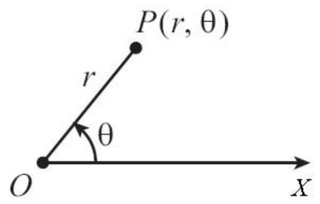
Chapter 7

Polar Coordinate System

7.1 Points in the Polar Coordinate System

The Polar Coordinate System consists of a horizontal ray that extends to the right known as the **polar axis** and the endpoint of the ray, called the **pole** as shown below.

A point P in the polar coordinate system is represented by a pair of numbers (r, θ).



- r is a directed distance from the pole to P. (r can be positive, negative, or zero)

  - $r > 0$ when the point lies on the terminal side of θ.

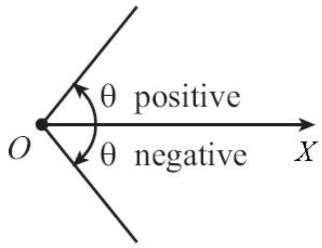
  - $r < 0$ when the point lies along the ray opposite the terminal side of θ.

  - $r = 0$ when the point lies at the pole.

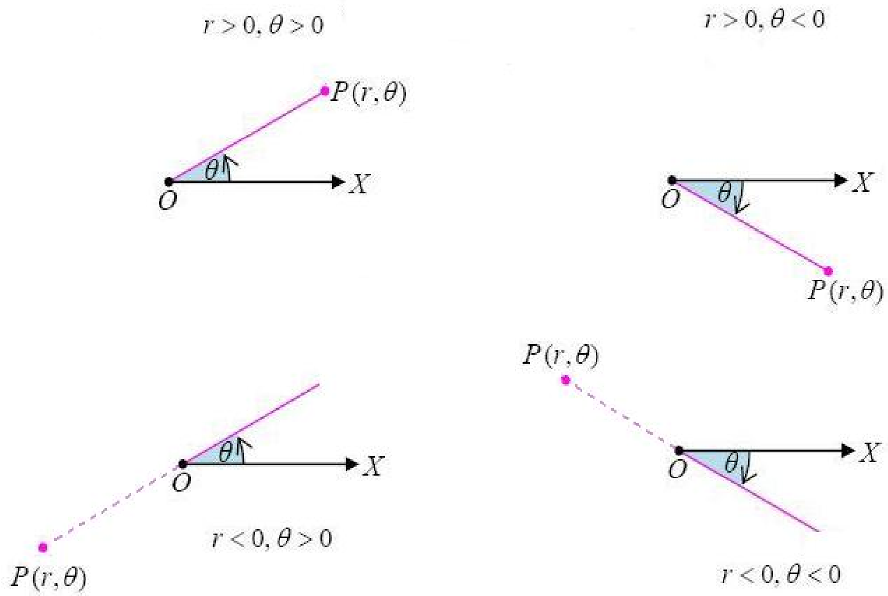
- θ is an angle from the polar axis to the line segment from the pole to P. This angle is measured in radians.

- $\theta > 0$ when the angle was measured counterclockwise from the polar axis.

- $\theta < 0$ when the angle was measured clockwise from the polar axis.

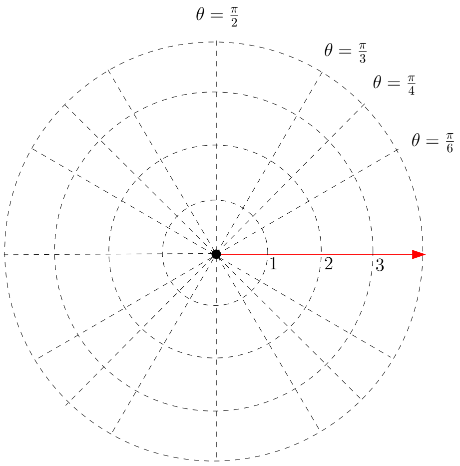


We refer to the ordered pair (r, θ) as the **polar coordinates** of P.

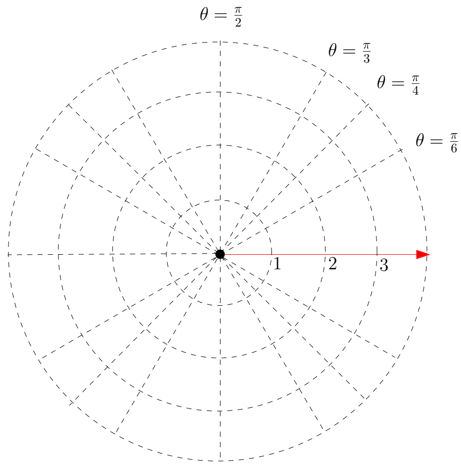


Example 7.1. Plot the points with the following polar coordinates:

