx} =$ ____________ $\qquad du =$ ____________

5.1.2.1 Integration by u-substitution

Consider

$$\int 2x\cos(x^2)\,dx.$$

In this case, let's notice that if we let $u = x^2$ and we compute the differential for this we get,

$$du = 2x\,dx.$$

Substitute u and dx, we obtain

$$\begin{aligned}\int 2x\cos(x^2)\,dx &= \int 2x\cos u\,\frac{du}{2x}\\ &= \int \cos u\,du\\ &= \sin u + C\\ &= \sin(x^2) + C.\end{aligned}$$

What we've done in the work above is called the **integration by u-substitution**.

3-Step of integration by u-substitution

$$\int 2x\cos(x^2)\,dx \underset{\substack{\Uparrow\\ \boxed{\text{Step 1}}}}{=} \int \cos u\,du \underset{\substack{\Uparrow\\ \boxed{\text{Step 2}}}}{=} \sin u + C \underset{\substack{\Uparrow\\ \boxed{\text{Step 3}}}}{=} \sin(x^2) + C.$$

Step 1 : Choose $u = x^2$, compute du. Substitute u and dx.

Step 2 : Integrate with respect to u.

Step 3 : Substitute x^2 for u.

Integration by u-substitution

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad \text{where} \quad u = g(x)$$

For integration by u-substitution to work, one needs to make an appropriate choice for the u substitution:

Strategy for choosing u

Identify a composition of functions in the integrand. If a rule is known for integrating the outside function, then let u equal the inside function.

In above example, the expression $\cos(x^2)$ is the composition of x^2 and $\cos x$. Since we had a rule for integrating the outside function $\cos x$, we choose to let u equal the inside function x^2.

Example 5.5. Find the following indefinite integrals.

1. $\int (2x-1)^{10}\,dx$

   Solution

2. $\int \cos(2x)\,dx$

   Solution

3. $\displaystyle\int xe^{2x^2+1}\,dx$

   Solution

4. $\displaystyle\int \cos x \sin^3 x\,dx$

   Solution

5. $\displaystyle\int x\sqrt{1+2x^2}\,dx$

   Solution

6. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

Solution

7. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

Solution

Example 5.6. Find the following indefinite integrals.

1. $\displaystyle\int 2\sin x\cos x\,dx$

   Solution

2. $\displaystyle\int \frac{\ln x}{x}\,dx$

   Solution

3. $\displaystyle\int \sin^2 x\,dx$

   Solution

Integration of Partial fractions

A **partial fraction** is a rational function of the forms

$$\frac{A}{(ax+b)^k}, \frac{Ax+B}{(ax^2+bx+c)^k}$$

where k is a positive integer, A, B, a, b, c are real numbers and $b^2-4ac<0$.

The condition $b^2-4ac<0$ means that the quadratic polynomial ax^2+bx+c has no real roots.

Example 5.7. Find the following indefinite integrals.

1. $\displaystyle\int \frac{2}{3x+1}\,dx$

   Solution

2. $\displaystyle\int \frac{2}{(3x+1)^5}\,dx$

   Solution

Integration of partial fractions $\boldsymbol{\dfrac{A}{(ax+b)^k}}$

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$$
$$\int \frac{A}{(ax+b)^k}\,dx = \frac{A}{a}\frac{(ax+b)^{-k+1}}{-k+1} + C, k > 1$$

Example 5.8. Find the following indefinite integrals.

1. $\displaystyle\int \frac{2}{x-1}\,dx =$

2. $\displaystyle\int \frac{3}{2x+1}\,dx =$

3. $\displaystyle\int \frac{4}{1-5x}\,dx =$

4. $\displaystyle\int \frac{1}{(x-3)^2}\,dx =$

5. $\displaystyle\int \frac{2}{(3x+1)^3}\,dx =$

Example 5.9. Find the following indefinite integrals.

1. $\displaystyle\int \frac{1}{1+4x^2}\,dx$

   Solution

2. $\displaystyle\int \frac{x}{x^2+1}\,dx$

   Solution

Exercise 5.1.2.1

Find the following indefinite integrals.

