Probability and Random Processes (15B11MA301)

Lecture-18

Department of Mathematics

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Course Content

- Earlang or Generalized Gamma Distribution
- Weibull distribution
- Statistical properties of Earlang and Weibull distribution
- Solved examples
- Practice Questions
- References

Continuous Distributions

(iii) Erlang or Generalized Gamma Distribution

A random variable X is said to have Erlang or generalized Gamma distribution with positive parameters λ and k when its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda^x}}{\lceil k}, x \geq 0 \\ 0, \text{ otherwise.} \end{cases}$$

Since $\int_0^{\infty} f(x)dx = 1,$ therefore $f(x)$ is a valid probability density function.

• On putting λ=1 Erlang distribution is called Gamma distribution and its pdf is given by

$$f(x) = \begin{cases} \dfrac{x^{k-1}e^{-x}}{\Gamma k}, x \geq 0 \\ 0, \text{ otherwise.} \end{cases}$$

• On putting $k = 1$ we obtain exponential distribution $f(x) = \lambda e^{-\lambda x}, x \geq 0.$

Mean and Variance of Erlang Distribution

$$E(X^r) = \int_0^\infty x^r \left(\frac{\lambda^k x^{k-1} e^{-\lambda^x}}{\Gamma k} \right) dx = \int_0^\infty \frac{\lambda^k x^{k+r-1} e^{-\lambda^x}}{\Gamma k} dx$$

Putting $\lambda x = t,$ we obtain

$$E(X^r) = \frac{\lambda^k}{(\Gamma k)} . \frac{1}{\lambda^{k+r}} \int_0^\infty t^{k+r-1} e^{-t} dt = \frac{1}{\lambda^r} . \frac{\Gamma k + r}{\Gamma k}$$

Use $\int_0^\infty t^{n-1}e^{-t}dt = \lceil n = (n-1)\lceil (n-1)$.

- Mean of $X = E(X) = \frac{1}{\lambda}.\frac{\lceil k+1}{\lceil k} = \frac{k}{\lambda}$.
- Variance of $X = E(X^2) - \{E(X)\}^2$

$$= \frac{1}{\lambda^2}.\frac{\lceil k+2}{\lceil k} - \left(\frac{k}{\lambda}\right)^2 = \frac{k}{\lambda^2}.$$

Moment Generating Function of Erlang Distribution

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\left(\frac{\lambda^k x^{k-1}e^{-\lambda^x}}{\lceil k}\right)dx$$

$$= \int_0^\infty \frac{\lambda^k x^{k-1}e^{tx-\lambda^x}}{\lceil k}dx$$

Putting $\lambda(x-t)=z,$ we obtain

$$M_X(t)=\frac{\lambda^k}{(\overline{|k})}\cdot\frac{1}{(\lambda-t)^k}\int_0^\infty z^{k-1}e^{-z}dt$$

$$=\frac{\lambda^k}{(\lambda-t)^k}=\left(1-\frac{t}{\lambda}\right)^{-k}.$$

Additive or Reproductive Property of Erlang Distribution

If $X_1, X_2, X_3, \ldots, X_n$ are n independent Erlang variables with parameters $k_1, k_2, k_3, \ldots, k_n$ and same λ then Z= $X_1+X_2+X_3+\ldots+X_n$ is also Erlang variable with parameters $k_1+k_2+k_3+\ldots+k_n$ and λ.

- If $M_{X_i}(t)$ be the MGF of the random variable X_i for $i = 1,2,\ldots,n$.

Then by the property of MGF, the moment generating function $M_Z(t)$ of the random variable $Z = X_1 + X_2 + X_3 + \ldots + X_n$ will be given by

$$M_Z(t) = M_{X_1}(t).M_{X_2}(t).M_{X_3}(t)\ldots.M_{X_n}(t)$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-k_1}.\left(1 - \frac{t}{\lambda}\right)^{-k}\ldots\left(1 - \frac{t}{\lambda}\right)^{-k_n}$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-(k_1+k_2+k_3+\ldots+k_n)}.$$

$\therefore Z = X_1 + X_2 + X_3 + \ldots + X_n$ is also an Erlang variate with parameters$k_1 + k_2 + k_3 + \ldots + k_n$ and λ.

Example 1: The daily consumption of electric power (in thousands megawatt-hours) of a city follows Erlang distribution with parameters

$\lambda = \frac{1}{3}$ and $k = 2$. If power plant of the city has a capacity of 18 thousands megawattt-hours, what is the probability that

(i) the power supply will be inadequate on any day?

(ii) the power consumption is up to 12 thousands megawatt-hours in any day.

Solution: Here $\lambda = \frac{1}{3}, k = 2$ and then pdf will be

$$f(x) = \frac{(1/3)^2 x^{2-1} e^{-x/3}}{\Gamma 2} = \frac{1}{9} x e^{-x/3}, x \geq 0.$$

(i)The power supply will be inadequate on any day if the demand is above the capacity of 18 thousands megawatt − hours, i.e.,

$$P(X > 18) = \int_{18}^{\infty} \frac{1}{9} x e^{-x/3} dx = \frac{1}{9}\left(x\left(-3e^{-x/3}\right) - 9e^{-x/3}\right)_{18}^{\infty} = 7/e^{6}.$$

$$(ii) P(X \leq 12) = \int_{0}^{12} \frac{1}{9} x e^{-x/3} dx = \frac{1}{9}\left(x\left(-3e^{-x/3}\right) - 9e^{-x/3}\right)_{0}^{12}$$

$$= \frac{1}{9}[(-36e^{-4} - 9e^{-4}) - (-9)] = 1 - \frac{5}{e^{4}}.$$

Example 2: Find the first three moments about the origin of the Erlang distribution from its moment generating function.

