Probability and Random Processes (15B11MA301)

Department of Mathematics
Jaypee Institute of Information Technology
Noida, India

References

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4.	**S. M.Ross,** A First Course in Probability, 6th edition, Pearson Education Asia, 2002.
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6	**P. L.Mayer,** Introductory Probability and Statistical Applications, Addison-Wesley, Second Edition, 1972.
7.	**R. E. Walpole, R H. Myers, S. L. Myers, and K. Ye,** Probability & Statistics for Engineers & Scientists, 9th edition, Pearson Education Limited, 2016.
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Course Content Covered

Joint Discrete Random variable

Bivariate Random Variable, Joint cumulative distribution function, Joint discrete density function, Joint probability density function.

Two dimensional random variable

Let S be the sample space associated with the random experiment E. Let X = $X(s)$ and Y = $Y(s)$ be two functions, each assigning a real number to each outcome $s \in$ S of the random experiment, then (X, Y) is called the *two dimensional random variable*.

If the possible values of (X, Y) are finite or countably infinite, then (X, Y) is called a *two-dimensional discrete random variable*. If (X, Y) can assume all values in a specified region R in the xy plane, then (X, Y) is called a *two- dimensional continuous random variable*.

Joint Cumulative Distribution Function

Let (X, Y) be the two dimensional random variable defined on the same probability space (S, ***A***, P). The joint cumulative distribution function of (X, Y) denoted by $F_{X,Y}(X,Y)$ is defined as $P[X \leq x, Y \leq y]$ for all (x, y).

Properties and Mathematical Definition of Bivariate Cumulative Distribution Function

Properties

1. $\lim_{x \to -\infty} F(x, y) = 0 = \lim_{y \to -\infty} F(x, y)$

2. $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = 1$

3. $F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \geq 0$

for all $x_1 < x_2, y_1 < y_2$

4. $\lim_{h \to 0^+} F(x + h, y) = F(x, y)$

$\lim_{h \to 0^+} F(x, y + h) = F(x, y)$

(right continuity with respect to each argument)

Mathematical Definition:
Any function satisfying these properties is Bivariave CDF.

JOINT PROBABILITY MASS FUNCTION (PMF)

Mathematically, pmf $f(x,y)$ of bivariate discrete random variables is a real valued function that satisfies the following properties:

1. $f(x, y) \in [0,1] \ \forall \ x, y \in R$
2. $\sum_{(x,y)} f(x, y) = 1$

where (x, y) belongs to range space of (X,Y).

Let (X, Y) be the two dimensional random variable defined on the same probability space (S, $\boldsymbol{A}$, P). The joint probability mass function of (X, Y) denoted by $f_{X,Y}(X,Y)$ is defined as $P[X=x_i, Y=y_j]$ for all (x_i, y_j) belongs to range of (X,Y)

BIVARIATE PROBABILITY DENSITY FUNCTION (PDF)

Mathematically, pdf $f(x,y)$ of biivariate random variable is a real valued function that satisfies the following properties:

1. $f(x,y) \geq 0 \;\forall\; x, y \in R$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$

Relation between Joint CDF and Joint PMF

1. $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,dudv,$
2. $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$

Relation between Joint CDF and Joint PMF

1. $F_{X,Y}(x,y) = \sum_{\substack{x_i \le x \\ y_j \le y}} f(x_i, y_j)$

2. $f(x_i, y_j) = \lim_{h \to 0^+} F_{X,Y}(x_i + h, y_j + h) - \lim_{h \to 0^+} F_{X,Y}(x_i + h, y_j)$

$- \lim_{h \to 0^+} F_{X,Y}(x_i, y_j + h) + \lim_{h \to 0^+} F_{X,Y}(x_i, y_j)$

where (x_i, y_j) belongs to range

space of (X,Y).

