Probability and Random Processes (15B11MA301)

Lecture- 9

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References

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Marginal probability distribution

Definition: Let X and Y denote two discrete random variables with joint probability function, $f_{(X,Y)}(x, y) = P[X = x, Y = y]$

Then

$f_X(x) = P[X = x]$ is called the marginal probability function of X.

and

$f_Y(y) = P[Y = y]$ is called the marginal probability function of Y.

Note: Let $y_1, y_2, y_3, \ldots$ denote the possible values of Y.

$$f_X(x) = P[X = x]$$

$$= P\left[\{X = x, Y = y_1\} \cup \{X = x, Y = y_2\} \cup \ldots\right]$$

$$= P\left[X = x, Y = y_1\right] + P\left[X = x, Y = y_2\right] + \ldots$$

$$= f(x, y_1) + f(x, y_2) + \cdots$$

$$= \sum_j f(x, y_j)$$

It is also written as

$$f_X(X = x_i) = f_X(x_i) = \sum_j f(x_i, y_j)$$

Thus the marginal probability function of X, $f_X(x)$ is obtained from the joint probability function of X and Y by summing $f(x,y)$ over the possible values of Y.

Also

$$f_Y(y) = P[Y = y]$$

$$= P\left[\{X = x_1, Y = y\} \cup \{X = x_2, Y = y\} \cup \ldots\right]$$

$$= P\left[X = x_1, Y = y\right] + P\left[X = x_2, Y = y\right] + \ldots$$

$$= f(x_1, y) + f(x_2, y) + \cdots$$

$$= \sum_i f(x_i, y)$$

It is also written as

$$f_Y(Y = y_j) = f_Y(y_j) = \sum_i f(x_i, y_j)$$

Thus the marginal probability function of *Y*, $f_Y(y)$ is obtained from the joint probability function of *X* and *Y* by summing $f(x,y)$ over the possible values of *X.*

Conditional probability distribution

Definition: Let (X, Y) be two dimensional discrete random variable with joint probability function, $f_{(X,Y)}(x, y) = P[X = x, Y = y]$

Then

$f_{X|Y}(x|y) = P[X = x|Y = y]$ is called the **conditional probability function of X** given $Y = y$

and

$f_{Y|X}(y|x) = P[Y = y|X = x]$ is called the **conditional probability function of Y** given $X = x$

Note:

$$f_{X|Y}(x|y) = P[X = x|Y = y]$$

$$P(X = x \mid Y = y) = \frac{P(X = x \text{ and } Y = y)}{P(Y = y)}$$

$$= \frac{P[X = x, Y = y]}{P[Y = y]}$$

and

$$f_{Y|X}(y|x) = P[Y = y|X = x]$$

$$P(Y = y \mid X = x) = \frac{P(X = x \text{ and } Y = y)}{P(X = x)}$$

$$= \frac{P[X = x, Y = y]}{P[X = x]}$$

Remarks:

- Marginal distributions describe how one variable behaves ignoring the other variable.
- Conditional distributions describe how one variable behaves when the other variable is held fixed

Example 1:

A die is rolled n = 5 times and X = the number of times a "six" appears and Y = the number of times a "five" appears. Joint probability distribution $f_{(X,Y)}(x,y)$ is given below. Find Marginal distribution (a)$f_X(x)$ and $f_Y(y)$ (b)$P(X \le 1), P(Y \le 3), P(X \le 1, Y \le 3), P(X \le 1|Y \le 3)$, $P(X+Y \le 2), P(X \le 1|Y = 1)$.

Solution: (a)

y

x	0	1	2	3	4	5	$f_X(x)$	
0	0.1317	0.1646	0.0823	0.0206	0.0026	0.0001	0.4019	$f_X(X=0) = P[X=0]$
1	0.1646	0.1646	0.0617	0.0103	0.0006	0	0.4019	$f_X(X=1)$
2	0.0823	0.0617	0.0154	0.0013	0	0	0.1608	$f_X(X=2)$
3	0.0206	0.0103	0.0013	0	0	0	0.0322	$f_X(X=3)$
4	0.0026	0.0006	0	0	0	0	0.0032	$f_X(X=4)$
5	0.0001	0	0	0	0	0	0.0001	$f_X(X=5)$
$f_Y(y)$	0.4019	0.4019	0.1608	0.0322	0.0032	0.0001		
	$f_Y(Y=0) = P[Y=0]$	$f_Y(Y=1)$	$f_Y(Y=2)$	$f_Y(Y=3)$	$f_Y(Y=4)$	$f_Y(Y=5)$		

Solution (b): $P(X \leq 1) = P(X = 0) + P(X = 1) = 0.4019 + 0.4019 = 0.8038$

$P(Y \leq 3) = P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) = 0.4019 + 0.4019 + 0.1608 + 0.0322 = 0.9968$

$P(X \leq 1, Y \leq 3) = P(X = 0, Y \leq 3) + P(X = 1, Y \leq 3) = 0.3992 + 0.4012 = 0.8004$