1. $\displaystyle\int (4x+1)^5 \, dx$
2. $\displaystyle\int \frac{\tan^{-1} x}{1+x^2} \, dx$
3. $\displaystyle\int \frac{1}{x\ln x} \, dx$
4. $\displaystyle\int \frac{\sin(\ln x)}{x} \, dx$
5. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$
6. $\displaystyle\int \frac{e^x}{1+e^x} \, dx$
7. $\displaystyle\int \frac{e^x+e^{-x}}{e^x-e^{-x}} \, dx$
8. $\displaystyle\int \frac{\cos x - \sin x}{\cos x + \sin x} \, dx$
9. $\displaystyle\int \frac{x^3+1}{\sqrt{x^4+4x}} \, dx$
10. $\displaystyle\int x\sqrt{1-x^2} \, dx$
11. $\displaystyle\int x(x^2+9)^9 \, dx$
12. $\displaystyle\int x^2 e^{3x^3-1} \, dx$
13. $\displaystyle\int (x^2-2x+5)^3(x-1) \, dx$
14. $\displaystyle\int \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}} \, dx$
15. $\displaystyle\int \frac{(1+\tan x)^3}{\cos^2 x} \, dx$
16. $\displaystyle\int \frac{\sin(2x)}{1-\cos(2x)} \, dx$
17. $\displaystyle\int e^{\sin x} \cos x \, dx$
18. $\displaystyle\int \frac{\sec^2 x}{\sqrt{1+\tan x}} \, dx$
19. $\displaystyle\int \tan^2 x \sec^2 x \, dx$
20. $\displaystyle\int \frac{2x+1}{x^2+x+1} \, dx$
21. $\displaystyle\int \frac{x+1}{x^2+1} \, dx$

5.1.2.2 Integration by parts

Consider the following indefinite integrals:

$$\int x\,dx = \frac{x^2}{2} + C,$$
$$\int x^2\,dx = \frac{x^3}{3} + C.$$

We see that

$$\int x \cdot x\,dx \neq \left(\int x\,dx\right)\left(\int x\,dx\right).$$

In general,

$$\int f(x)g(x)\,dx \neq \left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right).$$

However, there is a way to deal with products in integrals. This method is known as integration by parts. We use this technique with $\int F(x)\,dx$ when

1. $F(x)$ is a product of two functions, that is, $F(x) = f(x)g(x)$ or

2. $F(x) = \ln x$ or $F(x) =$ one of inverse trigonometric functions ($\sin^{-1} x, \tan^{-1} x$, etc).

Start with the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Thus,

$$d(uv) = udv + vdu,$$

and hence

$$\int d(uv) = \int u\,dv + \int v\,du.$$

Therefore

$$uv = \int u\,dv + \int v\,du.$$

Integration by parts Formula

$$\int u\,dv = uv - \int v\,du$$

Steps for Integration by Parts

1. First, separate the differential to be integrated into a product of a function u (easily differentiated) and a differential dv (easily integrated) such that the integral produced by the integration by parts formula is easier to evaluate than the original one.

   Normally, we choose u by the order "**LIATE**" or "**LIAET**".

   Whichever function comes first in the following list should be u:

L	Logarithmic function	$\ln x, \log_2 x$, etc
I	Inverse Trigonometric function	$\tan^{-1} x, \sin^{-1} x$, etc
A	Algebraic function	$\sqrt{x}, x^2 + 1$, etc
T	Trigonometric function	$\sin x, \cos x$, etc
E	Exponential function	$e^x, 3^x$, etc

2. Compute du and v where
$$du = \frac{du}{dx}\,dx \text{ and } v = \int dv.$$

3. Substitute u, v, du and dv into the integration by parts formula.

4. Find $\displaystyle\int vdu$.

Example 5.10. Find the following indefinite integrals.

1. $\displaystyle\int x \ln x \, dx$

   Solution

2. $\displaystyle\int e^x \cos x \, dx$

Solution

3. $\displaystyle\int \sec^3 x\,dx$

Solution

4. $\displaystyle\int \ln x \, dx$

Solution

5. $\displaystyle\int \tan^{-1} x \, dx$

Solution

Tabular Integration by Parts

The tabular integration by parts works well for integral of the form

$$\int f(x)g(x)\,dx$$

where f is a polynomial function and g is a function that can be integrate repeatedly.

Example 5.11. Find the following indefinite integrals.

