Probability and Random Processes (15B11MA301)

Lecture-17

Department of Mathematics

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Course Content

- Uniform Distribution
- Exponential Distributions
- Statistical Properties of Uniform and Exponential Distribution
- Solved Examples
- Practice Questions

Continuous Distributions

(i) Uniform distribution

If a random variable X is uniformly distributed over an interval (a, b), then its probability density function (pdf) $f(x)$ is constant in that interval, i.e.,

$$f(x) = \begin{cases} k(\text{constant}) \text{ when } a < x < b. \\ 0, \qquad \text{otherwise.} \end{cases}$$

As $f(x)$ is a valid probabilty density function (pdf), therefore

$$\int_a^b f(x)dx = 1.$$

$$\Rightarrow k\int_a^b 1.dx = k(b-a) \text{ or } k = \frac{1}{b-a}.$$

Hence, $$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

Example 1: If a random variable X is uniformly distributed over the interval [0, 5]. compute the following:

(i) $P(1< X <3)$ (ii) $P(X < 4)$ (iii) $P(X >1)$

(iv) $P(2 < X < 5/X< 5)$ (v) $P(0 < X < 5/X >1)$.

Solution

Here, $a = 0, b = 5\ f(x) = \frac{1}{5},\ 0 \le x \le 5 \quad \left(\because f(x) = \frac{1}{b-a} \right)$

$$(i) \quad P(1 < X < 3) = \int_1^3 f(x)dx = \int_1^3 \frac{1}{5} dx = 2/5 = 0.40$$

$$(ii) \quad P(X < 4) = \int_0^4 f(x)dx = \int_0^4 \frac{1}{5} dx = 4/5 = 0.80$$

$$(iii) \quad P(X > 1) = \int_1^5 f(x)dx = \int_1^5 \frac{1}{5} dx = 4/5 = 0.80$$

$$(iv) \quad P(2 < X < 5 / X < 5) = \frac{P(2 < X < 5)}{P(X < 5)} = \frac{3/5}{1} = 0.60$$

$$(v) \quad P(0 < X < 4 / X > 1) = \frac{P(1 < X < 4)}{P(X > 1)} = \frac{3/5}{4/5} = 0.75.$$

Mean and Variance of Uniform Distribution

(i) Mean of $X = E(X) = \int_a^b xf(x)dx = \int_a^b x\frac{1}{b-a}dx$

$$= \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{a+b}{2}.$$

(ii)Variance of $X = Var(X) = E(X^2) - E(X)^2$

We have $E(X) = \dfrac{a+b}{2}$ and

$$E(X^2) = \int_a^b x^2 f(x)dx = \frac{1}{3(b-a)}\left[x^3\right]_a^b$$

$$= \frac{\left(b^3 - a^3\right)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}.$$

$$Var(X) = \frac{(a-b)^2}{12}$$

Moment Generating Function (MGF) of Uniform Distribution

- Let X be a uniformly distributed random variable in the interval $[a, b]$, then its moment generating function is defined by

$$M_X(t) = E(e^{tX}) = \int_a^b f(x)e^{tx}dx$$

$$= \frac{1}{b-a}\int_a^b e^{tx}dx = \frac{1}{b-a}\left(\frac{e^{tx}}{t}\right)_a^b$$

$$= \frac{1}{b-a}\left(\frac{e^{tb}}{t} - \frac{e^{ta}}{t}\right) = \frac{e^{tb}-e^{ta}}{(b-a)t}.$$

$$= \frac{1}{(b-a)}\sum_{r=0}^{\infty}\frac{t^r}{(r+1)!}(b^{r+1}-a^{r+1}).$$

- Moments from MGF of Uniform Distribution

The rth moment of X about origin = Coefficient of $\dfrac{t^r}{r!}$ in $M_X(t)$.

That is,we have $E(X^r) = \dfrac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}$.

$\therefore$ Mean $= E(X) = \dfrac{b^2 - a^2}{(b-a)(1+1)} = (b+a)/2.$

$$E(X^2) = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}.$$

$\therefore$ Variance(X) $= E(X^2) - \{E(X)\}^2 = \dfrac{(b-a)^2}{12}$.

