# PLP - 41

## TOPIC 41 — DENUMERABLE SETS

Demirbaş & Rechnitzer

# DENUMERABLE SETS

# LIST OUT THE ELEMENTS

When a set $B$ is denumerable we can "list out" its elements.

- Since denumerable there is a bijection $f : \mathbb{N} \to B$
- So we can write $B$ as

$$\begin{aligned} B &= \{f(1), f(2), f(3), f(4), \ldots\} \\ &= \{b_1, b_2, b_3, b_4, \ldots\} \end{aligned} \qquad\qquad b_n = f(n)$$

This list has two nice properties

- Since $f$ is injective, the list does not repeat

$$k \neq n \implies b_k = f(k) \neq f(n) = b_n$$

- Since $f$ is surjective, any given $y \in B$ appears at some *finite* position

$$\forall y \in B, \exists n \in \mathbb{N} \text{ s.t. } y = f(n) = b_n$$

# A LIST GIVES A BIJECTION

Say we can write the elements of $B$ in a *nice* list

$$B = \{b_1, b_2, b_3, b_4, \ldots\}$$

then we can use this to construct a bijection: $g : \mathbb{N} \to B$.

What does *nice* mean? First define

$$g : \mathbb{N} \to B \qquad \text{by} \qquad g(k) = b_k$$

Then the list is *nice* when

- it does not repeat — so that $g$ is injective
- any given element $y \in B$ appears at a finite position

$$\forall y \in B, \exists n \in \mathbb{N} \text{ s.t. } y = g(n) = b_n$$

  so $g$ is surjective

So the construction of such a list proves a bijection from $\mathbb{N}$ to $B$, and so $B$ is denumerable.

# EXAMPLE 1

**PROPOSITION:**

The set of all integers is denumerable.

## Scratch

- We need to list out all the integers so that
  - the list does not repeat
  - any given integer appears at a finite position in the list
- Try $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ What $n \in \mathbb{N}$ gives $f(n) = 0$?
- Try $\mathbb{Z} = \{1, 2, 3, \ldots, 0, -1, -2, -3, \ldots\}$ What $n \in \mathbb{N}$ gives $f(n) = 0$?
- Try again: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

# PROOF $|\mathbb{N}| = |\mathbb{Z}|$

List $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ or equivalently

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \end{array}$$

**PROOF.**

List the elements $z \in \mathbb{Z}$ as above, so that

- if $z \geq 1$, then $z$ appears at position $2z$
- if $z \leq 0$, then $z$ appears at position $1 - 2z$

The list then

- does not repeat
- and any given $z \in \mathbb{Z}$ appears at some finite position

and thus the list defines a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

# NOTHING BETWEEN DENUMERABLE AND FINITE

**THEOREM:**

Let $A$, $B$ be sets with $A \subseteq B$. If $B$ is denumerable then $A$ is countable.

Proof sketch:

- If $A$ is finite then it is countable
- If $A$ is infinite then it suffices to construct a bijection $f : \mathbb{N} \to A$.
- Since $B$ is denumerable, list out its elements $B = \{b_1, b_2, b_3, b_4, b_5, b_6, \ldots\}$
- Since $A \subseteq B$, delete elements to get $A = \{b_1, b_2, b_3, b_4, b_5, b_6 \ldots\}$ (example only)
- Then $A = \{b_1, b_4, b_6, b_9, b_{13}, \ldots\}$
- Since the $B$-list did not repeat, this list does not repeat
- Since any given $a \in A$ is also in $B$, that $a$ appears at a finite position (earlier than in $B$-list)
- Hence $A$ is denumerable, and so countable

# EXAMPLE

**PROPOSITION:**

Let $k \in \mathbb{N}$, then following sets are denumerable:

$$k\mathbb{Z} = \{kn : n \in \mathbb{Z}\} \qquad \text{and} \qquad k\mathbb{N} = \{kn : n \in \mathbb{N}\}$$

We could establish bijections from those sets to $\mathbb{Z}$ or $\mathbb{N}$, or use previous theorem.