1. $\displaystyle\int x^2 e^x \, dx$

   Solution

2. $\displaystyle\int (x^3 + x + 1)\sin(2x) \, dx$

   Solution

Exercise 5.1.2.2

1. $\displaystyle\int x\sec^2 x\ dx$
2. $\displaystyle\int 2x\tan^{-1} x\ dx$
3. $\displaystyle\int x\sec x\tan x\ dx$
4. $\displaystyle\int \sqrt{x}\ln x\ dx$
5. $\displaystyle\int x^2\ln x\ dx$
6. $\displaystyle\int x^2\ln(x^2)\ dx$
7. $\displaystyle\int \frac{\ln(x^2+1)}{x^3}\ dx$
8. $\displaystyle\int \cos x\ln(\sin x)\ dx$
9. $\displaystyle\int \frac{\ln(\sin x)}{\cos^2 x}\ dx$
10. $\displaystyle\int 4x(\ln x)^2\ dx$
11. $\displaystyle\int e^{-x}\sin x\ dx$
12. $\displaystyle\int \sin(\ln x)\ dx$
13. $\displaystyle\int (x^2+x+1)\sin x\ dx$
14. $\displaystyle\int x^2\cos(2x)\ dx$
15. $\displaystyle\int x^2 e^{-x}\ dx$
16. $\displaystyle\int x^3 e^{x^2}\ dx$
17. $\displaystyle\int x\sin x\cos x\ dx$

5.1.2.3 Integration by partial fractions

We use this technique with $\int \frac{P(x)}{Q(x)}\,dx$, where $P(x)$ and $Q(x)$ are polynomials. By this method, we will rewrite a rational function as a sum of partial fractions.

Steps for Integration by partial fractions

Step 1: Check whether $\deg P(x) < \deg Q(x)$ or not!

- If $\deg P(x) < \deg Q(x)$, then moving on to Step 2.

- If $\deg P(x) \geq \deg Q(x)$, by long division algorithm for polynomials, we can write

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

  where $S(x)$ and $R(x)$ are polynomials and $\deg R(x) < \deg Q(x)$ and then decompose $\frac{R(x)}{Q(x)}$ into partial fractions according to the information in Step 2.
  For example,

$$\frac{x^4}{x^2+1} =$$

Step 2: Let's now assume that $\deg P(x) < \deg Q(x)$. Then

1. Factor $Q(x)$ as a product of linear factors $(ax+b)^m$ and quadratic factors $(ax^2+bx+c)^n$ where $b^2-4ac<0$.

   For example,

$$\frac{x-2}{x^2+3x+2} =$$

$$\frac{x^2+4}{(x+1)(x^2-1)} =$$

$$\frac{4x^2-3x-4}{x^3+x^2+2x} =$$

2. Decompose $\frac{P(x)}{Q(x)}$ into partial fractions, according to the following table.

Factor in $Q(x)$	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k, k>1$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k, k>1$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

   For example,

$$\frac{x-2}{(x+2)(x+1)} =$$

$$\frac{x^2+4}{(x-1)(x+1)^2} =$$

$$\frac{x^2+1}{(x+1)^2(x-1)^3} =$$

$$\frac{4x^2-3x-4}{x(x^2+x+2)} =$$

$$\frac{x}{(x^2+1)^3} =$$

Step 3: Find all unknown coefficients $A, B, C, \ldots$ by using the following method.

- Comparing coefficient
- Substitution
- Heaviside's Method

  If the denominator of the rational function factors into the product of distinct linear factors:

$$\frac{P(x)}{(a_1x-b_1)(a_2x-b_2)\ldots(a_px-b_p)} = \frac{A_1}{a_1x-b_1} + \frac{A_2}{a_2x-b_2} + \ldots + \frac{A_p}{a_px-b_p}$$

  where $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_p}{a_p}$ are all distinct, then A_i is found by covering up the factor a_ix-b_i on the left, and setting $x = \frac{b_i}{a_i}$ in the rest of the expression.

Example 5.12. Write the following as a sum of partial fractions.

1. $\dfrac{-2x+4}{(x^2+1)(x-1)^2} =$

2. $\dfrac{x^2+x+1}{(x^2-1)(x-2)(x+3)} =$

3. $\dfrac{x+1}{x^2(x-2)} =$

4. $\dfrac{2x+3}{x(x-1)(x^2+1)} =$

5. $\dfrac{5x-3}{x(x-1)^2(x^2+1)^3} =$

Example 5.13. Find the integral $\int \frac{5x+1}{x^2+x-2}\,dx$.

Solution

Example 5.14. Find the integral $\displaystyle\int \frac{2x^2 - x + 5}{(x-1)(x^2+1)}\,dx$.

Solution

Example 5.15. Find the integral $\displaystyle\int \frac{9x-1}{(x-1)(x+1)^2}\, dx$.

Solution

Exercise 5.1.2.3

Use the partial fractions to evaluate the following integrals.

