Probability and Random Processes (15B11MA301)

Lecture-38

(Content Covered: Poisson Random Process, Mean and Autocorrelation)

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Contents

Poisson Random Process: Introduction

- During certain practical situations, the event itself is not the area of interest for the researchers but the sequence of random time instants at which the events occur.
- An ensemble of discrete set of points from the time domain called a ***point process*** is used to model and analyse such phenomenon with the property that the number of occurrences in any finite collection of nonoverlapping time intervals are independent random variables, leads to Poisson Process.
- Example: (i) We may want to study the times of arrival of phone calls at a telephone exchange.

  (ii) Interest might be in the study of the times at which the components fail in a large system instead of how it failed.

Poisson Random Process: Definition

If X(t) represents the number of occurrences of a certain event in (0,t), then the discrete process {X(t)} is called the Poisson Process, provided the following postulates are satisfied:

(i) P[1 occurrence in $(t, t+\Delta t)$] $= \lambda\Delta t$+0(Δt)

(ii) P[0 occurrence in $(t, t+\Delta t)$] $= 1 - \lambda\Delta t$+0(Δt)

(iii) P[2 or more occurrences in $(t, t+\Delta t)$] =0(Δt)

(iv) X(t) is independent of the number of occurrences of the event in any interval prior and after the interval (0,t).

(v) The probability that the event occurs a specified number of times in $(t_0, t_0 + \Delta t)$ depends only on Δt, but not on t_0.

Probability Law for the Poisson Random Process {X(t)}

Let λ be the number of occurrences of the event in unit time.

Let $P_n(\mathrm{t}) = \mathrm{P}[\mathrm{X}(\mathrm{t}) = \mathrm{n}]$, therefore

$$P_n(t + \Delta t) = \mathrm{P}[\mathrm{X}(t + \Delta t) = \mathrm{n}]$$

$$= \mathrm{P}[(\mathrm{n} - 1)\text{calls in } (0, \mathrm{t})\text{and 1 call in } (t, t + \Delta)] + \mathrm{P}[(\mathrm{n})\text{calls in } (0, \mathrm{t}) \text{ and no call in } (t, t + \Delta t)]$$

$$\boldsymbol{P_n(t + \Delta t) = P_{n-1}(t)\lambda\, \Delta t + P_n(t)(1 - \lambda\, \Delta t)} \quad \text{(using postulates)}$$

Therefore, $\dfrac{\boldsymbol{P_n(t+\Delta t) - P_n(t)}}{\boldsymbol{\Delta t}} = \boldsymbol{\lambda\{P_{n-1}(t) - P_n(t)\}}$

Taking limits as $\boldsymbol{\Delta t}$ tends to 0, we have

$$\boldsymbol{\frac{d}{dt} P_n(t) = \lambda\{P_{n-1}(t) - P_n(t)\}} \qquad \ldots\ldots\ldots \qquad (1)$$

Let the solution of equation (1) be

$$\boldsymbol{P_n(t)} = \frac{(\boldsymbol{\lambda t})^{\boldsymbol{n}}}{\boldsymbol{n}\,!} \mathrm{f(t)} \qquad \ldots\ldots\ldots \qquad (2)$$

Differentiating (2) with respect to t,

$$P'_n(\mathrm{t}) = \frac{(\boldsymbol{\lambda})^{\boldsymbol{n}} \ [\boldsymbol{n t^{n-1} f(t) + t^n f'(t)}]}{\boldsymbol{n}\,!} \qquad \ldots\ldots\ldots.. \ (3)$$

Using (2) and (3) in (1), we get

$$\boldsymbol{f'(t)} = -\lambda f(t)$$

Therefore, $f(t) = ke^{-\lambda t}$ ……….(4)

From (2), at t =0, we have $f(0) = P[X(0) = 0] = 1$ … … … (5)

Using (5) in (4), we get k=1 and hence $\boldsymbol{f(t) = e^{-\lambda t}}$ ……….(6)

Using (6) in (2), we get

$$\boldsymbol{P_n(t)} = P[X(t)=n] = \frac{\boldsymbol{e^{-\lambda t}(\lambda t)^n}}{\boldsymbol{n}\,!}, \; n=0,1,2\ldots.$$

This gives the probability distribution of X(t), which is the Poisson distribution with parameter ($\boldsymbol{\lambda t}$).

Note: While deriving the probability law for the Poisson Process, the rate of occurrence of the event λ is a constant but it can be a function of t also.

