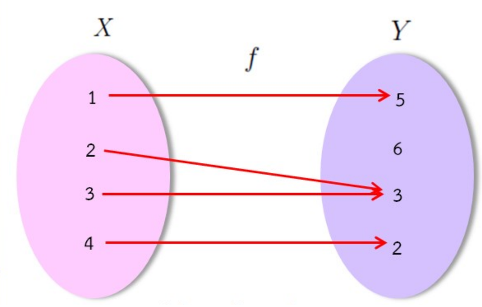
Chapter 1

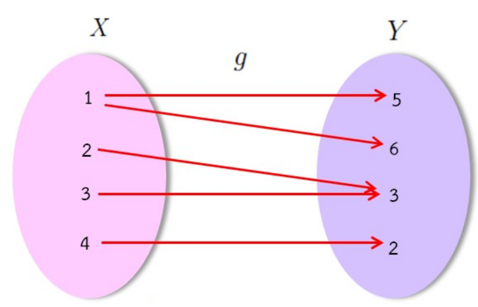
Functions and Graphs

Definition 1.1. A **function** from a set X to a set Y is a rule that assigns to each element x in X exactly one element, denoted by $f(x)$, in Y.

One way to demonstrate the meaning of this definition is by using arrow diagrams.



f is a function.



g is not a function.

The examples above can be described by the following sets of ordered pairs:

$$f = \{(1,5),(2,3),(3,3),(4,2)\},$$

$$g = \{(1,5),(1,6),(2,3),(3,3),(4,2)\}.$$

A function is usually denoted by a letter such as f, g, or h.

We can represent a function f from a set X to a set Y by the notation

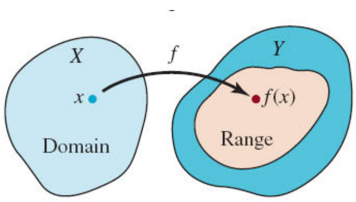
$$f : X \to Y.$$

The set X is called the **domain** of f. Denote the domain of f by D_f.

The unique element $f(x)$ in Y that corresponds to a selected element x in the domain X is called the **value** of f at x, or the **image** of x.

The **range** of f, denoted by R_f, is the subset of Y consisting of all possible values of $f(x)$ as x varies throughout the domain of f, that is,

$$R_f = \{f(x) \mid x \in D_f\}.$$

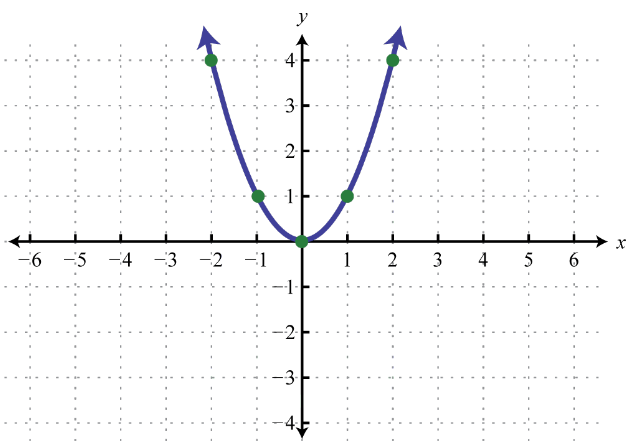


A function associates each element in its domain with one and only one element in its range.

We usually consider functions for which its domain and range are subsets of real numbers. We often describe a function using the rule , $y = f(x)$, and create a graph of that function by plotting the ordered pairs $(x, f(x))$ on the Cartesian Plane.

$f(x) = x^2$

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4



1.1 Domain and range of a function

The domain of a function f defined by a formula $y = f(x)$ is the set of real numbers x for which $f(x)$ is a real number, that is,

$$D_f = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}.$$

The range of $y = f(x)$ is

$$R_f = \{f(x) \mid x \in D_f\}.$$

Rules for Finding the Domain of $y = f(x)$

1. Denominator is nonzero.

   If $f(x) = \dfrac{a}{\square}$ where a is a nonzero real number, then $\square \neq 0$.

2. Terms in even roots must be nonnegative.

   If $f(x) = \sqrt[2n]{\square}$, then $\square \geq 0$.

3. Terms in logarithmic function must be positive.

   If $f(x) = \log_a(\square)$, then $\square > 0$.

We can write the domain and range in interval notation. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded.

Inequality	Interval Notation	Graph on Number Line
$x > a$	(a, ∞)	
$x < a$	$(-\infty, a)$	
$x \geq a$	$[a, \infty)$	
$x \leq a$	$(-\infty, a]$	
$a < x < b$	(a, b)	
$a \leq x < b$	$[a, b)$	
$a < x \leq b$	$(a, b]$	
$a \leq x \leq b$	$[a, b]$	

Example 1.1. Find the domain of the following functions.

1. $f(x) = \dfrac{1}{x-1}$

   Solution

2. $f(x) = \sqrt{2x+1}$

   Solution

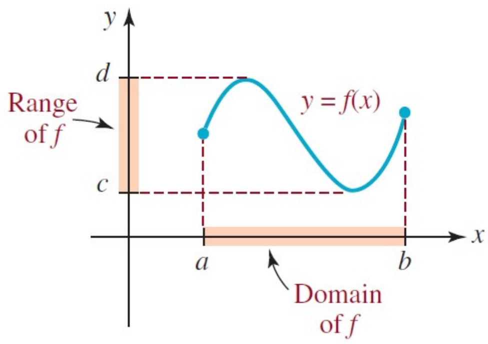
3. $f(x) = \ln(1 - x^2)$

   Solution

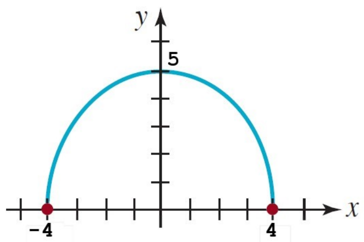
4. $f(x) = \dfrac{1}{\sqrt{x^2 + x - 6}}$

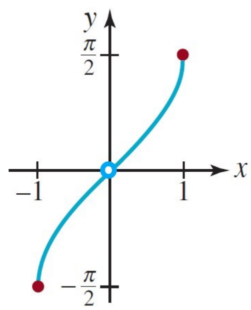
Solution

For a function defined by a graph, its domain consists of all the x-coordinate values (along x-axis) of the graph. The range is the set of all y- coordinate values (along y-axis) of the graph.

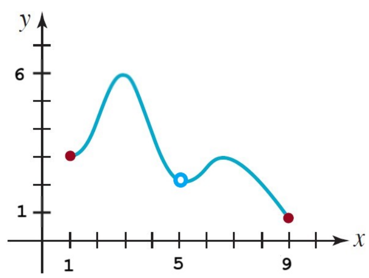


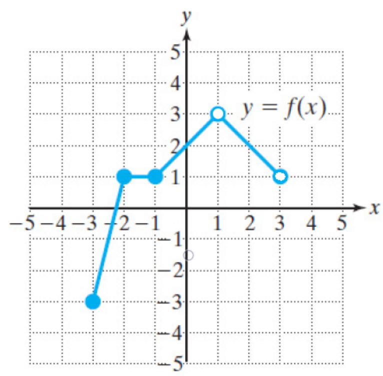
Example 1.2. Find the domain and range of the functions from the graphs given below.





$D_f =$ ____________ $R_f =$ ____________ $D_f =$ ____________ $R_f =$ ____________





$D_f =$ ____________ $R_f =$ ____________ $D_f =$ ____________ $R_f =$ ____________

Exercise 1.1

Find the domain of the given function f.

1. $f(x) = \sqrt{2x - 4}$
2. $f(x) = \sqrt[4]{15 - 5x}$
3. $f(x) = \dfrac{10}{\sqrt{1 - x}}$
4. $f(x) = \dfrac{2}{\ln(x - 2)}$
5. $f(x) = \dfrac{2x - 5}{x(x - 3)}$
6. $f(x) = \dfrac{x}{x^2 - 1}$
7. $f(x) = \dfrac{1}{\sqrt{25 - x^2}}$
8. $f(x) = \ln(x^2 - 4x - 12)$

1.2 Basic Functions

1.2.1 Polynomial Functions

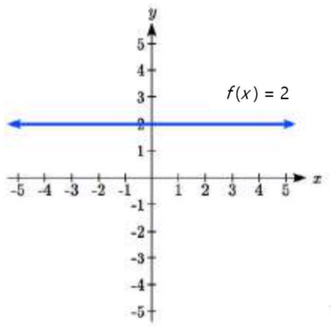
Definition 1.2. A **polynomial function** is a function of the form

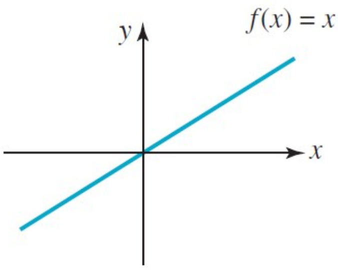
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

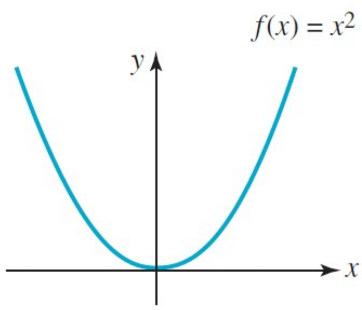
where $a_0, a_1, \ldots, a_n$ are real numbers and n is a nonnegative integer.

