Chapter 4

Application of Derivatives

4.1 Indeterminate Forms

The following expressions are all called **indeterminate forms**.

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0 \text{ and } 1^\infty.$$

Question	$\lim\limits_{x \to u} f(x)$	$\lim\limits_{x \to u} g(x)$	Indeterminate Form (I.F.)
$\lim\limits_{x \to u} \frac{f(x)}{g(x)}$	0	0	$\frac{0}{0}$
	∞ or $-\infty$	∞ or $-\infty$	$\frac{\infty}{\infty}$
$\lim\limits_{x \to u} f(x)g(x)$	0	∞ or $-\infty$	$0 \cdot \infty$
$\lim\limits_{x \to u} [f(x) - g(x)]$	∞	∞	$\infty - \infty$
	$-\infty$	$-\infty$	
$\lim\limits_{x \to u} f(x)^{g(x)}$	0	0	0^0
	∞	0	∞^0
	1	∞ or $-\infty$	1^∞

HERE: u stands for any of the symbols $a, a^-, a^+, \infty, -\infty$.

4.1.1 Indeterminate Form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

L'Hospital's Rule is a general method for evaluating the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hospital's Rule

Suppose

1. f and g are differentiable

2. $g'(x) \neq 0$ on an open interval I containing a.

3. $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ or
   $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$
   (In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$).

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or is ∞ or $-\infty$.

Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to \infty$ or as $x \to -\infty$.

Example 4.1. Calculate $\lim\limits_{x\to 0} \frac{3^x - 1}{x}$.

Solution

Example 4.2. Calculate $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$.

Solution

Example 4.3. Find $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}+1}$.

Solution

Example 4.4. Evaluate $\lim_{x\to\infty} \dfrac{e^{2x}}{x^2}$.

Solution

L'Hospital's Rule cannot be used to evaluate the following limit.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} &= \lim_{x\to-\infty} \frac{\frac{d}{dx}(\sqrt{x^2+1})}{\frac{d}{dx}x} \\
&= \lim_{x\to-\infty} \frac{\frac{1}{2\sqrt{x^2+1}}\frac{d}{dx}(x^2+1)}{1} \\
&= \lim_{x\to-\infty} \frac{\frac{1}{2\sqrt{x^2+1}}(2x)}{1} \\
&= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} \qquad \text{I.F.} \frac{\infty}{\infty} \\
&= \lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}(\sqrt{x^2+1})} \\
&= \lim_{x\to-\infty} \frac{1}{\frac{x}{\sqrt{x^2+1}}} \\
&= \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x}
\end{aligned}$$

L'Hospital's Rule does not help in this situation.

4.1.2 Indeterminate Form $0 \cdot \infty$

We can convert the indeterminate form $0 \cdot \infty$ to an indeterminate form of type $\dfrac{0}{0}$ by writing

$$f(x)g(x) = \frac{f(x)}{\dfrac{1}{g(x)}}$$

or to an indeterminate form of the type $\dfrac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{\dfrac{1}{f(x)}}.$$

Now check the three conditions of L'Hospital's rule are all satisfied with the two functions $f(x)$ and $\dfrac{1}{g(x)}$ or with $g(x)$ and $\dfrac{1}{f(x)}$.

Example 4.5. Evaluate $\lim\limits_{x \to 0^+} x^2 \ln x$.

Solution

Example 4.6. Evaluate $\lim_{x\to-\infty} x^2e^x$.

Solution

4.1.3 Indeterminate Form $\infty - \infty$

To evaluate a limit involving $\infty-\infty$, we also need to re-express the difference of two functions as a quotient, by finding some common denominator. So now check the three conditions of L'Hospital's rule are all satisfied with the two functions, those being the numerator and the denominator of the quotient produced.

Example 4.7. Evaluate $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$.

Solution

Example 4.8. Evaluate $\lim\limits_{x\to 0^+}(\operatorname{cosec} x - \cot x)$.

Solution

4.1.4 Indeterminate Forms $0^0, \infty^0$ and 1^∞

We need the following procedure to compute $\lim\limits_{x\to a} f(x)^{g(x)}$ of the indeterminate forms $0^0, \infty^0$ and 1^∞.

1. Let $y = f(x)^{g(x)}$.

2. Take the natural logarithm (ln) of both sides to get

$$\ln y = g(x)\ln(f(x)).$$

3. Compute the following limit, using the previous indeterminate forms,

$$\lim_{x\to a}\ln y = \lim_{x\to a}\big(g(x)\ln(f(x))\big).$$

4. If $\lim\limits_{x\to a}\ln y = L$, then

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = e^L.$$

Example 4.9. Find $\lim_{x\to\infty} x^{\frac{1}{x}}$.

Solution

Example 4.10. Evaluate $\lim_{x\to 0^+} x^{\tan x}$.

Solution

Example 4.11. Calculate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution

Exercise 4.1

Use L'Hospital's rule to evaluate the following limits.