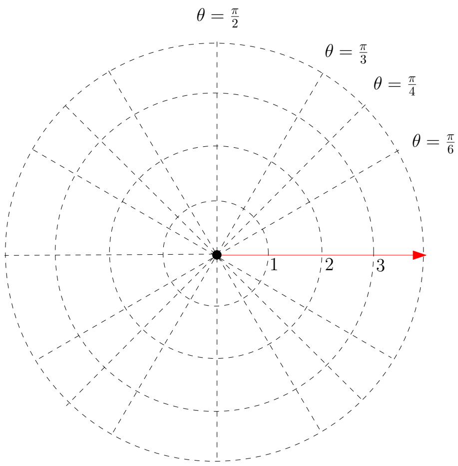
1. $\left(2, \dfrac{\pi}{3}\right)$



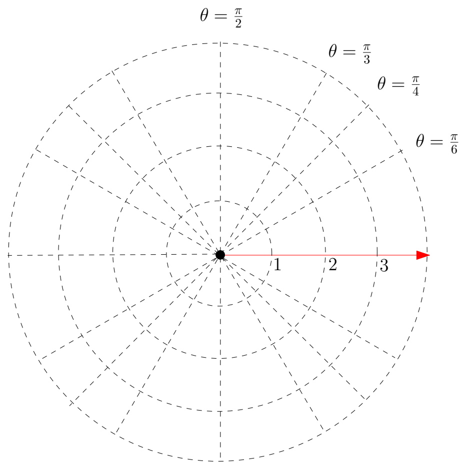
2. $\left(-2, \dfrac{\pi}{3}\right)$



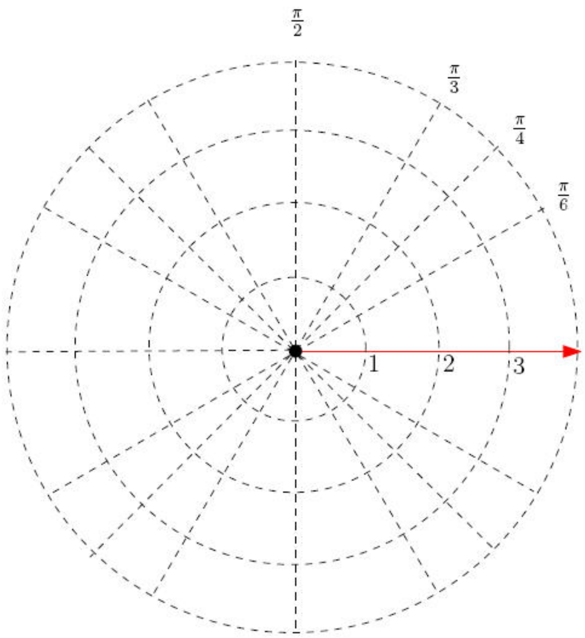
3. $\left(2, -\dfrac{\pi}{3}\right)$



4. $\left(-2, -\dfrac{\pi}{3}\right)$



Example 7.2. Plot the points $A\left(3, \frac{\pi}{4}\right)$, $B\left(-2, -\frac{5\pi}{3}\right)$, $C\left(3, -\frac{\pi}{6}\right)$, $D\left(-3, -\frac{\pi}{2}\right)$, $E(2, -\pi)$, $F\left(-2, \frac{7\pi}{4}\right)$, $G\left(1, -\frac{11\pi}{6}\right)$, $H\left(1, \frac{13\pi}{6}\right)$ on the grid below and label each point by the letter.



Multiple Representations of Points in the Polar Coordinate System

In rectangular coordinates, each point has exactly one representation but in the polar coordinate system each point can be represented in infinitely many ways.

For example, for any θ, the point $(0, \theta)$ represents the pole O.

We focus on 4 different forms of polar coordinates of a point (r, θ) :

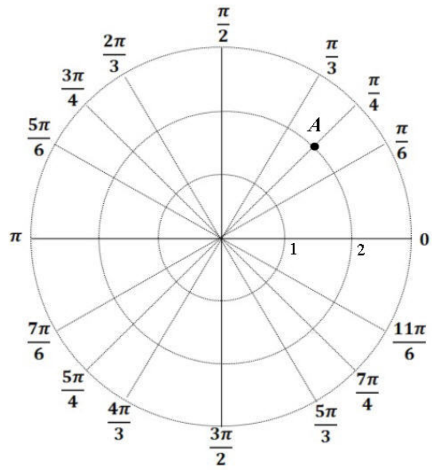
1. $r > 0, 0 \leq \theta < 2\pi$

2. $r > 0, -2\pi < \theta \leq 0$

3. $r < 0, 0 \leq \theta < 2\pi$

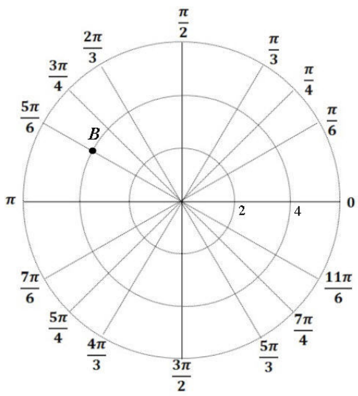
4. $r < 0, -2\pi < \theta \leq 0$

Example 7.3. What are the polar coordinates (r, θ) of the given point A under the following conditions?

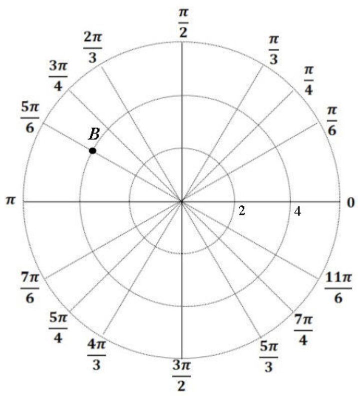


Condition of r	Condition of θ	Polar coordinates of the point A
$r > 0$	$0 \leq \theta < 2\pi$	$\Big(\ldots\ldots\ldots, \ldots\ldots\ldots\Big)$
$r > 0$	$-2\pi < \theta \leq 0$	$\Big(\ldots\ldots\ldots, \ldots\ldots\ldots\Big)$
$r < 0$	$0 \leq \theta < 2\pi$	$\Big(\ldots\ldots\ldots, \ldots\ldots\ldots\Big)$
$r < 0$	$-2\pi < \theta \leq 0$	$\Big(\ldots\ldots\ldots, \ldots\ldots\ldots\Big)$

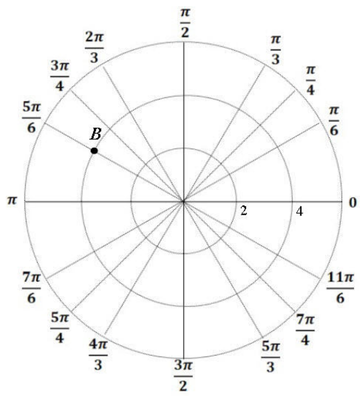
$r > 0, -2\pi < \theta \leq 0$



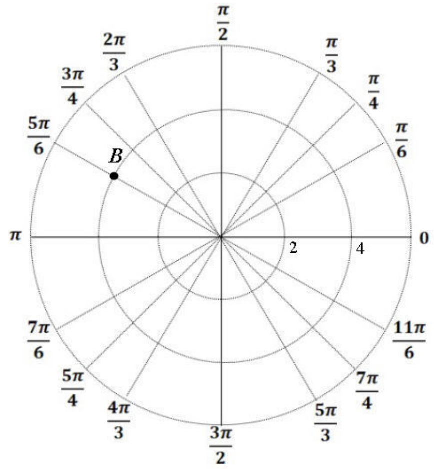
$r < 0, 0 \leq \theta < 2\pi$



$r < 0, -2\pi < \theta \leq 0$

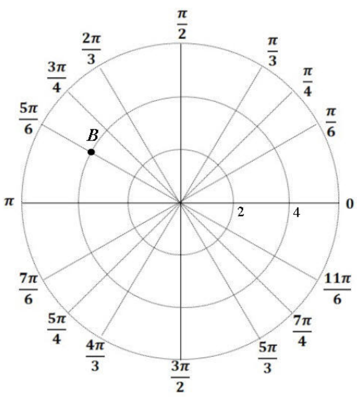


Example 7.4. Find the polar coordinates of the given point B under the following conditions.