- The Charateristic function of Uniform Distribution

$$\phi_X(\omega) = E(e^{i\omega X}) = \frac{1}{b-a}\int_a^b e^{i\omega x}dx = \frac{e^{i\omega b} - e^{i\omega a}}{i\omega(b-a)}.$$

Example 2: Find the characteristic function of a random variable X which is uniformly distributed over the interval [1, 6] and hence find the first three moments about origin form it.

Solution: Here $a = 1, b = 6$ and $f(x) = \frac{1}{b-a} = \frac{1}{5}$

$$\phi_X(w) = E(e^{i\omega x}) = \int_1^6 f(x) e^{i\omega x} dx$$

$$= \frac{1}{5}\int_1^6 e^{i\omega x} dx = \frac{e^{i\omega 6} - e^{i\omega}}{5 i\omega}.$$

$$= \frac{1}{5}\sum_{r=0}^{\infty} \frac{(i\omega)^r}{(r+1)!}(6^{r+1} - 1).$$

$E(X^r) =$ coefficient of $\frac{(i\omega)^r}{r!} = \frac{6^{r+1}-1}{r+1}$

Putting , $r = 1, 2, 3,$ we obtain

$E(X) = 3.50,\ E(X^2) = 215/15 = 14.33$ and $E(X^3) = 64.75.$

(ii) Exponential Distribution

- A random variable X is said to be exponentially distributed if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, \text{ if } x \geq 0 \\ 0 \qquad \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a parameter.

- Examples. Life time of a machine or its components; Arrival times of telephone calls or arrival times of train at a railway station etc.
- If the occurrences of events over nonoverlapping intervals are independent, then the waiting time distribution of these events can be shown to be exponential

Mean and Variance of Exponential Distribution

(i) Mean of $X = E(X) = \int_0^{\infty} x\lambda e^{-\lambda x} dx$

$$= \lambda\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - \left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} = \frac{1}{\lambda}.$$

(ii)Variance of $X = Var(X) = E(X^2) - \{E(X)\}^2$

$$E(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$$= \lambda\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right]_0^{\infty} = \frac{2}{\lambda^2}.$$

$$\therefore \quad Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

The rth Moment About Origin

$$\mu_r' = E(X^r) = \int_0^\infty x^r \lambda e^{-\lambda x} dx$$

Put $\lambda x = z$, we obtain

$$\mu_r' = E(X^r) = \frac{1}{\lambda^r}\int_0^\infty z^{(r+1)-1} e^{-z} dz = \frac{\Gamma(r+1)}{\lambda^r}.$$

[Use the definition of Gamma function, i.e., $\int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n)$]

- On putting $r = 1,\ 2,\ 3,\ \ldots$, all the moments about origin can be obtained.

Moment Generating Function (MGF) of Exponential Distribution

- Let X be an exponential random variable, then its moment generating function is defined by

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}(\lambda e^{-\lambda x})dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda - t)}dx = \lambda \left(\frac{e^{-x(\lambda-t)}}{t-\lambda} \right)_0^\infty$$

$$= \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} = 1 + \frac{t}{\lambda} + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^2 + \ldots.$$

The rth moment of X about origin = Coefficient of $\dfrac{t^r}{r!}$ in $M_X(t)$.

$$\therefore \quad \mu_r' = E(X^r) = \frac{\lfloor r}{\lambda^r}, \quad r = 1, 2, 3, \ldots .$$

- The Charateristic function of Exponential Distribution

$$\phi_X(\omega) = E(e^{i\omega X}) == \int_0^{\infty} e^{i\omega x}(\lambda e^{-\lambda x})dx$$

$$= \lambda \int_0^{\infty} e^{-x(\lambda - i\omega)} dx = \lambda \left(\frac{e^{-x(\lambda-i\omega)}}{i\omega - \lambda} \right)_0^{\infty}$$

$$= \frac{\lambda}{\lambda - i\omega} = \left(1 - \frac{i\omega}{\lambda}\right)^{-1}$$

$$= 1 + \frac{i\omega}{\lambda} + \left(\frac{i\omega}{\lambda}\right)^2 + \left(\frac{i\omega}{\lambda}\right)^2 + \ldots + \left(\frac{i\omega}{\lambda}\right)^r + \ldots.$$