1. $\lim\limits_{x\to\infty} \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

2. $\lim\limits_{x\to 0^+} \dfrac{x-\sin(\pi x)}{x+\sin(\pi x)}$

3. $\lim\limits_{x\to -2} \dfrac{x+2}{\ln(x+3)}$

4. $\lim\limits_{x\to\infty} \dfrac{3^x}{x^2+x-1}$

5. $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{\sec x}{1+\tan x}$

6. $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{x}$

7. $\lim\limits_{x\to\infty} \dfrac{e^x}{e^{2x}+1}$

8. $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$

9. $\lim\limits_{x\to 0} \dfrac{2\cos x - 2 + x^2}{x^4}$

10. $\lim\limits_{x\to 0^+} x^{-2}\ln(\cos x)$

11. $\lim\limits_{x\to\infty} 3^{-x}\ln(x+1)$

12. $\lim\limits_{x\to\infty} x\tan\left(\dfrac{1}{x}\right)$

13. $\lim\limits_{x\to 1^+} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$

14. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{xe^x} - \dfrac{1}{x}\right)$

15. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$

16. $\lim\limits_{x\to 1^+} \left(\dfrac{4}{x^2-1} - \dfrac{2}{x-1}\right)$

17. $\lim\limits_{x\to 1^+} \left(\dfrac{4}{\ln x} - \dfrac{4}{x-1}\right)$

18. $\lim\limits_{x\to 0^+} (1-2x)^{\frac{1}{x}}$

19. $\lim\limits_{x\to\infty} \left(\dfrac{x}{x+1}\right)^x$

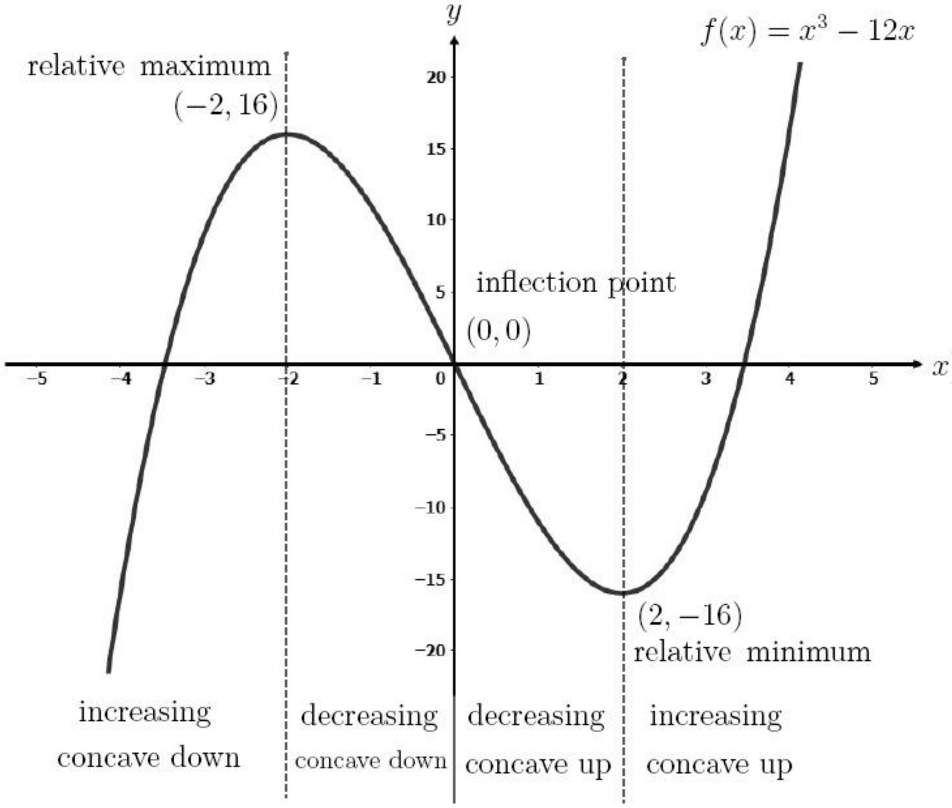
20. $\lim\limits_{x\to 0^+} x^{\sin x}$

21. $\lim\limits_{x\to 0^+} (2x)^x$

22. $\lim\limits_{x\to\infty} (3x+2)^{e^{-x}}$

23. $\lim\limits_{x\to\infty} (e^x+x)^{\frac{2}{x}}$

4.2 Curve Sketching : Graphs of polynomial functions



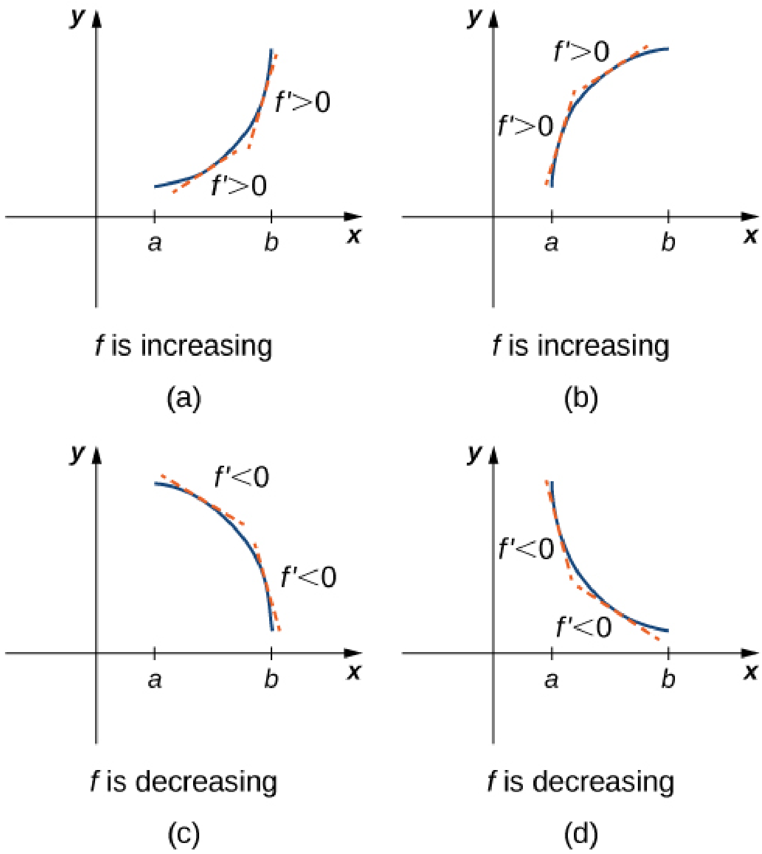
Guidelines for Sketching a graph of polynomial functions
1. $x-$Intercepts and $y-$Intercepts
2. Intervals of Increase or Decrease
3. Relative Maximum and Minimum
4. Concavity and Points of Inflection
5. Sketch the Curve : Using the information in step $1-4$, draw the graph

Increasing Functions and Decreasing Functions