1. $\displaystyle\int \frac{5x-13}{(x-3)(x-2)}\,dx$
2. $\displaystyle\int \frac{x+1}{x^3+x^2-2x}\,dx$
3. $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$
4. $\displaystyle\int \frac{1}{x(x+1)^2}\,dx$
5. $\displaystyle\int \frac{-x^2+2x+1}{(x^2+1)(x+1)}\,dx$
6. $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$
7. $\displaystyle\int \frac{x^2+x+1}{x^4+5x^2+4}\,dx$
8. $\displaystyle\int \frac{1}{x^2(x+1)}\,dx$
9. $\displaystyle\int \frac{x^2-1}{x(x^2+4)}\,dx$
10. $\displaystyle\int \frac{3x^2-4x+1}{(x-2)(x^2+1)}\,dx$
11. $\displaystyle\int \frac{x^2+10x+1}{(x-1)(x+2)(x^2+1)}\,dx$
12. $\displaystyle\int \frac{x^2-2x+4}{x(x-1)^2}\,dx$
13. $\displaystyle\int \frac{x^2+4}{x^2+x}\,dx$
14. $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

5.1.2.4 Integration by trigonometric substitution

We use this technique when an integrand contains the following expressions;

$$a^2 - x^2, \quad a^2 + x^2, \quad x^2 - a^2, \quad a > 0.$$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$a^2 - x^2$	$x = a\sin\theta, \; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$ $\sqrt{1 - \sin^2\theta} = \cos\theta$
$a^2 + x^2$	$x = a\tan\theta, \; -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$ $\sqrt{1 + \tan^2\theta} = \sec\theta$
$x^2 - a^2$	$x = a\sec\theta, \; 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$ $\sqrt{\sec^2\theta - 1} = \tan\theta$

Example 5.16. Find $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$.

Solution

Example 5.17. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}\,dx.$

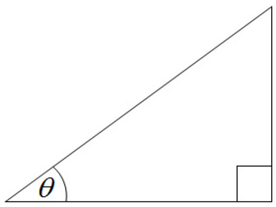
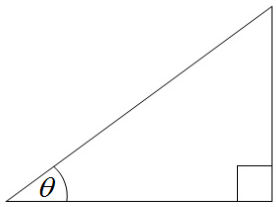
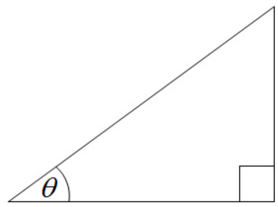
Solution

Example 5.18. Compute $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$.

Solution

Example 5.19. Find $\displaystyle\int \frac{\sqrt{x^2-1}}{x^2}\,dx$.

Solution

$f(x)$ contains $a^2 - x^2$ or $\sqrt{a^2 - x^2}$ where $a > 0$	$f(x)$ contains $a^2 + x^2$ or $\sqrt{a^2 + x^2}$ where $a > 0$	$f(x)$ contain $x^2 - a^2$ or $\sqrt{x^2 - a^2}$ where $a > 0$
Let $x =$ $dx =$ $\sqrt{a^2 - x^2} =$	Let $x =$ $dx =$ $\sqrt{a^2 + x^2} =$	Let $x =$ $dx =$ $\sqrt{x^2 - a^2} =$
 $\sin\theta =$ $\theta =$	 $\tan\theta =$ $\theta =$	 $\sec\theta =$ $\theta =$

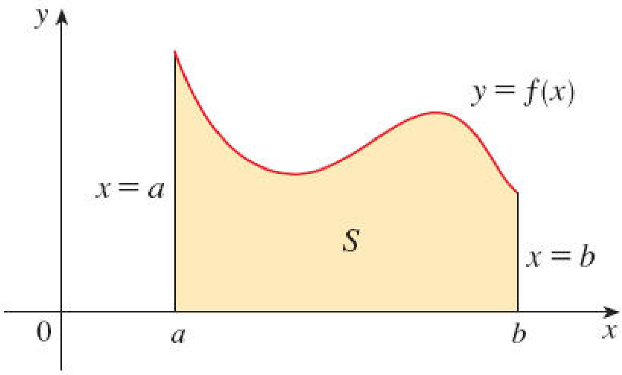
Exercise 5.1.2.4

Find the following integrals.

