# Probability and Random Processes (15B11MA301)

## Lecture-30

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# Contents

## Stationary Processes

**If certain probability distribution or averages do not depend on $\boldsymbol{t}$, then the random process $\{\boldsymbol{X(t)}\}$ is called stationary.**

Generally, stationarity refers to the degree to which the probabilistic model of the time-indexed random variables is "constant" with time.

First-Order Stationary Random process $\{X(t)\}$ is first-order stationary if the first order pdf does not depend on time:

$$f_{X(t)}(x) = f_X(x) \quad \forall\, t \in \mathcal{T}. \qquad (1)$$

<u>For this type of stationarity, we cannot say anything about moments across two or more time instants, such as correlation and covariance.</u>

Nth-Order Stationary Random process $\{X(t)\}$ is Nth-order stationary if the joint probability distribution of the process at N time instants does not change by any time shift '$\tau$' of all $\{t_1, t_2, \dots, t_N\}$:

$$f_{X(t_1),X(t_2),\dots X(t_N)}(x_1, x_2, \dots, x_N) = f_{X(t_1-\tau),X(t_2-\tau),\dots X(t_N-\tau)}(x_1, x_2, \dots, x_N) \qquad (2)$$

<u>This type of stationarity is obviously stronger than first-order stationarity.</u>

Strict Sense Stationary (SSS)Random process X(t) is strictly stationary if it is Nth-order stationary for all $N$ as $N \rightarrow \infty$.

This is the strongest form of stationarity: all statistics of the random process are constant for all time differences and all time shifts. **A strictly stationary process implies (2) which in turn implies (1).**

$$f(x,t) = f(x,t+h)$$

This is possible only when $f(x,t)$ is independent of $t$. Therefore, first-order densities (and hence distribution function) of a SSS process are independent of time.

$$E\{X(t)\} = \mu = constant$$

$$f(x_1,x_2,t_1,t_2) = f(x_1,x_2,t_1+h,t_2+h)$$

This is possible only if $f(x_1,x_2,t_1,t_2)$ is function of $\tau = t_1 - t_2$.

Therefore, second-order densities (and hence distribution functions) of a SSS process are functions of $\tau = t_1 - t_2$.

$R(t_1,t_2) = E\{X(t_1)X(t_2)\}$ is also a function of $\tau = t_1 - t_2$.

If $E\{X(t_1)\}$ is a constant and $R(t_1,t_2)$ is a function of $(t_1 - t_2)$, the random process $\{X(t)\}$ need not be a SSS process.

## Example of SSS Process

Let $\boldsymbol{X_n}$ denotes the presence or absence of a pulse at the nth time instant in a digital communication system or digital data processing system. If $\boldsymbol{P\{X_n = 1\} = p\ and\ P\{X_n = 0\} = 1 - p = q}$, then the random process (sequence) $\{\boldsymbol{X_n, n \geq 1}\}$, called the Bernoulli's Process, is a SSS process.

| $X_n = r$ | 1 | 0 |
|---|---|---|
| $P(X_n = r)$ | $p$ | $q$ |

| $X_r$ | $X_s$ | |
|---|---|---|
| | 1 | 0 |
| 1 | $p^2$ | $pq$ |
| 0 | $pq$ | $q^2$ |

It is interesting to observe that the mean and other statistical parameters in the adjoining example also remains the same for different values of parameter $n$. There are very few processes which satisfy this property.

The joint distribution is the same for the pair of members $X_r$ and $X_s$ and for the pair $X_{r+p}$ and $X_{s+p}$ of the process.

Similarly, the third order distribution of the process is same for $X_r, X_s, X_t$ and for$X_{r+p}, X_{s+p}, X_{t+p}$ of the processand so on, i.e. distributions of all orders are invariant under translation of time.

Two real-valued random processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the strict sense, if the joint distribution of $\{X(t)\}$ and $\{Y(t)\}$ are invariant under translation of time.

# The complex random process $\{Z(t)\}$, where $Z(t) = X(t) + iY(t)$, is said to be jointly stationary in the strict sense, if $\{X(t)\}$ and $\{Y(t)\}$, are jointly stationary in the strict sense.

Wide-sense stationarity (weakly stationary or covariance stationary process or WSS process)

A random process $\{X(t)\}$ with finite first and second-order moments is called a WSS process if its **mean is a constant and the autocorrelation depends only on the time difference**, i.e.,

$$E\{X(t)\} = \mu; \quad E\{X(t)X(t-\tau)\} = R(\tau)$$

A random process that is not stationary in any sense is called an evolutionary process.

Two random processes {X(t)} and {Y(t)} are said to be jointly stationary in the wide sense, if each process is individually a WSS process and $R_{XY}(t_1, t_2)$ is a function of $(t_1 - t_2)$ only.

Q. Prove that the random process $X(t) = A\cos(wt+\theta)$ is wide-sense stationary if it is assumed that $A$ and $w$ are constants and $\theta$ is uniformly distributed in $(0,2\pi)$.

$$E(X(t)) = \int_{-\infty}^{\infty} xf(x)dx.$$

for uniform distribution $f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$

$$E(X(t)) = \int_{0}^{2\pi} \frac{1}{2\pi} A\cos(wt+\theta)d\theta$$

$$= \frac{A}{2\pi}\int_{0}^{2\pi} A\cos(wt+\theta)d\theta = 0.$$

$$E(X(t_1)X(t_2)) = E\left(A^2 \cos(wt_1+\theta)\cos(wt_2+\theta)\right)$$

$$= \frac{A^2}{2} E\{\cos\left((t_1+t_2)w+2\theta\right)+\cos(t_1-t_2)w\}$$

$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos\left((t_1+t_2)w+2\theta\right)+\cos(t_1-t_2)w\right]d\theta$$

$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos(t_1-t_2)w\right]d\theta$$

$$= \frac{A^2}{2}\left[\cos(t_1-t_2)w\right]$$

Hence the random process is a WSS random process.

**Q.** Consider the random process $V(t) = \cos(wt + \theta)$, where $\theta$ is a RV with probability density $P(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi \le \theta \le \pi \\ 0 & otherwise \end{cases}$.

Show that the first and second moments of $V(t)$ are independent of time. If $\theta = constant$, will the ensemble mean of $V(t)$ be time independent?

**Solution**. $E\big(V(t)\big) = E[\cos(wt + \theta)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(wt + \theta)\, d\theta = 0.$

$$E[V^2(t)] = E[\cos^2(wt + \theta)] = \frac{1}{2}.$$

If $\theta$ is constant, then $E\big(V(t)\big) = E[\cos(wt + \theta)] = \cos(wt + \theta)$.

Therefore, the ensemble mean of $V(t)$will not be time independent.

## Practice questions

Consider the process $X(t) = Acos\, wt + Bsinwt$, where $A$ and $B$ are uncorrelated random variables each with mean 0 and variance 1 and $w$ is a positive constant. Show that the process $\{X(t)\}$is covariance stationary (WSS).

# References

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