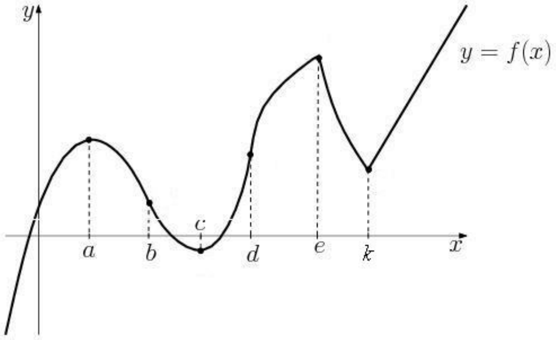
Definition 4.12. Let f be a function defined on an interval I and let x_1 and x_2 be any points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.

2. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.



Example 4.13. The following figure shows the graph of a function $y = f(x)$ which is defined on $\mathbb{R}$.



f is increasing on the interval ________________________________

f is decreasing on the interval ________________________________

Theorem 4.14. First Derivative Test for Increasing or Decreasing Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.

2. If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

Relative (Local) Maxima and Minima

Definition 4.15.

1. A function f has a **relative (local) maximum** at $x = c$ if

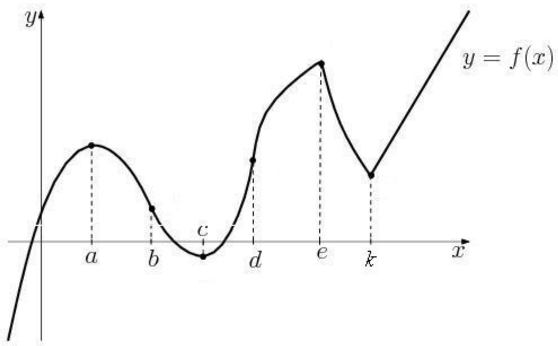
$$f(x) \leq f(c) \quad \text{for all } x \text{ in some open interval containing } c.$$

2. A function f has a **relative (local) minimum** at $x = c$ if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in some open interval containing } c.$$

A **relative extremum** is either a relative minimum or relative maximum.

Example 4.16. The following figure shows the graph of a function $y = f(x)$ which is defined on $\mathbb{R}$.



f has a relative maxima at ______________________________

f has a relative minima at ______________________________

Theorem 4.17. If f has a relative extremum at $x = c$, then $f'(c) = 0$ or $f'(c)$ does not exist.

Definition 4.18. We call $(c, f(c))$ a **critical point** of f if $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 4.19. First Derivative Test for Relative Extrema

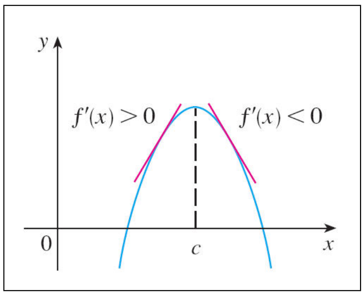
Suppose that f is continuous on $[a, b]$ and differentiable in (a, b) except at $c \in (a, b)$ where $(c, f(c))$ is a critical point.

