Poisson distribution:

Discrete probability distribution

Content:

- Poisson Distribution
- Limiting case of Binomial distribution
- Mean and Variance
- Moment Generating Function
- Some examples
- References

Poisson Distribution

If X is a discrete random variable that assumes only non-negative values such that its probability mass function is given by

$$p(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}; \quad k = 0, 1, 2, \ldots$$

$$and \qquad \lambda > 0$$

Then X is said to follow Poisson distribution with parameter λ.

Introduction to the **Poisson Distribution**

- Poisson distribution occurs when there are two events which do not occur as outcomes of a definite number of trials of an experiment but which occur at random points of time and space wherein our interest lies only in the number of occurrences of the event, not in its non-occurrences.

Poisson Distribution is a limiting case of Binomial Distribution:

- Poisson distribution is a limiting case of Binomial distribution under the following conditions:

1. The number of trials *n* is indefinitely large, i.e. $n \to \infty$.
2. The probability of success p for each trial is very small, i.e. $p \to 0$.
3. $np = \lambda$ is finite, where λ is a positive constant.

Proof:

Let X be a binomially distributed random variable. Then PMF of a binomial distribution is

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!};\quad k=0,1,2,\ldots$$

$$and\qquad \lambda>0$$

$$P(X=x)={}^nC_x p^x q^{n-x};\quad x=0,1,2,\ldots,n$$

$$and\qquad \lambda>0$$

$$P(X=x) = {}^nC_x p^x q^{n-x}; \quad x = 0,1,2,\ldots,n$$

$$= \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

$$= \frac{1.2.3\ldots[n-(x+1)](n-x)[n-(x-1)]\ldots(n-1)n}{1.2.3\ldots(n-x)x!} p^x (1-p)^{n-x}$$

We know that, mean of the binomial distribution is *np*.

Let $np = \lambda \Rightarrow p = \dfrac{\lambda}{n}$

and $q = 1 - p = 1 - \dfrac{\lambda}{n}$

$$\Rightarrow P(X=x) = \frac{n^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}}{x!}$$

as limit *n* tends to infinity

$$P(X = x) = \frac{\lim\limits_{n \to \infty} \lambda^x \left(1 - \frac{\lambda}{n}\right)^{n-x}}{x!} = \frac{e^{-\lambda} \lambda^x}{x!}$$

Mean:

$$E(X)=\sum_{x=0}^{\infty} xP(X=x)=\sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$=\lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$=\lambda e^{-\lambda}e^{\lambda}$$

$$=\lambda$$

Variance:

$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(X = x) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} [(x-1) + x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda^2 + \lambda$$

$$\mathrm{Var}(\mathrm{X}) = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

Moment Generating Function:

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} P(X = x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{e^{tx}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

Ques: Find the mean and variance of Poisson distribution using moment generating function.

Poisson Mean and Variance

- Mean

$$\boxed{\mu = \lambda}$$

For a Poisson random variable, the variance and mean are the same!

- Variance and Standard Deviation

$$\boxed{\sigma^2 = \lambda}$$

$$\boxed{\sigma = \sqrt{\lambda}}$$

where λ = expected number of hits in a given time period

Additive or Reproductive Property

Sum of independent Poisson variates is also a Poisson variate,

i.e. if $X_i (i = 1, 2, ..., n)$ are n independent Poisson variates with parameter $\lambda_i (i = 1, 2, ..., n)$, then $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$.

Proof We know that the MGF of the Poisson variate X_i is given by

$$M_{X_i}(t) = e^{\lambda_i(e^t - 1)}, \quad i = 1, 2, \ldots, n$$

$$M_{X_1 + X_2 + \ldots + X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_n}(t)$$

$$= e^{\lambda_1(e^t - 1)} \, e^{\lambda_2(e^t - 1)} \ldots e^{\lambda_n(e^t - 1)}$$

$$= e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)(e^t - 1)}$$

which gives the MGF of a Poisson variate $X_1 + X_2 + X_3 + \cdots + X_n$ with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Ques:

An insurance company found that only 0.005% of the population is involved in a certain type of accident each year. If its 2000 policyholders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year?