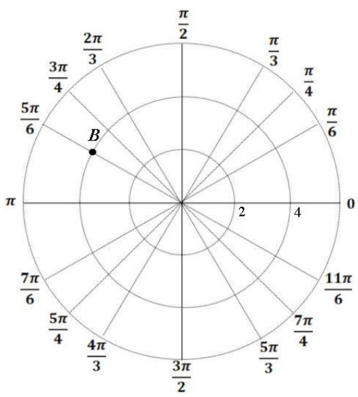


Condition of r	Condition of θ	Polar coordinates of the point B
$r > 0$	$0 \leq \theta < 2\pi$	$\Big(\dots\dots\dots, \dots\dots\dots\Big)$
$r > 0$	$-2\pi < \theta \leq 0$	$\Big(\dots\dots\dots, \dots\dots\dots\Big)$
$r < 0$	$0 \leq \theta < 2\pi$	$\Big(\dots\dots\dots, \dots\dots\dots\Big)$
$r < 0$	$-2\pi < \theta \leq 0$	$\Big(\dots\dots\dots, \dots\dots\dots\Big)$

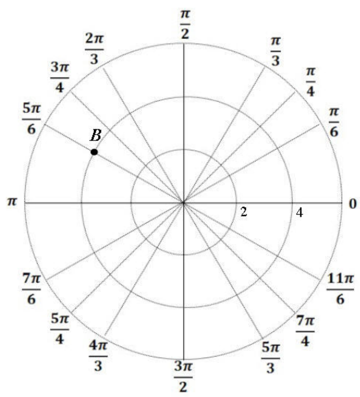
$r > 0, -2\pi < \theta \leq 0$



$r < 0, 0 \leq \theta < 2\pi$



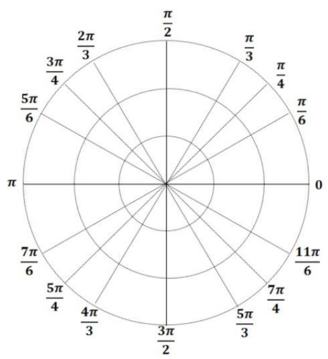
$r < 0, -2\pi < \theta \leq 0$



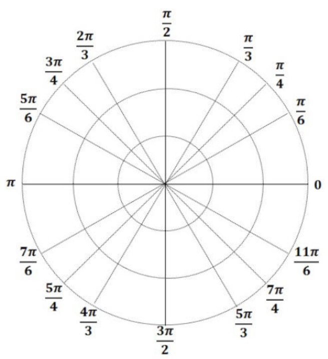
Example 7.5. Given the point $\left(-2, \dfrac{\pi}{3}\right)$, find its alternative polar coordinates satisfying the following conditions:

Condition of r	Condition of θ	Polar coordinates of the point $\left(-2, \dfrac{\pi}{3}\right)$
$r > 0$	$0 \leq \theta < 2\pi$	$\left(\ldots\ldots\ldots, \ldots\ldots\ldots\right)$
$r > 0$	$-2\pi < \theta \leq 0$	$\left(\ldots\ldots\ldots, \ldots\ldots\ldots\right)$
$r < 0$	$-2\pi < \theta \leq 0$	$\left(\ldots\ldots\ldots, \ldots\ldots\ldots\right)$

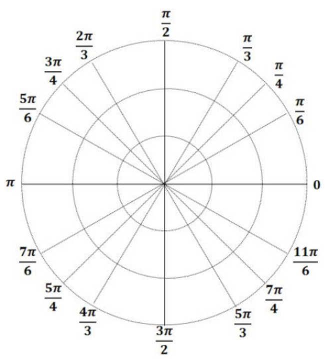
$r > 0, 0 \leq \theta < 2\pi$



$r > 0, -2\pi < \theta \leq 0$



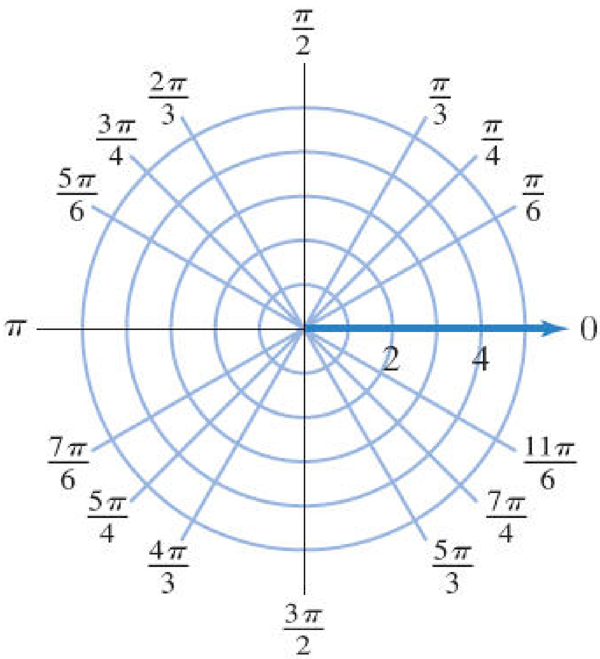
$r < 0, -2\pi < \theta \leq 0$



Exercise 7.1

1. Plot the following points.

Point	(r, θ)
A	$(1, 0)$
B	$\left(2, -\frac{11\pi}{6}\right)$
C	$\left(4, \frac{\pi}{4}\right)$
D	$\left(-4, \frac{\pi}{3}\right)$
E	$\left(4, \frac{2\pi}{3}\right)$
F	$\left(-3, -\frac{5\pi}{6}\right)$
G	$\left(1, \frac{4\pi}{3}\right)$
H	$(-3, -3\pi)$
I	$\left(1, \frac{5\pi}{6}\right)$
J	$\left(2, \frac{11\pi}{6}\right)$



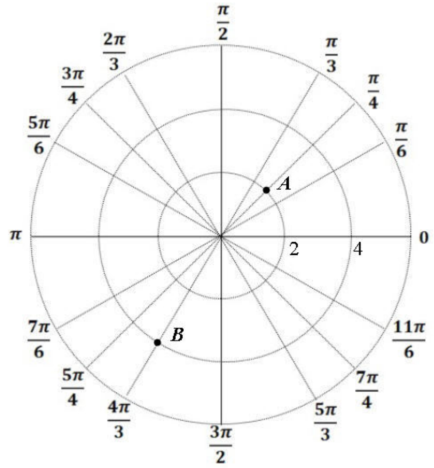
2. Given the point $\left(-1, -\frac{7\pi}{6}\right)$ in the polar coordinate system, find its alternative polar coordinates satisfying the following conditions

2.1) $r > 0, 0 \leq \theta < 2\pi$: ____________________

2.2) $r > 0, -2\pi < \theta \leq 0$: ____________________

2.3) $r < 0, 0 \leq \theta < 2\pi$: ____________________

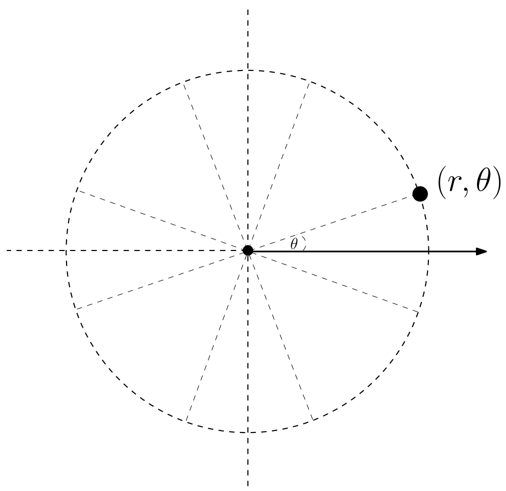
3. Find the polar coordinates of the given point under the following conditions.