1. If $f'(x) > 0$ for all $a < x < c$ and $f'(x) < 0$ for all $c < x < b$, then f has a relative maximum at c or $f(c)$ is a relative maximum.

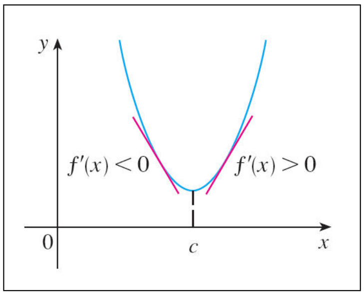
2. If $f'(x) < 0$ for all $a < x < c$ and $f'(x) > 0$ for all $c < x < b$, then f has a relative minimum at c or $f(c)$ is a relative minimum.

3. If $f'(x) > 0$ or $f'(x) < 0$ for all $x \in (a, b)$ except $x = c$, then f has no relative extremum at c or $f(c)$ is neither a relative maximum nor a relative minimum.

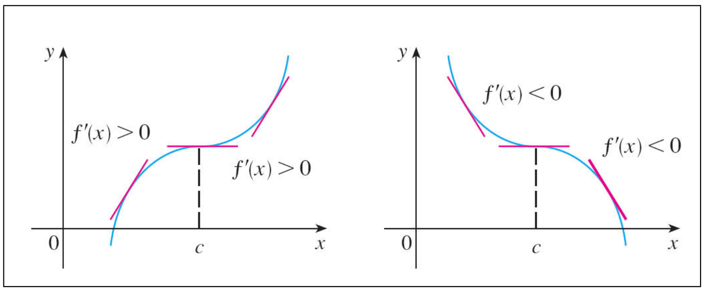
f has a relative maximum at c



f has a relative minimum at c



f has no relative extremum at c

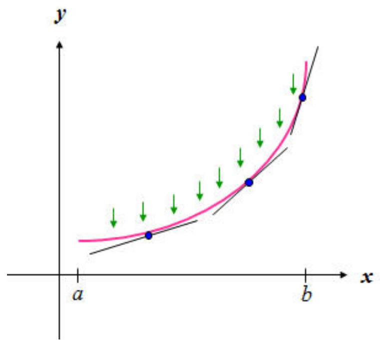


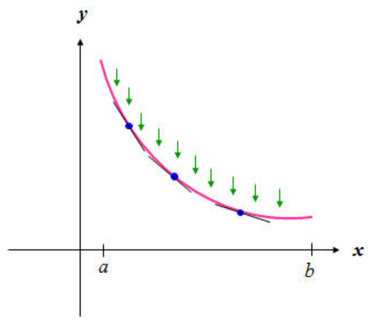
Concavity and Inflection Points

Definition 4.20. Let f be differentiable on (a, b).

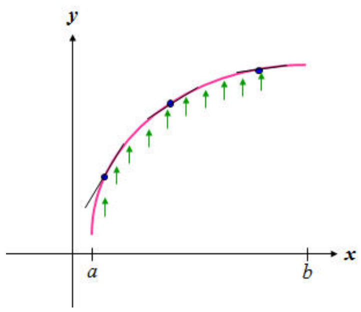
1. The graph of f is **concave up** on (a, b) if f' is increasing on (a, b).
2. The graph of f is **concave down** on (a, b) if f' is decreasing on (a, b).

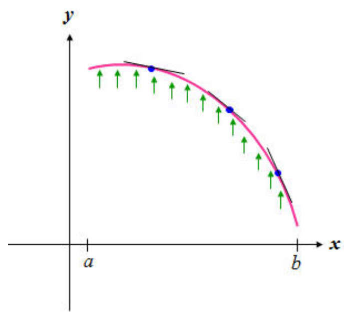
- The graph of f is concave up on (a, b).



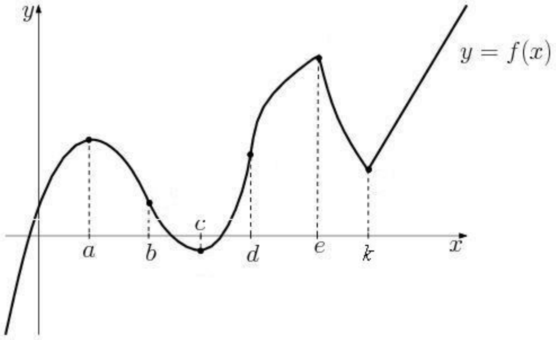


- The graph of f is concave down on (a, b).





Example 4.21. The following figure shows the graph of a function $y = f(x)$ which is defined on $\mathbb{R}$.



