Probability and Random Processes (15B11MA301)

Lecture-32

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Contents

Semi-Random telegraph signal process

Semi-Random Telegraph Signal Process: A random process X(t) defined as

$$X(t) = (-1)^{N(t)}$$

Where N(t) represents the number of occurrences of a specified event in (0, t). N(t) follows Poisson process.

i.e.,

$$P[N(t) = r] = \frac{e^{-\lambda t}(\lambda t)^r}{r!}, r = 0,1,2,.....$$

P[X(t)=1] = P [N(t) is even]

$$= P[N(t) = 0] + P[N(t) = 2] + P[N(t) = 4] + \dots\dots\dots\dots$$

$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \dots\dots\right]$$

$$= e^{-\lambda t} Cosh \lambda t$$

P[X(t)=-1] = P[N(t) is odd]

$$= P[N(t) = 1] + P[N(t) = 3] + \dots\dots\dots\dots$$

$$= e^{-\lambda t}\left[\frac{\lambda t}{1!} + \frac{(\lambda t)^3}{3!} + \dots\dots\right]$$

$$= e^{-\lambda t} \sinh \lambda t$$

Then means E[X(t)] = 1×P[X(t)=1] – 1 × P[X(t)= -1]

$$= e^{-\lambda t} Cosh\lambda t - e^{-\lambda t} \sinh \lambda t$$

$$= e^{-\lambda t} [Cosh\lambda t - Sinh\lambda t]$$

=

$$= e^{-\lambda t} . e^{-\lambda t}$$

$$E[X(t)] = e^{-2\lambda t}$$

Autocorrelation:

$$R(t_1, t_2) = E[X(t_1).X(t_2)]$$

$$X(t_1).X(t_2) \quad \text{may be} \quad \begin{array}{l} +1 \left[\begin{array}{c} X(t_1)=1, X(t_2)=1 \\ X(t_1)=-1, X(t_2)=-1 \end{array} \right. \\ -1 \left[\begin{array}{c} X(t_1)=1, X(t_2)=-1 \\ X(t_1)=-1, X(t_2)=1 \end{array} \right. \end{array}$$

$$R(t_1,t_2)=1\times P[X(t_1).X(t_2)=1]-1\times P[X(t_1).X(t_2)=-1]$$

$$= P[X(t_1)=1, X(t_2)=1]+P[X(t_1)=-1.X(t_2)=-1]$$
$$-P[X(t_1)=-1, X(t_2)=1]-P[X(t_1)=1, X(t_2)=-1]\ldots\ldots\ldots(1)$$

$$P[X(t_1)=1, X(t_2)=1]=P[X(t_1)=1 \mid X(t_2)=1]\times P[X(t_2)=1]$$

($t_2 \leq t_1$)

=P[an even no. of occurrences of event in (t_1-t_2)]×P[X(t_2)=1]

$= e^{-\lambda\tau} Cosh\lambda\tau \times e^{-\lambda t_2} Cosh\lambda t_2$ where $\tau = t_1 - t_2$

$$P[X(t_1)=1, X(t_2)=1]=e^{-\lambda\tau} Cosh\lambda\tau . e^{-\lambda t_2} Cosh\lambda t_2$$

$$P[X(t_1)=1, X(t_2)=1] = e^{-\lambda\tau} Cosh\lambda\tau . e^{-\lambda t_2} Cosh\lambda t_2$$

$$P[X(t_1)=-1, X(t_2)=-1] = P[X(t_1)=-1 \mid X(t_2)=-1] \times P[X(t_2)=-1]$$

=P [an even no. of occurrences of event in (t_1-t_2)].P[X(t_2)=1]

$= e^{-\lambda\tau} Cosh\lambda\tau \times e^{-\lambda t_2} Sinh\lambda t_2$

$$P[X(t_1)=-1, X(t_2)=-1] = e^{-\lambda\tau} Cosh\lambda\tau . e^{-\lambda t_2} Sinh\lambda t_2$$

