Probability and Random Processes (15B11MA301)

Lecture-33

(Content covered: Properties of autocorrelation and cross-correlation functions)

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Autocorrelation Function and its Properties

If the process $\{X(t)\}$ is stationary either in the strict sense or in the wide sense, then $E\{X(t)X(t-\tau)\}$ is a function of τ. It is denoted by $R_{xx}(\tau)$ or $R(\tau)$ or $R_x(\tau)$. This function is called the autocorrelation function of the process $\{X(t)\}$.

Properties of Autocorrelation Function

1. $R(\tau)$ is an even funciton of τ , i.e, $R(-\tau) = R(\tau)$.

Proof. We have

$$R(\tau) = E\{X(t) \times X(t-\tau)\}$$

If we replace τ by $-\tau$ in above, we get

$$R(-\tau) = E\{X(t) \times X(t+\tau)\}$$

$$= E\{X(t+\tau) \times X(t)\} \quad = R(\tau)$$

$\therefore$ $R(\tau)$ is an even funciton of τ.

2. The autocorrelation function $R(\tau)$ attains its maximum at $\tau = 0$.

Proof. If we take $X = X(t)$ and $Y = X(t-\tau)$ and use the following inequality (known as Cauchy-Schwarz inequality) :

$$\{E(X(t) \times Y(t))\}^2 \leq \{E(X(t))\}^2 \times \{E(Y(t))\}^2,$$

we obtain

$$\{E(X(t) \times X(t-\tau))\}^2 \leq \{E(X(t))\}^2 \times \{E(X(t-\tau))\}^2$$

$$\{R(\tau)\}^2 \leq \{E(X(t))\}^2 \times \{E(X(t-\tau))\}^2$$

$$\{R(\tau)\}^2 \leq \{E(X^2(t))\}^2 \qquad \ldots (*)$$

[$\because$ for a stationary process, mean and variance of $X(t)$ are constant.]

If we put $\tau = 0$ in $R(\tau) = E\{X(t) \times X(t-\tau)\}$, we get

$$R(0) = E(X^2(t)).$$

Using it in (*), we obtain

$$\{R(\tau)\}^2 \le \{R(0)\}^2.$$

Taking square root on both sides, we get

$$|R(\tau)| \le R(0) \quad (\because R(0) \ge 0).$$

3. If the autocorrelation function $R(\tau)$ of a real stationary process $\{X(t)\}$ is continuous at $\tau = 0$, it is continuous at every other point, i.e.,

$$\text{if } \lim_{\tau \to 0} R(\tau) = R(0), \text{ then } \lim_{\tau \to 0} R(\tau + h) = R(\tau).$$

Proof. We have

$$E\{(X(t) - X(t-\tau))^2\} = E\{X^2(t) - 2X(t)X(t-\tau) + X^2(t-\tau)\}$$
$$= E\{X^2(t)\} - 2E\{X(t)X(t-\tau)\} + E\{X^2(t-\tau)\}$$

$$= R(0) - 2R(\tau) + R(0) = 2\{R(0) - R(\tau)\}$$

Since $R(\tau)$ is continuous at $\tau = 0$, therefore, $\lim_{\tau \to 0} R(\tau) = R(0)$.

$$\therefore \qquad \lim_{\tau \to 0} E\{(X(t) - X(t-\tau))^2\} = 0 \quad \text{or}$$

$$\lim_{\tau \to 0} X(t-\tau) = X(t)$$

$$\Rightarrow \quad X(t) \text{ is continuous for all } t.$$

Again, consider

$$R(\tau + h) - R(\tau) = E\{X(t) \times X(t - \tau - h)\} - E\{X(t) \times X(t - \tau)\},$$

$$= E[X(t)\{X(t - \tau - h) - X(t - \tau)\}],$$

since $X(t)$ is continuous for all t, therefore, we have

$$\lim_{h \to 0} [X\{(t-\tau) - h\} - X(t-\tau)] = 0 \,.$$

This implies that

$$\lim_{h\to 0} R(\tau + h) = R(\tau).$$

$\therefore$ $R(\tau)$ is continuous for all values of τ.

4. If $R(\tau)$ is the autocorrelation function of a stationary process $\{X(t)\}$ with no periodic component, then

$$\lim_{\tau\to\infty} R(\tau) = \mu_x^2,$$

provided the limit exists.

