Probability and Random Processes (15B11MA301)

Lecture-13

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Contents

Characteristic Function

The characteristic function of a random variable X is given by

$\phi_X(\omega) = E\left(e^{i\omega X}\right)$.

If X is a discrete RV that can take the values $x_1, x_2, \dots,$ such that $P(X = x_r) = p_r$, then

$$\phi_X(\omega) = E\left(e^{i\omega X}\right) = \sum_r e^{i\omega x_r} p_r$$

If X is a continuous RV with probability density function $f(x)$, then

$$\phi_X(\omega) = E\left(e^{i\omega X}\right) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx$$

Remark: Characteristic function always exist even when moment-generating function may not exist.

Properties of Characteristic Function

1. $\mu_n^{'} = E(X^n) =$ the co-efficient of $\dfrac{i^n \omega^n}{n!}$ in the expansion of $\varphi_X(\omega)$ in series of ascending powers of $i\omega$.

$$\varphi_X(\omega) = E(e^{i\omega x}) = E\left(1 + \frac{i\omega X}{1!} + \frac{i^2\omega^2 X^2}{2!} + \ldots + \frac{i^n \omega^n X^n}{n!} + \ldots\right)$$

$$= 1 + \frac{i\omega}{1!}E(X) + \frac{i^2\omega^2}{2!}E(X^2) + \ldots + \frac{i^n\omega^n}{n!}E(X^n) + \ldots$$

$$= 1 + \frac{i\omega}{1!}\mu_1^{/} + \frac{i^2\omega^2}{2!}\mu_2^{/} + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{i^n\omega^n}{n!}\mu_n^{/}$$

2. $\mu_n^{/} = \frac{1}{i^n}\left[\frac{d^n}{d\omega^n}\varphi_X(\omega)\right]_{\omega=0}$

Differentiating both side of result (1) w.r.t ω, n times and then putting $\omega = 0$.

3. If the characteristic function of a RV X is $\varphi_{\mathrm{x}}(\omega)$ and if $Y = aX + b$, then

$$\varphi_Y(\omega) = E(e^{i\omega Y}) = E(e^{i\omega(aX+b)}) = e^{ib\omega}\varphi_X(a\omega)$$

4. If X and Y are independent RVs, then

$$\varphi_{X+Y}(\omega) = E(e^{i\omega(X+Y)}) = E(e^{i\omega X}.e^{i\omega Y}) = E(e^{i\omega X}).E(e^{i\omega Y}) = \varphi_X(\omega)\times\varphi_Y(\omega).$$

5. If the characteristic function of a continuous RV X with density function $f(x)$ is $\varphi_X(\omega)$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_X(\omega)e^{-ix\omega}d\omega.$$

Q. Find the characteristic function of the Poisson distribution given by

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}, r = 0,1,2,...\infty$$

and hence find the first three central moments.

Sol. $$\varphi_X(\omega) = \sum_{r=0}^{\infty} e^{i\omega r} \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= e^{-\lambda}\sum_{r=0}^{\infty}\frac{\left(e^{i\omega}\lambda\right)^r}{r!} = e^{-\lambda}e^{\lambda\left(e^{i\omega}\right)} = e^{-\lambda\left(1-e^{i\omega}\right)}$$

$$\frac{d}{d\omega}\varphi_X(\omega) = i\lambda e^{i\omega}e^{-\lambda\left(1-e^{i\omega}\right)}$$

$$E(X) = \frac{1}{i}\varphi_X^{/}(0) = \lambda$$

$$\frac{d^2}{d\omega^2}\varphi_X(\omega) = i^2\lambda e^{-\lambda}(e^{i\omega} + \lambda e^{i2\omega})e^{\lambda e^{i\omega}}$$

$$E(X^2) = \frac{1}{i^2}\varphi_X^{//}(0) = \lambda(1+\lambda)$$

$$E(X^3) = \frac{1}{i^3}\varphi_X^{///}(0) = \lambda(1+3\lambda+\lambda^2)$$

The central moments are given by

$$\mu_{\mathrm{k}} = E\{X - \mu_{\mathrm{k}}\}^k$$

$$\mu_1 = E\{X - \lambda\} = 0$$

$$\mu_2 = E\{X - \lambda\}^2 = \lambda$$

$$\mu_3 = E\{X - \lambda\}^3 = \lambda$$

Q. The characteristic function of a random variable X is given by

$$\varphi_x(\omega) = \begin{cases} 1-|\omega|, & |\omega| \le 1 \\ 0, & |\omega| > 1 \end{cases}$$

Find the pdf of X.