The graph of f is concave up on the interval ____________________

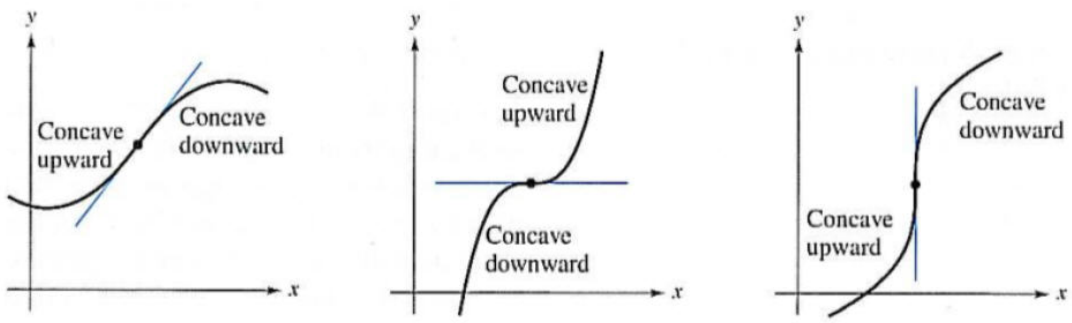
The graph of f is concave down on the interval ____________________

Theorem 4.22. The Second Derivative Test for Concavity

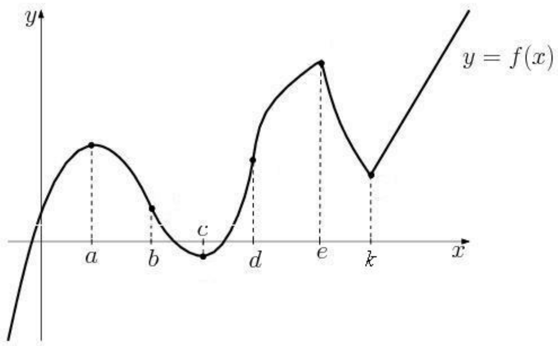
Let f be twice-differentiable on (a, b).

1. If $f''(x) > 0$ for all $x \in (a, b)$, then the graph of f is concave up on (a, b).
2. If $f''(x) < 0$ for all $x \in (a, b)$, then the graph of f is concave down on (a, b).

Definition 4.23. We call $(c, f(c))$ an **inflection point** of f if the curve of f changes from being concave up to concave down or from concave down to concave up at $x = c$.



Example 4.24. The following figure shows the graph of a function $y = f(x)$ which is defined on $\mathbb{R}$.



The inflection points are ____________________________________.

Theorem 4.25. If $(c, f(c))$ is an inflection point of f, then $f''(c) = 0$ or $f''(c)$ does not exist.

Theorem 4.26. Suppose that f is continuous on $[a, b]$ and twice differentiable on (a, b). Let $c \in (a, b)$. The point $(c, f(c))$ is an **inflection point** of f if one of the following two conditions holds.

1. $f''(x) > 0$ for all $a < x < c$ and $f''(x) < 0$ for all $c < x < b$.

2. $f''(x) < 0$ for all $a < x < c$ and $f''(x) > 0$ for all $c < x < b$.

Basic Principles for Graphing Polynomial Functions

Let $y = f(x)$ be a polynomial function.

1. **$x-$Intercepts and $y-$ Intercepts**

   To find the $x-$intercept, we set $y = 0$ and solve the equation for x.

   To find the $y-$intercept, we set $x = 0$ and find y.

2. **Intervals of Increase and Decrease**

   Calculate the first derivative $f'(x)$ and find the values of x where $f'(x) = 0$. These are critical points. Use the first derivative test for increasing or decreasing functions to find where the curve is increasing ($f''(x) > 0$) and where it is decreasing ($f''(x) < 0$).

3. **Relative Maximum and Minimum**

   Use the first derivative test for relative extremum to classify the critical points as relative maximum or relative minimum. Calculate the $y-$ values of the relative extrema points.

4. **Concavity and Points of Inflection**

   Calculate the second derivative $f''(x)$ and find the values of x where $f''(x) = 0$. These are potential points of inflection. If $x = c$ is a point of these kind and $f''(x)$ changes sign on two sides of c, then $(c, f(c))$ is an inflection point. Use the second derivative test for concavity to find where the graph of f is concave up ($f''(x) > 0$) and where it is concave down ($f''(x) < 0$).

5. **Sketch the Curve**

   Plot key points, such as the intercepts, the points found in steps $3-4$ and some additional points to get a nice shape of the graph, and sketch the graph of f using all the information obtained above.

Example 4.27. Given $f(x) = x^3 + 3x^2$.

1. Find the $x-$intercept and the $y-$intercept.

   $x-$intercepts are the points ______________ and ______________.

   $y-$intercept is the point ______________.

2. Find the critical points of f, and use the first derivative test for relative extremum to determine whether a critical point is a relative maximum point or a relative minimum point and use the first derivative test for increasing or decreasing functions to find where the curve is increasing and where it is decreasing.

