Probability and Random Processes (15B11MA301)

Lecture-14

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Contents

Bernoulli Trials

- Each trial of an experiment that has only two possible outcomes (success or failure) is called a "Bernoulli trial."
- If p is the probability of success, then $(1-p)$ is the probability of failure.
- The Bernoulli probability mass function is the density function of a discrete variable X having 0 and 1 as the only possible values, i.e. $P(X=1)=p$ and $P(X=0)=1-p$.
- A random variable X is said to be Bernoulli random variable if pmf satisfies above equation for some $p \in (0,1)$.

Characteristics of Bernoulli Distribution

$$(i)\ \mu_r' = E(X^r) = \sum_{i=0,1} x_i^r P(X = x_i) = 0^r.q + 1^r.p = p$$

$$\therefore \mu_r' = p \quad (r = 1, 2, \ldots)$$

$$(ii)\ Mean = E(X) = \mu_1' = p$$

$$(iii)\ Variance = \mu_2 = \mu_2' - \mu_1'^2 = p - p^2 = pq$$

$$(iv)\ M_X(t) = E(e^{tX}) = q + pe^t$$

$$(v)\ \varphi_X(\omega) = E(e^{i\omega X}) = q + pe^{i\omega}$$

Theorem

If the probability of occurrence of an event in a single trial of a Bernoulli's experiment is p, then the probability that the event occurs exactly r times out of n independent trials is equal to $\binom{n}{r}p^r q^{n-r},\ q = 1 - p$, the probability of failure of the event.

Proof: Getting exactly r successes means getting r successes and $(n-r)$ failures simultaneously. Therefore,

$P(getting\, r\, successes\, and\, n-r\, failures) = p^r q^{n-r}.$

The trials, from which the successes are obtained are not specified. There are $\binom{n}{r}$ ways of choosing r trials for successes. Once the r trials are chosen for r successes, the remaining $(n-r)$ trials should result in failures.

These $\binom{n}{r}$ ways are mutually exclusive. In each of these $\binom{n}{r}$ ways, probability of getting exactly r successes= $p^r q^{n-r}$.

Therefore, the required probability= $\binom{n}{r} p^r q^{n-r}$.

Binomial Distribution

Let A be some event associated with a random experiment E, such that $P(A) = p$ and $P(Ac) = 1 - p$. Assuming that p remains the same for all repetitions, if we consider n independent repetitions (or trials) and if the RV X denotes the number of times the event A has occurred, then X is called a binomial RV with parameters n and p denoted by $B(n, p)$, i.e., X can take values $0, 1, 2, 3, \ldots, n$ and its PMF is given by

$$P(X = r) = \binom{n}{r} p^r q^{n-r}; \;\; r = 0,1,2, \ldots, n; \;\; q = 1 - p$$

Here, $\binom{n}{r} = n_{C_r}$.

$$Mean = E(X) = \sum_{r=0}^{n} r\,{}^{n}C_r p^r (1-p)^{n-r} = \sum_{r=1}^{n} r\,{}^{n}C_r p^r (1-p)^{n-r}$$

$$= \sum_{r=1}^{n} r \frac{n(n-1)!}{r!(n-r)!} p^r (1-p)^{n-r}$$

$$= \sum_{r=1}^{n} r \frac{n(n-1)!}{r(r-1)!\left[(n-1)-(r-1)\right]!} p p^{r-1} (1-p)^{n-r}$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\left[(n-1)-(r-1)\right]!} p^{r-1} (1-p)^{n-r}$$

$$= np \sum_{r=1}^{n} {}^{(n-1)}C_{(r-1)} p^{r-1} (1-p)^{(n-1)-(r-1)}$$

$$= np(p+(1-p))^{n-1}, \text{since } \left[p+(1-p)\right]^n = \sum_{r=0}^{n} {}^{n}C_r p^r (1-p)^{n-r}$$

$$= np$$

$$Variance = E(X^2) - \{E(X)\}^2 = \sum_{r=0}^{n} r^2\, {}^nC_r p^r (1-p)^{n-r} - n^2 p^2 = np(1-p) = npq.$$

