Probability and Random Processes (15B11MA301)

Lecture-11

Department of Mathematics

Jaypee Institute of Information Technology, Noida

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Moments

If X is a random variable (discrete or continuous), the r^{th} moment of X about the origin denoted by μ_r' is defined as

$$\boldsymbol{\mu_r' = E[X^r], \quad r = 1, 2, 3}$$

Note: $\boldsymbol{\mu_1' = E[X] = \mu_x,}$ is the mean of X

Central Moments

If X is a random variable (discrete or continuous), and a be any point then the r^{th} central moment of X about a is defined as

$$\boldsymbol{\mu_r' = E[(X - a)^r], \quad r = 1, 2, 3}$$

If $\boldsymbol{a = \mu_x}$, then we get $\boldsymbol{r^{th}}$ **central moment of X about mean**, denoted by $\boldsymbol{\mu_r}$. So,

$$\boldsymbol{\mu_r = E[(X - \mu_x)^r],}$$

Thus, we may find following relationships easily…

1. $\mu_1 = E(X - \mu_x) = \mu_1' - \mu_1'$=0
2. $\mu_2 = E[(X - \mu_x)^2] = \mu_2' - (\mu_1')^2 \quad = \mathrm{Var(X)}$
3. $\mu_3 = E[(X - \mu_x)^3] = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

Example 1: Find the first four moments about the origin for a random variable X having the density function $f(x) = \frac{4x(9-x^2)}{81}, 0 \leq x \leq 3.$

Solution: Given, $f(x) = \frac{4x(9-x^2)}{81}, 0 \leq x \leq 3$

By the definition of moments,

$$\mu_r^{'} = E[X^r] = \int_{-\infty}^{\infty} x^r\, f(x)dx, r = 1,2,3, \ldots$$

$$\mu_1' = E[X] = \int_0^3 x^1\, f(x)dx = \int_0^3 x^1\, \frac{4x(9-x^2)}{81} dx = 8/5$$

$$\mu_2' = E[X^2] = \int_0^3 x^2\, f(x)dx = \int_0^3 x^2\, \frac{4x(9-x^2)}{81} dx = 3$$

$$\mu_3' = E[X^3] = \int_0^3 x^3\, f(x)dx = \int_0^3 x^3\, \frac{4x(9-x^2)}{81} dx = 299/35$$

$$\mu_4' = E[X^4] = \int_0^3 x^4\, f(x)dx = \int_0^3 x^4\, \frac{4x(9-x^2)}{81} dx = 27/2$$

Covariance

Let X and Y be any two random variables defined on same probability space. The ***covariance*** of X and Y, denoted by $\operatorname{cov}[X, Y]$ or $\sigma_{X,Y}$, is defined as

$$\operatorname{cov}[X, Y] = E[(X - \mu_x)(Y - \mu_Y)]$$

provided that the indicated expectation exists.

Correlation coefficient

The ***correlation coefficient***, denoted by $\rho[X, Y]$ or $\rho_{X,Y}$ of random variables X and Y is defined as

$$\rho_{X,Y} = \frac{\operatorname{cov}[X,Y]}{\sigma_X \sigma_Y}$$

provided that $\operatorname{cov}[X, Y]$, σ_X, and σ_Y exists and $\sigma_X, \sigma_Y > 0$.

Note: $-1 \leq \rho_{X,Y} \leq 1$.

Remark: $\mathbf{cov}[\boldsymbol{X},\boldsymbol{Y}] = \boldsymbol{E}[(\boldsymbol{X}-\boldsymbol{\mu_X})(\boldsymbol{Y}-\boldsymbol{\mu_Y})] = \boldsymbol{E}[\boldsymbol{XY}] - \boldsymbol{\mu_X}\boldsymbol{\mu_Y}$.

Proof:

$$\begin{aligned} E[(X-\mu_X)(Y-\mu_Y)] &= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\ &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\,\mu_Y \\ &= E[XY] - \mu_X\mu_Y \end{aligned}$$

Corollary: If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent, then $\mathbf{cov}[\boldsymbol{X},\boldsymbol{Y}] = \mathbf{0}$.

Proof:

$$\begin{aligned} \mathrm{Cov}[X,Y] &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E[g_1(X)g_2(Y)] \\ &= E[g_1(X)]\,E[g_2(Y)] \\ &= E[X-\mu_X].\,E[Y-\mu_Y] = 0 \end{aligned}$$

Remark: The converse of above corollary is not always true, i.e. $\mathrm{cov}[X,Y] = 0$ does not always imply that X and Y are independent.

Uncorrelated random variables

Random variables X and Y are said to be uncorrelated if and only if $\text{cov}[X, Y] = 0$.

