LECTURE 40
PROBABILITY THEORY AND RANDOM PROCESSES(15B11MA301)

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OUTLINE – MARKOV CHAIN, TPM AND RELATED EXAMPLES, REFRENCES

INTRODUCTION

Markov Chain

Stochastic processes for which the description of the present state fully captures all the information that could influence the future evolution of the process.

provides a powerful tool to draw samples from a distribution

Metropolis hasting algorithm

Created to address multi dimensional problems

Sampling from a probability distribution

MARKOV PROCESS

Present state depends on the immediately preceding state and only on it. X_n = 1, the outcome of the n^{th} toss is a HEAD, X_n = 0, the outcome of the n^{th} toss is a TAIL.

$$\Pr\left[X_n = 1\right] = 1/2 \qquad \Pr\left[X_n = 0\right] = 1/2$$

We Define $S_n = X_1 + X_2 + \ldots + X_n$ → Denotes the number of heads in first n tosses

$$\Pr\left[S_{n+1} = k+1 \mid S_n = k\right] = 1/2 \qquad \Pr\left[S_{n+1} = k+1 \mid S_n = k+1\right] = 1/2$$

S_n where $n \geq 1$, is a Markov process since the present state depends on the immediately preceding state and only on it .

MARKOV CHAIN

$$\Pr\left[X_n = a_n \mid X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_0 = a_0\right] = \Pr\left[X_n = a_n \mid X_{n-1} = a_{n-1}\right]$$

then the process X_n where $n \geq 1$ is called a Markov Chain *i.e.* a discrete parameter Markov Process is called a Markov Chain.

MARKOV CHAIN(CONT.)

One-step Transition Probability - The conditional transition probability $P\{X_n = a_j / X_{n-1} = a_i\}$ is called the one-step transition probability from state a_i to state a_j at the *n*th step and is denoted by $P_{ij}(n-1, n)$.

Homogeneous Markov Chain – If the one-step transition probability does not depend on the step i.e. $P_{ij}(n-1, n) = P_{ij}(m-1, m)$, the Markov chain is called a homogeneous Markov chain. That is, the chain is said to be stationary.

TRANSITION PROBABILITY MATRIX

When the Markov chain is homogeneous, the one-step transition probability is denoted by $\boldsymbol{P_{ij}}$. The matrix $\boldsymbol{P = (P_{ij})}$ is called the Transition Probability Matrix satisfying the conditions.

(i) $\boldsymbol{P_{ij} \geq 0, \ \forall\, i, j}$ and

(ii) $\boldsymbol{\sum_i P_{ij} = 1 \ \ \forall \ \ j}$

That is, the sum of the elements of any row of the TPM is 1.

N – Step Transition Probability

The conditional probability that the process is in the state a_j at step *n,* given that it was in state a_i at step 0, $P\{X_n = a_j/X_0 = a_i\}$ is called the n-step transition probability and is denoted by

$$P_{ij}{}^{(n)} = P\{X_n = a_j/X_0 = a_i\}$$

Note: $P_{ij}{}^{(1)} = P_{ij}$

TPM – Illustrative Example

Consider a Markov chain with three states 1, 2 and 3

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$P(0) = \begin{pmatrix} 0.7 & 0.2 & 0.1 \end{pmatrix}$$

Initial distribution if not given take uniform distribution.

Evaluate P(X_2 =3).

$$P^2 = P.P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

TPM – Illustrative Example(cont.)

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \qquad P^2 = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix} \qquad P(0) = \begin{pmatrix} 0.7 & 0.2 & 0.1 \end{pmatrix}$$

$$P\left(X_2 = 3\right) = P\left(X_2 = 3 / X_0 = 1\right) P\left(X_0 = 1\right) + P\left(X_2 = 3 / X_0 = 2\right) P\left(X_0 = 2\right) + P\left(X_2 = 3 / X_0 = 3\right) P\left(X_0 = 3\right)$$

$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1$$

$$= 0.279$$

TPM – ANOTHER ILLUSTRATIVE EXAMPLE

Consider a Markov chain with TWO states 1 and 2 with TPM and initial distribution given by

$$P = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} \qquad P(1) = \begin{pmatrix} 5/6 & 1/6 \end{pmatrix}$$

Evaluate P(2) and P(3) and the steady state distribution

$$P(2) = \begin{pmatrix} 5/6 & 1/6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/12 & 11/12 \end{pmatrix}$$

$$P(3) = \begin{pmatrix} 1/12 & 11/12 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 11/24 & 13/24 \end{pmatrix}$$

$$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$$

$$\frac{\pi_2}{2} = \pi_1; \pi_1 + \frac{\pi_2}{2} = \pi_2 \quad \Longrightarrow \quad \frac{1}{3} = \pi_1; \frac{2}{3} = \pi_2$$

ILLUSTRATIVE EXAMPLE TRY YOURSELF

Consider a Markov chain with three states 1, 2 and 3

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

$$P(1) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \qquad P(2) = ? \qquad P(3) = ?$$

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

Find the steady state distribution and check whether the Markov chain is periodic or not. (Hint. A state in a Markov chain is said to be periodic if return to that particular state occur at regular intervals)

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \quad P^3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \quad P^4 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$

REFERENCES

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