Where,

$P(X = 0, Y \leq 3) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2) + P(X = 0, Y = 3) =$
$0.1317 + 0.1646 + 0.0823 + 0.0206 = 0.3992$

$Similarly,$ $P(X = 1, Y \leq 3) = 0.4012$

$$P(X \leq 1 | Y \leq 3) = \frac{P(X \leq 1, Y \leq 3)}{P[Y \leq 3]} = \frac{0.8004}{0.9968} = 0.8029$$

$$P(X + Y \leq 2) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2) + P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 0) + P(X = 2, Y = 0) = 0.7901$$

$$P(X \leq 1 | Y = 1) = \frac{P(X \leq 1, Y = 1)}{P[Y = 1]} = \frac{P(X = 0, Y = 1) + P(X = 1, Y = 1)}{P[Y = 1]} = 0.3292/0.4019 = 0.8191$$

Example 2: Given the following probability distribution.

(i) Find the marginal distributions of X and Y.

(ii) Find the conditional distribution of X given Y= 2.

$Y \backslash X$	-1	0	1
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

Solution:

Marginal distribution of X:

$P(X = -1)$=6/15 ; $P(X = 0)$= 5/15; $P(X = 1)$= 4/15

Marginal distribution of Y:

$P(Y = 0)$=4/15 ; $P(Y = 1)$= 6/15; $P(Y = 2)$= 5/15

Solution (b): The conditional distribution of X given Y=2 is

$$P(X = x|Y = 2) = \frac{P(X=x \cap Y=2)}{P[Y=2]}$$

$$P(X = -1|Y = 2) = \frac{P(X=-1 \cap Y=2)}{P[Y=2]} = \frac{\frac{2}{5}}{\frac{1}{3}} = \frac{2}{15}$$

$$P(X = 0|Y = 2) = \frac{P(X=0 \cap Y=2)}{P[Y=2]} = \frac{1}{15}$$

$$P(X = 1|Y = 2) = \frac{P(X=1 \cap Y=2)}{P[Y=2]} = \frac{2}{15}$$

X	-1	0	1
	2/15	1/15	2/15

Marginal probability distribution for Continuous Random Variable

Definition: Let *(X ,Y)* denote two dimensional continuous random variables with joint probability density function *f*(*x,y*) then

the **marginal density of *X*** is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

the **marginal density of *Y*** is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Conditional probability distribution for Continuous Random Variable

Definition: Let (X ,Y) denote two dimensional continuous random variables with joint probability density function $f(x,y)$ and marginal densities $f_X(x)$, $f_Y(y)$ then the **conditional density** of Y given $X = x$

$$f_{Y|X}\left(y|x\right)=\frac{f\left(x,y\right)}{f_X\left(x\right)}$$

conditional density of X given $Y = y$

$$f_{X|Y}\left(x|y\right)=\frac{f\left(x,y\right)}{f_Y\left(y\right)}$$

Example 1: The joint PDF of the two-dimensional random variable is

$$f(x,y) = \begin{cases} \frac{8}{9}xy, & 1 < x < y < 2 \\ 0 & otherwise \end{cases}$$

(i) Find the marginal density functions of X and Y.

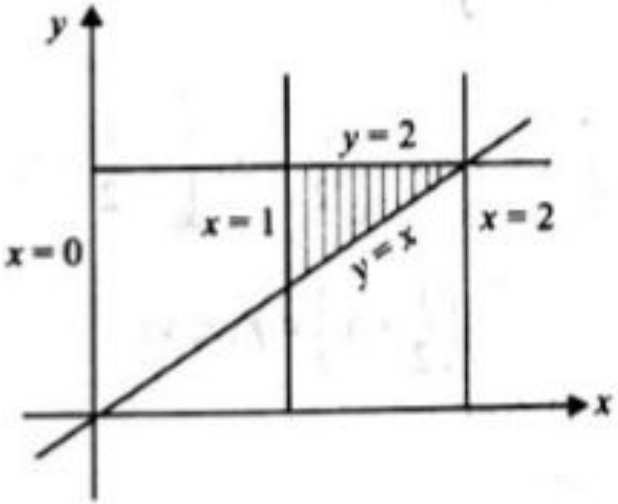
(ii) Find the conditional distribution of Y given X=x.

Solution:

(i) The marginal density functions of X is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= \int_x^2 \frac{8xy}{9}\,dy, \qquad (x \le y \le 2)$$



- The marginal density functions of Y is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

$$= \int_1^y \frac{8xy}{9}\,\mathrm{dx}, \qquad (1 \le \mathrm{x} \le y)$$

(ii) The conditional distribution of Y given X=x is

$$f_{Y|X}(y|x) = f(y/x) = \frac{f(x, y)}{f_X(x)}$$

$$f(y/x) = \frac{\frac{8xy}{9}}{\frac{4x}{9}(4 - x^2)} = \frac{2y}{4 - x^2}, \quad \mathrm{x} \le \mathrm{y} \le 2$$

Example 2: Given

$$f_{XY}(x, y) = \begin{cases} cx(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{elsewhere} \end{cases}$$

Evaluate

(i) c,

(ii) $f_X(x)$,

(iii) $f_{(Y/X)}(y/x)$, and

(iv) $f_Y(y)$.