Sol. The pdf of X is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(\omega) e^{-i\omega x} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{1}(1-|\omega|)e^{-i\omega x} d\omega\right] = \frac{1}{2\pi}\left[\int_{-1}^{0}(1+\omega)e^{-i\omega x} d\omega + \int_{0}^{1}(1-\omega)e^{-i\omega x} d\omega\right]$$

$$= \frac{1}{2\pi x^2}(2-e^{ix}-e^{-ix}) = \frac{1}{\pi x^2}\left(1-\cos x\right)$$

$$= \frac{1}{2\pi}\left[\frac{\sin\left(x/2\right)}{x/2}\right]^2, -\infty < x < \infty$$

Joint Characteristic Function

If (X, Y) is a two-dimensional RV, then $\mathrm{E}(\mathrm{e}^{\mathrm{i}\omega_1 X + \mathrm{i}\omega_2 Y})$ is called the joint characteristic function of (X, Y) and denoted by $\varphi_{XY}(\omega_1, \omega_2)$.

$$\varphi_{XY}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega_1 x + i\omega_2 y} f(x, y)\,dxdy$$

$$= \sum_i \sum_j e^{i\omega_1 x + i\omega_2 y} p(x_i, y_j)$$

Properties

$$(i)\varphi_{XY}(0,0) = 1.$$

$$(ii) E\{X^m Y^n\} = \frac{1}{i^{m+n}}\left[\frac{\partial^{m+n}}{\partial\omega_1{}^m \partial\omega_2{}^n}\varphi_{XY}(\omega_1, \omega_2)\right]_{\omega_1=0,\omega_2=0}.$$

$$(iii)\,\varphi_X(\omega) = \varphi_{XY}(\omega, 0) \text{ and } \varphi_Y(\omega) = \varphi_{XY}(0, \omega)$$

(iv) If X and Y are independent,

$$\varphi_{XY}(\omega_1, \omega_2) = \varphi_X(\omega_1)\varphi_Y(\omega_2) \text{ and conversely.}$$

Q. Two RVs X and Y have the joint characteristic function $\varphi_{XY}(\omega_1,\omega_2)=e^{\left(-2\omega_1^2-8\omega_2^2\right)}$. Show that X and Y are both zero mean RVs and also that they are uncorrelated.

Sol. By the property of joint CF

$$E\{X^m Y^n\}=\frac{1}{i^{m+n}}\left[\frac{\partial^{m+n}}{\partial\omega_1{}^m\partial\omega_2{}^n}\varphi_{XY}(\omega_1,\omega_2)\right]_{\omega_1=0,\omega_2=0}$$

$$E(X)=\frac{1}{i}\left[\frac{\partial}{\partial\omega_1}e^{\left(-2\omega_1^2-8\omega_2^2\right)}\right]_{\omega_1=0,\omega_2=0}=\left[e^{\left(-2\omega_1^2-8\omega_2^2\right)}4i\omega_1\right]_{\omega_1=0,\omega_2=0}=0$$

$$E(Y)=\left[e^{\left(-2\omega_1^2-8\omega_2^2\right)}16i\omega_2\right]_{\omega_1=0,\omega_2=0}=0$$

$$E(XY)=\frac{1}{i^2}\left[\frac{\partial^2}{\partial\omega_1\partial\omega_2}e^{\left(-2\omega_1^2-8\omega_2^2\right)}\right]_{\omega_1=0,\omega_2=0}=\left[\frac{\partial}{\partial\omega_1}e^{\left(-2\omega_1^2-8\omega_2^2\right)}16\omega_2\right]_{\omega_1=0,\omega_2=0}=\{-64\omega_1\omega_2e^{\left(-2\omega_1^2-8\omega_2^2\right)}\}_{\omega_1=0,\omega_2=0}$$

$$=0$$

$$\therefore C_{XY}=E(XY)-E(X)\times E(Y)=0 \text{ and hence } \rho_{X,Y}=0$$

Practice Question

Q. Show that the distribution for which the characteristic function is $e^{-|\omega|}$ has the density function $f(x) = \frac{1}{\pi} \times \frac{1}{1+x^2}, -\infty < x < \infty$

Hint. $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\omega) e^{-i\omega x} d\omega$

References

1. Veerarajan, T., Probability, Statistics and Random Processes, 3rd Ed. Tata McGraw-Hill, 2008.

2. Ghahramani, S., Fundamentals of Probability with Stochastic Processes, Pearson, 2005.

3. Papoulis, A. and Pillai, S.U., Probability, Random Variables and Stochastic Processes, Tata McGraw-Hill, 2002.

4. Miller, S., Childers, D., Probability and Random Processes, Academic Press, 2012.