Probability and Random Processes (15B11MA301)

Lecture-5

(Course content covered: Law of total probability, Bayes' theorem)

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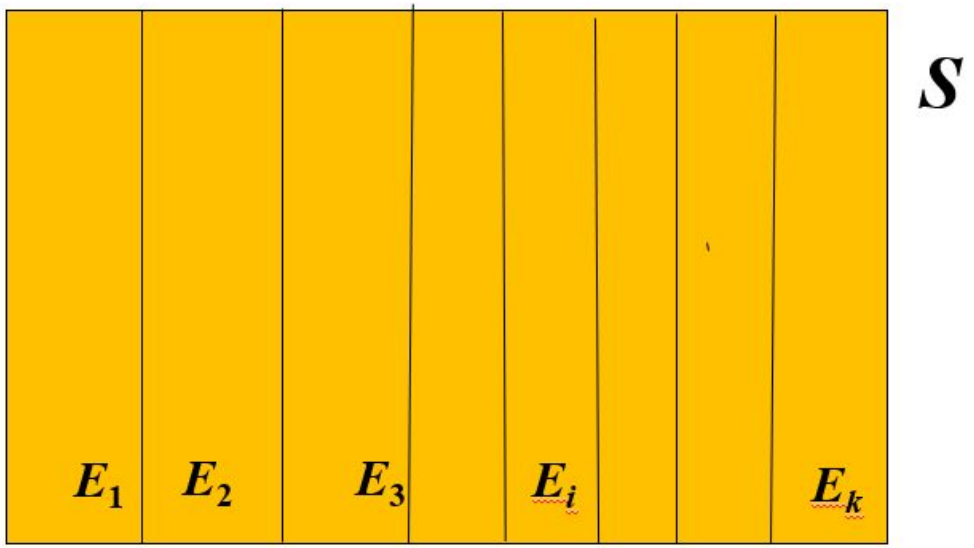
Partition of a Sample Space

The subsets $E_1, E_2, E_3, \ldots, E_k$ of the sample space S forms a partition of it if $S = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_k$, where $E_i \cap E_j = \phi \ \forall\, i \neq j$.

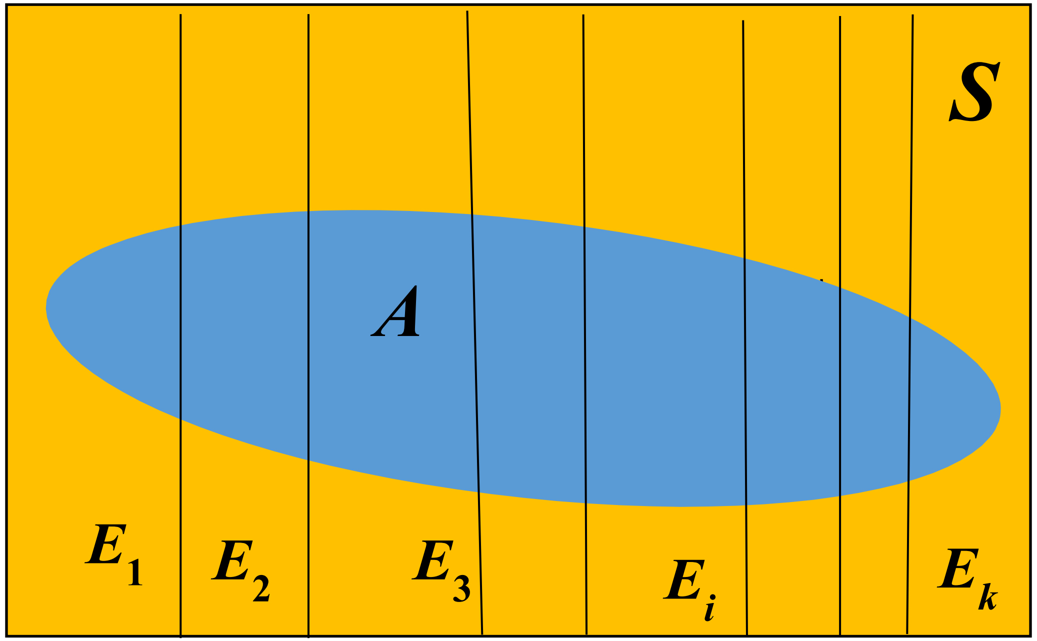


Example. When a fair die is thrown, its sample space is given by $S = \{1, 2, 3, 4, 5, 6\}$.