$$M_x(t) = E(e^{tX}) = \sum_{r=0}^{n} e^{tr}\, {}^nC_r p^r (1-p)^{n-r} = \sum_{r=0}^{n} {}^nC_r (pe^t)^r (1-p)^{n-r} = \left(q + pe^t\right)^n.$$

$$\varphi_X(\omega) = E(e^{i\omega X}) = \left(q + pe^{i\omega}\right)^n.$$

Example For a binomial distribution with mean6, standard deviation $\sqrt{2}$, find the first two terms of the distribution.

Sol. $((1/3)^9, 2/2187)$

Example The mean and variance of binomial distribution are 4 and 3 respectively. Find $P(X \geq 1)$.

Sol. $(1-(3/4)^{16}) = .9899$

Example Each of two persons A and B tosses 3 fair coins. What is the probability that they obtain the same number of heads?

Sol. P(A&B get the same no. of heads)

=P(they get no head each or 1 head each or 2 heads each or 3 heads each)

= P(A gets 0 head) P(B gets o head)+------

$$=\left[{}^{3}C_{0}\left(\frac{1}{2}\right)^{3}\right]^{2}+\left[{}^{3}C_{1}\left(\frac{1}{2}\right)^{3}\right]^{2}+\left[{}^{3}C_{2}\left(\frac{1}{2}\right)^{3}\right]^{2}+\left[{}^{3}C_{3}\left(\frac{1}{2}\right)^{3}\right]^{2}$$

Example An irregular 6-faced dice is such that the probability that it gives 3 even numbers in 5 throws is twice the probability that it gives 2 even numbers in 5 throws. How many sets of exactly 5 trials can be expected to give no even number out of 2500 sets?

Sol. Let the probability of getting an even number with the unfair dice be p. Let X denote the number of even numbers obtained in 5 trials. It is given that

$$P(x=3)=2P(X=2)\Rightarrow {}^5C_3p^3q^2=2\,{}^5C_2p^2q^3\Rightarrow p=2/3 \text{ and } q=1/3.$$

$$\text{Now, } P(\text{getting no even number})=P(X=0)={}^5C_0p^0q^5=\frac{1}{243}$$

$$\therefore \text{Number of sets having no success out of } N \text{ sets} = N\times P(X=0)$$

$$\therefore \text{Required no. of sets} = \frac{2500}{243}\approx 10.$$

Example. A game consists of 5 matches and two players, A and B. Any player who firstly wins 3 matches will be the winner of the game. If player A wins a match with probability 2/3. Suppose matches are independent. What will be the probability for player A to win the game?

Sol. Player A wins 3 matches or more out of the 5 matches. This event has probability

$$= {}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + {}^5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1 + {}^5C_5\left(\frac{2}{3}\right)^5$$

$$= \frac{64}{81}$$

Example. If the probability that a child is a boy is p, find the expected number of boys in a family with n children, given that there is at least one boy.

Sol. Let X be the number of boys out of n children. Then X follows a $B(n, p)$.

$$\text{Now, } E\{X / X \geq 1\} = \sum_{r} r.P(X = r / X \geq 1) = \sum_{r=1}^{n} r.\frac{P(X = r)}{P(X \geq 1)}$$

$$= \sum_{r=1}^{n} \frac{r.{}^{n}C_{r} p^{r} q^{n-r}}{1 - P(X = 0)} = \frac{1}{1 - q^{n}} \sum_{r=1}^{n} r.{}^{n}C_{r} p^{r} q^{n-r} = \frac{np}{1 - q^{n}}.$$

Practice Question

1. Find p for a binomial random variate X if n=6 and if 9P(X=4)=P(X=2).

Sol. p=1/4.

2. In a box of switches it is known 10% of the switches are faulty. A technician is wiring 30 circuits, each of which needs one switch. What is the probability that (a) all 30 work, (b) at most 2 of the circuits do not work?

Sol. (a) 0.04239 (b) 0.4113

References

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