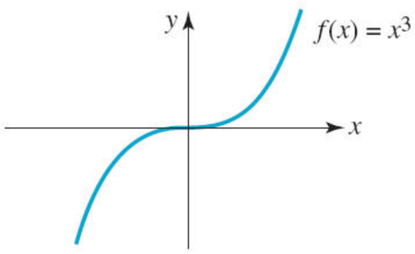
If $a_n \neq 0$, we say that f is a polynomial of degree n.

Here are some examples of polynomial functions.









The three polynomial functions considered in this subsection are

$$f(x) = a_0, \quad f(x) = a_1 x + a_0, \quad \text{and} \quad f(x) = a_2 x^2 + a_1 x + a_0.$$

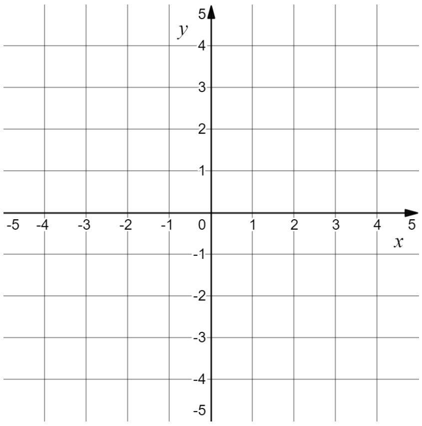
In the definitions that follow we change the coefficients of these functions to more convenient symbols.

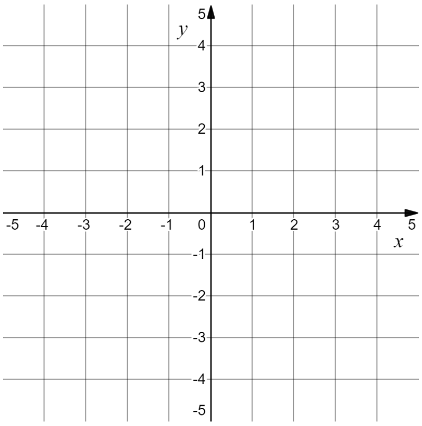
Definition 1.3. A **constant function** is a function of the form

$$y = c$$

where c is a real number.

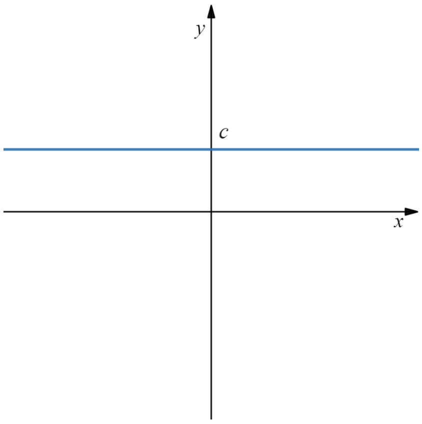
Example 1.3. Sketch the graph of $y = 2$ and $y = -1$.





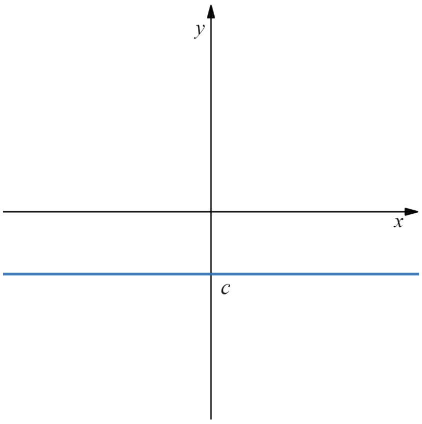
Graph of a Constant Function

The graph of the constant function $y = c$ is a horizontal line which passes through the y-axis at the point $(0, c)$. In particular, the graph of the function $y = 0$ is the x-axis.



Graph of $y = c$ where $c > 0$

$D_f = \mathbb{R}$ and $R_f = \{c\}$

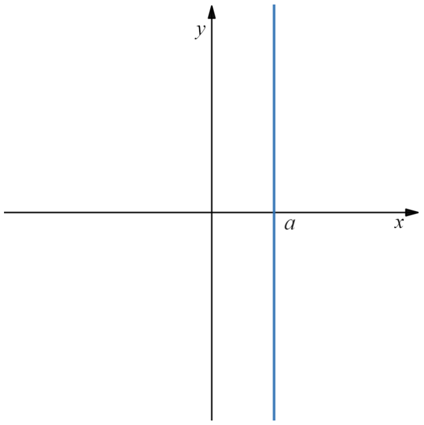


Graph of $y = c$ where $c < 0$

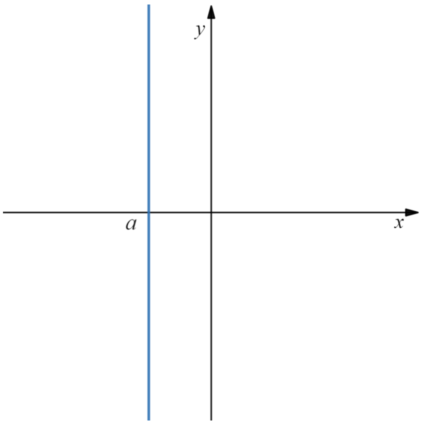
$D_f = \mathbb{R}$ and $R_f = \{c\}$

The Line $x = a$

The graph of $x = a$ is a vertical line which passes through the x-axis at the point $(a, 0)$. In particular, the graph of $x = 0$ is the y-axis.

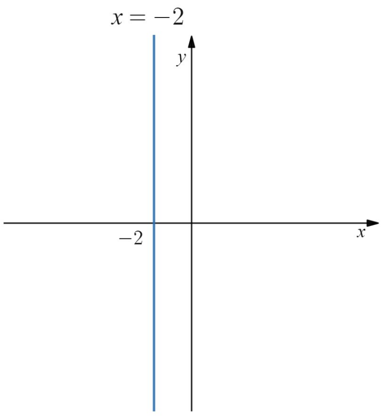


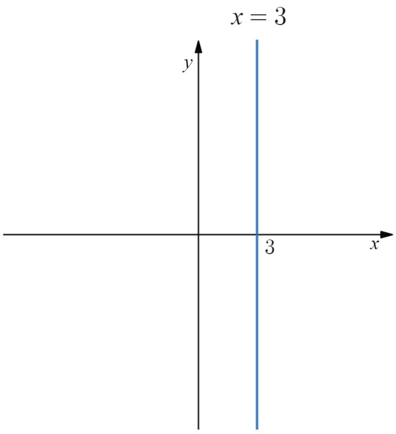
Graph of $x = a$ where $a > 0$



Graph of $x = a$ where $a < 0$

Example 1.4. The graph of $x = -2$ and $x = 3$ are as follows:.





Definition 1.4. A **linear function** is a function of the form

$$y = mx + b$$

where m and b are real number and $m \neq 0$.

To graph a linear function:

1. Find find the x-intercept and the y-intercept.
2. Plot the intercepts and draw the line.

Recall that

- The x-intercept is the point at which the graph crosses the x-axis. At this point, the y-coordinate is zero.

  To find the x-intercept:

  1. Set $y = 0$ in the equation.
  2. Solve for x. The value obtained is the x-coordinate of the x-intercept.
  3. The x-intercept is the point $(x, 0)$, with x the value found in step 2.

- The y-intercept is the point at which the graph crosses the y-axis. At this point, the x-coordinate is zero.