Then (i) $E_1 = \{1, 2, 3\}$, $E_2 = \{4, 5, 6\}$ and (ii) $E_1 = \{1, 2\}$, $E_2 = \{3\}$, $E_3 = \{4, 5, 6\}$ form two partitions of S.

The Law of Total Probability

Let the events $E_1, E_2, \ldots, E_k$ partition the finite discrete sample space S corresponding to a random experiment and let A be



an event defined on S and $P(E_i) \neq 0$ for $i = 1, 2, \ldots, k$. Then

$$P(A) = \sum_{i=1}^{k} P(A \mid E_i) P(E_i).$$

Proof. Since we have

$$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_k),$$

and

$$(A \cap E_i) \cap (A \cap E_j) = \phi \text{ for all } i, j = 1, 2, 3, \ldots, k \text{ and } i \neq j,$$

$$\begin{aligned}\therefore \quad P(A) &= P((A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_k)) \\ &= P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k) \\ &= P(A \mid E_1) \cdot P(E_1) + P(A \mid E_2) \cdot P(E_2) + \ldots + P(A \mid E_k) \cdot P(E_k)\end{aligned}$$

or

$$P(A) = \sum_{i=1}^{k} P(A \mid E_i) \cdot P(E_i).$$

Example 1. Three urns I, II and III contain 5 black, 5 white; 4 black, 6 white; and 7 black, 3 white balls, respectively. If a ball is drawn at random, what is the probability that it is a (i) white ball, (ii) black ball?

Solution: Define

E_1 : The event of selecting urn I,

E_2: The event of selecting urn II,

E_3: The event of selecting urn III,

A: The event of selecting a white ball,

B: The event of selecting a black ball.

Since each urns are equally likely to be selected, therefore $P(E_1)=P(E_2)=P(E_3)=1/3$.

Also we have,

$P(A \mid E_i) =$ The probability of getting a white ball from urn i, $i =$ I, II, III.

$P(B \mid E_i) =$ The probability of getting a black ball from urn i, $i =$ I, II, III.

Hence, we have from the law of total probability,

$$(i)\, P(A) = P(A \mid E_1)\cdot P(E_1) + P(A \mid E_2)\cdot P(E_2) + P(A \mid E_3)\cdot P(E_3)$$

$$= \left(\frac{5}{10}\right).\left(\frac{1}{3}\right) + \left(\frac{6}{10}\right).\left(\frac{1}{3}\right) + \left(\frac{3}{10}\right).\left(\frac{1}{3}\right) = \frac{7}{15}.$$

$$(ii)\, P(B) = P(B \mid E_1)\cdot P(E_1) + P(B \mid E_2)\cdot P(E_2) + P(B \mid E_3)\cdot P(E_3)$$

$$= \left(\frac{5}{10}\right).\left(\frac{1}{3}\right) + \left(\frac{4}{10}\right).\left(\frac{1}{3}\right) + \left(\frac{7}{10}\right).\left(\frac{1}{3}\right) = \frac{8}{15}.$$

Exercise. Three plants A, B and C produce respectively 40%, 10% and 50% of the total number of mobiles produced by a factory. The percentage of defective mobiles produced by these plants are 1%, 3% and 2%. If a mobile is selected at random, find the probability that the it is a defective one.

[Ans.: 0.017]

Bayes' Theorem

Suppose that the events $E_1, E_2, \ldots, E_k$ partition the sample space S corresponding to some random experiment S with $P(E_i) \neq 0$ for $i = 1, 2, \ldots, k$. Then, for any event A of S,

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{\sum_{j=1}^{k} P(A \mid E_j)P(E_j)}, \text{ where } i = 1, 2, \ldots, k.$$

- It is also known as a formula for the probability of causes.
- $P(E_i)$ is a priori probability known in advance of the experiment.
- $P(E_i \mid A)$, a posteriori probability determined after the experiment.