1. $\displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$
2. $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$
3. $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$
4. $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$
5. $\displaystyle\int \frac{\sqrt{1+x^2}}{x^4}\,dx$
6. $\displaystyle\int \sqrt{4+x^2}\,dx$
7. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^3}\,dx$
8. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$
9. $\displaystyle\int \sqrt{x^2-4}\,dx$
10. $\displaystyle\int \frac{1}{x^2\sqrt{4x^2-1}}\,dx$

5.2 Definite Integrals

Let f be a continuous function defined on the closed interval $[a, b]$. Let us suppose that $f(x) \geq 0$ for all $x \in [a, b]$, so that the graph of f is a curve above the $x-$axis.



Let S be the region that lies under the curve $y = f(x)$ between $x = a$ and $x = b$.

Now the problem is to find the area of the region S. If the graph of $y = f(x)$ is not a straight line we do not, at the moment, know how to calculate the area precisely.

We can find the area of the region S as follows:

Divide the interval $[a, b]$ into n subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Then the n subintervals are

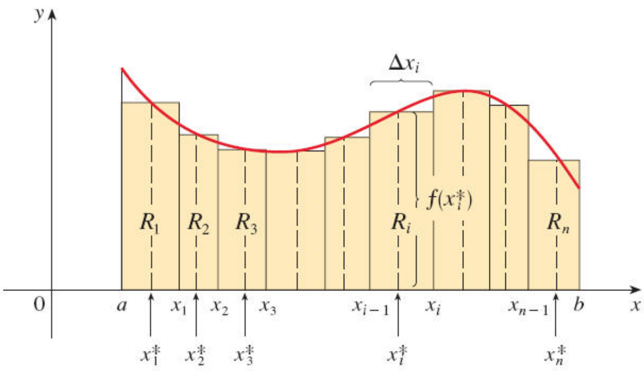
$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n].$$

The width of the ith subinterval $[x_{i-1}, x_i]$ is given by

$$\Delta x_i = x_i - x_{i-1}.$$

If we divide the interval $[a, b]$ into n subintervals of equal width, then the width of each subinterval is $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 0, 1, 2, \ldots, n$.

Choose a point x_i^* in each subinterval $[x_{i-1}, x_i]$ and construct a rectangle R_i with base Δx_i and height $f(x_i^*)$.



Each point x_i^* can be anywhere in its subinterval – at the right endpoint or at the left endpoint or somewhere between the endpoints.

The area of the ith rectangle R_i is $f(x_i^*)\Delta x_i$.

The area of the region S is approximated by the sum of the areas of these rectangles, which is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The summation is called a **Riemann sum**.

We can see that $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ is equal to the area of the region S.

If f is a continuous function on $[a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ exists, and is independent of the choices of x_i^* and the width of each subinterval Δx_i. Therefore, it is legitimate to define the following.

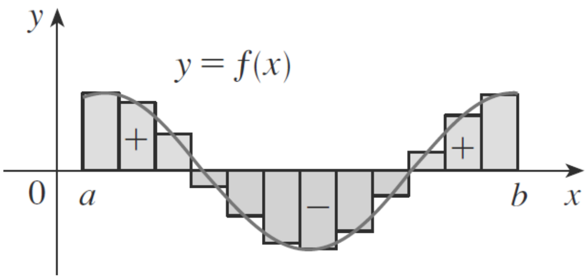
Riemann Integral

Let f be a continuous function on $[a, b]$. We define the **definite integral of** f **from** a **to** b by

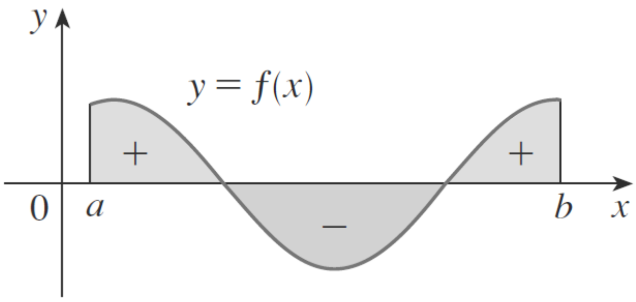
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

- If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx$ is equal to the area of the region that lies under the curve $y = f(x)$ between $x = a$ and $x = b$.

- If f takes both positive and negative values, then the Riemann sum is the sum of the area of the rectangles that lie above the $x-$axis and the negatives of the areas of the rectangles that lie below the $x-$axis.



When we take the limit of such Riemann sums, we get the situation illustrated in the figure below.



