Assignment #3 Thanarit Kanjanametawat ID: 6410322

Formal Proof

a. Let $w(T) = \sum_{i=1}^{V} w_i$, $(u_i, v_i, w_i) \in T$, (total weight of spanning tree T)

a sub-graph of G, named T^* be the optimal MST.

a sub-graph of G, named $T_{Kruskal}$ be a spanning tree by Kruskal Algorithm.

By our supposition, $w(T^*)$ should be less than or equal to any $w(ST)$.

so, $w(T^*) \leq w(T_{Kruskal})$ —— Fact #1

Prove that $w(T_{Kruskal}) = w(T^*)$

b. Assume that e_1 is the lightest edge such that

$$e_1 \in T^* \land e_1 \notin T_{Kruskal}$$

Adding e_1 to $T_{Kruskal}$ will create a cycle graph (C). Since $T_{Krurkal}$ is a spanning tree, adding e_1 to it will make a cycle on the graph.

c. In order to construct an MST on a graph with V vertices, the MST must have exactly $V-1$ edges. C was a spanning tree before, but when added e_1 to it, it creates a cycle of V edges. This mean that there must be an edge in C that is not T^* because T^* is a spanning tree. let that edge be f_1 where

$$f_1 \in C \land f_1 \notin T^*$$

d. Removing f_1 from C will create a new spanning tree since, e_1 and f_1 are a part of the cycle. Removing one of them will break the cycle, and thus, forming a spanning tree.

e. Let $C - \{f_1\} = T_1$

T_1 is more similar to T^* than $T_{Kruskal}$ now.

- Suppose that $w(T_1) < w(T_{Kruskal})$

  This means that $e_1 < f_1$

  But if it really is, Kruskal Algorithm would not have picked f_1, it would have picked e_1 instead.

  Therefore, such case is impossible, the fact must be that

$$e_1 \geq f_1$$

$$w(T_1) \geq w(T_{Kruskal})$$

f. Repeat the steps b) to e), until T_1 becomes T^*,

$$T_{k+1} = T_k + \{e_{k+1}\} - \{f_{k+1}\}$$

T_{k+1} will be more and more similar to T^*.

- Suppose that $w(T_{k+1}) < w(T_k)$

  This means that $e_{k+1} < f_{k+1}$

  But if it really is, Kruskal Algorithm would not have picked f_{k+1}, it would have picked e_{k+1} instead.

  Therefore, such case is impossible, the fact must be that

$$e_{k+1} \geq f_{k+1}$$

$$w(T_{k+1}) \geq w(T_k)$$

g. Ultimately, we can conclude that

$$w(T_{Kruskal}) \leq w(T_1) \leq w(T_2) \leq \ldots \leq w(T^*)$$

$$w(T^*) \geq w(T_{Kruskal}) \text{ —— Fact \#2}$$

h. $\text{Fact \#1} \wedge \text{Fact \#2} \equiv w(T^*) \leq w(T_{Kruskal}) \wedge w(T^*) \geq w(T_{Kruskal})$

$$\equiv w(T^*) = w(T_{Kruskal})$$

$\therefore$ Combining the finding in g), and the first supposition, the only possible conclusion is that the total weight of $T_{Kruskal}$ is equal to the total weight of T^*, meaning that the total weight of the spanning tree generated by Kruskal Algorithm is the least possible making it a Minimum Spanning Tree. PROVED!

Example

V E

6 10

$V = \{0, 1, 2, 3, 4, 5\}$

	1	2	3	4	5	6	7	8	9	10	11
u	0	0	1	1	1	1	2	3	4	0	4
v	1	3	2	3	4	5	5	4	5	4	2
w	2	4	4	4	3	1	5	2	5	3	4

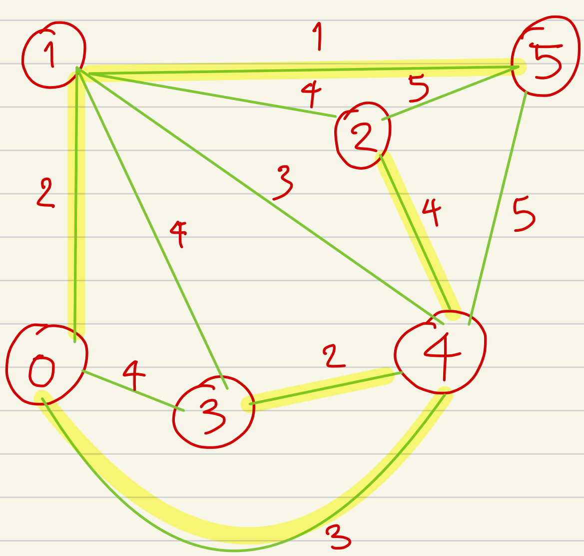
1 MST = 12

	6	8	1	10	5	3	2	4	11	7	9
u	1	3	0	0	1	1	0	1	4	2	4
v	5	4	1	4	4	2	3	3	2	5	5
w	1	2	2	3	3	4	4	4	4	5	5

