Probability and Random Processes (15B11MA301)

Lecture- 10

Department of Mathematics
Jaypee Institute of Information Technology
Noida, India

References

1.	**A. M. Mood, F. A. Graybill and D. C. Boes,** Introduction to the theory of statistics, 3rd Indian Ed., Mc Graw Hill, 1973.
2.	**R. V. Hogg and A. T. Craig,** Introduction to mathematical Statistics, Mc-Millan, 1995.
3.	**V. K. Rohatgi,** An Introduction to Probability Theory and Mathematical Statistics, Wiley Eastern, 1984.
4.	**S. M.Ross,** A First Course in Probability, 6th edition, Pearson Education Asia, 2002.
5	**S. Palaniammal,** Probability and Random Processes, PHI Learning Private Limited, 2012.
6	**P. L.Mayer,** Introductory Probability and Statistical Applications, Addison-Wesley, Second Edition, 1972.
7.	**R. E. Walpole, R H. Myers, S. L. Myers, and K. Ye,** Probability & Statistics for Engineers & Scientists, 9th edition, Pearson Education Limited, 2016.
8.	**I. Miller and M. Miller, John E. Freund's** Mathematical Statistics with Applications, 8th Edition, Pearson Education Limited 2014.

Contents:

- Independent Random Variables
- Conditional Means
- Conditional Variances
- Examples

Independent Random Variables

Let (X ,Y) be two dimensional random variable with joint probability function $f_{(X,Y)}(x, y)$= P[X = x, Y = y].Then (X,Y) is said to be independent if

$$f_{(X,Y)}(x, y) = f_X(x).\, f_Y(y)$$

Note: If X,Y are independent, then

a) $f_{X|Y}(x|y) = f(x)$,

b) $f_{Y|X}(y|x) = f(y)$,

Example 1: Let X,Y have following joint probability distribution, $f_{(X,Y)}(x,y)$ .Show that X,Y are independent.

	X=2	X= 4
Y=1	0.10	0.15
Y=3	0.20	0.30
Y=5	0.10	0.15

Solution:

$Y \backslash X$	2	4	$P(Y=y)$
1	0.10	0.15	0.25
3	0.20	0.30	0.50
5	0.10	0.15	0.25
$P(X=x)$	0.40	0.60	1

From the above table we can see that

$f_X(x).f_Y(y)$ = *P(X=x).P(Y=y)=* $f_{(X,Y)}(x,y)$ *for each pair of values (x, y).*

e.g for Pair (2,1), we have $f_{(X,Y)}(2,1)$ *=0.10 and P(X=2)=0.40 and P(Y=1)=0.25*

$f_X(x).f_Y(y)$ = *P(X=x).P(Y=y)=0.25*0.40=0.10=* $f_{(X,Y)}(x,y)$

Example 2: If X,Y have the joint PDF

$$f(x) = \begin{cases} x + y, 0 < x < 1, 0 < y < 1 \\ 0 \qquad otherwise \end{cases}$$

Check whether X and Y are independent or not.

Solution: The marginal density function of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 (x + y)dy = \ x + \frac{1}{2}, \qquad 0 < x < 1$$

The marginal density function of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 (x + y)dx = \ y + \frac{1}{2}, \qquad 0 < y < 1$$

Now,

$$f(x).f(y) = (x + 1/2)(y + 1/2) \neq f(x, y)$$

Ans: X and Y are not independent.

Conditional Mean and Variance

If (X,Y) is a two-dimensional random variable, then the mean or expectation of (X,Y) is defined as follows

- **Case 1:** when X,Y are discrete random variables, then

$$E(X) = \sum_{x_i} x_i . P(X = x_i)$$

$$E(Y) = \sum_{y_j} y_j . P(Y = y_j)$$

$$E(X/Y) = \sum_{x_i} x_i . P(X = x_i / Y = y_j)$$

$$E(Y/X) = \sum_{y_j} y_j . P(Y = y_j / X = x_i)$$

$$E(XY) = \sum_{x_i} \sum_{y_j} x_i y_j P(X = x_i, Y = y_j)$$

Case 2: When X,Y are continuous random variables, then

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy$$

Conditional Expected Values

$$E(X/Y) = \int_{-\infty}^{\infty} x f(x/y) dx$$

$$E(Y/X) = \int_{-\infty}^{\infty} y f(y/x) dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy$$

Conditional Variance: If (X,Y) is a two-dimensional random variable, then the conditional variance of (X,Y) is

$$\text{Var}(Y/X) = E(Y^2/X) - [E(Y/X)]^2$$

$$\text{Var}(X/Y) = E(X^2/Y) - [E(X/Y)]^2$$

Notes: If X and Y are independent random variables, then

$$E(X/Y) = E(X)$$

$$E(Y/X) = E(Y)$$

$$\mathrm{E}[E(Y/X)] = E(Y)$$

$$\mathrm{E}[E(X/Y)] = E(X)$$

Example 1: The joint distribution of X and Y is given by

$$f(x,y)=\frac{x+y}{21},\quad x=1,2,3,\quad y=1,2$$

Find the marginal distributions of X and Y. Find the mean of X and Y also.

