Probability and Random Processes (15B11MA301)

Lecture-29

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Average Values of Random Processes

Mean of the process $\{X(t)\}$ is the expected value of a typical number $X(t)$ of the process.

i.e., $\mu(t) = E\{X(t)\}$

Autocorrelation of the process $\{X(t)\}$, denoted by $R_{xx}(t_1, t_2)$ or $R_x(t_1, t_2)$ or $R(t_1, t_2)$, is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process.

$$R(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

Autocovariance of the process $\{X(t)\}$, denoted by $C_{xx}(t_1, t_2)$ or $C_x(t_1, t_2)$ or $C(t_1, t_2)$, is defined as

$$C(t_1, t_2) = E[\{X(t_1) - \mu(t_1)\}\{X(t_2) - \mu(t_2)\}] = R(t_1, t_2) - \mu(t_1)\mu(t_2).$$

Correlation co-efficient of the process $\{X(t)\}$, denoted by $\rho_{xx}(t_1, t_2)$ or $\rho(t_1, t_2)$, is defined as $\rho(t_1, t_2) = \frac{C(t_1,t_2)}{\sqrt{C(t_1,t_1)C(t_2,t_2)}}$, where $C(t_1, t_1)$ is the variance of $X(t_1)$.

Cross-correlation of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$R_{xy}(t_1, t_2) = E\{X(t_1)Y(t_2)\}$$

Cross-covariance of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1)\mu_y(t_2).$$

Cross correlation co-efficient of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$\rho_{xy}(t_1, t_2) = \frac{C_{xy}(t_1,t_2)}{\sqrt{C_{xx}(t_1,t_1)C_{yy}(t_2,t_2)}}$, where $C(t_1, t_1)$ is the variance of $X(t_1)$.

The above definitions are valid for both continuous-time and discrete-time random processes.

ILLUSTRATIVE EXAMPLE

Q1. Suppose that $\{X(t)\}$ is a process with $\mu(t) = 3,\ R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$. Find the mean, variance and covariance of the random variable $Z = X(5)$ and $W = X(8)$.

Solution.

$$E\{Z\} = \mu(5) = 3, \quad E\{W\} = \mu(8) = 3$$

$$E\{Z^2\} = R(5,5) = 13, \quad E\{W^2\} = R(8,8) = 13$$

$$\therefore Var(Z) = Var(W) = 13 - 9 = 4$$

$$C(5,8) = R(5,8) - \mu(5)\mu(8) = 9 + 4e^{-0.6} - 9 = 2.195$$

ILLUSTRATIVE EXAMPLE(CONT.)

Q 2. Consider the random process $X(t) = Acos(wt + \theta)$, where A and w are constants and θ is a uniformly distributed RV in $(0, 2\pi)$. Find mean and autocorrelation of the process $\{X(t)\}$.

Solution. Since θ is uniformly distributed in $(0, 2\pi)$

$$f_\theta(\theta) = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi$$

$$E\{X(t)\} = E\{A\cos(wt + \theta)\} = A\int_0^{2\pi} \frac{1}{2\pi}\cos(wt + \theta)d\theta = \frac{A}{2\pi}\{\sin(2\pi + wt) - \sin wt\} = 0$$

$$R(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{A^2\cos(wt_1 + \theta)\cos(wt_2 + \theta)\}$$

$$= \frac{A^2}{2}E\{\cos[(t_1 + t_2)w + 2\theta] + \cos[w(t_1 - t_2)]\}$$

$$= \frac{A^2}{2}\cos[w(t_1 - t_2)]$$

Illustrative Example(Cont.)

Q3. Given a RV Y with characteristic function $\boldsymbol{\phi(w) = E\{e^{iwY}\}}$ and a random process defined by $\boldsymbol{X(t) = \cos(\lambda t + Y)}$, find mean and auto correlation of the process $\boldsymbol{\{X(t)\}}$ if $\boldsymbol{\phi(1) = \phi(2) = 0}$.

Solution. $E\{X(t)\} = E\{\cos(\lambda t + Y)\}$

Given $\phi(1) = 0$ which implies $E(\cos Y) = 0 = E(\sin Y)$

$\therefore E\{X(t)\} = 0.$

$R(t_1, t_2) = E\{X(t_1)X(t_2)\}$

$$= \cos \lambda t_1 \cos \lambda t_2 E\left(\frac{1}{2} + \frac{1}{2}\cos 2Y\right) + \sin \lambda t_1 \sin \lambda t_2 E\left(\frac{1}{2} - \frac{1}{2}\cos 2Y\right) - \frac{1}{2}\sin \lambda(t_1 + t_2) E(\sin 2Y)$$

$\phi(2) = 0 \Rightarrow E(\cos 2Y) = E(\sin 2Y) = 0$

$$\therefore R(t_1, t_2) = \frac{1}{2}\cos[\lambda(t_1 - t_2)]$$

Illustrative Example(cont.)

Q4. Let Y_1, Y_2 ... Y_n, … be a sequence of identically independently distributed random variables with E(Y_i) =0 and Var(Y_i) = 4 for all *i*. We define the discrete time random process as {$X(n)$, $n \in$ N} as $X(n) = Y_1 + Y_2 + ... + Y_n$ for all $n \in$ N. Find the mean and autocorrelation function of {$X(n)$, $n \in$ N}.

- Solution
  - We have

$$\begin{aligned}\mu_X(n) &= E[X(n)] \\ &= E[Y_1 + Y_2 + \cdots + Y_n] \\ &= E[Y_1] + E[Y_2] + \cdots + E[Y_n] \\ &= 0.\end{aligned}$$

ILLUSTRATIVE EXAMPLE(CONT.)

Let $m \leq n$, then

$$\begin{aligned} R_X(m,n) &= E[X(m)X(n)] \\ &= E\left[X(m)\big(X(m) + Y_{m+1} + Y_{m+2} + \cdots + Y_n\big)\right] \\ &= E[X(m)^2] + E[X(m)]E[Y_{m+1} + Y_{m+2} + \cdots + Y_n] \\ &= E[X(m)^2] + 0 \\ &= \mathrm{Var}\big(X(m)\big) \\ &= \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \cdots + \mathrm{Var}(Y_m) \\ &= 4m. \end{aligned}$$

Similarly, for $m \geq n$, we have

$$\begin{aligned} R_X(m,n) &= E[X(m)X(n)] \\ &= 4n. \end{aligned}$$

We conclude

$$R_X(m,n) = 4\min(m,n).$$

Practice questions

Q 1. Given a RV Ω with density $f(w)$ and another RV ϕ uniformly distributed in $(-\pi, \pi)$ and independent of Ω and $X(t) = a\cos(\Omega t + \phi)$, prove that auto correlation function of the process $\{X(t)\}$ is $\frac{1}{2}a^2 E\{cos\Omega(t_1 - t_2)\}$.

Q2 . If $X(t) = P + Qt$, where P and Q are independent RVs with $E(P) = p, E(Q) = q, Var(P) = \sigma_1^2, Var\ (Q) = \sigma_2^2$, find $E\{X(t)\}, R(\mathrm{t}_1, \mathrm{t}_2), \mathrm{C}(\mathrm{t}_1, \mathrm{t}_2)$.

Ans. $p + qt$; $\sigma_1^2 + p^2 + (t_1 + t_2)pq + t_1 t_2(\sigma_2^2 + q^2)$; $\sigma_1^2 + t_1 t_2 \sigma_2^2$.

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