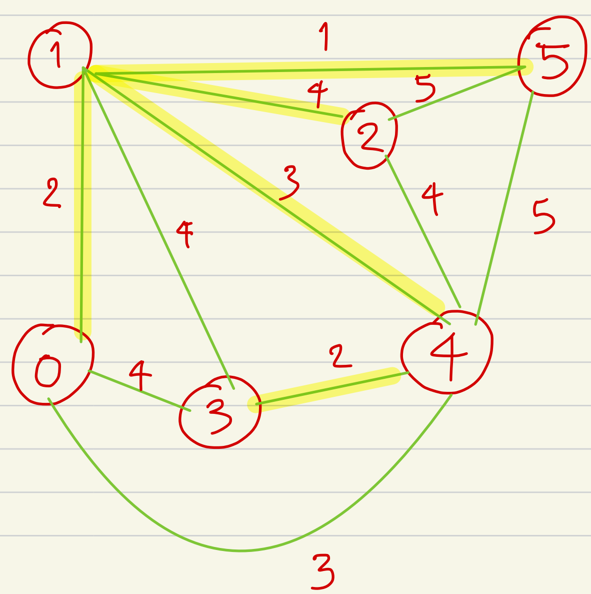
1 MST = 12

	6	8	1	10	5	3	2	4	11	7	9
u	1	3	0	0	1	1	0	1	4	2	4
v	5	4	1	4	4	2	3	3	2	5	5
w	1	2	2	3	3	4	4	4	4	5	5

1 T^*



2 T



Let the optimal Minimum spanning tree be

① $T^* = \{(1,5,1), (1,0,2), (4,3,2), (0,4,3), (4,2,4)\}$

Let Kruskal's Algorithm generated MST be

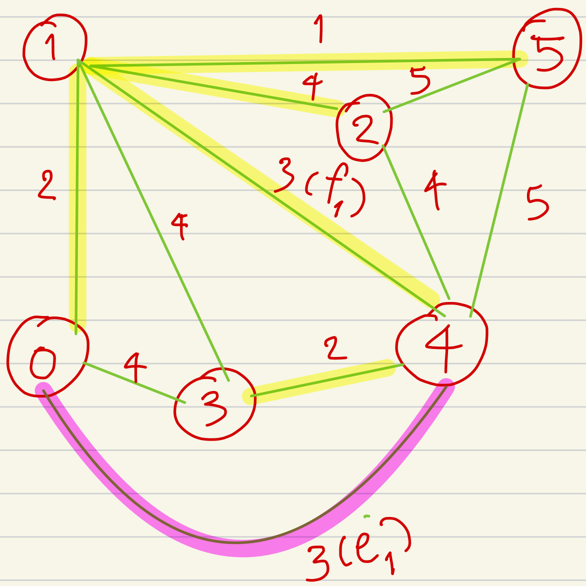
② $T = \{(1,5,1), (1,0,2), (4,3,2), (1,4,3), (1,2,4)\}$

Let $e_1 = (0,4,3)$ —— lightest edge such that $e_1 \in T^* \wedge e_1 \notin T$.

a.) Adding edge e_1 to T will create a cycle graph (C).

$C = \{(1,5,1), (1,0,2), (4,3,2), (1,4,3), (1,2,4), (0,4,3)\}$

③ C



b.) Since $|C| = 6$ and $|T^*| = 5$, there must be at least an edge f_1 that is in C but not be T^*. (Pigeon Hole Principle)

In this case, let's say f_1 should be $(1,4,3)$

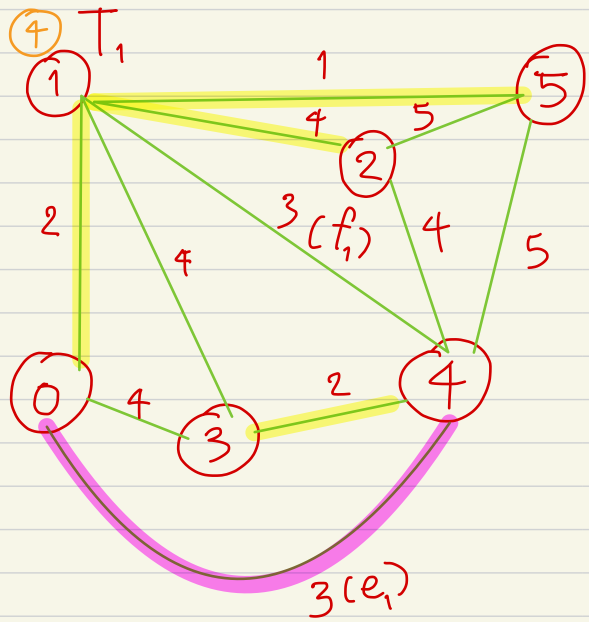
f_1 can be $(1,2,4)$, but if so, then e_1 must be $(4,2,4)$.