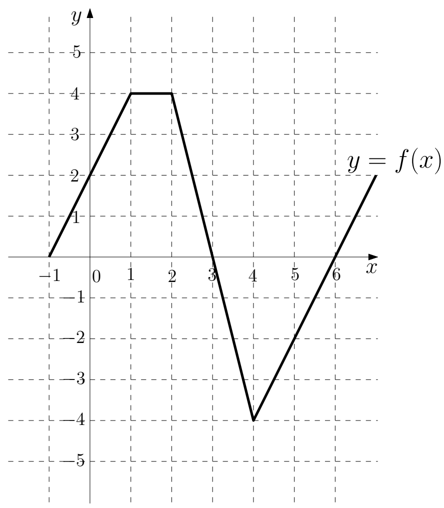
A definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and

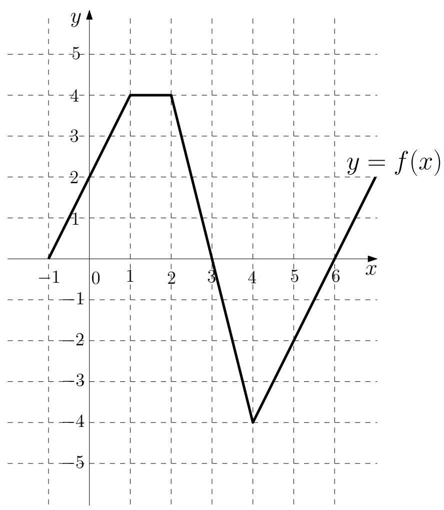
A_2 is the area of the region below the x-axis and above the graph of f.

Example 5.20. The graph of $y = f(x)$ is given as follows.

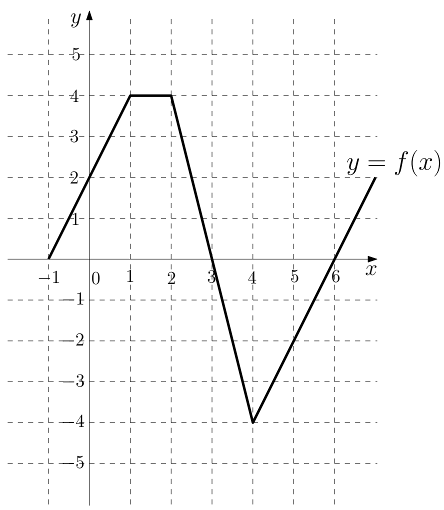


Evaluate the following definite integrals.

1. $\displaystyle\int_{-1}^{2} f(x)\, dx$



2. $\displaystyle\int_{2}^{6} f(x)\, dx$



Properties of the definite integrals

Let f and g be continuous functions on $[a, b]$ and let k be any constant.

1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx$ where $a \le c \le b$

6. If $f(x) \ge 0$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,dx \ge 0$

Example 5.21. Let f and g be continuous functions on the interval $[-2, 5]$ such that

$$\int_{-2}^{5} f(x)dx = 7, \int_{-2}^{5} g(x)dx = 2 \text{ and } \int_{3}^{5} f(x)dx = 3.$$

Evaluate the following definite integrals.

1. $\displaystyle\int_{-2}^{5} [2f(x) - 3g(x)]\,dx$

   Solution

2. $\displaystyle\int_{-2}^{3} f(x)\,dx$

   Solution

Fundamental Theorem of Calculus

The fundamental theorem of calculus will connect integration and differentiation, enabling us to compute definite integrals using an antiderivative of the integrand rather than by taking limits of Riemann sum.

The Fundamental Theorem of Calculus Part I

If f is continuous on $[a, b]$ and

$$F(x) = \int_a^x f(t)dt.$$

Then F is differentiable on (a, b) and $\frac{d}{dx}F(x) = f(x)$. That is,

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x).$$

Example 5.22. Find the following derivatives.

1. $\frac{d}{dx}\left(\int_0^1 e^{\sin t}\, dt\right) =$

2. $\frac{d}{dx}\left(\int_1^x e^{\sin t}\, dt\right) =$

3. $\frac{d}{dx^2}\left(\int_1^{x^2} e^{\sin t}\, dt\right) =$

4. $\frac{d}{dx}\left(\int_x^1 e^{\sin t}\, dt\right) =$

5. $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^2} e^{\sin t}\, dt\right)$

Solution

6. $\dfrac{d}{dx}\left(\displaystyle\int_x^{x^2} e^{\sin t}\, dt\right), x > 1$

Solution

Exercise 5.2.1

Find the following derivatives.