Example 2: Consider the experiment of tossing two tetrahedra. Find $\rho_{X,Y}$ for X, the number on the first, and Y,the larger of the two numbers.

Solution: In Lecture 8, example 2 we have calculated the pmf table for this experiment, as follows.

4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{4}{16}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	
2	$\frac{1}{16}$	$\frac{2}{16}$		
1	$\frac{1}{16}$			
y/x	1	2	3	4

So, $E(XY)$	$= \sum_i x_i y_i f_{X,Y}(x_i, y_i)$
	$= 1.1\left(\frac{1}{16}\right) + 1.2\left(\frac{1}{16}\right) + 1.3\left(\frac{1}{16}\right) + 1.4\left(\frac{1}{16}\right) + 2.2\left(\frac{2}{16}\right) + 2.3\left(\frac{1}{16}\right) + 2.4\left(\frac{1}{16}\right) + 3.3\left(\frac{3}{16}\right) + 3.4\left(\frac{1}{16}\right) + 4.4\left(\frac{4}{16}\right) = \left(\frac{135}{16}\right)$
$E(X)$	$= \sum_i x_i\ f_X(x_i) = 1.\left(\frac{4}{16}\right) + 2.\left(\frac{4}{16}\right) + 3.\left(\frac{4}{16}\right) + 4.\left(\frac{4}{16}\right) = \frac{5}{2}$
$E(Y)$	$= \sum_i y_i\ f_Y(y_i) = 1.\left(\frac{1}{16}\right) + 2.\left(\frac{3}{16}\right) + 3.\left(\frac{5}{16}\right) + 4.\left(\frac{7}{16}\right) = \frac{50}{16}$
$E(X^2)$	$= \sum_i x_i^2\ f_X(x_i) = 1.\left(\frac{4}{16}\right) + 4.\left(\frac{4}{16}\right) + 9.\left(\frac{4}{16}\right) + 16.\left(\frac{4}{16}\right) = \frac{30}{4}$
$E(Y^2)$	$= \sum_i y_i^2\ f_Y(y_i) = 1.\left(\frac{1}{16}\right) + 4.\left(\frac{3}{16}\right) + 9.\left(\frac{5}{16}\right) + 16.\left(\frac{7}{16}\right) = \frac{170}{16}$
$Var(X)$	$= E(X^2) - \left(E(X)\right)^2 = \frac{5}{4}$
$Var(Y)$	$= E(Y^2) - \left(E(Y)\right)^2 = \frac{55}{64}$

Now, $\rho_{X,Y} = \frac{\text{cov}[X,Y]}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = \frac{\frac{135}{16} - \frac{5}{2} \cdot \frac{50}{16}}{\sqrt{\frac{5}{4}} \cdot \sqrt{\frac{55}{64}}} = \frac{2}{\sqrt{11}}$

Example 3: Suppose $f_{X,Y}(x,y) = (x+y)I_{(0,1)}(x)\ I_{(0,1)}(y)$ then find $\rho_{X,Y}$?

Solution: $E(XY) = \int_0^1 \int_0^1 xy(x+y)dx\,dy = \frac{1}{3}$

Marginal densities are-

$f_X(x) = \int_0^1 (x+y)\ \ dy = x + \frac{1}{2}$, and $f_Y(y) = \int_0^1 (x+y)\ \ dx = y + \frac{1}{2}$

So, $E(X) = \int_0^1 x\, f_X(x)\,\mathrm{dx} = \frac{7}{12}$, and $E(Y) = \int_0^1 y\, f_Y(y)\,\mathrm{dy} = \frac{7}{12}$

$E(X^2) = \int_0^1 x^2\, f_X(x)\,\mathrm{dx} = \frac{5}{12}$, and $E(Y^2) = \int_0^1 y^2\, f_Y(y)\,\mathrm{dy} = \frac{5}{12}$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{11}{144}$, and $\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{11}{144}$

cov$[X,Y]$= $E[XY] - E[X]E[Y] = -1/144$

Hence, $\rho_{X,Y} = \frac{\mathrm{cov}[X,Y]}{\sigma_X \sigma_Y} = -\frac{1}{11}$.

Practice Questions:

1. Consider two sample random variables X and Y having a joint probability density function, $f_{X,Y}(x, y) = 6xyI_{(0,\sqrt{x})}(y)\, I_{(0,1)}(x)$, Find $\rho_{X,Y}$? [Ans:- $\rho_{X,Y}$=0.227]

2. Calculate the coefficient of correlation for the following heights (in inches) of fathers(X) and sons (Y):

[Ans. 0.603]

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

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