Probability and Random Processes (15B11MA301)

Lecture-2

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Contents

Basic probability theory

To start with, a few simple definitions are presented.

Experiment An experiment is a procedure we perform (quite often hypothetical) that produces some result. Often the letter E is used to designate an experiment (e.g., the experiment E_5 might consist of tossing a coin five times).

Outcome An outcome is a possible result of an experiment. e.g., an outcome of experiment E_5 might represent the sequence of tosses heads-heads-tails-heads-tails; however, the more concise HHTHT might also be used.

Event An event is a certain set of outcomes of an experiment (e.g., the event C associated with experiment E_5 might be C= {all outcomes consisting of an even number of heads}).

Sample Space The sample space is the collection or set of “all possible” distinct outcomes of an experiment. The letter S is used to designate the sample space, which is the universal set of outcomes of an experiment. A sample space is called discrete if it is a finite or a countably infinite set. It is called continuous or a continuum otherwise.

Examples

1. Consider the experiment of rolling two dice and observing the results.

The sample space consists of 36 outcomes, which may be labeled by the ordered pairs $x_1 = (1,1), x_2 = (1,2), \ldots, x_6 = (1,6), \ldots \ldots \ldots$; the first component in the ordered pair indicates the result of the toss of the first die and the second component indicates the result of the toss of the second die.

Many events can be defined from this experiment, such as:

= {the sum of the outcomes of the two rolls = 4},

= {the outcomes of the two rolls are identical},

= {the first roll was bigger than the second}.

2. Let us flip a coin until a tails occurs. The experiment is then terminated. The sample space consists of a collection of sequences of coin tosses. Label these outcomes as $x_n, n = 1, 2, 3, \ldots$ The final toss in any particular sequence is a tail and terminates the sequence. The preceding tosses prior to the occurrence of the tail must be heads. The possible outcomes that may occur are:

$$x_1 = T, x_2 = (H, T), x_3 = (H, H, T), \ldots .$$

In this example, the sample space is countably infinite, while the previous sample space was finite.

3. Consider a random number generator which selects a number in an arbitrary manner from the semi-closed interval $[0,1)$. The sample space consists of all real numbers, x, for which $0 \leq x < 1$. This is an example of an experiment with a continuous sample space. We can define events on a continuous space as well, such as:

$$A = \left\{x < \frac{1}{2}\right\}, \qquad B = \left\{\left|x - \frac{1}{2}\right| < \frac{1}{4}\right\}.$$

Classification of Events

Subset An event E is said to be a subset of the event F if, whenever E occurs, F also occurs. This means that all of the sample points of E are contained in F. Hence considering E and F solely as two sets, E is a subset of F in the usual set-theoretic sense: that is, $E \subseteq F$.

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Equality Events E and F are said to be equal if the occurrence of E implies the occurrence of F, and vice versa; that is, if $E \subseteq F$ and $F \subseteq E$, hence $E = F$.

Intersection An event is called the intersection of two events E and F if it occurs only whenever E and F occur simultaneously, denoted by EF or $E \cap F$ because it is the set containing exactly the common points of E and F.

Union An event is called the union of two events E and F if it occurs whenever at least one of them occurs. This event is $E \cup F$ since all of its points are in E or F or both.

Complement An event is called the complement of the event E if it only occurs whenever E does not occur. The complement of E is denoted by E^c.

Difference An event is called the difference of two events E and F if it occurs whenever E occurs but F does not. The difference of the events E and F is denoted by $E - F$. It is clear that $E^c = S - E$ and $E - F = E \cap F^c$.

Certainty An event is called certain if its occurrence is inevitable. Thus the sample space is a certain event.

Impossibility An event is called impossible if there is certainty in its nonoccurrence. Therefore, the empty set $\emptyset$, which is S^c, is an impossible event.

Mutually Exclusiveness If the joint occurrence of two events E and F is impossible, we say that E and F are mutually exclusive. Thus E and F are mutually exclusive if the occurrence of E precludes the occurrence of F, and vice versa.

Since the event representing the joint occurrence of E and F is EF, their intersection, E and F, are mutually exclusive if $EF = \emptyset$. A set of events $\{E_1, E_2, \dots\}$ is called mutually exclusive if the joint occurrence of any two of them is impossible, that is, if $\forall i \neq j, E_i E_j = \emptyset$. Thus $\{E_1, E_2, \dots\}$ is mutually exclusive if and only if every pair of them is mutually exclusive.

Example# At a busy international airport, arriving planes land on a first-come, first-served basis. Let

E $=$ there are at least five planes waiting to land,

F $=$ there are at most three planes waiting to land,

H $=$ there are exactly two planes waiting to land.

Then

1. E^c is the event that at most four planes are waiting to land.

2. F^c is the event that at least four planes are waiting to land.

3. E is a subset of F^c; that is, if E occurs, then F^coccurs. Therefore, $EF^c = E$.

4. H is a subset of F; that is, if H occurs, then F occurs. Therefore, $FH = H$.

5. E and F are mutually exclusive; that is, $EF = \emptyset$. E and H are also mutually exclusive since $EH = \emptyset$.

6. FH^c is the event that the number of planes waiting to land is zero, one, or three.

Classical Definition of Probability

- Let S be the Sample Space (the set of all possible outcomes which are assumed to be equally likely) and A be an event (a subset of S consisting of possible outcomes) associated with a random experiment.

- Let $n(S)$ and $n(A)$ be the number of elements of S and A. Then the probability of event A occurring denoted as $P(A)$ is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{No.of\ cases\ favourable\ to\ A}{Exhaustive\ number\ of\ cases\ in\ S}$$

e.g. If the experiment consists of tossing two dice, and E is the event that the sum of the two dice equals 7, then E is:

E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)} and $P(E) = \frac{1}{6}$.

Relative Frequency Approach

Let a random experiment be repeated n times and let an event A occur m times out of the n trials. The ratio $\frac{m}{n}$ is called the relative frequency of the event A. As n increases, $\frac{m}{n}$ shows a tendency to stabilize and to approach a constant value. This value, denoted by $P(A)$, is called the probability of the event A, i.e., $P(A) = \lim_{n\to\infty} \frac{m}{n}$.

Example

To find the probability that a spare part produced by a machine is defective, we study the record of defective items produced by the machine for a considerable period of time. If, out of 10,000 items produced, 500 are defective, it is assumed that the probability of a defective item is 0.05.

Subjective Approach

In the subjective approach, we define probability as the degree of belief that we hold in the occurrence of an event. Thus, judgment is used as the basis for assigning probabilities. Use of the subjective approach is usually limited to experiments that are unrepeatable.

Example

Consider a horse race with 8 horses running. What is the probability for a particular horse to win? Is it reasonable to assume that the probability is 1/8? Note that we can't apply the relative-frequency approach. People regularly place bets on the outcomes of such "onetime" experiments based on their judgment as to how likely it is for a particular horse to win.

Practice Questions

1. A deck of six cards consists of three black cards numbered 1, 2, 3, and three red cards numbered 1, 2, 3. First, Vann draws a card at random and without replacement. Then Paul draws a card at random and without replacement from the remaining cards. Let A be the event that Paul's card has a larger number than Vann's card. Let B be the event that Vann's card has a larger number than Paul's card.

(a) Are A and B mutually exclusive? Ans. Yes

(b) Are A and B complements of one another? Ans. No

2. A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, two officers of the sales department and 1 chartered accountant, Find the probability of forming the committee in the following manner:

(i) There must be one from each category. Ans. 8/70

(ii) It should have at least one from the purchase department. Ans. 13/14

(iii) The chartered accountant must be in the committee. Ans. 2/5

References

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