1. $\dfrac{d}{dx}\left(\displaystyle\int_{-1}^{1}\cos(t^3-1)\,dt\right)$
2. $\dfrac{d}{dx}\left(\displaystyle\int_{1}^{x}\cos(t^3-1)\,dt\right)$
3. $\dfrac{d}{dx}\left(\displaystyle\int_{x}^{1}\cos(t^3-1)\,dt\right)$
4. $\dfrac{d}{de^x}\left(\displaystyle\int_{1}^{e^x}\cos(t^3-1)\,dt\right)$
5. $\dfrac{d}{dx}\left(\displaystyle\int_{1}^{e^x}\cos(t^3-1)\,dt\right)$
6. $\dfrac{d}{dx}\left(\displaystyle\int_{x}^{e^x}\cos(t^3-1)\,dt\right)$
7. $\dfrac{d}{dx}\left(\displaystyle\int_{-1}^{1}(t^4-3t^2)\,dt\right)$
8. $\dfrac{d}{dx}\left(\displaystyle\int_{-1}^{x}(t^4-3t^2)\,dt\right)$
9. $\dfrac{d}{dx^4}\left(\displaystyle\int_{-1}^{x^4}(t^4-3t^2)\,dt\right)$
10. $\dfrac{d}{dx}\left(\displaystyle\int_{-1}^{x^4}(t^4-3t^2)\,dt\right)$
11. $\dfrac{d}{dx}\left(\displaystyle\int_{x}^{1}(t^4-3t^2)\,dt\right)$
12. $\dfrac{d}{dx}\left(\displaystyle\int_{x}^{x^4}(t^4-3t^2)\,dt\right), x>1$

The Fundamental Theorem of Calculus Part II

If f is continuous on $[a, b]$ and $\int f(x)\,dx = F(x) + C$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example 5.23. Find the following definite integrals.

1. $\displaystyle\int_1^3 2\;dx$

   Solution

2. $\displaystyle\int_0^1 x^2\;dx$

   Solution

3. $\displaystyle\int_{\pi}^{2\pi} \sin x\;dx$

   Solution

4. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$

Solution

5. $\displaystyle\int_{-1}^1 |x|\, dx$

Solution

Exercise 5.2.2

1. Find the following definite integrals.

1.1. $\displaystyle\int_1^e \frac{1}{x}\,dx$

1.2. $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$

1.3. $\displaystyle\int_0^\pi \cos x\,dx$

1.4. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

1.5. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

2. Given $F(x) = \displaystyle\int_{\frac{1}{2}}^{x} f(t)\,dt$.

2.1. Find $\dfrac{d}{dx}F(\sqrt{x})$.

2.2. Suppose $f(t) = 1 + \dfrac{1}{t^2}$. Find $F(1)$.

5.3 Improper Integrals

The conditions for definite integral $\int_a^b f(x)\,dx$ are

1. a and b are real numbers,

2. f is a continuous function on $[a, b]$.

Consider $f(x) = e^x$. The function e^x is continuous on $[1, \infty)$ but the domain $[1, \infty)$ of the integral $\int_1^\infty e^x\,dx$ is infinite. The integral $\int_1^\infty e^x\,dx$ is called an ***improper integral of type I*** because the function does not satisfy the first condition.
Consider another case, the function $f(x) = \frac{1}{x}$ is not continuous at $x = 0$. In this case, we say that $\int_0^1 \frac{1}{x}\,dx$ is called an ***improper integral of type II*** because the function does not satisfy the second condition.

Improper Integrals of Type I

1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx =$$

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx =$$

3. If f is continuous on $(-\infty, \infty)$, then

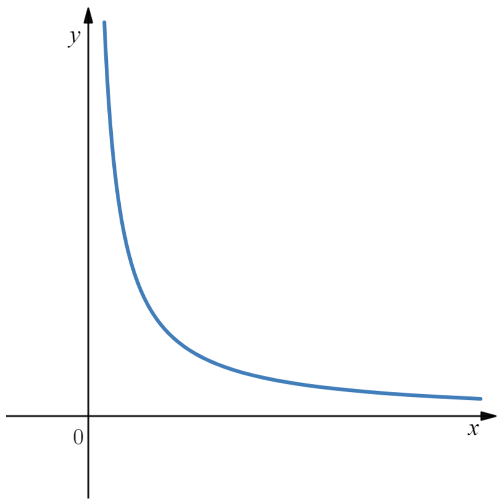
$$\int_{-\infty}^\infty f(x)\,dx =$$

In each case, if the limit is finite, we say that the improper integral **converges** and that the limit is the value of the integral. If the limit fails to exist, the improper integral **diverges**.

Example 5.24. Find the following improper integrals.

