Probability and Random Processes (15B11MA301)

Lecture-3

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Contents of the Lecture:

- Axiomatic approach of Probability
- Addition Theorem
- Related Results and Questions

Event Space

The sample space of an experiment contains all possible outcomes, the event space contains all *sets of outcomes*; all subsets of the sample space.

Example: **Coin Flip**

For a simple coin flip, the two possible outcomes are either heads or tails, so the **sample space** is given by

S = {H, T}

The **event space** is a little different. The possible events are:

- {H}— getting heads,
- {T}— getting tails,
- {H,T}— getting either heads *or* tails.

Because each of these are different subsets of the sample space, they count as different events, even though {H} (heads) would imply {H, T} (either H or T). The event space contains all three of those events:

$$\boldsymbol{A}=\{\{H\}, \{T\}, \{H, T\}\}$$

Axiomatic Definition of Probability:

Let $\boldsymbol{S}$ be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by $\boldsymbol{P}(A)$, is designed as a real number satisfying the following axioms.

(i) $0 \leq \boldsymbol{P}(A) \leq 1$

(ii) $\boldsymbol{P}(\boldsymbol{S}) = 1$

(iii) If A and $\boldsymbol{B}$ are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$ and

(iv) If $A_1, A_2, \ldots, A_n, \cdots$ are a set of mutually exclusive events,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n \ldots) = P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$$

Revision

Definition: Mutually Exclusive Events

A set of events is said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the others.

Two events A and B are mutually exclusive if A occurs and B does not occur and vice versa.

In other words, A and B cannot occur simultaneously, i.e. $P(A \cap B) = 0.$

***Some results derived using axioms of probability*:**

Result 1: The probability of the impossible event is zero, i.e., if ϕ is the subset (event) containing no sample point, $P(\phi)=0$.

Proof: The certain event **S** and the impossible event ϕ are mutually exclusive.

Hence $P(S \cup \phi) = P(S) + P(\phi)$ [Axiom (iii)]

But $S \cup \phi = S$.

$\therefore \quad P(S) = P(S) + P(\phi)$

$\therefore \quad P(\phi) = 0$

Result 2: If $\overline{A}$ is the complementary event of A, $P(\overline{A})=1-P(\mathrm{A})\leq 1$.

Proof: A and $\overline{A}$ are mutually exclusive events, such that $A\cup\overline{A}=S$.

$\therefore \quad P(A\cup\overline{A}) = P(S)$

$P(A\cup\overline{A}) = P(S)=1$ **[Axiom (ii)]**

i.e., $P(A)+P(\overline{A})=1$ **[Axiom (iii)]**

$\therefore \quad \mathrm{P}(\overline{A}) = 1-\mathrm{P}(\mathrm{A})$

Since $\mathrm{P}(A)\geq 0$, it follows that $\mathrm{P}(\overline{A})\leq 1$.

ADDITION LAW OF PROBABILITY

Result 3: If A and B are any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: A is the union of the mutually exclusive events $A \cap \bar{B}$ and $A \cap B$ and B is the union of the mutually exclusive events $\bar{A} \cap B$ and $A \cap B$.

$\therefore \quad P(A) = P(A \cap \bar{B}) + P(A \cap B)$ **[Axiom(iii)]**

and $\quad P(B) = P(\bar{A} \cap B) + P(A \cap B)$ **[Axiom(iii)]**

$$\therefore \quad P(A) + P(B) = [P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B) + P(A \cap B)]$$

$$= P(A \cup B) + P(A \cap B)$$

$$As,\ A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B)$$

By Axiom (iii)

$$\Rightarrow P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B)$$

This implies $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Observations:

- If A and B are any two events, $P(A \cup B) \leq P(A) + P(B)$.

  It follows from Addition Law of Probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  as $P(A \cap B) \geq 0$.

  This implies $P(A \cup B) \leq P(A) + P(B)$.

- If A, B and C are any three events, then

  $P(A \cup B \cup C) = P$ (at least one of A, B and C occurs)

  $$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Result 4: If $B \subseteq A,\ P(B) \leq P(A)$.

Proof: B and $A \cap \bar{B}$ are mutually exclusive events

such that $B \cup (A \cap \bar{B}) = A.$

$\therefore \quad P(B \cup (A \cap \bar{B})) = P(A)$

i.e. $P(B) + P(A \cap \bar{B}) = P(A)$ **[Axiom (iii)]**

$\therefore \quad P(B) \leq P(A)$

Example: Determine the range for $P(A \cup B)$ and $P(A \cap B)$ when $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$.

Example: Determine the range for $P(A \cup B)$ and $P(A \cap B)$ when $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$.

Sol: Since

$A \subseteq A \cup B$ and $B \subseteq A \cup B$

$\Rightarrow P(A) \leq P(A \cup B)$ and $P(B) \leq P(A \cup B)$

$\Rightarrow \max[P(A), P(B)] \leq P(A \cup B)$

Here, $P(A \cup B) \geq \frac{3}{4}$

$\Rightarrow \frac{3}{4} \leq P(A \cup B) \leq 1$

$A \supseteq A \cap B$ and $B \supseteq A \cap B$

$\Rightarrow P(A) \geq P(A \cap B)$ and $P(B) \geq P(A \geq B)$

$\Rightarrow \min[P(A), P(B)] \geq P(A \cap B)$

Here, $P(A \cap B) \leq \frac{5}{8}$

By addition law: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow \frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$

Practice Questions

1. If A and B are any two events, prove that $P(\overline{A} \cap B) = P(B) - P(A \cap B)$.
2. If $A \cap B = \phi$, then show that $P(A) \leq P(\overline{B})$.

References

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3. Papoulis, A. and Pillai, S.U., Probability, Random Variables and Stochastic Processes, Tata McGraw-Hill, 2002.

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