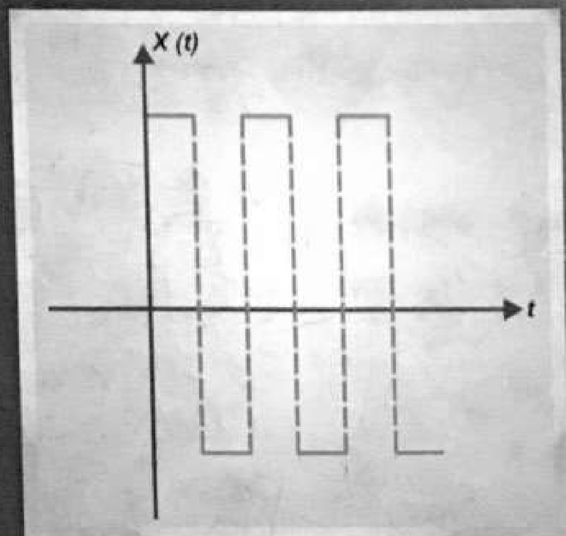
Eastern Economy Edition

S. PALANIAMMAL



PROBABILITY and RANDOM PROCESSES

PHI

To
My beloved Parents
who made me what I am today

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Preface

Most of the engineering students find the theory of probability very difficult due to inadequate knowledge and understanding of the basic concepts. The aim of this book is to provide a thorough understanding of the fundamental concepts and applications of probability and random processes for the undergraduate and postgraduate students of Engineering and, in particular, Electronics and Communication Engineering, Computer Science Engineering and Information Technology. This book is an outcome of my long teaching experience. It is written in a simple and an easy-to-understand language. The book covers the syllabi of most Indian universities and, in particular, the entire syllabus of Anna University.

The text is divided into eight chapters. Chapter 1 provides a detailed discussion on probability theory. Chapter 2 presents the concepts of random variables, probability mass and density functions, cumulative distribution function and moments. Chapter 3 discusses discrete and continuous distributions such as binomial, Poisson, geometrical, uniform, exponential, Gamma, Weibull and Normal distributions. Chapter 4 provides functions of a random variable. Chapter 5 analyses the concepts of two-dimensional discrete and continuous random variables, joint probability mass and density functions, conditional probability function and cumulative distribution function. It also discusses covariance, correlation, regression, transformation of random variables and central limit theorem.

Chapter 6 deals with random processes such as Poisson, Bernoulli, Sine wave, Ergodic and Gaussian random processes. It also explains Markov processes and Markov chain. Chapter 7 discusses in detail the correlation and spectral densities and the last Chapter 8 covers the linear systems with random inputs.

All the topics are covered exhaustively ... explanations are clear and lucid in this book. It provides definitions, examples, theorems, proofs and exercises to reinforce the students' understanding of the subject matter. Numerous solved university questions including those of Anna University are included. It is written in such a manner that even the beginners will find the subject easy and interesting. They will find this book very useful to pursue their studies.

I am greatly indebted and wish to express my sincere gratitude to our Managing Trustee and Chairperson, Smt. S. Malarvizhi, V.L.B. Janakiammal College of Engineering and Technology, Coimbatore for the encouragement and support extended to me during this project.

I sincerely express my thanks and gratitude to Dr. S. Subramanyan, Advisor and Dr. V. Soundararajan, Principal, V.L.B. Janakiammal College of Engineering and Technology, Coimbatore for their appreciative interest shown and constant encouragement given to me during the preparation of this manuscript.

I express my sincere thanks to all my colleagues in the Department of Science and Humanities for their support and encouragement.

I owe a lot to my husband A.M. Balakrishnan and my sons B. Gowtham Prassath and B. Ajay Rahul for motivating me to take this project and I appreciate their patience during my long hours of work.

Special thanks are due to my publisher PHI Learning, New Delhi, in particular the editorial and production team for their cooperation in bringing out this book in such a fine form. I hope that the comprehensive treatment of the various topics covered in this book will be welcomed by the students, faculty and other readers.

I welcome constructive comments and valuable suggestions for the improvement of the book and it will be highly appreciated and gratefully acknowledged.

S. Palaniammal

1

Probability Theory

The theory of probability was originated by two famous mathematicians Blaise and Pierre de Fermat in 1654. They made a study of the gambling problems to formulate the fundamental principles of probability theory. The theory of probability was developed rapidly in the 18th century. The basic mathematical techniques of probability were introduced by Laplace, Chebyshev, Markov, Von Mises and Kolmogorov. They made an important contribution to the theory of probability. In the recent decades, the important feature of its growth has been in the field of mathematical statistics such as statistical inference and decision theory. In the recent years, the theory of probability has provided the modern techniques such as operations research, game theory, queueing theory, statistical quality control and reliability.

1.1 PROBABILITY CONCEPT

The statisticians are basically concerned with drawing conclusions from experiments involving uncertainties. For these inferences to be reasonably accurate, an understanding of the theory of probability is essential.

The concept of probability is extremely important, as it has very extensive applications in the development of all physical sciences. Starting with games of chance 'probability' today has become one of the fundamental tools of statistics. The word 'probability' or 'chance' is very commonly used in day to day conversation. We come across statements like "probably Rahul is right", "it is possible to catch the bus in time today", "the chances of team A and B winning the football match are equal", "it is likely that Gowtham may attend the party tomorrow". All these terms—probably, possible chance, likely, etc. convey the same meaning. That is, the event is not certain to take place or in other words, there is uncertainty about happening of the event in question.

When we make statements like as above, we are expressing an outcome about which we are not certain, but because of past information we have some degree of confidence in the validity of the statement. The mathematical theory of probability provides a mean of evaluating the uncertainty, likelihood, or chance of happening of outcomes resulting from a statistical experiment. The mathematical measure of uncertainty or likelihood, or chance is called *probability*. In mathematics and statistics, we try to present conditions under which we can make sensible numerical statements about uncertainty and apply certain methods of calculating numerical values of probabilities and expectations.

1.2 BASIC NOTATIONS OF SET THEORY

A fundamental concept of probability and statistics is the set. A set is a well-defined collection of objects or members. For example, the letters in the alphabet, first ten natural numbers, the number of points on the dice, etc. Each object in a set is called an *element* or *member of the set*. Usually, the capital letters $A, B, C, \ldots$ denote the sets and the small letters $a, b, c, d, \ldots$ denote the elements of the set.

The sets can be written in two ways.

One way is, if the set has a finite number of members, we may list the members separating by commas and enclose in brackets. For example,

$$A = \{1, 2, 3, 4, 5\}$$

The second way is, the set may be described by a statement or rule. For example,

$$A = \{x/x \text{ is a natural number from 1 to 10}\}$$
$$B = \{y/y \text{ is a vowel in English alphabet}\}$$

1.3 DEFINITIONS

Null Set or Empty Set

A set which contains no element is called a null set or empty set. It is denoted by ϕ. For example, the set of people who have travelled from Earth to Mars is a null set.

Subsets

From a given set, we may form new sets, called the subsets of a given set B. If each element of a set A is also an element of B, then A is called a subset of B and we write $A \subset B$. For example, if A is a set of all positive integers and B is a set of all integers, then A is contained in B.

Union of Sets

Union of two sets A and B is the set of all elements that belongs to either A or B, or both.

$$A \cup B = \{x/x \in A \quad \text{or} \quad x \in B\}$$

It is shown in Figure 1.1.

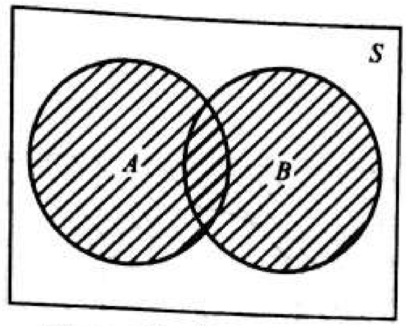

Figure 1.1 Union of sets.

Intersection of Sets

The intersection of two sets A and B is the set of all elements common to both A and B.

$$A \cap B = \{x/x \in A \quad \text{and} \quad x \in B\}$$

It is shown in Figure 1.2.

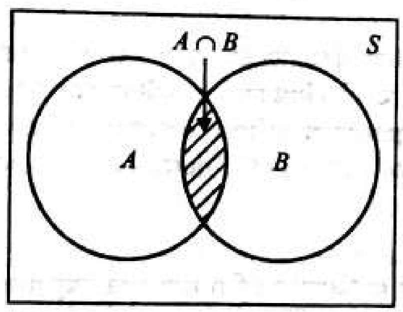

Figure 1.2 Intersection of sets.

Mutually Exclusive or Disjoint Sets

Two sets A and B are said to be disjoint or mutually exclusive if they have no elements in common. That is,

$$A \cap B = \{\ \} = \phi$$

They are shown in Figure 1.3.

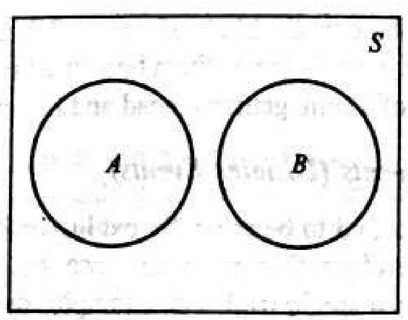

Figure 1.3 Mutually exclusive or disjoint sets.

Complement of a Set

If A is a subset of the set S, then the set of all elements in S, that are not in A is called the complement of A and is denoted by

$$\bar{A} = \{x/x \notin A\}$$

It is shown in Figure 1.4.

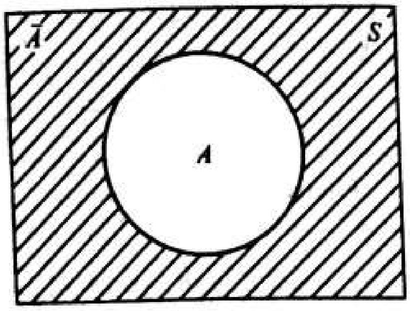

Figure 1.4 Complement of a set.

1.4 RANDOM EXPERIMENT

If an experiment is repeated under the same conditions, any number of times, it does not give unique results but may result in any one of the several possible outcomes. Thus, an experiment whose outcome cannot be predicted is called a **random** experiment or trial and the outcomes are known as events or cases.

Sample Space

The set of all possible outcomes of a random experiment is called a sample space and is denoted by S.

Favourable Events

The trials which entail the happening of an event are said to be favourable to the event. For example, in tossing of a die, the number of favourable events to the appearance of a multiple of 2 are 2, 4 and 6.

Equally Likely Events

The events are said to be equally likely if none of them is expected to occur in preference to other, i.e. each one of them has an equal chance of happening. For example, in tossing of a coin, getting a head and tail are equally likely events.

Mutually Exclusive Events (Disjoint Events)

Two events A and B are said to be mutually exclusive if the occurrence of one prevents the occurrence of another and vice versa, i.e. both the events cannot occur simultaneously in a single trial. For example, in tossing of a coin, both head and tail cannot occur in a single trial.

Independent Events

Two events A and B are said to be independent if the occurrence of one event does not affect the occurrence of the other event. That is, both the events can occur simultaneously. For example, in successive tossing of a coin, the event of getting a head or tail in the first toss does not affect the event of getting a head or tail in the second toss.

Exhaustive Events

The total number of possible outcomes in any trial is known as exhaustive events.

Dependent Events

The events are said to be dependent if the occurrence or non-occurrence of one event in any one trial affects the occurrence of other events in other trials.

1.5 MATHEMATICAL OR CLASSICAL, OR A PRIORI DEFINITION OF PROBABILITY

If a trial results in n exhaustive mutually exclusive and equally likely cases and m of them are favourable to the happening of an event A, then the probability of happening of A is given by

$$P(A) = p = \frac{\text{Number of favourable cases}}{\text{Exhaustive number of cases}} = \frac{n(A)}{n(S)} = \frac{m}{n}$$

For example, in throwing a die, the possible cases are:

$$S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$$

If A is an event of getting a number 5, then

$$n(A) = 1$$

and
$$P(A = 5) = \frac{1}{6}$$

Note:

(i) The probability p of the happening of an event is also known as the probability of success.
(ii) The probability $q = 1 - p$ of the non-happening of the event is known as the probability of failure.
(iii) If $P(A) = 1$, A is called a certain event.
(iv) If $P(A) = 0$, A is called an impossible event.
(v) If the exhaustive number of cases in a trial is infinite, then this definition of classical probability breaks down.

1.6 STATISTICAL OR A POST PRIORI DEFINITION OF PROBABILITY

If a trial is repeated n number of times under essentially homogeneous and identical conditions and let an event A occur m times out of n trials, n becomes indefinitely large, then the probability p of the happening of A is given by

$$P(A) = p = \lim_{n \to \infty} \frac{m}{n}$$

1.7 AXIOMATIC DEFINITION OF PROBABILITY

Let S be the sample space and A be an event associated with a random experiment. Then, the probability of the event A, denoted by $P(A)$ is a real number satisfying the following axioms:

(i) $0 \le P(A) \le 1$.

(ii) $P(S) = 1$, $P(\phi) = 0$.

(iii) If A and B are any two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

(iv) If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events, then $P(A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots) = \sum_{n=1}^{\infty} P(A_n)$.

Note:

(i) If A and B are mutually exclusive events, then $P(A \cap B) = 0$.

(ii) If A and B are independent events, then $P(A \cap B) = P(A)\,P(B)$.

THEOREM 1 The probability of an impossible event is zero, i.e. if ϕ is a null set, then $P(\phi) = 0$.

Proof Let S be the sample space, then $S = S \cup \phi$

$$P(S) = P(S \cup \phi) \tag{1.1}$$

The certain events S and ϕ are mutually exclusive.

$\therefore$ $P(S \cup \phi) = P(S) + P(\phi)$ [using axiom (iii)]

Substituting in Eq. (1.1), we get

$$P(S) = P(S) + P(\phi) \Rightarrow P(\phi) = 0$$

THEOREM 2 If $\bar{A}$ is the complementary event of A, then $P(\bar{A}) = 1 - P(A)$.

Proof The events A and $\bar{A}$ are mutually exclusive and

$$S = A \cup \bar{A}$$

Then, $P(S) = P(A \cup \bar{A}) = P(A) + P(\bar{A})$

But, $P(S) = 1$

$\therefore \qquad 1 = P(A) + P(\bar{A}) \quad \Rightarrow \quad P(A) = 1 - P(\bar{A})$

Note: As $P(A) \geq 0$, $P(\bar{A}) \leq 1$.

EXAMPLE 1.1 A bag contains 3 red, 6 white and 7 blue balls. What is the probability that 2 balls drawn are white and blue?

Solution Let A be the event of choosing 1 white and 1 blue ball.

The total number of balls = 3 + 6 + 7 = 16

Out of 16 balls, 2 balls can be drawn in $16C_2$ ways.

$\therefore$ The exhaustive number of cases = $16C_2 = \dfrac{16 \times 15}{1 \times 2} = 120 = n(S)$

Out of 6 white balls, 1 ball can be drawn in $6C_1$ ways and out of 7 blue balls 1 ball can be drawn in $7C_1$ ways. Therefore, the total number of favourable cases = $6C_1 \times 7C_1 = 6 \times 7 = 42 = n(A)$

$\therefore$ The required probability is

$$P(A) = \frac{n(A)}{n(S)} = \frac{42}{120} = \frac{7}{20}$$

EXAMPLE 1.2 What is the chance that a leap year selected at random will contain 53 Sundays? **[AU December '07]**

Solution Let A be the event that there are 53 Sundays in a leap year. In a leap year (which consists of 366 days), there are 52 complete weeks and 2 days over. The possible combinations for these two days are:

(i) Sunday and Monday
(ii) Monday and Tuesday
(iii) Tuesday and Wednesday
(iv) Wednesday and Thursday
(v) Thursday and Friday
(vi) Friday and Saturday
(vii) Saturday and Sunday.

$\therefore$ The required probability is

$$P(A) = \frac{\text{Number of favourable cases}}{\text{Number of possible cases}} = \frac{2}{7}$$

EXAMPLE 1.3 If you twice flip a balanced coin, what is the probability of getting at least one head? **[AU May '04]**

Solution When we twice flip a balanced coin, the sample space

$$S = \{HH, HT, TH, TT\}$$

i.e. $\qquad n(S) = 4$

Let A be the event of getting at least one head.

$$A = \{HH, HT, TH\}$$
$$n(A) = 3$$

$$\therefore \qquad P(A) = \frac{n(A)}{n(S)} = \frac{3}{4}$$

EXAMPLE 1.4 Two unbiased dice are thrown and the difference between the number of spots turned up is noted. Find the probability that the difference between the number is 4. **[AU November '09]**

Solution The sample space S in throwing two dice

$$S = \begin{Bmatrix} (1, 1) & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\ (2, 1) & (2, 2) & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 1) & (3, 2) & (3, 3) & (3, 4) & (3, 5) & (3, 6) \\ (4, 1) & (4, 2) & (4, 3) & (4, 4) & (4, 5) & (4, 6) \\ (5, 1) & (5, 2) & (5, 3) & (5, 4) & (5, 5) & (5, 6) \\ (6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & (6, 6) \end{Bmatrix}$$

$$n(S) = 36$$

Let A be the event that the difference between the numbers is 4.

$$A = \{(5, 1), (1, 5), (2, 6), (6, 2)\}$$

$$n(A) = 4$$

$$\therefore \qquad P(A) = \frac{4}{36} = \frac{1}{9}$$

EXAMPLE 1.5 A room has 3 electric lamps. From a collection of 10 electric bulbs of which 6 are good, 3 are selected at random and put in the lamps. Find the probability that the room is lighted. **[AU November '07]**

Solution From a collection of 10 electric bulbs, 3 can be selected in $10C_3$ ways

$$n(S) = 10C_3$$

There are 6 good bulbs and 4 defective. The room is lighted only when at least 1 good electric bulb is put in the lamps. This can happen in the following three mutually exclusive cases:

(i) One good and 2 defective bulbs
(ii) Two good and 1 defective bulbs
(iii) All the 3 good bulbs

$$n(A) = 6C_1 \times 4C_2 + 6C_2 \times 4C_1 + 6C_3$$

The required probability is

$$P(A) = \frac{n(A)}{n(S)} = \frac{6C_1 \times 4C_2 + 6C_2 \times 4C_1 + 6C_3}{10C_3} = \frac{29}{30}$$

Aliter

It can also be done as follows:

$$1-P[\text{all 3 defective}] = 1 - \frac{4C_3}{10C_3}$$

$$= 1 - \frac{1}{30} = \frac{29}{30}$$

EXAMPLE 1.6 A sample space contains three sample points with associated probabilities given by $2p$, p^2 and $4p - 1$. Find the value of p.

Solution We know that the total probability is always 1. Therefore,

$$2p + p^2 + 4p - 1 = 1$$
$$p^2 + 6p - 2 = 0$$

Solving the quadratic equation,

$$p = \frac{-6 \pm \sqrt{36+8}}{2} = -3 \pm \sqrt{11}$$

But the probability is always positive.

$\therefore$ $$p = \sqrt{11} - 3 = 0.3166$$

EXAMPLE 1.7 What is the chance of getting two sixes in two rollings of a single die? **[AU June '07]**

Solution In two rollings of a single die i.e when the die is thrown twice

$$n(S) = 36$$

Let A be the event of getting two sixes, i.e. 6 in the first throw and again 6 in the second throw

$$A = \{(6, 6)\} \Rightarrow n(A) = 1$$

$\therefore$ $$P(A) = \frac{n(A)}{n(S)} = \frac{1}{36}$$

EXAMPLE 1.8 A coin is tossed thrice. What is the chance of getting all heads? **[AU May '04]**

Solution When a coin is tossed thrice

$$n(S) = 2^3 = 8$$

i.e. $$= \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let A be the event of getting all heads

$$A = \{HHH\} \Rightarrow n(A) = 1$$

$\therefore$ $$P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$$

EXAMPLE 1.9 A coin is tossed until the first head occurs. Assuming that the tosses are independent and the probability of a head occurring is p, find the value of p so that the probability that an odd number of tosses is required is equal to 0.6. Can you find a value of p so that the probability is 0.5 that an odd number of tosses is required? **[AU May '04]**

Solution When a coin is tossed, the probability of getting a head is p(given). The probability of not getting a head is $1 - p = q$.

Given that the coin is tossed until the first head occurs. The first head can occur in the first toss or in the second toss, or in the third toss, etc.

To find the probability that the first head occurs in the odd number of tosses, i.e. in the first toss or in the third toss, or in the fifth toss, etc. is

$$p + q^2p + q^4p + \cdots = 0.6 \text{ (given)}$$

$$p(1 + q^2 + q^4 + \cdots) = 0.6$$

$$p(1 - q^2)^{-1} = 0.6$$

$$\frac{p}{(1-q^2)} = 0.6 \quad \Rightarrow \quad \frac{p}{(1-q)(1+q)} = 0.6$$

$$\frac{p}{p(1+q)} = 0.6 \quad \Rightarrow \quad \frac{1}{1+q} = 0.6$$

i.e. $$(1 + q) = \frac{1}{0.6} = 1.667 \quad \Rightarrow \quad q = 0.667$$

$\therefore$ $$p = 1 - q = 0.333$$

If $p + q^2p + q^4p + \cdots = 0.5$, then

$$\frac{p}{(1-q^2)} = 0.5 \quad \Rightarrow \quad \frac{1}{(1+q)} = 0.5$$

$$1 + q = \frac{1}{0.5} = 2$$

$\therefore$ $$q = 1 \quad \Rightarrow \quad p = 1 - q = 0$$

$\therefore$ $$p = 0$$

EXAMPLE 1.10 Find the probability of a card drawn at random from an ordinary pack, is a diamond. **[AU May '05]**

Solution There are 52 cards in a pack of which 13 are diamonds.

$\therefore$ The total number of ways of getting 1 card $= 52C_1$

The number of ways of getting 1 diamond card $= 13C_1$

$\therefore$ The required probability $= \dfrac{\text{Number of favourable events}}{\text{Number of exhaustive events}}$

$$= \frac{13C_1}{52C_1} = \frac{13}{52} = \frac{1}{4}$$

EXAMPLE 1.11 From a pack of 52 cards, 1 card is drawn at random. Find the probability of getting a queen. **[AU November '07]**

Solution A pack contains 4 queens.

A queen may be chosen in 4 ways = $4C_1 = 4$

The total number of ways of selecting a card = $52C_1 = 52$

$$\therefore \quad P(\text{getting a queen}) = \frac{\text{Favourable events}}{\text{Total number of ways}}$$

$$= \frac{4}{52} = \frac{1}{13}$$

EXAMPLE 1.12 Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is 10/21.

Solution The total number of persons in the group = (3 + 2 + 4) = 9.

4 persons can be chosen out of 9 persons in $9C_4$ ways.

$$9C_4 = \frac{9\times8\times7\times6}{1\times2\times3\times4} = 126 \text{ ways}$$

$\therefore$ We have 126 exhaustive number of cases.

We need to choose 4 persons. Out of these 4 persons, 2 persons should be children and, therefore, the remaining 2 persons are either men or women.

The number of ways of choosing 2 children out of 4 children in $4C_2$ ways.

$$4C_2 = 6$$

The remaining 2 persons can be chosen from 5 persons (3 men + 2 women) in $5C_2$ ways, i.e. 10 ways.

$\therefore$ The number of favourable cases = 6 × 10 = 60 ways

$\therefore$ P(group consists of exactly two children)

$$= \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$= \frac{60}{126} = \frac{10}{21}$$

EXAMPLE 1.13 A fair coin is tossed four times. Define the sample space corresponding to this random experiment. Also give the subsets corresponding to the following events and find the probability of each:

(i) more heads than tails are obtained,
(ii) tails occur on the even number of tosses.

Solution The number of possible outcomes in each trial = 2

$\therefore$ When the trial is repeated four times, the sample space S contains $2^4 = 16$ elements.

The sample space S is defined as

$$S = \{HHHH, HHHT, HHTH, HTHH, THHH, HTTH, TTHH, THTH, THHT, HTHT, HHTT, TTTH, TTHT, THTT, HTTT, TTTT\}$$

i.e. $n(S) = 16$

(i) Let A be the event that more heads than tails.

$$A = \{HHHH, HHHT, HHTH, HTHH, THHH\}$$

$$n(A) = 5$$

$$\therefore \quad P(A) = \frac{5}{16}$$

(ii) Let B be the event of getting tails in the even numbered tosses.

$$B = \{HTHT, TTHT, HTTT, TTTT\}$$

$$n(B) = 4$$

$$\therefore \quad P(B) = \frac{4}{16} = \frac{1}{4}$$

EXAMPLE 1.14 From a group of 3 Indians, 4 Pakistanis and 5 Americans, a sub-committee of 4 people is selected by lots. Find the probability that the sub-committee will consist of

(i) 2 Indians and 2 Pakistanis,
(ii) 1 Indian 1 Pakistani and 2 Americans, and
(iii) 4 Americans. **[AU May '05, April '07]**

Solution Given that the selected committee should consist of 4 people.

The total number of people = 3 + 4 + 5 = 12

$\therefore$ 4 people can be chosen from 12 people in $12C_4$ ways.

i.e. $$12C_4 = \frac{12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4} = 495 \text{ ways}$$

(i) 2 Indians can be chosen from 3 Indians in $3C_2$ ways.
2 Pakistanis can be chosen from 4 Pakistanis in $4C_2$ ways.
The number of ways of choosing 2 Indians and 2 Pakistanis
$= 3C_2 \times 4C_2$ ways.
$\therefore$ The number of favourable cases $= 3C_2 \times 4C_2 = 3 \times 6 = 18$
$\therefore$ P (The sub-committee consists of 2 Indians and 2 Pakistanis)

$$= \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$= \frac{18}{495} = \frac{6}{165}$$

(ii) 1 Indian can be chosen from 3 Indians in $3C_1$ ways.
1 Pakistani can be chosen from 4 Pakistanis in $4C_1$ ways.

2 Americans can be chosen from 5 Americans in $5C_2$ ways.

$\therefore$ The number of ways of choosing 1 Indian, 1 Pakistani and 2 Americans

$$= 3C_1 \times 4C_1 \times 5C_2$$
$$= 3 \times 4 \times 10 = 120 \text{ ways}$$

$\therefore$ The number of favourable cases = 120

P(committee consists of 1 Indian, 1 Pakistani and 2 Americans)

$$= \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$= \frac{120}{495} = \frac{24}{99}$$

(iii) 4 Americans can be chosen from 5 Americans in $5C_4$ ways.

P(committee consists of only 4 Americans)

$$= \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$= \frac{5C_4}{495} = \frac{5}{495} = \frac{1}{99}$$

EXAMPLE 1.15 When two dice are thrown, find the probability that

(i) the number 3 is in the first die,

(ii) the sum of the numbers on the faces is 10, and

(iii) the sum of the numbers on the faces is 15.

Solution There are six possible outcomes in a single throw of a die.

When it is thrown twice the possible outcomes are $6^2 = 36$

The sample space S in throwing two dice

$$S = \begin{Bmatrix} (1, 1) & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\ (2, 1) & (2, 2) & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 1) & (3, 2) & (3, 3) & (3, 4) & (3, 5) & (3, 6) \\ (4, 1) & (4, 2) & (4, 3) & (4, 4) & (4, 5) & (4, 6) \\ (5, 1) & (5, 2) & (5, 3) & (5, 4) & (5, 5) & (5, 6) \\ (6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & (6, 6) \end{Bmatrix}$$

$n(S) = 36$

(i) Let A be the event that the number 3 is in the first die.

$$A = \{(3, 1)\ (3, 2)\ (3, 3)\ (3, 4)\ (3, 5)\ (3, 6)\}$$
$$n(A) = 6$$

$$\therefore \quad P(A) = \frac{6}{36} = \frac{1}{6}$$

(ii) Let B be the event that the sum of the faces is 10.

$$B = \{(4, 6)\ (5, 5)\ (6, 4)\}$$
$$n(B) = 3$$
$$\therefore \quad P(B) = \frac{3}{36} = \frac{1}{12}$$

(iii) Let C be the event that the sum of the faces is 15.

$$C = \{\phi\} \Rightarrow P(C) = 0$$

EXAMPLE 1.16 Two cards are drawn at random from a well-shuffled pack of 52 cards. Show that the chance of drawing 2 aces is 1/221.

Solution Let A be the event of drawing 2 aces from a pack of 52 cards. 2 cards can be drawn in $52C_2$ ways, all being equally likely.

$\therefore$ The exhaustive number of cases = $52C_2$

In a pack, there are 4 aces and therefore, 2 aces can be drawn in $4C_2$ ways.

$\therefore$ The required probability is

$$P(A) = \frac{4C_2}{52C_2} = \frac{1}{221}$$

EXAMPLE 1.17 One card is drawn from a standard pack of 52. What is the chance that it is either a king or a queen?

Solution There are 4 kings and 4 queens in a pack of 52 cards.

The probability that the card drawn is a king = $\frac{4C_1}{52C_1} = \frac{4}{52}$

The probability that the card drawn is a queen = $\frac{4C_1}{52C_1} = \frac{4}{52}$

Since the events are mutually exclusive, the probability that it is either a king or a queen = $\frac{4}{52} + \frac{4}{52} = \frac{2}{13}$

EXAMPLE 1.18 From a pack of 52 cards, 3 are drawn at random. Find the chance that they are a king, a queen and a knave.

Solution Let A be the event that the 3 cards chosen are a king, a queen and a knave.

A pack of cards contains 4 kings, 4 queens and 4 knaves.

$\therefore$ A king, a queen and a knave can each be drawn in $4C_1$ ways.

The total number of favourable cases = $4C_1 \times 4C_1 \times 4C_1$

The exhaustive number of cases = $52C_3$

$\therefore$ The required probability is

$$P(A) = \frac{4C_1 \times 4C_1 \times 4C_1}{52C_3}$$

$$= \frac{4 \times 4 \times 4 \times 3 \times 2 \times 1}{52 \times 51 \times 50} = \frac{16}{5525}$$

EXAMPLE 1.19 One bag contains 6 white and 4 black balls. Another contains 4 white and 8 black balls. One of the bags is chosen at random and a draw of 2 balls is made from it. Find the probability that one is white and the other is black.

Solution There are 50% chances of choosing either bag. The probability of selecting the bag I = 1/2 and the probability of drawing 1 white and 1 black balls from the bag I $= \frac{1}{2} \times \frac{6C_1 \times 4C_1}{10C_2} = \frac{4}{15}$

The probability of selecting the bag II = 1/2 and the probability of drawing 1 white and 1 black balls from the bag II $= \frac{1}{2} \times \frac{4C_1 \times 8C_1}{12C_2} = \frac{8}{33}$

Since the events are mutually exclusive, the probability of the event = P(drawing 1 white and 1 black balls from the bag I or II) $= \frac{4}{15} + \frac{8}{33} = \frac{84}{165} = 0.51$

EXAMPLE 1.20 One bag contains 4 white and 2 black balls. Another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that

(i) both are white,

(ii) one is white and the other is black.

Solution

(i) The probability of drawing a white ball from the bag I $= \frac{4}{6}$

The probability of drawing a white ball from the bag II $= \frac{3}{8}$

The probability that both balls drawn are white (independent events)

$$= \frac{4}{6} \times \frac{3}{8} = \frac{1}{4}$$

(ii) The probability of drawing a black ball from the bag I $= \frac{2}{6}$

The probability of drawing a black ball from the bag II $= \frac{5}{8}$

The event of drawing 1 white and 1 black balls can happen in the following two mutually exclusive cases:

(a) Drawing a black ball from the bag I and a white ball from the bag II $= \frac{4}{6} \times \frac{5}{8}$

(b) Drawing a white ball from the bag I and a black ball from the bag II $= \frac{2}{6} \times \frac{3}{8}$

$\therefore$ The required probability $= \frac{4}{6} \times \frac{5}{8} + \frac{2}{6} \times \frac{3}{8} = \frac{13}{24}$

EXAMPLE 1.21 A bag contains 10 white, 6 red, 4 black and 7 blue balls. Five balls are drawn at random. What is the probability that 2 of them are red and 1 is black? **[AU May '05]**

Solution The total number of balls = 10 + 6 + 4 + 7 = 27

5 balls can be drawn from these 27 balls in $27C_5$ ways

$$= \frac{27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5} = 80730 \text{ ways}$$

$\therefore$ The total number of exhaustive events = 80730

2 red balls can be drawn from 6 red balls in $6C_2$ ways $= \frac{6 \times 5}{1 \times 2} = 15$ ways

1 black ball can be drawn from 4 black balls in $4C_1$ ways = 4

$\therefore$ The number of favourable cases

(2 red balls and 1 black ball) = 15 × 4 = 60

The remaining 2 balls can be chosen from 4 black and 7 blue balls in $11C_2$ ways $= \frac{11 \times 10}{1 \times 2} = 55$

$$\therefore \quad \text{The required probability} = \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$= \frac{60 \times 55}{80730} = \frac{330}{8073}$$

EXAMPLE 1.22 What is the probability of having a king and a queen when 2 cards are drawn from a pack of 52 cards? **[AU June '04]**

Solution 2 cards can be drawn from a pack of 52 cards in $52C_2$ ways.

$$= \frac{52 \times 51}{1 \times 2} = 1326 \text{ ways}$$

1 queen card can be drawn from 4 queen cards in $4C_1$ ways.

1 king card can be drawn from 4 king cards in $4C_1$ ways.

The number of favourable ways of drawing 1 queen and 1 king cards

$$= 4 \times 4 = 16 \text{ ways}$$

$\therefore$ P(drawing 1 queen and 1 king cards)

$$= \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$= \frac{16}{1326} = \frac{8}{663}$$

EXAMPLE 1.23 What is the probability that out of 6 cards taken from a full pack, 3 will be black and 3 will be red? **[AU April '06]**

Solution A full pack contains 52 cards. Out of 52 cards, 26 cards are red and 26 are black cards.

6 cards can be chosen from 52 cards in $52C_6$ ways.

$\therefore$ The total number of mutually exclusive events are $52C_6$.

3 black cards can be chosen from 26 black cards in $26C_3$ ways.

3 red cards can be chosen from 26 black cards in $26C_3$ ways.

$\therefore$ The total number of ways of choosing 3 red and 3 black balls

$$= 26C_3 \times 26C_3$$

$\therefore$ The required probability $= \dfrac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$

$$= \frac{26C_3 \times 26C_3}{52C_6} = 0.332$$

EXAMPLE 1.24 Find the probability that a hand at bridge will consist of 3 spades, 5 hearts, 2 diamonds and 3 clubs? **[AU May '04]**

Solution From 52 cards, 13 (3 + 5 + 2 + 3) cards are chosen in $52C_{13}$ ways.

In a pack of 52 cards, there are 13 cards of each type.

3 spades can be chosen from 13 spades in $13C_3$ ways.

5 hearts can be chosen from 13 hearts in $13C_5$ ways.

2 diamonds can be chosen from 13 diamonds in $13C_2$ ways.

3 clubs can be chosen from 13 clubs in $13C_3$ ways.

Hence the total number of favourable cases $= 13C_3 \times 13C_5 \times 13C_2 \times 13C_3$

$\therefore$ The required probability $= \dfrac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$

$$= \frac{13C_3 \times 13C_5 \times 13C_2 \times 13C_3}{52C_{13}} = 0.01293$$

EXAMPLE 1.25 A bag contains 30 balls numbered from 1 to 30. One ball is drawn at random. Find the probability that the number of the drawn ball will be a multiple of

(i) 5 or 9, and

(ii) 5 or 6.

Solution Out of 30 balls, 1 ball can be drawn in $30C_1$ ways.

$\therefore$ The exhaustive number of cases $= 30C_1 = 30$

(i) Multiples of 5 or 9 = (5, 10, 15, 20, 25, 30, 9, 18, 27)
The total number of favourable cases = 9

$\therefore$ The required probability $= \dfrac{9}{30} = \dfrac{3}{10}$

(ii) Multiples of 5 or 6 = (5, 10, 15, 20, 25, 30, 6, 12, 18, 24)
The total number of favourable cases = 10

$\therefore$ The required probability $= \dfrac{10}{30} = \dfrac{1}{3}$

EXAMPLE 1.26 A bag contains 8 white and 4 red balls. Five balls are drawn at random. What is the probability that 2 of them are red and 3 of them are white?

Solution The total number of balls in the bag = 8 + 4 = 12

The number of balls drawn = 5

5 balls can be drawn from 12 balls in $12C_5$ ways.

2 red balls can be drawn from 4 red balls in $4C_2$ ways and 3 white balls can be drawn from 8 white balls in $8C_3$ ways.

$\therefore$ The number of favourable cases $= 4C_2 \times 8C_3$

$$\text{The required probability} = \frac{4C_2 \times 8C_3}{12C_5}$$

$$= \frac{\left(\dfrac{4\times 3}{1\times 2}\right)\left(\dfrac{8\times 7\times 6}{1\times 2\times 3}\right)}{\left(\dfrac{12\times 11\times 10\times 9\times 8}{1\times 2\times 3\times 4\times 5}\right)}$$

$$= \frac{14}{33} = 0.424$$

EXAMPLE 1.27 A six-faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is even?

Solution Let A be the event of getting an even number and B be the event of getting an odd number in a single throw of a die.

$\therefore$ $$P(A) = \frac{2}{3}$$

and $$P(B) = \frac{1}{3}$$

There are two mutually exclusive cases in which the event that the sum of the two numbers thrown is even can occur:

(i) An odd number in the first throw and an odd number in the second throw with probability $= \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

(ii) An even number in the first throw and an even number in the second throw with probability $= \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$\therefore$ The required probability $= \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$

EXAMPLE 1.28 From a bag containing 4 white and 6 black balls, 2 balls are drawn at random. If the balls are drawn one after the other without replacement, find the probaility that

(i) both balls are white,
(ii) both balls are black,
(iii) the first ball is white and the second ball is black,
(iv) one ball is white and the other is black. **[AU April '04]**

Solution The total number of balls $= 4 + 6 = 10$
The number of white balls $= 4$
The number of black balls $= 6$

Two balls are chosen one by one without repalcement

(i) $$P(\text{first ball is white}) = \frac{4}{10}$$

$$P(\text{second ball is white}) = \frac{3}{9}$$

$$\therefore \quad P(\text{both balls are white}) = \frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$$

(ii) $$P(\text{first ball is black}) = \frac{6}{10}$$

$$P(\text{second ball is black}) = \frac{5}{9}$$

$$\therefore \quad P(\text{both balls are white}) = \frac{6}{10} \times \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

(iii) $$P(\text{first ball is white}) = \frac{4}{10}$$

$$P(\text{second ball is black}) = \frac{6}{9}$$

$\therefore$ $P(\text{first ball is white and second ball is black}) = \frac{4}{10} \times \frac{6}{9} = \frac{24}{90}$

(iv) It can happen in any one of the following mutually exclusive cases:

(a) First white and second black, or

(b) First black and second white.