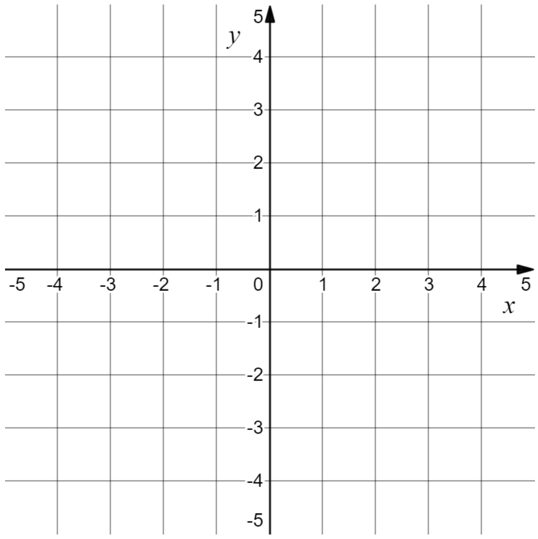
  To find the y-intercept:

  1. Set $x = 0$ in the equation.
  2. Solve for y. The value obtained is the y-coordinate of the y-intercept.
  3. The y-intercept is the point $(0, y)$, with y the value found in step 2.

Example 1.5. Sketch the graph of the following linear functions.

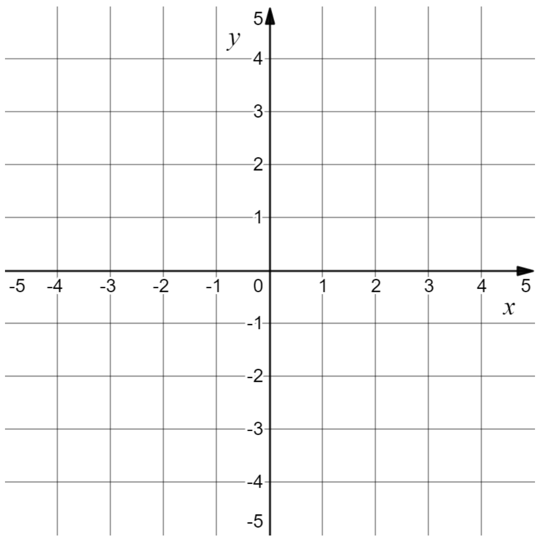
1. $y = 2x - 4$

   - The x-intercept is ___________.
   - The y-intercept is ___________.



2. $x + y = 2$

   - The x-intercept is ___________.
   - The y-intercept is ___________.



Graph of a Linear Function

The graph of $y = mx + b, m \neq 0$ is a line with slope m, and the graph crosses the y-axis at the point $(0, b)$.

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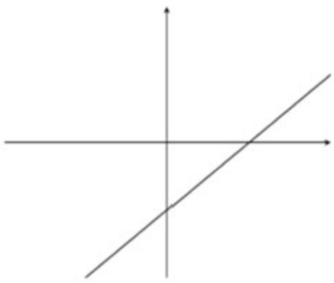
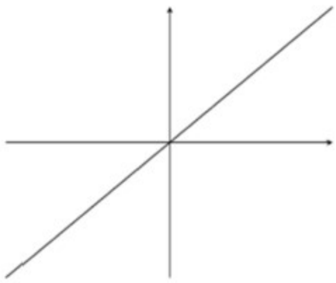
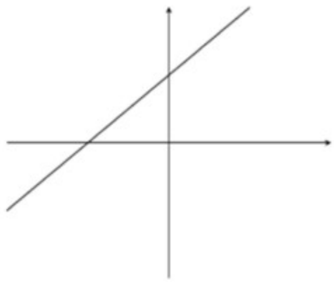
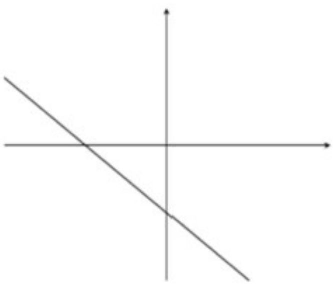
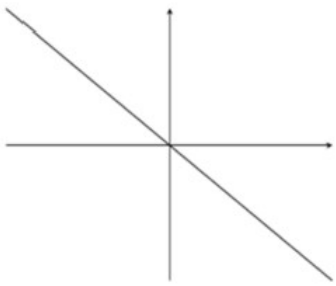
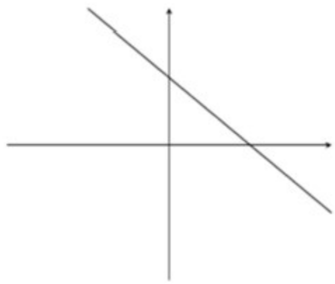
$m > 0$			
$m < 0$			

Equations for Lines

1. **Point-slope form :** An equation for the line that passes through the point (a, b) with slope m is
$$y - b = m(x - a).$$

2. **Point-point form:** An equation for the line that passes through the points (a_1, b_1) and (a_2, b_2) is
$$y - b_1 = m(x - a_1) \text{ where } m = \frac{b_2 - b_1}{a_2 - a_1}.$$

Example 1.6.

1. Find an equation for a line that passes through the point $(9, 2)$ with slope $-\frac{2}{3}$.

   Solution An equation for the line that passes through the point $(9, 2)$ with slope $-\frac{2}{3}$ is

2. Find an equation for the line that passes through the points $(2, 1)$ and $(9, 5)$.

   Solution The slope of the line that passes through the points $(2, 1)$ and $(9, 5)$ is
$$m =$$

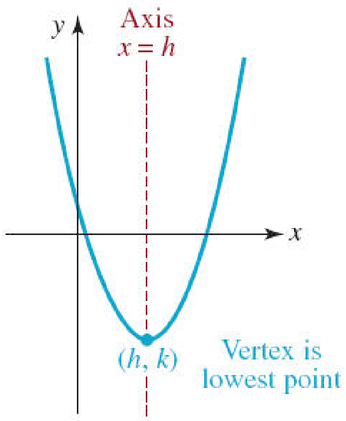
   Then an equation for the line that passes through the points $(2, 1)$ and $(9, 5)$ is

Definition 1.5. A **quadratic function** is a function of the form

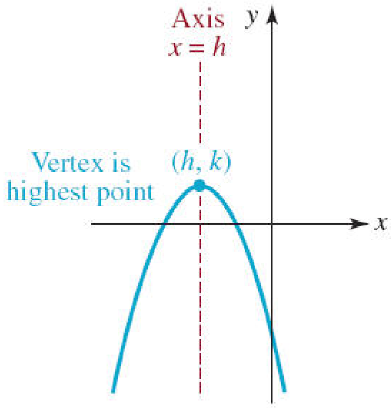
$$y = ax^2 + bx + c,$$

where $a, b,$ and c are real constants with $a \neq 0$.

The graph of any quadratic function is called a **parabola**. The parabola opens upward if $a > 0$ and downward if $a < 0$. The lowest (highest) point (h, k) on the parabola is called **vertex**. All parabolas are symmetric with respect to a vertical line through the vertex (h, k). The line $x = h$ is called the **axis of symmetry**.



$y = ax^2 + bx + c, a > 0$

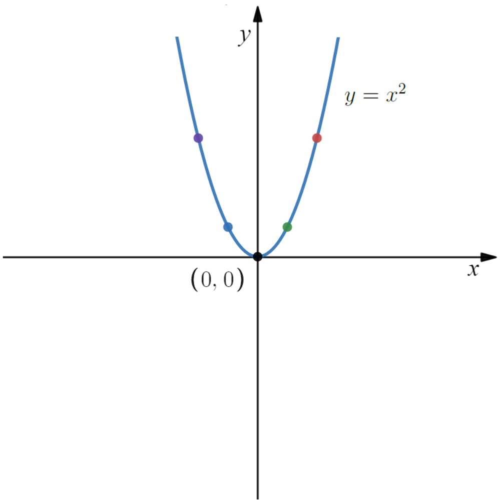


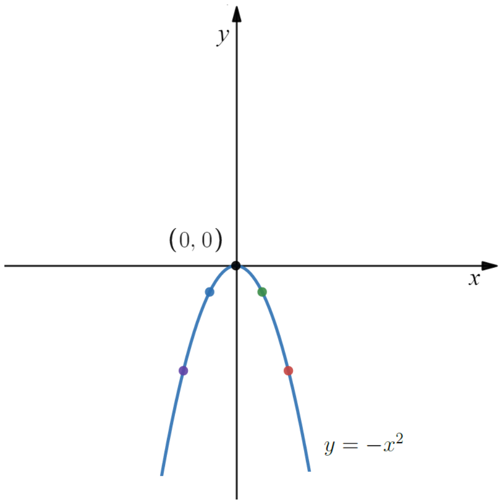
$y = ax^2 + bx + c, a < 0$

We can graph a quadratic equation if we know the following:

1. The location of the vertex
   - The vertex of $y = a(x - h)^2 + k$ is (h, k).
   - The vertex of $y = f(x) = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
2. Whether it opens upward or downward
3. A few points (including x-intercept, y-intercept)

Example 1.7. The graph of $y = x^2$ and $y = -x^2$ are as follows:

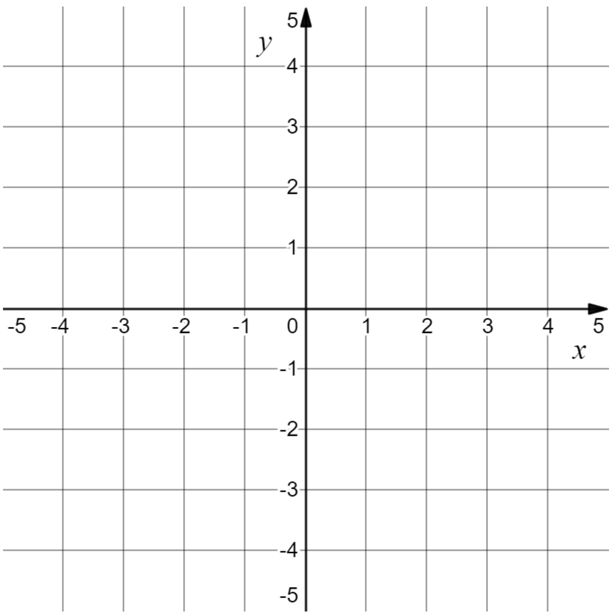




Example 1.8. Sketch the graph of the following quadratic functions.

