Probability and Random Processes (15B11MA301)

Lecture: 42

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Contents

Ex: A man either drives a car or catches a train to go to office each day. He never goes two days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work if a 6 appeared. Find

(i) the probability that he takes a train on the third day and

(ii) the probability that he drives to work in the long run.

Sol: TPM

$$P = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

Initial (first day) state probability distribution $p^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$

Second day state probability distribution

$$p^{(2)} = p^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

Third day state probability distribution

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \left(\frac{\mathbf{11}}{\mathbf{24}}, \frac{13}{24}\right)$$

P (travel by train on the third day) = 11/24

Long run

Let $\pi = (\pi_1, \pi_2)$ be the long run/ steady state distribution. Then,

$$\pi P = \pi$$

$$(\pi_1, \pi_2)\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = (\pi_1, \pi_2)$$

This gives

$$\pi_2 = 2\pi_1 \qquad (1)$$

We also have,

$$\pi_1 + \pi_2 = 1 \qquad (2)$$

From (1) and (2)

$$\pi_1 = \frac{1}{3}, \pi_2 = \frac{2}{3}$$

So,

P (travel by car in long run) = 2/3

Classification of states of a Markov chain

Irreducible chain: if every state can be reached from every other state [when $p_{ij}^{(n)} > 0$ for some n and all i, j]

Return state: state *i* is return state if $p_{ii}^{(n)} > 0$ for some n>1.

- **Period** of a return state $d_i = GCD\{m: p_{ii}^{(m)} > 0\}$
- State *i* is said to be **periodic** if $d_i > 1$ and **aperiodic** if $d_i = 1$.
- The probability that the chain returns to state *i* , having started from state *i* , for the first time at the nth step is called as **first return time probability or recurrence time probability (**$f_{ii}^{(n)}$).

  If $F_{ii} = \sum f_{ii}^{(n)} = 1$, the return to state *i* is certain.

  $\mu_{ii} = \sum n f_{ii}^{(n)}$ is called as **mean recurrence time** of the state i.

- State i is said to be **persistent or recurrent** if the return to state i is certain i.e. if $F_{ii} = 1$.
- State i is said to be **transient** if the return to state i is uncertain i.e. if $F_{ii} < 1$.
- State i is said to be **non-null persistent** if its mean recurrence time μ_{ii} is finite and **null persistent** if μ_{ii} is infinite.
- A non-null persistent and aperiodic state is called as **ergodic**.

Theorem 1: ***If a Markov chain is irreducible, all its states are of the same type.*** They are all transient, all null persistent or all non-null persistent. All its states are either aperiodic or periodic with same period.

Theorem 2: ***If a Markov chain is finite irreducible, all its states are non-null persistent.***

Ex: Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Find TPM of the Markov process and classify the states.

Sol: <u>TPM</u>

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

Here, $p_{12} > 0$, $p_{23} > 0$, $p_{31} > 0$, $p_{32} > 0$ for m=1.

Now,

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Here, $p_{13}^{(2)} > 0, p_{21}^{(2)} > 0,\ p_{22}^{(2)} > 0, p_{33}^{(2)} > 0$ for m=2

Now,

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

here, $p_{11}^{(2)} > 0$ for m=3.

So, we have $p_{ij}^{(m)} > 0$ for some m and all i, j.

So, Markov chain is irreducible.

Also, chain is finite. Therefore, for finite irreducible chain all its states are non-null persistent.

Now, observe $p_{ii}^{(m)}$

$$P^4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, P^5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

For state 1,

$$p_{11}^{(3)},\ p_{11}^{(5)} \ldots > 0$$

So, period d_i=*GCD{3, 5,…}=1*

Therefore, state 1 is aperiodic.

For state 2,

$$p_{22}^{(2)},\ p_{22}^{(3)},\ p_{22}^{(4)} \ldots > 0$$

So, period d_i=*GCD{2,3, 4,…}=1*

Therefore, state 2 is aperiodic.

Similarly, state 3 is also aperiodic.

So, **all states being non-null persistent and aperiodic states are ergodic.**

Example Find the nature of the states of the Markov chain with the tpm

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Solution $P^2 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$; $P^3 = P$

$\therefore \qquad P^4 = P^2$

and so on. In general, $P^{2n} = P^2, P^{2n+1} = P$

We note that $p_{00}^{(2)} > 0, p_{01}^{(1)} > 0, p_{02}^{(2)} > 0$

$p_{10}^{(1)} > 0, p_{11}^{(2)} > 0, p_{12}^{(1)} > 0$

$p_{20}^{(2)} > 0, p_{21}^{(1)} > 0, p_{22}^{(2)} > 0$

Therefore, the Markov chain is irreducible.

Also $p_{ii}^{(2)} = p_{ii}^{(4)} = p_{ii}^{(6)} \ldots > 0$, for all i, all the states of the chain are periodic, with period 2.

Since the chain is finite and irreducible, all its states are nonnull persistent. All states are not ergodic.

References

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