P(one is white and the other is black)

$= P$(first white and second black) $+ P$(first black and second white)

$$= \frac{4}{10} \times \frac{6}{9} + \frac{6}{10} \times \frac{4}{9} = \frac{48}{90}$$

EXAMPLE 1.29 Five men in a company of 20 are graduates. Three men are picked out of 20 at random.

(i) What is the probability that all are graduates?

(ii) What is the probability of at least 1 graduate?

Solution The total number of ways of picking 3 men out of 20 men = $20C_3$

The number of ways of picking 3 graduate men out of 5 graduate = $5C_3$

(i) The probability of picking 3 men, all being graduates out of

$$20 \text{ men} = \frac{5C_3}{20C_3} = \frac{1}{114}$$

(ii) The probability of picking 3 non-graduate men out of 15 non-graduate is $15C_3$

$$P(\text{no graduate}) = \frac{15C_3}{20C_3} = \frac{91}{228}$$

$$P(\text{at least 1 graduate}) = 1 - P(\text{no graduate})$$

$$= 1 - \frac{91}{228} = \frac{137}{228}$$

EXAMPLE 1.30 A box contains 3 white and 2 black balls. We remove at random 2 balls in succession. What is the probability that the first removed ball is white and the second is black.

Solution The required probability $= \frac{3C_1}{5C_1} \times \frac{2C_1}{4C_1} = \frac{6}{20} = \frac{3}{10}$ (if the ball is not replaced)

$$= \frac{3C_1}{5C_1} \times \frac{2C_1}{5C_1} = \frac{6}{25} \quad \text{(if the ball is replaced)}$$

EXAMPLE 1.31 An urn contains 3 white, 4 red and 5 black balls. Two balls are drawn from the urn at random. Find the probability that,

(i) both are of the same colour,
(ii) they are of different colours.

Solution The total number of balls in the urn = 12
The number of ways the 2 balls can be drawn = $12C_2$

(i) P(both are of same colour) = P(both red or both white or both black)

$$= \frac{3C_2 + 4C_2 + 5C_2}{12C_2} = \frac{19}{66}$$

(ii) P(both are of different colours) = P(1white and 1 red or 1white and 1 black or 1 red and 1 black)

$$= \frac{3C_1 4C_1 + 3C_1 5C_1 + 5C_1 4C_1}{12C_2} = \frac{47}{66}$$

EXAMPLE 1.32 A lot consists of 10 good ornaments, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least one is good, (iv) at most one is good, (v) exactly one is good, (vi) neither has major defects, (vii) neither is good.

Solution We are drawing 2 ornaments from the lot without replacement. Therefore, in $16C_2$ ways we can draw those 2 ornaments.

(i) The number of good ornaments = 10
The number of ornaments with minor defects = 4
The number of ornaments with major defects = 2
$\therefore$ The number of defective ornaments = 4 + 2 = 6
The total number ornaments = 16
The total number of ways to choose any 2 from 16 = $16C_2$
The total number of ways to choose 2 from 10 good ornaments = $10C_2$

(ii) P(both good) = $\dfrac{10C_2}{16C_2} = \dfrac{3}{8}$

(ii) P(both having major defects) = $\dfrac{2C_2}{16C_2} = \dfrac{1}{120}$

(iii) P(at least one good) = P(one defective and one good, or both good)

$$= \frac{10C_1 \times 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$$

(iv) $P(\text{at most one good}) = P(\text{one bad and one good or both bad})$

$$= \frac{10C_1 \times 6C_1 + 6C_2}{16C_2} = \frac{5}{8}$$

(v) $P(\text{exactly one good}) = P(\text{one good and one defective})$

$$= \frac{10C_1 \times 6C_1}{16C_2} = \frac{1}{2}$$

(vi) $P(\text{neither has major defects}) = P$(one good and one minor defective or both good, or both having minor defects)

$$= \frac{10C_1 \times 4C_1 + 10C_2 + 4C_2}{16C_2} = \frac{91}{120}$$

(vii) $P(\text{neither good}) = P(\text{both defective})$

$$= \frac{6C_2}{16C_2} = \frac{1}{8}$$

EXAMPLE 1.33 From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is positive?

Solution The number of possible ways to choose 4 numbers from 14 numbers $= 14C_4 = 1001$

If the product is to be positive, then it can happen in the following three mutually exclusive cases:

(i) All the 4 numbers chosen are positive.
(ii) All the 4 numbers chosen are negative, or
(iii) Two positive and two negative numbers

$$\therefore \qquad P(\text{product is positive}) = \frac{6C_4 + 8C_4 + 6C_2 \times 8C_2}{14C_4} = \frac{505}{1001}$$

EXAMPLE 1.34 A box contains tags marked, 1, 2,..., n. Two tags are chosen at random without replacement. Find the probability that the numbers on the tags will be consecutive numbers.

Solution The total number of possible ways to select two tags is nC_2, i.e. we can choose 2 tags out of n in nC_2 ways.

$$n(S) = nC_2$$

The numbers of favourable cases that the numbers on the tags are consecutive numbers are

$= \{(1, 2)\ (2, 3)\ (3, 4)\ (4, 5) \ldots (n-2, n-1)\ (n-1, n)\}$
$(n-1)$ number of favourable cases

$n(A) = n - 1$

$\therefore$ P(numbers on the tags are 2 consecutive numbers) $= \dfrac{n-1}{nC_2} = \dfrac{n-1}{\dfrac{n(n-1)}{1\times 2}} = \dfrac{2}{n}$

EXAMPLE 1.35 Four persons are chosen at random from a group consisting of 4 men, 3 women and 2 children. Find the chance that the selected group contains at least 1 child.

Solution The total number of persons = 4 + 3 + 2 = 9
The number of possible ways to select 4 persons out of 9 = $9C_4$
The total number of men and women = 4 + 3 = 7
The number of favourable ways to select no child = $7C_4$

$$P(\text{selecting no child}) = \frac{7C_4}{9C_4} = \frac{7\times 6\times 5\times 4}{9\times 8\times 7\times 6} = \frac{5}{18}$$

$$P(\text{selecting at least 1 child}) = 1 - P(\text{selecting no child})$$

$$= 1 - \frac{5}{18} = \frac{13}{18}$$

Aliter

A be the event of choosing at least 1 child
$\therefore$ $\bar{A}$ is the event of choosing no children

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{7C_4}{9C_4}$$

$$= -\frac{7\times 6\times 5\times 4}{9\times 8\times 7\times 6} = 1 - \frac{5}{18} = \frac{13}{18}$$

EXAMPLE 1.36 Ten chips numbered 1 through 10 are mixed in a bowl. Two chips are drawn from the bowl successively and without replacement. What is the probability that their sum is 10?

Solution The number of possible ways to choose 2 chips out of 10 chips

$$= 10C_2 \text{ ways}$$

$$n(S) = 10C_2$$

$$= \frac{10\times 9}{1\times 2} = 45$$

A = Event of choosing 2 chips such that their sum is 10

$A = \{(1, 9), (2, 8), (3, 7), (4, 6)\}$

$$n(A) = 4$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{45}$$

EXAMPLE 1.37 A bag contains 10 tickets numbered 1, 2, 3, ..., 10. Three tickets are drawn at random and arranged in ascending order of magnitude. What is the probability that the middle number is 5?

Solution The number of possible ways to select 3 tickets out of $10 = 10C_3$

$$n(S) = 10C_3 = \frac{10\times9\times8}{1\times2\times3} = 120$$

A = the number of the tickets are arranged in the ascending order and the middle number is 5

$$\therefore \quad A = \left\{\begin{matrix} (1,5,6) & (1,5,7) & (1,5,8) & (1,5,9) & (1,5,10) \\ (2,5,6) & (2,5,7) & (2,5,8) & (2,5,9) & (2,5,10) \\ (3,5,6) & (3,5,7) & (3,5,8) & (3,5,9) & (3,5,10) \\ (4,5,6) & (4,5,7) & (4,5,8) & (4,5,9) & (4,5,10) \end{matrix}\right\}$$

$\therefore \; n(A) = 20$

$$P(A) = \frac{n(A)}{n(S)} = \frac{20}{120} = \frac{1}{6}$$

THEOREM 3 For any two events A and B, $P(\bar{A}\cap B) = P(B) - P(A\cap B)$.

Proof To prove that $P(\bar{A}\cap B) = P(B) - P(A\cap B)$

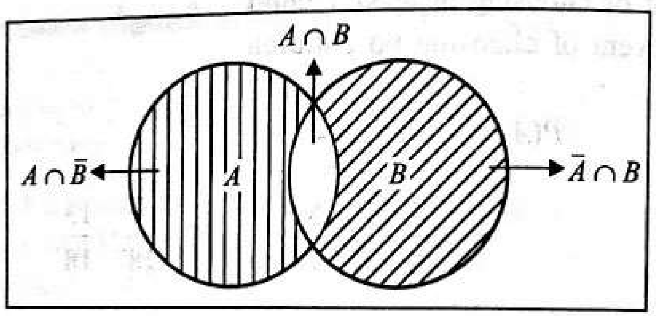

Figure 1.5 Venn diagram.

Using the Venn diagram (Figure 1.5),

$$B = (\bar{A}\cap B)\cup(A\cap B)$$

Since $\bar{A}\cap B$ and $A\cap B$ are mutually exclusive events

$$\therefore \quad P(B) = P[(\bar{A}\cap B)\cup(A\cap B)] = P(\bar{A}\cap B) + P(A\cap B)$$

$$\therefore \quad P(\bar{A}\cap B) = P(B) - P(A\cap B)$$

Hence the proof.

Note: Similarly, it can be shown that $P(A\cap\bar{B}) = P(A) - P(A\cap B)$.

THEOREM 4 If $B\subset A$, then

(i) $P(B) \le P(A)$

(ii) $P(A\cap\bar{B}) = P(A) - P(B)$

Proof From Figure 1.6, $B \subset A$, B and $A \cap \bar{B}$ are two mutually exclusive events and $A = B \cup (A \cap \bar{B})$, then

$$P(A) = P\left[B \cup (A \cap \bar{B})\right]$$
$$P(A) = P(B) + P(A \cap \bar{B})$$
$$P(A \cap \bar{B}) = P(A) - P(B)$$

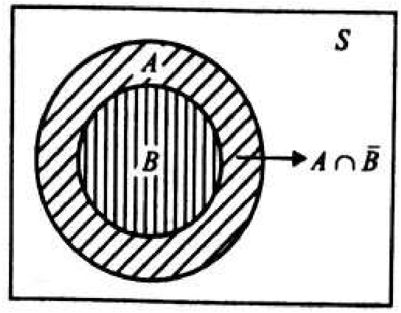

Figure 1.6 $A = B \cup (A \cap \bar{B})$ or venn diagram.

But, $P(A \cap \bar{B}) \geq 0$

$\therefore$ $$P(B) \leq P(A)$$

Hence the proof.

Note: From the above theorem, it can be shown that

- If $A \cap B \subset A$, then $P(A \cap B) \leq P(A)$, and
- If $A \cap B \subset B$, then $P(A \cap B) \leq P(B)$

THEOREM 5 *(*Additive Law of Probability) If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof From the Venn diagram (Figure 1.5), $A \cap \bar{B}$ and $A \cap B$ are two mutually exclusive events and $A = (A \cap \bar{B}) \cup (A \cap B)$, then

$$P(A) = P\left[(A \cap \bar{B}) \cup (A \cap B)\right] = P(A \cap \bar{B}) + P(A \cap B)$$
$$B = (\bar{A} \cap B) \cup (A \cap B)$$
$$P(B) = P(\bar{A} \cap B) + P(A \cap B)$$
$$P(A) + P(B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B) + P(A \cap B) \qquad (1.3)$$

From Figure 1.5,

$$A \cup B = (A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (A \cap B)$$

Using it in Eq. (1.3)

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

$$\Rightarrow \qquad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Hence the proof.

Note: From the above result,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

i.e. $$P(A \cup B) \leq P(A) + P(B) \qquad [\because P(A \cap B) \geq 0]$$

THEOREM 6 If $A \cap B = \phi$, then show that $P(A) \leq P(\bar{B})$.

Proof We have $A = (A \cap B) \cup (A \cap \bar{B})$, from Figure 1.5

i.e. $$A = \phi \cup (A \cap \bar{B}) = A \cap \bar{B}$$

$$\Rightarrow \qquad A \subset \bar{B}$$

$$\therefore \qquad P(A) \leq P(\bar{B})$$

EXAMPLE 1.38 A is known to hit the target is 2 out of 5 shots, whereas B is known to hit the target is 3 out of 4 shots. Find the probability of the target being hit when they both try. **[AU December '03]**

Solution Given:

$$P(A) = 2/5 \text{ and } P(B) = 3/4$$

Since A and B are trying independently

$$\therefore \qquad P(A \cap B) = P(A)\, P(B)$$

To find the probability of the target being hit when they both try, we use additive law of probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A) \cdot P(B)$$

$$= \frac{2}{5} + \frac{3}{4} - \frac{2}{5} \times \frac{3}{4} = \frac{8 + 15 - 6}{20}$$

$$= \frac{17}{20}$$

EXAMPLE 1.39 If $P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$, find $P(\bar{A} \cap \bar{B})$ and $P(\bar{A} \cup \bar{B})$. **[AU May '04, November '07]**

Solution Given:

$$P(A) = 0.4,\ P(B) = 0.7 \text{ and } P(A \cap B) = 0.3$$

$$\therefore \qquad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.4 + 0.7 - 0.3 = 0.8$$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - 0.8 = 0.2$$

$$P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = 1 - 0.3 = 0.7$$

EXAMPLE 1.40 If $P(A) = 3/4$ and $P(B) = 5/8$, prove that $P(A \cap B) \geq 3/8$.

Solution

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{4} + \frac{5}{8} - P(A \cap B)$$
$$P(A \cup B) = \frac{11}{8} - P(A \cap B)$$

But,

$$P(A \cup B) \leq 1 \Rightarrow \frac{11}{8} - P(A \cap B) \leq 1$$
$$\frac{11}{8} - 1 \leq P(A \cap B) \Rightarrow \frac{3}{8} \leq P(A \cap B)$$
$$\Rightarrow P(A \cap B) \geq \frac{3}{8}$$

EXAMPLE 1.41 If A and B are two events such that $P(A) = 3/4$ and $P(B) = 5/8$, show that

(i) $P(A \cup B) \geq \frac{3}{4}$

(ii) $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$

Solution We know that

$$A \subset A \cup B$$
$$\Rightarrow P(A) \leq P(A \cup B) \Rightarrow \frac{3}{4} \leq P(A \cup B)$$
$$\Rightarrow P(A \cup B) \geq \frac{3}{4}$$

Since

$$A \cap B \subseteq B \Rightarrow P(A \cap B) \leq P(B) = \frac{5}{8} \Rightarrow P(A \cap B) \leq \frac{5}{8} \quad \text{(i)}$$

But,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$$

i.e.

$$\frac{3}{4} + \frac{5}{8} - P(A \cap B) \leq 1 \Rightarrow \frac{3}{4} + \frac{5}{8} - 1 \leq P(A \cap B)$$
$$\frac{6+5-8}{8} \leq P(A \cap B) \Rightarrow \frac{3}{8} \leq P(A \cap B) \quad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$$

EXAMPLE 1.42 When A and B are two mutually exclusive events, are the values $P(A) = 0.6$ and $P(A \cap \bar{B}) = 0.5$ consistent? Why?

When A and B are mutually exclusive, $P(A \cap B) = 0$

Solution We know that

$$A = (A \cap B) \cup (A \cap \bar{B})$$

From Figure 1.5, $(A \cap B)$ and $(A \cap \bar{B})$ are mutually exclusive

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$\therefore$

$$P(A) = 0 + P(A \cap \bar{B}) \Rightarrow P(A) = P(A \cap \bar{B})$$

But,

$$0.6 \neq 0.5$$

Therefore, these values are not consistent.

EXAMPLE 1.43 If A and B are events with $P(A) = 3/8$, $P(B) = 1/2$ and $P(A \cap B) = 1/4$, find $P(A^C \cap B^C)$.

Solution $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{5}{8}$$

$$P(A^C \cap B^C) = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$$

EXAMPLE 1.44 A person is known to hit the target is 3 out of 4 shots, whereas another person is known to hit the target 2 out of 3 shots. Find the probability of the target being hit at all when they both try.

[AU November '03]

Solution $$P(\text{first person hits the target}) = \frac{3}{4}$$

$$P(\text{second person hits the target}) = \frac{2}{3}$$

$$P(\text{both persons hit the target}) = \frac{3}{4} \times \frac{2}{3}$$

The events are not mutually exclusive because both of them may hit the target. Using the additive law of probability, the required probability

$$= \frac{3}{4} + \frac{2}{3} - \frac{3}{4} \times \frac{2}{3} = \frac{11}{12}$$

Multiplication Law of Probability

If A and B are two independent events, then

$$P(A \cap B) = P(A)\,P(B)$$

THEOREM 7 If A and B are independent events, then $\bar{A}$ and B are also independent.

Proof Given: A and B are independent events.

$\therefore$
$$P(A \cap B) = P(A)\, P(B)$$

To prove $\bar{A}$ and B are also independent, we have to prove that

$$P(\bar{A} \cap B) = P(\bar{A})\, P(B)$$

From Figure 1.5, $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive events and $B = (A \cap B) \cup (\bar{A} \cap B)$

$$\begin{aligned} P(B) &= P[(A \cap B) \cup (\bar{A} \cap B)] = P(A \cap B) + P(\bar{A} \cap B) \\ &= P(A)\, P(B) + P(\bar{A} \cap B) \qquad [\because A \text{ and } B \text{ are independent}] \\ P(B) - P(A)\, P(B) &= P(\bar{A} \cap B) \\ P(B)\, [1 - P(A)] &= P(\bar{A} \cap B) \\ P(B) \cdot P(\bar{A}) &= P(\bar{A} \cap B) \qquad [\because P(A) + P(\bar{A}) = 1] \end{aligned}$$

Therefore, $\bar{A}$ and B are independent.
Hence the proof.

Note: Similarly, we can prove that A and $\bar{B}$ are also independent.

THEOREM 8 If A and B are independent events, then prove that $\bar{A}$ and $\bar{B}$ are also independent events. **[AU May '07]**

Proof Given: A and B are independent events.

$\therefore$
$$P(A \cap B) = P(A)\, P(B)$$

We have

$$\begin{aligned} \bar{A} \cap \bar{B} &= A \cup B \\ P(\bar{A} \cap \bar{B}) &= P(A \cup B) = 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cup B)] \\ &= 1 - [P(A) - P(B) + P(A)\, P(B)] \\ &= [1 - P(A)] - P(B)\, [1 - P(A)] \\ &= [1 - P(B)]\, [1 - P(A)] \\ P(\bar{A} \cap \bar{B}) &= P(\bar{A})\, P(\bar{B}) \end{aligned}$$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent events.
Hence the proof.

EXAMPLE 1.45 Can two events be simultaneously independent and mutually exclusive? Explain.

Solution If A and B are mutually exclusive, then $P(A \cap B) = 0$ and if A and B are independent, then $P(A \cap B) = P(A)\, P(B)$.

Hence A and B can be both independent and mutually exclusive, provided either of the events is an impossible event, (i.e.) either $P(A) = 0$ or $P(B) = 0$.

EXAMPLE 1.46 The odds against A solving a certain problem are 4 to 3 and odds in favour of B solving the same problem are 7 to 5. What is the probability that the problem is solved if they both try independently?

Solution The probability that A solves the problem is

$$P(\bar{A}) = \frac{4}{4+3} = \frac{4}{7} \Rightarrow P(A) = 1 - \frac{4}{7} = \frac{3}{7}$$

The probability that B solves the problem is

$$P(B) = \frac{7}{7+5} = \frac{7}{12} \Rightarrow P(\bar{B}) = 1 - \frac{7}{12} = \frac{5}{12}$$

$\therefore$ The probability that the problem is solved

$$= P(\text{at least one of } A \text{ and } B \text{ solves the problem})$$
$$= 1 - P(\text{none solves the problem})$$
$$= 1 - P(\bar{A} \cap \bar{B})$$

If A and B are independent then $\bar{A}$ and $\bar{B}$ are also independent

$\therefore$ The required probability $= 1 - P(\bar{A})\,P(\bar{B}) = 1 - \frac{4}{7} \times \frac{5}{12} = 1 - \frac{5}{21} = \frac{16}{21}$

EXAMPLE 1.47 From a group of 8 children, 5 boys and 3 girls, 3 children are selected at random. Calculate the probabilities that the selected group contains (i) no girl, (ii) only 1 girl, (iii) 1 particular girl, (iv) at least one girl and (v) more girls than boys.

Solution There are 8 children. Out of 8 children, 3 can be selected in $8C_3$ ways.

The number of boys = 5
The number of girls = 3

To find the probability that the selected group contains

(i) No girl, i.e. all the 3 are boys

$$\text{The required probability} = \frac{5C_3}{8C_3} = \frac{5}{28}$$

(ii) Only 1 girl, i.e. 1 girl and two boys

$$\text{The required probability} = \frac{3C_1 \times 5C_2}{8C_3} = \frac{15}{28}$$

(iii) One particular girl, i.e. 2 boys and 1 particular girl

$$\text{The required probability} = \frac{5C_2 \times 1}{8C_3} = \frac{5}{28}$$

(iv) At least 1 girl can happen in the following mutually exclusive cases:

(a) One girl and 2 boys or
(b) Two girls and 1 boy, or
(c) All the 3 girls.

$$\text{The required probability} = \frac{3C_1 \times 5C_2 + 3C_2 \times 5C_1 + 3C_3}{8C_3} = \frac{23}{28}$$

(v) More girls than boys. It can happen in the following two mutually exclusive cases:

(a) Two girls and 1 boy, or
(b) Three girls.

$$\text{The required probability} = \frac{3C_2 \times 5C_1}{8C_3} + \frac{3C_3}{8C_3} = \frac{2}{7}$$

EXAMPLE 1.48 A is one of the 6 horses entered for a race and is to be ridden by one of the 2 jockeys B and C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win, with rider C, A's chances tripled

(i) Find the probability that A wins.
(ii) What are odds against A's winning?

Solution The chance for A to win can happen in one of the two mutually exclusive cases:

(a) B rides and A wins
(b) C rides and A wins.

The chance for B to ride the horse A is 2 to 1.

i.e. $$P(B) = \frac{2}{3} \Rightarrow P(C) = 1 - \frac{2}{3} = \frac{1}{3} \text{ (only } B \text{ or } C \text{ rides)}$$

The chance for A to win is equally likely if B rides, i.e. $\frac{1}{6}$

The chance for A to win if C rides is tripled, i.e. $\frac{3}{6}$

$$\text{The probability of } A\text{'s win} = \frac{2}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{3}{6} = \frac{5}{18}$$

To find the odds against A winning:

$$\text{The probability of } A \text{ losing} = 1 - \frac{5}{18} = \frac{13}{18}$$

i.e. 13 out of 18

$\therefore$ Odds against A's winning are 13 : 5 (18 − 13 = 5).

EXAMPLE 1.49 From 100 tickets numbered 1, 2, ..., 100, 4 are drawn at random. What is the probability that 3 of them will be a numbered from 1 to 20 and the fourth will bear any number from 21 to 100?

Solution There are 100 tickets. Out of 100, 4 can be drawn in $100C_4$ ways. 3 tickets to be selected from 1 to 20. Therefore, the favourable cases are $20C_3$ and 1 ticket from 21 to 100 is $80C_1$

$$\therefore \qquad \text{The required probability} = \frac{20C_3 \times 80C_1}{100C_4}$$

$$= \frac{91200}{3921225} = \frac{1216}{52283} = 0.02326$$

EXAMPLE 1.50 Two-third of the students in a class are boys and the rest are girls. It is known that the probability of a girl getting first class is 0.25 and that of a boy getting first class is 0.28. Find the probability that a student chosen at random will get first class marks in the subject.

Solution In a class, $\frac{2}{3}$ are boys and the rest, i.e. $\frac{1}{3}$ are girls.

$$P(B) = \frac{2}{3}$$

and
$$P(G) = \frac{1}{3}$$

The probability of a boy getting first class is 0.25 and the probability of a girl getting first class is 0.28

A student is chosen at random. It may be a girl or a boy.

$$\therefore \qquad \text{The required probability} = P(G) \times 0.25 + P(B) \times 0.28$$

$$= \frac{1}{3} \times 0.25 + \frac{2}{3} \times 0.28 = 0.27$$

EXAMPLE 1.51 A man and woman appear in an interview for two vacancies in the same post. The probability of man's selection is 1/7 and that of woman's selection is 1/5. What is the probability that only one of them will be selected?

Solution Given: $P(\text{man's selection}) = \frac{1}{7} \Rightarrow P(\text{man's rejection}) = \frac{6}{7}$

$$P(\text{woman's selection}) = \frac{1}{5} \Rightarrow P(\text{woman's rejection}) = \frac{4}{5}$$

Only one of them will be selected can happen in the following two mutually exclusive ways:

(i) Man selected and woman rejected, or
(ii) Woman selected and man rejected.

$$\text{The required probability} = \frac{1}{7}\times\frac{4}{5}+\frac{1}{5}\times\frac{6}{7}=\frac{4+6}{35}=\frac{2}{7}$$

EXAMPLE 1.52 A town has two doctors X and Y operating independently. If the probability that doctor X is available is 0.9 and that for Y is 0.8, what is the probability that at least one doctor is available when needed?

Solution Given:

$$P(X \text{ available}) = P(\bar{X}) = 0.9$$

and
$$P(Y \text{ available}) = 0.8 = P(Y)$$

$$P(X \text{ not available}) = 1 - 0.9 = 0.1 = P(\bar{X})$$

$$P(Y \text{ not available}) = 0.2 = P(\bar{Y})$$

X and Y are independently operating

$$\begin{aligned}\therefore \quad P(\text{at least one is available}) &= 1 - P(\text{both not available})\\ &= 1 - P(\bar{X}\cap\bar{Y})\\ &= 1 - P(\bar{X})\,P(\bar{Y})\\ &= 1 - (0.1)\,(0.2)\\ &= 0.98\end{aligned}$$

EXAMPLE 1.53 The odds that a book will be favourably reviewed by three independent critics are 5 to 2, 4 to 3 and 3 to 4 respectively. What is the probability that of the three reviews, a majority will be favourable?

Solution Let the three critics be A, B and C. The odds that a book will be reviewed favourably by A, B and C are 5 to 2, 4 to 3 and 3 to 4 respectively.

i.e.
$$P(A) = \frac{5}{7},\ P(B) = \frac{4}{7},\ P(C) = \frac{3}{7}$$

$$\Rightarrow \quad P(\bar{A}) = \frac{2}{7},\ P(\bar{B}) = \frac{3}{7},\ P(\bar{C}) = \frac{4}{7}$$

To find the probability that a majority will be favourable is the same as the probability that at least two of them are favourable.

$$\begin{aligned}\therefore \text{ The required probability} &= P(A\cap B\cap\bar{C}) + P(A\cap\bar{B}\cap C)\\ &\quad + P(\bar{A}\cap B\cap C) + P(A\cap B\cap C)\\ &= P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(\bar{B})\,P(C)\\ &\quad + P(\bar{A})\,P(B)\,P(C) + P(A)\,P(B)\,P(C)\end{aligned}$$

$$= \frac{2}{7}\times\frac{4}{7}\times\frac{4}{7}+\frac{5}{7}\times\frac{3}{7}\times\frac{3}{7}+\frac{2}{7}\times\frac{4}{7}\times\frac{3}{7}+\frac{5}{7}\times\frac{4}{7}\times\frac{3}{7}$$

$$= \frac{209}{343}$$

EXAMPLE 1.54 The odds against a certain event are 5 to 2 and the odds in favour of another (independent) event are 6 to 5. Find the chance that at least one of the events will happen.

Solution The odds against a certain event (A) are 5 to 2,

i.e. $$P(\bar{A}) = \frac{5}{7} \Rightarrow P(A) = \frac{2}{7}$$

The odds in favour of another event (B) are 6 to 5,

i.e. $$P(B) = \frac{6}{11} \Rightarrow P(\bar{B}) = \frac{5}{11}$$

The chance that at least one of the events will happen (i.e. either A and $\bar{B}$ or $\bar{A}$ and B, or both A and B are in favour)

$$= P(A\cap\bar{B}) + P(\bar{A}\cap B) + P(A\cap B)$$

$$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B) + P(A)\,P(B)$$

$$= \frac{2}{7}\times\frac{5}{11}+\frac{5}{7}\times\frac{6}{11}+\frac{2}{7}\times\frac{6}{11} = \frac{52}{77}$$

The required probability $= \dfrac{52}{77}$

EXAMPLE 1.55 The probability that a 50 years old man will be alive at 60 is 0.83 and the probability that a 45 years old woman will be alive at 55 is 0.87. What is the probability that a man who is 50 and his wife who is 45 will both be alive 10 years hence?

Solution P(man alive after 10 years) = 0.83

P(woman alive after 10 years) = 0.87

P(both alive 10 years hence) = 0.83 × 0.87

= 0.7221 ($\because$ they are independent)

EXAMPLE 1.56 Suppose the events $A_1, A_2, ..., A_n$ are independent and that

$$P(A_i) = \frac{1}{i+1}, 1 \le i \le n$$

Find the probability that none of the n event occurs, justifying each step in year calculation.

Solution Given, the probability that the independent events $A_1, A_2, \ldots, A_n$ occur is

$$P(A_i) = \frac{1}{i+1}, 1 \le i \le n$$

The probability that the event does not occur is

$$P(\bar{A}_i) = 1 - P(A_i) = 1 - \frac{1}{i+1} = \frac{i}{i+1}, 1 \le i \le n$$

If $A_1, A_2, \ldots, A_n$ are independent, then $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ are independent.

$\therefore$ The probability that none of the event occurs is

$$\begin{aligned} P(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \bar{A}_n) &= P(\bar{A}_1)\, P(\bar{A}_2) \cdots P(\bar{A}_n) \\ &= \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \cdots \frac{n-2}{n-1}, \frac{n-1}{n} \end{aligned}$$

$\therefore$ The required probability $= \dfrac{1}{n}$

EXAMPLE 1.57 A man has a bunch of n keys, exactly one of which fits a lock. If the man tries to open the lock by trying the keys at random, what is the probability that he requires exactly k attempts if he rejects the keys already tried? Find the same probability if he does not reject the keys already tried.

Solution A man has n keys, out of which, only one key opens the lock and all the remaining $(n - 1)$ keys are not fit .

The probability that the man tries and opens the lock at the kth attempt by replacing the keys already tried is 1st, 2nd, ..., $(k - 1)$th keys which are unfit and are rejected. At the kth trial, there will be $[n - (k - 1)]$ keys of which one will be fit.

$$\text{The required probability} = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{n-(k-1)}{n-(k-2)} \cdot \frac{1}{n-(k-1)}$$

The man does not reject the keys already tried. Each time he tries with n keys and in the first $(k - 1)$ trials, he fails and in the kth trial, he opens the lock.

$$\text{The required probability} = \frac{n-1}{n} \cdot \frac{n-1}{n} \ldots (k-1)\text{times} \times \frac{1}{n} = \left(\frac{n-1}{n}\right)^{k-1} \frac{1}{n}$$

THEOREM 9 If A, B, C are mutually independent events, then $A \cup B$ and C are also independent.

Proof Given: A, B, and C are mutually independent events.

From the Venn diagram (Figure 1.7),

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$$
$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

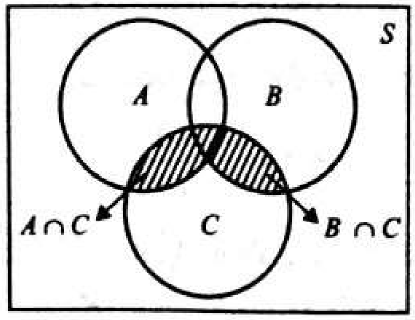

Figure 1.7 Venn diagram.

Since A, B and C are independent, we have

$\therefore$ $$P(A \cup B \cap C) = P(A)\,P(C) + P(B)\,P(C) - P(A)\,P(B)\,P(C)$$
$$= P(C)\,[P(A) + P(B) - P(A)\,P(B)]$$
$$= P(C)\,[P(A) + P(B) - P(A \cap B)]$$
$\therefore$ $$P[(A \cup B) \cap C] = P(C)\,P(A \cup B)$$

$\Rightarrow$ $A \cup B$ and C are also independent.
Hence the proof.

THEOREM 10 If A, B and C are random events in a sample space and if A, B and C are pairwise independent and A is independent of $B \cup C$, then A, B and C are mutually independent.

Proof Given A, B and C are pairwise independent.

$\therefore$ $$P(A \cap B) = P(A)\,P(B),\ P(B \cap C) = P(B)\,P(C)$$

and $$P(A \cap C) = P(A)\,P(C)$$
And also $$P[A \cap (B \cup C)] = P(A)\,P(B \cup C)$$

To prove A, B and C are mutually independent, we have to prove

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

Now $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, from Figure 1.7

$$P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$
$$P(A)\,P(B \cup C) = P(A)\,P(B) + P(A)\,P(C) - P(A \cap B \cap C)$$
$$P(A)\,[P(B) + P(C) - P(B \cap C)] = P(A)\,[P(B) + P(C) - P(A \cap B \cap C)]$$
$$P(A)[P(B) + P(C)] - P(A)\,P(B)\,P(C) = P(A)[P(B) + P(C)] - P(A \cap B \cap C)$$
$\Rightarrow$ $$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$
Hence the proof.

THEOREM 11 If A, B and C are independent, prove that A and $B \cup C$ are also independent.

Proof From Figure 1.7,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\begin{aligned} P(A \cap B \cup C) &= P[(A \cap B) \cup (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C) \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\ &= P(A)\, P(B) + P(A)\, P(C) - P(A)\, P(B)\, P(C) \\ &= P(A)\, [P(B) + P(C) - P(B)\, P(C)] \end{aligned}$$

$$P[(A \cap (B \cup C)] = P(A)\, P(B \cup C)$$

Hence the proof.

EXAMPLE 1.58 Two defective tubes get mixed up with two good ones. The tubes are tested one by one until both defectives are found. What is the probability that the last defective tube is obtained on

(i) the second test,
(ii) the third test, and
(iii) the fourth test?

Solution Let D_1 be the event of choosing the first defective tube, D_2 be the event of choosing the second defective tube.

$$P(D_1) = \frac{2C_1}{4C_1} = \frac{1}{2}$$

$$P(D_2) = \frac{1}{3}$$

Let G_i denote the event of choosing ith good tube.

(i) The second defective tube is obtained on the second test means that the first defective tube is chosen in the first test

$$P(D_1 \cap D_2) = P(D_1)\, P(D_2) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

(ii) The second defective tube occurs in the third test can happen in two mutually exclusive ways:

$$G_1 \cap D_1 \cap D_2$$

or

$$D_1 \cap G_1 \cap D_2$$

The required probability $= P[(G_1 \cap D_1 \cap D_2) \cup (D_1 \cap G_1 \cap D_2)]$

$$= P(G_1)\, P(D_1)\, P(D_2) + P(D_1)\, P(G_1)\, P(D_2)$$

$$= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

(iii) The second defective tube occurs in the fourth test can happen in three mutually exclusive ways:

$$G_1 \cap G_2 \cap D_1 \cap D_2$$

or $$G_1 \cap D_1 \cap G_2 \cap D_2$$

or $$D_1 \cap G_1 \cap G_2 \cap D_2$$

The required probability $= P(G_1 \cap G_2 \cap D_1 \cap D_2) + P(G_1 \cap D_1 \cap G_2 \cap D_2) + P(D_1 \cap G_1 \cap G_2 \cap D_2)$

$$= P(G_1)\,P(G_2)\,P(D_1)\,P(D_2) + P(G_1)\,P(D_1)\,P(G_2)\,P(D_2) + P(D_1)\,P(G_1)\,P(G_2)\,P(D_2)$$

$$= \frac{2}{4}\cdot\frac{1}{3}\cdot\frac{2}{2}\cdot 1 + \frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot 1 + \frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{2}$$

EXAMPLE 1.59 If A, B and C are any three events such that $P(A) = P(B) = P(C) = 1/4$, $P(A \cap B) = P(B \cap C) = 0$ and $P(C \cap A) = 1/8$, find the probability that at least one of the events A, B and C occurs.

Solution We know that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Given: $$P(A \cap B) = 0 \Rightarrow A \cap B = \phi$$

$$P(B \cap C) = 0 \Rightarrow B \cap C = \phi$$

and $$A \cap B \cap C = \phi \Rightarrow P(A \cap B \cap C) = 0$$

$\therefore$

$$P(A \cup B \cup C) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 0 - 0 - \frac{1}{8} + 0$$

$$= \frac{3}{4} - \frac{1}{8} = \frac{5}{8}$$

EXAMPLE 1.60 In a shooting test, the probability of hitting the target is 1/2 for A, 2/3 for B and 3/4 for C. If all of them fire at the target, find the probability that

(i) none of them hits the target,
(ii) at least one of them hits the target.

Solution Given:

$$P(A) = \frac{1}{2},\ P(B) = \frac{2}{3} \text{ and } P(C) = \frac{3}{4}$$

$\therefore$ The probability of not hitting the target is

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(\bar{B}) = 1 - P(B) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$P(\bar{C}) = 1 - P(C) = 1 - \frac{3}{4} = \frac{1}{4}$$

(i) The probability that none of them hits the target

$$P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A})\,P(\bar{B})\,P(\bar{C}) \; [\because \text{events are independent}]$$

$$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}$$

(ii) The probability that at least one of them hits the target

$$= 1 - P(\text{none of them hits the target})$$

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$

$$= 1 - \frac{1}{24} = \frac{23}{24}$$

EXAMPLE 1.61 A and B toss a fair coin alternately with the understanding that the one who obtains the head first wins. If A starts, what is his chance of winning?