Proof. By the definition of conditional probabilty, we get

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(A \mid E_i) \times P(E_i)}{P(A)}$$

From the law of total probability, we have

$$P(A) = P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \ldots + P(A \mid E_k)P(E_k)$$

$$= \sum_{j=1}^{k} P(A \mid E_j)P(E_j)$$

Hence

$$P(E_i \mid A) = \frac{P(A \mid E_i) \times P(E_i)}{\sum_{j=1}^{k} P(A \mid E_j) \times P(E_j)}, \text{ for } i = 1,\ 2,\ \ldots,\ k.$$

Example 2. Three urns I, II and III contain 5 black, 5 white; 4 black, 6 white; and 7 black, 3 white balls, respectively. If a ball is drawn at random and found to be black what is the probability that it is drawn from urn II ?

Solution: Define

E_1 : The event of selecting urn I,

E_2: The event of selecting urn II,

E_3: The event of selecting urn III,

A: The event of selecting a black ball,

Since each urns are equally likely to be selected, therefore $P(E_1)=P(E_2)=P(E_3)=1/3$.

Also we have,

$P(A \mid E_i) =$ The probability of getting a black ball from urn i, $i =$ I, II, III.

(i) Then, we have

$$P(A) = P(A \mid E_1) \cdot P(E_1) + P(A \mid E_2) \cdot P(E_2) + P(A \mid E_3) \cdot P(E_3)$$

$$= \left(\frac{5}{10}\right).\left(\frac{1}{3}\right) + \left(\frac{4}{10}\right).\left(\frac{1}{3}\right) + \left(\frac{7}{10}\right).\left(\frac{1}{3}\right) = \frac{8}{15}.$$

From Bayes' theorem,

$$\mathrm{P}(E_2 \mid A) = \frac{P(A \mid E_2)P(E_2)}{\sum_{j=1}^{3} P(A \mid E_j)P(E_j)} = \frac{P(A \mid E_2)P(E_2)}{P(A)}$$

$$= \frac{\left(\frac{4}{10}\right).\left(\frac{1}{3}\right)}{\frac{8}{15}} = \frac{1}{4}.$$

Example 3: Three factories P, Q and R produce 2000, 3000 and 5000 laptops per week respectively. P produces 1% defective, Q produces 0.5% defective and R produces 1 % defective items. A laptop is checked at the end of a week and is found to be defective. What is the probability that it is manufactured by factory Q?

Solution: Let E_1, E_2, E_3 represent the event that a laptop is produced by the factory P, Q and R respectively. Let A be the event that it is defective. Then, we have

$$P(E_1) = \frac{2000}{10000} = \frac{1}{5}, P(E_2) = \frac{3000}{10000} = \frac{3}{10}, P(E_3) = \frac{5000}{10000} = \frac{1}{2},$$

and $P(A) = P(A \mid E_1) \cdot P(E_1) + P(A \mid E_2) \cdot P(E_2) + P(A \mid E_3) \cdot P(E_3)$

$$= \left(\frac{1}{5}\right)(0.01) + \left(\frac{3}{10}\right)(0.005) + \left(\frac{1}{2}\right).(0.01) = 0.0085.$$

From Bayes' theorem, $\mathrm{P}(E_2 \mid A) = \dfrac{P(A \mid E_2)P(E_2)}{P(A)} = \dfrac{\left(\frac{3}{10}\right).(0.005)}{0.0085} = \dfrac{3}{17}.$

Practice Questions

1. Three urns I, II and III contain 8 red, 7 white; 10 red, 5 white; and 7 red, 8 white balls, respectively. (i) If a ball is drawn at random, what is the probability that it is a white ball? (ii) If a ball drawn at random and found to be red what is the probability that it is drawn from the urn III?

   [Ans: (i) 4/9, (ii) 7/25]

2. Coins '1' and '2' are tossed. The probabilities of obtaining heads by them are respectively 1/3 and 3/5. Suppose that one coin is selected randomly and tossed twice. If both tosses show heads, what is the probability that coin '2' was selected? [Ans: 81/106]

References/Further Reading

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