Solution:

$X \backslash Y$	1	2	$P(X=x_i)$	
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{5}{21}$	$P(X=1)$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{7}{21}$	$P(X=2)$
3	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{9}{21}$	$P(X=3)$
$P(Y=y_j)$	$\frac{9}{21}$	$\frac{12}{21}$	1	
	$P(Y=1)$	$P(Y=2)$		

The marginal distributions of X are

$$P(X=1)=\frac{5}{21},\ P(X=2)=\frac{7}{21},\ P(X=3)=\frac{9}{21}$$

The marginal distributions of Y are

$$P(Y=1)=\frac{9}{21},\ P(Y=2)=\frac{12}{21}$$

$$E(X)=\sum_{x=1}^{3} xP(X=x) = 1P(X=1)+2P(X=2)+3P(X=3)$$

$$=\frac{5}{21}+\frac{14}{21}+\frac{27}{21}=\frac{46}{21}$$

$$E(Y)=\sum_{y=1}^{2} yP(Y=y) = 1P(Y=1)+2P(Y=2)$$

$$=\frac{9}{21}+\frac{24}{21}=\frac{33}{21}=\frac{11}{7}$$

Example 2: The joint PDF of (X,Y) is given by

$$f(x,y)=\begin{cases}24xy, 0<x, 0<y, x+y\le 1\\ 0 \qquad otherwise\end{cases}$$

Find the conditional mean and variance of Y given X.

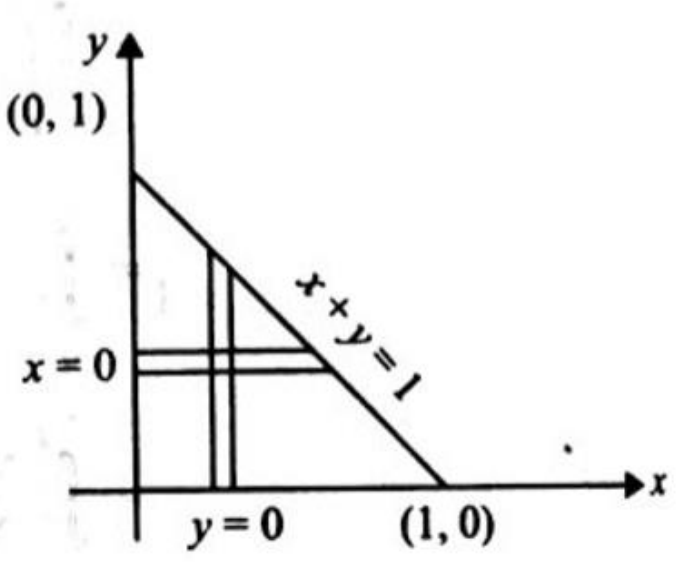
Solution:

Given: $f(x, y) = 24xy, \quad x > 0, y > 0, x + y \le 1$

$$\therefore \quad f_X(x) = \int_0^{1-x} 24xy\, dy = 24x \int_0^{1-x} y\, dy,$$

$$= 24x\left[\frac{y^2}{2}\right]_0^{1-x} = 24x\frac{(1-x)^2}{2}$$

$$= 12x\,(1-x)^2, \qquad 0 < x < 1$$



$$f(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{(1-x)^2}, \qquad 0 < y < 1 - x$$

$$E(Y/X) = \int_0^{1-x} y f(y/x)\,dy$$

$$= \int_0^{1-x} \frac{2y^2}{(1-x)^2}\,dy = \frac{2}{(1-x)^2}\left[\frac{y^3}{3}\right]_0^{1-x} = \frac{2}{3}(1-x), \qquad x > 0$$

$$E(Y^2/X = x) = \int_0^{1-x} y^2 f(y/x)\,dy$$

$$= \int_0^{1-x} y^2 \frac{2y}{(1-x)^2}\,dy = \frac{2}{(1-x)^2}\left[\frac{y^4}{4}\right]_0^{1-x} = \frac{1}{2}(1-x)^2, \qquad x > 0$$

$$\mathrm{Var}(Y/X) = E(Y^2/X) - [E(Y/X)]^2$$

$$= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2 = \frac{1}{18}(1-x)^2, \qquad x > 0$$

THANK YOU