Solution

$$P(A) = \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

$$P(\bar{A}) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(\bar{B}) = 1 - \frac{1}{2} = \frac{1}{2}$$

A wins if he obtains head in the first toss or in the third toss, or in the fifth toss, etc.

The required probability $= P(A) + P(\bar{A})\,P(\bar{B})\,P(A)$

$$+ P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(\bar{B})\,P(A) + \cdots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots\right]$$

$$= \frac{1}{2}\left[1-\left(\frac{1}{2}\right)^2\right]^{-1} = \frac{1}{2}\left(1-\frac{1}{4}\right)^{-1}$$

$$= \frac{1}{2}\left(\frac{3}{4}\right)^{-1} = \frac{1}{2}\times\frac{4}{3} = \frac{2}{3}$$

EXAMPLE 1.62 A and B alternately throw a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61. **[AU April '03]**

Solution A pair of dice thrown.

$\therefore$ $n(S) = 6^2 = 36$

A = the event of throwing the sum of the numbers in the two dice is 6

$A = [(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)]$

$\therefore$ $$P(A) = \frac{5}{36}$$

and $$P(\bar{A}) = 1-\frac{5}{36} = \frac{31}{36}$$

B = the event of throwing the sum of the numbers in the two dice is 7.

$B = [(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)]$

$\therefore$ $$P(B) = \frac{6}{36}$$

and $$P(\bar{B}) = 1-\frac{6}{36} = \frac{30}{36} = \frac{5}{6}$$

If A begins, then he plays first, third, fifth, ... trials. Therefore, A will win if he throws 6 in the first trial or A does not throws 6 in the first trial and B, does not throw 7 in the second, and A throws 6 in the third trial and so on. That is, the probability of A winning the game is

$$= P(A) + P(\bar{A}\cap\bar{B}\cap A) + P(\bar{A}\cap\bar{B}\cap\bar{A}\cap\bar{B}\cap A) + P(\bar{A}\cap\bar{B}\cap\bar{A}\cap\bar{B}\cap\bar{A}\cap\bar{B}\cap A) \ldots$$

$$= P(A) + P(\bar{A})\,P(\bar{B})\,P(A) + P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(\bar{B})\,P(A) + P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(\bar{B})\,P(A) + \cdots$$

$$= \frac{5}{36} + \left(\frac{31}{36}\times\frac{5}{6}\right)\times\frac{5}{36} + \left(\frac{31}{36}\times\frac{5}{6}\right)^2\times\frac{5}{36} + \left(\frac{31}{36}\times\frac{5}{6}\right)^3\times\frac{5}{36} + \cdots$$

$$= \frac{5}{36}\left[1-\left(\frac{31}{36}\times\frac{5}{6}\right)\right]^{-1} = \frac{5}{36}\times\left[1-\left(\frac{155}{216}\right)\right]^{-1}$$

$$= \frac{5}{36}\times\left(\frac{216-155}{216}\right)^{-1} = \frac{5}{36}\times\left(\frac{61}{216}\right)^{-1} = \frac{30}{61}$$

EXAMPLE 1.63 Two fair dice are thrown independently. Three events A, B and C are respectively defined as follows:

(i) odd face with the first die,
(ii) odd face with the second die, and
(iii) the sum of the two numbers in the two dice is odd.

Are the events A, B and C mutually independent or pairwise independent?

Solution Two dice are thrown.

$$n(S) = 6^2 = 36$$

$\therefore$

(i) Let A be the event that the odd face in the first die. Therefore,

$$A = \begin{bmatrix} (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\ (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6) \end{bmatrix}$$

$$n(A) = 18$$

$$\therefore \qquad P(A) = \frac{18}{36} = \frac{1}{2}$$

(ii) Let B be the event that the odd face in the second die. Therefore,

$$B = \begin{bmatrix} (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), \\ (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3), \\ (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5) \end{bmatrix}$$

$$n(B) = 18$$

$$\therefore \qquad P(B) = \frac{18}{36} = \frac{1}{2}$$

(iii) Let C be the event that the sum of the numbers in the two dice is odd. Therefore,

$$C = \begin{bmatrix} (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), \\ (3, 2), (3, 4), (3, 6), (4, 1), (4, 3), (4, 5), \\ (5, 2), (5, 4), (5, 6), (6,1), (6, 3), (6, 5) \end{bmatrix}$$

$$n(C) = 18$$

$$\therefore \qquad P(C) = \frac{18}{36} = \frac{1}{2}$$

$$\because \qquad A \cap B \cap C = \{\ \} = \phi, \text{ we have}$$

$$P(A \cap B \cap C) = 0$$

$$P(A)\, P(B)\, P(C) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$P(A \cap B \cap C) \neq P(A)\, P(B)\, P(C)$$

$\therefore$ They are not mutually independent.

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

$$P(B \cap C) = \frac{9}{36} = \frac{1}{4}$$

$$P(A \cap C) = \frac{9}{36} = \frac{1}{4}$$

$$P(A)\ P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A \cap B)$$

$$P(B \cap C) = P(B) \cdot P(C) = \frac{1}{4}$$

$$P(A \cap C) = P(A) \cdot P(C) = \frac{1}{4}$$

Hence they are pairwise independent.

EXAMPLE 1.64 A problem is given to 3 students whose chances of solving it are 1/2, 1/3 and 1/4 respectively. What is the probability that the

(i) problem will be solved,
(ii) exactly 2 of them will solve the problem? **[AU May '04]**

Solution Given:

$$P(A) = \frac{1}{2} \Rightarrow P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(B) = \frac{1}{3} \Rightarrow P(\bar{B}) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{1}{4} \Rightarrow P(\bar{C}) = 1 - P(C) = 1 - \frac{1}{4} = \frac{3}{4}$$

(i) P(problem will be solved) = 1 − P(problem will not be solved)

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - P(\bar{A})P(\bar{B})P(\bar{C})$$

$$= 1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{3}{4}$$

(ii) P(exactly 2 of them will solve the problem)

$$= P(A \cap B \cap \bar{C}) + P(\bar{A} \cap B \cap C) + P(A \cap \bar{B} \cap C)$$

$$= P(A)\ P(B)\ P(\bar{C}) + P(\bar{A})\ P(B)\ P(C) + P(A)\ P(\bar{B})\ P(C)$$

$$= \left(\frac{1}{2} \times \frac{1}{3} \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{2}{3} \times \frac{1}{4}\right) = \frac{1}{4}$$

EXAMPLE 1.65 A pair of dice is thrown simultaneously. If A denotes the number on the first die is 1 and B be the event that the number on the second die is 6 and C be the event that the sum of the two numbers on the dice is 7. Find whether A, B and C are mutually independent.

Solution A pair of dice is thrown.

$$\therefore \quad n(S) = 6^2 = 36$$

A be the event that first die has 1 on it.

$$A = [(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)]$$

$$n(A) = 6$$

$$\therefore \quad P(A) = \frac{6}{36} = \frac{1}{6}$$

B be the event that the second die has 6 on it.

$$B = [(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)]$$

$$n(B) = 6$$

$$\therefore \quad P(B) = \frac{6}{36} = \frac{1}{6}$$

C be the event that the sum is 7

$$C = [(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)]$$

$$n(C) = 6$$

$$\therefore \quad P(C) = \frac{6}{36} = \frac{1}{6}$$

$$A \cap B = \{(1, 6)\} \Rightarrow n(A \cap B) = 1$$

$$A \cap C = \{(1, 6)\} \Rightarrow n(A \cap C) = 1$$

$$B \cap C = \{(1, 6)\} \Rightarrow n(B \cap C) = 1$$

$$P(A \cap B) = \frac{1}{36} = P(A)\ P(B)$$

$$P(B \cap C) = \frac{1}{36} = P(B)\ P(C)$$

$$P(A \cap C) = \frac{1}{36} = P(A)\ P(C)$$

$\therefore$ They are pairwise independent.

$$P(A \cap B \cap C) = \frac{1}{36} \neq P(A)\ P(B)\ P(C) = \frac{1}{216}$$

$\therefore$ They are not mutually independent.

1.8 CONDITIONAL PROBABILITY

If the probability of the event A provided the event B has already occurred is called the conditional probability and is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \text{ provided } P(B) \neq 0$$

The probability of an event B provided A has occurred already is given by

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \text{ provided } P(A) \neq 0$$

Note:

(i) If the events A and B are independent then

$$P(A/B) = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

$$P(B/A) = \frac{P(A)\,P(B)}{P(A)} = P(B)$$

(ii) If A and B are mutually exclusive events, then
$P(B/A) = 0$ and $P(A/B) = 0$, since $P(A \cap B) = 0$

Multiplication Law of Probability

If A and B are two dependent events, then

$$P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)$$

THEOREM 12 If $B \subset A$, then $P(B) \leq P(A)$.

Proof $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} \leq 1 \quad [\because B \subset A, A \cap B = B]$$

i.e. $$P(B) \leq P(A)$$

Hence proved.

Note: Since $A \subset S$, $P(A) \leq P(S) = 1$ [$\because$ total probability = 1]

i.e. $P(A) \leq 1$

THEOREM 13 If $A \subset B$, then $P(A/B) \geq P(A)$ and $P(B/A) = 1$.
If $B \subset A$, then $P(B/A) \geq P(B)$. **[AU June '03]**

Proof Given: $A \subset B$.

Therefore,

$$A \cap B = A$$
$$P(A \cap B) = P(A)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A)}{P(B)} \leq 1 \Rightarrow P(A/B) \geq P(A)$$

By definition,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$\therefore$ $P(B/A) = 1$

Hence proved.

Again, if $B \subset A$, $A \cap B = B$

$\therefore$ $P(A \cap B) = P(B)$

By definition,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}$$

$\Rightarrow$ $P(B/A) \geq P(B)$ $[\because P(A) \leq 1]$

Hence the proof.

THEOREM 14 For any three events A, B and C

$$P(A \cup B/C) = P(A/C) + P(B/C) - P(A \cap B/C)$$

Proof For any three events A, B and C

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \quad \text{from Figure 1.7}$$

$$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$$

$$= [P(A \cap C) + P(B \cap C) - P(A \cap C)(B \cap C)]$$

Dividing both sides by $P(C) > 0$, we get

$$\frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

Using the definition of conditional probability,

$$P(A \cup B/C) = P(A/C) + P(B/C) - P(A \cap B/C)$$

Hence the proof.

THEOREM 15 For any three events A, B and C

$$P(A \cap B/C) + P(A \cap \overline{B}/C) = P(A/C)$$

Proof From Figure 1.7,

$$A \cap C = (A \cap \overline{B} \cap C) \cup (A \cap B \cap C)$$

$$P(A \cap C) = P(A \cap \overline{B} \cap C) + P(A \cap B \cap C)$$

$(\because$ they are mutually exclusive)

Dividing by $P(C) > 0$, we get

$$\frac{P(A \cap C)}{P(C)} = \frac{P(A \cap \bar{B} \cap C)}{P(C)} + \frac{P(A \cap B \cap C)}{P(C)}$$

By definition,

$$P(A/C) = P(A \cap \bar{B}/C) + P(A \cap B/C)$$

Hence the proof.

THEOREM 16 For any three events A, B and C defined on the sample space S such that $B \subset C$, prove that $P(B/A) \leq P(C/A)$.

Proof From Figure 1.7,

$$A \cap C = (A \cap \bar{B} \cap C) \cup (A \cap B \cap C)$$

where $(A \cap \bar{B} \cap C)$ and $(A \cap B \cap C)$ are mutually exclusive

$$P(A \cap C) = P(A \cap \bar{B} \cap C) + P(A \cap B \cap C)$$

$$P(C/A) = \frac{P(A \cap C)}{P(A)} = \frac{P(A \cap \bar{B} \cap C) + P(A \cap B \cap C)}{P(A)}$$

$$P(C/A) = \frac{P(\bar{B} \cap C \cap A)}{P(A)} + \frac{P(B \cap C \cap A)}{P(A)}$$

By definition,

$$P(C/A) = P(\bar{B} \cap C/A) + P(B \cap C/A)$$

Given:

$$B \subset C \Rightarrow B \cap C = B$$

$\therefore$ $$P(C/A) = P(\bar{B} \cap C/A) + P(B/A)$$

But, $$P(\bar{B} \cap C/A) \geq 0 \Rightarrow P(C/A) \geq P(B/A)$$

$\therefore$ $$P(B/A) \leq P(C/A)$$

Hence the proof.

THEOREM 17 If $P(A) > P(B)$, then $P(A/B) > P(B/A)$.

Proof By definition,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(B) = \frac{P(A \cap B)}{P(A/B)}$$

$$P(B/A) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A) = \frac{P(A \cap B)}{P(B/A)}$$

If $P(A) > P(B)$, then

$$\frac{P(A \cap B)}{P(B/A)} > \frac{P(A \cap B)}{P(A/B)} \Rightarrow P(A/B) > P(B/A)$$

EXAMPLE 1.66 If $P(A) = 0.5$, $P(B) = 0.3$ and $P(A \cap B) = 0.15$, find $P(A/\bar{B})$.

Solution

$$P(A \cap B) = P(A) \cdot P(B) = 0.15$$

Therefore, A and B are independent. Hence A and $\bar{B}$ are also independent.

$$P(A/\bar{B}) = P(A) = 0.5$$

EXAMPLE 1.67 A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Solution Let A be the event that the first tube is good and B be the event that the second is also good.

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P \text{ (both tubes are good)}$$

$$P(A \cap B) = \frac{6C_2}{10C_2} = \frac{1}{3}$$

$$P(A) = P(\text{first tube is good})$$

$$= \frac{6C_1}{10C_1} = \frac{3}{5}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{3/5} = \frac{5}{9}$$

EXAMPLE 1.68 A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are black.

Solution The probability of drawing a black ball in the first draw

$$P(A) = \frac{3}{5+3} = \frac{3}{8}$$

The probability of drawing the second black ball given that the first ball drawn is black

$$P(B/A) = \frac{2}{5+2} = \frac{2}{7}$$

$\therefore$ The probability that both balls drawn are black is

$$P(A \cap B) = P(A)\, P(B/A) = \frac{3}{8} \times \frac{2}{7} = \frac{6}{56} = \frac{3}{28}$$

EXAMPLE 1.69 Find the probability of drawing a queen, a king and a knave in that order from a pack of cards in three consecutive draws, the cards drawn not being replaced.

Solution The probability of drawing a queen $= \dfrac{4}{52}$

The probability of drawing a king after a queen has been drawn $= \dfrac{4}{51}$

The probability of drawing a knave given that a king and a queen have been drawn $= \dfrac{4}{50}$

$\therefore$ The required probability $= \dfrac{4}{52} \cdot \dfrac{4}{51} \cdot \dfrac{4}{50} = \dfrac{64}{1,32,600} = 0.0005$

EXAMPLE 1.70 In a random experiment, $P(A) = 1/12$, $P(B) = 5/12$ and $P(B/A) = 1/15$, find $P(A \cup B)$.

Solution
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{(i)}$$

We know that

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow \frac{1}{15} = \frac{P(A \cap B)}{\frac{1}{12}}$$

i.e.
$$P(A \cap B) = \frac{1}{15} \times \frac{1}{12} = \frac{1}{180}$$

$\therefore$
$$P(A \cup B) = \frac{1}{12} + \frac{5}{12} - \frac{1}{180} = \frac{90}{180} - \frac{1}{180} = \frac{89}{180}$$

EXAMPLE 1.71 If the probability that a communication system has high selectivity is 0.54 and the probability that it will have high fidelity is 0.81 and the probability that is will have both is 0.18. Find the probability that a system with high fidelity will have high selectivity. **[AU December '07]**

Solution Let A be the event that the system has high selectivity

$\therefore$
$$P(A) = 0.54$$

Let B be the event that the system has high fidelity

$\therefore$
$$P(B) = 0.81$$

$$P(\text{system will have both}) = P(A \cap B) = 0.18$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.18}{0.81} = 0.222$$

EXAMPLE 1.72 For a certain binary channel, the probability that a transmitted 0 is correctly received as 0 is 0.94 and the probability that a transmitted 1 was received as 1 is 0.91. Further, the probability of transmitting a 0 is 0.45. If a signal is sent, determine

(i) the probability that a 0 was received,
(ii) the probability that a 0 was transmitted given that a 0 was received, and
(iii) the probability of an error. **[AU December '07]**

Solution Let A be the event of transmitting a 0 and B be the event of receiving a 0. $\bar{A}$ be the event of transmitting a 1 and $\bar{B}$ be the event of receiving a 1.

Given:

$$P(A) = 0.45$$
$$P(B/A) = 0.94$$

and
$$P(\bar{B}/\bar{A}) = 0.91$$

∴
$$P(\bar{A}) = 1 - P(A) = 1 - 0.45 = 0.55$$
$$P(\bar{B}/A) = 1 - P(B/A) = 0.06$$

and
$$P(B/\bar{A}) = 1 - P(\bar{B}/\bar{A}) = 0.09$$

(i) The probability that a 0 was received. It can happen in two mutually exclusive cases:

(a) 0 transmitted and received as 0
(b) 1 transmitted and received as 0

$$P(B) = P(A)\,P(B/A) + P(\bar{A})\,P(B/\bar{A})$$
$$= (0.45)\,(0.94) + (0.55)\,(0.09) = 0.4725$$

(ii) The probability that a 0 was transmitted given that a 0 was received is

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B/A)}{P(B)}$$

$$= \frac{(0.94)\,(0.45)}{0.4725} = 0.8952$$

(iii) Error can occur in the following two mutually exclusive cases:

(a) 0 transmitted and received as 1
(b) 1 transmitted and received as 0

$$P(\text{error}) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$
$$= P(A)\,P(\bar{B}/A) + P(\bar{A})\,P(B/\bar{A})$$
$$= (0.45)\,(0.06) + (0.55)\,(0.09) = 0.0765$$

EXAMPLE 1.73 A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one and examined. The ones examined are not put back. What is the probability that the ninth one examined is the last defective?

Solution Let A be the event of getting exactly 3 defectives in examination of 8 record players and let B be the event that the ninth piece examined is a defective one.

Since it is a problem of sampling without replacement and there are 4 defectives out of 15 record players, we have

$$P(A) = \frac{4C_3 \times 11C_5}{15C_8}$$

Since there is only one defective piece left among the remaining $15 - 8 = 7$ record players.

$P(B/A)$ = probability that the ninth examined record player is defective $= \dfrac{1}{7}$

$\therefore$ The required probability $= P(B/A)\,P(A) = \dfrac{1}{7} \times \dfrac{4C_3 \times 11C_5}{15C_8} = \dfrac{8}{195}$

EXAMPLE 1.74 Two weak students attempt to write a programme. Their chances of writing the programme successfully are 1/8 and 1/12 and the chance of making a common error is 1/1001. Find the chance that the programme is correctly written. **[AU June '07]**

Solution Let A be the event that two students get the same answer, i.e. either both are correct or both are wrong by committing a common mistake.

Let B be the event that their answer was correct.

$$P(A) = \left(\frac{1}{8}\cdot\frac{1}{12}\right) + \left(1-\frac{1}{8}\right)\left(1-\frac{1}{12}\right)\left(\frac{1}{1001}\right)$$

$$= \frac{1}{96} + \frac{77}{96 \times 1001} = \frac{1078}{96 \times 1001}$$

Since A and B are independently writing the programme

$$P(A \cap B) = P(\text{both correct}) = P(A)\,P(B)$$

$$= \frac{1}{8}\cdot\frac{1}{12} = \frac{1}{96}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{96} \times \frac{96 \times 1001}{1078} = \frac{1001}{1078} = \frac{13}{14}$$

EXAMPLE 1.75 A manufacturer of airplane parts knows that the probability is 0.8 that an order will be ready for shipment on time and it is 0.7 that an order will be ready for shipment and will be delivered on time. What is the probability that such an order will be delivered on time given that it was also ready for shipment on time?

Solution Let A be the event that an order will be ready for shipment on time and let B be the event that an order will be delivered on time

Given: $$P(A) = 0.8$$

Given that an order will be ready for shipment and will be delivered on time is 0.7, i.e.

$$P(A \cap B) = 0.7$$

Therefore, the probability that such an order will be delivered on time given that it was also ready for shipment on time is

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = 0.7/0.8 = 7/8$$

EXAMPLE 1.76 In a certain group of engineers 60% have insufficient background of information theory, 50% have inadequate knowledge of probability and 80% are in either one or both of two adequate categories. What is the percentage of people who have adequate knowledge of probability among those who have a sufficient background of information theory?

[AU May '08]

Solution Let B_1 be the event that the engineers have sufficient knowledge of information theory and B_2 be the event that the engineers have sufficient knowledge of probability.

Given:

$$P(\bar{B}_1) = 60\% = \frac{60}{100} = 0.6 \quad \text{and} \quad P(\bar{B}_2) = 50\% = \frac{50}{100} = 0.5$$

$$P(\bar{B}_1 \cup \bar{B}_2) = 80\% = \frac{80}{100} = 0.8$$

$$\therefore \quad P(B_1) = 40\% = 0.4$$

$$P(B_2) = \frac{1}{2} = 0.5$$

$$P(B_1 \cap B_2) = 1 - P(\bar{B}_1 \cup \bar{B}_2) = 1 - 0.8 = 0.2$$

$$P(B_1/B_2) = \frac{P(B_1 \cap B_2)}{P(B_2)} = \frac{0.2}{0.5} = \frac{2}{5} = 0.4$$

$\therefore$ 40% of people have adequate knowledge of probability among those who have a sufficient background of information.

1.9 TOTAL PROBABILITY THEOREM

If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive set of events of a sample space S and A is any event associated with (or caused by) the events $B_1, B_2, \ldots, B_n$, then

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + \cdots + P(B_n)\,P(A/B_n)$$

i.e. $$P(A) = \sum_{i=1}^{n} P(B_i)\,P(A/B_i)$$

Proof Since $B_1, B_2, \ldots, B_n$ are mutually exclusive events, $A \cap B_1, A \cap B_2, \ldots, A \cap B_n$ are also mutually exclusive (Figure 1.8).

$$\begin{aligned} A &= (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cup (A \cap B_n) \\ P(A) &= P[(A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cup (A \cap B_n)] \\ &= P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \cdots + P(A \cap B_n) \qquad (1.3) \end{aligned}$$

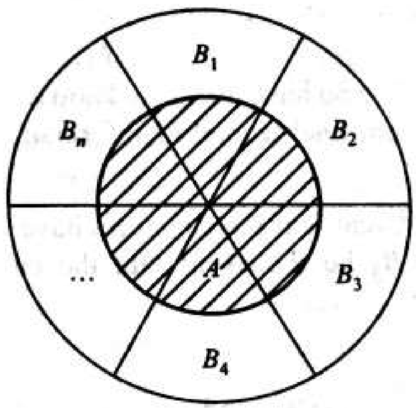

Figure 1.8

By the definition of conditional probability,

$$P(A/B_i) = \frac{P(A \cap B_i)}{P(B_i)}, \quad i = 1, 2, \ldots, n$$

$$\Rightarrow \qquad P(B_i)\,P(A/B_i) = P(A \cap B_i)\,, \; i = 1, 2, \ldots, n$$

Using it in Eq. (1.3), we get

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + \cdots + P(B_n)\,P(A/B_n)$$

i.e. $$P(A) = \sum_{i=1}^{n} P(B_i)\,P(A/B_i)$$

Hence proved.

1.10 BAYES' THEOREM

If $B_1, B_2, \ldots, B_n$ are a set of exhaustive and mutually exclusive events of a sample space S and A is any event associated with $B_1, B_2, \ldots, B_n$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_i \text{ then } P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}$$

Proof Since $B_1, B_2, \ldots, B_n$ are mutually exclusive events, $A \cap B_1, A \cap B_2, A \cap B_3, \ldots, A \cap B_n$ are also mutually exclusive.

Given: $$A \subseteq \bigcup_{i=1}^{n} B_i$$

$$\therefore \qquad A = A \cap \left(\bigcup_{i=1}^{n} B_i\right)$$

$$P(A) = P\left(A \cap \bigcup_{i=1}^{n} B_i\right) = P\left(\bigcup_{i=1}^{n} A \cap B_i\right)$$

$$= P[(A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cup (A \cap B_n)]$$

$$= P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \ldots + P(A \cap B_n) \quad (1.4)$$

[$\because$ they are mutually exclusive]

Using the definition of conditional probability,

$$P(A/B_i) = \frac{P(A \cap B_i)}{P(B_i)},\ i = 1, 2, \ldots, n$$

$$P(B_i)\,P(A/B_i) = P(A \cap B_i),\ i = 1, 2, \ldots, n \quad (1.5)$$

Using it in Eq. (1.4)

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + \cdots + P(B_n)\,P(A/B_n)$$

i.e. $$P(A) = \sum_{i=1}^{n} P(B_i)\,P(A/B_i) \quad (1.6)$$

Again, $$P(B_i/A) = \frac{P(A \cap B_i)}{P(A)}$$

From Eqs. (1.5) and (1.6), we get

$$\therefore \qquad P(B_i/AJ) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}$$

Hence proved.

EXAMPLE 1.77 The contents of urns I, II and III are as follows:

(i) 1 white, 2 black and 3 red balls,
(ii) 2 white, 1 black and 1 red balls, and
(iii) 4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from urns I, II and III?
[AU May '06]

Solution There are three urns. The probability of choosing one urn is $\frac{1}{3}$.

Let B_1 be the event of choosing urn I, B_2 be the event of choosing urn II, and B_3 be the event of choosing urn III.

$$\therefore \quad P(B_1) = \frac{1}{3},\ P(B_2) = \frac{1}{3},\ P(B_3) = \frac{1}{3}$$

Let A be the event of choosing 2 balls are white and one red. If the urn I is chosen, then

$$P(A/B_1) = \frac{1C_1 \times 3C_1}{6C_2} = \frac{1}{5}$$

If the urn II is chosen, then

$$P(A/B_2) = \frac{2C_1 \times 1C_1}{4C_2} = \frac{1}{3}$$

If the urn III is chosen, then

$$P(A/B_3) = \frac{4C_1 \times 3C_1}{12C_2} = \frac{2}{11}$$

$$\therefore \quad P(A) = P(B_1)\ P(A/B_1) + P(B_2)\ P(A/B_2) + P(B_3)\ P(A/B_3)$$

$$= \frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11} = \frac{118}{495}$$

Assume A has happened, i.e. 1 white and 1 red balls are chosen.

The probability that they come from urn I is

$$P(B_1/A) = \frac{P(B_1)\ P(A/B_1)}{P(A)} = \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{118}{495}} = 0.2797$$

The probability that they oome from urn II is

$$P(B_2/A) = \frac{P(B_2)\ P(A/B_2)}{P(A)} = \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{118}{495}} = 0.4661$$

The probability that they come from urn III is

$$P(B_3/A) = \frac{P(B_3)\,P(A/B_3)}{P(A)} = \frac{\frac{1}{3}\times\frac{2}{11}}{\frac{118}{495}} = 0.2542$$

EXAMPLE 1.78 A box contains 7 red and 13 blue balls. Two balls are selected at random and are discarded without their colours being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue? **[AU November '07]**

Solution Given: a box contains 7 red and 13 blue balls. Therefore, the number of balls in the box is 20.

Two balls are selected from the box. Those two balls may be:

(i) Red and red—B_1
(ii) Red and blue—B_2
(iii) Blue and blue—B_3

The probability of selecting 2 red balls from the box is

$$P(B_1) = \frac{7C_2}{20C_2} = \frac{21}{190}$$

The probability of selecting 1 red ball and 1 blue ball from the box is

$$P(B_2) = \frac{7C_1 \times 13C_1}{20C_2} = \frac{91}{190}$$

The probability of selecting 2 red balls from the box is

$$P(B_3) = \frac{13C_2}{20C_2} = \frac{78}{190}$$

B_1, B_2 and B_3 are exhaustive events because

$$P(B_1) + P(B_2) + P(B_3) = \frac{21+91+78}{190} = 1$$

The number of remaining balls in the box = 18 (since 2 balls are already selected and discarded)

Let A be the event that the third drawn ball is red.

If 2 red balls are selected and discarded (B_1) already, then

$$P(A/B_1) = \frac{5C_1}{18C_1} = \frac{5}{18}$$

If 1 red ball and 1 blue ball are selected and discarded (B_2) already, then

$$P(A/B_2) = \frac{6C_1}{18C_1} = \frac{6}{18}$$

If two blue balls are selected and discarded (B_3) already, then

$$P(A/B_3) = \frac{7C_1}{18C_1} = \frac{7}{18}$$

If the third ball drawn is red, then

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)$$
$$= \frac{21}{190} \times \frac{5}{18} + \frac{91}{190} \times \frac{6}{18} + \frac{78}{190} \times \frac{7}{18} = \frac{1197}{190 \times 18} = 0.35$$

If the third ball drawn is red, then the probability that both of the discarded balls were blue is

$$\therefore \quad P(B_3/A) = \frac{P(B_3)\,P(A/B_3)}{P(A)} = \frac{\frac{78}{190} \times \frac{7}{18}}{\frac{1197}{190 \times 18}} = \frac{546}{1197} = 0.456$$

EXAMPLE 1.79 A factory produces its entire output with three machines. Machines I, II and III produce 50%, 30% and 20% of the output, but 4%, 2% and 4% of their outputs are defective respectively. What fraction of the total output is defective? **[AU December '09]**

Solution Let B_1 be the event of production of machine I, B_2 be the event of production of machine II, and B_3 be the event of production of machine III.

$$\therefore \quad P(B_1) = \frac{50}{100} = \frac{5}{10}$$

$$P(B_2) = \frac{30}{100} = \frac{3}{10}$$

$$P(B_3) = \frac{20}{100} = \frac{2}{10}$$

B_1, B_2 and B_3 are exhaustive events because

$$P(B_1) + P(B_2) + P(B_3) = 1$$

Let A be the event of defective output. Then,

$$P(A/B_1) = 4\% = \frac{4}{100}$$

$$P(A/B_2) = 2\% = \frac{2}{100}$$

$$P(A/B_3) = 4\% = \frac{4}{100}$$

The fraction of the total defective output

$$P(A) = P(B_1)\ P(A/B_1) + P(B_2)\ P(A/B_2) + P(B_3)\ P(A/B_3)$$

$$= \frac{5}{10} \times \frac{4}{100} + \frac{3}{10} \times \frac{2}{100} + \frac{2}{10} \times \frac{4}{100} = \frac{34}{1000} = 0.034$$

Therefore, 3.4% of the total output is defective.

EXAMPLE 1.80 A box contains 5 red and 4 white balls. Two balls are drawn successively from the box without replacement and it is noted that the second one is white. What is the probability that the first is also white?

[AU December '05]

Solution The total number of balls in the box = 9

Let B_1 be the event that the first drawn ball is white, and B_2 be the event that the first drawn ball is red.

$$\therefore \qquad P(B_1) = \frac{4}{9}$$

$$P(B_2) = \frac{5}{9}$$

B_1 and B_2 are exhaustive events because

$$P(B_1) + P(B_2) = 1$$

Let A be the event that the second drawn ball is white.

If one white ball is drawn already and is not replaced, then the remaining balls in the box are 3 white and 5 red balls.

$$P(A/B_1) = \frac{3}{8}$$

If one red ball is drawn already and is not replaced, then the remaining balls in the box are 4 white and 4 red balls.

$$P(A/B_2) = \frac{4}{8}$$

The probability that the second drawn ball is white is

$$P(A) = P(B_1)\ P(A/B_1) + P(B_2)\ P(A/B_2)$$

$$= \frac{4}{9} \times \frac{3}{8} + \frac{5}{9} \times \frac{4}{8} = \frac{32}{72}$$

The probability that the first drawn ball is also white given that the second drawn ball is white

$$P(B_1/A) = \frac{P(B_1)\,P(A/B_1)}{P(A)} = \frac{4 \times 3}{32} = \frac{12}{32} = \frac{3}{8}$$

EXAMPLE 1.81 A company has two plants to manufacture scooters. Plant I manufactures 80% of the scooters and plant II the rest. At plant I, 85 out of 100 scooters are rated higher quality and at plant II, only 65 out of 100 scooters are rated higher quality. A scooter is chosen at random. What is the probability that the scooter came from plant II, if it is known that the scooter is of higher quality?

[AU June '06]

Solution Let B_1 be the event that the scooters are manufactured by plant I, and B_2 be the event that the scooters are manufactured by plant II.

$$\therefore \quad P(B_1) = 80\% = 0.8$$
$$P(B_2) = 20\% = 0.2$$

B_1 and B_2 are exhaustive events because

$$P(B_1) + P(B_2) = 1$$

Let A be the event that the scooters are rated higher quality.

Given:

$$P(A/B_1) = \frac{85}{100} = 0.85$$

$$P(B_2) = \frac{65}{100} = 0.65$$

The probability that the scooter is rated higher quality (A)

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2)$$
$$= \frac{80}{100} \times \frac{85}{100} + \frac{20}{100} \times \frac{65}{100} = \frac{8100}{100 \times 100}$$

The probability that the scooter came from plant II, if it is known that the scooter is of higher quality

$$P(B_2/A) = \frac{P(B_2)\,P(A/B_2)}{P(A)} = \frac{20 \times 65}{8100} = \frac{13}{81} = 0.1605$$

EXAMPLE 1.82 In a coin tossing experiment if the coin shows head, one die is thrown and the number is recorded. If the coin shows tail, two dice are thrown and their sum is recorded. What is the probability that the recorded number will be 2? **[AU November '04]**

Solution If a coin is tossed, then

$$P(H) = \frac{1}{2} = P(T)$$

If the coin shows head, then a single die is thrown,

$$n(S) = 6$$

Let A be the event of getting the recorded number 2.

$$A = \{2\},\ n(A) = 1$$

$$P(\text{getting the number 2}) = P(A/H) = \frac{1}{6}$$

If the coin shows tail, then two dice are thrown,

$$n(S) = 36$$

$$A = \{(1, 1)\},\ n(A) = 1.$$

$$P(\text{getting the sum 2}) = P(A/T) = \frac{1}{36}$$

Therefore, the probability that recorded number will be 2 is

$$P(A) = P(H)\ P(A/H) + P(T)\ P(A/T)$$

$$= \frac{1}{2}\times\frac{1}{6}+\frac{1}{2}\times\frac{1}{36}=\frac{7}{72}$$

EXAMPLE 1.83 A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that all the balls in the bag are white?

[AU November '08]

Solution A bag contains 5 balls. Two balls are drawn at random and found to be white.

Let B_1 be the event that the bag contains 2 white and 3 different colour balls, B_2 be the event that the bag contains 3 white and 2 different colour balls, B_3 be the event that the bag contains 4 white and 1 different colour balls, and B_4 be the event that the bag contains 5 white balls.

Let A be the event that the 2 balls drawn are white.

Since the chances for B_1, B_2, B_3 and B_4 are equally likely, we have

$$P(B_1) = \frac{1}{4},\ P(B_2) = \frac{1}{4},\ P(B_3) = \frac{1}{4},\ \text{and } P(B_4) = \frac{1}{4}$$

$$P(A/B_1) = \frac{2C_2}{5C_2}=\frac{1}{10},\ P(A/B_2) = \frac{3C_2}{5C_2}=\frac{3}{10},$$

$$P(A/B_3) = \frac{4C_2}{5C_2}=\frac{6}{10},\ \text{and } P(A/B_4) = \frac{5C_2}{5C_2}=1$$

The chances for 2 balls drawn are white

$$P(A) = P(B_1)\ P(A/B_1) + P(B_2)\ P(A/B_2) + P(B_3)\ P(A/B_3) + P(B_4)\ P(A/B_4)$$

$$= \frac{1}{4}\times\frac{1}{10}+\frac{1}{4}\times\frac{3}{10}+\frac{1}{4}\times\frac{6}{10}+\frac{1}{4}\times 1=\frac{20}{40}$$

Two balls are drawn and found to be white (A). The chance that all the balls in the bag are white (B_4)

$$P(B_4/A) = \frac{P(B_4)\,P(A/B_4)}{P(A)} = \frac{(1/4)\times 1}{20/40} = \frac{1}{2} = 0.5$$

EXAMPLE 1.84 Two shipments of parts are received. The first shipment contains 1000 parts with 10% defective and the second shipment contains 2000 parts with 5% defective. One shipment is selected at random. Two parts are tested and found good. Find the probability (a posterior) that the tested parts were selected from the first shipment.

Solution Let B_1 be the event of selectig shipment I, and B_2 be the event of selectig shipment II.

$$\therefore \qquad P(B_1) = \frac{1}{2} \quad \text{and} \quad P(B_2) = \frac{1}{2}$$

In shipment I, 10% of 1000 parts are defective

$$= 1000 \times \frac{10}{100} = 100$$

Therefore, in shipment I, out of 1000, 100 are defective parts and 900 are good parts.

In shipment II, 5% of 2000 parts are defective

$$= 2000 \times \frac{5}{100} = 100$$

Therefore, in shipment II, out of 2000, 100 are defective parts and 1900 are good parts.

Let A be the event of selecting 2 good parts.

$$P(A/B_1) = \frac{900C_2}{1000C_2} = 0.8099$$

$$P(A/B_2) = \frac{1900C_2}{2000C_2} = 0.9025$$

The probability that the 2 parts selected are good is

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2)$$

$$P(A) = \frac{1}{2}\times 0.8099 + \frac{1}{2}\times 0.9025 = 0.8562$$

Given that the 2 parts are tested and found good (A). The probability that the tested good parts were selected from the shipment I is

$$P(B_1/A) = \frac{P(B_1)\,P(A/B_1)}{P(A)} = \frac{\frac{1}{2}\times 0.8099}{0.8562} = 0.4729$$

EXAMPLE 1.85 Three urns are given. Each containing red and white balls as indicated:

- Urn I: 4 white and 6 red
- Urn II: 6 white and 2 red
- Urn III: 8 white and 1 red

(i) An urn is chosen at random and a ball is drawn from this urn and is found to be red. Find the probability that the urn chosen was urn I.

