Probability and Random Processes (15B11MA301)

Lecture-31

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Contents

RANDOM WALK

- Suppose we toss a fair coin every T seconds and instantly after each toss we move a distance d to the right if heads show and to the left if tails show. If the process starts at t = 0, our position at time t = nT is a random process/sequence $\{X(nT)\}$ called as a random walk.

- Suppose that r heads and (n-r) tails have occurred in the first n tosses of the coin. Then the random walk consists of r steps to the right and (n-r) steps to the left.

Therefore,

$$X(nT) = rd - (n-r)d$$
$$= (2r-n)d$$
$$= md \text{ (net gain)}, \quad \text{where } m = 2r - n$$

Now

$$P\{X(t)\} = md\} = P\,\{\text{object is at the point 'md' at stage 'n'}\}$$
$$= P\,\{\text{getting r heads in n tosses}\}$$

$$= nC_r \; p^r \; q^{n-r}$$

$$= nC_r \left(\frac{1}{2}\right)^n, \text{where } r = \frac{m+n}{2}, \text{since } 2r - n = m$$

$$= nC_{(m+n)/2} \left(\frac{1}{2}\right)^n$$

expressing as a sum,

$$X(nT) = X_1 + X_2 + \ldots + X_n$$

where X_i represents the distance moved in the ith step.

The random variables X_i are independen t, taking the values $\pm$d with equal probabilit y

$$\text{so, } E\{X(nT)\} = \sum_{i=0}^{n} E(X_i) = 0$$

$$E\{X^2(nT)\} = \sum_{i=1}^{n} E(X_i^2) = \sum_{i=0}^{n}\left(\frac{1}{2}\times d^2 + \frac{1}{2}\times d^2\right) = nd^2$$

Wiener Process

(As limiting form of random walk)

- A wiener process is the scaling limit of random walk. This means that if you take a random walk with very small steps, you get an approximation to a wiener process (or Brownian motion).

- In other words, as the step size d tends to 0 and the no. of steps (n) tends to infinity (and therefore T tends to 0), random walk converges to wiener process in an approximate sense.

When n is very large

limiting form of the binomial distribution with mean np and variance npq as $n \to \infty$ is the normal distribution $N(np, \sqrt{npq})$

$$nC_r\, p^r q^{n-r} \cong \frac{1}{\sqrt{2\pi npq}} e^{-(r-np)^2/2npq}$$

So, by putting $p = q = 1/2$ and $r = \dfrac{m+n}{2}$,

$$P\{X(nT) = md\} = \frac{1}{\sqrt{2\pi \frac{n}{4}}} e^{-m^2/2n} \qquad (1)$$

Now, put nT = t , md = x

$$E\left\{X^2(t)\right\} = n\mathrm{d}^2 = \frac{t}{T}\mathrm{d}^2$$

$$\because\ d \to 0 \ and\ \mathrm{T} \to 0 \ \therefore\ \mathrm{d}^2 = \alpha T$$

$$\therefore E\left\{X^2(t)\right\} = \frac{t}{T}\alpha T = \alpha t.$$

Also,

$$\text{variance}\, X(nT) = E\left\{X^2(nT)\right\} - [E\left\{X(nT)\right\}]^2$$

$$npq = E\left\{X^2(t)\right\}$$

$$\frac{n}{4} = \alpha t$$

Now,

$$\frac{\mathrm{m}^2}{2n} = \frac{x^2/d^2}{2t/T} = \frac{x^2}{2\alpha t}$$

using these in (1), we get pdf of Weiner process as

$$\mathrm{f}_{\mathrm{x(t)}}(x) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-x^2/2\alpha t}, \qquad -\infty < x < \infty$$

which is $\mathrm{N}(0, \sqrt{\alpha \mathrm{t}})$

Note:(i)

The random walk $\{X(nT)\}$ is a process with independent increments i.e. $\{X(n_2T) - X(n_1T)\}$ *and* $X(n_1T) - X(0)$ are independent .

and since Wiener process {X(t)} is the limiting form of random walk, $\{X(t_2) - X(t_1)\}$ and $X(t_1)$ are independent.

Note (ii) : Autocorrelation of Wiener process{X(t)} is given by

$$R(t_1, t_2) = \alpha \min(t_1, t_2)$$

for,

Let $t_1 < t_2$

$$E\left[\left\{X\left(t_2\right) - X\left(t_1\right)\right\} X\left(t_1\right)\right]$$

$$= E\{X\left(t_2\right) - X\left(t_1\right)\} E\left\{X\left(t_1\right)\right\} \qquad \text{(note(i))}$$

$$= 0 \qquad (E\left\{X\left(t\right)\right\} = 0)$$

so, $\quad E\{X(t_1) X(t_2)\} = E\{X^2(t_1)\} = \alpha t_1$

$$R(t_1, t_2) = \alpha t_1$$

similarly, when $t_2 < t_1$

$$R(t_1, t_2) = \alpha t_2$$

$$\therefore R(t_1, t_2) = \alpha \min(t_1, t_2)$$

$$\text{Note : Autocovariance } C(t_1, t_2) = R(t_1, t_2) - \mu(t_1)\mu(t_2)$$
$$= \alpha \min(t_1, t_2)$$

- **PRACTICE QUESTION:**

Ques 1:
Give the one- dimensional DENSITY FUNCTION OF WIENER PROCESS. WHAT ARE ITS MEAN AND VARIANCE.

References

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2. Ghahramani, S., Fundamentals of Probability with Stochastic Processes, Pearson, 2005.

3. Papoulis, A. and Pillai, S.U., Probability, Random Variables and Stochastic Processes, Tata McGraw-Hill, 2002.

4. Miller, S., Childers, D., Probability and Random Processes, Academic Press, 2012.