Analyzing the Efficiency in Asset Allocation of the UCSB Investment Advisory Committee's Positions in AAPL, COST, and Other Hypothetical Investments Using Markowitz Portfolio Theory

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MOTIVATION

When an investor sets out to construct his or her portfolio of assets, they take into consideration a number of different criteria to achieve an optimal allocation of said assets. There is a multitude of approaches and theories that an investor can look to in this regard. One of the most prominent theories in finance is Modern Portfolio Theory, also known as the Markowitz Portfolio Theory. The overall goal of the Markowitz Portfolio Theory is to construct an "efficient frontier" of optimal portfolios that produce maximal expected returns for an investor's given tolerance of risk. In essence, the theory utilizes statistical measures such as mean, variance, and covariance to analyze how different assets interact with the portfolio as a whole and maximize the expected returns while minimizing the riskiness involved in obtaining those returns versus experiencing a loss. We will implement Markowitz Portfolio Theory to analyze the current real life positions UCSB's Investment Advisory Committee has in Apple **(AAPL)** and Costco **(COST)** stocks by comparing their current allocation to two hypothetical portfolios, one under the constraint of an eight percent expected return and another that minimizes the portfolio's overall riskiness. In addition to this comparison, we will provide similar analysis on the inclusion of economies of scale to a portfolio of assets. The following section provides a brief overview of the analysis we will perform and components necessary in applying numerical methods to Markowitz Portfolio Theory.

PROBLEM STATEMENT

UCSB Investment Advisory Committee Asset Allocation Analysis

We begin by considering two assets, Apple stock **(AAPL)** and Costco stock **(COST)** with expected returns denoted μ_A, μ_C, variances σ_A^2, σ_C^2, and covariance $\sigma_{A,C}$. Currently, the UCSB Investment Advisory Committee has \$18,706.37 invested between the two assets with the exact breakdown being \$13,025.25 (69.63%) allotted to **(AAPL)** and \$5,681.12 (30.37%) allotted to **(COST)** which we will use in our comparison to the found optimal allocations. Using historical data for the monthly closing prices of each asset ranging from July 1, 1986 to December 1, 2018, we calculated the respective values for expected returns using the arithmetic mean, the sample variance, and the sample covariance between the two assets. We define the expected return for each asset i as

$$\mu_i = E(r_i),$$

where r_i is the return of asset i at each data point and $E(r_i)$ is the average of all returns over the range of entries. We found $\mu_A = 0.0240$ and $\mu_C = 0.0119$. Next, we will calculate the riskiness of each asset, defined as variance (σ_i), the measure of spread, or deviation, from the mean between returns in the data set. Variance is defined as

$$\sigma_i^2 = var(r_i) = E(|r_i - \mu_i|^2).$$

From the data, the variances for each asset were calculated to be $\sigma_A^2 = 0.01622$ and $\sigma_C^2 = 0.00651$. When finding an optimal portfolio of multiple assets, it is pertinent to consider how much the assets' prices are dependent on one another. This relationship is measured by the covariance of two assets ($\sigma_{i,j}$). Covariance is defined as

$$\sigma_{i,j} = E((r_i - \mu_i)(r_j - \mu_j)).$$

In this portfolio, the covariance was found to be $\sigma_{A,C} = 0.00301418$. With the calculation of the statistical measurements out of the way, we can move to introduce the equations necessary to Markowitz Portfolio Theory.

Our specific problem boils down to an optimization problem of the following equations from Markowitz Portfolio Thoery which will be used to derive two optimally allocated portfolios to be compared to the current real life allocation of the two assets in the UCSB Investment Advisory Committee's live portfolio: one given a desired expected return of 8% and another possessing the minimal risk parity. Since we have the constraint $g_2(w_A, w_C) = w_A + w_c - 1 = 0$ we can re-parameterize the terms w_A, w_C, which represent the fraction of total wealth invested in the respective assets, as $w_A = (1-\alpha)$ and $w_C = \alpha$. *Remark:* w_A, w_C can take negative values, which corresponds to going short on the negatively weighted asset. Moving forward, our portfolio variance, the portfolio risk, can then be written as,

$$\sigma_p^2 = ((1-\alpha)^2\sigma_A^2 + 2(\alpha)(1-\alpha)\sigma_{A,C} + (\alpha)^2\sigma_C^2)$$