Solution Given: $p = 0.005\% = \frac{0.005}{100} = \frac{1}{20000}$

and $n = 2000$

$$\text{Mean} = \lambda = n \times p = 2000 \times \frac{1}{20000} = 0.1$$

Let the random variable X denote the number of clients involved in certain type of accident each year.

Then using Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.1}(0.1)^x}{x!}, x = 0, 1, 2, \ldots$$

$$\begin{aligned}
P(\text{not more than two clients}) &= P(X \leq 2) \\
&= P(X = 0) + P(X = 1) + P(X = 2) \\
&= \frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!} \\
&= e^{-0.1}\left(1 + 0.1 + \frac{0.01}{2}\right) \\
&= 0.9048 \times 1.105 = 0.9998
\end{aligned}$$

Ques:

If X is a Poisson random variable with $P(X=1)=P(X=2)$, find $P(X>3)$.

Example

- For example, if new cases of West Nile Virus in New England are occurring at a rate of about 2 per month, then these are the probabilities that: 0,1, 2, 3, 4, 5, 6, to 1000 to 1 million to… cases will occur in New England in the next month:

Poisson Probability table

X	P(X)
0	$\frac{2^0 e^{-2}}{0!} = .135$
1	$\frac{2^1 e^{-2}}{1!} = .27$
2	$\frac{2^2 e^{-2}}{2!} = .27$
3	$\frac{2^3 e^{-2}}{3!} = .18$
4	$= .09$
5	
…	…

Example: Poisson distribution

Suppose that a rare disease has an incidence of 1 in 1000 person-years. Assuming that members of the population are affected independently, find the probability of k cases in a population of 10,000 (followed over 1 year) for k=0,1,2.

The expected value (mean) =λ = .001*10,000 = 10
10 new cases expected in this population per year□

$$P(X=0)=\frac{(10)^0 e^{-(10)}}{0!}=.0000454$$
$$P(X=1)=\frac{(10)^1 e^{-(10)}}{1!}=.000454$$
$$P(X=2)=\frac{(10)^2 e^{-(10)}}{2!}=.00227$$

Example

For example, if new cases of West Nile in New England are occurring at a rate of about 2 per month, then what's the probability that exactly 4 cases will occur in the next 3 months?

X ~ Poisson (λ=2/month)

$$P(X = 4 \text{ in } 3 \text{ months}) = \frac{(2*3)^4 e^{-(2*3)}}{4!} = \frac{6^4 e^{-(6)}}{4!} = 13.4\%$$

Exactly 6 cases?

$$P(X = 6 \text{ in } 3 \text{ months}) = \frac{(2*3)^6 e^{-(2*3)}}{6!} = \frac{6^6 e^{-(6)}}{6!} = 16\%$$

Practice problems

1a. If calls to your cell phone are a Poisson process with a constant rate $\lambda=2$ calls per hour, what's the probability that, if you forget to turn your phone off in a 1.5 hour movie, your phone rings during that time?

1b. How many phone calls do you expect to get during the movie?

Answer

1a. If calls to your cell phone are a Poisson process with a constant rate λ=2 calls per hour, what's the probability that, if you forget to turn your phone off in a 1.5 hour movie, your phone rings during that time?

X ~ Poisson (λ=2 calls/hour)

P(X≥1)=1 – P(X=0)

$$P(X=0)=\frac{(2*1.5)^{0}e^{-2(1.5)}}{0!}\frac{(3)^{0}e^{-3}}{0!}=e^{-3}=.05$$

∴P(X≥1)=1 – .05 = 95% chance

1b. How many phone calls do you expect to get during the movie?

E(X) = λt = 2(1.5) = 3

References

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Thank You