# Probability and Random Processes (15B11MA301)

## Lecture-34

(Content covered: Ergodicity, mean ergodic theorem, examples)


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# Ergodicity and its Properties

Time Average of a Random Process: If $\{X(t)\}$ is a random process, then the time average of $\{X(t)\}$ over the time interval $(-T, T)$ is given by

$$\frac{1}{2T}\int_{-T}^{T} X(t)dt,$$

and it is denoted by $\overline{X}_T$.

- The time-average over large length of time is

$$\mu_{\mathrm{T}} = \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T} X(t)dt\right)$$

- The concept of ergodicity deals with the equality of time averages and ensembled averages.
- Time averages and ensebmle averages are not equal in general.

# Ergodic Processes

**Definition:** A random process $\{X(t)\}$ is said to be ergodic, if its ensemble averages (or statistical averages) are equal to appropriate time averages.

- Any ensemble averages of $\{X(t)\}$ can be obtained from a single sample function of $\{X(t)\}$ .

**Mean Ergodic Process.** If the random process $\{X(t)\}$ has a constant mean $E\{X(t)\} = \mu$, and if

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt \to \mu, \quad \text{as } T \to \infty,$$

then $\{X(t)\}$ is said to be a mean ergodic process.

- Every ergodic process is stationary but the converse need not be true.

## Mean Ergodic Theorem

If $\{X(t)\}$ is a random process with constant mean $\mu$ and if

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt,$$

then $\{X(t)\}$ is mean ergodic, provided $\lim_{T\to\infty}\left\{Var\left(\overline{X}_T\right)\right\} = 0.$

- This theorem provides a sufficient condition for the mean ergodicity of a random process.

Example1. Consider a random process $\{X(t) = \cos(wt + \theta)\}$, where $w$ is constant and $\theta$ is a random variable with a probability density function given by

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, 0 \leq \theta \leq 2\pi \\ 0, \quad \text{otherwise.} \end{cases}$$

Test whether $X(t)$ is a mean ergodic random process or not.

Solution: We have

$$E\{X(t)\} = E\{\cos(wt + \theta)\} = \int_0^{2\pi} \frac{1}{2\pi} \cos(wt + \theta) d\theta$$

$$= \frac{1}{2\pi}\{\sin(2\pi + wt) - \sin wt\} = 0,$$

and so the ensemble average of $\{X(t)\}$, $\mu = 0$.

Now, we calculate the time average of $\{X(t)\}$ over (-$T$, $T$)

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt = \frac{1}{2T}\int_{-T}^{T}\cos(wt+\theta)dt$$

$$= \frac{1}{2T}\left(\frac{\sin(wT+\theta)}{w}\right)_{-T}^{T}$$

$$\therefore \quad \overline{X}_T = \frac{1}{2wT}[\sin(wT+\theta)-\sin(-wT+\theta)].$$

So, $$\mu_T = \lim_{T\to\infty}\left(\overline{X}_T\right) = \lim_{T\to\infty} \frac{1}{2wT}[\sin(wT+\theta)-\sin(-wT+\theta)] = 0.$$

Therefore , we have

$$E\{X(t)\} = \mu_T .$$

$\therefore$ $\{X(t)\}$ is a mean ergodic process.

- If $\overline{X_T}$ is the time-average of a stationary random process $\{X(t)\}$ over $(-T,T)$, then

$$\operatorname{Var}(\overline{X_T}) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\,dt_1dt_2$$

$$= \frac{1}{T}\int_{0}^{2T} C(\tau)\left[1-\frac{|\tau|}{2T}\right]d\tau$$

- If $\overline{X_T}$ is the time-average of a stationary random process $\{X(t)\}$ over $(0,T)$, then

$$\operatorname{Var}(\overline{X_T}) = \frac{1}{T}\int_{-T}^{T} C(\tau)\left[1-\frac{|\tau|}{T}\right]d\tau,$$

where

$$C(t_1,t_2) = E\{X(t_1)X(t_2)\} - E\{X(t_1)\}E\{X(t_2)\},$$

$$C(\tau) = E\{X(t)X(t+\tau)\} - E\{X(t)\}E\{X(t+\tau)\}.$$

Example 2. If $\{X(t)\}$ is a WSS process with autocorrelation function $R_{xx}(\tau) = 4 + e^{-|\tau|/10}$, find the mean and variance of the time average of $\{X(t)\}$ over the interval $(-T,\ T)$ and test it for mean ergodicity.