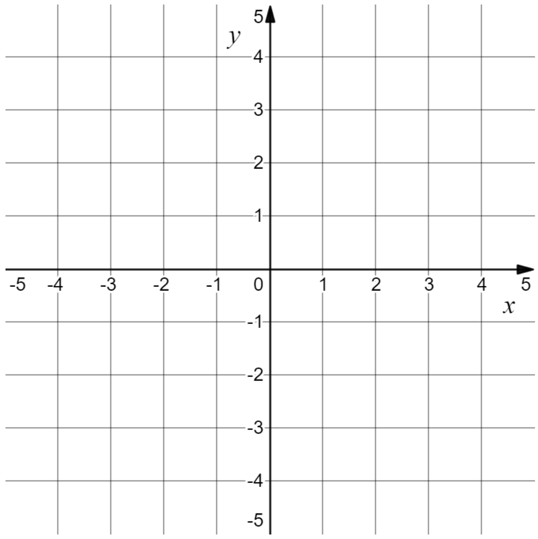
1. $y = -x^2 + 1$

   - The vertex is ______________.



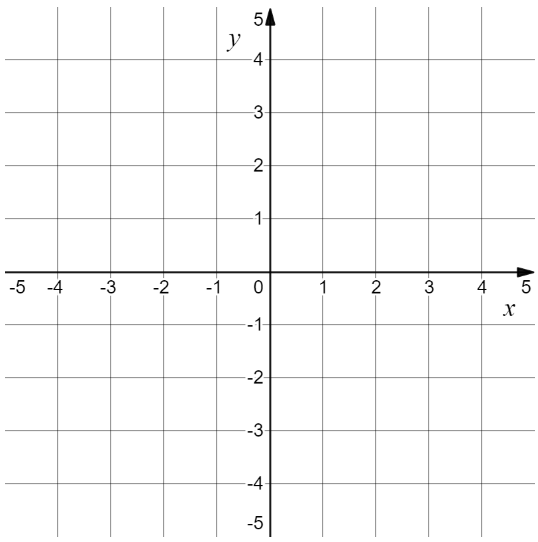
2. $y = (x+2)^2$

   - The vertex is ____________.



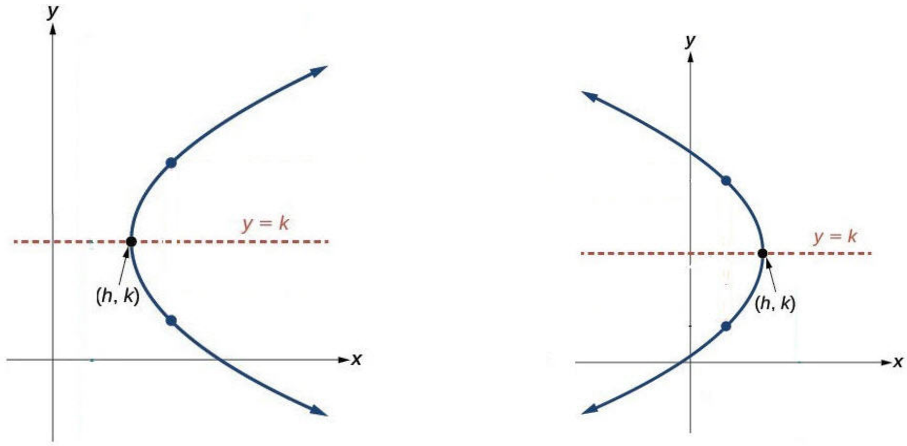
3. $y = -x^2 + 2x + 3$

   - The vertex is ____________.



Parabolas of the Form $x = ay^2 + by + c$

The parabola $x = ay^2 + by + c, a \neq 0$ opens to the right if $a > 0$ and opens to the left if $a < 0$.

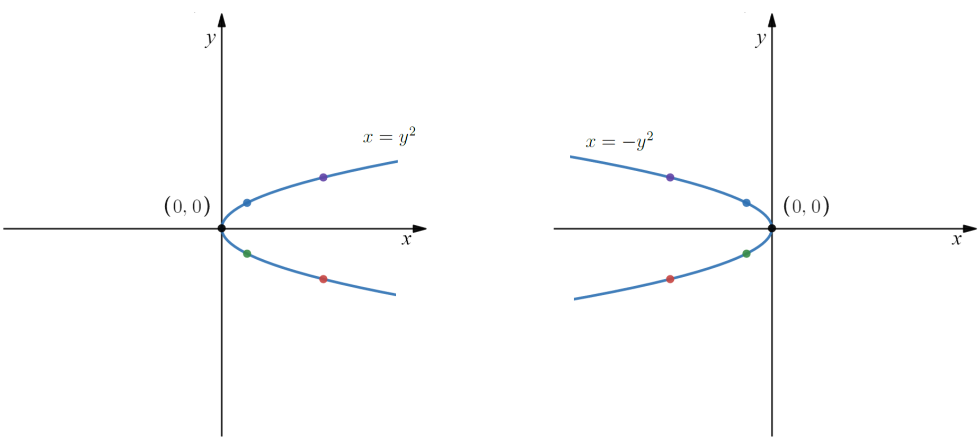


$x = ay^2 + by + c, a > 0$ $\qquad$ $x = ay^2 + by + c, a < 0$

We can sketch the graph of $x = ay^2 + by + c, a \neq 0$ if we know the following:

1. The location of the vertex
   - The vertex of $x = a(y-k)^2 + h$ is (h, k).
   - The vertex of $x = g(y) = ay^2 + by + c$ is $\left(g\left(-\dfrac{b}{2a}\right), -\dfrac{b}{2a}\right)$.
2. Whether it opens to the right or opens to the left
3. A few points (including x-intercept, y-intercept)

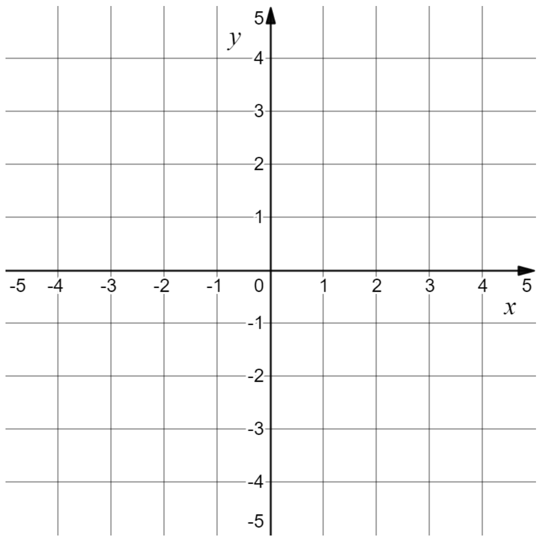
Example 1.9. The graph of $x = y^2$ and $x = -y^2$ are as follows:



Example 1.10. Graph the following parabolas.

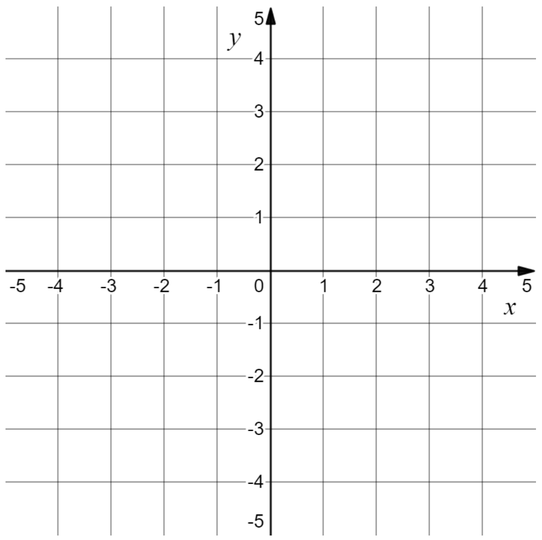
1. $x = y^2 + 4y + 3$

    - The vertex is ____________.