EXAMPLE 1 Consider the experiment of tossing two tetrahedra (regular four-sided polyhedron) each with sides labeled 1 to 4. Let X denote the number on the downturned face of the first tetrahedron and Y the larger of the downturned numbers. The goal is to find $F_{X,Y}(\cdot, \cdot)$, .

TABLE OF VALUES OF $F_{X,Y}(x,y)$

$4 \le y$	0	$\frac{4}{16}$	$\frac{8}{16}$	$\frac{12}{16}$	1
$3 \le y < 4$	0	$\frac{3}{16}$	$\frac{6}{16}$	$\frac{9}{16}$	$\frac{9}{16}$
$2 \le y < 3$	0	$\frac{2}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$
$1 \le y < 2$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
$y < 1$	0	0	0	0	0
	$x < 1$	$1 \le x < 2$	$2 \le x < 3$	$3 \le x < 4$	$4 \le x$

EXAMPLE 2 Let X denote the number on the downturned face of the first tetrahedron and Y the larger of the downturned numbers in the experiment of tossing two tetrahedra. The values that (X, Y) can take on are (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), and (4, 4); hence X and Y are jointly discrete. The joint discrete density function of X and Y

4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{4}{16}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	
2	$\frac{1}{16}$	$\frac{2}{16}$		
1	$\frac{1}{16}$			
y / x	1	2	3	4

Validate the following formula

$$F_{X,Y}(x, y) = \sum f_{X,Y}(x_i, y_i),$$

4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{4}{16}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	
2	$\frac{1}{16}$	$\frac{2}{16}$		
1	$\frac{1}{16}$			
y/x	1	2	3	4

PMF table

Validate the following formula

$$f_{X,Y}(x_i, y_i) = F_{X,Y}(x_i, y_i) - \lim_{0<h\to 0} F_{X,Y}(x_i - h, y_i) - \lim_{0<h\to 0} F_{X,Y}(x_i, y_i - h) + \lim_{0<h\to 0} F_{X,Y}(x_i - h, y_i - h).$$

TABLE OF VALUES OF $F_{X,Y}(x,y)$

$4 \leq y$	0	$\frac{4}{16}$	$\frac{8}{16}$	$\frac{12}{16}$	1
$3 \leq y < 4$	0	$\frac{3}{16}$	$\frac{6}{16}$	$\frac{9}{16}$	$\frac{9}{16}$
$2 \leq y < 3$	0	$\frac{2}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$
$1 \leq y < 2$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
$y < 1$	0	0	0	0	0
	$x < 1$	$1 \leq x < 2$	$2 \leq x < 3$	$3 \leq x < 4$	$4 \leq x$

Consider the bivariate function

$$f(x, y) = K(x + y)I_{(0,1)}(x)I_{(0,1)}(y) = K(x + y)I_U(x, y),$$

where $U = \{(x, y): 0 < x < 1 \text{ and } 0 < y < 1\}$, a unit square. Can the constant K be selected so that $f(x, y)$ will be a joint probability density function? If K is positive, $f(x, y) \geq 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Kf(x, y)\, dx\, dy &= \int_0^1\int_0^1 K(x + y)\, dx\, dy \\
&= K\int_0^1\int_0^1 (x + y)\, dx\, dy \\
&= K\int_0^1 (\tfrac{1}{2} + y)\, dy \\
&= K(\tfrac{1}{2} + \tfrac{1}{2}) \\
&= 1
\end{aligned}$$

Practice Problem

Hint: K=1/32

Find K if the joint PDF of a bivariate random variable (X, Y) is given by

$$f(x, y) = \begin{cases} K(1-x)(1-y), & \text{if } 0 < x < 4; 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$$

Find joint pdf, If joint CDF is

$$F(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Hint

$$f(x, y) = \begin{cases} e^{-(x+y)}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Reference for self study

1. Chapter 3 of **A. M. Mood, F. A. Graybill and D. C. Boes,** Introduction to the theory of statistics, 3rd Indian Ed., Mc Graw Hill, 1973