Probability and Random Processes (15B11MA301)

Lecture-6

(Course content covered: One dimensional discrete random variable)

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Random Variable

- A random variable X is a function that assigns a real number to each outcome of the sample space S, i.e., $X: S \rightarrow R$ is a mapping from the sample space S to the set of real numbers R. Obviously, a random variable is a deterministic function.
- The sample space S is the domain of the random variable and the set of all values taken on by the random variable is the range of the random variable X.
- A random variable is usually denoted by an uppercase letter such as X and the corresponding lowercase letter, such as x, denotes a possible value of the random variable X.
- A random variable may assumes values from a countable or uncountable set.
- If only one characteristics is considered corresponding to the sample space of a random experiment, then the random variable is called one dimensional random variable.

Random Variable

- More than one characteristics may be considered corresponding to the sample space of a random experiment, e.g., suppose that S consists of a large group of students of a college. Let the experiment consist of choosing a student at random. Let X denotes the weight of the student and Y denotes the students' height. Then (X, Y) is a two dimensional random variable. The same idea may be extended further.

Examples

(i) The "number of heads obtained" in an experiment of tossing three unbiased coins simultaneously, so the random variable X can take values 0, 1, 2 and 3, i.e.,

Domain of $X = S = \{TTT, TTH, THT, HTT, HHT, HTH, THH, HHH\}$,

and

Range of $X = \{0, 1, 2, 3\}$

(i) The 'temperature gained' by a fan after one hour of operation.

Discrete Random Variable

- A random variable which can assume some specific values is called a discrete random variable.
- The main characteristic of a discrete random variable is that the set of possible values in the range can all be listed and the list may be a finite list or a countably infinite list.

Examples

- Number of heads obtained when 10 coins are tossed.
- Number of phone calls received, per day at a telephone booth,
- number of sixes obtained when a pair of dice are thrown.
- Number of spade cards when 4 cards are chosen from a well shuffled pack of 52 playing cards.
- The number of odd numbers selected out of the set of positive integers.

Continuous Random Variable

- A random variable X which can take any value in an interval of real numbers is said to be a continuous random variable.
- The range of X can take infinitely many real values within one or more intervals of real numbers.
- The set of all possible values in the range cannot be listed in case of a continuous variable as the list is uncountably infinite.

Examples

- The duration of a phone call received.
- The time to failure of a machine.
- The temperature gained by an electric motor after one hour of operation.
- The amount of rainfall in a day.

Probability Mass Function

Let a discrete random variable X takes the values $x_1, x_2, x_3, \ldots, x_n$, then the probability function or the probability mass function (PMF) of X is denoted by

$$f(x_i) = P(X = x_i) = p_i \text{; for } i = 2, 3, \ldots, n,$$

i.e.,

$$\begin{array}{llllll} X: & x_1 & x_2 & x_3 & x_4 \ldots & x_n \\ P(X = x_i): & p_1 & p_2 & p_3 & p_4 \ldots & p_n \end{array}$$

such that

(i) $P(X = x_i) \geq 0;$

(ii) $\sum_{x_i} P(X = x_i) = p_1 + p_2 + p_3 + p_4 + \ldots + p_n = 1.$

Probability Distribution

If p_i represents the probability corresponding to $X = x_i$, for $i=1$, 2, 3,…,n; then the collection of pairs $(x_i\ ,\ p_i)$ is called the probability distribution of the discrete random variable X.

Example. If a pair of coins is tossed and random variable 'X' is 'Number of heads', then its probability distribution is:

$(X=x_i)$:	0	1	2
$P(X=x_i) = p_i$:	1/4	2/4	1/4

Example. Let X represents the number of heads when three fair coins are tossed. Find

(a) the probability distribution of the number of heads,

(b) $P(0 < X < 3)$, (c) $P(X > 1)$, (d) $P(X < 2)$.

88 Sol.: S = {TTT, TTH,THT, HTT,HHT,HTH,THH,HHH}

(a) The probability distribution is as follows:

X (Number of Heads):	0	1	2	3
Probability:	1/8	3/8	3/8	1/8

(b) $P(0 < X < 3) = P(X = 1) + P(X = 2) = 6/8 = 0.75$,

(c) $P(X > 1) = P(X = 2) + P(X = 3) = 4/8 = 0.50$,

(d) $P(X < 2) = P(X = 0) + P(X = 1) = 0.50$.

Cumulative Distribution Function

- Cumulative distribution function (CDF) of a discrete random variable X is denoted by $F(x)$. It is defined as follows:

$$F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i).$$

- Properties of cumulative distribution function (CDF)

$$
\begin{array}{ll}
(i) & 0 \le F(x) \le 1, \\
(ii) & F(-\infty) = 0, \\
(iii) & F(\infty) = 1, \\
(iv) & \text{If } x_1 < x_2, \text{ then } F(x_1) < F(x_2), \\
(v) & P(X = x_i) = F(x_i) - F(x_{i-1}).
\end{array}
$$

Example. The probability mass function of a random variable is as follows:

X	1	2	3	4	5	6
P	k	$2k/3$	$3k$	1/3	$k/3$	1/6

Determine the following:

(a) value of k (b) $F(4)$ (c) $F(6)$ (d) $P(X=3)$

Solution:

(a) Since $k + (2k/3) + 3k + (1/3) + (k/3) + (1/6) = 1$, so
$k = 1/10 = 0.1$,

(b) $F(4) = P(X \leq 4) = P(1) + P(2) + P(3) + P(4) = 0.8$,

(c) $F(6) = P(X \leq 6) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$,

(d) $P(X = 3) = 3k = 0.3$, or $P(X = 3) = F(3) - F(2) = 0.3$.

Practice Questions

1. The probability mass function of a random variable X is as follows:

X	1	2	3	4	5	6
P	a	$a/2$	a	$a/3$	$2a/3$	$a/2$

Determine the following:

(i) value of a (ii) $F(4)$ (iii) $F(6)$ (iv) $P(X = 4)$.

[Ans: (i) ¼ (ii) 17/24 (iii) 1 (iv) 1/12]

2. A pair of fair dice is thrown. Let X be the number of sixes appearing. Find (i) the probability distribution of X, (ii) $P(X < 2)$, (iii) $P(X < 1)$, (iv) $F(x)$.

[Ans: (ii)35/36 (iii) 25/36]

References/Further Reading

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