Mean and Autocorrelation of the Poisson Process

The probability law of the Poisson process $\{X(t)\}$ is the same as that of a Poisson distribution with parameter λt.

$$\therefore \quad E\{X(t)\} = \text{Var}\{X(t)\} = \lambda t$$

$$\therefore \quad E\{X_2(t)\} = \lambda t + \lambda^2 t^2 \tag{1}$$

$$\begin{aligned} R_{xx}(t_1, t_2) &= E\{X(t_1)\, X(t_2)\} \\ &= E[X(t_1)\, \{X(t_2) - X(t_1) + X(t_1)\}] \\ &= E[X(t_1)\, \{X(t_2) - X(t_1)\}] + E\{X^2(t_1)\} \\ &= E[X(t_1)]\, E[X(t_2) - X(t_1)] + E\{X^2(t_1)\}, \end{aligned}$$

since $\{X(t)\}$ is a process of independent increments.

$$\begin{aligned} &= \lambda t_1, \lambda\, (t_2 - t_1) + \lambda\, t_1 + \lambda\, t^2{}_1, \text{ if } t_2 \geq t_1 && \text{[by (1)]} \\ &= \lambda^2\, t_1 t_2 + \lambda t_1, \text{ if } t_2 \geq t_1 \end{aligned}$$

or $\quad R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min\, (t_1, t_2)$

Autocovariance and Correlation Coefficient of the Poisson Process

The autocovariance $\{C_{xx}(t_1, t_2)\}$ and Correlation coefficient $\{r_{xx}(t_1, t_2)\}$ are given as:

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - E\{X(t_1)\}\, E\{X(t_2)\}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2$$

$$= \lambda t_1, \text{ if } t_2 \geq t_1$$

or

$$= \min(t_1, t_2)$$

$$r_{xx}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sqrt{\operatorname{var}\{X(t_1)\}\operatorname{var}\{X(t_2)\}}}$$

$$= \frac{\lambda t_1}{\sqrt{\lambda t_1\, \lambda t_2}} = \sqrt{\frac{t_1}{t_2}}, \text{ if } t_2 \geq t_1$$

Example Queries presented in a computer database are following a Poisson process of rate λ= 6 queries per minute. An experiment consists of monitoring the database for 'm' minutes and recording N(m) the number of queries presented.

(i) What is the probability that no queries arriving in a one-minute interval?

(ii) What is the probability that exactly 6 queries arriving in a one-minute interval?

(iii) What is the probability that less than 3 queries arriving in a half-minute interval?

Solution: Given λ=6

N(m)= number of queries presented in 'm' minutes

$$P[N(t) = x] = \frac{e^{-6t}(6t)^x}{x\,!}, x = 0,1,2, \ldots.$$

(i) $P[N(1) = 0] = e^{-6} = 0.00248$

(ii) $P[N(1) = 6] = \frac{e^{-6}.6^6}{6!} = 0.1607$

(iii) $P\left[N\left(\frac{1}{2}\right) < 3\right] = e^{-3}(1 + \frac{3}{1!} + \frac{3^2}{2!})$ =0.4231

Example A fisherman catches fish at a Poisson rate of 2 per hour from a large lake with lots of fish. If he starts fishing at 10:00 am, what is the probability that he catches one fish by 10:30 am and three fishes by noon?

Solution: Let X(t) be the total number of fishes caught at or before time t.

$$P[X(t) = n] = \frac{e^{-2t}(2t)^n}{n\,!}, n = 0,1,2, \ldots.$$

Given: λ = 2

t = 10 am – 10:30 am = 30 minutes =1/2 hour,   n = 1 fish

t = 10 am – 12 noon = 2 hours,   n= 3 fishes

$$P\left[X\left(\frac{1}{2}\right) = 1 \; and \; X(t) = 3\right] = P\left[X\left(\frac{1}{2}\right) = 1\right] P\left[X\left(2 - \frac{1}{2}\right) = 3 - 1\right]$$

$$= \left\{\frac{1.e^{-1}}{1!}\right\} \{e^{-3}.\frac{3^2}{2!}\}$$

$$= 0.082$$

Practice Questions

1. A machine goes out of order, whenever a component fails. The failure of this part follows a Poisson process with a mean rate of 1 per week. Find the probability that 2 weeks have elapsed since last failure. If there are 5 spare parts of this component in an inventory and that the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in the next 10 weeks. (Ans. 0.135, 0.068)

2. A radioactive source emits particles at a rate of 6 per minute in accordance with Poisson process. Each particle emitted has a probability of 1/3 of being recorded. Find the probability that at least 5 particles are recorded in a 5-minute period. (Ans. 0.9707)

References

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