Probability and Random Processes (15B11MA301)

Lecture-7

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References

1.	**A. M. Mood, F. A. Graybill and D. C. Boes,** Introduction to the theory of statistics, 3rd Indian Ed., Mc Graw Hill, 1973.
2.	**R. V. Hogg and A. T. Craig,** Introduction to mathematical Statistics, Mc-Millan, 1995.
3.	**V. K. Rohatgi,** An Introduction to Probability Theory and Mathematical Statistics, Wiley Eastern, 1984.
4.	**S. M.Ross,** A First Course in Probability, 6th edition, Pearson Education Asia, 2002.
5	**S. Palaniammal,** Probability and Random Processes, PHI Learning Private Limited, 2012.
6	**P. L.Mayer,** Introductory Probability and Statistical Applications, Addison-Wesley, Second Edition, 1972.
7.	**R. E. Walpole, R H. Myers, S. L. Myers, and K. Ye,** Probability & Statistics for Engineers & Scientists, 9th edition, Pearson Education Limited, 2016.
8.	**I. Miller and M. Miller, John E. Freund's** Mathematical Statistics with Applications, 8th Edition, Pearson Education Limited 2014.

Course Content Covered

continuous random variable, cumulative distribution function, probability density function, mean, variance

Cumulative Distribution Function (CDF)

The *cumulative distribution function* of a random variable X, denoted by $F_X(.)$, is defined to be that function with domain the real line and counter-domain the interval [0, 1] which satisfies

$F_X(x) = P[X \leq x] = P[\{w: X(w) \leq x\}]$ for every real number x.

Cumulative Distribution Function (CDF)

Properties of CDF

1. $\lim_{x \to -\infty} F_X(x) = 0$
2. $\lim_{x \to \infty} F_X(x) = 1$
3. $F_X(a) \leq F_X(b)$ for all $a < b$. (monotone non-decreasing)
4. $\lim_{h \to 0^+} F_X(x+h) = F_X(x)$ (right continuity)

Second definition of CDF: Any *function* $F_X(.)$, with domain the real line and counter-domain the interval [0, 1] which satisfies above four properties is defined to be CDF.

Mean and Variance of discrete random variable

Mean: If X is a discrete random variable, then mean of X denoted by μ_X or E[X] (also read as expectation of X) is defined by

$$E[X] = \sum_{x_j} x_j f_X(x_j)$$

Variance: It is defined as follows:

$$E\left[\left(X-\mu_X\right)^2\right] = \sum_{x_j}\left(x_j-\mu_X\right)^2 f_X(x_j)$$

It is denoted by

$\sigma_X{}^2$, var$[X]$ or $E\left[\left(X-\mu_X\right)^2\right]$ (read as expectation of $X-\mu_X$ square)

Mean and Variance of discrete random variable

Example: Let X be the total of the two dice in the experiment of tossing two balanced dice. Find mean and variance of X.

S={11 12 13 14 15 16
21 22 23 24 25 26
31 32 33 34 35 36
41 42 43 44 45 46
51 52 53 54 55 56
61 62 63 64 65 66}

X(S)={2 3 4 5 6 7
3 4 5 6 7 8
4 5 6 7 8 9
5 6 7 8 9 10
6 7 8 9 10 11
7 8 9 10 11 12}
={2 3 4 5 6 7 8 9 10 11 12}

X	2	3	4	5	6	7	8	9	10	11	12
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Mean and Variance of discrete random variable

Example: Let X be the total of the two dice in the experiment of tossing two unbalanced dice. Find mean and variance of X.

X	2	3	4	5	6	7	8	9	10	11	12
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Mean: $E[X] = \sum_{x_j} x_j f_X(x_j) = 7$

$$\begin{aligned}\text{var}[X] &= \sum (x_j - \mu_X)^2 f_X(x_j) \\ &= (2-7)^2 \tfrac{1}{36} + (3-7)^2 \tfrac{2}{36} + (4-7)^2 \tfrac{3}{36} + (5-7)^2 \tfrac{4}{36} \\ &\quad + (6-7)^2 \tfrac{5}{36} + (7-7)^2 \tfrac{6}{36} + (8-7)^2 \tfrac{5}{36} + (9-7)^2 \tfrac{4}{36} \\ &\quad + (10-7)^2 \tfrac{3}{36} + (11-7)^2 \tfrac{2}{36} + (12-7)^2 \tfrac{1}{36} = \tfrac{210}{36}.\end{aligned}$$

PROBABILITY DENSITY FUNCTION (PDF)

Consider the small interval $\left(x-\dfrac{\Delta x}{2}, x+\dfrac{\Delta x}{2}\right)$ of length Δx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)dx$ represents the probability that x falls in the infinitesimal interval $\left(x-\dfrac{\Delta x}{2}, x+\dfrac{\Delta x}{2}\right)$, which is denoted by

$$P\left(x-\frac{\Delta x}{2} \leq x \leq x+\frac{\Delta x}{2}\right) = f(x)\, dx.$$

Let $f(x)dx$ represent the area bounded by the curve $y = f(x)$, x axis and the ordinates at the points $x-\dfrac{\Delta x}{2}$ and $x+\dfrac{\Delta x}{2}$. The function $f(x)$ so defined is known as probability density function or density function of the random variable X.