2. $x = -y^2 - 2y - 1$

    - The vertex is ____________.



Exercise 1.2.1

1. Find an equation for the line that passes through the point $(4, 1)$ with slope -3.

2. Find an equation for the line that passes through the two given points.

   2.1) $(2, 1)$ and $(3, -4)$

   2.2) $(-4, -3)$ and $(2, 3)$

   2.3) $(3, -3)$ and $(0, 3)$

   2.4) $(-4, 2)$ and $(4, 4)$

3. Sketch the graph of the following equations.

   3.1) $y = 2x + 6$

   3.2) $3x + 2y = 6$

   3.3) $y = x^2 - 4x - 5$

   3.4) $y = -2x^2 - 4x - 3$

   3.5) $y = x^2 - 9$

   3.6) $y = -x^2 + 12x - 32$

   3.7) $x = y^2 + 1$

   3.8) $x = -y^2 - 10y - 24$

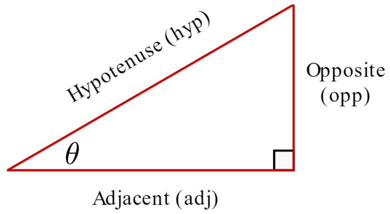
   3.9) $x = y^2 - 2y - 8$

   3.10) $x = -2y^2 + 4y - 5$

1.2.2 Trigonometric Functions

Right-angled triangle definitions

Consider the right triangle shown below, where angle θ is one of the acute angles.



With reference to the above triangle, for an acute angle $(0 < \theta < \frac{\pi}{2})$, the six trigonometric functions can be described as follows:

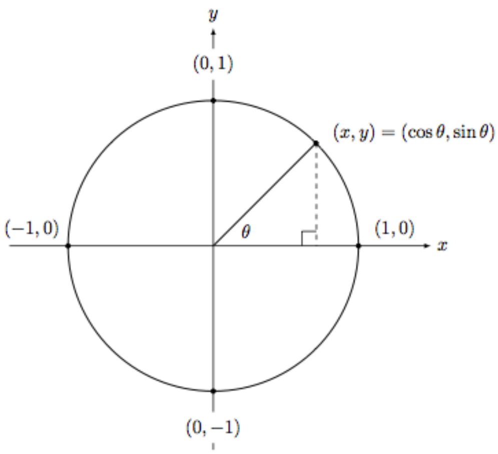
Definition 1.6. Basic Six Trigonometric Functions.

$$\sin\theta = \frac{opp}{hyp} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{adj}{opp}$$

Unit-circle definitions



In the unit circle, one can define the trigonometric functions cosine and sine as follows:

$$\cos\theta = \frac{x}{1} = x$$
$$\sin\theta = \frac{y}{1} = y$$

Then, each point (x, y) on the unit circle can be written as $(\cos\theta, \sin\theta)$.

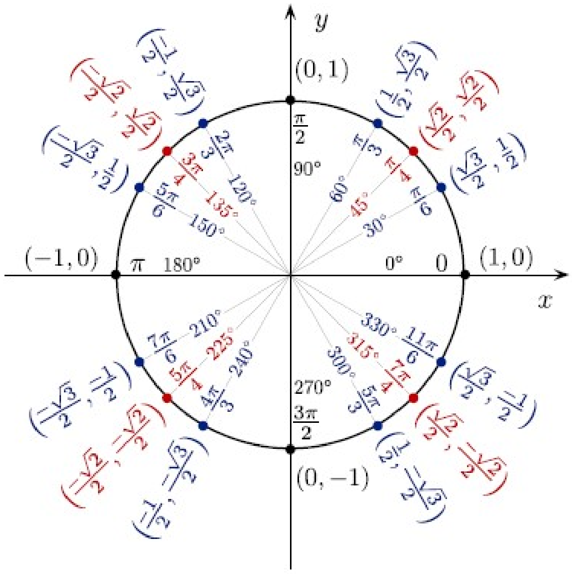
Combined with the equation $x^2 + y^2 = 1$, the definitions above give the relationship

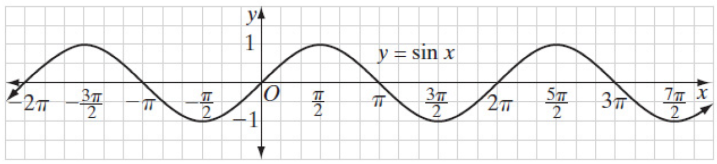
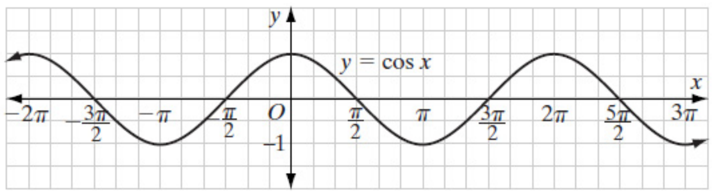
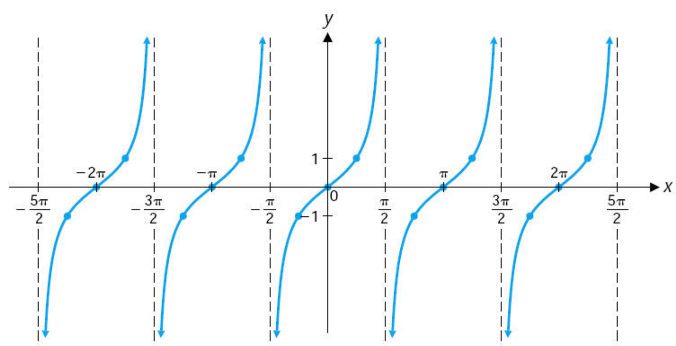
$$\sin^2\theta + \cos^2\theta = 1.$$

In addition, other trigonometric functions can be defined in terms of x and y:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{y}$$

A unit circle with certain exact values marked on it is below:



Function	Domain	Range	Graph
$y = \sin x$	$\mathbb{R}$	$[-1, 1]$	
$y = \cos x$	$\mathbb{R}$	$[-1, 1]$	
$y = \tan x$	$\mathbb{R} - \left\{ \frac{(2k+1)\pi}{2} \mid k \in \mathbb{Z} \right\}$	$\mathbb{R}$	

Trigonometric Identities

Pythagoras Type Formulas.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Double-angle Formulas.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Half-angle Formulas.

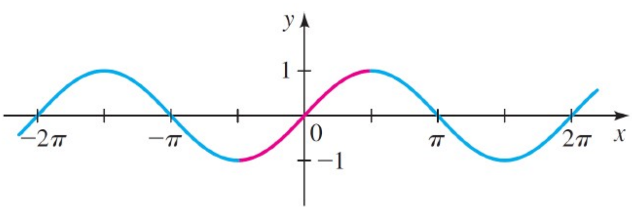
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

1.2.3 Inverse Trigonometric Functions

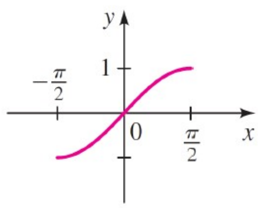
Recall that if f is a one-to-one function with domain A and range B, then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

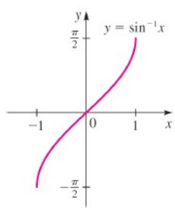
The trigonometric functions are not one-to-one. By restricting their domains, we can construct one-to-one functions from them. For example, if we restrict the domain of $\sin x$ to the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ we have a one-to-one function with domain $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and range $[-1, 1]$ which has an inverse denoted by $\sin^{-1} x$ or $\arcsin x$ and it is called the **inverse sine function**.



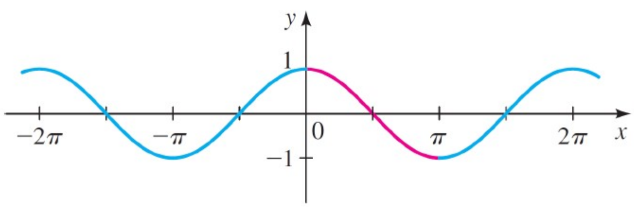
$y = \sin x$



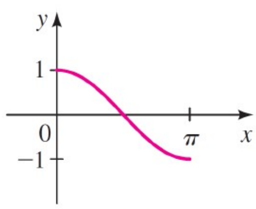
$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



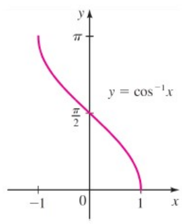
The **inverse cosine function**, denoted by $\cos^{-1} x$ or $\arccos x$, is defined to be the inverse of the restricted cosine function $\cos x, x \in [0, \pi]$.