The optimization problem becomes,

$$\text{minimize } \lambda(\alpha) = \sigma_p^2 = ((1-\alpha)^2\sigma_A^2 + 2(\alpha)(1-\alpha)\sigma_{A,C} + (\alpha)^2\sigma_C^2)$$

$$\text{subject to } g_1(w_A, w_C) = (1-\alpha)\mu_A + (\alpha)\mu_C - \mu_p = 0$$

$$\text{and } g_2(w_A, w_C) = (1-\alpha) + (\alpha) - 1 = 0$$

The first constraint equation computes the expected return of the portfolio as a weighted average of each respective asset's weight and expected return while the final constraint equation ensures all available wealth is invested in the portfolio's two assets. Once the optimal allocations are found, we will model the typical fluctuations in future value of the portfolios for a specified period of time T to be

$$V_1(T) = W_0 e^{\mu_p T - \sigma_p \sqrt{T}}$$

$$V_2(T) = W_0 e^{\mu_p T + \sigma_p \sqrt{T}}$$

where W_0 denotes the total available capital. Finally, the optimally allocated portfolios' fluctuations will be compared to investing the same amount of capital in a bank account yielding a continuously compounded interest rate of 4%, which bares an amount function,

$$P(t) = P_0 e^{0.04t}$$

Economy of Scale Asset Allocation Analysis

Similar analysis will be performed on a different basket of assets which has an economy of scale, meaning expected returns increase as more inputs are received. The expected returns of these assets shall be defined

$$\mu_1(w_1) = 0.0005e^{3w_1}$$

$$\mu_2(w_2) = 0.07$$

with variances

$$\sigma_1^2(w_1) = e^{-3w_1}$$

$$\sigma_2^2(w_2) = 0.04.$$

The asset weights will again be defined by $w_1 = (1-\alpha), w_2 = \alpha$. The main difference in the analysis of this particular portfolio arises in the variance equation to be minimized, which in this case is represented analogously to the previous portfolio's as,

$$\sigma_p^2 = w_1^2\sigma_1^2(w_1) + 2w_2w_1\sigma_{1,2}(w_1, w_2) + w_2^2\sigma_2^2(w_2)$$

Upon performing a similar optimization problem as above using this portfolio's parameters, we will determine the optimal allocations for two portfolios, one under the constraint of a desired 5% expected return and another possessing the minimal risk parity.

METHODS

UCSB Investment Advisory Committee Asset Allocation Analysis

Having gathered all the necessary components, we shall proceed in applying the methods of Numerical Analysis to this problem of Markowitz Portfolio Theory. The main tools that will be implemented are root finding and fixed-point algorithms including, the Bisection Method, Newton's Method and the Secant Method. The first task is to find the optimal allocation of weights, w_A, w_C, that produce our desired expected return $\mu_p = 0.08$. We will approach this by employing the Bisection Method algorithm, solving for α in the portfolio expected return equation

$$0 = (1-\alpha)\mu_A + (\alpha)\mu_C - 0.08$$

which you will notice is modified by subtracting the desired $\mu_p = 0.08$ to the other side of the equality. This found value of α will be compared to the analytical solution to guarantee correctness. In order to move forward with our fixed point and root finding algorithms for the optimization problem determining the minimal risk allocations, it must be verified that their solutions will result in an absolute minimum of the given variance equation. This is done through Fermat's Theorem (interior extremum theorem) by taking the partial derivative of the portfolio variance with respect to α,

$$\lambda(\alpha) = \frac{\partial \sigma_p^2}{\partial \alpha} = 2(-(1-\alpha)\sigma_A^2 + ((1-\alpha)+\alpha)\sigma_{A,C} + \alpha\sigma_C^2)$$

since this equation is continuous and differentiable on the interval $[0,1]$, given that this function evaluations at these endpoints have differing signs we know finding alpha will supply us with the minimum of σ_p^2 in this range. Therefore, we will perform the Secant Method algorithm on $\lambda(\alpha)$ within the range $[0,1]$ resulting in the value of α to be used in calculating the optimal weights of assets. We further validate this result through the use of the Bisection Method, applicable since the function evaluations at the endpoints are opposite signed, to verify the correct root, α, was found.

Economy of Scale Asset Allocation Analysis

Similar numerical methods will be applied to the analysis of the portfolio containing an economy of scale. Root finding and fixed-point algorithms will be our main tools in this new context. However, some further development of the methods is necessary due to the nature of the assets included in this portfolio. Regarding the first set of allocations to be found, the weighting of assets that achieve an expected return of 5%, we will utilize the Secant Method as well as Newton's Method. Turning this into a root finding problem again necessitates the subtraction of our desired return value to the other side of the equality. In addition, Newton's Method requires the derivative of the function on which the algorithm will be applied to the portfolio's expected return equation. This is given as,

$$f'(\alpha) = e^{-3\alpha}(0.0301283\alpha + 0.07e^{3\alpha} - 0.0401711)$$

In the calculation to find the α value in the portfolio with minimized risk, the Secant Method will be applied to the partial derivative with respect to α of the portfolio variance equation on the range $[-1,1]$,

$$\frac{\partial \sigma_p^2}{\partial \alpha} = \lambda(\alpha) + 3w_1^2\sigma_1^2$$

