Modeling the Spread of COVID-19 in Daegu, South Korea Through Numerical Methods to Solve the SIR Model of Ordinary Differential Equations

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MOTIVATION

Diseases are a ubiquitous component of the human experience. As long as the species has been around, it has played host to an uncountable number of viruses and infections. The current COVID-19 global pandemic has triggered panic throughout the world's population and sent shock waves through financial markets. The unseen and seemingly unpredictable spread of this disease has been further fueling the public's hysteria surrounding it. Therefore, it is of particular interest to be able to model the spread of COVID-19 and identify potential avenues of spread mitigation to buy time while vaccines or other medical solutions are developed. We will utilize the deterministic model known as the SIR Model developed in the 1920s by Kermack and McKendrick. This compartmental model depends on three separate compartments, S: the number of individuals in a population who are not yet sick but susceptible to the infection, I: the number of individuals who are sick and infectious, and R: the number of individuals who have contracted the disease but recovered therefore have become immune to the virus as well as parameters A: the initial population of the modeled area, r: the rate of transmission of the disease or the rate at which someone susceptible becomes infected, a: the rate of recovery, and μ: the overall death rate. A visual representation of the model can be seen in the diagram below.

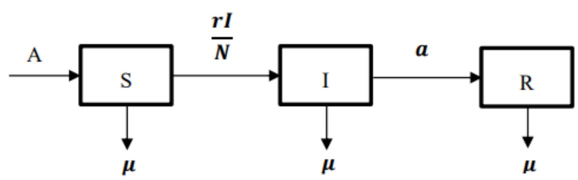

Figure 1. Diagram of the SIR Model

The following analysis will provide a numerical solution of the SIR Model applied to the South Korean city Daegu in the 50 days from Jan. 20th to Mar. 10th. The initial values and parameters will be estimated using a data set containing information regarding the COVID-19 outbreak in various South Korean cities. Our results will be obtained through the implementation of two numerical methods, Euler's Method and the Runge-Kutta Method of Order 4, for solving initial value problems of the ordinary differential equations defined by the SIR model.

PROBLEM STATEMENT

With respect to the initial values and parameters defined above, the ODEs to be solved are given by the SIR model with vital dynamics and constant population as follows;

$$\frac{dS}{dt} = f(t,S,I,R) = A - \mu S - \frac{rI}{N}S$$
$$\frac{dI}{dt} = g(t,S,I,R) = \frac{rI}{N}S - (\mu + a)I$$
$$\frac{dR}{dt} = h(t,S,I,R) = aI - \mu R$$

where, $N = S + I + R$.

As for the numerical methods to be utilized, Euler's Method will be performed first. Euler's method is an approximation technique for well-posed initial-value problems defined as follows. For the IVP

$$\frac{dy}{dt} = f(t, y),\ a \leq t \leq b,\ y(a) = \alpha$$

Euler's Method can be expressed explicitly at each mesh point with step size h as,

$$y_0 = \alpha_i$$

$$y_{i+1} = y_i + hf(t_i, y_i),\ for\ each\ i = 0, 1, ..., N-1$$

Given that this is an elementary technique, it will be considered as more of an illustration and a baseline model to aid in the construction of a more accurate approximation method. More reputable results will be arrived at through the application of the Runge-Kutta Method of Order 4 (RK4).This is the most widely used variant of the Runge-Kutta methods. It has the form as follows,

$$y_0 = \alpha$$
$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

with,

$$k_1 = hf(t_i,\ y_i)$$
$$k_2 = hf(t_i + \frac{h}{2},\ y_i + \frac{1}{2}k_1)$$
$$k_3 = hf(t_i + \frac{h}{2},\ y_i + \frac{1}{2}k_2)$$
$$k_4 = hf(t_i + \frac{h}{2},\ y_i + \frac{1}{2}k_3)$$

$for\ each\ i = 0, 1, ..., N-1$

It can easily be seen that RK4 is an extension of Euler's Method with the introduction of $k_1,\ k_2,\ k_3,\ k_4$ to eliminate the need for successive nesting in the second variable of $f(t, y)$. The RK4 method has a local truncation error of $O(h^4)$ and requires four evaluations per step whereas Euler's Method only requires one evaluation per step. Therefore, RK4 will give us more accurate approximations with one fourth the step size as Euler's.