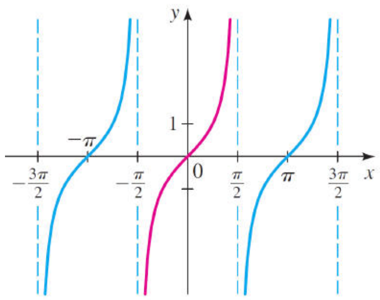
$y = \cos x$



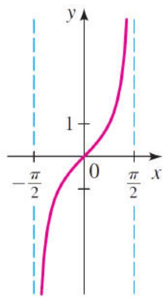
$y = \cos x,\ 0 \le x \le \pi$



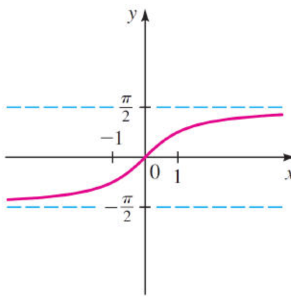
The **inverse tangent function**, denoted by $\tan^{-1} x$ or $\arctan x$, is defined to be the inverse of the restricted tangent function $\tan x, x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.



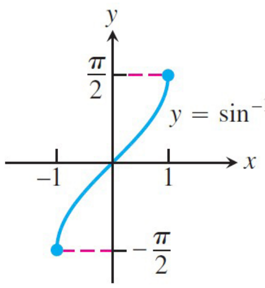
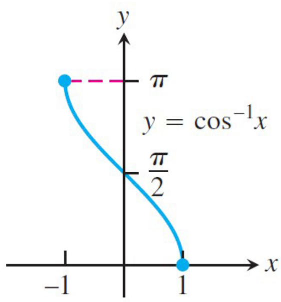
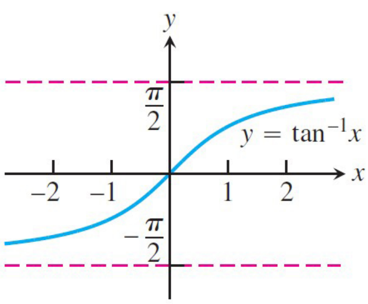
$y = \tan x$



$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1} x$

Function	Domain	Range	Graph
$y = \sin^{-1} x \Leftrightarrow x = \sin y$	$[-1, 1]$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$	
$y = \cos^{-1} x \Leftrightarrow x = \cos y$	$[-1, 1]$	$[0, \pi]$	
$y = \tan^{-1} x \Leftrightarrow x = \tan y$	$\mathbb{R}$	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$	

<u>IMPORTANT:</u> Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x$$

with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}$$

Example 1.11.

x	1	-1	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
$\sin^{-1} x$						
$\cos^{-1} x$						

x	1	-1	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
$\tan^{-1} x$				

1.2.4 Exponential Functions

Definition 1.7. An **exponential function with base a** is a function of the form

$$y = a^x$$

where $a > 0, a \neq 1$, and x is any real number.

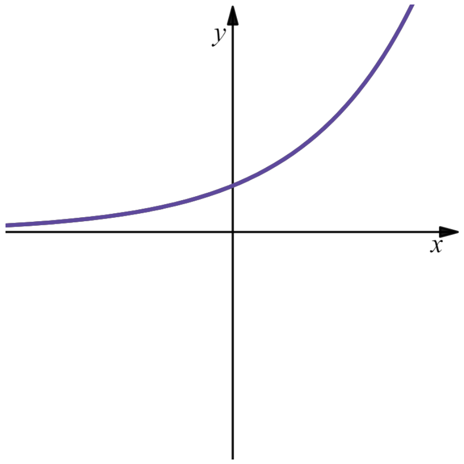
Rules of exponents

Let a and b be positive real numbers, and let x and y be real numbers. Then

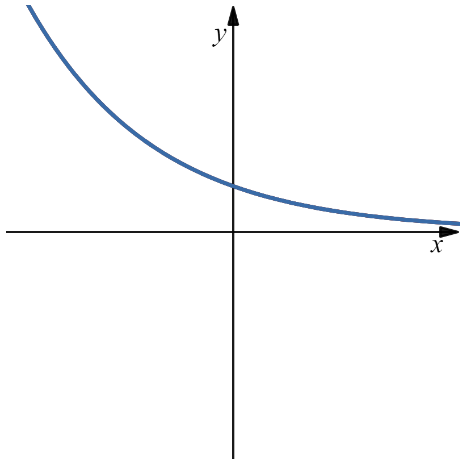
1. $a^0 = 1$
2. $a^{-x} = \dfrac{1}{a^x} = \left(\dfrac{1}{a}\right)^x$
3. $a^x a^y = a^{x+y}$
4. $\dfrac{a^x}{a^y} = a^{x-y}$
5. $(a^x)^y = a^{xy}$
6. $(ab)^x = a^x b^x$
7. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
8. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$, where m and n are positive integers

Graphs of Exponential Functions

We know that $a^0 = 1$ for any $a > 0$. Then for any base $a > 0$, the graph of $y = a^x$ always passes through the point $(0, 1)$.

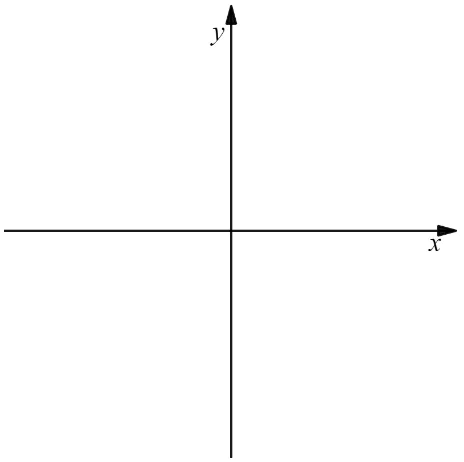


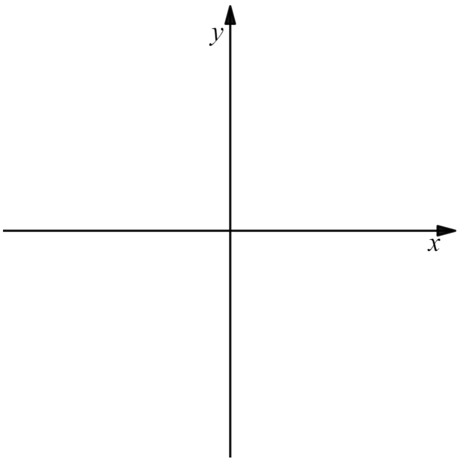
$y = a^x, a > 1$



$y = a^x, 0 < a < 1$

Example 1.12. Sketch the graphs of $y = e^x$ and $y = e^{-x}$.





1.2.5 Logarithmic Functions

Definition 1.8. A **logarithmic function with base *a*** is a function of the form

$$y = \log_a x$$

where $a > 0, a \neq 1$, and $x > 0$.

The function

$$y = \ln x = \log_e x$$

is called the **natural logarithmic function**.

Note that

$$y = \log_a x \quad \Leftrightarrow \quad x = a^y.$$

Properties of Logarithms

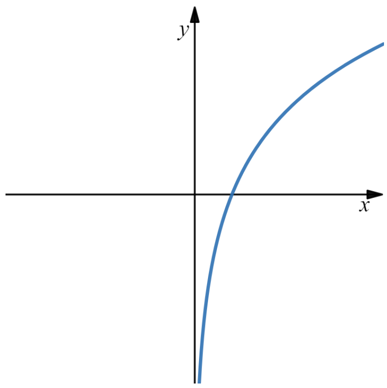
Let a, x, y be positive real numbers such that $a \neq 1$ and let k be a real number.

The following properties are true.

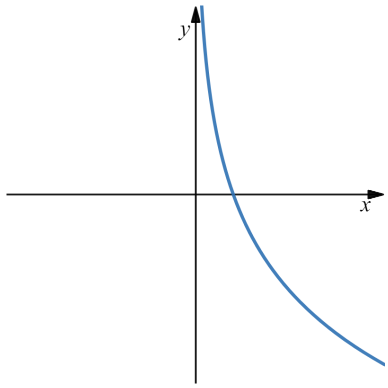
1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$
5. $\log_a(xy) = \log_a x + \log_a y$
6. $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
7. $\log_a(x^k) = k \log_a x$
8. $\log_a x = \dfrac{\log_b x}{\log_b a}$, where $b > 0$ and $b \neq 1$

Graphs of Logarithmic Functions

For any $a > 0, a \neq 1$, the graph of $y = \log_a x$ always passes through the point $(1, 0)$.