   $f'(x) =$ ___________________________

   $f'(x) = 0$ when $x =$ ______________.

   The critical points of f are ______________ and ______________.

   Divide the real line by $x =$ ______________.

   Then check the sign of $f'(x)$ on the subintervals __________________________________.

   ____

   Sign of $f'(x)$

   Behavior of f

   - f is increasing on the interval __________________________.
   - f is decreasing on the interval __________________________.
   - The relative maximum point of f is __________________________.
   - The relative minimum point of f is__________________________.

3. Find the points of inflection and use the second derivative test for concavity to find where the graph of f is concave up and where it is concave down.

$f''(x) =$ __________________________.

$f''(x) = 0$ when $x =$ _______________.

Divide the real line by $x =$ ______________.

Then check the sign of $f''(x)$ on the subintervals __________________________________.

Sign of $f''(x)$

Behavior of f

- The graph of f is concave up on __________________________.
- The graph of f is concave down on __________________________.
- The inflection point on the graph of f is the point __________________________.

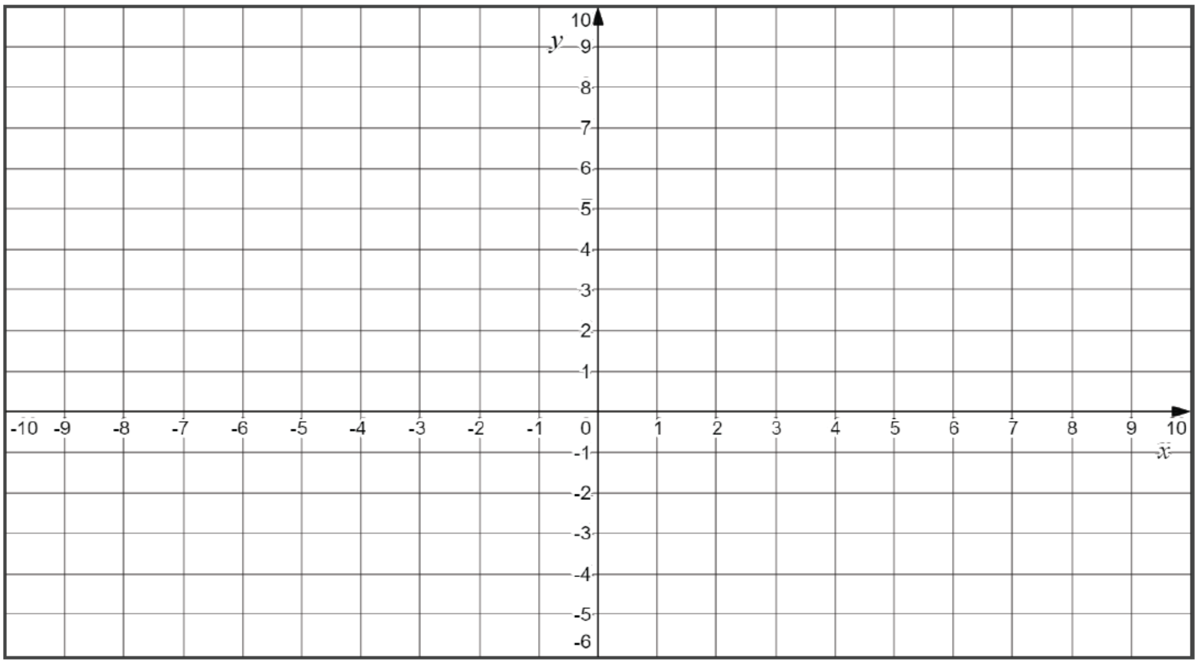
4. Summarizing the information in steps $2-3$, we obtain

sign of $f'(x)$

sign of $f''(x)$

graph of f

5. Sketch the graph of $f(x) = x^3 + 3x^2$.



Example 4.28. Given $f(x) = x^4 - 4x^3$.

1. $x-$intercepts are the points ____________ and ____________.

   $y-$intercept is the point ____________.

2. $f'(x) =$ ________________________

   $f'(x) = 0$ when $x =$ ____________.

   The critical points of f are ____________ and ____________.

   Sign of $f'(x)$

   Behavior of f

   - f is increasing on the interval ________________________.
   - f is decreasing on the interval ________________________.
   - The relative minimum is the point ________________________.

3. $f''(x) =$ ________________________.

   $f''(x) = 0$ when $x =$ ____________.

   Sign of $f''(x)$

   Behavior of f

   - The graph of f is concave up on ________________________.
   - The graph of f is concave down on ________________________.
   - The inflection point on the graph of f are the points ____________ and ____________.

