Geometric & Negative Binomial Distribution :

Discrete probability distribution

Content:

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Geometric Distribution

A random variable ***X*** is said to have a geometric distribution, if it assumes only non-negative values such that its probability mass function is given by

$$P(X = x) = q^{x-1}p; \quad x = 1, 2, \ldots$$

where p is the probability of success and q = 1-p.

Here $P(X = x) = q^{x-1}p$ gives the probability that the first success occurs only at the **(*x*)th** trial, the previous x-1 trial are failure.

Mean:

$$E(X) = \sum_{x=1}^{\infty} xP(X = x) = \sum_{x=1}^{\infty} xq^{x-1}p;$$

$$= p\sum_{x=1}^{\infty} xq^{x-1}$$

$$= p(1-q)^{-2}$$

$$= \frac{1}{p}$$

Second moment about origin:

$$E(X^2) = \sum_{x=1}^{\infty} x^2 P(X = x)$$

$$= \sum_{x=1}^{\infty} [x(x-1) + x] P(X = x)$$

$$= \sum_{x=1}^{\infty} [x(x-1) + x] P(X = x)$$

$$= \sum_{x=1}^{\infty} [x(x-1) + x] q^{x-1} p;$$

$$= pq \sum_{x=2}^{\infty} x(x-1) q^{x-2} + \frac{1}{p}$$

$$= 2pq(1-q)^{-3} + \frac{1}{p}$$

$$= \frac{2q}{p^2} + \frac{1}{p}$$

Variance:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{q}{p^2}$$

Moment Generating Function:

$$
\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \sum_{x=1}^{\infty} e^{tx} p(x) \\
&= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p \\
&= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} \\
&= pe^t [1 - qe^t]^{-1} \\
&= \frac{pe^t}{(1 - qe^t)}
\end{aligned}
$$

Memoryless Property of Geometric Distribution:

Prove that the geometric distribution possesses memoryless property.

Memoryless Property of Geometric Distribution:

Sol: We have to show that

$$P(X > s+t \,/\, X > t) = P(X > s)$$

Now,

$$P(X > k) = p\sum_{x=k+1}^{\infty} q^{x-1} = p(q^k + q^{k+1} + \ldots.)$$

$$= pq^k(1+q+q^2+\ldots)$$

$$= q^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$P(X > s+t \,/\, X > t) = \frac{P(X > s+t \cap X > t)}{P(X > t)}$$

$$= \frac{P(X > s+t)}{P(X > t)}$$

$$= \frac{q^{s+t}}{q^t} = q^s = P(X > s)$$

Thus geometric distribution possesses memoryless property.

Examples:

1. If the probability that the target is destroyed on any one shot is 0.5, what is the probability that it will be destroyed on the 6th attempt?

Ans: $(0.5)^6$

2. If the probability that an applicant for a driver's license will pass road test on any given trial is 0.8, what is the probability that he will finally pass the test.

(i) On the fourth trial,

(ii) Fewer than 4 trials?

Ans: 0.0064, 0.992

Negative Binomial Distribution:

A **negative binomial experiment** is a random experiment that has the following properties:

- The experiment consists of x repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.
- The probability of success, denoted by p, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.
- The experiment continues until r successes are observed, where r is specified in advance.

Negative Binomial Distribution:

- Suppose a negative binomial experiment where ***x*th** trial is ***r*th** success. If the probability of success on an individual trial is p, then the negative binomial probability is:

- $b^*(x; r, p) = {}_{x\text{-}1}C_{r\text{-}1} * p^r * (1 - p)^{x-r}$

$$where\ \ x = r, r+1, \ldots,$$

Moment Generating Function(MGF):

$$E(e^{tX}) = \sum_{x=r}^{\infty} e^{tx}\,{}^{x-1}C_{r-1}\,p^r q^{x-r}$$

$$= (pe^t)^r \sum_{x=r}^{\infty} \frac{(x-1)!}{(r-1)!(x-r)!}(e^t q)^{x-r}$$

$$= (pe^t)^r \sum_{y=0}^{\infty} \frac{(y+r-1)!}{(r-1)!(y)!}(e^t q)^k$$

$$= (pe^t)^r [1-(q)e^t]^{-r}$$

$$= \frac{(pe^t)^r}{(1-qe^t)^r}$$

FIRST MOMENT ABOUT ORIGIN:

$$E(X) = \left. \frac{dM_X(t)}{dt} \right|_{t=0}$$

$$= r\,\frac{(p^2 + pq)}{p^2}$$

$$= \frac{r}{p}$$

Negative Binomial Distribution:

- Mean = r/p
- Variance, Var(X) = rq/p^2
- MGF:

$$M_X(t) = \left(\frac{\mathrm{pe}^t}{1-\mathrm{qe}^t} \right)^r$$

Negative Binomial Distribution:

- Bob is a high school basketball player. He is a 70% free throw shooter. That means his probability of making a free throw is 0.70. During the season, what is the probability that Bob makes his third free throw on his fifth shot?

- What is the probability that Bob makes his first free throw on his fifth shot?

- **Ans**: 0.18522, 0.00567

References

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Thank You