RESULTS

UCSB Investment Advisory Committee Asset Allocation Analysis

Upon performing the Bisection Method (17 iterations) and Secant Method (1 iteration), we found the value of α that obtained a portfolio with an expected return of 8% to be $\alpha = -4.628$. This corresponds to $w_A = 5.628$ and $w_C = -4.628$. The negative percentage represents short selling the **(COST)** asset. In essence this means we should leverage our portfolio by borrowing shares of the stock then immediately sell them at the market price in hopes that we can re-purchase these shares at a lower price before the specified return date. In our case, we would use the funds from the sale of the borrowed shares to create more capital to be invested in **(AAPL)**. This is not surprising due to the stellar historical performance of the **(AAPL)** stock. The variance of this particular portfolio was then calculated to be $\sigma_p^2 = 0.8441$ or 84.41%. Evaluating the functions $V_1(T), V_2(T)$ with $T = 1$ gives a fluctuation in future value of our original investment of \$18,706.37 after one year $[\$8,085.68, \$50,786.64]$. Using the same initial investment, a bank account with a continuously compounded interest rate of 4% yields \$27,906.62 after 10 years. As we can see, even after 10 years, the return on the initial investment is just over half of what the maximum return of our portfolio investment could achieve after 1 year. Although this allocation obtains the desired expected return of 8%, the risk involved in achieving said return seems far too high to be viable in a real life scenario in that it has the potential to result in a catastrophic loss. Not to mention that investors are usually charged a fee from the entity that they borrow the shares from, further siphoning away returns.

Next, we calculated the optimal allocation of capital between the two assets such that the portfolio variance was minimized. After 14 iterations of the Bisection Method algorithm and 1 iteration of the Secant Method algorithm, we found the approximate value of α for this optimal portfolio to be $\alpha = 0.7907$ with corresponding weights $w_A = 0.2093$ and $w_C = 0.7907$. The portfolio variance was found to be $\sigma_p^2 = 0.005045$. However, the expected return for this portfolio came out to be substantially lower than the 8% of the previous, coming in at $\mu_p = 0.01443253$ or a mere 1.44%. The given range of fluctuation in future value at $T = 1$ was $[\$17,677.11, \$20,375.29]$ which provides a much higher lower bound but a substantially lower potential gain. If the investor's main goal in their portfolio was to protect their wealth at all costs, this is considered the optimal allocation. The discrepancy of these two portfolios illustrates an inherent tenet of investing, achieving greater returns necessitates taking on more risk.

While neither of these two portfolio's deemed the UCSB Investment Advisory Committee's allocations as optimal under their respective criterion, their constructions provide meaningful analysis into the Markowitz Portfolio Theory. We have seen that obtaining an optimal allocation of assets hinges on the selection of constraints that embody your investment goals. If the committee's goals were solely to protect the invested funds, the current allocation $w_A = .6963, w_C = .3037$ will overexpose them to risk. However, they would miss out on some of the potential gain.

Economy of Scale Asset Allocation Analysis

Concerning the portfolios including economies of scale, our Secant Method algorithm returned an $\alpha = 0.06017$ to give an expected return of 5%. This was verified through Newton's Method then used to determine the weights $w_1 = 0.9398, w_2 = 0.06017$. Assuming a hypothetical investment of \$1,000,000 this represents an allocation of \$93,980 in asset 1 and \$6,020 in asset 2. This portfolio provides a variance $\sigma_p^2 = 0.0529$ and a fluctuation in future after 1 year model by the range [\$99,7011.52, \$1,108,483.59]. This allocation makes intuitive sense since it is heavily overweighted in the asset that is an economy of scale. This seems like a sound investment in that it not only captures the desired level of expected return but also contains a decently small risk metric.

Moving to minimize the risk in a portfolio of the same two assets, Secant Method on the partial derivative of the variance equation with respect to α yield a solution $\alpha = -0.0373$ giving $w_1 = 1.0373, w_2 = -0.0373$. This allocation provided an expected return $\mu_p = 0.00904$ The minimized variance was found to be $\sigma_p^2 = 0.2302$ and a typical fluctuation in future value of the same \$1,000,000 initial investment is modeled by the range [\$801,583.76, \$1,270,281.02]. We can see that this allocation will obtain returns at a much smaller rate in order to provide the minimal risk.

The construction of these two optimized portfolios again captures the nature of Markowitz Portfolio Thoery and how it captures the trade-offs in differing allocations to achieve an investor's desired outcomes. If our imposed hypothetical criterion dictated their current asset allocation between these two assets, we found it to be suboptimal according to Markowitz Portfolio Theory.