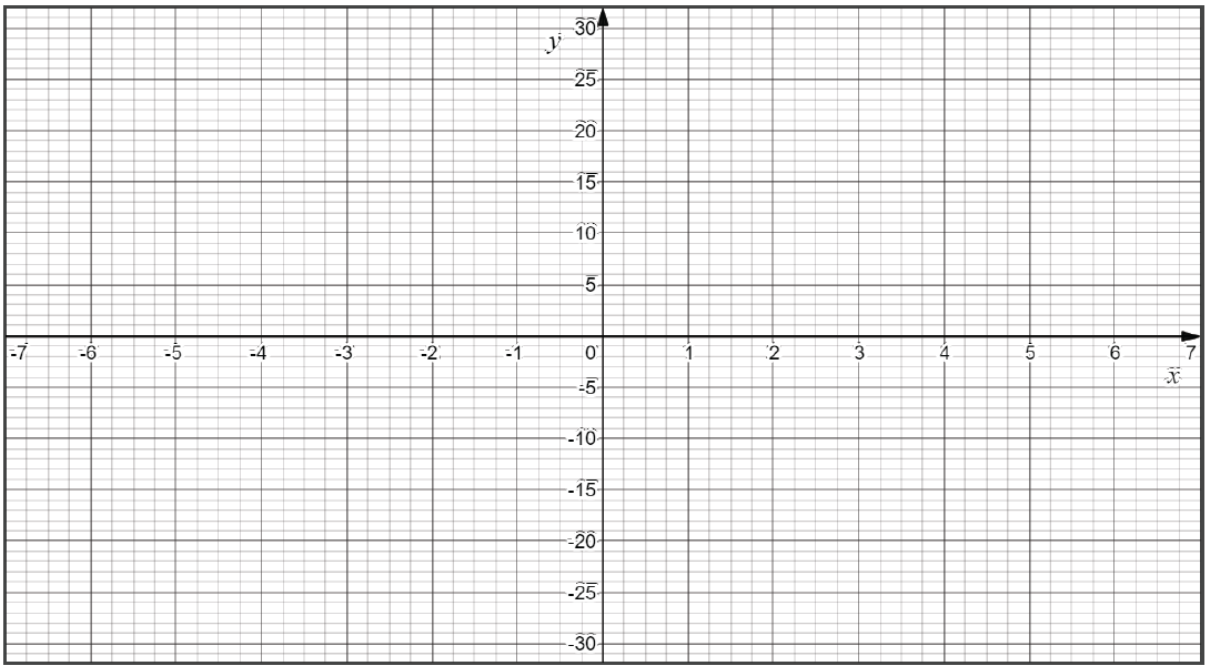
4. Summarizing the information in steps $2-3$, we obtain

sign of $f'(x)$

sign of $f''(x)$

graph of f

5. Sketch the graph of $f(x)=x^4-4x^3$.



Exercise 4.2

1. Let $y = f(x)$ be a continuous function on $\mathbb{R}$ satisfying the following properties:

- $f(-2) = 6, f(0) = 3, f(2) = 0 = f(-4)$
- $f'(x) > 0$ for $x < -2$ or $x > 2$
- $f'(x) < 0$ for $-2 < x < 2$
- $f'(2) = f'(-2) = 0$
- $f''(x) < 0$ for $x < 0$
- $f''(x) > 0$ for $x > 0$

Answer the following questions.

1.1) $x-$ intercept are the points __________________ and __________________.

$y-$ intercept is the point __________________.

1.2) The critical points of f are __________________.

1.3) f is increasing on __________________.

f is decreasing on __________________.

1.4) The relative maximum is the point __________________.

The relative minimum is the point __________________.

1.5) The graph of f is concave up on __________________.

The graph of f is concave down on __________________.

1.6) The inflection point on the graph of f is the point __________________.

2. Let $f(x) = x^3 - 12x$.

2.1) Fill in the blanks.

- $f'(x) =$ ______________________ and $f''(x) =$ ______________________.
- $x-$ intercepts are the points ____________________________.

  $y-$ intercept is the point ________________.
- The critical points of f are ______________________.
- f is increasing on ______________________.
- f is decreasing on ______________________.
- The relative maximum is the point ______________________.
- The relative minimum is the point ______________________.
- The graph of f is concave up on ______________________.
- The graph of f is concave down on ______________________.
- The inflection point on the graph of f is the point ______________________.

2.2) Sketch the graph of $f(x) = x^3 - 12x$.

3. Let $f(x) = -\dfrac{x^3}{3} + 3x^2$.

3.1) Fill in the blanks.

- $f'(x) =$ ________________________ and $f''(x) =$ ________________________.
- $x-$ intercepts are the points ________________ and ________________.

  $y-$ intercept is the point ________________.
- The critical points of f are ________________________.
- f is increasing on ________________________.
- f is decreasing on ________________________.
- The relative maximum is the point ________________________.
- The relative minimum is the point ________________________.
- The graph of f is concave up on ________________________.
- The graph of f is concave down on ________________________.
- The inflection point on the graph of f is the point ________________________.

3.2) Sketch the graph of $f(x) = -\dfrac{x^3}{3} + 3x^2$.