Solution:

(i) To find the value of c, we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

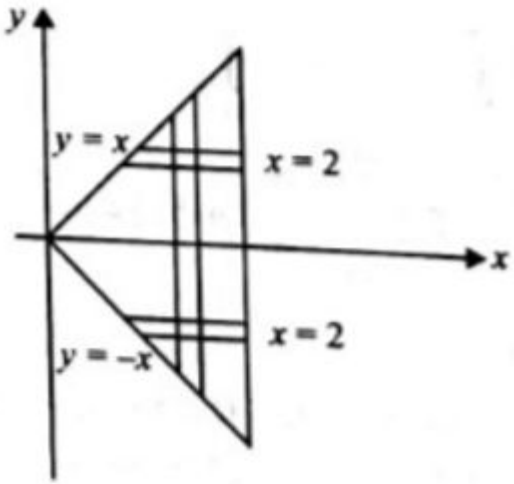
$$\therefore \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = \int_{0}^{2}\int_{-x}^{x} cx(x-y)dydx = 1 \qquad \text{(i)}$$

$$c\int_{0}^{2}\int_{-x}^{x}(x^2 - xy)dydx = c\int_{0}^{2}\left[\left(x^2 y - \frac{xy^2}{2}\right)\right]_{-x}^{x} dx = 1$$

$$\Rightarrow \quad c\int_0^2\left[\left(x^3-\frac{x^3}{2}+x^3+\frac{x^3}{2}\right)\right]dx = c\int_0^2 2x^3 dx = 1$$

$$\Rightarrow \quad 2c\left[\frac{x^4}{4}\right]_0^2 = 2c\times\frac{16}{4} = 1$$

$$\Rightarrow \quad 8c = 1 \quad\Rightarrow\quad c = \frac{1}{8}$$



Figure

(ii) $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$

$$f_X(x) = \frac{1}{8}\int_{-x}^{x}(x^2 - xy)dy = \frac{1}{8}\left[\left(x^2y - \frac{xy^2}{2}\right)\right]_{-x}^{x}.$$

$$= \frac{1}{8}\left(x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2}\right) = \frac{2x^3}{8} = \frac{x^3}{4}, \quad 0 < x < 2$$

(iii) $f_{Y/X}(y/x) = \dfrac{f(x, y)}{f_X(x)} = \dfrac{\dfrac{x(x-y)}{8}}{\dfrac{x^3}{4}} = \dfrac{x-y}{2x^2}, \qquad -x < y < x$

(iv) $f_Y(y)\int_{-\infty}^{\infty} f(x, y)dx = \int_{-y}^{2} \frac{1}{8}x(x-y)dx, \qquad -2 \le y \le 0$

$$\frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{-y}^{2} = \frac{1}{8}\left[\frac{8}{3} - 2y - \left(\frac{-y^3}{3} - \frac{y^3}{2}\right)\right]$$

$$= \frac{1}{3} - \frac{y}{4} + \frac{5y^3}{48}$$

$$f_Y(y) = \int_{y}^{2} \frac{1}{8} x(x-y)dx, \qquad 0 \le y \le 2$$

$$= \frac{1}{8}\left(\frac{x^3}{3} - \frac{x^2 y}{2}\right)_{y}^{2} = \frac{1}{8}\left(\frac{8}{3} - \frac{4y}{2} - \frac{y^3}{3} + \frac{y^3}{2}\right)$$

$$= \frac{1}{3} - \frac{y}{4} + \frac{y^3}{48}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{3} - \dfrac{y}{4} + \dfrac{5y^3}{48}, & -2 < y < 0 \\ \dfrac{1}{3} - \dfrac{y}{4} + \dfrac{y^3}{48}, & 0 < y < 2 \end{cases}$$

Marginal Densities and Distribution Functions

- The marginal (cumulative) distribution function of X is

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(u, y)\,dy\,du$$

- The marginal density of X is then

$$f_X(x) = F_X^{'}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy$$

- Similarly the marginal density of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx$$

Practice Problems

Q.1 Consider the joint density

$$f_{X,Y}(x,y)=\begin{cases}\lambda^2 e^{-\lambda y} & 0\le x\le y\\ 0 & otherwise\end{cases}$$

where λ is a positive parameter.

a) Check if it is a valid density.

b) Find the marginal densities of X and Y and identify them.

c) Find conditional density function for X given Y=y

Q. 2 Roll a die twice. Let X: number of 1's and Y: total of the 2 die.

a) Find the joint distribution of X and Y

b) The marginal probability mass function of X and Y .

c) Find $P(X\le 1$ and $Y\le 4)$

THANK YOU