Proof. We know that $R(\tau) = E\{X(t) \times X(t-\tau)\}$.

If τ is very large, then the sample functions $X(t)$ and $X(t-\tau)$ of the random process $\{X(t)\}$ are observed at long interval of time and and so the may tend to become independent for large τ.

Therefore,

$$E\{X(t)\times X(t-\tau)\} = E\{X(t)\}\times E\{X(t-\tau)\},$$

when $\tau\to\infty$, and hence we have

$$\lim_{\tau\to\infty} R(\tau) = \lim_{\tau\to\infty}[E\{X(t)\}\times E\{X(t-\tau)\}] = \mu_x^2.$$

($\because$ $E\{X(t)\}$ is constant, not depending on t.)

or $$\mu_x = \sqrt{\lim_{\tau\to\infty} R(\tau)}\ .$$

Remark. When $X(t)$ contains a periodic component, then $X(t)$ and $X(t-\tau)$ may be dependent.

Cross-Correlation Function and Its Properties

If the processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary then $E\{X(t)\times Y(t-\tau)\}$ is a functon of τ, denoted by $R_{xy}(\tau)$.

Properties

1. $R_{yx}(\tau) = R_{xy}(-\tau)$.
2. $\left|R_{xy}(\tau)\right| \le \sqrt{R_{xx}(0)\times R_{yy}(0)}$.
3. $\left|R_{xy}(\tau)\right| \le \frac{1}{2}\left\{R_{xx}(0)+R_{yy}(0)\right\}$.
4. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal, then $R_{xy}(\tau)=0$.
5. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are independent, then $R_{xy}(\tau)=\mu_x \times \mu_y$.

Example1. A stationary process has an autocorrelation function given by $R(\tau)=\frac{25\tau^2+36}{6.25\tau^2+4}$. Find the mean and variance of the process.

Solution. From the property of autocorrelation function,

Mean of the process, $\mu_x=\sqrt{\lim_{\tau\to\infty} R(\tau)}$.

$$\mu_x=\sqrt{\lim_{\tau\to\infty}\left(\frac{25\tau^2+36}{6.25\tau^2+4}\right)}=\sqrt{\left(\frac{25}{6.25}\right)}=2.$$

Variance of the process = $E\{X^2(t)\}-\mu_x^2$

$$=R(0)-\mu_x^2=9-4=5.$$

Example 2. Find the standard deviation of a stationary process $\{X(t)\}$, if its autocorrelation function is given by n$R(\tau)=2+4e^{-2|\tau|}$. Also find the maximum value of $R(\tau)$.

Solution. From the property of autocorrelation function, we have

Mean of the process, $\mu_x=\sqrt{\lim_{\tau\to\infty} R(\tau)}$.

$$\mu_x=\sqrt{\lim_{\tau\to\infty}\left(2+4e^{-2|\tau|}\right)}=\sqrt{2}$$

Standard deviation of the process = $\sqrt{\left(E\{X^2(t)\}-\mu_x^2\right)}$.

$$=\sqrt{R(0)-\mu_x^2}=\sqrt{6-2}=2.$$

The maximum value of $R(\tau)$ is given by $R(0)=6$.

Practice Questions

Question 1. A stationary process has an autocorrelation function given by $R(\tau) = \dfrac{9\tau^3 + 2\tau + 4}{4\tau^3 + 3\tau + 1}$. Find the mean and variance of the process. [Ans: mean $= \pm 1.5$, variance $= 1.75$]

Question 2. Find the maximum value of the autocorrelation function of a stationary process $\{X(t)\}$ when $R(\tau) = 2 + 4e^{-2|\tau|}$.

References/Further Reading

1. Veerarajan, T., Probability, Statistics and Random Processes, 3rd Ed. Tata McGraw-Hill, 2008.
2. Ghahramani, S., Fundamentals of Probability with Stochastic Processes, Pearson, 2005.
3. Papoulis, A. and Pillai, S.U., Probability, Random Variables and Stochastic Processes, Tata McGraw-Hill, 2002.
4. Miller, S., Childers, D., Probability and Random Processes, Academic Press, 2012.
5. https://nptel.ac.in/courses/117/105/117105085/