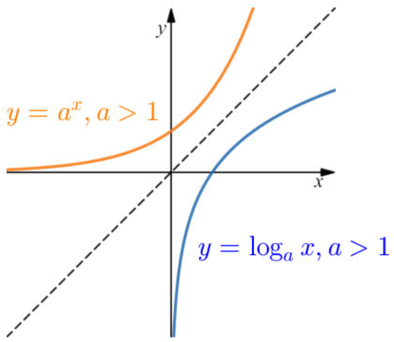


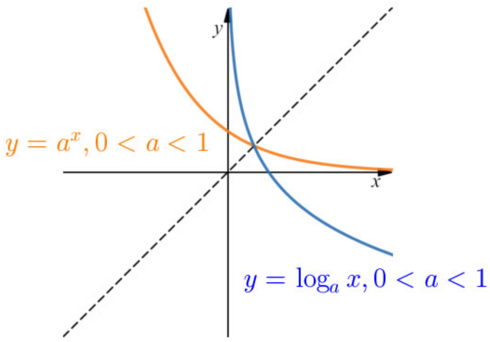
$y = \log_a x, a > 1$



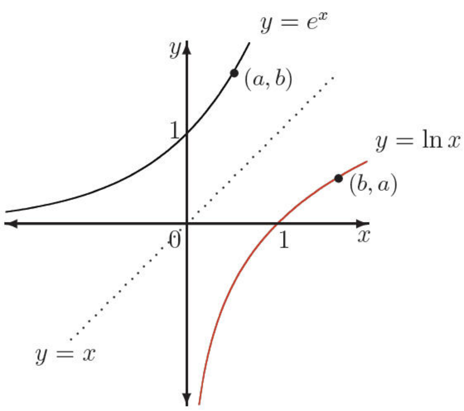
$y = \log_a x, 0 < a < 1$

Recall that $y = \log_a x$ is the inverse of $y = a^x$. The graph of $y = \log_a x$ is symmetric to the graph of $y = a^x$ about the line $y = x$.





Example 1.13. The graph of $y = e^x$ and $y = \ln x$ are as follows:



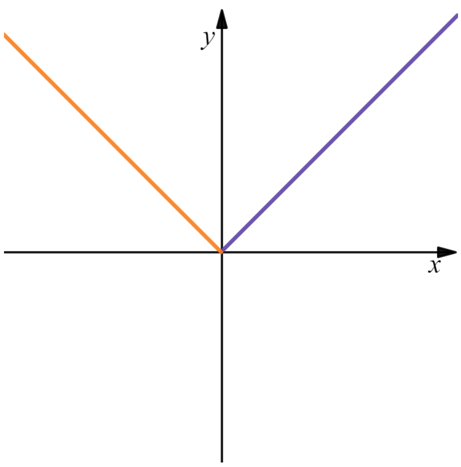
1.2.6 Piecewise-Defined Functions

Definition 1.9. A **piecewise-defined function** is a function which involves two or more expressions or formulas, which are defined on different parts of the domain.

For example,

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The graph of f is below.



This function is called the **absolute-valued function**.

Example 1.14. Sketch the graph of

$$f(x) = \begin{cases} x^2, & x < 1 \\ x + 1, & x \geq 1. \end{cases}$$

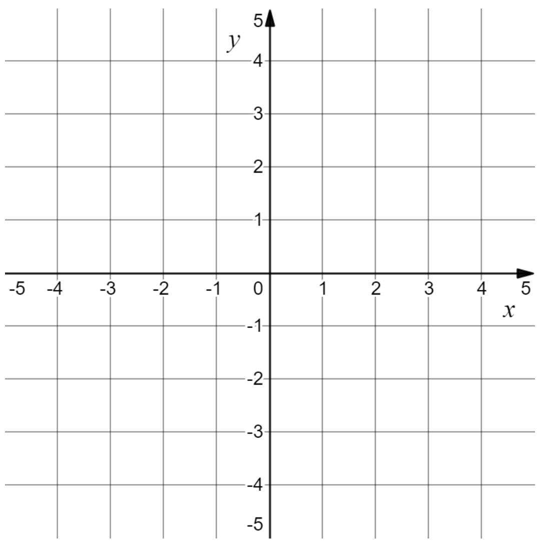
Solution

- Graph $f(x) = x^2$ and $f(x) = x + 1$ on the same plane.

- When $x < 1$, this function is defined by the square function, $f(x) = x^2$, so darken the graph $y = x^2$ to the left of $x = 1$. An open circle is used at the right endpoint, $x = 1$.

- When $x \geq 1$, the function is defined by the linear function $f(x) = x + 1$, so darken the graph $y = x + 1$ to the right of $x = 1$. The circle is filled in at the left endpoint, $x = 1$.

- Erase everything that is not darkened, and the resulting graph of the piecewise-defined function is given below.



Example 1.15. Sketch the graph of

$$f(x) = \begin{cases} -2, & x < -1 \\ x^3, & -1 \leq x < 1 \\ x - 1, & x \geq 1 \end{cases}$$

Solution

Exercise 1.2.6

Sketch the graph of the following piecewise-defined functions.

1. $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ x+1, & x > 0 \end{cases}$

2. $f(x) = \begin{cases} x^2, & x < -1 \\ 1, & -1 \leq x \leq 1 \\ x, & x > 1 \end{cases}$

3. $f(x) = \begin{cases} 1-x, & x < 0 \\ x, & 0 \leq x < 2 \\ -1, & x > 2 \end{cases}$

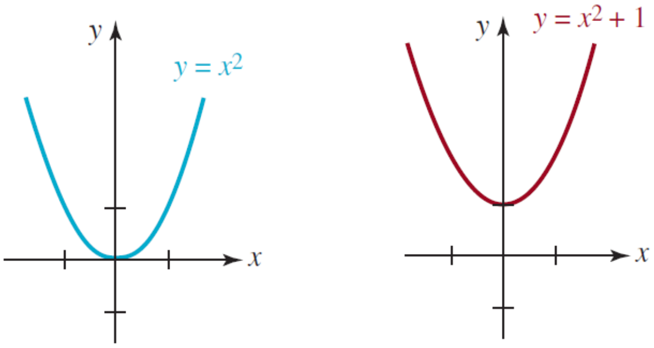
4. $f(x) = \begin{cases} \frac{1}{x}, & x \leq 2 \\ 2, & x > 2 \end{cases}$

5. $f(x) = \begin{cases} x^2, & x \leq -1 \\ 3, & 1 < x \leq 2 \\ x, & x > 2 \end{cases}$

6. $f(x) = \begin{cases} x^2+1, & x < 2 \\ -x+3, & x \geq 2 \end{cases}$

1.3 Graphing functions using vertical and horizontal shifts

Consider

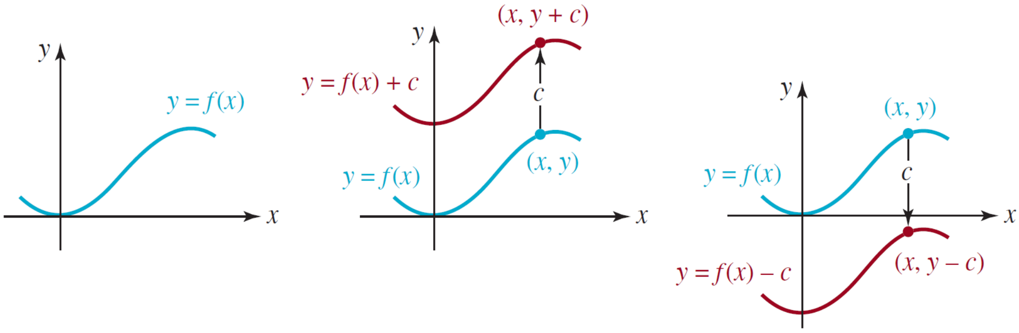


Vertical Shifts

Suppose that the graph of a function $y = f(x)$ is given and $c > 0$.

1. The graph of $y = f(x) + c$ is the graph of f shifted vertically **up** c units.

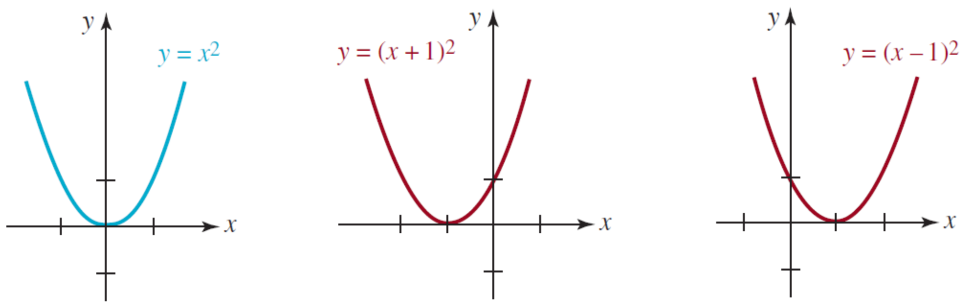
2. The graph of $y = f(x) - c$ is the graph of f shifted vertically **down** c units.