The rth moment of X about origin can be obtained as:

the coefficient of $\dfrac{(i\omega)^r}{r!}$ in $\phi_X(\omega)$ or $\dfrac{1}{i^r}\left(\dfrac{d^r}{d\omega^r}(\phi_X(\omega))\right)_{\omega=0}$

$$\therefore \mu_r' = E(X^r) = \frac{\lfloor r}{\lambda^r}, r = 1,2,3,\ldots.$$

Example 3: Let a random variable X follows an exponential distribution with mean of 20. Find (i) $P(X < 30)$ (ii) $P(X < 10/ X > 1)$.

Solution: (i) We have mean $= E(X) = \dfrac{1}{\lambda} = 20, \Rightarrow \lambda = 1/20 = 0.05$

$$P(X < 30) = \frac{1}{20}\int_0^{30} e^{-x/20} dx = \frac{1}{20}\left(\frac{e^{-x/20}}{-1/20}\right)_0^{30} = 1 - e^{-1.5}.$$

$$(ii)\ P(X < 10/X > 1) = \frac{P(1 < X < 10)}{P(X > 1)} = \frac{1}{20}\int_1^{10} e^{-x/20} dx$$

$$P(1 < X < 10) = \frac{1}{20}\left(\frac{e^{-x/20}}{-1/20}\right)_1^{10} = e^{-0.05} - e^{-0.5}.$$

Memoryless Property of Exponential Distribution

If X is exponentially distributed, then for any $s, t > 0$

$$P(X > s + t \,/\, X > s) = P(X > t).$$

Proof: We have, for any $k > 0$

$$P(X > k) = \int_k^\infty \lambda e^{-\lambda x} dx = \left(-e^{-\lambda x}\right)_k^\infty = e^{-\lambda k}.$$

$$\therefore P(X > s + t \,/\, X > s) = \frac{P\{X > s + t \text{ and } X > s\}}{P\{X > s\}}$$

$$= \frac{P(X > s + t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

Example 4: The time X (in hours) to repair a mobile is exponentially distributed with mean 1/10. Find the probability that (i) the repair time is less than 30 minutes, (ii) the repair time is between 1 to 2 hours (iii) a repair time is greater than 5 hours given that its duration exceeds 3 hours.

Solution: We have mean $= E(X) = \dfrac{1}{\lambda} = \dfrac{1}{10}, \Rightarrow \lambda = 10$

$$(i)\, P(X > 0.5\,) = 10\int_{0.5}^{\infty} e^{-10x}dx = 10\left(\frac{e^{-10x}}{-10}\right)_{0.5}^{\infty} = e^{-5}.$$

$$(ii)\ P(1 < X < 2) = 10\int_{1}^{2} e^{-10x}dx = e^{-10} - e^{-20}.$$

$$(iii)\ P(X > 5 / X > 3) = P(X > 3 + 2 / X > 3)$$

$$= P(X > 2) = 10\int_{2}^{\infty} e^{-10x}dx = e^{-20}.$$

Practice Questions

1. If X is uniformly distributed over the interval (-3, 3), then evaluate the following:

   (*i*) $P(X > 1)$, (*ii*) $P(0< X<3)$, (*iii*) Var(X), $(iv) P(|X| > 2)$.

   [Ans: (i) 1/3, (ii) 1/2 , (iii) 3, (iv) 1/3]

2. The time in hours required to repair a motor is exponentially distributed with parameter $\lambda = 1/3$. What is the probability that (i) the repair time exceeds 3 hours (ii) a repair takes 10 hours given that its duration exceeds 7 hours. [Ans: (i) 1/e (ii) 1/e]

3. If X follows exponential distribution with $P(X < 1) = P(X > 1)$, find the mean and variance of X.

   [Ans: Mean =1.45,Var(X) = 2.08]

References/Further Reading

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