(ii) An urn is chosen at random and two balls are drawn without replacement from this urn. If both balls are red, find the probability that urn III was chosen.

Solution There are 3 urns. The probability of choosing one urn is $\frac{1}{3}$.

Let B_1 be the event of choosing urn I (10 balls), B_2 be the event of choosing urn II (8 balls), and B_3 be the event of choosing urn III (9 balls)

$$\therefore \qquad P(B_1) = \frac{1}{3},\ P(B_2) = \frac{1}{3} \text{ and } P(B_3) = \frac{1}{3}$$

(i) Let A be the event of choosing 1 red ball.
If the urn I is chosen, then

$$P(A/B_1) = \frac{6C_1}{10C_1} = 0.6$$

If the urn II is chosen, then

$$P(A/B_2) = \frac{2C_1}{8C_1} = 0.25$$

If the urn III is chosen, then

$$P(A/B_3) = \frac{1}{9} = 0.111$$

$\therefore$ The probability to draw 1 red ball is

$$P(A) = P(B_1)\ P(A/B_1) + P(B_2)\ P(A/B_2) + P(B_3)\ P(A/B_3)$$

$$= \frac{1}{3} \times 0.6 + \frac{1}{3} \times 0.25 + \frac{1}{3} \times 0.111 = \frac{0.961}{3} = 0.320$$

Given that 1 red ball is drawn (A). The probability that urn I (B_1) was chosen is

$$P(B_1/A) = \frac{P(B_1)\ P(A/B_1)}{P(A)} = \frac{(1/3) \times 0.6}{0.961/3} = 0.624$$

(ii) Let A be the event of choosing 2 red balls.
If the urn I is chosen, then

$$P(A/B_1) = \frac{6}{10} \times \frac{5}{9} = 0.3333$$

If the urn II is chosen, then

$$P(A/B_2) = \frac{2}{6} \times \frac{1}{5} = 0.0667$$

If the urn III is chosen, then

$$P(A/B_3) = \frac{1}{9} \times 0 = 0$$

$\therefore$ The probability to draw 2 red balls is

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)$$

$$= \frac{1}{3} \times 0.3333 + \frac{1}{3} \times 0.0667 + \frac{1}{3} \times 0 = \frac{0.4}{3}$$

Given that 2 red balls are drawn (A).
The probability that urn III (B_3) was chosen is

$$P(B_3/A) = \frac{P(B_3)\,P(A/B_3)}{P(A)} = \frac{\frac{1}{3} \times 0}{\frac{0.4}{3}} = 0$$

EXAMPLE 1.86 There are 4 candidates for the Office of the Highway Commissioner. The respective probabilities that they will be selected are 0.3, 0.2, 0.4 and 0.1 and the probabilities for a project approval are 0.35, 0.85, 0.45 and 0.15, depending on which of the 4 candidates are selected. What is the probability of the project getting approved.

Solution Let B_1, B_2, B_3 and B_4 be the events of selecting the 1st, 2nd, 3rd and 4th candidates respectively and let D be the event of the project getting approved.

Given: $P(B_1) = 0.3$, $P(B_2) = 0.2$, $P(B_3) = 0.4$ and $P(B_4) = 0.1$
$P(D/B_1) = 0.35$, $P(D/B_2) = 0.85$, $P(D/B_3) = 0.45$ and $P(D/B_4) = 0.15$

P(project getting approved) = P(any one of them gets the approval)

i.e. $P(D) = P(B_1)\,P(D/B_1) + P(B_2)\,P(D/B_2) + P(B_3)\,P(D/B_3) + P(B_4)\,P(D/B_4)$
$= 0.3 \times 0.35 + 0.2 \times 0.85 + 0.4 \times 0.45 + 0.1 \times 0.15$
$= 0.47$

EXAMPLE 1.87 A box contains 2000 components of which 5% are defective. The second box contains 50 components of which 40% are defective. Two other boxes contain 1000 components each with 10% defects. We select at random one of the above boxes and remove from it at random a single component.

(i) What is the probability that the component is defective?
(ii) What is the probability that it is drawn from the second box?

Solution Let B_1, B_2, B_3, B_4 be the events of selecting box 1, box 2, box 3 and box 4 and D be the event of selecting the defective component.

Given:

$$P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$$

The number of defective components in $B_1 = 2000 \times \frac{5}{100} = 100$

$$\therefore \qquad P(D/B_1) = \frac{100C_1}{2000C_1} = \frac{5}{100}$$

The number of defective components in $B_2 = 50 \times \frac{40}{100} = 20$

$$\therefore \qquad P(D/B_2) = \frac{20C_1}{50C_1} = \frac{20}{50} = \frac{40}{100}$$

The number of defective components in $B_3 = 1000 \times \frac{10}{100} = 100$

$$\therefore \qquad P(D/B_3) = \frac{100C_1}{1000C_1} = \frac{10}{100}$$

Similarly,

$$P(D/B_4) = \frac{10}{100}$$

(i) P(component is defective)

$$P(D) = P(B_1)\,P(D/B_1) + P(B_2)\,P(D/B_2) + P(B_3)\,P(D/B_3) + P(B_4)\,P(D/B_4)$$

$$= \frac{1}{4}\left(\frac{5}{100} + \frac{40}{100} + \frac{10}{100} + \frac{10}{100}\right) = \frac{65}{400} = \frac{13}{80}$$

(ii) By Bayes' theorem,

$$P(B_2/D) = \frac{P(B_2)\,P(D/B_2)}{\sum_{i=1}^{n} P(B_i)\,P(D/B_i)} = \frac{\frac{1}{4} \times \frac{40}{100}}{\frac{13}{80}} = \frac{8}{13}$$

EXAMPLE 1.88 A bag contains 7 red and 3 black marbles. Another bag contains 4 red and 5 black marbles. One marble is transferred from the first bag into the second bag and then a marble is taken out from the second bag at random. If this marble happens to be red, find the probability that a black marble was transferred. [AU June '05]

Solution One marble is transferred from bag I to bag II. It may be a red or black marble.

Let B_1 be the event of transferring 1 red marble from bag I.

$$P(B_1) = \frac{7C_1}{10C_1} = 0.7$$

Let B_2 be the event of transferring 1 black marble from bag I.

$$P(B_2) = \frac{3C_1}{10C_1} = 0.3$$

$$P(B_1) + P(B_2) = 1$$

Now, the total marbles in bag II are 10.

Then, let A be the event of selecting a red marble from bag II. If a red marble is transferred from bag I, the number of red marbles in bag II is $(4 + 1) = 5$.

If a black marble is transferred from bag I, the number of red marbles in bag II is $(4 + 0) = 4$.

$$\begin{aligned}P(A) &= P(\text{selecting a red/a red marble was transferred}) \\ &\quad + P(\text{selecting a red/a black marble was transferred}) \\ &= P(B_1)\, P(A/B_1) + P(B_2)\, P(A/B_2) \\ &= 0.7 \times \frac{5C_1}{10C_1} + 0.3 \times \frac{4C_1}{10C_1} \\ &= 0.7 \times \frac{5}{10} + 0.3 \times \frac{4}{10} = 0.35 + 0.12 = 0.47 = \frac{47}{100}\end{aligned}$$

The probability that a black marble is transferred from bag I given that a red marble is chosen from bag II.

i.e. $$P(B_2/A) = \frac{P(B_2)\, P(A/B_2)}{P(A)}$$

$$= \frac{P(B_2)\, P(A/B_2)}{P(B_1)\, P(A/B_1) + P(B_2)\, P(A/B_2)} = \frac{\left(\frac{3}{10}\right)\left(\frac{4}{10}\right)}{\frac{47}{100}} = \frac{12}{47}$$

EXAMPLE 1.89 The chances of A, B and C becoming the G.M. of a company are in the ratio 4 : 2 : 3. The probabilities that the bonus scheme will be introduced in the company if A, B, and C become G.M. are 0.3, 0.7 and 0.8 respectively. If the bonus scheme has been introduced, what is the probability that A has been appointed as G.M.?

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Solution Given that number of possible cases = 4 + 2 + 3 = 9

$$P(A) = \frac{4}{9},\ P(B) = \frac{2}{9} \text{ and } P(C) = \frac{3}{9},\ P(A) + P(B) + P(C) = 1$$

Let D be the event that the bonus scheme has been introduced.

Given:

$$P(D/A) = 0.3,\ P(D/B) = 0.7 \text{ and } P(D/C) = 0.8$$

The probability that A has been appointed as G.M. given that bonus scheme has been introduced is

$$P(A/D) = \frac{P(A)\,P(D/A)}{P(D)} = \frac{P(A)\,P(D/A)}{\sum P(A)\,P(D/A)}$$

$$= \frac{\frac{4}{9}\times 0.3}{\frac{4}{9}\times 0.3 + \frac{2}{9}\times 0.7 + \frac{3}{9}\times 0.8} = \frac{6}{25}$$

EXAMPLE 1.90 Box 1 contains 1000 bulbs of which 10% are defective. Box 2 contains 2000 bulbs of which 5% are defective. Two bulbs are drawn (without replacement) from a randomly selected box.

(i) Find the probability that both bulbs are defective.

(ii) Assuming that both are defective, find the probability that they came from box 1.

Solution Let B_1 and B_2 be the events of selecting bulbs randomly from the box 1 and box 2 respectively. Let D be the event of selecting the two defective bulbs.

The probability of selecting box 1 and box 2 are equal.

$$\therefore \qquad P(B_1) = \frac{1}{2},\ P(B_2) = \frac{1}{2}$$

$$P(B_1) + P(B_2) = 1$$

The number of defective bulbs in box 1 is equal to

$$10\% \text{ of } 1000 = \frac{10}{100}\times 1000 = 100$$

Assuming box 1 is chosen randomly, the probability of choosing two defective bulbs from box 1 is

$$P(D/B_1) = \frac{100C_2}{1000C_2} = 0.0099$$

The number of defective bulbs in box 2 is equal to

$$5\% \text{ of } 2000 = \frac{5}{100} \times 2000 = 100$$

$$P(D/B_2) = \frac{100C_2}{2000C_2} = 0.0024$$

The probability of selecting two defective bulbs is

$$P(D) = P(B_1)\ P(D/B_1) + P(B_2)\ P(D/B_2)$$

$$= \frac{1}{2} \times 0.0099 + \frac{1}{2} \times 0.0024 = 0.00615$$

The probability of choosing the defective bulbs from box 1 given that two defective bulbs have been drawn is

$$P(B_1/D) = \frac{P(B_1)\ P(D/B_1)}{P(D)} = \frac{0.0099 \times 0.5}{0.00615} = 0.8048$$

EXAMPLE 1.91 The chance that doctor A will diagnose a disease x correctly is 60%. The chance that a patient will die by his treatment after correct diagnosis is 40% and the chance of death of wrong diagnosis is 70%. A patient of doctor A who had disease x died. What is the chance that his disease was diagonsed correctly?

Solution Let B_1 be the event that the doctor diagnosed the disease correctly.

$$P(B_1) = \frac{60}{100} = 0.6$$

Let B_2 be the event that the doctor diagnosed the disease wrongly.

$$P(B_2) = \frac{40}{100} = 0.4$$

$$P(B_1) + P(B_2) = 1$$

Let D be the event that the patient is dead.

The probability that the disease was diagnosed correctly and the patient dead is

$$P(D/B_1) = 40\% = \frac{40}{100} = 0.4$$

The probability that the disease was diagnosed wrongly and the patient dead is

$$P(D/B_2) = 70\% = \frac{70}{100} = 0.7$$

The probability that the disease was diagnosed correctly given that the patient dead.

$$P(B_1/D) = \frac{P(B_1)\,P(D/B_1)}{P(D)} = \frac{P(B_1)\,P(D/B_1)}{P(B_1)\,P(D/B_1) + P(B_2)\,P(D/B_2)}$$

$$= \frac{0.6 \times 0.4}{0.6 \times 0.4 + 0.4 \times 0.7} = \frac{6}{13} = 0.4615$$

EXAMPLE 1.92 The probability that a student passes a certain exam is 0.9 given that he studied. The probability that he passes the exam without studying is 0.2. Assume that the probability that student studies for an exam is 0.75. Given that the student passed the exam, what is the probability that he studied?

Solution Let B_1 be the event that the student studied for the exam.

$$P(B_1) = 0.75$$

Let B_2 be the event that the student has not studied for the exam.

$$P(B_2) = 0.25$$

$$P(B_1) + P(B_2) = 1$$

Let D be the event that the student has passed the exam.
The probability that he passes the exam given that he has studied.

$$P(D/B_1) = 0.9$$

The probability that he passes the exam without studying is

$$P(D/B_2) = 0.2$$

The probability that he studied given that he passed the exam is

$$P(B_1/D) = \frac{P(B_1)\,P(D/B_1)}{P(D)} = \frac{P(B_1)\,P(D/B_1)}{P(B_1)\,P(D/B_1) + P(B_2)\,P(D/B_2)}$$

$$= \frac{0.75 \times 0.9}{0.75 \times 0.9 + 0.25 \times 0.2} = \frac{27}{29} = 0.9310$$

EXAMPLE 1.93 A computer centre has three printers A, B and C, which print at different speeds. Programmes are routed to the first available printer. The probability that the programmes are routed to the printers A, B and C are 0.6, 0.3 and 0.1 respectively. Occasionally, a printer will jam and destroy a print out. The probability that printers A, B and C will jam are 0.01, 0.05 and 0.04 respectively. Your programme is destroyed when a printer jams. What is the probability that printer A is involved? **[AU November '09]**

Solution Given:

$$P(A) = 0.6,\ P(B) = 0.3 \text{ and } P(C) = 0.1$$

Let D be the event that the programme is destroyed when a printer jams. Then,

$$P(D/A) = 0.01,\ P(D/B) = 0.05 \text{ and } P(D/C) = 0.04$$

Given the programme is destroyed because of printer jam.

To find the conditional probability that the programme destroyed because of the printer A. Using Bayes' theorem,

$$P(A/D) = \frac{P(A)\,P(D/A)}{\sum P(A)\,P(D/A)} = \frac{0.01 \times 0.6}{0.01 \times 0.6 + 0.05 \times 0.3 + 0.04 \times 0.3} = 0.24$$

EXAMPLE 1.94 Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews and 10% of poor products received good reviews. In addition, 40% of products have been successful, 35% have been moderately successful and 25% have been poor products.

(i) What is the probability that a product attains a good review?

(ii) If a new design attains a good review, what is the probability that it will be a highly successful product?

(iii) If a product does not attain a good review, what is the probability that it will be a highly successful product? **[AU December '09]**

Solution Given: 40% of products have been successful, 35% have been moderately successful and 25% have been poor products, i.e.

$$P(H) = 0.4,\ P(M) = 0.35 \text{ and } P(B) = 0.25$$

In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews and 10% of poor products received good reviews.

Let G be the event that the product receives good reviews and $\bar{G}$ be the event that the product does not attain good reviews. Therefore,

$$P(G/H) = 0.95,\ P(G/M) = 0.60 \text{ and } P(G/B) = 0.10$$

(i) The probability that a product attains a good review

$$P(G) = P(H)\,P(G/H) + P(M)\,P(G/M) + P(B)\,P(G/B) = 0.615$$

(ii) If a new design attains a good review, the probability that it will be a highly successful product is

$$P(H/G) = \frac{P(G/H)\,P(H)}{P(G)}$$

$$= \frac{0.95 \times 0.40}{0.615} = 0.618$$

(iii) If a product does not attain a good review, the probability that it will be a highly successful product is

$$P(H/\bar{G}) = \frac{P(\bar{G}/H)\,P(H)}{P(\bar{G})} = \frac{[1 - P(G/H)]\,P(H)}{1 - P(G)}$$

$$= \frac{(1 - .95)\,(0.40)}{1 - 0.615} = \frac{0.05(0.40)}{0.385} = 0.052$$

EXAMPLE 1.95 A box contains 5 red and 4 white balls. A ball from the box is taken out at random and kept outside. If once again a ball is drawn from the box, what is the probability that the drawn ball is red? **[AU April '08]**

Solution Let B_1 be the event that the first ball drawn is red and B_2 be the event that the first ball drawn is white

$$P(B_1) = \frac{5}{9},\ P(B_2) = \frac{4}{9}$$

A be the event that the second ball drawn is red.

The event of selecting a red ball in the second draw depends on whether the first ball drawn is white or red.

If the first ball drawn is red, then the box contains 4 red and 4 white balls

$$P(A/B_1) = 4C_1/8C_1 = \frac{4}{8}$$

If the first ball drawn is white, then the box contains 5 red and 3 white balls

$$P(A/B_2) = 5C_1/8C_1 = \frac{5}{8}$$

The probability that the drawn ball is red

$$P(A) = P(B_1)\ P(A/B_1) + P(B_2)\ P(A/B_2) = \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8} = \frac{40}{72} = \frac{5}{9}$$

EXAMPLE 1.96 A bin contains three different types of disposable flashlights. The probability that a type 1 flashlight will give over 100 hours of use is 0.7 with the corresponding probabilities for type 2 and type 3 flashlights being 0.4 and 0.3 respectively. Suppose that 20% of the flashlights in the bin are type 1, 30% are type 2 and 50% are type 3.

(i) Find the probability that a randomly chosen flashlight will give more than 100 hours of use.

(ii) Given the flashlight lasted over 100 hours, what is the conditional probability that it was type j flashlight, $j = 1, 2, 3$?

[AU November '07]

Solution Let B_1 denote the type 1 disposable flashlight, B_2 denote the type 2 disposable flash light and B_3 denote the type 3 disposable flashlight, then

$$P(B_1) = 0.2,\ P(B_2) = 0.3 \text{ and } P(B_3) = 0.5 \text{ (given)}$$

Let D be the event that the flash light will give over 100 hours of use

$$P(D/B_1) = 0.7,\ P(D/B_2) = 0.4 \text{ and } P(D/B_3) = 0.3 \text{ (given)}$$

(i) The probability that a randomly chosen flashlight will give more than 100 hours of use

$$P(D) = P(B_1)\,P(D/B_1) + P(B_2)\,P(D/B_2) + P(B_3)\,P(D/B_3)$$
$$= 0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3 = \frac{41}{100}$$
$$= 0.41$$

(ii) Given the flashlight lasted over 100 hours, the conditional probability that it was type j flashlight, $j = 1, 2, 3$ is

$$P(B_1/D) = \frac{P(B_1)\,P(D/B_1)}{P(D)} = \frac{0.2 \times 0.7}{0.41} = \frac{14}{41} = 0.3413$$

$$P(B_2/D) = \frac{P(B_2)\,P(D/B_2)}{P(D)} = \frac{12}{41} = 0.293$$

$$P(B_3/D) = \frac{P(B_3)\,P(B_3/D)}{P(D)} = \frac{15}{41} = 0.366$$

EXAMPLE 1.97 In answering a question on a multiple choice test, an examinee either knows the answer with probability p or he guesses with probability $1 - p(= q)$. Assume that the probability of answering a question correctly is unity for an examinee who knows the answer and $1/n$ for the examinee who guesses, where n is the number of multiple choice alternatives. Supposing an examinee answers a question correctly, what is the probability that he really knows the answer?

Solution Let B_1 be the event that the examinee knows the answer, B_2 be the event that the examinee guesses the answer and A be the event that the examinee answers correctly.

$\therefore$
$$P(B_1) = p,\ P(B_2) = 1 - p = q$$
$$P(A/B_1) = 1,\ P(A/B_2) = \frac{1}{n} \quad \text{(given)}$$

The probability that an examinee answers a question correctly is

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) = p \times 1 + (1 - p) \times \frac{1}{n} = p + q \times \frac{1}{n}$$

Given that an examinee answers a question correctly (A). Then the probability that he really knows the answer (B_1) is

$$P(B_1/A) = \frac{P(B_1)\,P(A/B_1)}{P(A)} = \frac{p}{p + (q/n)} = \frac{np}{np + q}$$

EXAMPLE 1.98 An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are drawn at random from the first urn and placed in the second urn and then one ball is taken at random from the latter. What is the probability that it is a white ball? **[AU June '05]**

Solution The 2 balls transferred may be both white or both black or 1 white and 1 black.

Let B_1 be the event that the balls chosen from the first urn are white, B_2 be the event that the balls chosen from the first urn are black, and B_3 be the event that the balls chosen from the first urn are 1 white and 1 black.

Then B_1, B_2 and B_3 are mutually exclusive events.

$$P(B_1) = \frac{10C_2}{13C_2} = \frac{15}{26},\ P(B_2) = \frac{3C_2}{13C_2} = \frac{1}{26} \text{ and } P(B_3) = \frac{10C_1 \times 3C_1}{13C_2} = \frac{10}{26}$$

$$P(B_1) + P(B_2) + P(B_3) = 1$$

Let A be the event that the ball chosen from the second urn is white.

P(drawing white ball/2 white balls have been transferred)

i.e.
$$P(A/B_1) = \frac{5C_1}{10C_1} = \frac{1}{2}$$

P(drawing white ball/2 black balls have been transferred)

i.e.
$$P(A/B_2) = \frac{3C_1}{10C_1} = \frac{3}{10}$$

P(drawing white ball/1 white and 1 black balls have been transferred)

i.e.
$$P(A/B_3) = \frac{4C_1}{10C_1} = \frac{4}{10} = \frac{2}{5}$$

$$P(A) = P(B_1)\, P(A/B_1) + P(B_2)\, P(A/B_2) + P(B_3)\, P(A/B_3)$$

$$P(A) = \frac{15}{26} \times \frac{1}{2} + \frac{1}{26} \times \frac{3}{10} + \frac{10}{26} \times \frac{2}{5} = \frac{1}{26}\left(\frac{15}{2} + \frac{3}{10} + 4\right)$$

$$= \frac{1}{26}\left(\frac{75+3+40}{10}\right) = \frac{1}{26}\left(\frac{118}{10}\right) = \frac{59}{130}$$

EXAMPLE 1.99 In a bolt factory, machines A, B and C produce 25%, 35% and 40% of the total output respectively. Of their outputs, 5%, 4% and 2% respectively are defective bolts. If a bolt is chosen at random from the combined output, what is the probability that it is defective? What is the probability that it was produced by B or C, if a bolt chosen is found to be defective? **[AU June '04]**

Solution Given:

$$P(A) = \frac{25}{100} = 0.25,\ P(B) = \frac{35}{100} = 0.35 \text{ and } P(C) = \frac{40}{100} = 0.4$$

$$P(A) + P(B) + P(C) = 1$$

Let D be the event that the bolt chosen is defective

$$\therefore \quad P(D/A) = \frac{5}{100} = 0.05$$

$$P(D/B) = \frac{4}{100} = 0.04$$

$$P(D/C) = \frac{2}{100} = 0.02$$

The probability that the bolt chosen is defective = probability that it may be chosen from A, B or C.

$\therefore$ The total probability is

$$P(D) = P(A)P(D/A) + P(B)\,P(D/B) + P(C)\,P(D/C)$$

$$= \frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100} = \frac{69}{2000} = 0.0345$$

The probability that it was produced by B or C if the bolt chosen is found to be defective.

$$P(B/D) = \frac{P(B)\,P(D/B)}{P(D)}$$

$$P(C/D) = \frac{P(C)\,P(D/C)}{P(D)}$$

$$P(B/D) + P(C/D) = \frac{P(B)\,P(D/B) + P(C)\,P(D/C)}{P(D)}$$

$$= \frac{0.35 \times 0.04 + 0.4 \times 0.02}{0.0345} = 0.6377$$

EXAMPLE 1.100 A given lot of product contains 2% defective products. Each product is tested before delivery. The probability that the product is good given that it is actually good is 0.95 and the probability that the product is defective given that it is actually defective is 0.94. If a tested product is defective, what is the probability that it is actually defective?

[AU December '04, '07]

Solution Let A be the event that the product is actually defective and B be the event that the product is actually good.

Let D be the event that the tested product is defective.

Given:

$$P(A) = 0.02,\ P(B) = 0.98$$

and $\qquad P(A) + P(B) = 1$

$$P(\bar{D}/B) = 0.95 \ [\bar{D}\text{—not defective or good}]$$

and $\qquad P(D/A) = 0.94$

$$\therefore \quad P(D/B) = 1 - P(\bar{D}/B) = 1 - 0.95 = 0.05$$

$$P(A/D) = \frac{P(A)\,P(D/A)}{P(A)\,P(D/A) + P(B)\,P(D/B)}$$

$$= \frac{(0.02)\,(0.94)}{(0.02)\,(0.94) + (0.98)\,(0.05)} = 0.2773$$

EXAMPLE 1.101 Each of two tanks fired independently at a target. The probability of the first tank destroying the target is $p_1 = 0.8$ that of the second $p_2 = 0.4$. The target is destroyed by a single hit. Determine the probability that it was destroyed by the first tank. **[AU June '07]**

Solution Prior to the firing, we have the following four possible cases:

(i) First tank hit and second tank missed
(ii) First tank missed and second tank hit
(iii) Both tanks first and second hit
(iv) Both tanks first and second missed

Let A be the event that the target is destroyed at a single hit.

Let B_1 be the event that the first tank hit and the second tank missed and the B_2 be the event that the first tank missed and the second tank hit

$$\therefore \quad P(B_1) = p_1(1 - p_2) = 0.8 \times (1 - 0.4) = 0.8 \times 0.6 = 0.48$$
$$P(B_2) = (1 - p_1)p_2 = (1 - 0.8) \times 0.4 = 0.2 \times 0.4 = 0.08$$
$$P(B_3) = p_1 \times p_2 = 0.8 \times 0.4 = 0.32$$
$$P(B_4) = (1 - p_1)\,(1 - p_2) = 0.2 \times 0.6 = 0.12$$

Since the target is destroyed by a single hit

$$P(A/B_1) = P(A/B_2) = 1$$

and $\qquad P(A/B_3) = P(A/B_4) = 0$

$$\therefore \quad P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)P(A/B_3) + P(B_4)\,P(A/B_4)$$

$$= 0.48 \times 1 + 0.08 \times 1 + 0 + 0 = 0.56$$

$$P(B_1/A) = \frac{P(B_1)\,P(A/B_1)}{P(A)} = \frac{0.48 \times 1}{0.56} = \frac{0.48}{0.56} = \frac{6}{7}$$

EXAMPLE 1.102 There are 3 true coins and 1 false coin with head on both sides. A coin is chosen at random and tossed 4 times. If head occurs all the 4 times, what is the probability that the false coin has chosen and used?

[AU November '04]

Solution Let T be the event of choosing a true coin and F be the event of choosing a false coin.

$$P(T) = \frac{\text{Number of true coins}}{\text{Total number of coins}} = \frac{3}{4}$$

$$P(F) = \frac{\text{Number of false coins}}{\text{Total number of coins}} = \frac{1}{4}$$

Let A be the event of getting head in all the 4 tosses.

When a true coin is chosen, the probability of getting a head $= \frac{1}{2}$

$$\therefore \qquad P(A/T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

When a false coin is chosen, the probability of getting a head = 1 (since both sides are head)

$$\therefore \qquad P(A/F) = 1 \times 1 \times 1 \times 1 = 1$$

Given: head occurs 4 times.

To find the probability that the false coin is chosen.

$$\therefore \qquad P(F/A) = \frac{P(F)\,P(A/F)}{P(A)} = \frac{P(F)\,P(A/F)}{P(T)\,P(A/T) + P(F)\,P(A/F)}$$

$$= \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{16}} = \frac{16}{19}$$

EXERCISES

1. State the axioms of probability.
2. Define mutually exclusive events with an example.
   (**Example:** Getting an odd number and getting an even number, when a six-faced die is tossed are two mutually exclusive events.)
3. Give the definition of probability with an example.
4. Define probability of an event.
5. Three coins are tossed together. Find the probability that there are exactly two tails. **[Ans. 1/4]**

6. Define the sample space and an event associated with a random experiment with an example.
7. Give the axiomatic definition of probability.
8. What is a random experiment? Give an example.
9. What do you infer from the statements $P(A) = 0$ and $P(A) = 1$?
10. From a bag containing 3 red and 2 black balls, 2 balls are drawn at random. Find the probability that they are of the same colour.
[Ans. 2/5]
11. When two cards are drawn from a well-shuffled pack of playing cards, what is the probability that they are of the same suit? [Ans. 4/17]
12. When A and B are two mutually exclusive events such that $P(A) = 1/2$ and $P(B) = 1/3$, find $P(A \cup B)$ and $P(A \cap B)$. [Ans. 5/6, 0]
13. If $P(A) = 0.29$ and $P(B) = 0.43$, find $P(A \cap \bar{B})$ if A and B are mutually exclusive.
[Ans. 0.29]
14. Prove that the probability of an impossible event is zero or prove that $P(\phi) = 0$.
15. Prove that $P(\bar{A}) = 1 - P(A)$, where $\bar{A}$ is the complement of A.
16. A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace? [Ans. 4/13]
17. The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will get an electric contract is 4/9. If the probability of getting at least one contract is 4/5, what is the probability that he will get both?
[Ans. 14/45]
18. If $P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$, find $P(\bar{A} \cap \bar{B})$.
[Ans. 0.2]
19. If $P(A) = 0.35$, $P(B) = 0.75$ and $P(A \cup B) = 0.95$, find $P(\bar{A} \cup \bar{B})$.
[Ans. 0.85]
20. Prove that $P(A \cup B) \leq P(A) + P(B)$. When does the equality hold good?
21. Give the definitions of joint and conditional probabilities with examples.
22. Events A and B are such that $P(A + B) = 3/4$, $P(AB) = 1/4$ and $P(\bar{A}) = 2/3$, find $P(B)$. [Ans. 2/3]
23. Give the definition of conditional probability and deduce the product theorem of probability.
24. When are two events said to be independent? Give an example for two independent events.
25. What is the probability of getting at least 1 head when 2 coins are tossed?
[Ans. 3/4]
26. Distinguish between unconditional and conditional probabilities.

27. When 2 dice are tossed, what is the probability of getting 4 as the sum of the face numbers? [Ans. 1/12]

28. If the probability that A solves a problem is 1/2 and that for B is 3/4 and if they aim at solving a problem independently, what is the probability that the problem is solved? [Ans. 7/8]

29. If $P(A) = 0.65$, $P(B) = 0.4$ and $P(A \cap B) = 0.24$, can A and B be independent events? [Ans. No]

30. Fifteen per cent of a firm's employees are BE degree holders, 25% are MBA degree holders and 5% have both the degrees. Find the probability of selecting a BE degree holder, if the selection is confined to MBAs. [Ans. 0.2]

31. In a random experiment, $P(A) = 1/12$, $P(B) = 5/12$ and $P(B/A) = 1/15$, find $P(A \cup B)$. [Ans. 89/180]

32. What is the difference between total independence and mutual independence?

33. Can two events be simultaneously independent and mutually exclusive? Explain.

34. A is known to hit the target 2 out of 5 shots whereas B is known to hit the target 3 out of 4 shots. Find the probability that the target is hit when both of them try. [Ans. 17/20]

35. If $P(A) = 0.29$ and $P(B) = 0.43$, find $P(A \cap \bar{B})$ if A and B are mutually exclusive. [Ans. $P(A \cap \bar{B}) = P(A) = 0.29$]

36. If $P(A) = 0.65$, $P(B) = 0.4$ and $P(A \cap B) = 0.24$, can A and B be independent events. [Ans. No. $P(A \cap B) \neq P(A)\ P(B)$]

37. If $P(A) = 0.5$, $P(B) = 0.3$ and $P(A \cap B) = 0.15$, find $P(A/\bar{B})$.
[Ans. $P(A \cap B) = P(A)\ P(B)$. Therefore, A and B are independent. $P(A/\bar{B}) = P(A) = 0.5$]

38. If A and B are independent events, prove that $P(A \cup B) = 1 - P(\bar{A})\ P(\bar{B})$.

39. State total probability theorem.

40. Suppose that 25% of the population of a country are unemployed women and a total of 35% are unemployed. What per cent of the unemployed are women? [Ans. 5/7]

41. State Bayes' theorem on inverse probability.

42. Bag I contains 2 red and 1 black balls and bag II contains 3 red and 2 black balls. What is the probability that a ball drawn from one of the bags is red? [Ans. 19/30]

43. Bag I contains 2 white and 3 black balls and bag II contains 4 white and 1 black balls. A ball chosen at random from one of the bags is white. What is the probability that it has come from bag I? [Ans. 1/3]

44. Five men out of 100 and 25 women out of 1000 are colour-blind. A colour-blind person is chosen at random. What is the probability that the person is a male? (Assume male and female are in equal numbers)
[Ans. 2/3]

45. State under which situation total probability theorem could be used.
[Ans. When B_1, B_2, ..., B_n are mututally exclusive and exhaustive events]

46. If there are 4 persons A, B, C and D and if A tossed with B, then C tossed with D and then the winners tossed. This process continues till the prize is won. What are the probabilities of each of the 4 to win?
[Ans. 1/4]

47. If $P(A + B) = 5/6$, $P(AB) = 1/3$ and $P(\bar{B}) = 1/2$, prove that the events A and B are independent.

48. Players X and Y roll a pair of dice alternately. The player who rolls 11 first wins. If X starts, find his chance of winning. [Ans. 18/35]

49. Three persons A, B and C draw in succession from a bag containing 8 red and 4 white balls until a white ball is drawn. What is the probability that C draws the white ball? [Ans. 7/33]

50. Write the sample space associated with the experiment of tossing three coins at a time and the event of getting heads from the first two coins. Also find the corresponding probability. [Ans. 1/4]

51. An urn contains 2 white and 4 black balls. Two balls are drawn one by one without replacement. Write the sample space corresponding to this experiment and the subsets corresponding to the following events:
(i) the first ball drawn is white
(ii) both the balls drawn are black.
Also find the probabilities of the above events.
[Ans. (i) 1/3, (ii) 2/5]

52. One integer is chosen at random from the numbers 1, 2, 3, ..., 100. What is the probability that the chosen number is divisible by
(i) 6 or 8, and
(ii) 6 or 8 or both? [Ans. (i) 1/5, (ii) 6/25]

53. Two fair dice are thrown independently. Four events A, B, C and D are defined as follows:
A: Even face with the first die.
B: Even face with the second die.
C: Sum of the points on the 2 dice is odd.
D: Product of the points on the 2 dice exceeds 20.
Find the probabilities of the four events.
[Ans. $P(A) = 1/2 = P(B) = P(C)$, $P(D) = 1/6$]

54. A committee of 6 is to be formed from 5 lecturers and 3 professors. If the members of the committee are chosen at random, what is the probability that there will be a majority of lecturers in the committee? [Ans. 9/14]

55. Twelve balls are placed at random in three boxes. What is the probability that the first box will contain 3 balls? [Ans. $12C_3 \times 2^9/3^{12}$]

56. If A and B are any 2 events, show that $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$.

57. A and B are two events associated with an experiment. If $P(A) = 0.4$ and $P(A \cup B) = 0.7$, find $P(B)$ when
    (i) A and B are mutually exclusive, and
    (ii) A and B are independent. [Ans. (i) 0.3, (ii) 0.5]

58. If $P(A + B) = 5/8$, $P(AB) = 1/8$ and $P(\bar{B}) = 1/2$, prove that the events A and B are independent.

59. If $A \subset B$, $P(A) = 1/4$ and $P(B) = 1/3$, find $P(A/B)$ and $P(B/A)$. [Ans. 3/4]

60. An electronic assembly consists of two subsystems A and B. From previous testing procedures, the following probabilities are assumed to be known: $P(A$ fails$) = 0.20$, $P(A$ and B fail$) = 0.15$ and $P(B$ fails alone$) = 0.15$. Evaluate (i) $P(A$ fails alone) and (ii) $P(A$ and B fail). [Ans. (i) 0.05, (ii) 0.50]

61. A card is drawn from a 52-card deck, and without replacing it, a second card is drawn. The first and second cards are not replaced and a third card is drawn.
    (i) If the first card is a heart, what is the probability of second card being a heart?
    (ii) If the first and second cards are hearts, what is the probability that the third card is the king of clubs? [Ans. (i) 12/51, (ii) 1/50]

62. A pair of dice is rolled once. Let A be the event that the first die has a 2 on it, B the event that the second die has a 4 on it and C the event that the sum is 6. Are A, B and C independent? [Ans. Pairwise independent, but not totally independent]

63. A, B and C are independent witnesses of an event which is known to have occurred. A speaks the truth three times out of four, B speaks the truth four times out of five and C speaks the truth five times out of six. What is the probability that the occurrence will be reported truthfully by majority of three witnesses? [Ans. 31/60]

64. A man who seeks advice regarding one of two possible courses of action from three advisers who arrived at their recommendations independently. He follows the recommendations of the majority. The probability that the

individual advisers are wrong are 0.1, 0.05 and 0.05 respectively. What is the probability that the man takes incorrect advice? [Ans. 0.012]

65. It is 8 : 5 against a male who is 55 years old living till he is 75 and 4 : 3 against a female who is now 48, living till she is 68. Find the probability that
    (i) they both will be active 20 years hence, and
    (ii) at least one of them will be active 20 years hence.
    [Ans. (i) 15/91, (ii) 59/91]

66. An urn contains 10 red and 3 black balls while another urn contains 3 red and 5 black balls. Two balls are drawn from the first urn and put into the second urn, then a ball is drawn from the latter and is found to be red. What is the probability that 1 red and 1 black were transferred?
    [Ans. 20/59]

67. The chances of a cricket match to be played between India and Pakistan in one of the three selected venues X, Y, Z are 0.25, 0.35 and 0.40 respectively. The probability of the match being disturbed at these venues are 0.05, 0.04 and 0.02 respectively. A match is played and is disturbed. What is the probability that the match was played at X? [Ans. 0.362]

68. Only 1 in 1000 adults is affected with a rare disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease a positive result will occur 99% of the time, whereas an individual without the disease will show a positive result only 2% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?
    [Ans. 0.047]

69. A bolt is manufactured by 3 machines A, B and C. A turns out twice as many items as B and machines B and C produce equal number of items. Two per cent of bolts produced by A and B are defective and 4% of bolts produced by C are defective. All bolts are put into one stock pile and one is chosen from this pile. What is the probability that it is defective?
    [Ans. 1/40]

70. Suppose that coloured balls are distributed in 3 boxes as follows : box 1 contains 2 red, 3 white and 5 blue balls, box 2 contains 4 red, 1 white and 3 blue balls, and box 3 contains 3 red, 4 white and 5 blue balls. A box is selected at random, one ball is chosen from it and it is observed to be red. What is the probability that box 3 was selected? [Ans. 5/19]

2

Random Variables

Let S be the sample space associated with a given random experiment. A real-valued function defined on S and taking values in $R(-\infty, \infty)$ is called a *one-dimensional* random variable. By a random variable, we mean a real number X connected with the outcome of a random experiment, i.e. a random variable is a variable which assigns a real value to each out come of a random experiment.