Condition of r	Condition of θ	Polar coordinates of the point A	Polar coordinates of the point B
$r > 0$	$0 \leq \theta < 2\pi$	(........, ...)	(........, ...)
$r > 0$	$-2\pi < \theta \leq 0$	(........, ...)	(........, ...)
$r < 0$	$0 \leq \theta < 2\pi$	(........, ...)	(........, ...)
$r < 0$	$-2\pi < \theta \leq 0$	(........, ...)	(........, ...)

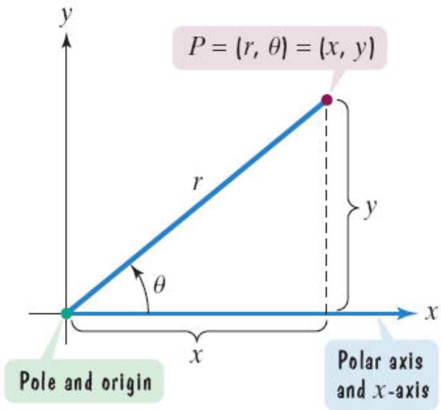
4. Let (r, θ) be a point in the polar coordinate system where $r > 0$ and $0 < \theta < \dfrac{\pi}{2}$ as in the following figure. Plot the points A, B, C and D whose polar coordinates are given below.

point	coordinates
A	$(r, -\theta)$
B	$(-r, -\theta)$
C	$(-r, \theta + \pi)$
D	$(r, \theta - \dfrac{\pi}{2})$



7.2 Relations between Polar and Rectangular Coordinates

We now consider both polar and rectangular coordinates simultaneously. The polar axis coincides with the positive x-axis and the pole coincides with the origin. A point P has rectangular coordinates (x, y) and polar coordinates (r, θ) as indicated in the figure.



From the figure, we see that

$$x^2 + y^2 = r^2$$
$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r} \text{ and } \tan\theta = \frac{y}{x}.$$

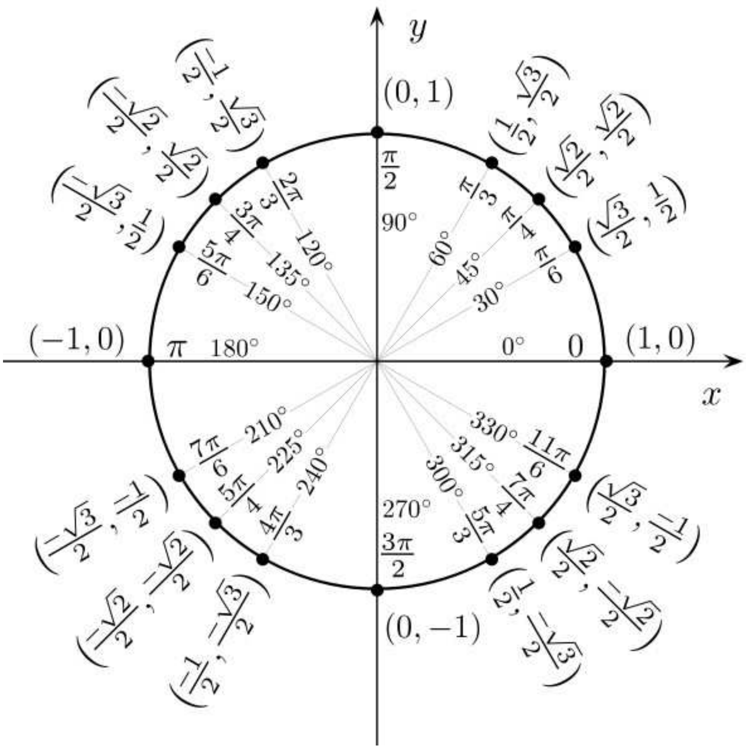
Relations between Polar and Rectangular Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Point Conversion from Polar to Rectangular Coordinates

To convert a point from polar coordinates (r, θ) to rectangular coordinates (x, y), use the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$



Example 7.6. Find the rectangular coordinates of the points with the following polar coordinates:

1. $\left(2, \frac{\pi}{2}\right)$ 2. $\left(-2, -\frac{\pi}{3}\right)$ 3. $\left(2, -\frac{3\pi}{4}\right)$ 4. $\left(2, \frac{11\pi}{6}\right)$

Solution 1. $\left(2, \frac{\pi}{2}\right)$

$$x = r\cos\theta = \underline{\hspace{8cm}}$$

$$y = r\sin\theta = \underline{\hspace{8cm}}$$

The rectangular coordinates of $\left(2, \frac{\pi}{2}\right)$ are ____________.

2. $\left(-2, -\dfrac{\pi}{3}\right)$

$$x = r\cos\theta = \underline{\hspace{6cm}}$$

$$y = r\sin\theta = \underline{\hspace{6cm}}$$

The rectangular coordinates of $\left(-2, -\dfrac{\pi}{3}\right)$ are ____________.

3. $\left(2, -\dfrac{3\pi}{4}\right)$

$$x = r\cos\theta = \underline{\hspace{6cm}}$$

$$y = r\sin\theta = \underline{\hspace{6cm}}$$

The rectangular coordinates of $\left(2, -\dfrac{3\pi}{4}\right)$ are ____________.

4. $\left(2, \dfrac{11\pi}{6}\right)$

$$x = r\cos\theta = \underline{\hspace{6cm}}$$

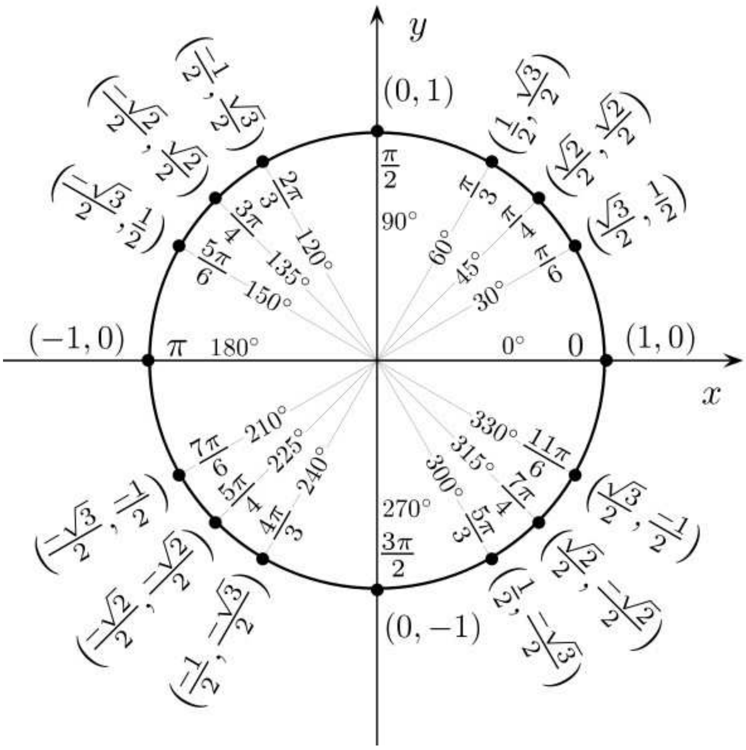
$$y = r\sin\theta = \underline{\hspace{6cm}}$$

The rectangular coordinates of $\left(2, -\dfrac{11\pi}{6}\right)$ are ____________.