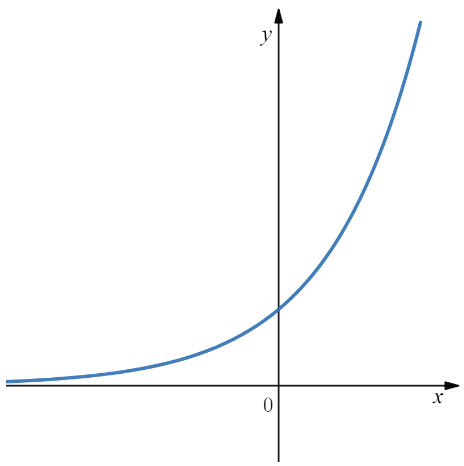
1. $\displaystyle\int_1^\infty \frac{1}{x}\, dx$

   Solution



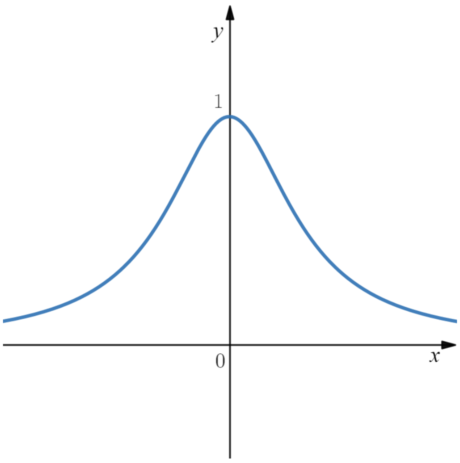
2. $\displaystyle\int_{-\infty}^0 e^x\, dx$

   Solution



3. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Solution



Improper Integrals of Type II

1. If f is continuous on $(a, b]$ and $\lim_{x \to a^+} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx =$$

2. If f is continuous on $[a, b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx =$$

3. If f is not continuous at c, where $a < c < b$, and

$$\lim_{x \to c^-} f(x) = \pm\infty \text{ and } \lim_{x \to c^+} f(x) = \pm\infty,$$

then

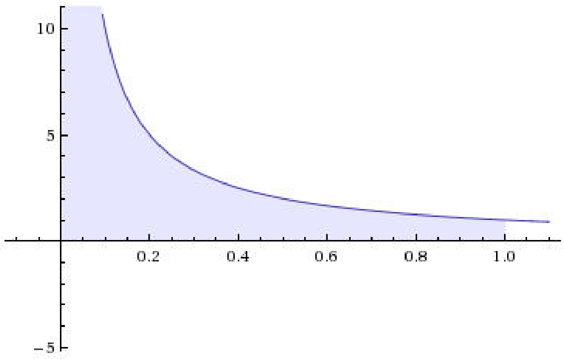
$$\int_a^b f(x)\,dx =$$

In each case, if the limit is finite, we say that the improper integral **converges** and that the limit is the value of the integral. If the limit fails to exist, the improper integral **diverges**.

Example 5.25. Find the following improper integrals.

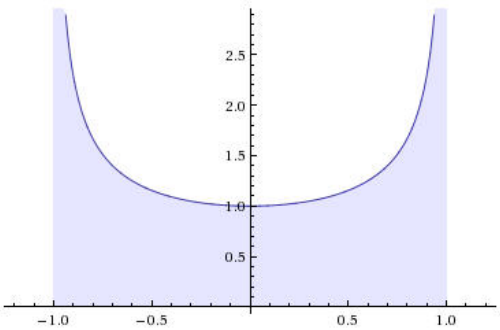
1. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

   Solution



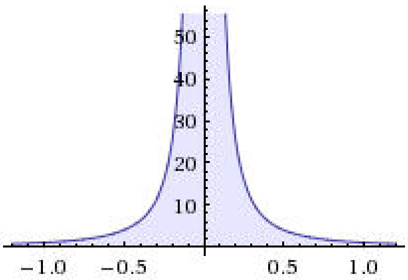
2. $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

   Solution



3. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$

Solution



Exercise 5.3

Find the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$
2. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$
3. $\displaystyle\int_{-\infty}^1 \frac{1}{(x-2)^3}\,dx$
4. $\displaystyle\int_{-\infty}^0 x^2 e^x\,dx$
5. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$
6. $\displaystyle\int_0^{\pi/2} \frac{1}{\cos^2 x}\,dx$
7. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
8. $\displaystyle\int_{-1}^1 x^{-\frac{2}{5}}\,dx$
9. $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$