Graph of $y = f(x)$ Vertical shift up Vertical shift down

In a vertical shift, the x-coordinates of points on the shifted graph are the same as those on the original graph.

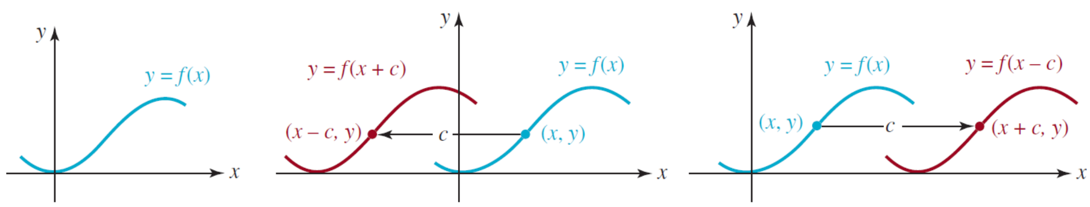
Consider



Horizontal Shifts

Suppose that the graph of a function $y = f(x)$ is given and $c > 0$.

1. The graph of $y = f(x+c)$ is the graph of f shifted horizontally to the **left** c units.
2. The graph of $y = f(x-c)$ is the graph of f shifted horizontally to the **right** c units.



Graph of $y = f(x)$ Horizontal shift left Horizontal shift right

In a horizontal shift, the y-coordinates of points on the shifted graph are the same as those on the original graph.

Summary of vertical and horizontal shifts

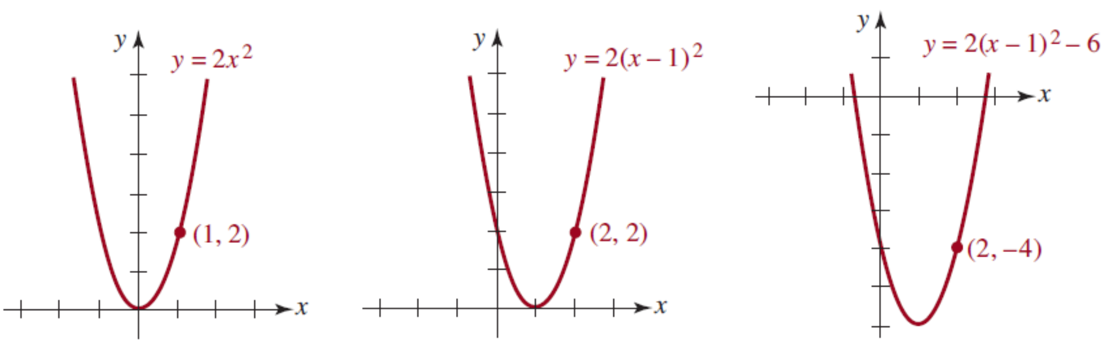
Transformation Rules for Functions		
Function Notation	Type of Transformation	Change to Coordinate Point
f(x) + d	Vertical translation up d units	(x, y) → (x, y + d)
f(x) – d	Vertical translation down d units	(x, y) → (x, y – d)
f(x + c)	Horizontal translation left c units	(x, y) → (x – c, y)
f(x – c)	Horizontal translation right c units	(x, y) → (x + c, y)

Combining Shifts

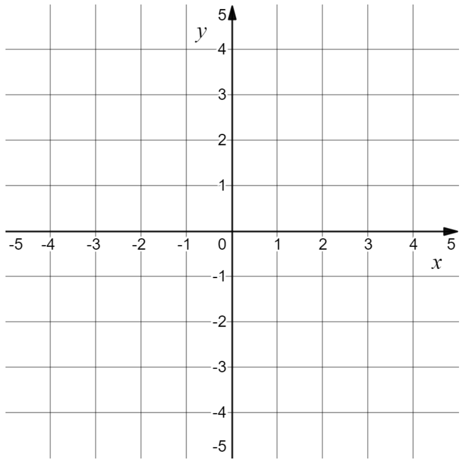
Suppose that the graph of a function $y = f(x)$ is given and $a, b > 0$.

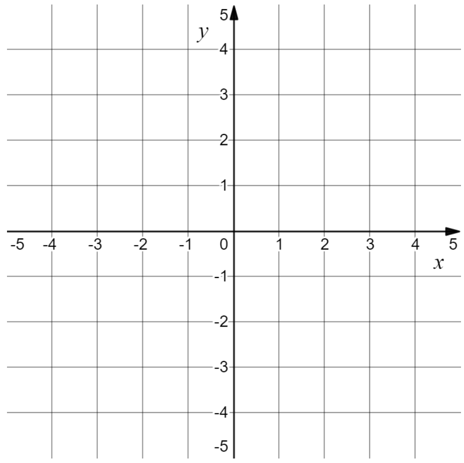
1. The graph of $y = f(x + a) + b$ is the graph of $y = f(x)$ shifted a units to the left and then b units up.
2. The graph of $y = f(x + a) - b$ is the graph of $y = f(x)$ shifted a units to the left and then b units down.
3. The graph of $y = f(x - a) + b$ is the graph of $y = f(x)$ shifted a units to the right and then b units up.
4. The graph of $y = f(x - a) - b$ is the graph of $y = f(x)$ shifted a units to the right and then b units down.

Example 1.16. Sketch the graph of $y = 2(x - 1)^2 - 6$.



Example 1.17. Sketch the graph of $y = |x+2| + 4$.



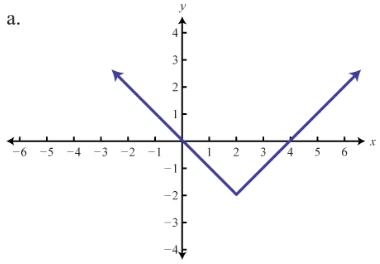


Example 1.18. Sketch the graph of $y = \sqrt{x+1} + 1$.

Exercise 1.3

1. Match the equation on the right hand side with its graph.

a.



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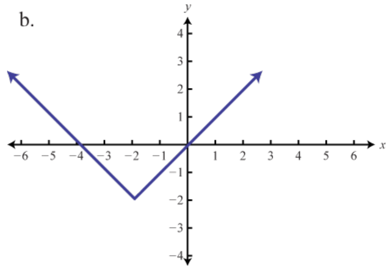
b.



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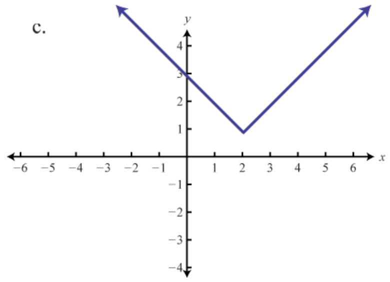
c.



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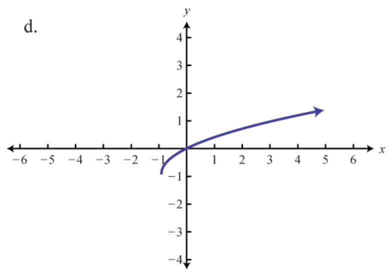
d.



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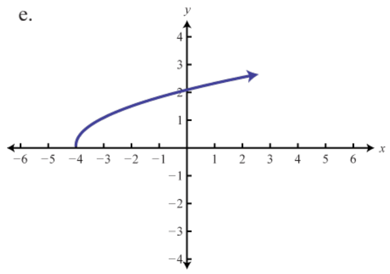
e.



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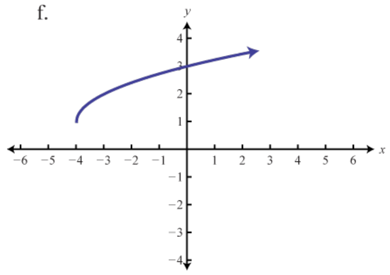
f.



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(1) $f(x) = \sqrt{x+4}$

(2) $f(x) = |x-2| - 2$

(3) $f(x) = \sqrt{x+1} - 1$

(4) $f(x) = |x-2| + 1$

(5) $f(x) = \sqrt{x+4} + 1$

(6) $f(x) = |x+2| - 2$

2. Sketch the graph of the following functions.

   2.1) $y = 4 + \sqrt{x-3}$

   2.2) $y = |x+1| - 3$

   2.3) $y = \dfrac{1}{x-2}$

   2.4) $y = \ln(x+2)$