Point Conversion from Rectangular to Polar Coordinates

1. Find r using $r^2 = x^2 + y^2$.

2. Find θ using $\tan\theta = \dfrac{y}{x}$.



$\tan\theta$	0	$\dfrac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-\dfrac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$
$0 \le \theta < 2\pi$		Q I	Q I	Q I	Q II	Q II	Q II
		Q III	Q III	Q III	Q IV	Q IV	Q IV

Example 7.7. Find the polar coordinates (r, θ) where $\underline{r > 0 \text{ and } 0 \le \theta < 2\pi}$ of the points with the following rectangular coordinates:

1. $(-1, \sqrt{3})$

   - Find r.

     $$r^2 = x^2 + y^2 = \underline{\hspace{6cm}}$$

     Since $r > 0$, we get $r =$ ___________.

   - Find θ.

     $$\tan\theta = \frac{y}{x} = \underline{\hspace{2.5cm}}$$

     Since $0 \le \theta < 2\pi$ and the point$(-1, \sqrt{3})$ lies in quadrant _____ , we get $\theta =$ _____.

   The polar coordinates of $(-1, \sqrt{3})$ where $r > 0$ and $0 \le \theta < 2\pi$ are _________.

2. $(-1, -1)$

   - Find r.

     $$r^2 = x^2 + y^2 = \underline{\hspace{6cm}}$$

     Since $r > 0$, we get $r =$ ___________.

   - Find θ.

     $$\tan\theta = \frac{y}{x} = \underline{\hspace{2.5cm}}$$

     Since $0 \le \theta < 2\pi$ and the point$(-1, -1)$ lies in quadrant _____ , we get $\theta =$ _____.

   The polar coordinates of $(-1, -1)$ where $r > 0$ and $0 \le \theta < 2\pi$ are _________.

3. $(2\sqrt{3}, -2)$

- Find r.

$$r^2 = x^2 + y^2 = \rule{6cm}{0.4pt}$$

Since $r > 0$, we get $r =$ ___________.

- Find θ.

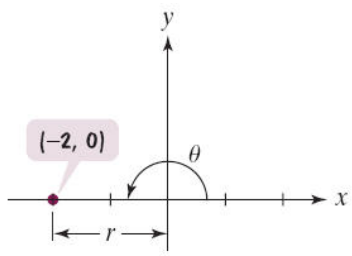
$$\tan\theta = \frac{y}{x} = \rule{2.5cm}{0.4pt}$$

Since $0 \le \theta < 2\pi$ and the point$(2\sqrt{3}, -2)$ lies in quadrant _____ , we get $\theta =$ _____.

The polar coordinates of $(2\sqrt{3}, 2)$ where $r > 0$ and $0 \le \theta < 2\pi$ are __________.

4. $(-2, 0)$

Solution



From the figure, we see that $r =$ _______ and $\theta =$ _______.

The polar coordinates of $(-2, 0)$ where $r > 0$ and $0 \le \theta < 2\pi$ are __________.

5. $(0, -2)$

Solution

From the figure, we see that $r =$ _______ and $\theta =$ _______.

The polar coordinates of $(0, -2)$ where $r > 0$ and $0 \le \theta < 2\pi$ are __________.

Equation Conversion from Polar to Rectangular Coordinates

A polar equation is an equation whose variables are r and θ. Two examples of polar equations are

$$r = \frac{5}{\cos\theta + \sin\theta} \quad \text{and} \quad r = 3\sec\theta.$$

To convert a polar equation in r and θ to a rectangular equation in x and y, we use one or more of the following equations:

$$r^2 = x^2 + y^2 \qquad r\cos\theta = x \qquad r\sin\theta = y \qquad \tan\theta = \frac{y}{x}.$$

Example 7.8. Convert each polar equation to a rectangular equation in x and y.

1. $r = 5$

   Solution

2. $\theta = \dfrac{\pi}{4}$

   Solution

3. $r = 3 \operatorname{cosec} \theta$

Solution

4. $r = 2 \cos \theta$

Solution

5. $r = \dfrac{12}{3 \cos \theta + 2 \sin \theta}$

Solution

6. $r = \dfrac{2}{\cos\theta - \operatorname{cosec}\theta}$

Solution

7. $r^2 \sin 2\theta = 4$

Solution

Equation Conversion from Rectangular to Polar Coordinates

To convert a rectangular equation in x and y to a polar equation in r and θ, use the formulas

$$x = r\cos\theta,\ y = r\sin\theta \text{ and } x^2 + y^2 = r^2.$$

Example 7.9. Convert each rectangular equation to a polar equation.

1. $y = 1$

   Solution

2. $x = 2$

   Solution

3. $x + y = 1$

   Solution

4. $x^2 + y^2 = 1$

Solution

5. $y = 0$

Solution

6. $y = x$

Solution

7. $x^2 + y^2 - 4x = 0$

Solution

8. $x^2 + (y - 1)^2 = 1$

Solution

Exercise 7.2

1. Find the rectangular coordinates of the points with the following polar coordinates:

   1.1 $(1, 0)$

   1.2 $\left(2, -\dfrac{11\pi}{6}\right)$

   1.3 $\left(3, \dfrac{\pi}{4}\right)$

   1.4 $\left(-4, \dfrac{\pi}{3}\right)$

   1.5 $\left(5, \dfrac{2\pi}{3}\right)$

   1.6 $\left(-6, -\dfrac{5\pi}{6}\right)$

   1.7 $\left(7, \dfrac{4\pi}{3}\right)$

   1.8 $(-7, 7\pi)$

   1.9 $\left(8, \dfrac{5\pi}{6}\right)$

   1.10 $\left(6, \dfrac{11\pi}{6}\right)$

2. Find the polar coordinates (r, θ) where $r > 0, 0 \leq \theta < 2\pi$ of the points with the following rectangular coordinates:

   2.1 $(-1, 0)$

   2.2 $(4, -4)$

   2.3 $(1, -\sqrt{3})$

   2.4 $(\sqrt{2}, \sqrt{2})$

   2.5 $(-4, -4\sqrt{3})$

   2.6 $(2\sqrt{3}, -2)$

   2.7 $(0, 4)$

   2.8 $(-4, 4\sqrt{3})$

3. Convert each polar equation to a rectangular equation.

   3.1 $r = 7$

   3.2 $\theta = \dfrac{\pi}{3}$

   3.3 $r \sin\theta = 3$

   3.4 $r \cos\theta = 7$

   3.5 $r = 4 \sec\theta$

   3.6 $r = 2 \operatorname{cosec}\theta$

   3.7 $r = 16 \sin\theta$

   3.8 $r = 4 \sin\theta + 2 \cos\theta$

4. Convert each rectangular equation to a polar equation.

   4.1 $x = -5$

   4.2 $y = 4$

   4.3 $x + y = 0$

   4.4 $x^2 + y^2 = 25$

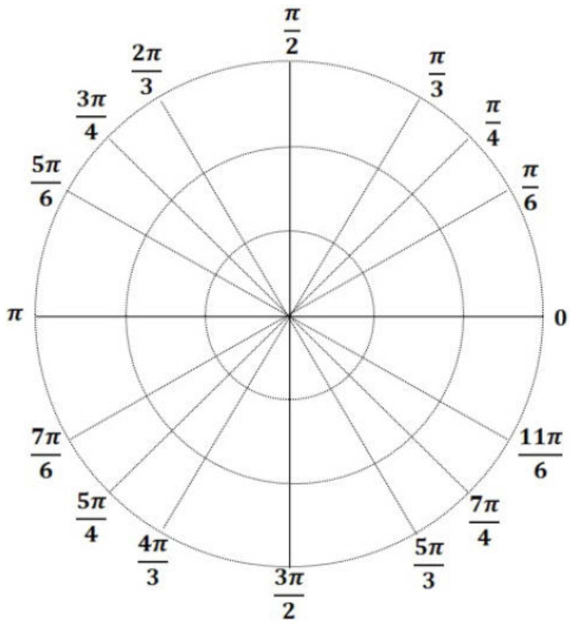
   4.5 $x^2 + y^2 - 8y = 0$

   4.6 $x^2 + y^2 - 10x = 0$

   4.7 $x^2 + y^2 - 4x + 6y = 0$

7.3 Graphing in Polar Coordinates

Recall that a polar equation is an equation whose variables are r and θ. The graph of a polar equation is the set of all points whose polar coordinates satisfy the equation. We use polar grids like the one shown in the following figure to graph polar equations.



Graphing a Polar Equation by Point Plotting

One method for graphing a polar equation is the point-plotting method.

Point-plotting method

1. Make a table of values that satisfy the equation.

2. Plot these ordered pairs as points in the polar coordinate system.

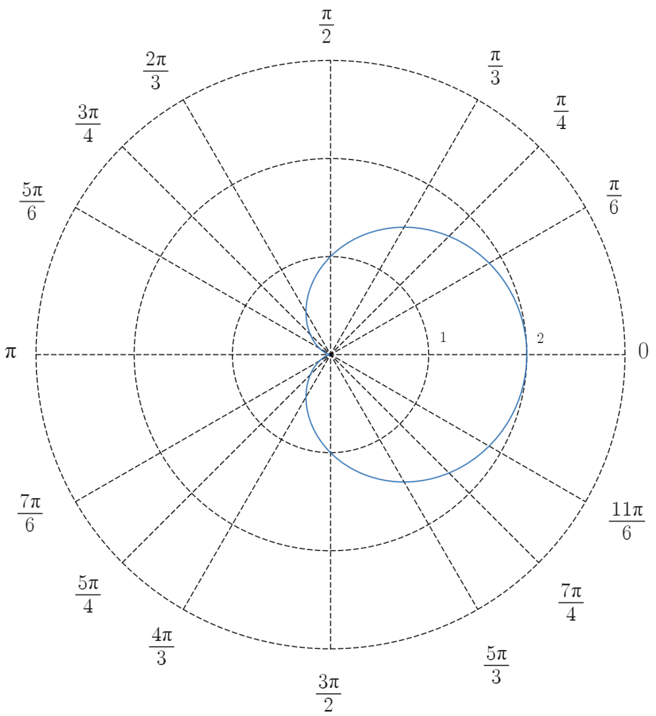
3. Connect the points with a smooth curve.

Example 7.10. Sketch the graph $r = 1 + \cos\theta$.

Solution Note that $\dfrac{\sqrt{2}}{2} \approx 0.7$ and $\dfrac{\sqrt{3}}{2} \approx 0.9$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
r								
θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
r								

Plotting points obtains the graph below.

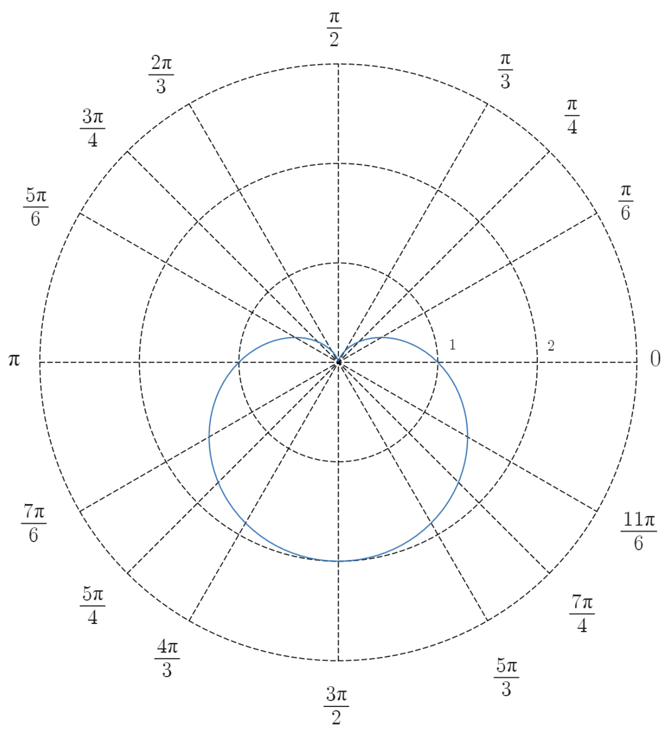


Example 7.11. Plot the graph of the equation $r = 1 - \sin\theta$.

Solution

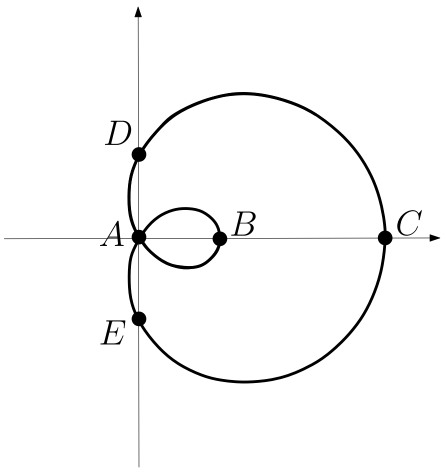
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
r								
θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
r								

Plotting points obtains the graph below.



