Probability and Random Processes (15B11MA301)

Lecture-39

(Content Covered: Properties of Poisson Process, Examples)

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Contents

Properties of Poisson Process

❑ **Property 1: Poisson Process is not a stationary process.**

Proof: Let {X(t)} be a Poisson Random Process, then

$$\boldsymbol{P_n(t)} = P[X(t)=n] = \frac{\boldsymbol{e^{-\lambda t}(\lambda t)^n}}{\boldsymbol{n}\,!}, \ n = 0,1,2\ldots.$$

$$E[X(t)] = \sum_{n=0}^{\infty} n\, P_n(t)$$

$$= \sum_{n=0}^{\infty} n \frac{\boldsymbol{e^{-\lambda t}(\lambda t)^n}}{\boldsymbol{n}\,!} = \sum_{n=0}^{\infty} \frac{\boldsymbol{e^{-\lambda t}(\lambda t)^n}}{\boldsymbol{(n-1)}\,!}$$

$E[X(t)] \ = \boldsymbol{\lambda t}$, which is a function of time.

Therefore, it is not a stationary process.

Property 2: Additive Property :- Sum of two independent Poisson processes is also a Poisson process.

Proof: Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be two independent Poisson Random Processes with mean arrival rates as λ_1 and λ_2 respectively.

Let us define X(t) = $X_1(t)+X_2(t)$. Since $\{X_1(t)\}$ and $\{X_2(t)\}$ are Poisson Random Processes,

$$E[X_1(t)] = \lambda_1 t,\ E[X_2(t)] = \lambda_2 t$$

and $E[X_1^2(t)] = \lambda_1^2 t + \lambda_1 t$ and $E[X_2^2(t)] = \lambda_2^2 t + \lambda_2 t$

Now, $E[X(t)] = E[X_1(t)+X_2(t)]$

$$= \lambda_1 t + \lambda_2 t = \boldsymbol{(\lambda_1+\lambda_2)t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(1)}$$

$$E[X^2(t)] = E[X_1(t)+X_2(t)]^2$$

$$= \boldsymbol{(\lambda_1+\lambda_2)^2 t^2} + \boldsymbol{(\lambda_1+\lambda_2)t} \quad \ldots\ldots\ldots \quad \mathbf{(2)}$$

Therefore, $X_1(t)+X_2(t)$ is also a Poisson process with mean arrival rate $\boldsymbol{\lambda_1+\lambda_2}$.

Note: (i) It can be extended to any number of independent Poisson processes.

(ii) This property can also be proved by using characteristic function of Poisson distribution.

Property 3: Difference of two independent Poisson processes is not a Poisson process.

Proof: Let $\{X_1(\mathrm{t})\}$ and $\{X_2(\mathrm{t})\}$ be two independent Poisson Random Processes with mean arrival rates as λ_1 and λ_2 respectively.

Let us define X(t) = X_1(t) - X_2(t). Since $\{X_1(\mathrm{t})\}$ and $\{X_2(\mathrm{t})\}$ are Poisson Random Processes,

$$E[X_1(t)] = \lambda_1 t,\ E[X_2(t)] = \lambda_2 t$$

and $E[X_1^2(t)] = \lambda_1^2 t + \lambda_1 t$ and $E[X_2^2(t)] = \lambda_2^2 t + \lambda_2 t$

Now, $E[X(t)] = E[X_1(\mathrm{t}) - X_2(\mathrm{t})]$

$$= \lambda_1 t - \lambda_2 t = \boldsymbol{(\lambda_1 - \lambda_2)t} \quad \textbf{................(1)}$$

$$E[X^2(t)] = E[X_1(\mathrm{t}) + X_2(\mathrm{t})]^2$$

$$= \boldsymbol{(\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 + \lambda_2)t} \quad \textbf{.......... (2)}$$

$$\neq \boldsymbol{(\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 - \lambda_2)t}$$

Since, the parameter is not $(\lambda_1 - \lambda_2)$ in equation (2), therefore, difference of two independent Poisson processes is not a Poisson process.