4.3 Maximum and Minimum Problems

What are the dimensions of a rectangle with fixed perimeter having maximum area? This is called the maximum and minimum problem.

How to solve the maximum and minimum problems?

1. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized (maximized or minimized)?
2. Draw picture if you can and label variables in the picture and any variables in the problem.
3. Identify the known variables and the unknown variable that is to be found.
4. Construct an equation relating the quantities.
5. Differentiate both sides of the equation with respect to the dependent variable. Set the derivative to be zero and solve for the unknown variable.
6. Verify that your result is a maximum or minimum value using the first or second derivative test for relative extrema.

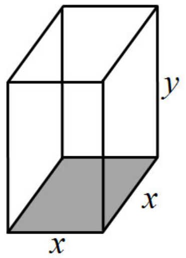
Second Derivative Test for Relative Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function may have a relative maximum, a relative minimum, or neither.

Example 4.29. A box having a square base and an open top is to have a volume of 32 cubic feet. Find the dimensions of the box that require the least material.

Solution We begin by drawing the picture and defining all variables.



Let x be the side length of square base, y be the height of the box, and A be the area of the area of all sides (total amount of material). Then

$$A = x^2 + 4xy. \tag{4.1}$$

We express y in terms of x by using the fact that the volume of the box must be 32 cubic feet, that is,

$$x^2 y = 32.$$

This implies that

$$y = \frac{32}{x^2}.$$

Substitute $\frac{32}{x^2}$ for y into (4.1) getting

$$A(x) = x^2 + \frac{128}{x}.$$

Hence

$$A'(x) = 2x - \frac{128}{x^2}.$$

Let $A'(x) = 0$. We obtain

$$2x - \frac{128}{x^2} = 0.$$

Multiplying both sides by x^2 yields

$$\begin{aligned} 2x^3 - 128 &= 0 \\ x^3 &= 64 \\ x &= 4. \end{aligned}$$

Using the second derivative test for relative extrema:

$$\begin{aligned} A''(x) &= 2 + \frac{256}{x^3} \\ A''(4) &= 6 > 0 \end{aligned}$$

Then A has a relative minimum at $x = 4$.

We can now evaluate y:

$$y = \frac{32}{x^2} = \frac{32}{4^2} = \frac{32}{16} = 2.$$

Hence the dimensions of the box that require the least amount of material are a length and width of 4 feet and a height of 2 feet.

Example 4.30. Find two numbers whose sum is 42 and whose product will be the largest.

Solution Let x be the first number, y be the second number and P be the product of x and y. Then

$$P = xy. \tag{4.2}$$

We express y in terms of x by using the fact that $x + y = 42$, that is, $y = 42 - x$.

Substitute $42 - x$ for y into (4.2) getting

$$P(x) = x(42 - x) = 42x - x^2.$$

Thus

$$P'(x) = 42 - 2x.$$

Let $P'(x) = 0$. We obtain $42 - 2x = 0$, that is, $x = 21$.

Using the second derivative test for relative extrema:

$$P''(x) = -2$$

$$P''(21) = -2 < 0$$

Then P has a relative maximum at $x = 21$.

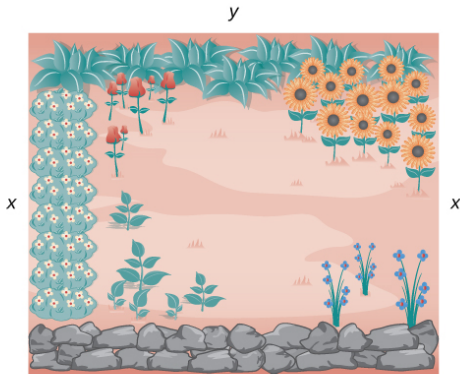
We can now evaluate y:

$$y = 42 - x = 42 - 21 = 21.$$

Hence the two numbers whose sum is 42 and whose product will be the largest are 21 and 21.

Example 4.31. A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 100 feet of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

Solution



Let x denote the length of the side of the garden perpendicular to the rock wall, y denote the length of the side parallel to the rock wall and A denote the area of the garden. Then

$$A = xy.$$

We express y in terms of x by using the fact that the total fencing is 100 feet, that is,

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x(100 - 2x) = 100x - 2x^2.$$

Then

$$A'(x) = 100 - 4x.$$

Let $A'(x) = 0$. We get $x = 25$.

Using the second derivative test for relative extrema:

$$A''(x) = -4$$

$$A''(25) = -4 < 0$$

We conclude that the maximum area must occur when $x = 25$. Then we have $y = 100-2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ feet and $y = 50$ feet. The area of this garden is $1,250$ square feet.

Exercise 4.3

1 Two nonnegative numbers have a sum of 9. What is the maximum product of one number times the square of the second number?

2 Find the dimensions of a rectangle with perimeter $1,000$ metres so that the area of the rectangle is a maximum.

3 An open rectangular box with square base is to be made from 48 square feet of material. What dimensions will result in a box with the largest possible volume?