Exercise 7.3

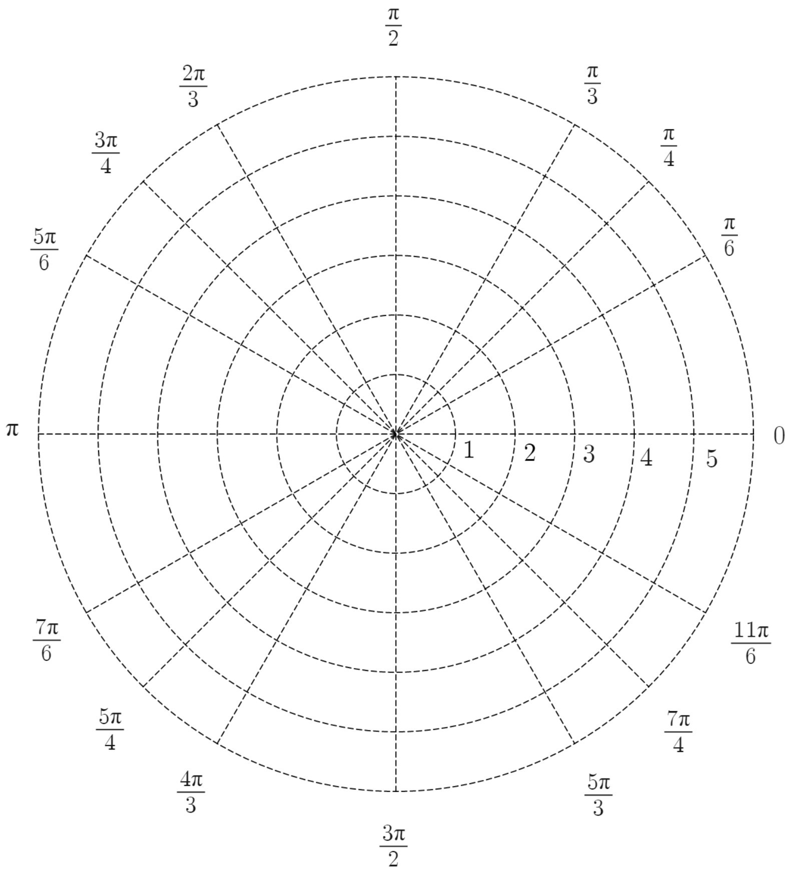
1. Given the graph of $r = 1 + 2\cos\theta$ as in the following figure, find the polar coordinates of the point A, B, C, D and E on the graph where $0 \le \theta < 2\pi$.



point	polar coordinates
A	(.......... , ...), (.......... , ...)
B	(.......... , ...)
C	(.......... , ...)
D	(.......... , ...)
E	(.......... , ...)

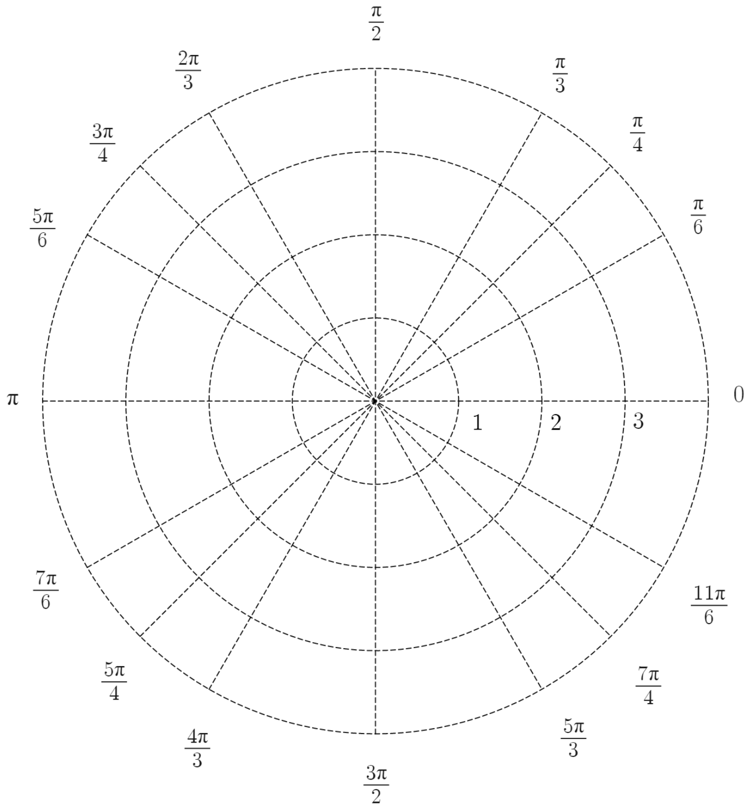
2. Plot the graph of the equation $r = 3 + 2\cos\theta$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 3 + 2\cos\theta$									



3. Plot the graph of the equation $r = 1 + 2\sin\theta$.

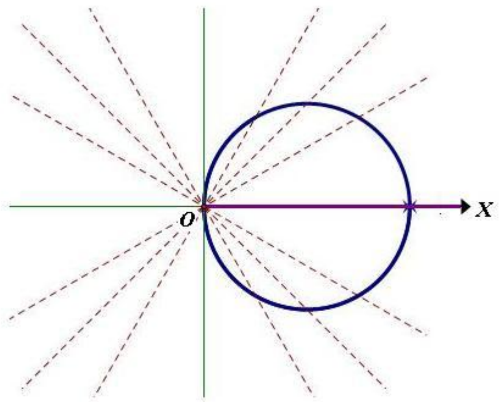
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r = 1 + 2\sin\theta$									



