Probability and Random Processes (15B11MA301)

Lecture: 41

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Contents

Markov Chain

If $P\{X_n = a_n/X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots, X_0 = a_0\} = P\{X_n = a_n/X_{n-1} = a_{n-1}\}\ \forall\, n$, then the process $\{X_n\}, n = 0, 1, 2, \dots$ is called a Markov Chain i.e. a discrete parameter Markov Process is called a Markov Chain.

One-step Transition Probability- The conditional transition probability $P\{X_n = a_j/X_{n-1} = a_i\}$ is called the on e-step transition probability from state a_i to state $a_j$at the nth step and is denoted by $P_{ij}(n-1, n)$.

Homogeneous Markov Chain – If the one-step transition probability does not depend on the step i.e. $P_{ij}(n-1, n) = P_{ij}(m-1, m)$, the Markov chain is called a homogeneous Markov chain. That is, the chain is said to be stationary.

Transition Probability Matrix (TPM)– When the Markov chain is homogeneous, the one-step transition probability is denoted by P_{ij}. The matrix $P = (P_{ij})$is called the Transition Probability Matrix satisfying the conditions.

(i) $P_{ij} \geq 0, \ \forall \, i, j$ and

(ii) $\sum_i P_{ij} = 1 \ \ \forall \ \ j$

That is, the sum of the elements of any row of the TPM is 1.

n-Step Transition Probability- The conditional probability that the process is in the state a_j at step n, given that it was in state a_i at step 0, $P\{X_n = a_j/X_0 = a_i\}$ is called the n-step transition probability and is denoted by

$$P_{ij}{}^{(n)} = P\{X_n = a_j/X_0 = a_i\}$$

Note: $P_{ij}{}^{(1)} = P_{ij}$

Example- This example explain how the tpm is formed for a Markov chain.

Assume that a man is at an integral point of the x-axis between the origin and the point $x = 3$, He takes a unit step either to the right with prob. 0.7 or to the left with prob. 0.3, unless he is at the origin when he takes a step to the right to reach $x = 1$ or he is at the point $x = 3$, when he takes a step to the left to reach $x = 2$. The chain is called 'Random walk with reflecting barriers'.

The tpm is given below:

$$\begin{array}{cc} & \text{States of } X_n \\ & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \text{States of } X_{n-1} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note: p_{23} = the element in the 2^{nd} row, 3^{rd} column of this tpm = 0.7. This means that, if the process is at state 2 at step (n-1), the probability that it moves to state 3 at step n = 0.7, where n is any positive integer.

Probability Distribution of the Process- If the probability that the process in the state a_i is $p_i (i = 1, 2, \dots, n)$ at any arbitrary step, then the row vector $P = (p_1, p_2, \dots, p_n)$ is called the probability distribution of the process at that time.

In particular, $P^{(0)} = ({p_1}^{(0)}, {p_2}^{(0)}, \dots, {p_n}^{(0)})$ is the initial probability distribution,

where ${p_1}^{(0)} + {p_2}^{(0)} + \dots + {p_n}^{(0)} = 1$.

The nth step probability distribution of the Markov chain is given by $P^{(n)}$ and is computed from the initial probability distribution $P^{(0)}$ and the TPM *P*.

i.e. $P^{(1)} = P^{(0)} P$, $P^{(2)} = P^{(1)} P, \dots,$ $P^{(n)} = P^{(n-1)} P$

Chapman-Kolmogrov Theorem

If P is the TPM of a homogenous Markov chain, then n-step TPM $P^{(n)}$ is equal to P^n, i.e.

$$[p_{ij}{}^{(n)}] = [p_{ij}]^n$$

Example- Consider the problem of Random walk with reflecting barriers, discussed previously, for which the tpm is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 0.51 & 0 & 0.49 \\ 0.09 & 0 & 0.91 & 0 \\ 0 & 0.3 & 0 & 0.7 \end{pmatrix}$$

From this matrix, we see that $p_{11}{}^{(2)} = 0.51$. This is so, because

$$p_{11}{}^{(2)} = p_{10}\,p_{01} + p_{11}P_{11} + p_{12}\,p_{21} + p_{13}\,p_{31}$$
$$= (0.3)\,(1) + (0)\,(0) + (0.7)\,(0.3) + (0)\,(0) = 0.51$$

Regular Matrix

A stochastic matrix P is said to be a regular matrix, if all the entries of P^m(for some positive integer m) are positive. A homogeneous Markov chain is said to be regular if its TPM is regular.

Theorems

1. If $p = \{p_i\}$ is the state probability distribution of the process at an arbitrary time, then that after one step is pP , where P is the TPM of the chain and that after n steps is pP^n.

2. *Steady-state Distribution-* If a homogenous Markov chain is regular, then every sequence of state probability distributions approaches a unique fixed probability distribution called the *stationary (state) distribution* or *steady-state distribution* of the Markov chain.

That is, $\lim_{n\to\infty}\{p^{(n)}\} = \pi$, where the state probability distribution at step n, $p^{(n)} = ({p_1}^{(n)}, {p_2}^{(n)}, \ldots, {p_k}^{(n)})$ and the stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ are the row vectors. Moreover, if P is the TPM of the regular chain, then $\pi P = \pi$ (π is a row vector). Using this property of π, it can be found out.

Example- Check whether $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$ is a probability vector.

Solution- Since

(i) all the elements are ≥ 0, and

(ii) the row sum is equal to 1, the given vector is a probability vector.

Example- Let A $= \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ be a stochastic matrix, check whether it is regular.

Solution- A matrix A is regular if all the entries of some power are > 0 i.e. all entries of A^n are positive for some n.

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Since all the entries in A^2 are positive (> 0), A is regular.

Example- At an intersection, a working traffic light will be out of order the next day with probability 0.07, and an out-of-order traffic light will be working the next day with probability 0.88. Let $X_n = 1$ if on day n, the traffic light will work, $X_n = 0$if on day n, the traffic light will not work. Is $\{X_n : n = 0, 1, 2, ...\}$ a Markov chain ? If so, write the transition probability matrix.

Solution- The traffic light will work on the next day depends on whether it work or not today.

Since, the states of X_n depend only on X_{n-1} but not on X_{n-2}, X_{n-3}, X_{n-4},… or earlier states $\{X_n : n = 0, 1, 2, ...\}$ is a Markov chain.

The required TPM is $$\mathrm{P} = \begin{bmatrix} 0.12 & 0.88 \\ 0.07 & 0.93 \end{bmatrix}$$

Find the steady state distribution of Markov chain whose TPM is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Solution:

If $\pi = (\pi_1, \pi_2)$ is the steady state distribution of the Markov chain, then

$$\pi P = \pi \quad \text{and} \quad \pi_1 + \pi_2 = 1$$

$$(\pi_1, \pi_2).\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2) \Rightarrow \frac{1}{2}\pi_2 = \pi_1 \Rightarrow \pi_2 = 2\pi_1.$$

Solving the above two expressions we get

$\pi_2 = \frac{2}{3}$ and $\pi_1 = \frac{1}{3}$. Hence, the steady state distribution is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

References

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