**PROOF.**

For any $k \in \mathbb{N}$ the sets are subsets of $\mathbb{Z}$. Since $\mathbb{Z}$ is denumerable, it follows that the sets are countable (by the previous theorem). Further, since the sets are not finite, it follows that they must be denumerable.

# UNION AND INTERSECTION PRESERVE COUNTABLE

## PROPOSITION:

Let $A$, $B$ be countable sets, then $A \cap B$ and $A \cup B$ are all countable.

### Proof sketch

- If $A$, $B$ are finite, then all are finite, so countable
- Since $A \cap B \subseteq A$, by the previous theorem, this is countable.
- Since $A$, $B$ countable, $B - A$ is countable. Then list carefully

$$A = \{a_1, a_2, a_3, \ldots\} \qquad (B - A) = \{b_1, b_2, b_3, \ldots\}$$

  then combine the lists by alternating

$$A \cup B = A \cup (B - A) = \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}$$

  If $A$ finite, then $A \cup B = \{a_1, a_2, \ldots, a_n, b_1, b_2, b_3, \ldots\}$

# CARTESIAN PRODUCT PRESERVES COUNTABLE

## PROPOSITION:

Let $A$, $B$ be countable sets, then $A \times B$ is countable.

**Scratchwork** — If neither finite then $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ and so

| $\times$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $\cdots$ |
|---|---|---|---|---|---|
| $b_1$ | $(a_1, b_1)$ | $(a_2, b_1)$ | $(a_3, b_1)$ | $(a_4, b_1)$ | $\cdots$ |
| $b_2$ | $(a_1, b_2)$ | $(a_2, b_2)$ | $(a_3, b_2)$ | $(a_4, b_2)$ | $\cdots$ |
| $b_3$ | $(a_1, b_3)$ | $(a_2, b_3)$ | $(a_3, b_3)$ | $(a_4, b_3)$ | $\cdots$ |
| $b_4$ | $(a_1, b_4)$ | $(a_2, b_4)$ | $(a_3, b_4)$ | $(a_4, b_4)$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

Construct list of pairs by careful sweep of the table.

# PROOF

**PROOF.**

Since $A$, $B$ are denumerable we can construct the following table

| $\times$ | $a_1$ | $a_2$ | $a_3$ | $\cdots$ |
|---|---|---|---|---|
| $b_1$ | $(a_1, b_1)$ | $(a_2, b_1)$ | $(a_3, b_1)$ | $\cdots$ |
| $b_2$ | $(a_1, b_2)$ | $(a_2, b_2)$ | $(a_3, b_2)$ | $\cdots$ |
| $b_3$ | $(a_1, b_3)$ | $(a_2, b_3)$ | $(a_3, b_3)$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

By sweeping through diagonals $\swarrow\swarrow\swarrow$ we list all the elements of $A \times B$:

$$A \times B = \left\{(a_1, a_1), (a_2, a_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \ldots\right\}$$

This list does not repeat, and any given $(a_k, b_n)$ appears at finite position, so $A \times B$ is denumerable.

# RATIONALS ARE DENUMERABLE

## PROPOSITION:

The set of all rational numbers $\mathbb{Q}$ is denumerable.

*Very strange* since $\mathbb{Q}$ is **dense**: between any two rationals you can always find another rational.

**Proof-sketch**

- Note that any $q \in \mathbb{Q}$ can be written uniquely as $q = \frac{a}{b}$ with $a \in \mathbb{Z}, b \in \mathbb{N}$ and $\gcd(a, b) = 1$
- We can rewrite rationals as $P = \{(a, b) \in \mathbb{Z} \times \mathbb{N} \text{ s.t. } \gcd(a, b) = 1\}$
- There is a bijection $f : \mathbb{Q} \to P$ given by $f(a/b) = (a, b)$, where $a/b$ is the reduced fraction
- Since $P \subseteq \mathbb{Z} \times \mathbb{N}$, we know $P$ is denumerable.
- Thus since $|P| = |\mathbb{Q}|$ we have that $\mathbb{Q}$ is denumerable also.