Solution: The moment generating function of the Erlang distribution is

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-k}. \text{Use} E(X^r) = \left(\frac{d^r}{dt^r} M_X(t)\right)_{t=0}, r = 1, 2, 3.$$

(iv) Weibull Distribution

A random variable X is said to follow a Weibull distribution with parameters $\alpha > 0, \beta > 0$ if the variable $Y = \alpha X^{\beta}$ follows an exponential distribution with density function

$$f(y) = e^{-y}, y > 0$$

Then, the probabilty density function of X is given by

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, x > 0.$$

On putting $\beta = 1$ in the Weibull distribution, we obtain

$$f(x) = \alpha e^{-\alpha x}, x > 0,$$

which is the exponential distribution with parameter α.

The rth Moment About Origin of the Weibull Distribution

$$\mu_r{}' = E(X^r) = \int_0^\infty x^r(\alpha\beta x^{\beta-1}e^{-\alpha x^\beta})dx$$

$$= \alpha\beta\int_0^\infty x^{r+\beta-1}e^{-\alpha x^\beta}dx$$

Put $\alpha x^\beta = y$ or $x = \left(\frac{y}{\alpha}\right)^{1/\beta}$

$\Rightarrow dx = \left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}-1} dy,$ then we get

$$\mu_r{}' = E(X^r) = \alpha^{-r/\beta}\int_0^\infty \left(\frac{y}{\alpha}\right)^{\left(\frac{r}{\beta}+1\right)-1} e^{-y}dy = \alpha^{-r/\beta}\Gamma\left(\frac{r}{\beta}+1\right).$$

The Mean and Variance of the Weibull Distribution

- Mean of $X = E(X) = \alpha^{-1/\beta}\,\Gamma\left(\frac{1}{\beta}+1\right)$,

  $E(X^2) = \alpha^{-2/\beta}\,\Gamma\left(\frac{2}{\beta}+1\right)$

- Varianceof $X = E(X^2) - \{E(X)\}^2$

$$= \alpha^{-2/\beta}\left\{\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2\right\}.$$

Example 3: The life (in years) of a Motor bike of certain brand follows Weibull distribution with parameter α=1/100. If the probability that the life of the bike exceeds 10 years is e^{-10}, find the value of the parameter β. Also find the probability that the life of the bike is at most 5 years.

Solution: Here $\alpha = 1/100$, and the probabilty density function of X is $f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$, So we have

$$f(x) = \frac{\beta}{100}x^{\beta-1}e^{-\frac{x^{\beta}}{100}}, x > 0.$$

$$(i) P(X > 10) = \int_{10}^{\infty} \frac{\beta}{100}x^{\beta-1}e^{-\frac{x^{\beta}}{100}}dx$$

Put $\frac{x^{\beta}}{100} = z \Rightarrow \frac{\beta}{100}x^{\beta-1}dx = dz$ and then, we get

$$P(X > 10) = \int_{\frac{10^{\beta}}{100}}^{\infty} e^{-z}dz = (-e^{-z})_{10^{\beta-2}}^{\infty} = e^{-10^{\beta-2}}$$

$$\therefore e^{-10^{\beta-2}} = e^{-10} \Rightarrow \beta - 2 = 1 \quad \text{or} \quad \beta = 3.$$

The probabilty that the life of the bike is at most 5 years is

$$P(X \leq 5) = \int_0^5 \frac{3}{100} x^2 e^{-\frac{x^3}{100}} dx \; (\because \beta = 3)$$

$$= \int_0^{1.25} e^{-z} dz = (-e^{-z})_0^{1.25} = 1 - e^{-1.25}.$$

Example 4: The lifetime of a machine is (in months) follows Weibull distribution with parameters $\alpha = 1/5$ and $\beta = 1/2$. What is the mean life of the machine?

[Hint: Use mean of $X = \alpha^{-1/\beta} \sqrt{\left(\frac{1}{\beta} + 1\right)}$, Ans: 50 months.]

Practice Questions

1. The daily consumption of milk in a town in excess of 20000L is approximately distributed as an Erlang variate with parameters $\lambda = 1/10000$ and $k = 2$. The town has a daily stock of 30000L. What is the probability that the stock is insufficient on a particular day? [Ans: 2/e]
2. If the service life, in hours, of a semiconductor is a RV having a Weibull distribution with the parameters $\alpha = 0.025$ and $\beta = 0.5$, (i) How long can such a semiconductor be expected to last? (ii) What is the probability that such a semiconductor will still be in operating condition after 4000h? [Ans:(i) 3200 (ii) 0.21]
3. The life X of a car of certain brand follows Weibull distribution with parameter $\beta=2$, If the probability that the life of the car exceeds 5 years is given to be $e^{-.25}$, find (i) the value of the parameter α (ii) expected life of the car (iii) the variance of X. [Ans: (i) $\alpha = 1/100$ (ii) 8.86 yrs (iii) 21.46]
4. Find the mean and variance of Gamma distribution from its characteristic function.

References/Further Reading

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