the order of replacement ($e_i \xrightarrow{\text{replace}} f_i$) does not matter, as long as

weight of e_i = weight of f_i

so that $e_i \xrightarrow{\text{safely replace}} f_i$

c.) Removing f from T will create a new spanning tree



e_i is a part of optimal solution (T^*).

f_i is a part of solution generated by Kruskal (T).

Since adding e_1 to C causes a cycle in the graph, and f_1 is one of the path in the cycle, removing f will break the cycle and form the MST.

T_1 is more similar to T^* now than T.

d.) - Suppose that weight of T_1 is less than weight of T.

- That would mean that at least $e_1 < f_1$

- Such case is impossible. If it really is, Kruskal would not have picked f_1, it would have picked e_1 instead.

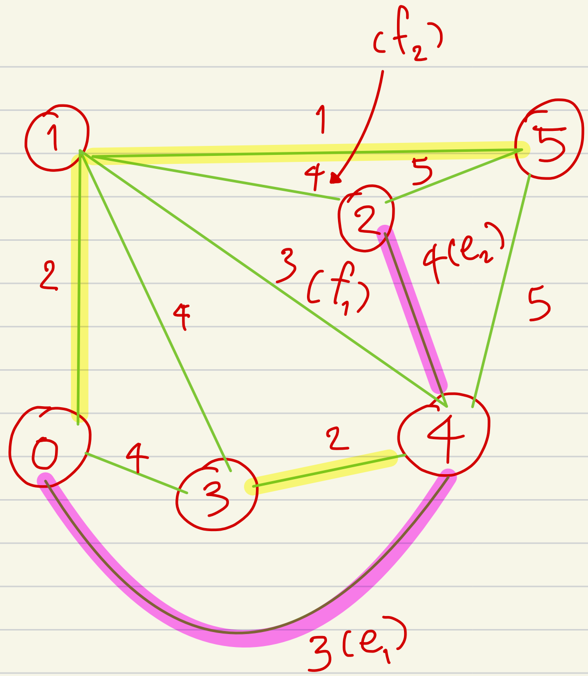
- Therefore, our supposition is incorrect, meaning that

$$e_1 \geq f_1$$

Kruskal picked f_1, meaning that e_1 is more than or equal than f_1. since Kruskal will pick the least weight first.

$\therefore$ Therefore, $T_1 \geq T$

e.) 5 T_2



continue to transform T_1 to T_2 by letting

$e_2 = (4, 2, 4)$ $\quad$ $f_2 = (1, 2, 4)$

using the same logic in c.) and d.) we know that MST is formed by adding e_2 and removing f_2, and

$$e_2 \geq f_2$$

weight of $T_2 \geq T_1 \geq T$ $\longrightarrow$ $e_k \geq f_k \;,\; 1 \leq k \leq V-1$

$T_k = T^*$

$T_2 \geq T$ $\quad$ $T^* > T$

Let $E = \{e_1, e_2, \ldots, e_k\}$, and $F = \{f_1, f_2, \ldots, f_k\}$

1 $e_1 \geq f_1$ $\quad$ 2 $e_2 \geq f_2$ $\quad \cdots\cdots$ k $e_k \geq f_k$

1 + 2 + ... + k ; $e_1 + e_2 + \ldots + e_k \geq f_1 + f_2 + \ldots + f_k$

It is impossible that $T^* < T$, since every f_i and e_i, it is impossible that $e_i < f_i$ because if it is true, Kruskal would not have picked f_i, it would have picked e_i instead. Therefore, the only possible conclusion is that T_k or T^* must be greater than or equal to T.

f.) from e.) we know that

$$T^* \geq T \quad \text{——— ①}$$

But by our supposition, T^* is the optimal solution. Meaning that T^* is less than or equal to any T generated by any algorithm.

$$T^* \leq T \quad \text{——— ②}$$

By combining these 2 facts, there is only one possible conclusion.

$$T^* = T \quad \text{——— ①} \wedge \text{②}$$

∴ The MST generated by Kruskal Algorithm has the total weight equal to the optimal MST. Hence, Kruskal Algorithm yields the least total weight to form a spanning tree. PROVED!