Solution. The time average $\overline{X}_T$ of $\{X(t)\}$ over the interval $(-T,\ T)$ is given by

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt, \qquad \text{so}$$

$$E\{\overline{X}_T\} = E\left(\frac{1}{2T}\int_{-T}^{T} X(t)dt\right) = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}dt = 2.$$

Also, here we have $C_{xx}(\tau) = R_{xx}(\tau) - 4.$ (taking +ve value of mean)

If $\overline{X_T}$ is the time-average of a stationary random process $\{X(t)\}$ over $(-T,T)$, then

$$\text{Var}\,(\overline{X_T}) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\,dt_1 dt_2$$

$$= \frac{1}{T}\int_{0}^{2T} C_{xx}(\tau)\left(1-\frac{|\tau|}{2T}\right)d\tau$$

$$= \frac{1}{T}\int_{0}^{2T} (R_{xx}(\tau)-4)\left(1-\frac{|\tau|}{2T}\right)d\tau$$

$$= \frac{1}{T}\int_{0}^{2T} (R_{xx}(\tau)-4)\left(1-\frac{\tau}{2T}\right)d\tau$$

$$= \frac{1}{T}\int_{0}^{2T} (4+e^{-|\tau|/10}-4)\left(1-\frac{\tau}{2T}\right)d\tau = \frac{1}{T}\int_{0}^{2T} e^{-\tau/10}\left(1-\frac{\tau}{2T}\right)d\tau$$

$$= \frac{1}{T}\left\{\left(1-\frac{\tau}{2T}\right)\frac{e^{-\tau/10}}{(-1/10)} - \left(-\frac{1}{2T}\right)\frac{e^{-\tau/10}}{(1/100)}\right\}_0^{2T}$$

$$= \frac{1}{T}\left\{\frac{50}{T}e^{-T/5}\right\} - \left\{-10+\frac{50}{T}\right\}$$

$$= \frac{1}{T}\left\{10-\frac{50}{T}\left(1-\frac{e^{-T/5}}{T}\right)\right\}.$$

Thus, here we have

$$\lim_{T\to\infty}(\text{Var }(\overline{X_T})) = \lim_{T\to\infty}\frac{1}{T}\left\{10-\frac{50}{T}\left(1-\frac{e^{-T/5}}{T}\right)\right\} = 0$$

$\therefore$ Using the mean ergodic theorem, the given process is mean ergodic

as $\lim_{T\to\infty}(\text{Var }(\overline{X_T})) = 0$ .

# Practice Questions

Question 1. If $\{X(t)\}$ is a WSS process with mean 8 and autocovariance function

$$C_{xx}(\tau) = \begin{cases} 4\left(1 - \dfrac{|\tau|}{3}\right), & 0 \le |\tau| \le 3 \\ 0, & \text{when } |\tau| > 3. \end{cases}$$

Find the mean and variance of the time average of $\{X(t)\}$ over $(0, T)$. Also examine if the process $\{X(t)\}$ is mean ergodic.

[Ans: Mean= 8, variance = 4(1-(T/9)), yes, mean ergodic]

Question 2. Given two mean-ergodic processes $\{X(t)\}$ and $\{Y(t)\}$ with means $\mu_x$ and $\mu_y$ respectively and let the process $\{Z(t)\}$ be defined as follows:

$$Z(t)\} = a\{X(t)\} + \{Y(t)\},$$

where $a$ is a random variable independent of $\{X(t)\}$ taking the values 0 and 2 with equal probability. Is the process $\{Z(t)\}$ mean ergodic? Justify.

# References/Further Reading


1. Veerarajan, T., Probability, Statistics and Random Processes, 3rd Ed. Tata McGraw-Hill, 2008.
2. Ghahramani, S., Fundamentals of Probability with Stochastic Processes, Pearson, 2005.
3. Papoulis, A. and Pillai, S.U., Probability, Random Variables and Stochastic Processes, Tata McGraw-Hill, 2002.
4. Miller, S., Childers, D., Probability and Random Processes, Academic Press, 2012.
5. https://nptel.ac.in/courses/117/105/117105085/