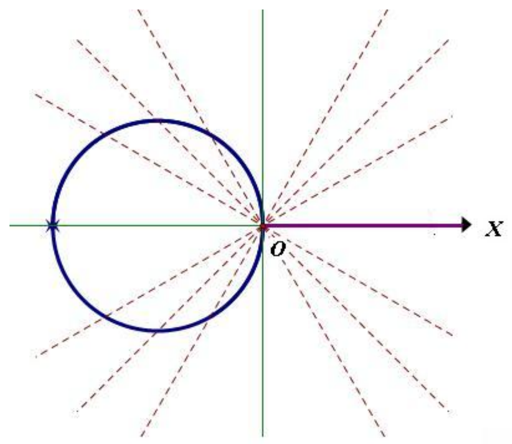
7.4 Some Standard polar curves

Circles in Polar Coordinates

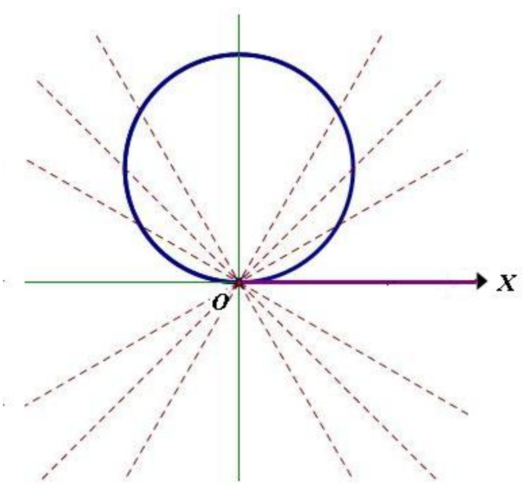
$r = 2a\cos\theta,\ a > 0$



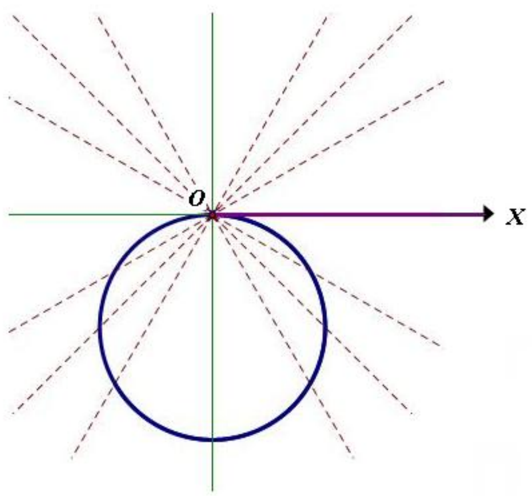
$r = 2a\cos\theta,\ a < 0$



$r = 2a\sin\theta,\ a > 0$



$r = 2a\sin\theta,\ a < 0$



Limacons

The graphs of

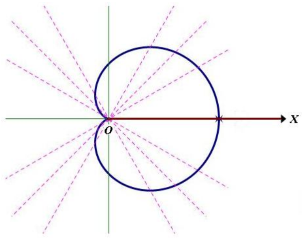
$$r = a + b\cos\theta, \quad r = a - b\cos\theta,$$

$$r = a + b\sin\theta, \quad r = a - b\sin\theta, \quad a > 0, b > 0$$

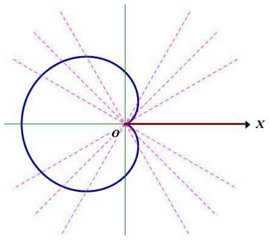
are called **Limacons**.

$\boxed{a = b}$

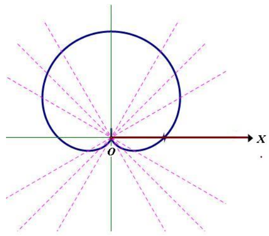
Cardioids



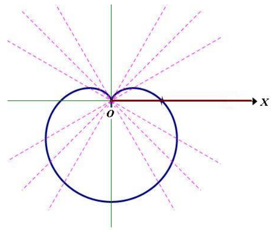
$r = a(1 + \cos\theta)$



$r = a(1 - \cos\theta)$

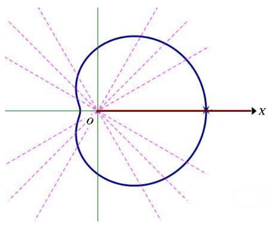


$r = a(1 + \sin\theta)$

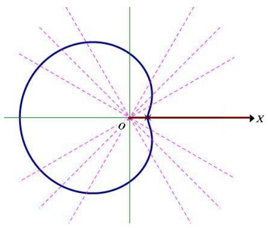


$r = a(1 - \sin\theta)$

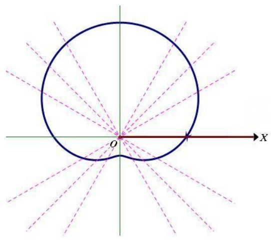
$\boxed{a > b}$



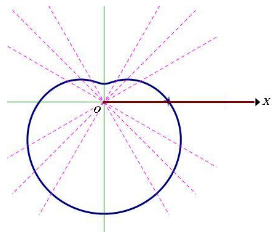
$r = a + b\cos\theta$



$r = a - b\cos\theta$

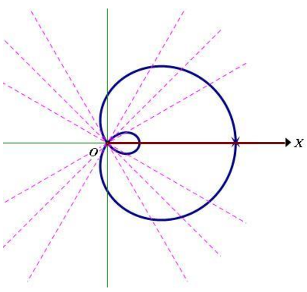


$r = a + b\sin\theta$

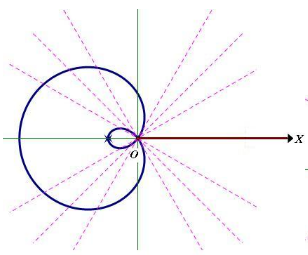


$r = a - b\sin\theta$

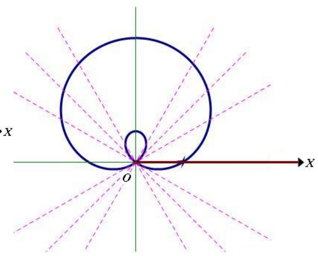
$\boxed{a < b}$



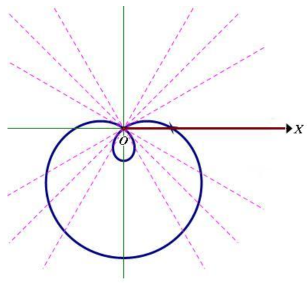
$r = a + b\cos\theta$



$r = a - b\cos\theta$



$r = a + b\sin\theta$



$r = a - b\sin\theta$

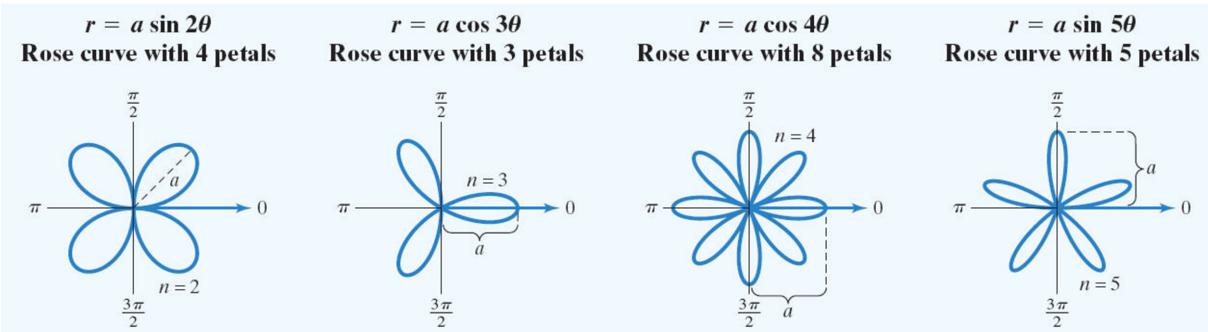
Rose Curves

The graphs of

$$r = a \sin n\theta \quad \text{and} \quad r = a \cos n\theta, \quad a \neq 0,$$

are called **rose curves**.

- If n is even, the rose has $2n$ petals.
- If n is odd, the rose has n petals.



Lemniscates

The graphs of

$$r^2 = a^2 \sin 2\theta \qquad \text{and} \qquad r^2 = a^2 \cos 2\theta, \quad a \neq 0$$

are called **lemniscates**.