PROBABILITY DENSITY FUNCTION (PDF)

Mathematically, pdf $f(x)$of univariate random variable is a real valued function that satisfies the following properties:

1. $f(x) \geq 0 \ \forall \ x \in R$

2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Cumulative Distribution Function (CDF: $F_X(x)$)
X: continuous random variable

1. $F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(u)du, \quad -\infty < x < \infty$

2. $f_X(x) = \dfrac{dF_X(x)}{dx}$

$P[a \leq X \leq b] = P[a < X \leq b]$

$= P[a \leq X < b] = P[a < X < b] =$

3. $\int_{a}^{b} f_X(x)dx = F_X(b) - F_X(a)$

Mean and Variance of continuous random variable

Mean: If X is a continuous random variable, then mean of X denoted by μ_X or E[X] (also read as expectation of X) is defined by

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

Variance: It is defined as follows:

$$E\left[\left(X-\mu_X\right)^2\right] = \int_{-\infty}^{\infty} \left(x-\mu_X\right)^2 f_X(x)\,dx$$

It is denoted by

$\sigma_X{}^2$, $\mathrm{var}[X]$ or $E\left[\left(X-\mu_X\right)^2\right]$ (read as expectation of $X-\mu_X$ square)

Properties of Variance

If X is a random variable (discrete or continuous), then

1. $\mathrm{Var}[X] = E\left[\left(X-\mu_X\right)^2\right] = E[X^2] - \left(E[X]\right)^2$

Provided $E[X^2]$ exists.

2. $\mathrm{Var}[aX+b] = a^2\mathrm{Var}[X]$

Example Suppose that the error in the reaction temperature, in °*C,* for a controlled laboratory experiment is a continuous random variable *X* having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3} & \text{if} \quad -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that *f(x)* is a density function.

(b) Find $P(0 < X \leq 1)$

Solution (a) (i) $f(x) \geq 0$ for all x.

(ii) $$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{2} \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1.$$

(b) $$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3} dx = \left.\frac{x^3}{9}\right|_0^1 = \frac{1}{9}.$$

Example If a continuous random variable X having the probability density function

$$f(x)=\begin{cases}\dfrac{x^2}{3} & \text{if} \quad -1<x<2\\ 0 & \text{otherwise}\end{cases}$$

(a) Find CDF of X.

(b) Find $P(0<X\le 1)$ by using the CDF.

Solution (a) For $-1<x<2$,

$$F(x)=\int_{-\infty}^{x} f(t)\,dt=\int_{-1}^{x}\frac{t^2}{3}dt=\left.\frac{t^3}{9}\right|_{-1}^{x}=\frac{x^3+1}{9}.$$

So

$$F(x)=\begin{cases}0, & x<-1,\\ \frac{x^3+1}{9}, & -1\le x<2,\\ 1, & x\ge 2.\end{cases}$$

(b) $P(0<X\le 1)=F(1)-F(0)=\dfrac{2}{9}-\dfrac{1}{9}=\dfrac{1}{9},$

Example

The CDF of a random variable X is given by

$$F(x)=\begin{cases}0, & x<0\\ x^2, & 0\le x<\frac{1}{2}\\ 1-\frac{3}{25}(3-x)^2, & \frac{1}{2}\le x<3\\ 1, & x\ge 3\end{cases}$$

Find the PDF of X

Formula used

$$f(x)=\frac{d}{dx}[F(x)]=F'(x)$$

Solution

$$f(x)=\begin{cases}0, & x<0\\ 2x, & 0\le x<\frac{1}{2}\\ \frac{6}{25}(3-x), & \frac{1}{2}\le x<3\\ 0, & x\ge 3\end{cases}$$

For the triangular distribution

$$f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance.

Solution:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot x\, dx + \int_1^2 x(2-x)dx = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2(2-x)dx = \frac{7}{6}$$

$$Var(X) = E[X^2] - \left(E[X]\right)^2 = \frac{7}{6} - 1^2 = \frac{1}{6}$$

Practice Problem: If a continuous random variable X having the probability density function

$$f_2(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find CDF of X.

Hint

$$F(t) = \begin{cases} 0, & -\infty < t < 0 \\ \frac{t^2}{2}, & 0 \le t < 1 \\ -1 + 2t - \frac{t^2}{2}, & 1 \le t < 2 \\ 1, & t \ge 2. \end{cases}$$

Practice Problem:

A continuous random variable X is defined as

$f(x)=(ax+bx^2)\ I_{(0,1)}(x)$,

where I is the indicator function.

if E[X]=0.6, then find

(i) P[X<0.5],

(ii) variance of X.

a=3.6, b=-2.4, P=0.35, var=0.06

Thank You