For example: Let E be the random experiment consisting of two tosses of a coin.

$$S = \{HH\ HT\ TH\ TT\}$$

We may define the random variable X which denotes the number of heads (0, 1 or 2).

$$X = \{2\ 1\ 1\ 0\}$$

If the function values are ordered pairs of real numbers (vectors in two-space), the function is said to be a *two-dimensional random variable*.

In general, an *n-dimensional random variable* is a function whose domain is S and whose range is a collection of n-tuples of real numbers (vectors in n-space).

2.1 THEOREMS ON RANDOM VARIABLES

THEOREM 1 If X_1 and X_2 are random variables and if C_1 and C_2 are constants, then $C_1X_1, X_1 + X_2, X_1X_2, C_1X_1 + C_2X_2, X_1 - X_2$ are also random variables.

THEOREM 2 If X is a random variable, then $1/X$, $|X|$ are also random variables.

THEOREM 3 If X_1 and X_2 are random variables, then max $[X_1, X_2]$ and min $[X_1, X_2]$ are also random variables.

THEOREM 4 If X is a random variable and $f(\ )$ is a continuous function, then $f(x)$ is a random variable.

2.2 DISTRIBUTION FUNCTION

Let X be a random variable, then the function $F_x(x) = F(x) = P(X \leq x) = P\{w : x(w) \leq x\}$, $-\infty < x < \infty$ is called the distribution function of X.

2.3 PROPERTIES OF DISTRIBUTION

1. If $F(x)$ is the distribution function of the random variable X and if $a < b$, then $P(a < X \leq b) = F(b) - F(a)$.

Proof The events $a < X \leq b$ and $X \leq a$ are disjoint and their union is the event $X \leq b$.

$$(a < X \leq b) + (X \leq a) = (X \leq b)$$

Using addition theorem of probability,

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

Hence the proof.

Note:

(i) $P(a \leq X \leq b) = P[(X = a)\ (a < X \leq b)]$
$\qquad = P(X = a) + P(a < X \leq b)$
$\qquad = P(X = a) + F(b) - F(a)$

(ii) $P(a < X < b) = P(a < X \leq b) - P(X = b)$
$\qquad = F(b) - F(a) - P(X = b)$

(iii) $P(a \leq X \leq b) = P(a < X \leq b) + P(X = a) - P(X = b)$
$\qquad = F(b) - F(a) + P(X = a) - P(X = b)$

(iv) When $P(X = a) = 0 = P(X = b)$, all the four probabilities $P(a \leq X \leq b)$, $P(a < X < b)$, $P(a \leq X < b)$ and $P(a < X \leq b)$ have the same probability $F(b) - F(a)$.

2. If $F(x)$ is the distribution function of one-dimensional random variable X, then

(i) $0 \leq F(x) \leq 1$
(ii) $F(x) \leq F(y)$, if $x < y$

i.e. all distribution functions are monotonically non-decreasing and lie between 0 and 1.

Proof

(i) Using axioms of certainty and non-negativity for the probability function, it is trivial that $0 \leq F(x) \leq 1$.

(ii) For $x < y$,

$$F(y) - F(x) = P(x < X \le y) \ge 0$$

$$\Rightarrow \quad F(y) \ge F(x)$$

$$\Rightarrow \quad F(x) \le F(y)$$

when $x < y$

Hence the proof.

3. If $F(x)$ is a distribution function of one-dimensional random variable X, then $F(\infty) = \lim_{x \to \infty} F(x) = 1$ and $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$.

Combining all these properties, if $F(x)$ is a distribution function of a random variable X, then

(i) $0 \le F(x) \le 1$

(ii) $F(\infty) = 1$, $F(-\infty) = 0$

(iii) $F(x) \le F(y)$ for $x < y$

2.4 DISCRETE RANDOM VARIABLE

If a random variable X takes at most a countable number of values or countably infinite number of values, it is called a *discrete random variable*. In other words, a real valued function defined on a discrete sample space is called a *discrete random variable*.

2.5 PROBABILITY MASS FUNCTION (PMF)

Suppose X is an one-dimensional discrete random variable taking atmost a countably infinite number of values $x_1, x_2, \ldots$. With each possible outcome x_i, we associate a number p_i, $P(X = x_i) = p(x_i) = p_i$, called the probability of x_i.

The function $p(x_i)$, $i = 1, 2, \ldots$ satisfying the conditions

(i) $p(x_i) \ge 0 \; \forall i$

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

is called the probability mass function or probability function of the random variable X. The collection of pairs $\{x_i, p_i\}$, $i = 1, 2, 3, \ldots$ is called the probability distribution of the random variable X.

Note: The set of values which X takes is called the *spectrum* of the random variable.

2.6 DISCRETE DISTRIBUTION FUNCTION

The distribution function of the random variable X with PMF $p(x_i)$, $i = 1, 2, 3, \ldots$, is defined as

$$F(x_i) = \sum_{i : x_i \le x} p(x_i)$$

Note:

(i) $p(x_i) = P(X = x_i) = F(x_i) - F(x_{i-1})$, where F is the distribution function of the random variable X.

(ii) Mean of the random variable $X = E(X) = \sum_x xP(x)$

(iii) Variance of the random variable X

$$\text{Var}(X) = \sum x^2 p(x) - \left[\sum xp(x)\right]^2$$

EXAMPLE 2.1 If X is a discrete random variable having the probability distribution

$X = x$	1	2	3
$P(X = x)$	k	$2k$	k

Find $P(X \le 2)$.

Solution We know that

$$\sum P(X = x) = 1 \quad \Rightarrow \quad 4k = 1$$

$$\therefore \qquad k = \frac{1}{4}$$

$$P(X \le 2) = P(X = 1) + P(X = 2) = 3k \quad \Rightarrow \quad P(X \le 2) = \frac{3}{4}$$

EXAMPLE 2.2 If X is a discrete random variable having the PMF

x	-1	0	1
$P(x)$	k	$2k$	$3k$

Find $P(X \ge 0)$.

Solution We know that

$$\sum P(X = x) = P(X = -1) + P(X = 0) + P(X = 1)$$

$$k + 2k + 3k = 1 \Rightarrow 6k = 1$$

$$k = \frac{1}{6}$$

$$P(X \geq 0) = P(X = 0) + P(X = 1) = 2k + 3k = 5k$$

$$\Rightarrow \quad P(X \geq 0) = \frac{5}{6}$$

EXAMPLE 2.3 If X is a discrete random variable with the following probability distribution

x	1	2	3	4
$P(x)$	a	$2a$	$3a$	$4a$

find $P(2 < X < 4)$

Solution We know that

$$\sum P(X = x) = 1$$

$$\Rightarrow \quad P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1$$

$$\Rightarrow \quad 10a = 1$$

$$\therefore \quad a = \frac{1}{10}$$

$$P(2 < X < 4) = P(X = 3) = 3a = \frac{3}{10}$$

EXAMPLE 2.4 If the probability distribution of X is given as:

x	1	2	3	4
$p(x)$	0.4	0.3	0.2	0.1

find $P(1/2 < X < 7/2/X > 1)$.

Solution By definition,

$$P(1/2 < X < 7/2/X > 1) = \frac{P\left(\frac{1}{2} < X < \frac{7}{2} \cap X > 1\right)}{P(X > 1)}$$

$$= \frac{P\left(1 < X < \frac{7}{2}\right)}{P(X > 1)}$$

$$= \frac{P(X = 2) + P(X = 3)}{1 - P(X \leq 1)}$$

$$= \frac{P(X = 2) + P(X = 3)}{1 - P(X = 1)}$$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

: following

Solution By definition,

$$F(x) = P(X \le x)$$

$$= \int_0^x x\,dx = \frac{x^2}{2}, 0 \le x \le 1$$

$$F(x) = \int_0^1 x\,dx + \int_1^x \frac{3}{2}(x-1)^2 dx, \qquad 1 \le x \le 2$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \frac{3}{2}\left[\frac{x^3}{3} - \frac{2x^2}{2} + x\right]_1^x$$

$$= \frac{1}{2} + \frac{3}{2}\left(\frac{x^3}{3} - x^2 + x - \frac{1}{3} + 1 - 1\right)$$

$$= \frac{1}{2} + \frac{3}{2}\left(\frac{x^3}{3} - x^2 + x - \frac{1}{3}\right)$$

$$F(x) = \frac{x^3}{2} - \frac{3x^2}{2} + \frac{3}{2}x, \ 1 \le x \le 2$$

$$F(x) = 1, \qquad x > 2$$

$\therefore$ The cumulative distribution function is

$$= \begin{cases} \dfrac{x^2}{2}, & 0 \le x \le 1 \\ \dfrac{x^3}{2} - \dfrac{3x^2}{2} + \dfrac{3x}{2}, & 1 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

To find $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$.

$$P\left(\frac{3}{2} < X < \frac{5}{2}\right) = F\left(\frac{5}{2}\right) - F\left(\frac{3}{2}\right)$$

$$= 1 - \left(\frac{27}{16} - \frac{27}{8} + \frac{9}{4}\right) = \frac{7}{16}$$

EXAMPLE 2.41 Verify whether $f(x) = xe^{-\frac{x^2}{2}}, 0 < x < \infty$ is a probability density function of a random variable X. If so, find its distribution function. **[AU April '04, June '07]**

Solution If $f(x)$ is a PDF, then $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \qquad \int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} x e^{-\frac{x^2}{2}}\,dx$$

Put $\frac{x^2}{2} = t \Rightarrow x^2 = 2t,$

$\therefore \quad x = \sqrt{2t},\ 2x\,dx = 2\,dt$ and $dx = \frac{dt}{x} = \frac{dt}{\sqrt{2t}}$

when $x = 0,\ t = 0$, and $x = \infty,\ t = \infty$

$$= \int_{0}^{\infty} \sqrt{2t}\, e^{-t} \frac{dt}{\sqrt{2t}} = \int_{0}^{\infty} e^{-t}\,dt = \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= \frac{e^{-\infty}}{-1} + \frac{e^{0}}{1} = 1 \quad (e^{-\infty} = 0)$$

i.e. $$\int_{-\infty}^{\infty} f(x)dx = 1$$

$\therefore$ It is a PDF.

To find the distribution function:

$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}}\,dx = \int_{0}^{\frac{x^2}{2}} e^{-t}\,dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^{\frac{x^2}{2}}$$

$$= 1 - e^{-\frac{x^2}{2}},\ x \ge 0$$

EXAMPLE 2.42 The diameter of an electric cable say X is assumed to be a continuous random variable with probability distribution function $f(x) = 6x(1 - x),\ 0 \le x \le 1$. Determine a number b, such that $P(X < b) = P(X > b)$. Also, find its cumulative distribution function.

[AU June '04, '07]

Solution By definition

$$P(X < b) = \int_{-\infty}^{b} f(x)\,dx = \int_{0}^{b} 6x(1 - x)dx$$

$$P(X > b) = \int_{0}^{\infty} f(x)\,dx = \int_{b}^{1} 6x(1 - x)dx$$

Given: $P(X < b) = P(X > b)$

$$\therefore \quad \int_0^b 6x(1-x)dx = \int_b^1 6x(1-x)dx$$

$$\int_0^b (6x-6x^2)dx = \int_b^1 (6x-6x^2)dx$$

$$\left[\frac{6x^2}{2}-\frac{6x^3}{3}\right]_0^b = \left[\frac{6x^2}{2}-\frac{6x^3}{3}\right]_b^1$$

$$3b^2 - 2b^3 = 3 - 2 - 3b^2 + 2b^3$$

$$4b^3 - 6b^2 + 1 = 0$$

i.e. $$(2b-1)(2b^2-2b-1) = 0$$

$\Rightarrow$ $$b = 0.5$$

or $$b = \frac{1 \pm i}{2}$$

$\therefore$ $b = 0.5$ is the only real value satisfying the condition.

To find the cumulative distribution function:

$$F(x) = P(X \le x)$$

$$= \int_0^x f(x)dx = 6\int_0^x x(1-x)dx = 6\int_0^x (x-x^2)dx$$

$$= 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^x = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right] = 3x^2-2x^3$$

$$F(x) = \begin{cases} 0, & x<0 \\ 3x^2-2x^3, & 0 \le x < 1 \\ 1, & x > 1 \end{cases}$$

EXAMPLE 2.43 A continuous random variable X has the PDF

$$f(x) = \frac{k}{1+x^2}, -\infty < x < \infty.$$

Find

(i) k,
(ii) distribution function of X, and
(iii) $P(X \le 0)$. **[AU December '07, May '08]**

Solution

(i) Since $f(x)$ is a PDF of the random variable X,

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}dx = 1$$

$$k\left[\tan^{-1}(x)\right]_{-\infty}^{\infty} = 1 \Rightarrow k\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

i.e.
$$k\pi = 1 \Rightarrow k = \frac{1}{\pi}$$

(ii) The distribution functions of X is

$$F(x) = P(X \le x)$$

$$\therefore \quad F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+x^2}dx = \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left(\tan^{-1}(x) + \frac{\pi}{2}\right), -\infty < x < \infty$$

(iii) $$P(X \ge 0) = \int_{0}^{\infty} f(x)dx = \frac{1}{\pi}\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{0}^{\infty}$$

$$= \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

EXAMPLE 2.44 The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon with probability function specified by the PDF

$$f(x) = \begin{cases} Ae^{-\frac{x}{5}}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of A that makes $f(x)$ a PDF. **[AU November '04]**

Solution If $f(x)$ is a PDF, then

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{0}^{\infty} Ae^{\frac{-x}{5}}dx = 1$$

$$= \left[\frac{e^{\frac{-x}{5}}}{-\frac{1}{5}}\right]_{0}^{\infty} = 1 \Rightarrow A(5) = 1$$

i.e.
$$A = \frac{1}{5}$$

EXAMPLE 2.45 Verify whether

$$f(x) = \begin{cases} 0, & \text{if } x < 2 \\ \dfrac{3+2x}{18}, & \text{if } 2 \le x \le 4 \\ 0, & \text{if } x > 4 \end{cases}$$

is a PDF. If so, find the value of $P(2 \le X \le 3)$. **[AU December '06]**

Solution

(i) $f(x) \ge 0 \; \forall \; x$

(ii) $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_2^4 \frac{3+2x}{18}dx = \frac{1}{18}\int_2^4 (3+2x)dx$

$$= \frac{1}{18}[3x + x^2]_2^4 = \frac{1}{18}(12 + 16 - 6 - 4) = \frac{18}{18} = 1$$

Since $f(x) \ge 0 \; \forall \; x$ and $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$, the given $f(x)$ is a PDF

$$P(2 \le X \le 3) = \int_2^3 \frac{3+2x}{18}dx = \frac{1}{18}\int_2^3 (3+2x)dx$$

$$= \frac{1}{18}[3x + x^2]_2^3 = \frac{8}{18}$$

EXAMPLE 2.46 If $f(x) = kx$, $0 < x < 1$ is a PDF of a random variable, find the value of k and $P(X > 0.5)$. **[AU December '06]**

Solution If $f(x)$ is a PDF, then

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx\,dx = 1 \;\Rightarrow\; k\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\frac{k}{2} = 1 \;\Rightarrow\; k = 2$$

$$P(X > 0.5) = \int_{0.5}^1 kx\,dx = k\left[\frac{x^2}{2}\right]_{0.5}^1 = 2\left(\frac{1}{2} - \frac{0.5}{2}\right)$$

$$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

EXAMPLE 2.47 The CDF of a random variable X is given by

$$F(x)=\begin{cases}0, & x<0\\ x^2, & 0\le x<\frac{1}{2}\\ 1-\frac{3}{25}(3-x)^2, & \frac{1}{2}\le x<3\\ 1, & x\ge 3\end{cases}$$

Find the PDF of X and evaluate $P(|X|\le 1)$ and $P\left(\frac{1}{3}\le X\le 4\right)$ using both PDF and CDF.

Solution Given the CDF $F(x)$. We know that

$$f(x)=\frac{d}{dx}[F(x)]=F'(x)$$

The PDF of X is

$$f(x)=\begin{cases}0, & x<0\\ 2x, & 0\le x<\frac{1}{2}\\ \frac{6}{25}(3-x), & \frac{1}{2}\le x<3\\ 0, & x\ge 3\end{cases}$$

Case (i): Using PDF

$$P(|X|\le 1)=P(-1\le X\le 1)=\int_{-1}^{1}f(x)dx$$

$$=\int_0^{\frac{1}{2}}2xdx+\int_{\frac{1}{2}}^{1}\frac{6}{25}(3-x)dx=[x^2]_0^{\frac{1}{2}}+\frac{6}{25}\left[3x-\frac{x^2}{2}\right]_{\frac{1}{2}}^{1}$$

$$=\frac{1}{4}+\frac{6}{25}\left(3-\frac{1}{2}-\frac{3}{2}+\frac{1}{8}\right)=\frac{13}{25}$$

$$P(1/3<X\le 4)=\int_{\frac{1}{3}}^{\frac{1}{2}}2xdx+\int_{\frac{1}{2}}^{3}\frac{6}{25}(3-x)dx$$

$$=[x^2]_{\frac{1}{3}}^{\frac{1}{2}}+\frac{6}{25}\left[3x-\frac{x^2}{2}\right]_{\frac{1}{2}}^{3}=\frac{8}{9}$$

Case (ii): Using CDF

(i) $P(|X| \le 1) = P(-1 \le X \le 1)$

$$= F(1) - F(-1) = 1 - \frac{3}{25}(3-1) - 0 = 1 - \frac{12}{25} = \frac{13}{25}$$

(ii) $P\left(\frac{1}{3} \le X \le 4\right) = F(4) - F\left(\frac{1}{3}\right) = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$

h PDF

EXAMPLE 2.48 If a random variable X has the PDF

$$f(x) = \begin{cases} \frac{1}{4}, & |x| < 2 \\ 0, & \text{otherwise} \end{cases}$$

find

(i) $P(X < 1)$,
(ii) $P(|X| > 1)$, and
(iii) $P(2X + 3 > 5)$. **[AU June '07]**

Solution

(i) $P(X < 1) = \int_{-\infty}^{1} f(x)dx$

$$= \int_{-2}^{1} \frac{1}{4} dx = \frac{1}{4}[x]_{-2}^{1} = \frac{1}{4}(1+2) = \frac{3}{4}$$

(ii) $P(|X| > 1) = 1 - P(|X| < 1) = 1 - P(-1 < X < 1)$

$$= 1 - \int_{-1}^{1} \frac{1}{4} dx = 1 - \frac{1}{4}[x]_{-1}^{1} = 1 - \frac{2}{4} = \frac{1}{2}$$

(iii) $P(2X + 3 > 5) = P(2X > 5 - 3) = P(2X > 2) = P(X > 1)$

$$= 1 - P(X < 1) = 1 - \frac{3}{4} = \frac{1}{4}$$

EXAMPLE 2.49 A continuous random variable has the PDF $f(x) = kx^4$, $-1 < x < 0$. Find the value of k and also $P\left(X > \frac{-1}{2} / X < \frac{-1}{4}\right)$.

Solution Given: $f(x) = kx^4$, $-1 < x < 0$ is a PDF.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-1}^{0} (kx^4)dx = 1$$

i.e. $$k\left[\frac{x^5}{5}\right]_{-1}^{0} = k\left[0-\left(\frac{-1}{5}\right)\right] \Rightarrow k=5$$

$$P\left(X>\frac{-1}{2}/X<\frac{-1}{4}\right) = \frac{P\left(X>\frac{-1}{2}\cap X<\frac{-1}{4}\right)}{P\left(X<\frac{-1}{4}\right)}$$

$$= \frac{P\left(\frac{-1}{2}<X<\frac{-1}{4}\right)}{P\left(X<\frac{-1}{4}\right)}$$

$$= \frac{\int_{\frac{-1}{2}}^{\frac{-1}{4}} x^4 dx}{\int_{-1}^{\frac{-1}{4}} x^4 dx} = \frac{\left[\frac{x^5}{5}\right]_{\frac{-1}{2}}^{\frac{-1}{4}}}{\left[\frac{x^5}{5}\right]_{-1}^{\frac{-1}{4}}} = \frac{1}{33}$$

EXAMPLE 2.50 A continuous random variable that can assume any value between $x = 2$ and $x = 5$ has the density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$. **[AU May '06, June '07]**

Solution Given: $f(x) = k\,(1 + x)$, $2 < x < 5$, and 0, otherwise

$\therefore$ $$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{2}^{5} k(1+x)dx = 1$$

i.e. $$k\int_{2}^{5}(1+x)dx = 1 \Rightarrow k\left[x+\frac{x^2}{2}\right]_2^5 = 1$$

$$k\left(5+\frac{25}{2}-2-\frac{4}{2}\right) = 1$$

i.e. $$\frac{27}{2}k = 1 \Rightarrow k = \frac{2}{27}$$

$\therefore$ $$k = \frac{2}{27}$$

$$P(X<4) = \int_{-\infty}^{4} f(x)dx = \int_{-\infty}^{2} f(x)dx + \int_{2}^{4} f(x)dx$$

2.9 CUMULATIVE DISTRIBUTION FUNCTION (CDF)

The cumulative distribution $F(x)$ of a continuous random variable X with PDF $f(x)$ is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx, -\infty < x < \infty$$

Note: $P(a < x < b) = F(b) - F(a)$

The relation between the CDF and PDF is

$$f(x) = \frac{d}{dx} F(x)$$

If X is a continuous random variable with PDF $f(x)$, then

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

EXAMPLE 2.23 Verify whether $f(x) = \begin{cases} |x|, -1 \le x \le 1 \\ 0, \text{elsewhere} \end{cases}$ can be the PDF of a continuous random variable.

Solution For $f(x)$ to be a PDF, it should satisfy

(i) $f(x) = |x| \ge 0, \forall\, x$ (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Given:

(i) $f(x)|x| \ge 0, \forall\, x$

(ii) $\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} |x|\, dx = 2\int_{0}^{1} xdx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$

Therefore, $f(x)$ can be the PDF of X.

EXAMPLE 2.24 A random variable X has the PDF $f(x)$ given by

$$f(x) = \begin{cases} cxe^{-x}, x > 0 \\ 0, x \le 0 \end{cases}$$

Find the value of c and CDF of x. **[AU April/May '08]**

Solution If $f(x)$ is a PDF, then

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \qquad \int_{0}^{\infty} cxe^{-x}dx = 1$$

$$c\left[x\left(\frac{e^{-x}}{-1}\right)-1\left(\frac{e^{-x}}{1}\right)\right]_0^{\infty} = 1$$

$$\Rightarrow \qquad c(0+1) = 1 \Rightarrow c = 1$$

$$\therefore \qquad f(x) = xe^{-x}, x > 0$$

$$\text{The CDF of } X = F(x) = P(X \le x) = \int_0^x xe^{-x}dx$$

$$= \left[x\left(\frac{e^{-x}}{-1}\right)-1\left(\frac{e^{-x}}{1}\right)\right]_0^x$$

$$= (-xe^{-x} - e^{-x}) - (0 - 1)$$

$$F(x) = 1 - (1 + x)e^{-x}, x > 0$$

$$= 0, \text{ otherwise}$$

EXAMPLE 2.25 A continuous random variable X follows the probability law $f(x) = ax^2$, $0 \le x \le 1$. Determine a and find the probability that x lies between 1/4 and 1/2.

Solution If $f(x)$ is a PDF, then

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^{\infty} f(x)dx = 1 \Rightarrow \int_0^1 f(x)dx = 1$$

$$\Rightarrow \qquad \int_0^1 ax^2dx = 1 \Rightarrow a\left[\frac{x^3}{3}\right]_0^1 = 1$$

i.e. $$\frac{a}{3} = 1 \Rightarrow a = 3$$

$$P\left(\frac{1}{4} \le x \le \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} 3x^2 dx$$

$$= 3\left[\frac{x^3}{3}\right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= 3\left[\frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{4}\right)^3}{3}\right] = \frac{7}{64}$$

EXAMPLE 2.26 If the PDF of a random variable X is $f(x) = \frac{x}{2}$ in $0 \leq x \leq 2$, find $P(X > 1.5/X > 1)$.

Solution We know that

$$P(X > 1/5/X > 1) = \frac{P(X > 1.5 \cap x > 1)}{P(X > 1)} = \frac{P(X > 1.5)}{P(X > 1)}$$

$$= \frac{\int_{1.5}^{2} \frac{x}{2} dx}{\int_{1}^{2} \frac{x}{2} dx}$$

$$= \frac{4 - 2.25}{4 - 1} = 0.5833$$

EXAMPLE 2.27 If $f(x) = kx^2$, $0 < x < 3$ is to be the density function, find the value of k.

Solution If $f(x)$ is a PDF, then

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \qquad \int_{0}^{3} kx^2\, dx = 1 \quad \Rightarrow \quad 9k = 1$$

$$\therefore \qquad k = \frac{1}{9}$$

EXAMPLE 2.28 If the CDF of a random variable X is given by $F(x) = 0$ for $x < 0$; $= \frac{x^2}{16}$ for $0 \le x < 4$, and $= 1$ for $x \ge 4$, find $P(X > 1/X < 3)$.

Solution Given: $$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{16}, & 0 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$

Using $P(A/B) = \frac{P(A \cap B)}{P(B)}$, we get

$$P(X > 1/X < 3) = \frac{P(X > 1 \cap X < 3)}{P(X < 3)}$$

$$= \frac{P(1 < X < 3)}{P(0 < X < 3)}$$

$$= \frac{F(3) - F(1)}{F(3) - F(0)}$$

$$P(X > 1/X < 3) = \frac{\frac{8}{16}}{\frac{9}{16}} = \frac{8}{9}$$

EXAMPLE 2.29 The cumulative distribution of X is $F(x) = \frac{x^3 + 1}{9}$, $-1 < X < 2$ and $= 0$, otherwise. Find $P(0 < X < 1)$.

Solution Using $P(a < X < b) = F(b) - F(a)$, we get

$$P(0 < X < 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

EXAMPLE 2.30 The CDF of X is given by $F(x) = \begin{cases} 0, & x > 0 \\ x^2, & 0 \le x \le 1. \\ 1, & x > 1 \end{cases}$

Find the PDF of X and obtain $P(X > 0.75)$.

Solution Given: $$F(x) = \begin{cases} 0, & x > 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$\therefore \qquad f(x) = \frac{d}{dx}F(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} P(X > 0.75) &= 1 - P(X \le 0.75) \\ &= 1 - F(0.75) = 1 - (0.75)^2 = 0.4375 \end{aligned}$$

EXAMPLE 2.31 Verify whether $f(x) = 1 - |1 - x|$, for $0 < x < 2$ is a PDF of a random variable X.

Solution If $f(x)$ is a PDF, then

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) &= \int_0^1 [1-(1-x)]dx + \int_{1}^{2} [1+(1-x)]dx \\ &= \int_0^1 x dx + \int_1^2 (2-x)dx \\ &= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{1}{2} + \left(2 - \frac{3}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

$\therefore$ $f(x)$ is a PDF.

EXAMPLE 2.32 Check whether $f(x) = \frac{1}{4}xe^{-x/2}$ for $0 < x < \infty$ can be the PDF of X.

Solution If $f(x)$ is a PDF, then

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_0^{\infty} \frac{x}{4}e^{-\frac{x}{2}}dx = \int_0^{\infty} te^{-t}dt \qquad \left(\text{put } \frac{x}{2} = t, dx = 2dt\right) \\ &= \left[-te^{-t} - e^{-t}\right]_0^{\infty} = -(0-1) = 1 \end{aligned}$$

$\therefore$ $f(x)$ is the PDF of X.

EXAMPLE 2.33 Find the cumulative distribution function $F(x)$ corresponding to the PDF $f(x) = \dfrac{1}{\pi(1+x^2)}, -\infty < x < \infty.$

Solution Given: $f(x) = \dfrac{1}{\pi(1+x^2)}, -\infty < x < \infty$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dx}{1+x^2} = \frac{1}{\pi}[\tan^{-1} x]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} x\right)$$

EXAMPLE 2.34 The diameter of an electric cable, say X is assumed to be a continuous random variable with PDF given by $f(x) = kx(1 - x)$, $0 \le x \le 1$. Determine k and $P\left(X \le \frac{1}{3}\right)$.

Solution Given: $f(x)$ is a PDF

$$\therefore \qquad \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \qquad \int_{0}^{1} kx(1-x)dx = 1 \;\Rightarrow\; k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$$

$$\Rightarrow \qquad k = 6$$

$$P\left(X < \frac{1}{3}\right) = 6\int_{0}^{\frac{1}{3}} (x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{1}{3}}$$

$$= \left[3x^2 - 2x^3\right]_0^{\frac{1}{3}}$$

$$= \frac{1}{3} - \frac{2}{27}$$

$$= \frac{7}{27}$$

EXAMPLE 2.35 If the distribution function of a random variable X is given by $F(x) = 1 - \frac{4}{x^2}$, for $x > 2$ and $F(x) = 0$ for $x \le 2$, find $P(4 < X < 5)$.

Solution Given: $f(x) = \begin{cases} 1 - \dfrac{4}{x^2}, & x > 2 \\ 0, & x \le 2 \end{cases}$

$$\therefore \qquad P(4 < X < 5) = F(5) - F(4) = \frac{21}{25} - \frac{3}{4} = 0.09$$

EXAMPLE 2.36 If the CDF of a random variable X is given by $F(x) = \frac{1}{2}e^{kx}, x \le 0$ and $F(x) = 1 - \frac{1}{2}e^{-kx}, x > 0$, find $P\left(|X| \le \frac{1}{k}\right)$.

Solution Using the definition

$$P\left(|X| \le \frac{1}{k}\right) = P\left(-\frac{1}{k} < X < \frac{1}{k}\right)$$

$$= F\left(\frac{1}{k}\right) - F\left(-\frac{1}{k}\right)$$

$$= 1 - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-1} = 1 - \frac{1}{e}$$

EXAMPLE 2.37 If X is a continuous random variable whose PDF is given by $f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$.

Find
(i) the value of c, and
(ii) $P(X > 1)$.

Solution If $f(x)$ is a PDF, then $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \qquad \int_0^2 c(4x - 2x^2)dx = 1 \Rightarrow c\left[\frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2 = 1$$

$$c\left(8 - \frac{16}{3}\right) = 1 \Rightarrow c\left(\frac{8}{3}\right) = 1$$

$$\therefore \qquad c = \frac{3}{8}$$

$$P(X > 1) = \int_{-\infty}^{\infty} f(x)dx = c\int_1^2 (4x - 2x^2)dx$$

$$= c\left[\frac{4x^2}{2} - \frac{2x^3}{3}\right]_1^2$$

$$= \frac{3}{8}\left[\left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)\right]$$

$$= \frac{3}{8} \times \frac{4}{3} = \frac{1}{2}$$

EXAMPLE 2.38 Verify whether the function $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is a probability density function. If so, determine the variate having this density will fall in the interval (1, 2). **[AU May '04]**

Solution To verify $f(x)$ is a PDF, we have to check

(i) $f(x) \geq 0 \;\forall\; x$ in $(-\infty, \infty)$

(ii) $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$

$$f(x) = e^{-x} \geq 0 \;\forall\; x \text{ in } (-\infty, \infty)$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 0-(-1) = 1$$

$\therefore$ The given $f(x)$ is a PDF.

$$P(1 \leq X \leq 2) = \int_{1}^{2} f(x)dx = \int_{1}^{2} e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_1^{2} = e^{-1} - e^{-2} = 0.233$$

EXAMPLE 2.39 Show that the function $f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ is a PDF.

Solution

(i) $f(x) \geq 0 \;\forall\; x$

(ii) $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx$$

$$= \int_{-1}^{2} \frac{x^2}{3}dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{9}(8+1) = 1$$

$\therefore$ $f(x)$ is a PDF

EXAMPLE 2.40 If X is a continous random variable with PDF

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ \dfrac{3}{2}(x-1)^2, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

find the comulative distribution function $F(x)$ of X and use it to find $P\left(\dfrac{3}{2} < X < \dfrac{5}{2}\right)$.

[AU May '08]

EXAMPLE 2.16 A random variable X has the following probability function:

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k.
(ii) Evaluate $P(X < 6)$, and $P(0 < X < 5)$.
(iii) If $P(X \leq k) > 1/2$, find the minimum value of k and determine the distribution function of X. **[AU April '05, December '08]**

Solution

(i) If $p(x)$ is the PMF, then

$$\sum_{x=0}^{7} p(x) = 1$$

$$\therefore \quad k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0 \quad \Rightarrow \quad (10k - 1)(k + 1) = 0$$

$$\Rightarrow \quad k = \frac{1}{10} \qquad (\because k = -1 \text{ is not possible})$$

(ii) $P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7)$$

$$= \frac{2}{100} + \frac{7}{100} + \frac{1}{10} = \frac{19}{100}$$

$$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$$

(iii) Given: $P(X \leq k) > 1/2$.
To find the minimum value of k:

$$P(X \leq 1) = \frac{1}{10} < \frac{1}{2}$$

and

$$P(X \leq 2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} < \frac{1}{2}$$

$$P(X \leq 3) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$$

$$P(X \leq 4) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{3}{10} = \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$$

$$\therefore \qquad P(X \le 4) \ge \frac{1}{2} \text{ i.e. } k = 4$$

The distribution function is

$$F(x) = 0,\ x < 1$$
$$= k = \frac{1}{10}, 1 \le x < 2$$
$$= 3k = \frac{3}{10}, 2 \le x < 3$$
$$= 5k = \frac{5}{10}, 3 \le x < 4$$
$$= 8k = \frac{8}{10}, 4 \le x < 5$$
$$= 8k + k^2 = \frac{81}{100}, 5 \le x < 6$$
$$= 8k + 3k^2 = \frac{83}{100}, 6 \le x < 7$$
$$= 9k + 10k^2 = 1,\ x \ge 7$$

EXAMPLE 2.17 From a lot of 10 items containing 3 defective items, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Answer the following when the sample is drawn without replacement:

(i) Find the probability distribution of X.
(ii) Find $P(X \le 1)$, $P(X < 1)$ and $P(0 < X < 2)$.

Solution

(i) To find the probability distribution of X which denotes the number of defective items. As there are only 3 defective, X takes the values 0, 1, 2 and 3.
Out of 10 items, 4 items are chosen in $10C_4$ ways.
$\because$ Out of 10, 3 are defective and 7 are good, we have
$X = 0 \Rightarrow$ 0 defective, 4 good

$$\therefore \qquad P(X = 0) = \frac{7C_4}{10C_4} = \frac{1}{6}$$

$X = 1 \Rightarrow$ 1 defective, 3 good

$$\therefore \qquad P(X = 1) = \frac{7C_3 \times 3C_1}{10C_4} = \frac{1}{2}$$

$X = 2 \Rightarrow$ 2 defective, 2 good

$$\therefore \quad P(X = 2) = \frac{7C_2 \times 3C_2}{10C_4} = \frac{3}{10}$$

$X = 3 \Rightarrow$ 3 defective, 1 good

$$\therefore \quad P(X = 3) = \frac{7C_1 \times 3C_3}{10C_4} = \frac{1}{30}$$

Therefore, the probability distribution of X is

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$

$$P(X < 1) = P(X = 0) = \frac{1}{6}$$

$$P(0 < X < 2) = P(X = 1) = \frac{1}{2}$$

EXAMPLE 2.18 A random variable X takes the values $-2, -1, 0, 1, 2$ such that $P(X = 0) = P(X > 0) = P(X < 0)$. Obtain the probability distribution and the distribution function of X. **[AU December '06]**

Solution Given: $P(X = 0) = P(X > 0) = P(X < 0) = k$ (say)

Then, we know that

$$P(X < 0) + P(X = 0) + P(X > 0) = 1$$

$$k + k + k = 1 \Rightarrow 3k = 1, \text{ i.e. } k = \frac{1}{3}$$

Let
$$P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = b$$
$$P(X = -2) + P(X = -1) + P(X = 1) + P(X = 2) = 4b$$

$$P(X < 0) = P(X = -2) + P(X = -1) = k = \frac{1}{3}$$

$$P(X > 0) = P(X = 1) + P(X = 2) = k = \frac{1}{3}$$

$$\frac{1}{3} + \frac{1}{3} = 4b \Rightarrow \frac{2}{3} = 4b, \text{ i.e. } b = \frac{1}{6}$$

The probability mass function of X

x	-2	-1	0	1	2
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

The distribution function of X is $F(x) = P(X \le x)$

$$
\begin{aligned}
F(x) &= 0, \; x < -2 \\
&= \frac{1}{6}, \; -2 \le x < -1 \\
&= \frac{1}{3}, \; -1 \le x < 0 \\
&= \frac{2}{3}, \; 0 \le x < 1 \\
&= \frac{5}{6}, \; 1 \le x < 2 \\
&= 1, \; x \ge 2
\end{aligned}
$$

EXAMPLE 2.19 A random variable X may assume four values with probability $\frac{1+3x}{4}, \frac{1-x}{4}, \frac{1+2x}{4}$ and $\frac{1-4x}{4}$. Find the condition on X so that these values represent the probability function of X.