Property 4: Poisson Process is a Markov Process (memoryless property)

Proof: Let $\{X_1(\mathrm{t})\}$ and $\{X_2(\mathrm{t})\}$ be two independent Poisson Random Processes with parameter λ. Then,

$$P[X(t)=n_1] = \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!}$$

and $$P[X(t)=n_2] = \frac{e^{-\lambda t_2}(\lambda t_2)^{n_2}}{n_2!}, \; t_2 > t_1$$

The second order probability function of a homogenous Poisson process, is

$$P[X(t_1) = n_1, X(t_2) = n_2] = P[X(t_2) = n_2/X(t_1) = n_1].\, P[X(t_1) = n_1] \ldots. \quad \textbf{(4.1)}$$

$$= P[X(t_1) = n_1]\mathrm{P}[(n_2 - n_1) \text{ number of occurrences in } (t_2 - t_1)]$$

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!} . \frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^{(n_2-n_1)}}{(n_2-n_1)!}$$

$$= \frac{e^{-\lambda t_2}(\lambda)^{n_2}((t_2-t_1))^{(n_2-n_1)}(t_1)^{n_1}}{n_1!(n_2-n_1)!}, \; n_2 \geq n_1 \qquad \ldots\ldots\ldots\textbf{(4.2)}$$

The third order probability function of a Poisson random process is

$$P[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3]$$

$$= P[X(t_3) = n_3 / X(t_2) = n_2] . P[X(t_1) = n_1, X(t_2) = n_2]$$

$$= \frac{e^{-\lambda(t_3-t_2)}(\lambda(t_3-t_2))^{(n_3-n_2)}}{(n_3-n_2)!} . \frac{e^{-\lambda t_2}(\lambda)^{n_2}((t_2-t_1))^{(n_2-n_1)}(t_1)^{n_1}}{n_1!(n_2-n_1)!}, n_3 \geq n_2 \geq n_1$$

$$= \frac{e^{-\lambda t_3}(\lambda)^{n_3}((t_2-t_1))^{(n_2-n_1)}((t_3-t_2))^{(n_3-n_2)}(t_1)^{n_1}}{n_1!(n_2-n_1)!(n_3-n_2)!}, n_3 \geq n_2 \geq n_1 \text{........} \mathbf{(4.3)}$$

Now to prove Poisson process is a Markov Process:

$$P[X(t_3) = n_3 \,/ X(t_2) = n_2 \,, X(t_1) = \; n_1]$$

$$= \frac{P[X(t_3) = n_3, X(t_2) = n_2 \,, X(t_1) = n_1]}{P[X(t_2) = n_2 \,, X(t_1) = n_1]}$$

Using results obtained in (4.2) and (4.3), we get

$$= \frac{\boldsymbol{e}^{-\lambda(t_3 - t_2)}(\boldsymbol{\lambda}(t_3 - t_2))^{(n_3 - n_2)}}{(n_3 - n_2)!}$$

$$= \mathrm{P}[(n_3 - n_2) \text{ number of occurrences in } (t_3 - t_2)]$$

$$= P[X(t_3) = n_3 \,/ X(t_2) = n_2 \,]$$

Therefore, by the Markov property , Poisson process is a Markov process.

Property 5: The interarrival time of a Poisson process, i.e., the interval between two successive occurrences of a Poisson process with parameter λ has an exponential distribution with mean 1/ λ.

Proof: Let two consecutive occurrences of the event be E_i and E_{i+1} .

Let E_i take place at time instant t_i and T be the interval between the occurrences of E_i and E_{i+1}, where T is a continuous random variable.

$$P(T > t) = P\{E_{i+1} \text{ did not occur in } (t_i, t_{i+1})\}$$

$$= P\{\text{No event occurs in a n interval of length } t\}$$

$$= \mathrm{P}\{\mathrm{X(t)=0}\} = e^{-\lambda t}$$

Therefore, the CDF of T is given by $F(t) = P\{T \leq t\} = 1 - e^{-\lambda t}$

Therefore, the pdf of T is given by

$$f(t) = \lambda e^{-\lambda t}, \{0 \leq t\}$$

Which is an exponential distribution with mean 1/ λ.