METHODS

The Runge-Kutta method of order 4 presented in the previous section will now be given in terms of the equations defined by the SIR model. A python algorithm will be used to perform the calculations involved. The explicit form of the evaluations to be performed by the algorithm at each mesh point are given below:

$$S_{i+1} = S_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$I_{i+1} = I_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$
$$R_{i+1} = R_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

where

$$k_1 = A - \mu S_i - \frac{rI_i}{N}S_i$$

$$l_1 = \frac{rI_i}{N}S_i - (\mu + a)I_i$$
$$m_1 = aI_i - \mu R_i$$
$$k_2 = A - \mu(S_i + k_1\frac{h}{2}) - \frac{r}{N}(I_i + l_1\frac{h}{2})(S_i + k_1\frac{h}{2})$$
$$l_2 = \frac{r}{N}(I_i + l_1\frac{h}{2})(S_i + k_1\frac{h}{2}) - (\mu + a)(I_i + l_1\frac{h}{2})$$
$$m_2 = a(I_i + l_1\frac{h}{2}) - \mu(R_i + m_2\frac{h}{2})$$

and so on. The explicit derivation of the Euler's method algorithm will be excluded from the report as it can easily be seen that it is identical to a Runge-Kutta method of order one, where only k_1 is utilized at each step and the averaging term, $\frac{1}{6}$, is removed. For both methods, a step size $h = \frac{1}{2}$, which falls into the stability constraints for both methods will be used.

The parameters and initial values estimated from the data set are:

N = 2,489,802; S = (N - I); I = 10; R = 0; A = N

r = $\frac{1}{10}$

a = $\frac{1}{10}$

μ = 3.4%

Therefore the exact equations to be used for the SIR model with vital dynamics and constant population estimating the spread of COVID-19 in Daegu, South Korea are:

$$\frac{dS}{dt} = 2489802 - (0.034)S - \frac{(.1)}{2489802}IS$$
$$\frac{dI}{dt} = \frac{(.1)}{2489802}IS - (0.034 + 0.1)I$$
$$\frac{dR}{dt} = (0.1)I - 0.034R$$

RESULTS

The results of the Euler's and RK4 methods algorithms are be plotted in figures 2 and 3 to aid in the interpretation of the approximated interactions within the system of ODEs.

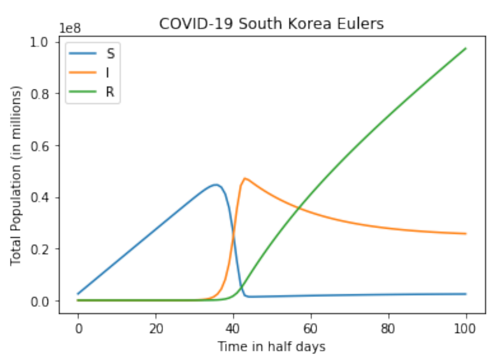

Figure 2. Results of Euler's Method

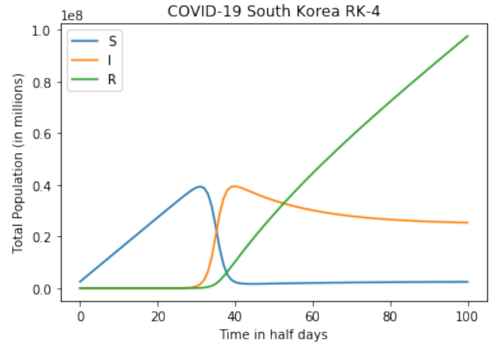

Figure 3. Results of Runge-Kutta Method of Order 4

It can be seen that the graph of results from the RK4 algorithm appears more smooth which is especially evident at inflection points along the curves. In addition, we know that RK4 produces a more accurate approximation. Thus, we will only interpret the results from RK4.

Over the 50 days that the model spans, the susceptible individuals steadily increases until approximately 15.5 days where it sharply declines as the quantity of infected individuals begins to increase and eventually remaining at a value of 0. This increase in the infected individual group can be attributed to the transmission rate being greater than the rate of recovery. The number of infections achieves a local maximum close to 400,000 at 20 days. Upon reaching the maximum, the infected curve begins a slow decline where it will theoretically approach an asymptote of 0. The infected maximum coincides with the steady, almost linearly, increasing uptick of the recovered curve which increases for the remainder of the model.

The interaction between the three ODEs can be succinctly described as such: to begin, the number of susceptible individuals slowly increases which is dictated by the fact that as this number grows, the rate of transmission grows, making it more likely for people to be infected. However, with the inclusion of a recovered group and if recovery confers immunity, the individuals that successfully recover are subsequently removed from the susceptible group. If it is assumed that no new individuals are added to the population, then over enough time, the transmission rate will decrease significantly enough for the number of infections to become negligible.

Overall, the SIR model seems most dependent on one parameter r, the rate of transmission, or basic reproductive ratio that determines the rate at which someone susceptible becomes infected. The reduction of this number either through decreasing those susceptible or through vaccination is the key to stagnating the spread of a disease.

This variation of the SIR model is extremely simplified as it does not account for the various confounding parameters in the real world such a travel or certain individuals being at higher risk of infection. However, it provides meaningful insight into how to mitigate the spread of an infectious disease as well as coming to a hopeful conclusion that over enough time and given certain precautions, the presence of almost any disease shall pass.