Solution If these probabilities represent a PMF, then total probability is equal to 1 and each probability is greater than or equal to 0, then

$$\frac{1+3x}{4} + \frac{1-x}{4} + \frac{1+2x}{4} + \frac{1-4x}{4} = 1$$

$$\frac{1+3x+1-x+1+2x+1-4x}{4} = \frac{4}{4} = 1$$

Since $p(x) \ge 0$, we have

$$\frac{1+3x}{4} \ge 0 \Rightarrow 1+3x \ge 0 \Rightarrow x \ge -\frac{1}{3}$$

$$\frac{1-x}{4} \ge 0 \Rightarrow 1-x \ge 0 \Rightarrow x \le 1$$

$$\frac{1+2x}{4} \ge 0 \Rightarrow 1+2x \ge 0 \Rightarrow x \ge -\frac{1}{2}$$

$$\frac{1-4x}{4} \ge 0 \Rightarrow 1-4x \ge 0 \Rightarrow x \le \frac{1}{4}$$

All these conditions will be satisfied when $\frac{1}{-3} \le x \le \frac{1}{4}$

$\therefore$ The condition on x for which $p(x) \ge 0$ is $\frac{1}{-3} \le x \le \frac{1}{4}$

EXAMPLE 2.20 If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$, find the probability distribution function and cumulative distribution function of X.

[AU June '04]

Solution Let $P(X = 3) = k$, then

$$2P(X = 1) = k \Rightarrow P(X = 1) = \frac{k}{2}$$

$$3P(X = 2) = k \Rightarrow P(X = 2) = \frac{k}{3}$$

$$5P(X = 4) = k \Rightarrow P(X = 4) = \frac{k}{5}$$

$$\sum_{i=1}^{4} p(x_i) = 1 \Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

i.e.
$$\frac{61k}{30} = 1 \Rightarrow k = \frac{30}{61}$$

The probability distribution function is

$x = i$	1	2	3	4
$P(X = i)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

The CDF $F(x) = P(X \le x)$ is defined as follows:

$F(x) = 0, x < 1$

$= \frac{15}{61}, 1 \le x < 2$ $\qquad \left[\because P(X = 1) = \frac{15}{61}\right]$

$= \frac{25}{61}, 2 \le x < 3$ $\qquad [P(X = 1) + P(X = 2)]$

$= \frac{55}{61}, 3 \le x < 4$ $\qquad [P(X = 1) + P(X = 2) + P(X = 3)]$

$= 1, x \ge 4$ $\qquad [P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4]$

EXAMPLE 2.21 The probability mass function of a random variable is zero except at the points $x = 0, 1, 2$. At these points it has the values $P(0) = 3a^3$, $P(1) = 4a - 10a^2$ and $P(2) = 5a - 1$ for some $a > 0$.

(i) Determine the value of a.
(ii) Compute the probabilites $P(X < 2)$ and $P(1 < X \leq 2)$.
(iii) Describe the distribution function.
(iv) Find the largest x such that $F(x) < 1/2$.
(v) Find the smallest x such that $F(x) < 1/3$.

Solution Given:

$$P(0) = 3a^3,\ P(1) = 4a - 10a^2 \text{ and } P(2) = 5a - 1$$

and 0 for all the other values

(i) We know that if $p(x)$ is a PMF, then $\sum_x P(x) = 1$

$$\therefore \quad P(0) + P(1) + P(2) = 3a^3 + 4a - 10a^2 + 5a - 1 = 1$$

$$3a^3 - 10a^2 + 9a - 2 = 0$$

$$(a - 1)(3a^2 - 7a + 2) = 0$$

$$(a - 1)(3a - 1)(a - 2) = 0$$

$$\Rightarrow \quad a = 1, 2, \frac{1}{3}$$

If $a = 1$, then $P(0) = 3 > 1$, which is not possible. Similarly, $a \neq 2$

$$\therefore \quad a = \frac{1}{3}$$

$$P(0) = 3\left(\frac{1}{3}\right)^3 = \frac{1}{9}$$

$$P(1) = 4a - 10a^2 = \frac{4}{3} - \frac{10}{9} = \frac{2}{9}$$

$$P(2) = 5a - 1 = \frac{5}{3} - 1 = \frac{2}{3} = \frac{6}{9}$$

The probability distribution function is

x	0	1	2
$P(X = x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{6}{9}$

(ii) $P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$

$$P(1 < X \leq 2) = P(X = 2) = \frac{6}{9} = \frac{2}{3}$$

(iii) The distribution function is

$$F(x) = 0,\ x < 0$$
$$= \frac{1}{9}, 0 \le x < 1$$
$$= \frac{3}{9}, 1 \le x < 2$$
$$= 1,\ x \ge 2$$

(iv) Since $F(x) = \frac{1}{3} < \frac{1}{2}$, the largest value of x for which $f(x) < \frac{1}{2}$ is $x = 1$.

Since $F(x) = \frac{1}{3}$ for $x = 1$ and $F(x) = 1$ for $x \ge 2$, the smallest value of x for which $f(x) \ge \frac{1}{3}$ is $x = 1$.

EXAMPLE 2.22 If the probability distribution of a discrete random variable X is given by $P(X = x) = ke^{-t}(1 - e^{-t})^{x-1}$, $x = 1, 2, 3, \ldots, \infty$, find the value of k, the mean and variance of X.

Solution Given the probability distribution

$$P(X = x) = p(x) = ke^{-t}(1 - e^{-t})^{x-1},\ x = 1, 2, 3, \ldots, \infty$$

To find the value of k, we know that

$$\sum_{x=1}^{\infty} p(x) \Rightarrow \sum_{x=1}^{\infty} ke^{-t}(1-e^{-t})^{x-1} = 1$$

$$\Rightarrow ke^{-t}[1 + (1 - e^{-t}) + (1 - e^{-t})^2 + (1 - e^{-t})^3 + \cdots] = 1$$
$$\Rightarrow ke^{-t}[1 - (1 - e^{-t})]^{-1} = 1 \quad \Rightarrow \quad ke^{-t}(e^{-t})^{-1} = 1 \quad \Rightarrow \quad k = 1$$

$$\text{Mean} = E(X) = \sum_{x=1}^{\infty} xp(x)$$
$$= e^{-t}[1 + 2(1 - e^{-t}) + 3(1 - e^{-t})^2 + 4(1 - e^{-t})^3 + \cdots]$$
$$= e^{-t}[1 - (1 - e^{-t})]^{-2} = e^{-t}(e^{-t})^{-2} = e^{t}$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 p(x) = \sum_{x=1}^{\infty} [x(x+1) - x]e^{-t}(1-e^{-t})^{x-1}$$

$$= \sum_{x=1}^{\infty} x(x+1)e^{-t}(1-e^{-t})^{x-1} - \sum_{x=1}^{\infty} xe^{-t}(1-e^{-t})^{x-1}$$
$$= e^{-t}[1 \times 2 + 2 \times 3(1 - e^{-t}) + 3 \times 4(1 - e^{-t})^2 + 4 \times 5(1 - e^{-t})^3 + \cdots] - e^{t}$$
$$= 2e^{-t}[1 - (1 - e^{-t})]^{-3} - e^{t} = 2e^{-t}(e^{-t})^{-3} - e^{t} = 2e^{2t} - e^{t}$$

$\therefore$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 2e^{2t} - e^{t} - e^{2t} = e^{2t} - e^{t}$$

2.7 CONTINUOUS RANDOM VARIABLE

A random variable X is said to be continuous if it can take all possible values between certain limits. In otherwords, a random variable is said to be cotinuous when its different values cannot be put in one to one correspondence with a set of positive integers. Examples of continuous random variable are height, weight, age, etc.

Note: The sample space of the continuous random variable must be continuous and cannot be discrete. In most of the practical problems, continuous random variable represent measured data, such as all possible heights, weights, temperature, etc. whereas discrete random variables represent count data such as the number of defectives in a sample and so on.

2.8 PROBABILITY DENSITY FUNCTION (PDF)

Consider the small interval $\left(x-\frac{\Delta x}{2}, x+\frac{\Delta x}{2}\right)$ of length Δx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)dx$ represents the probability that x falls in the infinitesimal interval $\left(x-\frac{\Delta x}{2}, x+\frac{\Delta x}{2}\right)$, which is denoted by $P\left(x-\frac{\Delta x}{2} \le x \le x+\frac{\Delta x}{2}\right) = f(x)\,dx.$

Let $f(x)dx$ represent the area bounded by the curve $y = f(x)$, x axis and the ordinates at the points $x-\frac{\Delta x}{2}$ and $x+\frac{\Delta x}{2}$. The function $f(x)$ so defined is known as probability density function or density function of the random variable X.

The probability density function of a random variable X denoted by $f(x)$ has the following properties:

(i) $f(x) \ge 0, \forall x \in R$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a < X < b) = \int_a^b f(x)dx$

Note: In case of continuous random variables, the probability at a point is always zero, i.e. $P(X = a) = 0$ for all possible values of a.

EXAMPLE 2.5 A random variable X has the probability function

x	-2	-1	0	1
$p(x)$	0.4	k	0.2	0.3

Find k and the mean value of X.

Solution $\sum p(x) = 1 \Rightarrow 0.9 + k = 1 \Rightarrow k = 0.1$

$\therefore$ $$\text{Mean} = \sum xp(x) = -0.8 - 0.1 + 0 + 0.3 = -0.6$$

EXAMPLE 2.6 A random variable X has the following probability function:

x	0	1	2	3	4
$p(x)$	k	$3k$	$5k$	$7k$	$9k$

Find
(i) the value of k,
(ii) $P(X < 3)$ and $P(0 < X < 4)$, and
(iii) the distribution function of X. **[AU May '08]**

Solution

(i) We know that

$$\sum p(x) = 1$$

$$k + 3k + 5k + 7k + 9k = 1 \Rightarrow 25k = 1$$

$$k = \frac{1}{25}$$

(ii) $$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= k + 3k + 5k = 9k = \frac{9}{25}$$

$$P(0 < X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$= 3k + 5k + 7k = 15k = \frac{15}{25}$$

(iii) The distribution function of X is

$$F(x) = 0, x < 0$$

$$= \frac{1}{25}, 0 \le x < 1$$

$$= \frac{4}{25}, 1 \le x < 2$$

$$= \frac{9}{25}, 2 \le x < 3$$

$$= \frac{16}{25}, 3 \le x < 4$$

$$= 1, x \ge 4$$

EXAMPLE 2.7 After a coin is tossed two times, if X is the number of heads, find the probability distributions of X. **[AU December '05]**

Solution The coin is tossed 2 times,

$$n(S) = 2^2 = 4$$

$$S = \{HH, HT, TH, TT\}$$

If X is the number of heads, then X takes the values 0, 1 or 2

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

EXAMPLE 2.8 If $P(X = x) = \begin{cases} kx, & x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$

represents a probability, find

(i) k,
(ii) $P(X$ being a prime number),
(iii) $P(1/2 < X < 5/2 \,/X > 1)$, and
(iv) the distribution function. **[AU December '09]**

Solution

(i) Given:

x	1	2	3	4	5
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$

Since $p(x)$ is a probability function, $\sum p(x) = 1$

$$k + 2k + 3k + 4k + 5k = 1$$

$$15k = 1 \Rightarrow k = \frac{1}{15}$$

(ii) $P(X$ being a prime number$) = P(X = 1, 2, 3, 5)$

$$= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 5)$$

$$= k + 2k + 3k + 5k = 11k = \frac{11}{15}$$

(iii) $P\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right) = \dfrac{P\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right)}{P(X > 1)}$

$$= \frac{P\left(1 < X < \frac{5}{2}\right)}{P(X > 1)}$$

$$= \frac{P(X = 2)}{P(X=2)+P(X=3)+P(X=4)+P(X=5)}$$

$$= \frac{2k}{14k} = \frac{2}{14} = \frac{1}{7}$$

(iv) The distribution function is

$$f(x) = 0,\ x < 1$$
$$= \frac{1}{15},\ 1 \le x < 2$$
$$= \frac{3}{15},\ 2 \le x < 3$$
$$= \frac{6}{15},\ 3 \le x < 4$$
$$= \frac{10}{15},\ 4 \le x < 5$$
$$= 1,\ x \ge 5$$

EXAMPLE 2.9 If the CDF of a random variable is given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x^2}{16} & \text{for } 0 \le x \le 4 \\ 1 & \text{for } 4 < x \end{cases}$$

find $P(X > 1/X < 3)$. **[AU November/December '09]**

Solution We know that

$$P(X > 1/X < 3) = \frac{P(X > 1 \cap X < 3)}{P(X < 3)}$$

$$= \frac{P(1 < X < 3)}{P(X < 3)}$$

$$= \frac{F(3) - F(1)}{F(3)}$$

$$= \frac{\frac{9}{16} - \frac{1}{16}}{\frac{9}{16}} = \frac{8}{9}$$

EXAMPLE 2.10 Suppose that the random variable assumes three values 0, 1 and 2 with probabilities 1/3, 1/6 and 1/2 respectively. Obtain the distribution function of X.

Solution Given:

x	0	1	2
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$

The distribution function $F(x) = P(X \leq x)$ is

$$F(x) = 0,\ x < 0$$
$$= \frac{1}{3},\ 0 \leq x < 1$$
$$= \frac{3}{6},\ 1 \leq x < 2$$
$$= 1,\ x \geq 2$$

EXAMPLE 2.11 If $P(X = x) = \begin{cases} \frac{x}{15}, & x = 1, 2, 3, 4, 5 \\ 0, & \text{elsewhere} \end{cases}$

find

(i) $P(X = 1 \text{ or } 2)$, and
(ii) $P[(1/2 < X < 5/2)/X > 1]$.

Solution

(i) $P(X = 1 \text{ or } 2) = P(X = 1) + P(X = 2)$

$$= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$$

(ii) $P[(1/2 < X < 5/2/X > 1] = \dfrac{P\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right)}{P(X > 1)}$

$$= \frac{P(X = 1 \text{ or } 2 \cap X > 1)}{P(X > 1)}$$

$$= \frac{P(X = 1 \text{ or } 2 \cap X = 2, 3, 4, 5)}{P(X > 1)}$$

$$= \frac{P(X = 2)}{P(X = 2, 3, 4, 5)}$$

$$= \frac{\frac{2}{15}}{\frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15}} = \frac{1}{7}$$

EXAMPLE 2.12 If the probability mass function of a random variable is given by $P(X = r) = kr^3$, $r = 1, 2, 3, 4$, find

(i) the value of k,
(ii) $P[(1/2 < X < 5/2/X > 1]$,
(iii) the mean and variance of X, and
(iv) the distribution function of X.

Solution Given:

$$P(X = r) = kr^3, \quad r = 1, 2, 3, 4$$

r	1	2	3	4
$p(r)$	k	$8k$	$27k$	$64k$

(i) To find the value of k, we know that

$$\sum_{r=1}^{4} p(r) = 1$$

$$= k + 8k + 27k + 64k = 1$$

$$\Rightarrow \qquad 100k = 1 \quad \Rightarrow \quad \frac{1}{100}$$

(ii) $P[(1/2 < X < 5/2)/X > 1)]$

$$= \frac{P(X = 1, X = 2) \cap P(X = 2, X = 3, X = 4)}{P(X = 2) + P(X = 3) + P(X = 4)}$$

$$= \frac{P(X = 2)}{P(X = 2) + P(X = 3) + P(X = 4)}$$

$$= \frac{\frac{8}{100}}{\frac{8}{100} + \frac{27}{100} + \frac{64}{100}} = \frac{8}{99}$$

(iii) Mean $= E(X) = \sum_{r=1}^{4} rp(r) = 1 \times k + 2 \times 8k + 3 \times 27k + 4 \times 64k$

$$= 354k = \frac{354}{100} = 3.54$$

(iv) $E(X^2) = \sum_{r=1}^{4} r^2 p(r) = 1 \times k + 4 \times 8k + 9 \times 27k + 16 \times 64k$

$$= 1300k$$

$$= \frac{1300}{100} = 13$$

$\therefore$ $\quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 13 - (3.54)^2 = 0.4684$

The distribution function of X is

$$F(x) = 0,\ x < 1$$
$$= \frac{1}{100}, 1 \le x \le 2$$
$$= \frac{9}{100}, 2 \le x < 3$$
$$= \frac{36}{100}, 3 \le x < 4$$
$$= 1, x \ge 4$$

EXAMPLE 2.13 A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7	8
$P(X)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

find

(i) a,
(ii) $P(X \le 2)$,
(iii) the distribution function of X, and
(iv) the mean of X. **[AU May '04]**

Solution

(i) If $p(x)$ is a probability mass function, then $\sum p(x) = 1$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$81a = 1 \Rightarrow a = \frac{1}{81}$$

(ii) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$= a + 3a + 5a$

$= 9a = 9 \times \frac{1}{81} = \frac{1}{9}$

(iii) The distribution function of X is

$$F(x) = 0, \; x < 0$$

$$= P(X = 0) = a \;= \frac{1}{81}, \; 0 \leq x < 1$$

$$= P(X \leq 1) = 4a \;= \frac{4}{81}, \; 1 \leq x < 2$$

$$= P(X \leq 2) = 9a \;= \frac{9}{81}, \; 2 \leq x < 2$$

$$= P(X \leq 3) = 16a = \frac{16}{81}, \; 3 \leq x < 4$$

$$= P(X \leq 4) = 25a = \frac{25}{81}, \; 4 \leq x < 5$$

$$= P(X \leq 5) = 36a = \frac{36}{81}, \; 5 \leq x < 6$$

$$= P(X \leq 6) = 49a = \frac{49}{81}, \; 6 \leq x < 7$$

$$= P(X \leq 7) = 64a = \frac{64}{81}, \; 7 \leq x < 8$$

$$= P(X \leq 8) = 81a = 1, \; x \geq 8$$

(iv) Mean of $x = \Sigma x\, p(x)$

$= 0 + 3a + 10a + 21a + 36a + 55a + 78a + 105a + 136a$

$= 444a = \frac{444}{81} = 5.48$

EXAMPLE 2.14 The probability distribution of a random variable X is

x	0	1	2	3
$p(x)$	0.1	0.3	0.5	0.1

If $Y = X^2 + 2X$, find the probability distribution, CDF, mean, and variance of Y.

Solution Given:

$$Y = X^2 + 2X$$

y	0	3	8	15
$P(y)$	0.1	0.3	0.5	0.1

(i) $F(y) = P(Y \le y) = \sum_{y_j < y} P(y)$

$$\begin{aligned} F(y) &= 0, \; y < 0 \\ &= 0.1, \; 0 \le y < 3 \\ &= 0.4, \; 3 \le y < 8 \\ &= 0.9, \; 8 \le y < 15 \\ &= 1.0, \; y \ge 15 \end{aligned}$$

(ii) $\text{Mean} = E(Y) = \Sigma y\, p(y) = 0 \times 0.1 + 3 \times 0.3 + 8 \times 0.5 + 15 \times 0.1$

$= 0 + 0.9 + 4.0 + 1.5$

$= 6.4$

$E(Y^2) = \Sigma\, y^2 p(y) = 0 \times 0.1 + 9 \times 0.3 + 64 \times 0.5 + 225 \times 0.1$

$= 57.2$

$\therefore \quad \text{Var}\,(Y) = E(Y^2) - [E(Y)]^2 = 16.24$

EXAMPLE 2.15 A random variable X has the following probability function:

x	-2	-1	0	1	2	3
$P(x)$	0.1	k	0.2	$2k$	0.3	k

(i) Find the value of k and calculate mean and variance.

[AU December '07, June '07]

Solution Since $p(x)$ is PMF of a random variable X, $\sum_{x=-2}^{3} p(x) = 1$

$$\Rightarrow \quad 0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$4k + 0.6 = 1 \quad \Rightarrow \quad k = \frac{0.4}{4} = 0.1$$

$$E(X) = \text{mean} = \sum_{x=-2}^{3} xP(x)$$

$= (-2 \times 0.1) + (-1 \times 0.1) + (0 \times 0.2) + (1 \times 0.2) + (2 \times 0.3) + (3.0 \times 0.1)$

$= 0.8$

$$E(X^2) = \sum_{x=-2}^{3} x^2 P(x) = (4 \times 0.1) + (1 \times 0.1) + (0 \times 0.2) + (1 \times 0.2) + (4 \times 0.3) + (9 \times 0.1)$$

$= 0.4 + 0.1 + 0.2 + 1.2 + 0.9$

$= 2.8$

$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2$

$= 2.8 - (0.8)^2$

$= 2.8 - 0.64 = 2.16$

$$= \int_2^4 k(1+x)dx = \frac{2}{27}\int_2^4 (1+x)dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4 = \frac{16}{27}$$

EXAMPLE 2.51 If the density function of a continues random variable X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

find

(i) the value of a,
(ii) the CDF of X, and
(iii) $P(X \le 1.5)$. **[AU December '07]**

Solution

(i) Since $f(x)$ is a PDF of X,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$\frac{a}{2} + a + \frac{9a}{2} - 4a = 1$$

$$\frac{4a}{2} = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$

(ii) The CDF of the random variable X is $F(x) = P(X \le x)$

$$F(x) = 0, \ x < 0$$

For $0 \le x < 1$,

$$F(x) = \frac{1}{2}\int_0^x x dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{4}, 0 \le x < 1$$

For $1 \le x < 2$,

$$F(x) = \frac{1}{2}\int_0^1 x\,dx + \int_1^x \frac{1}{2}\,dx = \frac{x}{2} - \frac{1}{4}, 1 \le x < 2$$

For $2 \le x < 3$,

$$F(x) = \frac{1}{2}\int_0^1 x dx + \int_1^2 \frac{1}{2} dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right) dx$$

For $x \ge 3$,

$$= \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}, 2 \le x < 3$$

$$F(x) = 1,\ x \ge 3$$

$$\therefore \quad F(x) = 0,\ x < 0$$

$$= \frac{x^2}{4},\ 0 \le x < 1$$

$$= \frac{x}{2} - \frac{1}{4},\ 1 \le x < 2$$

$$= \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4},\ 2 \le x < 3$$

$$= 1,\ x > 3$$

(iii) $P(X \le 1.5) = P(0 \le X \le 1) + P(1 \le X \le 1.5)$

$$\int_0^1 \frac{x}{2} dx + \int_1^{1.5} \frac{1}{2} dx = \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{2}[x]_1^{1.5}$$

$$= \frac{1.5}{2} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

Using CDF,

$$P(X \le 1.5) = F(1.5) = \frac{1}{2} \qquad [\because P(X \le x) = F(x)]$$

EXAMPLE 2.52 If X is a continuous random variable with PDF $f(x) = 6x(1-x)$, $0 \le x \le 1$. Find $P\left(X \le \frac{1}{2} / \frac{1}{3} < X < \frac{2}{3}\right)$.

[AU November/December '09]

Solution By definition,

$$P\left(X \le \frac{1}{2} / \frac{1}{3} < X < \frac{2}{3}\right) = \frac{P\left(X \le \frac{1}{2} \cap \frac{1}{3} < X < \frac{2}{3}\right)}{P\left(\frac{1}{3} < X < \frac{2}{3}\right)}$$

$$= \frac{P\left(\frac{1}{3} < X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} < X < \frac{2}{3}\right)} \qquad \text{(i)}$$

$$P\left(\frac{1}{3} < X < \frac{2}{3}\right) = 6\int_{\frac{1}{3}}^{\frac{1}{2}} (x - x^2)dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{3}}^{\frac{1}{2}}$$

$$= 6\left(\frac{1}{8} - \frac{1}{24} - \frac{1}{18} + \frac{1}{81}\right) = \frac{13}{54}$$

$$P\left(\frac{1}{3} < X < \frac{2}{3}\right) = 6\int_{\frac{1}{3}}^{\frac{2}{3}} (x - x^2)\, dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{3}}^{\frac{2}{3}}$$

$$= 6\left(\frac{4}{18} - \frac{8}{81} - \frac{1}{18} + \frac{1}{81}\right) = \frac{13}{27}$$

Substituting in Eq. (i) gives

$$P\left(X \le \frac{1}{2} / \frac{1}{3} < X < \frac{2}{3}\right) = \frac{\frac{13}{54}}{\frac{13}{27}} = \frac{1}{2}$$

EXAMPLE 2.33 Suppose the life in hours of a certain kind of a radio tube has the PDF

$$f(x) = \begin{cases} \dfrac{100}{x^2}, & \text{when } x \ge 100 \\ 0, & \text{when } x < 100 \end{cases}$$

Find the distribution function. What is the probability that none of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation? What is the probability that all three of the original tubes lasts for the first 150 hours.

Solution Let X be a continuous random variable which denotes the life in hours of a certain kind of radio tube.

Then probability that a tube will not last for first 150 hours, i.e. the probability that it has to be replaced during the first 150 hours is

$$P(X < 150) = P(0 < X < 100) + P(100 \le X < 150)$$

$$= 0 + \int_{100}^{150} \frac{100}{x^2} dx$$

$$= \left[\frac{-100}{x}\right]_{100}^{150}$$

$$= \frac{-100}{150} + \frac{100}{100}$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

$\therefore$ The probability that all the three tubes will have to be replaced during the first 150 hours is

$$= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

The probability that a tube lasts for the first 150 hours is

$$P(X > 150) = 1 - P(X < 150) = 1 - \frac{1}{3} = \frac{2}{3}$$

The probability that all the three tubes last for the first 150 hours is

$$= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

EXAMPLE 2.54 A continuous random variable has a PDF $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that

(i) $P(X \le a) = P(X > a)$, and
(ii) $P(X > b) = 0.05$.

Solution Given: $f(x) = 3x^2$, $0 \le x \le 1$

By definition,

$$P(X \le a) = \int_{-\infty}^{a} f(x)dx = \int_{0}^{a} 3x^2 dx$$

$$P(X > a) = \int_{a}^{1} f(x)dx = \int_{a}^{1} 3x^2 dx$$

(i) $P(X \le a) = P(X > a) \Rightarrow \int_0^a 3x^2 dx = \int_a^1 3x^2 dx$

$$\Rightarrow \left[\frac{3x^3}{3}\right]_0^a = \left[\frac{3x^3}{3}\right]_a^1$$

$$\Rightarrow a^3 = 1 - a^3$$

i.e. $a^3 = \frac{1}{2}$

or $a = \left(\frac{1}{2}\right)^{\frac{1}{3}} = 0.79$

(ii) $P(X > b) = 0.05$

$$P(X > b) = \int_b^1 3x^2 dx = 0.05$$

$$= \left[\frac{3x^3}{3}\right]_b^1 = 0.05$$

$$\Rightarrow 1 - b^3 = 0.05 \text{ i.e. } -b^3 = 0.05 - 1$$

$$b^3 = 0.95 \quad \Rightarrow \quad b = 0.983$$

EXAMPLE 2.55 The function $f(t) = \begin{cases} 3\times 10^{-9} t^2 (100-t)^2, & 0 \le t \le 1000 \\ 0, & \text{otherwise} \end{cases}$.

If the probability that a person will die in the time interval (t_1, t_2) is given by

$P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)dt$ determine

(i) the probability that a person will die between the ages 60 and 70, and
(ii) the probability that he will die between those ages, assuming he lived up to 60.

Solution

(i) Given that

$$P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)dt$$

$$P(60 \le t \le 70) = \int_{60}^{70} f(t)dt = 3\times 10^{-9} \int_{60}^{70} t^2 (100-t)^2 dt$$

$$= 3\times10^{-9}\int_{60}^{70}(100^2t^2 - 200t^3 + t^4)dt$$

$$= 3\times10^{-9}\left[100^2\frac{t^3}{3} - 200\frac{t^4}{4} + \frac{t^5}{5}\right]_{60}^{70}$$

$$= 3\times10^{-9}\left[100^2\left(\frac{70^3 - 60^3}{3}\right) - 50(70^4 - 60^4) + \frac{70^5 - 60^5}{5}\right]$$

$$= 0.1544$$

(ii) $P(60 < t < 70/t \geq 60) = P(60 < t < 70/60 \leq t \leq 100)$

$$= \frac{P(60<t<70)}{P(60<t<100)}$$

$$= \frac{\int_{60}^{70} f(t)dt}{\int_{60}^{100} f(t)dt}$$

$$= \frac{3\times10^{-9}\int_{60}^{70} t^2(100-t)^2\,dt}{3\times10^{-9}\int_{60}^{100} t^2(100-t)^2\,dt} = \frac{0.15436}{0.31744}$$

$$= 0.4863$$

EXAMPLE 2.56 A random variable X has the PDF

$$f(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

Find

(i) $P\left(X < \frac{1}{2}\right)$,

(ii) $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$ and

(iii) $P\left(X > \frac{3}{4} \middle/ X > \frac{1}{2}\right)$. **[AU June '07]**

Solution By definition,

(i) $$P\left(X<\frac{1}{2}\right)=\int_0^{\frac{1}{2}} f(x)dx=\int_0^{\frac{1}{2}} 2x\,dx$$

$$=2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}}=2\times\frac{1}{8}=\frac{1}{4}$$

(ii) $$P\left(\frac{1}{4}<X<\frac{1}{2}\right)=\int_{\frac{1}{4}}^{\frac{1}{2}} f(x)dx=\int_{\frac{1}{4}}^{\frac{1}{2}} 2x\,dx$$

$$=2\left[\frac{x^2}{2}\right]_{\frac{1}{4}}^{\frac{1}{2}}=\frac{1}{4}-\frac{1}{16}=\frac{3}{16}$$

(iii) $$P\left(X>\frac{3}{4}\Big/X>\frac{1}{2}\right)=\frac{P\left(X>\frac{3}{4}\cap X>\frac{1}{2}\right)}{P\left(X>\frac{1}{2}\right)}=\frac{P\left(X>\frac{3}{4}\right)}{P\left(X>\frac{1}{2}\right)} \qquad \text{(i)}$$

$$P(X>3/4)=\int_{\frac{3}{4}}^{1} f(x)dx=\int_{\frac{3}{4}}^{1} 2x\,dx$$

$$=2\left[\frac{x^2}{2}\right]_{\frac{3}{4}}^{1}=1-\frac{9}{16}=\frac{16-9}{16}=\frac{7}{16} \qquad \text{(ii)}$$

$$P(X>1/2)=\int_{\frac{1}{2}}^{1} f(x)dx=\int_{\frac{1}{2}}^{1} 2x\,dx$$

$$=2\left[\frac{x^2}{2}\right]_{\frac{1}{2}}^{1}=1-\frac{1}{4}=\frac{3}{4} \qquad \text{(iii)}$$

Substituting Eqs. (ii) and (iii) in Eq. (i), we get

$$P\left(X>\frac{3}{4}\Big/X>\frac{1}{2}\right)=\frac{P\left(X>\frac{3}{4}\right)}{P\left(X>\frac{1}{2}\right)}=\frac{\frac{7}{16}}{\frac{3}{4}}=\frac{7}{12}$$

Note: $P\left(X>\frac{1}{2}\right)$ can also be found from $P\left(X>\frac{1}{2}\right)=1-P\left(X<\frac{1}{2}\right)=1-\frac{1}{4}=\frac{3}{4}$

EXAMPLE 2.57 A continuous random variable X is distributed over the interval (0, 1) with PDF $ax^2 + bx$, where a, b are constants. If the arithmetic mean of X is 0.5, find the values of a and b. **[AU June '05, December '07]**

Solution Since $f(x)$ is a PDF,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = \int_0^1 (ax^2 + bx)dx = 1$$

$$\left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = 1$$

$$a\left(\frac{1}{3}\right) + b\left(\frac{1}{2}\right) = 1$$

$$2a + 3b = 6 \qquad \text{(i)}$$

Now,

$$\text{Mean} = \int_0^1 xf(x)dx = \int_0^1 x(ax^2 + bx)dx$$

$$= \int_0^1 (ax^3 + bx^2)dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3}\right]_0^1 = \frac{a}{4} + \frac{b}{3}$$

Given: $\text{Mean} = 0.5 = \dfrac{1}{2}$

$$\therefore \quad \frac{a}{4} + \frac{b}{3} = \frac{1}{2} \Rightarrow \frac{3a + 4b}{12} = \frac{1}{2}$$

i.e. $$3a + 4b = 6 \qquad \text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$a = -6$$
$$b = 6$$

EXAMPLE 2.58 Verify whether the following is a distribution function:

$$F(x) = \begin{cases} 0, & x < -a \\ \dfrac{1}{2}\left(\dfrac{x}{a} + 1\right), & -a < x < a \\ 1, & x > 1 \end{cases}$$

Solution Given:

$$F(x) = 0, x < -a \Rightarrow F(-\infty) = 0$$
$$F(x) = 1, x > 1 \Rightarrow F(\infty) = 1$$

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Clearly, $0 \le F(x) \le 1$

Again,
$$f(x) = \frac{dF(x)}{dx}$$
$$= \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

Also,
$$\int_{-a}^{a} f(x)dx = \int_{-a}^{a} \frac{1}{2a} dx = \frac{1}{2a}[x]_{-a}^{a}$$
$$= \frac{1}{2a}(2a) = 1$$

$\therefore$ $F(x)$ is a PDF.
$\therefore$ $F(x)$ satisfies all the conditions of a distribution function.
$\therefore$ $F(x)$ is a distribution function.

EXAMPLE 2.59 If X be a continuous random variable with PDF

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find CDF of the random variable. **[AU December '07, November '05]**

Solution If $x < 0$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} dx$$
$$= 0 + [x]_0^x = x$$

If $x > 1$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{0}^{1} 0dx + \int_{1}^{x} dx$$
$$= 0 + [x]_0^1 = 1$$

$\therefore$
$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 1, & x > 1 \end{cases}$$

EXAMPLE 2.60 The PDF of a continuous random variable X is $f(x) = ke^{-|x|}$. Find k and the CDF $F(x)$. **[AU April '04, May '08]**

Solution Since $f(x)$ is a PDF, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \qquad \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ke^{-|x|}dx = 1$$

$$ke^{-|x|} = \begin{cases} ke^{x}, & -\infty < x < 0 \\ ke^{-x}, & 0 < x < \infty \end{cases}$$

Since $|x|$ is an even function,

$$2k\int_{0}^{\infty} e^{-|x|}dx = 1$$

$$2k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1 \Rightarrow 2k \times 1 = 1 \Rightarrow k = \frac{1}{2}$$

When $x \le 0$,

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{1}{2}e^{x}dx$$

$$= \frac{1}{2}[e^{x}]_{-\infty}^{x} = \frac{1}{2}(e^{x} - e^{-\infty}) = \frac{1}{2}e^{x}$$

When $x > 0$,

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} \frac{1}{2}e^{x}dx + \int_{0}^{x} \frac{1}{2}e^{-x}dx$$

$$= \frac{1}{2}[e^{x}]_{-\infty}^{0} + \frac{1}{2}[-e^{-x}]_{0}^{x}$$

$$= \frac{1}{2}(1 - e^{-\infty}) + \frac{1}{2}(-e^{-x} + e^{0}) = 1 - \frac{1}{2}e^{-x}$$

$$\therefore \qquad F(x) = \begin{cases} \frac{1}{2}e^{x}, & x \le 0 \\ 1 - \frac{1}{2}e^{-x}, & x > 0 \end{cases}$$

EXAMPLE 2.61 The PDF of a random variable X is

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

Find the cumulative distribution function of X. **[AU June '07]**

Solution The CDF of $X = F(x) = \int_{-\infty}^{x} f(x)dx$

When $0 < x < 1$,

$$F(x) = \int_{-\infty}^{x} f(x)dx$$
$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$
$$= 0 + \int_{0}^{x} xdx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

When $1 < x < 2$,

$$F(x) = \int_{-\infty}^{x} f(x)dx$$
$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)dx$$
$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)dx$$
$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x = \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$
$$= 2x - \frac{x^2}{2} - 1$$

When $x > 2$,

$$F(x) = \int_{-\infty}^{x} f(x)dx$$
$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx$$
$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)dx + 0$$
$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{1}{2} + 4 - 2 - 2 + \frac{1}{2} = 1$$

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$$\therefore \quad F(x) = \begin{cases} \dfrac{x^2}{2}, & 0 < x < 1 \\ 2x - \dfrac{x^2}{2} - 1, & 1 < x < 2 \\ 1, & x > 2 \end{cases}$$

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EXAMPLE 2.62 Given the PDF of a continuous random variable X as follows:

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find CDF for X. **[AU June '07]**

Solution The CDF is

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

(i) When $x < 0$,

$$F(x) = \int_{-\infty}^{x} f(x)dx = 0$$

(ii) When $0 \le x \le 1$,

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$

$$= 0 + \int_{0}^{x} 6x(1-x)dx = 6\int_{0}^{x} (x - x^2)dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^x = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) = 3x^2 - 2x^3$$

(iii) When $x > 1$,

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx$$

$$= 0 + \int_{0}^{1} 6x(1-x)dx + 0 = 6\int_{0}^{1} (x - x^2)dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 6\left(\frac{1}{2} - \frac{1}{3}\right) = 1$$

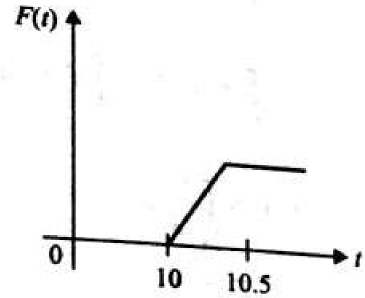
$$\therefore \quad F(x) = \begin{cases} 0, & x < 0 \\ 3x^2 - 2x^3, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

EXAMPLE 2.63 Suppose that a bus arrives a station everyday between 10.00 a.m. and 10.30 a.m. at random. Let X be the arrival time, find the distribution function of X and sketch its graph. **[AU December '07]**

Solution

$$F(t) = P(X \le t) = \begin{cases} 0, & t < 10 \\ 2(t-10), & 10 \le t < 10.30 \\ 1, & x \ge 10.30 \end{cases}$$

Its graph is



EXAMPLE 2.64 In a continuous random variable X having the PDF

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Verify $\int_{-\infty}^{\infty} f(x)dx = 1$,

(ii) Find $P(0 < X \le 1)$, and

(iii) Find $F(x)$ (CDF). **[AU June '06]**

Solution

(i) To verify $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{2} \frac{x^2}{3} dx$$

$$= \frac{1}{3}\int_{-1}^{2} x^2 dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{9}(8+1) = 1$$

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(ii) $P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3} dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\left(\frac{1}{3} - 0\right) = \frac{1}{9}$

(iii)
$$F(x) = \int_{-\infty}^{x} f(x)dx$$
$$= \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{x} f(x)dx$$
$$= 0 + \int_{-1}^{x} \frac{x^2}{3} dx$$
$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{x}$$
$$= \frac{1}{9}(x^3 + 1)$$

$\therefore$
$$F(x) = \begin{cases} 0, & x \leq -1 \\ \frac{x^3 + 1}{9}, & -1 < x < 2 \\ 1, & x \geq 2 \end{cases}$$

2.10 MOMENTS

If X is a random variable which is discrete or continuous, the moments about the origin denoted by μ_r' is defined as

$$\mu_r' = E(X^r), \text{ for } r = 1, 2, 3, \ldots$$

The moments about the mean or central moments denoted by μ_r is defined as

$$\mu_r = E[(X - \bar{X})^r], \text{ for } r = 1, 2, 3, \ldots$$

If X is a discrete random variable which can assume any of the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p(x_1), p(x_2), \ldots, p(x_n)$, then

$$\mu_r' = E(X^r) = \sum_{r=1}^{\infty} x^r p(x)$$

and
$$\mu_r = E[(X - \bar{X})^r] = \sum_{r=1}^{\infty} (x - \bar{x})^r p(x_r)$$

If X is a continuous random variable with PDF $f(x)$, then

$$\mu'_r = \int_{-\infty}^{\infty} x^r f(x)dx, \quad r = 1, 2, 3, \ldots$$

and
$$\mu_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f(x)dx, \quad r = 1, 2, 3, \ldots$$

2.10.1 Relation between Moments about Origin and Moments about Mean $\bar{X}$

By definition,

$$\begin{aligned}\mu_r &= [E(X - \bar{X})^r] \\ &= E[X^r - rC_1 X^{r-1}\bar{X} + rC_2 X^{r-2}\bar{X}^2 - rC_3 X^{r-3}\bar{X}^3 + \cdots \\ &\qquad + (-1)^{r-1} rC_{r-1} X\bar{X}^{r-1} + (-1)^r \bar{X}^r] \\ &= E(X^r) - rE(X^{r-1})\bar{X} + \frac{r(r-1)}{2!} E(X^{r-2})\bar{X}^2 - \frac{r(r-1)(r-2)}{3!} \\ &\qquad E(X^{r-3})\bar{X}^3 + \cdots + (-1)^r \bar{X}^r\end{aligned}$$

$\because \quad \bar{X} = E(X) = \mu'_1$, we have

$$\mu_r = \mu'_r - r\mu'_{r-1}\mu'_1 + \frac{r(r-1)}{2!}\mu'_{r-2}\mu_1'^2 - \frac{r(r-1)(r-2)}{3!}\mu'_{r-3}\mu_1'^3 + \cdots + (-1)^r \mu_1'^r$$

$$[E(X - \bar{X})] = \mu_1 = \mu'_1 - \mu'_1 = 0$$

$\Rightarrow$ The first moment about the mean is always zero.

$$[E(X - \bar{X})^2] = \mu_2 = \mu'_2 - 2\mu_1'^2 + \mu_1'^2$$

i.e.
$$\mu_2 = \mu'_2 - \mu_1'^2$$
$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$[E(X - \bar{X})^3] = \mu_3 = \mu'_3 - 3\mu'_2\,\mu'_1 + \frac{6}{2!}\mu'_1\mu_1'^2 - \mu_1'^3$$

$$= \mu'_3 - 3\mu'_2\,\mu'_1 + 3\mu_1'^3 - \mu_1'^3$$

i.e.
$$\mu_3 = \mu'_3 - 3\mu'_2\,\mu'_1 + 2\mu_1'^3$$

Similarly, $\mu_4 = \mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu_1'^2\mu'_2 - 3\mu_1'^4$ and so on.