Property 6: **If the number of occurrences of an event E is an interval of length t is a Poisson process {X(t)} with parameter λ and if each outcome of E has a constant probability p of being recorded and the recordings are independent of each other, then the number N(t) of the recorded occurrences in t is also a Poisson process with parameter λp.**

Proof:

$$P[N(t)=n]=\sum_{r=0}^{\infty}P(\text{the event } E \text{ occurs } (n+r) \text{ times in } t \text{ and } n \text{ of them being recorded})$$

$$=\sum_{r=0}^{\infty}P[\text{the event } E \text{ occurs } (n+r) \text{ times}]\times P[n \text{ of them being recorded out of } (n+r) \text{ occurrences}]$$

$$=\sum_{r=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!}(n+r)C_n p^n q^r, (q=1-p)$$

$$=e^{-\lambda t}p^n(\lambda t)^n\sum_{r=0}^{\infty}\frac{(\lambda t)^r q^r}{(n+r)!}\frac{(n+r)!}{r!n!}$$

$$\left[\text{since } nC_r=\frac{n!}{(n-r)!r!}\right]$$

On solving, we get

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^{n+r}}{\lfloor n+r} \frac{\lfloor n+r}{\lfloor n \lfloor r} p^n q^r$$

$$= \frac{e^{-\lambda t} (\lambda p t)^n}{\lfloor n} \sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{\lfloor r}$$

$$= \frac{e^{-\lambda t} (\lambda p t)^n}{\lfloor n} e^{\lambda q t}$$

$$= \frac{e^{-\lambda p t} (\lambda p t)^n}{\lfloor n}$$

which is the PDF of the Poisson Process with parameter $\boldsymbol{\lambda p}$.

Example **If {X(t)} is a Poisson process, then prove that correlation coefficient between X(t) and X(t+s) is $\sqrt{\frac{t}{t+s}}$.**

Solution: Since, {X(t)} follows Poisson process, therefore

$$E[X(t)] = \lambda t \text{ and } E[X(t+s)] = \lambda(t+s)$$

The autocorrelation function is

$$\begin{aligned} R(t_1, t_2) &= E[X(t)X(t+s)] \\ &= E[X(t)\{X(t+s) - X(t) + X(t)\}] \\ &= E[X(t)\{X(t+s) - X(t)\}] + E[X^2(t)] \\ &= \lambda t[\lambda(t+s) - \lambda t] + \lambda^2 t^2 + \lambda t \\ &= \lambda^2 t^2 + \lambda^2 ts + \lambda t \end{aligned}$$

$$Cov(t, t+s) = R(t_1, t_2) - E[X(t)]\, E[X(t+s)] = \lambda t$$

Correlation coefficient between X(t) and X(t+s) is

$$r = \frac{Cov(t,t+s)}{\sqrt{Var\,[X(t+s)]}\sqrt{Var\,[X(t)]}} = \frac{\lambda t}{\sqrt{\lambda(t+s)}\sqrt{\lambda t}} = \sqrt{\frac{t}{t+s}}$$

Example If {X(t)} is a Poisson process, prove that

$$\boldsymbol{P[X(s) = r / X(t) = n] = nCr \left(\frac{s}{t}\right)^{r} \left(1 - \frac{s}{t}\right)^{n-r}, s < t}$$

Solution: By the property of Poisson process,

$$P[X(s) = r / X(t) = n] = \frac{P[X(s)=r \cap X(t)=n]}{P[X(t)=n]}$$

$$= \frac{P[X(s)=r \cap X(t-s)=n-r]}{P[X(t)=n]}$$

Since, X(t) and X(t-s) are independent

$$P[X(s) = r / X(t) = n] = \frac{P[X(s)=r]P[X(t-s)=n-r]}{P[X(t)=n]}$$

$$= \frac{\left(\frac{e^{-\lambda s}(\lambda s)^{r}}{r!}\right)\left(\frac{e^{-\lambda(t-s)}[\lambda(t-s)]^{n-r}}{(n-r)!}\right)}{\left(\frac{e^{-\lambda t}(\lambda t)^{n}}{n!}\right)} = \frac{s^{r}(t-s)^{n-r} n!}{t^{n}\, r!(n-r)!}$$

$$= nCr \left(\frac{s}{t}\right)^{r} \left(\frac{t-s}{t}\right)^{n-r} = nCr \left(\frac{s}{t}\right)^{r} \left(1 - \frac{s}{t}\right)^{n-r}$$

References

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