Similarly,

$$P[X(t_1)=1, X(t_2)=-1] = e^{-\lambda\tau} Sinh\lambda\tau . e^{-\lambda t_2} Sinh\lambda t_2$$

and

$$P[X(t_1)=-1, X(t_2)=1] = e^{-\lambda\tau} Sinh\lambda\tau . e^{-\lambda t_2} Cosh\lambda t_2$$

Then by equation 1,

$$R(t_1,t_2)=e^{-\lambda\tau}Cosh\lambda\tau\times e^{-\lambda t_2}Sinh\lambda t_2$$

$$+e^{-\lambda\tau}Cosh\lambda\tau.e^{-\lambda t_2}Sinh\lambda t_2-e^{-\lambda\tau}Sinh\lambda\tau.e^{-\lambda t_2}Cosh\lambda t_2$$

$$-e^{-\lambda\tau}Sinh\lambda\tau.e^{-\lambda t_2}Sinh\lambda t_2$$

$$R(t_1,t_2)=e^{-\lambda\tau}Cosh\lambda\tau\times e^{-\lambda t_2}.e^{-\lambda t_2}-e^{-\lambda\tau}Sinh\lambda\tau e^{-\lambda t_2}e^{\lambda t_2}$$

$$=e^{-\lambda\tau}[Cosh\lambda\tau-\sinh\lambda\tau]=e^{-\lambda\tau}.e^{-\lambda\tau}=e^{-2\lambda\tau}$$

$$R(t_1,t_2)=e^{-2\lambda(t_1-t_2)}=e^{-2\lambda\tau}$$

In semi random telegraph signal process,

Mean $\neq$ constant & $R(t_1, t_2) = f(t_1 - t_2)$

= {X(t)} is evolutionary process

Imp Formulas:

$$
\left\{
\begin{aligned}
&\sinh x = \frac{e^{x} - e^{-x}}{2} \\
&\cosh x = \frac{e^{x} + e^{-x}}{2} \\
&\frac{\sinh x + \cosh x}{2} = e^{x} \\
&\frac{\cosh x - \sinh x}{2} = e^{-x}
\end{aligned}
\right.
$$

Random telegraph Signal process:

A random process {Y(t)} defined as:

$$Y(t)=\alpha X(t)$$

Where α is a random variable independent of X(t) defined as:

$$\alpha = \{1,-1\} with\, P[\alpha = 1] = \frac{1}{2}\, and\;\; P[\alpha = -1] = \frac{1}{2}.$$

and X(t) is semi random telegraph signal process.

MEAN:

E[Y(t)] = E[α.X(t)]

=E[α]. E[X(t)]

$$\left\{\begin{aligned} E[\alpha] &= 1\times P[\alpha=1]-1\times P[\alpha=-1] \\ &= 1\times\frac{1}{2}-1\times\frac{1}{2}=\frac{1}{2}-\frac{1}{2}=0 \end{aligned}\right\}$$

E[Y(t)] = E[α]. E[X(t)]

$$= 0 \times e^{-2\lambda t} = 0$$

E[Y(t)] = 0

$$E[Y(t_1).Y(t_2)] = R_y(t_1, t_2)$$

= $R_y(t_1, t_2)$ = $E[\alpha X(t_1).\alpha X(t_2)]$

$$= E[\alpha^2 X(t_1).X(t_2)]$$

$$= E[\alpha^2].E[X(t_1).X(t_2)]$$

$$= E[\alpha^2].R_x(t_1, t_2)$$

=

$$= 1.e^{-2\lambda\tau}$$

$$= e^{-2\lambda\tau}$$

$$R_y(t_1, t_2) = e^{-2\lambda(t_1 - t_2)} = e^{-2\lambda\tau}$$

$$\left\{ \begin{aligned} E[\alpha^2] \quad &= (1)^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned} \right\}$$

In random telegraph signal process

Mean = E[(t)}=0

& $R(t_1, t_2) = e^{-2\lambda\tau}$

That implies {Y(t)} is WSS process.

References

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3. Papoulis, A. and Pillai, S.U., Probability, Random Variables and Stochastic Processes, Tata McGraw-Hill, 2002.

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