2.10.2 Relation between Moments about any Point A and Moments about Mean $\bar{X}$

We know that

$$\mu_r' = E[(X - A)^r]$$

Putting $r = 1$,

$$\mu_1' = E(X - A) = \bar{X} - A \quad \Rightarrow \quad \text{Mean} = \bar{X} = \mu_1' + A$$

Putting $r = 2$,

$$\mu_2' = \mu_2 + \mu_1'^2$$

Similarly, we get

$$\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3$$

$$\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + \mu_1'^4 \text{ etc.}$$

2.10.3 Properties of Moments

1. If X is a random variable, then $E(aX + b) = aE(X) + b$.

Proof By definition,

$$E(aX + b) = \sum_x (ax + b)\, p(x)$$

$$= a\sum_x xp(x) + b\sum_x p(x)$$

$$= aE(X) + b \qquad \left[\because \sum_x p(x) = 1\right]$$

$$\therefore \quad E(aX + b) = aE(X) + b$$

Note: $E(X \pm Y) = E(X) \pm E(Y)$

2. If X is a random variable, then $\text{Var}(aX + b) = a^2\, \text{Var}(X)$.

Proof Let $Y = aX + b$, $E(Y) = E(aX + b) = aE(X) + b$

$$\therefore \quad Y - E(Y) = (aX + b) - [aE(X) + b] = a[X - E(X)]$$

$$[Y - E(Y)]^2 = a^2[X - E(X)]^2 \quad \Rightarrow \quad (Y - \bar{Y})^2 = a^2(X - \bar{X})^2$$

$$E(Y - \bar{Y})^2 = E[a^2(X - \bar{X})^2] = a^2\, E[(X - \bar{X})^2]$$

$$\text{Var}(Y) = a^2\, \text{Var}(X)$$

$$\therefore \quad \text{Var}(aX + b) = a^2\, \text{Var}(X) \quad [\text{Var}(b) = 0]$$

3. It can easily be proved that if X and Y are independent, then

$$\text{Var}(aX \pm bY) = a^2\, \text{Var}(X) + b^2\, \text{Var}(Y).$$

4. If X and Y are independent random variables, then $E(XY) = E(X)\, E(Y)$.

5. If X and Y are any two random variables such that $Y \leq X$, then $E(Y) \leq E(X)$.

Proof Given: $Y \leq X \Rightarrow Y - X \leq 0$

i.e. $X - Y \geq 0$

$\therefore$ $E(X - Y) \geq 0 \Rightarrow E(X) - E(Y) \geq 0$

i.e. $E(X) \geq E(Y) \Rightarrow E(Y) \leq E(X)$

2.11 COVARIANCE (X, Y)

The covariance of the two random variables is denoted by

$\rho(x, y) = \rho_{xy} = \text{Cov}(X, Y)$ which is defined as

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y)$$

Important note:

(i) If X and Y are independent random variables, then Cov $(X, Y) = 0$.

(ii) If X and Y are any two random variables, then
$\text{Var}(aX \pm bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) \pm 2ab\,\text{Cov}\,(X, Y)$

(iii) $\text{Cov}(aX + b, cY + d) = ab\,\text{Cov}(X, Y)$.

EXAMPLE 2.65 The number of hardware failures of a computer sytem in a week of operations has the following PMF:

Number of failures	0	1	2	3	4	5	6
Probability	0.18	0.28	0.25	0.18	0.16	0.04	0.01

Find the mean of the number of failures in a week. **[AU June '06]**

Solution We know that

$$E(X) = \sum_{x=0}^{6} xp(x)$$

$$= 0 \times 0.18 + 1 \times 0.28 + 2 \times 0.25 + 3 \times 0.18 + 4 \times 0.06 + 5 \times 0.04 + 6 \times 0.01$$

$$E(X) = 1.82$$

EXAMPLE 2.66 If X has the distribution function

$$f(x) = \begin{cases} 0, & x < 1 \\ \dfrac{1}{3}, & 1 \leq x < 4 \\ \dfrac{1}{2}, & 4 \leq x < 6 \\ \dfrac{7}{8}, & 6 \leq x < 10 \\ 1, & x \geq 0 \end{cases}$$

Find

(i) the probability distribution function of X,
(ii) $P(2 < X < 6)$,
(iii) mean of X, and
(iv) variance of X. **[AU April/May '08]**

Solution

(i) The probaility distribution of X using $p(x) = P(X = x) = F(x_i) - F(x_{i-1})$

x	1	4	6	10
$P(X = x)$	1/3	1/6	1/3	1/6

(ii) $P(2 < X < 6) = P(X = 4) = \dfrac{1}{6}$

(iii) Mean of $X = E(X) = \Sigma xp(x) = 1 \times \dfrac{1}{3} + 4 \times \dfrac{1}{6} + 6 \times \dfrac{1}{3} + 10 \times \dfrac{1}{6} = \dfrac{14}{3}$

$$E(X^2) = \Sigma x^2 p(x) = 1 \times \frac{1}{3} + 4^2 \times \frac{1}{6} + 6^2 \times \frac{1}{3} + 10^2 \times \frac{1}{6} = \frac{95}{3}$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{89}{9}$$

EXAMPLE 2.67 If X and Y are independent random variables with means 2, 3 and variance 1, 2 respectively, find the mean and variance of the random variable $Z = 2X - 5Y$. **[AU May '07]**

Solution Given:

$$E(X) = 2,\ E(Y) = 3,\ \text{Var}(X) = 1,\ \text{Var}(Y) = 2,\ Z = 2X - 5Y$$

$$\therefore \quad E(Z) = E(2X - 5Y) = 2E(X) - 5E(Y)$$

$$E(Z) = (2 \times 2) - (5 \times 3) = -11$$

If X and Y are independent, then $\text{Var}(aX + bY) = a^2\, \text{Var}(X) + b^2\, \text{Var}(Y)$

$$\therefore \quad \text{Var}(2X - 5Y) = 4\ \text{Var}(X) + 25\ \text{Var}(Y)$$

$$= (4 \times 1) + (25 \times 2) = 54$$

EXAMPLE 2.68 If the range of X is the set $\{0, 1, 2, 3, 4\}$ and $P(X = x) = 0.2$, determine the mean and variance of the random variable. **[AU December '09]**

Solution Given:

x	0	1	2	3	4
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

By definition,

$$E(X) = \sum_{x=0}^{4} xp(x) = 0.2(0+1+2+3+4)$$

$$= 0.2 \times 10 = 2$$

$$E(X^2) = \sum_{x=0}^{4} x^2 p(x) = 0.2(0+1+4+9+16) = 6$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$$

EXAMPLE 2.69 A fair coin is tossed three times. Let X be the number of tails appearing. Find the probability distribution of X and also calculate $E(X)$.
[AU December '09]

Solution Given: X denotes the number of tails. Since the coin is tossed 3 times, the number of tails may be 0, 1, 2 or 3

$$n(S) = 2^3 = 8.$$

$$S = \{HHH, HTH, THH, HHT, TTH, THT, HTT, TTT\}$$

Given:

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The probability distribution is

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \le x < 1 \\ \frac{4}{8}, & 1 \le x < 2 \\ \frac{7}{8}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

$$E(X) = \sum_{x=0}^{3} xp(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{2}$$

EXAMPLE 2.70 Given the following probability distribution of X, compute

(i) $E(X)$,
(ii) $E(X^2)$,

(iii) $E(2X \pm 3)$, and

(iv) $\text{Var}(2X \pm 3)$

x	–3	–2	–1	0	1	2	3
$p(x)$	0.05	0.10	0.30	0	0.30	0.15	0.10

[AU June '07]

Solution We know that for a discrete random variable X

(i) $E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$

$$= (-3)(0.05) - 2(0.1) - 1(0.30) + 0 + 1(0.30) + 2(0.15) + 3(0.10)$$

$$= 0.25$$

(ii) $E(X^2) = \sum_{i=1}^{n} x_i^2\, P(x_i)$

$$= (-3)^2(0.05) + (-2)^2(0.1) + (-1)^2(0.30) + 0 + (1)^2(0.30) + (2)^2(0.15) + (3)^2(0.10)$$

$$= 2.95$$

(iii) $E(2X \pm 3) = 2E(X) \pm 3$

$$= 2(0.25) \pm 3$$

$$= 0.5 \pm 3$$

(iv) $\text{Var}(2X \pm 3) = 2^2\, \text{Var}(X)$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 2.95 - (0.25)^2$$

$$= 2.8875$$

$$\text{Var}(2X \pm 3) = 4(2.8875)$$

$$= 11.55$$

EXAMPLE 2.71 The monthly demand for Allwyn watches is known to have the following probabilty distribution:

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches. Also compute the variance.

[AU December '06]

Solution We know that for a discrete random variable X

(i) $E(X) = \sum_{i=1}^{8} x_i\, p(x_i)$

$$= (1)(0.08) + 2(0.12) + 3(0.19) + 4(0.24) + 5(0.16) + 6(0.10) + 7(0.07) + 8(0.04)$$

$$= 4.06$$

(ii) $$E(X^2) = \sum_{i=1}^{8} x_i^2\, p(x_i)$$

$$= 1(0.08) + 4(0.12) + 9(0.19) + 16(0.24) + 25(0.16) + 36(0.10) + 49(0.07) + 64(0.04)$$

$$= 19.7$$

(iii) $$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 19.7 - (4.06)^2 \\ &= 19.7 - 16.48 \\ &= 3.22 \end{aligned}$$

EXAMPLE 2.72 Suppose that the random variable X is equal to the number of hits obtained by a certain base ball player in his next 3 bats and if $P(X = 1) = 0.3$, $P(X = 2) = 0.2$ and $P(X = 0) = 3P(X = 3)$, find $E(X)$.

[AU December '03]

Solution The random variable X is the number of hits obtained and $i = 3$ bats, $i = 0, 1, 2, 3$.

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1 \quad \text{(i)}$$

Given: $$P(X = 0) = 3\,P(X = 3) \quad \text{(ii)}$$

Substituting Eq. (ii) in Eq. (i), we get

$$3P(X = 3) + 0.3 + 0.2 + P(X = 3) = 1$$

i.e. $$4P(X = 3) = 1 - 0.5$$

$$P(X = 3) = \frac{0.5}{4} = 0.125$$

$\therefore$ $$\begin{aligned} P(X = 0) &= 3P(X = 3) \\ &= 3 \times 0.125 \\ &= 0.375 \end{aligned}$$

We know that

$$\begin{aligned} E(X) &= \sum_{i=0}^{3} x_i\, p(x_i) \\ &= (1)\,P(X = 1) + (2)\,P(X = 2) + (3)\,P(X = 3) \\ &= 1(0.3) + 2(0.2) + 3(0.125) \\ &= 1.075 \end{aligned}$$

EXAMPLE 2.73 The cumulative distribution function (CDF) of a random variable X is $F(x) = 1 - (1 + x)\,e^{-x}, x > 0$. Find the probability density function of X, mean and variance. **[AU June '07, May '06]**

Solution Given: $F(x) = 1 - (1 + x)e^{-x}, \quad 0 < x < \infty$

$$\text{PDF} = f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}[1-(1+x)e^{-x}]$$

$$= 0 + e^{-x} - (-xe^{-x} + e^{-x}) = xe^{-x},\ x > 0$$

$$\text{Mean} = E(X) = \int_0^{\infty} xf(x)\,dx = \int_0^{\infty} x \cdot xe^{-x}dx = \int_0^{\infty} x^2e^{-x}dx$$

$$= \left[x^2(-e^{-x}) - (2x)(e^{-x}) + 2(-e^{-x})\right]_0^{\infty}$$

$$= 0 + 2 = 2$$

$$E(X^2) = \int_0^{\infty} x^2 f(x)dx = \int_0^{\infty} x^2 \cdot xe^{-x}dx = \int_0^{\infty} x^3e^{-x}dx$$

$$= \left[x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^{\infty}$$

$$= 6$$

$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$

EXAMPLE 2.74 If $dF = kx^2e^{-x}\,dx,\ x \geq 0$, find k, mean and variance.

[AU May '06]

Solution Given: $\dfrac{dF}{dx} = kx^2\,e^{-x} = f(x),\ x \geq 0,$

(i) Since $f(x)$ is a PDF

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \quad \int_0^{\infty} kx^2e^{-x}dx = 1 \Rightarrow k\int_0^{\infty} x^2e^{-x}dx = 1$$

$$k\left[\frac{x^2e^{-x}}{-1} - \frac{2xe^{-x}}{1} + \frac{2e^{-x}}{-1}\right]_0^{\infty} = 1 \Rightarrow k(0+2) = 1 \Rightarrow k = \frac{1}{2}$$

(ii) $E(X) = \displaystyle\int_0^{\infty} xf(x)dx = \int_0^{\infty} \frac{1}{2}x^3e^{-x}dx$

$$= \frac{1}{2}\left[x^3\frac{e^{-x}}{-1} - 3x^2\cdot\frac{e^{-x}}{1} + 6x\frac{e^{-x}}{-1} - \frac{6e^{-x}}{1}\right]_0^{\infty}$$

$$= \frac{1}{2}(6) = 3$$

(iii) $E(X^2) = \int_0^\infty x^2 f(x)dx = \frac{1}{2}\int_0^\infty x^4 e^{-x}dx$

$$= \frac{1}{2}\left[\frac{x^4e^{-x}}{-1} - \frac{4x^3e^{-x}}{1} + \frac{12x^2e^{-x}}{-1} - \frac{24x\,e^{-x}}{1} + \frac{24e^{-x}}{-1}\right]_0^\infty$$

$= 12$

$\therefore \quad \text{Var}(X) = E(X) - [E(X)]^2 = 12 - 9 = 3$

EXAMPLE 2.75 Let X be a random variable with $E(X) = 10$ and $\text{Var}(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1. **[AU June '06, May '07]**

Solution Given: $E(X) = 10$

and $\text{Var}(X) = 25$

Now, $E(Y) = E(aX - b) = 0$ (given)

$aE(X) - b = 0 \Rightarrow a(10) - b = 0$

$10a - b = 0$ (i)

Given: $\text{Var}(Y) = 1$

$\text{Var}(aX - b) = a^2\,\text{Var}(X) = 1 \Rightarrow 25a^2 = 1$

i.e. $a = \frac{1}{5}$

Substituting $a = \frac{1}{5}$ in Eq. (i), we get

$$b = 2$$

EXAMPLE 2.76 For the triangular distribution

$$f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance. **[AU May '06, June '07]**

Solution Mean $= E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)dx$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2(2 - x) dx$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3) dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[2\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{7}{6}$$

$$\text{Var}(X) = E(X^2) - E(X) = \frac{7}{6} - (1)^2 = \frac{1}{6}$$

EXAMPLE 2.77 Find the value of

(i) c, and

(ii) mean of the following distribution given

$$f(x) = \begin{cases} c(x - x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$ **[AU December '06]**

Solution

(i) Since $f(x)$ is a PDF, $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_0^1 c(x - x^2) dx = 1 \Rightarrow c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$c\left[\frac{1}{2} - \frac{1}{3}\right]_0^1 = 1 \Rightarrow c\left(\frac{3-2}{6}\right) = 1$$

$$\therefore \quad c = 6$$

(ii) Mean $= E(X) = \int_0^1 x \cdot 6(x - x^2)dx = \int_0^1 6(x^2 - x^3)dx$

$$= \left[\frac{6x^3}{3} - \frac{6x^4}{4}\right]_0^1 = \left[\frac{6}{3} - \frac{6}{4}\right] = 2 - \frac{3}{2} = \frac{1}{2}$$

EXAMPLE 2.78 If X has the probability density function

$$f(x) = \begin{cases} ke^{-3x}, \text{ for } x > 0 \\ 0, \quad \text{otherwise} \end{cases}$$

find k, $P(0.5 \le X \le 1)$ and the mean of X.

Solution Since $f(x)$ is a PDF, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \quad \int_0^\infty ke^{-3x}dx = 1 \Rightarrow k\left[\frac{e^{-3x}}{-3}\right]_0^\infty = 1 \Rightarrow \frac{k}{3} = 1, \text{ i.e. } k = 3$$

$$\therefore \quad f(x) = 3e^{-3x}, \ x > 0$$

$$P(0.5 \le X \le 1) = \int_{0.5}^1 f(x)dx = \int_{0.5}^1 3e^{-3x}dx$$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_{0.5}^1 = 3\left(\frac{e^{-1.5} - e^{-3}}{3}\right) = 0.17331$$

$$E(X) = \int_{-\infty}^\infty xf(x)dx = 3\int_0^\infty xe^{-3x}dx$$

$$= 3\left[x\left(\frac{e^{-3x}}{-3}\right) - 1\left(\frac{e^{-3x}}{9}\right)\right]_0^\infty = 3\left(\frac{0+1}{9}\right) = \frac{1}{3}$$

$$\text{Mean} = E(X) = \frac{1}{3}$$

EXAMPLE 2.79 Suppose the duration x in minutes of long distance calls from your home, follows exponential law with PDF $f(x) = \frac{1}{5}e^{-x/5}$, for $x > 0$, 0 otherwise. Find

(i) $P(X > 5)$,
(ii) $P(3 \le X \le 6)$,

(iii) mean of X, and

(iv) variance of X. **[AU November '05]**

Solution Given: $f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

(i) $$P(X > 5) = \int_5^{\infty} \frac{1}{5} e^{-\frac{x}{5}} dx = \frac{1}{5}\left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right]_5^{\infty} = e^{-1} = 0.3679$$

(ii) $$P(3 \le X \le 6) = \int_3^{6} \frac{1}{5} e^{-\frac{x}{5}} dx = \frac{1}{5}\left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right]_3^{6} = e^{-\frac{3}{5}} - e^{-\frac{6}{5}} = 0.2476$$

(iii) $$E(X) = \int_{-\infty}^{\infty} xf(x)\frac{1}{5} e^{-\frac{x}{5}} dx = \frac{1}{5}\int_{-\infty}^{\infty} xe^{-\frac{x}{5}} dx = \frac{1}{5}\left[x\left(\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right) - 1\frac{e^{-\frac{x}{5}}}{\left(-\frac{1}{5}\right)^2}\right]_0^{\infty}$$

$$= 0 + \frac{1}{5} \times 25 = 5$$

$$E(X^2) = \frac{1}{5}\int_0^{\infty} x^2 e^{-\frac{x}{5}} dx = \frac{1}{5}\left\{x^2\left(\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right) - 2x\left[\frac{e^{-\frac{x}{5}}}{\left(-\frac{1}{5}\right)^2}\right] + 2\left[\frac{e^{-\frac{x}{5}}}{\left(-\frac{1}{5}\right)^3}\right]\right\}_0^{\infty}$$

$$= \frac{1}{5} \times 250 = 50$$

(iv) $\therefore$ $\text{Var}(X) = 50 - 5^2 = 25$

EXAMPLE 2.80 Let X be a continuous random variable with PDF $f_x(x) = x^2$, $1 < x < 2$. Find $E(\log X)$. **[AU December '08]**

Solution By definition,

$$E(\log X) = \int_{-\infty}^{\infty} \log x \, f(x) dx$$

$$= \int_1^2 x^2 \log x \, dx$$

$$= \left[\log x \frac{x^3}{3}\right]_1^2 - \int_1^2 \frac{x^3}{3} \times \frac{1}{x} dx$$

$$= \log 2\left(\frac{2^3}{3}\right) - \log 1 \times \left(\frac{1}{3}\right) - \int_1^2 \frac{x^2}{3} dx$$

$$= \frac{8}{3}\log 2 - 0 - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{8}{3}\log 2 - \frac{1}{9}(8-1)$$

$$= \frac{8}{3}\log 2 - \frac{7}{9}$$

EXAMPLE 2.81 If a random variable X has the probability density function

$$f(x) = \begin{cases} \frac{1}{2}(x+1), & \text{if } -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the mean and variance. **(AU December '07]**

Solution Mean $= E(X) = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{2}\int_{-1}^{1} x(x+1)dx$

$$= \frac{1}{2}\int_{-1}^{1}(x^2 + x)dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3} - \frac{1}{2}\right) = \frac{1}{3}$$

$$E(X^2) = \frac{1}{2}\int_{-1}^{1} x^2(x+1)dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\right) = \frac{1}{3}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

EXAMPLE 2.82 If $f(x)$ is a density function defined by $f(x) = ae^{-|x|}$, $-\infty < x < \infty$, find

(i) the value of a,
(ii) mean and variance. **[AU April '04, June '06, December '05]**

Solution

(i) If $f(x)$ is a PDF, then $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ae^{-|x|}\, dx = 1$$

Since $|x|$ is an even function,

$$2\int_0^{\infty} ae^{-|x|}\, dx = 1$$

$$2\int_0^{\infty} ae^{-x} dx = 1 \qquad (\because |x| = x, \text{ if } x > 0)$$

$$2a\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1 \Rightarrow 2a \cdot 1 = 1 \Rightarrow a = \frac{1}{2}$$

(ii) $\quad$ Mean $= E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} xe^{-|x|}dx = 0 \;[\because xe^{-|x|} \text{ is an odd function}]$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}dx$$

$$= \frac{1}{2}\cdot 2\int_0^{\infty} x^2 e^{-x}dx \qquad [\because x^2e^{-|x|} \text{ is an even function}]$$

$$= \left[x^2\frac{e^{-x}}{-1} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty}$$

$$= [0 - 0 - 2(0 - 1)] = 2$$

$$\therefore \text{Var}(X) = E(X^2) - E(X) = 2 - 0 = 2$$

EXAMPLE 2.83 The first four moments of the distribution about the value 4 of the variable are −1.5, 17, −30 and 108 respectively. Find the mean and moments about the mean.

Solution Given: $\mu_1' = -1.5,\ \mu_2' = 17,\ \mu_3' = -30,\ \mu_4' = 108$ and $A = 4$

$$\text{Mean} = \bar{X} = A + \mu_1' = 4 + (-1.5) = 2.5$$

The moments about mean are

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'$, $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$$\mu_2 = \mu_2' - \mu_1'^2 = 17 - (-1.5)^2 = 14.75$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2\mu_1'^3 = -30 + 76.5 - 6.75 = 39.75$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4 = 108 - 180 + 229.5 - 15.1875 = 142.313$$

EXAMPLE 2.84 The first four moments of a distribution about $X = 4$ are 1, 4, 10 and 45 respectively. Show that the mean is 5, variance is 3, $\mu_3 = 0$, and $\mu_4 = 26$.

[AU November '04]

Solution Given: $\mu_1' = 1, \mu_2' = 4, \mu_3' = 10, \mu_4' = 45$

and $A = 4$

$$E(X) = \text{mean} = \bar{X} = A + \mu_1' = 1 + 4 = 5$$

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 4 - 1^2 = 3$$

$$\text{Var} = \mu_2 = 3$$

$$\begin{aligned}\mu_3 &= \mu_3' - 3\mu_2'\,\mu_1' + 2\mu_1'^3\\ &= 10 - 3 \times 4 \times 1 + 2 \times 1^3\\ &= 10 - 12 + 2 = 0\end{aligned}$$

$$\begin{aligned}\mu_4 &= \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4\\ &= 45 - 4 \times 10 \times 1 + 6 \times 1^2 \times 4 - 3 \times 1^4\\ &= 45 - 40 + 24 - 3\\ &= 26\end{aligned}$$

EXAMPLE 2.85 Calculate the first four moments of the following distribution about the mean.

x_i	0	1	2	3	4	5	6	7	8
y_i	1	8	28	56	70	56	28	8	1

Solution

x_i	f_i	$d_i = x_i - 4$	f_id_i	$f_id_i^2$	$f_id_i^3$	$f_id_i^4$
0	1	−4	−4	16	−64	256
1	8	−3	−24	72	−216	648
2	28	−2	−56	112	−224	448
3	56	−1	−56	56	−56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	256		0	512	0	2816

$$N = \sum f_i = 256, \quad \mu_1' = \frac{1}{N}\sum_i f_i d_i = 0$$

$$\mu_2' = \frac{1}{N}\sum_i f_i d_i^2 = \frac{512}{256} = 2$$

$$\mu_3' = \frac{1}{N}\sum_i f_i d_i^3 = \frac{0}{256} = 0$$

$$\mu_4' = \frac{1}{N}\sum_i f_i d_i^4 = \frac{2816}{256} = 11$$

The moments about the mean are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 2 - 0 = 2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 0 - 3(2)0 + 2(0) = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4 = 11 - 0 = 11$$

2.12 MOMENT GENERATING FUNCTION (MGF): $M_X(t)$

The moment generating function of a random variable denoted by $M_X(t)$ is defined as

$$M_X(t) = E(e^{tX})$$

$$\therefore \quad M_X(t) = E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots + \infty$$

$$\therefore \quad M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots + \infty \qquad (2.1)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

which gives the MGF in terms of moments.

$\therefore$ The coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ is μ_r', where $r = 1, 2, 3\ldots$ and $\mu_r' = E(X^r)$, moment about the origin.

Moment generating function of X about any point $X = a$ is defined as

$$M_X(t) = E[e^{t(X-a)}] = E\left[1 + \frac{t}{1!}(X-a) + \frac{t}{2!}(X-a)^2 + \frac{t}{3!}(X-a)^3 + \cdots\right]$$

$$M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t}{2!}\mu_2' + \frac{t}{3!}\mu_3' + \cdots + \infty, \text{ where } E[(X-a)^r] = \mu_r'$$

Since $M_X(t)$ generates moments it is called *moment generating function*.

If X is a discrete random variable with PMF $p(x)$, then

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable with PDF $f(x)$, then

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

2.12.1 Moments Using Moment Generating Function

Differentiating Eq. (2.1) with respect to t and then putting $t = 0$, gives

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

In general,

$$\mu_r' = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}, \quad r = 1, 2, 3, \ldots$$

Note: Moment generating function $M_X(t)$ is used to calculate the higher moments.

2.12.2 Limitations of Moment Generating Function

1. A random variable X may have no moments although its moment generating function exist.
   For example:

$$f(x) = \begin{cases} \dfrac{1}{x(x+1)}, & x = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

2. A random variable X can have MGF and some or all moments, yet the MGF does not generate the moments.

For example:

$$P(X = \pm 2^x) = \frac{e^{-1}}{x!}, \quad x = 0, 1, 2, \ldots$$

3. A random variable X can have all or some moments, but MGF does not exist, except perhaps at one point.
   For example:

$$P(X = \pm 2^x) = \begin{cases} \dfrac{e^{-1}}{2x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

2.12.3 Theorems on Moment Generating Function

THEOREM 5 $M_{aX}(t) = M_X(at)$, a being a constant.

Proof By definition,

$$M_{aX}(t) = E(e^{taX}) = E(e^{atX})$$

$$\therefore \quad M_{aX}(t) = M_X(at)$$

THEOREM 6 The moment generating function of the sum of n independent random variables is equal to the product of their respective moment generating functions, i.e.

$$M_{X_1+X_2+X_3+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$$

Proof By definition,

$$\begin{aligned} M_{X_1+X_2+X_3+\cdots+X_n}(t) &= E\left[e^{t(X_1+X_2+X_3+\cdots+X_n)}\right] \\ &= E(e^{tX_1})\, E(e^{tX_2})\, E(e^{tX_3}) \ldots E(e^{tX_n}) \end{aligned}$$

(Since $X_1, X_2, \ldots, X_n$ are independent)

$$\therefore \quad M_{X_1+X_2+X_3+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$$

Hence the proof.

2.12.4 Effect of Change of Origin and Scale on Moment Generating Function

Let a random variable X be transformed to a new variable U by changing both the origin and scale in X as

$$U = \frac{X - a}{h}$$

where a and h are constants.

The MGF of U (about origin) is given by

$$M_U(t) = E(e^{tU})$$

$$= E\left[e^{t\left(\frac{X-a}{h}\right)}\right]$$

$$= E\left(e^{\frac{tX}{h}} e^{-\frac{ta}{h}}\right) = e^{\frac{-at}{h}} E\left(e^{\frac{tX}{h}}\right)$$

$$\therefore \quad M_{\frac{X-a}{h}}(t) = e^{\frac{-at}{h}} M_X\left(\frac{t}{h}\right)$$

Note: If $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$

EXAMPLE 2.86 If X represents the outcome when a fair die is tossed, find the MGF of X and hence, find $E(X)$ and $\text{Var}(X)$. **[AU May '06, '07]**

Solution When a fair die is tossed

$$P(X = x) = \frac{1}{6}, \; x = 1, 2, 3, 4, 5, 6$$

$$\therefore \quad MX(t) = \sum_{X=1}^{6} e^{tx} p(X = x)$$

$$= \frac{1}{6}\sum_{x=1}^{6} e^{tx}$$

$$= \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0}$$

$$= \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$$

$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$$

$$\therefore \quad \text{Mean} = E(X) = \frac{7}{2}$$

$$E[X^2]_{t=0} = \left\{\frac{d^2}{dt^2}[M_X(t)]\right\}_{t=0}$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

$$\therefore \qquad \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

EXAMPLE 2.87 Find the MGF of the random variable X whose probability function $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$. Hence find its mean. **[AU May '05]**

Solution By definition,

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{x=1}^{\infty} e^{tx}\left(\frac{1}{2^x}\right) = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$$

$$= \left[\frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2}\left(\frac{2 - e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{2}\left(\frac{2}{2 - e^t}\right)$$

$$= \frac{e^t}{2 - e^t}$$

$$\therefore \qquad M_X(t) = \frac{e^t}{2 - e^t}$$

$$\mu_1' = \text{mean} = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{e^t}{2 - e^t}\right)\right]_{t=0}$$

$$= \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}\right]_{t=0}$$

$$= \frac{(2-1)1 + 1}{(2-1)^2} = 2$$

$$\therefore \qquad E(X) = \text{mean} = 2$$

EXAMPLE 2.88 If the moments of a random variable X are defined by $E(X^r) = 0.6$, $r = 1, 2, \ldots$. Show that $P(X = 0) = 0.4$, $P(X = 1) = 0.6$, and $P(X \geq 2) = 0$.

Solution We know that

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

where $$\mu_r' = E(X^r) = 0.6$$

$\therefore$ $$M_X(t) = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} \mu_r' = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6)$$

$$= 1 + (0.6)\left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

$$= 1 + (0.6)(e^t - 1)$$

$$= 1 - 0.6 + 0.6e^t = 0.4 + 0.6e^t \tag{i}$$

But by definition,

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$M_X(t) = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \cdots \tag{ii}$$

From Eqs. (i) and (ii), we have

$$0.4 + 0.6e^t = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \cdots$$

Equating the coefficients of like terms on both sides,

$$P(X = 0) = 0.4, \quad P(X = 1) = 0.6$$

$$P(X = 2) = P(X = 3) = \cdots = 0 \quad \Rightarrow \quad P(X \geq 2) = 0$$

EXAMPLE 2.89 Let X be a random variable with $E(X) = 1$ and $E[X(X - 1)] = 4$. Find $\text{Var}(X/2)$ and $\text{Var}(2 - 3X)$.

Solution Given $E(X) = 1$, $E(X^2 - X) = 4 \quad \Rightarrow \quad E(X^2) - E(X) = 4$

$\Rightarrow$ $$E(X^2) = 4 + E(X) = 4 + 1 = 5$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 5 - 1^2 = 4$$

By the property,

$$\text{Var}\left(\frac{X}{2}\right) = \frac{1}{4}\text{Var}(X) = \frac{1}{4} \times 4 = 1$$

$$\text{Var}(2 - 3X) = \text{Var}(2) + 3^2\,\text{Var}(X)$$

$$= 0 + 9 \times 4 = 36$$

EXAMPLE 2.90 Find the first four moments about the origin for a random variable X having density function $f(x) = \dfrac{4x(9-x^2)}{81}, 0 \le x \le 3$.

Solution Given: The density function

$$f(x) = \frac{4x(9-x^2)}{81}, 0 \le x \le 3$$

Using the definition,

$$\mu_1^1 = E(X) = \int_0^{\infty} xf(x)dx = \frac{4}{81}\int_0^3 x^2(9-x^2)dx = \frac{4}{81}\left[\frac{9x^3}{3} - \frac{x^5}{5}\right]_0^3 = \frac{8}{5}$$

$$\mu_2^1 = E(X^2) = \int_0^{\infty} x^2 f(x)dx = \frac{4}{81}\int_0^3 x^3(9-x^2)dx = \frac{4}{81}\left[\frac{9x^4}{4} - \frac{x^6}{6}\right]_0^3 = 3$$

$$\mu_3^1 = E(X^3) = \frac{4}{81}\int_0^3 x^4(9-x^2)dx = \frac{4}{81}\left[\frac{9x^5}{5} - \frac{x^7}{7}\right]_0^3 = \frac{299}{35}$$

$$\mu_4^1 = E(X^4) = \int_0^{\infty} x^4 f(x)dx = \frac{4}{81}\int_0^3 x^5(9-x^2)dx = \frac{4}{81}\left[\frac{9x^6}{6} - \frac{x^8}{8}\right]_0^3 = \frac{27}{2}$$

EXAMPLE 2.91 Find the MGF of a random variable whose moments are $\mu_r = (r+1)!\, 2^r$. **[AU December '07]**

Solution By definition,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)!\,2^r = \sum_{r=0}^{\infty} (r+1)\,(2t)^r \qquad [\because\ (r+1)! = (r+1)r!]$$

$$= 1 + 2(2t) + 3(2t)^2 + \cdots = (1-2t)^{-2} = \frac{1}{(1-2t)^2}$$

$$\therefore \qquad M_X(t) = \frac{1}{(1-2t)^2}$$

EXAMPLE 2.92 If a random variable X has the MGF $M_X(t) = \dfrac{3}{3-t}$, find the standard deviation of X. **[AU December '04]**

Solution Given: $M_X(t) = \dfrac{3}{3-t} = \dfrac{3}{3\left(1-\frac{t}{3}\right)} = \dfrac{1}{1-\frac{t}{3}}$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \cdots$$

$$1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots$$

$$= 1 + \frac{1}{3}\frac{t}{1!} + \frac{2!}{9}\frac{t^2}{2!} + \cdots$$

$\therefore$ Equating the coefficients of $\frac{t}{1!}$ and $\frac{t^2}{2!}$, we get

$$\mu_1' = \frac{1}{3},\ \mu_2' = \frac{2}{9}$$

$\therefore$
$$\text{Var}(X) = \mu_2' - (\mu_1')^2$$
$$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$\therefore$ Standard deviation of $X = \frac{1}{3}$

EXAMPLE 2.93 Find the MGF of a random variable X having the density function

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

Solution By definition,

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_0^2 e^{tx} \frac{x}{2} dx$$

$$= \frac{1}{2}\left[x\frac{e^{tx}}{t} - (1)\frac{e^{tx}}{t^2}\right]_0^2$$

$$= \frac{1}{2}\left[\left(2\frac{e^{2t}}{t} - 0\right) - \left(\frac{e^{2t}}{t^2} - \frac{1}{t^2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2}\right)$$

$\therefore$
$$M_X(t) = \frac{1}{2t^2}(1 + 2te^{2t} - e^{2t})$$

EXAMPLE 2.94 A random variable X has the PDF given by

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find

(i) the MGF, and

(ii) the first four moments about the origin. **[AU December '07]**

Solution By definition,

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$M_X(t) = \int_0^{\infty} e^{tx} 2e^{-2x} dx = 2\int_0^{\infty} e^{-(2-t)x} dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty}$$

$$\therefore \qquad M_X(t) = \frac{2}{2-t}$$

We know that

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' = \frac{2}{2-t} = \frac{2}{2\left(1-\frac{t}{2}\right)}$$

$$\therefore \quad 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots = \left(1-\frac{t}{2}\right)^{-1}$$

$$= 1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots$$

$$= 1 + \frac{1}{2}\frac{t}{1!} + \frac{2!}{4}\frac{t^2}{2!} + \frac{3!}{8}\frac{t^3}{3!} + \frac{4!}{16}\frac{t^4}{4!} + \cdots$$

Equating the coefficients of $\frac{t}{1!}, \frac{t^2}{2!}, \frac{t^3}{3!} \cdots$, we get

$$\mu_1' = \frac{1}{2},\ \mu_2' = \frac{1}{2},\ \mu_3' = \frac{6}{8} = \frac{3}{4} \text{ and } \mu_4' = \frac{4!}{2^4} = \frac{3}{2}$$

EXAMPLE 2.95 Show that the MGF of the random variable X having the PDF $f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ is given by $M_X(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$.

[AU December '04]

Solution By definition,

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_{-1}^{2} e^{tx} \frac{1}{3} dx$$

$$= \frac{1}{3}\left[\frac{e^{tx}}{t}\right]_{-1}^{2}$$

$$= \frac{1}{3}\left(\frac{e^{2t} - e^{-t}}{t}\right), \; t \neq 0$$

If $t = 0$,

$$M_X(t) = \lim_{t \to 0} \frac{e^{2t} - e^{-t}}{3t} = \frac{0}{0}$$

$$= \lim_{t \to 0} \frac{2e^{2t} - e^{t}}{3} = \frac{3}{3} = 1 \quad \text{(using L'Hospitals rule)}$$

$\therefore$

$$M_X(t) = \frac{e^{2t} - e^{-t}}{3t}, \quad t \neq 0$$

$$= 1, \qquad t = 0$$

EXAMPLE 2.96 Find the MGF of the random variable X having PDF

$$f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

[AU December '06, May '09]

Solution We know that

$$M_X(t) = \int_0^{\infty} e^{tx} f(x)dx$$

$$= \int_0^1 e^{tx} f(x)dx + \int_1^2 e^{tx} f(x)dx$$

$$= \int_0^1 e^{tx} \cdot x\,dx + \int_1^2 e^{tx}(2 - x)dx$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) - \left(\frac{e^{tx}}{t^2}\right)\right]_0^1 + \left[(2 - x)\left(\frac{e^{tx}}{t}\right) - (-1)\left(\frac{e^{tx}}{t^2}\right)\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t}}{t^2} + \frac{1}{t^2} - \frac{2e^t}{t^2}$$

$$= \frac{(e^t - 1)^2}{t^2}$$

EXAMPLE 2.97 Let X be a random variable with PDF

$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find

(i) $P(X > 3)$,

(ii) MGF of X,

(iii) $E(X)$ and Var (X). **[AU June '07, December '05]**

Solution Using the definition,

(i) $$P(X > 3) = \int_3^\infty f(x)dx = \int_3^\infty \frac{1}{3} e^{-\frac{x}{3}} dx$$

$$= \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_3^\infty = -(0 - e^{-1}) = e^{-1} = \frac{1}{e}$$

(ii) $$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx} \frac{1}{3} e^{-\frac{x}{3}} dx = \frac{1}{3} \int_0^\infty e^{\left(t - \frac{1}{3}\right)x} dx$$

$$= \frac{1}{3} \int_0^\infty e^{-\left(\frac{1}{3} - t\right)x} dx = \frac{1}{3} \left[\frac{e^{-\left(\frac{1}{3} - t\right)x}}{-\left(\frac{1}{3} - t\right)} \right]_0^\infty$$

$$= \frac{1}{3} \left[0 - \frac{1}{-\left(\frac{1}{3} - t\right)} \right] = \frac{1}{3} \left[\frac{1}{\left(\frac{1 - 3t}{3}\right)} \right]$$

$$M_X(t) = \frac{1}{1-3t} = (1-3t)^{-1}$$

$$\frac{d}{dt}[M_X(t)] = -(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$$

(iii) $$E(X) = \text{Mean} = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = 3$$

$$\frac{d^2}{dt^2}[M_X(t)] = -6(1-3t)^{-3}(-3) = 18(1-3t)^{-3}$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = 18$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 18 - 3^2 = 9$$

EXAMPLE 2.98 Consider a discrete random variable X with a probability function

$$p(x) = \begin{cases} \dfrac{1}{x(x+1)}, & x = 1, 2 \cdots \\ 0, & \text{otherwise} \end{cases}$$

Show that $E(X)$ does not exist even though MGF exist.

Solution $$E(X) = \sum_{x=1}^{\infty} xp(x) = \sum_{x=1}^{\infty}\frac{1}{x+1} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{x=1}^{\infty}\frac{1}{x} - 1$$

But $\displaystyle\sum_{x=1}^{\infty}\frac{1}{x}$ is a divergent series.

$\therefore$ $E(X)$ does not exist and hence, no moment exists.
Now, MGF of X is given by

$$M_X(t) = \sum_{x=1}^{\infty} p(x)e^{tx} = \sum_{x=1}^{\infty}\frac{e^{tx}}{x(x+1)}$$

Using $z = e^t$

$$M_X(t) = \sum_{x=1}^{\infty}\frac{z^x}{x(x+1)} = \frac{z}{1\cdot 2}+\frac{z^2}{2\cdot 3}+\frac{z^3}{3\cdot 4}+\cdots$$

$$= z\left(1-\frac{1}{2}\right)+z^2\left(\frac{1}{2}-\frac{1}{3}\right)+z^3\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$

$$= \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right) - \frac{z}{2} - \frac{z^2}{3} - \frac{z^3}{4} \cdots$$

$$= -\log(1-z) - \frac{1}{z}\left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} \cdots\right)$$

$$= -\log(1-z) + 1 + \frac{1}{z}\log(1-z), \ |z| < 1$$

$$= 1 + \left(\frac{1}{z} - 1\right)\log(1-z), \ |z| < 1$$

$$M_X(t) = \begin{cases} 1 + (e^{-t} - 1)\log(1 - e^t), t < 0 \\ 1, \ \text{for } t = 0 \end{cases}$$

and $M_X(t)$ does not exist for $t > 0$.

EXAMPLE 2.99 A random variable X has density function given by

$$f(x) = \begin{cases} \frac{1}{k}, & \text{for } 0 < x < k \\ 0, & \text{otherwise} \end{cases}$$

Find

(i) MGF,
(ii) rth moment,
(iii) mean, and
(iv) variance. **[AU April '07]**

Solution

$$\text{MGF} = M_X(t) = E(e^{tx}) = \int_0^k \frac{1}{k} e^{tx} dx = \frac{1}{k}\left[\frac{e^{tx}}{t}\right]_0^k = \frac{1}{kt}(e^{tk} - 1)$$

$$= \frac{1}{kt}\left[1 + \frac{(kt)}{1!} + \frac{(kt)^2}{2!} + \cdots - 1\right]$$

$$= 1 + \frac{(kt)}{2!} + \cdots + \frac{(kt)^r}{(r+1)!} + \cdots$$

$\therefore$ $$\mu_r' = \text{coefficient of } t^r = \frac{k^r}{(r+1)!}$$

When $r = 1$, we get

$$\mu_1' = \text{mean} = \frac{k}{2}$$

When $r = 2$, we get

$$\mu_2' = \frac{2k^2}{3!} = \frac{k^2}{3}$$

$$\text{Var} = \mu_2' - \mu_1'^2$$

$$= \frac{k^2}{3} - \frac{k^2}{4} = \frac{k^2}{12}$$

EXERCISES

1. Define discrete and continuous random variables with examples.
2. Define distribution function of a random variable.
3. If $F(x)$ is a distribution function, then what is the value of $F(x)$ at $x = \infty$ and at $x = -\infty$?
4. Define probability mass function.
5. What are the important two conditions that a PMF $p(x)$ should satisfy.
6. Define PDF of a continuous random variable X.
7. Define probability curve or probability distribution curve.
8. If $f(x)$ is a PDF of a continuous random variable X, then what is the value of $\int_{-\infty}^{\infty} f(x)dx$?
9. If X is a continuous random variable, prove that $P(X = c) = 0$.
10. If X is a continuous random variable for the following density function $f(x) = cx^2(1 - x)$, $0 < x < 1$, find
    (i) the constant c, and
    (ii) mean.

    $\left[\textbf{Ans.} \quad c = 12, \text{mean} = \frac{3}{5}\right]$
11. If the PDF of a random variable X is $f(x) = y_0(x - x^2)$ in $0 \le x \le 1$, find mean.

    $\left[\textbf{Ans.} \quad \frac{1}{2}\right]$
12. Check whether the following are PDF or not
    (i) $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$, $\lambda > 0$ **[Ans.** PDF]

(ii) $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 4 - 2x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ [**Ans.** not a PDF]

13. Given the PDF of a random variable X is $f(x) = y_0 e^{-b(x-a)}$, $a \le x < \infty$, $b > 0$, where y_0, a and b are constants, show that $y_0 = b$.

14. Find k, so that $f(x)$ given below may be a PDF

$$f(x) = \begin{cases} \dfrac{1}{k}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Find also the distribution function.

$$\left[\textbf{Ans. } k = b - a, F(x) = 0, \text{ if } x < a \quad = \frac{x-b}{b-a}, \text{ if } a \le x \le b \quad = 1, \text{ if } x > b\right]$$

15. Define raw moment and central moment of a random variable.
16. Define expected value of a random variable X.
17. Define the mean and variance of a random variable.
18. If X is a discrete/continuous random variable prove that $E(aX + b) = aE(X) + b$ and $\text{Var}(aX) = a^2\,\text{Var}(X)$.
19. Define MGF of a discrete and continuous random variable X and state any two of its properties.
20. Define moments of a random variable X about any point and about origin.
21. If $M_X(t)$ is the MGF of a random variable X about the origin, then show that

$$\left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$

22. Find the relation between $M_X(t)$ and $M_Y(t)$ when $Y = aX + b$.
23. Define the raw and central moments of a random variable and state the relation between them.
24. The probability distribution of a random variable X is given by

x	0	1	2	3
$p(x)$	0.1	0.3	0.4	0.2

find $E(Y)$, where $Y = X^2 + X$.

25. The PMF of a random variable X is given by

$$p(x) = \frac{a\lambda^x}{x!}, \lambda = 0, 1, 2, \ldots$$

where λ is a positive value. Find

(i) $P(X = 0)$, and

(ii) $P(X > 2)$.

$$\left[\textbf{Ans.} \quad \text{(i) } e^{-\lambda}, \text{(ii) } 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2 e^{-\lambda}}{2}\right]$$

26. The CDF $F(x)$ of a random variable X is given by

(i) $F(x) = \begin{cases} 0, & x \le 0 \\ c(1 - e^{-x}), & x > 0 \end{cases}$

(ii) $F(x) = \begin{cases} 0, & \text{if } x \le 1 \\ c(x-1)^4, & \text{if } 1 < x \le 3 \\ 1, & \text{if } x > 3 \end{cases}$

Find the PDF $f(x)$.

$$\left[\begin{array}{ll} \textbf{Ans.} \ \text{(i) } f(x) = \begin{cases} 0 & x \le 0 \\ ce^{-x} & x > 0 \end{cases} & \\ \text{(ii) } f(x) = 0 & x \le 1 \\ \quad = 4c(x-1)^3, & 1 < x \le 3 \\ \quad = 0 & x > 3 \end{array}\right]$$

27. A continuous random variable has the PDF $f(x) = kx^4$, $-1 < x < 0$.

Find the value of k and also $P\left(X > -\frac{1}{2} / X < -\frac{1}{4}\right)$.

$$\left[\textbf{Ans.} \quad k = 5, p = \frac{1}{33}\right]$$

28. If the probability distribution function of a random variable X is $f(x) = Ce^{-\alpha x}$, $0 \le x < \infty$, $\alpha > 0$, find

(i) the value of C,

(ii) mean, and

(iii) variance

$$\left[\textbf{Ans.} \quad \text{(i) } C = \alpha, \text{(ii) mean} = \frac{1}{\alpha^2}, \text{(iii) variance} = \frac{2}{\alpha^3}\right]$$

29. If the PDF of a random variable X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$, find $P(X > 1.5/X > 1)$.

$\left[\text{Ans. } \frac{7}{12}\right]$

30. Find $P(X = 0)$, if $P(X = 0) = P(X < 0) = P(X > 0)$.

$\left[\text{Ans. } \frac{1}{3}\right]$

31. A continuous distribution of a random variable X in the range $(-3, 3)$ is defined by

$$f(x) = \begin{cases} \frac{1}{16}(3 + x)^2, & -3 \le x - 1 \\ \frac{1}{16}(2 - 6x^2), & -1 \le x \le 3 \\ \frac{1}{16}(3 - x)^2, & 1 \le x \le 3 \end{cases}$$

Show that the area under the curve is unity and the mean is zero.

32. Verify that each of the following functions is a probability density function:

(i) $f(x) = 1 - |1 - x|$, for $0 < x < 2$.

(ii) $f(x) = \frac{1}{\pi} \frac{c}{c^2 + (x - a)^2}$, for $-\infty < x \le \infty$

(iii) $f(x) = \frac{1}{2\sigma} e^{\frac{-(x-\mu)}{\sigma}}$, for $-\infty < x < \infty$

(iv) $f(x) = \frac{1}{4} x e^{\frac{-x}{2}}$, for $-\infty < x < \infty$.

33. The amount of bread (in hundreds of kilos) that a bakery sells in a day is a random variable with PDF

$$f(x) = \begin{cases} cx, & \text{for } 0 \le x < 3 \\ c(6 - x), & \text{for } 3 \le x < 6 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the value of c which makes $f(x)$ a PDF.

(ii) What is the probability that the number of kilos of bread that will be sold in a day is

(a) more than 300 kilos

(b) between 150 and 450 kilos.

$\left[\text{Ans. (i) } c = \frac{1}{9}, \text{(ii) } \frac{3}{4}\right]$

34. A continuous random variable X has a PDF $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that
 (i) $P(X \le a) = P(X > a)$, and
 (ii) $P(X > b) = 0.05$.

$$\left[\textbf{Ans.}\ \ \text{(i) } a = \left(\frac{1}{2}\right)^{\frac{1}{3}}, \text{(ii) } b = (0.95)^{\frac{1}{3}}\right]$$

35. A continuous random variable X has a PDF $f(x) = kx^2e^{-x}$; $x \ge 0$. Find k, mean and variance.

$$\left[\textbf{Ans.}\ \ k = \frac{1}{2}, \text{mean} = 3, \text{variance} = 3\right]$$

36. A continuous random variable X that can assumes values between $x = 2$ and $x = 5$ has a density function given by $f(x) = \frac{2}{27}(1 + x)$. Find $P(3 < X < 4)$.

$$\left[\textbf{Ans.}\ \ \frac{1}{3}\right]$$

37. Find the MGF if it exists, given the PDF

$$f(x)\begin{cases} xe^{-x}, x > 0 \\ 0, \quad \text{otherwise} \end{cases} \qquad \left[\textbf{Ans.}\ \ \frac{1}{1-t}\right]$$

38. Find the MGF of the random variable X with PDF

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and also find μ_1, μ_2.
$$\left[\textbf{Ans.}\ \ \frac{(e^t - 1)^2}{t^2}, \mu_1 = 0 = \mu_2\right]$$

39. Find the MGF of a random variable whose density function is given by $f(x) = \lambda e^{-\lambda(x - a)}$, $x \ge a$. Hence find its mean and variance.

$$\left[\textbf{Ans.}\ \ \text{MGF} = \frac{\lambda e^{at}}{\lambda - t}, \text{mean} = \frac{a\lambda + 1}{\lambda}, \text{Var} = \frac{1}{\lambda^2}\right]$$

40. A continuous random variable has the PDF $f(x) = \begin{cases} \frac{1}{2}(x + 1), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find the mean and variance.

$$\left[\textbf{Ans.}\ \ \text{mean} = \frac{1}{3}, \text{Var} = \frac{2}{9}\right]$$

41. Given the following probability distribution of X:

x	-3	-2	-1	0	1	2	3
$p(x)$	0.05	0.10	0.30	0	0.30	0.15	0.10

Compute

(i) $E(X)$,
(ii) $E(X^2)$,
(iii) $E(2X + 3)$, and
(iv) Var $(2X + 3)$

[**Ans.** (i) 0.25, (ii) 2.95, (iii) 0.5 + 3, and (iv) 11.55]

42. Let X be a random variable with $E(X) = 10$ and $\text{Var}(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.

$\left[\textbf{Ans.} \quad a = \frac{1}{5}, b = 2\right]$

43. If $f(x) = \frac{1}{2}e^{-|x|}$, find MGF of X and, hence, find its mean and variance.

$\left[\textbf{Ans.} \quad \frac{1}{1-t^2}, \text{mean} = 0, \text{Var} = 2\right]$

44. If the MGF of a random variable X is $\frac{2}{2-t}$, find the standard deviation of X.

$\left[\textbf{Ans.} \quad \frac{1}{2}\right]$

45. Find the MGF of a random variable X whose density function is

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

[**Ans.** 1]

46. If $\bar{X}$ is the mean of n independent identically distributed variables, then show that

$$M_{\bar{X}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n.$$

47. (i) If $Y = aX + b$, show that $\sigma_y = a\sigma_x$, and
(ii) If $Y = (X - \mu_X)/\sigma_X$, find σ_X and σ_Y.

[**Ans.** (ii) 0, 1]

48. Show that

(i) $E(aX + bY) = aE(X) + bE(Y)$, and
(ii) $\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2abC(X, Y)$, where $C(X, Y)$ is the covariance of (X, Y).

49. If two random variables are uncorrelated, prove that the variance of their sum is equal to the sum of their variances.

50. Let X be a random variable with mean value = 3 and variance = 2. Find the second moment of X about the origin. Another random variable Y is defined by $Y = -6x + 22$. Find the mean value of Y.

[Ans. 11, 4]

51. If the continuous random variable X has the density function $f(x) = 2xe^{-\frac{x^2}{2}}$, $x \geq 0$, and if $Y = X^3$, find the mean and variance Y.

[Ans. 1, 1]

3

Standard Distributions

This chapter is devoted to the study of theoretical discrete and continuous distributions in which variables are distributed according to some definite probability law and can be expressed mathematically. The present study will also enable us to fit a mathematical model or a function of the form $y = p(x)$ to the observed data. While constructing probabilistic models for observable phenomena, these distributions play important roles in many engineering applications.

Certain probability distributions arise more frequently than do others. In this chapter, we will deal such distributions as special discrete distributions like binomial, Poisson, geometrical and negative binomial distributions, and continuous distributions like uniform, exponential, Gamma, Weibull and normal distributions.

3.1 DISCRETE DISTRIBUTIONS

3.1.1 Bernoulli Trials and Bernoulli Distribution

Let A be an event (trial) associated with a random experiment such that $P(A)$ remains the same for the repetitions of that random experiment, then the events (trials) are called *Bernoulli trials*.

A random variable X which takes only two values either 1 (success) or 0 (failure), with probability p and q respectively, i.e. $P(X = 1) = p$, $P(X = 0) = q$, $p + q = 1$ is called *Bernoulli variate* and is said to have a *Bernoulli distribution*.

Note: Sometime the two values 1 and 0 are taken as 1 and -1 respectively.

Moments of Bernoulli Distribution

$$\mu_r' = E(X^r) = 1^r \cdot p + 0^r \cdot q = p$$

$$\mu_1' = E(X) = p, \quad \mu_2' = E(X^2) = p$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = p - p^2 = p(1-p) = pq$$

$\therefore$ $$\text{Mean} = p$$

$\therefore$ and $$\text{Var} = pq$$

3.1.2 Binomial Distribution

Binomial distribution was discovered by James Bernoulli in the year 1700.

Let a random experiment be performed repeatedly and let the occurrence of an event A in any trial be called a success and its non-occurrence $\bar{A}$, a failure (Bernoulli trial). Consider a series of n independent Bernoullian trials (n being finite) in which the probability of success $P(A) = p$ or failure $P(\bar{A}) = 1 - p = q$ in any trial is constant for each trial.

Let the random variable X denote the number of successes. The x successes in n trials can occur in nC_x ways and the probability for each of these ways is $p^x q^{n-x}$. Hence the probability of x successes [$(n-x)$ failures] in n trials in any order whatsoever is given by

$$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, ..., n$$

Since the probabilities of 0, 1, 2, 3, ..., n successes, viz. q^n, $nC_1 q^{n-1} p$, $nC_2 q^{n-2} p^2$, ..., p^n are the successive terms of the binomial expansion $(q + p)^n$, the probability distribution so obtained is called the *binomial probability distribution.*

A random variable X is said to follow binomial distribution denoted by $B(n, p)$ if it assumes only non-negative values and its probability mass function is given by

$$p(x) = P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, ..., n, \; q = 1 - p$$
$$= 0, \text{ otherwise.}$$

where n and p are known as the *parameters.*

Note:

(i) n is also sometimes, known as the degree of the distribution.

(ii) $\sum_{x=0}^{n} nC_x p^x q^{n-x} = (q + p)^n = 1$

(iii) Binomial distribution is important not only because of its wide applicability, but also because it gives rise to many other probability distributions.

(iv) Any variable which follows binominal distribution is known as binomial variate.

3.1.3 Binomial Frequency Distribution

Suppose that n trials constitute an experiment and if this experiment is repeated N times, the frequency function of the binomial distribution is given by

$$Np(x) = N \times nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

Properties of Binomial Frequency Distribution

A binomial experiment must possess the following properties:

1. Each trial results in two mutually disjoint outcomes, termed success and failure.
2. The trials must be independent of each other.
3. All trials have same constant probability of success.
4. The number of trials n is finite.

Note: Let p denote the probability of success.

$$P(\text{all successes}) = p^n$$

and

$$P(\text{all failures}) = q^n$$

$$P(\text{at least one success}) = 1 - q^n$$

$$P(\text{at least one failure}) = 1 - p^n$$

Mean of Binomial Distribution*

The PMF of the binomial distribution is

$$\text{Mean} = E(X) = \sum_x xp(x)$$

$$= \sum_{x=0}^{n} x \cdot nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1)!p \cdot p^{x-1} q^{n-x}}{(x-1)!(n-x)!}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} (n-1)C_{x-1} p^{x-1} q^{n-x}$$

$$= np\,(q + p)^{n-1}$$

$$\therefore \quad \text{Mean} = np$$

Variance of Binomial Distribution

The PMF of a binomial distribution is,

$$P(X = x) = p(x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, ..., n$$

By definition,

$$\text{Var}(X) = E(X^2) - [E(X)]^2 \tag{3.1}$$

$$E(X^2) = \sum_{x=0}^{n} x^2 p(x)$$

$$= \sum_{x=0}^{n} x^2 nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1) + x] \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1)(n-2)!}{(x-2)!(n-x)!} p^2 \cdot p^{x-2} q^{n-x} + E(X)$$

$$= n(n-1)p^2 \sum_{x=0}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 (q + p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

Substituting in Eq. (3.1), we get

$$\text{Var}(X) = n(n-1)p^2 + np - (np)^2 = n^2p^2 - np^2 + np - n^2p^2$$

$$\text{Var}(X) = np - np^2 = np\,(1 - p) = npq$$

Note:

(i) Similarly, it can be shown that $\mu_3 = npq(q - p)$

$$\mu_4 = npq(1 - 6pq + 3npq)$$

(ii) Mean of the binomial distribution is always greater than its variance.

Moment Generating Function of Binomial Distribution

EXAMPLE 3.1 Find the moment generating function of a binomial distribution and hence, find the mean and variance.

[AU December '03; '07, May '04; '05, June '06, '07]

Solution The probability mass function of a binomial distribution is

$$P(X = x) = nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

where n is the number of independent trials and x is the number of successes.

By definition of the moment generating function,

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{x=0}^{n} e^{tx} nc_x p^x q^{n-x} \\ &= \sum_{x=0}^{n} nc_x (pe^t)^x q^{n-x} \\ &= q^n + nC_1(pe^t)^1 q^{n-1} + nC_2(pe^t)^2 q^{n-2} + \cdots + (pe^t)^n \\ &= (q + pe^t)^n \end{aligned}$$

To find mean:

$$\begin{aligned} E(X) = \mu_1' &= \left[\frac{d}{dt} M_X(t)\right]_{t=0} \\ &= \left[\frac{d}{dt}(q + pe^t)^n\right]_{t=0} \\ &= [n(q + pe^t)^{n-1} pe^t]_{t=0} \\ &= np(p + q)^{n-1} = np \qquad [\because p + q = 1] \end{aligned}$$

$$\begin{aligned} E(X^2) = \mu_2' &= \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} \\ &= \left[\frac{d^2}{dt^2}(q + pe^t)^n\right]_{t=0} \\ &= [npe^t(q + pe^t)^{n-1} + n(n-1)(q + pe^t)^{n-2}(pe^t)^2]_{t=0} \\ &= np(q + p)^{n-1} + n(n-1)\, p^2(q + p)^{n-2} \\ &= np + n(n-1)p^2 = np + (n^2 - n)p^2 \end{aligned}$$

$$\begin{aligned} \text{Var} &= E(X^2) - [E(X)]^2 \\ &= np + n^2p^2 - np^2 - (np)^2 \\ &= np - np^2 = np(1 - p) \\ &= npq \qquad [\because p + q = 1] \end{aligned}$$

Moment Generating Function of Binomial Distribution about Mean (np)

[AU November '03; '07; '08]

By definition,

$$M_{X-np}(t) = E[e^{t(X-np)}] = E(e^{tX}e^{-npt}) = e^{-npt}E(e^{tX})$$
$$M_X(t) = e^{-npt}(q + pe^t)^n$$

Additive or Reproductive Property of Binomial Distribution

If X_1 and X_2 are two independent binomial variates with parameters (n_1, p) and (n_2, p) respectively, then $X_1 + X_2$ is a binomial variate with parameter $(n_1 + n_2, p)$.

Proof The MGF of the random variable $X_1 + X_2$ is

$$M_{X_1+X_2}(t) = M_{X_1}(t)\,M_{X_2}(t) \qquad (\because X_1 \text{ and } X_2 \text{ are independent})$$
$$= (q + pe^t)^{n_1}(q + pe^t)^{n_2}$$
$$= (q + pe^t)^{n_1+n_2}$$

This shows that $X_1 + X_2$ is also a binomial variate with parameters $n_1 + n_2$ and p.

Note: If X_1 and X_2 are two independent binomial variates with parameter (n_1, p_1) and (n_2, p_2), then $X_1 + X_2$ is not a binomial variate.

Generalization

If X_i, $(i = 1, 2, ..., k)$ are independent binomial variates with parameters (n_i, p), $(i = 1, 2, ..., k)$, then $X_1 + X_2 + ... + X_k$ is a binomial variate with parameters $(n_1 + n_2 + ... + n_k, p)$.

Recurrence Formula for Central Moments of Binomial Distribution

By definition, the kth order central moment μ_k is given by

$$\mu_k = E[X - E(X)]^k = \sum_{x=0}^{n}(x-np)^k nC_x p^x q^{n-x} \tag{3.2}$$

Differentiating Eq. (3.2) with respect to p, we get

$$\frac{d\mu_k}{dp} = \sum_{x=0}^{n} nC_x\{[-nk(x-np)^{k-1}p^x q^{n-x}] + (x-np)^k[xp^{x-1}q^{n-x} + p^x(n-x)q^{n-x-1}(-1)]\}$$
$$= -nk\mu_{k-1} + \sum_{x=0}^{n} nC_x(x-np)^k p^{x-1}q^{n-x-1}[xq - (n-x)p]$$

$$= -nk\mu_{k-1} + \frac{1}{pq}\sum_{x=0}^{n} nC_x p^x q^{n-x}(x-np)^{k+1} \qquad [\because p+q=1]$$

$$= -nk\mu_{k-1} + \frac{1}{pq}\mu_{k+1}$$

$$\therefore \qquad \mu_{k+1} = pq\left(\frac{d\mu_k}{dp} + nk\mu_{k-1}\right) \qquad (3.3)$$

Using recurrence relation (3.3), we can compute moments of higher order, provided the moments of lower order are known.

Putting $k = 1$ in Eq. (3.3)

$$\mu_2 = pq\left(\frac{d\mu_1}{dp} + n\mu_0\right) = npq \qquad (\mu_0 = 1 \text{ and } \mu_1 = 0)$$

Putting $k = 2$ in Eq. (3.3)

$$\mu_3 = pq\left(\frac{d\mu_2}{dp} + 2n\mu_1\right) = pq\frac{d}{dp}[np(1-p)] = npq(1-2p)$$

$$\mu_3 = npq\ (q - p)$$

Putting $k = 3$ in Eq. (3.3) and simplifying, we get

$$\mu_4 = npq[1 + 3pq(n-2)]$$

EXAMPLE 3.2 The mean and variance of a binomial distribution are 4 and 3 respectively. Find $P(X = 0)$, $P(X = 1)$, and $P(X \geq 2)$. **[AU April '04]**

Solution Mean of binomial distribution = $np = 4$

Variance of binomial distribution = $npq = 3$

$$\frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}$$

But, $p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

$$\text{Mean} = np = 4 \Rightarrow n = \frac{4}{p} = \frac{4}{\frac{1}{4}}$$

$$\therefore \qquad n = 16$$

$$P(X = x) = 16C_x p^x q^{n-x} = 16C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{16-x}$$

$$P(X = 0) = 16C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{16} = 1 \times 1 \times 0.01 = 0.01$$

$$P(X = 1) = 16C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{15} = 16 \times \frac{1}{4} \times \left(\frac{3}{4}\right)^{15} = 0.053$$

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - [P(X = 0) + P(X = 1)] \\ &= 1 - [0.01 + 0.053] = 0.93 \end{aligned}$$

EXAMPLE 3.3 The mean and variance of a binomial distribution are 16 and 8. Find $P(X \geq 3)$. **[AU April '04]**

Solution Mean of binomial distribution = $np = 16$

Variance of binomial distribution = $npq = 8$

$$np = 16$$

and $$npq = 8 \Rightarrow \frac{npq}{np} = \frac{8}{16}$$

$\Rightarrow$ $$q = \frac{1}{2}, \quad p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$$

$$np = 16 \Rightarrow \frac{n}{2} = 16 \Rightarrow n = 32$$

$\therefore$ $$P(X = x) = nC_x p^x q^{n-x} = 32C_x p^x q^{32-x}, \quad x = 0, 1, 2, ..., 32$$

$$P(X = 0) = 32C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{32} = 0.2328 \times 10^{-9}$$

$$P(X = 1) = 32C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{31} = 0.0074 \times 10^{-6}$$

$$P(X = 2) = 32C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{30} = 0.1155 \times 10^{-6}$$

$$\begin{aligned} P(X \geq 3) &= 1 - P(X \leq 3) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] = 0.9999 \end{aligned}$$

EXAMPLE 3.4 The mean and SD of a binomial distribution are 6 and 2 respectively. Determine the distribution.

Solution Given: mean = $np = 6$

$$\text{SD} = 2 \Rightarrow \text{Var} = (\text{SD})^2 = 4 = npq$$

$\therefore$ $$\frac{npq}{np} = \frac{4}{6} \Rightarrow q = \frac{2}{3} \Rightarrow p = 1 - q = \frac{1}{3}$$

$$np = 6 \Rightarrow \frac{n}{3} = 6 \Rightarrow n = 18$$

The binomial distribution is

$$P(x) = P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, 3, \ldots, n$$

$$= 18C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{18-x}, \quad x = 0, 1, 2, \ldots, 18$$

EXAMPLE 3.5 If the mean is 3 and variance is 4 of a random variable X, check whether X follows binomial distribution. [AU April '04]

Solution No. Because, for a binomial distribution mean should be greater than the variance.

Note: If Mean $= np = 3$

and Variance $= npq = 4$

Hence $\dfrac{npq}{np} = \dfrac{4}{3} \Rightarrow q = \dfrac{4}{3} > 1$ (but probability is always < 1)

$\therefore$ Mean should be greater than the varriance for a binomial distribution.

EXAMPLE 3.6 Four coins were tossed simultaneously. What is the probability of getting (i) 3 heads, (ii) at least 3 heads and (iii) at most 3 heads.

Solution When a coin is tossed,

$$P(\text{getting a head}) = \frac{1}{2}$$

$$P(\text{getting a tail}) = \frac{1}{2}$$

and

$$n = 4$$

If random variable X denotes the number of heads, then

$$p(x) = P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, 3, \ldots, n$$

$$= 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} = 4C_x \left(\frac{1}{2}\right)^4, \; x = 0, 1, 2, 3, 4$$

(i) The probability of getting 3 heads

$$P(X = 3) = 4C_3 \left(\frac{1}{2}\right)^4 = 4 \times \frac{1}{16} = \frac{1}{4}$$

(ii) The probability of getting at least 3 heads

$$P(X \geq 3) = P(X = 3) + P(X = 4)$$

$$= 4C_3 \left(\frac{1}{2}\right)^4 + 4C_4 \left(\frac{1}{2}\right)^4 = \frac{4}{16} + \frac{1}{16} = \frac{5}{16}$$

(iii) The probability of getting at most 3 heads

$$P(X \le 3) = 1 - P(X > 3)$$
$$= 1 - P(X = 4)$$
$$= 1 - 4C_4\left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16} = \frac{15}{16}$$

EXAMPLE 3.7 Find the probability that in tossing a fair coin 5 times, there will appear

(i) 3 heads,
(ii) 3 heads and 2 tails,
(iii) at least 1 head and
(iv) not more than 1 tail. **[AU December '03]**

Solution In tossing a coin,

$$p = \frac{1}{2}$$
$$q = \frac{1}{2},$$

and given that $n = 5$
If a random variable X denotes the number of heads, then

$$P(X = x) = nC_x p^x q^{n-x} = 5C_x\left(\frac{1}{2}\right)^5, \; x = 0, 1, 2, ..., 5$$

(i) P(getting 3 heads)

$$P(X = 3) = 5C_3\left(\frac{1}{2}\right)^5 = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} \times \left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

(ii) P(getting 3 heads and 2 tails)

$$P(\text{getting 3 heads}) = 5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} \times \left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

(iii) P(at least 1 head)

$$P(X \ge 1) = 1 - P(X < 1)$$
$$= 1 - P(X = 0) = 1 - 5C_0\left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}$$

(iv) P(not more than 1 tail)

$$= P(0 \text{ tail and } 5 \text{ heads}) + P(1 \text{ tail and } 4 \text{ heads})$$
$$= P(X = 5) + P(X = 4)$$
$$= 5C_5\left(\frac{1}{2}\right)^5 + 5C_4\left(\frac{1}{2}\right)^5 = \frac{1}{32}(1 + 5) = \frac{6}{32} = \frac{3}{16}$$

EXAMPLE 3.8 A pair of dice is thrown 4 times. If getting a doublet is considered as a success, find the probability of 2 successes.

Solution The possible doublets are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6)

The number of possible cases are 36 (pair of dice = $6^2 = 36$)

$$P(\text{getting doublets}) = p = \frac{6}{36} = \frac{1}{6}$$

$\therefore$

$$q = 1 - p = \frac{5}{6}$$

$\therefore$

and dice are thrown four times $n = 4$.

If a random variable X denotes the number of successes (i.e. getting doublets), then

$$P(X = x) = nC_x p^x q^{n-x} = 4C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{n-x}, \quad x = 0, 1, 2, ..., 4$$

The probability of 2 successes is

$$P(X = 2) = 4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{4-2} = \frac{4 \times 3}{1 \times 2} \times \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = \frac{25}{216}$$

EXAMPLE 3.9 Given X is a binomial variate such that $P(X = 0) = 1 - P(X = 1)$ and $E(X) = 3\ \text{Var}(X)$, find $P(X = 0)$.

Solution Given: mean = 3 × variance $\Rightarrow$ $np = 3npq$

$\therefore$

$$3q = 1 \Rightarrow q = \frac{1}{3}$$

$$p = 1 - q = \frac{2}{3}$$

$$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, ..., n$$

$$P(X = 0) = 1 - P(X = 1) \quad \Rightarrow \quad P(X = 0) + P(X = 1) = 1$$

$$\Rightarrow \quad nC_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{n-0} + nC_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{n-1} = 1$$

$$\left(\frac{2}{3}\right)^n + n \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = 1$$

$$\Rightarrow \quad \left(\frac{2}{3}\right)^n \left(1 + \frac{n}{2}\right) = 1$$

$\frac{2+n}{2} = \left(\frac{3}{2}\right)^n$ which will be true only when $\Rightarrow$ $n = 0$ or 1

$\therefore$ When $n = 1$,

$$P(X = 0) = 1C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{1-0} = \frac{1}{3}$$

EXAMPLE 3.10 If 6 dice are thrown simultaneously 720 times, getting 3 or 5 is considered as success. Find the number of times at least 3 dice to show 3 or 5. **[AU December '03, January '04]**

Solution First, we find the probability for getting 3 or 5 from one throw

$$p = \text{probability of getting 3 or 5}$$
$$p = P(\text{getting 3}) + P(\text{getting 5})$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$\therefore$ $$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

and $$n = 6$$

We know that

$$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, 3, ..., n$$
$$= 6C_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{6-x}, \quad x = 0, 1, 2, ..., 6$$

$\therefore$ $P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$ (i)

$$P(X = 0) = 6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^6 = 0.087$$

$$P(X = 1) = 6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5 = 0.26$$

$$P(X = 2) = 6C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = 0.329$$

Substituting in Eq. (i) gives

$$P(X \geq 3) = 1 - 0.677 = 0.323$$

Therefore, in 720 throws, the number of times to get 3 or 5 at least in 3 dice is

$$720 \times 0.323 = 233 \text{ times (approx.)}$$

EXAMPLE 3.11 Sixteen coins are thrown simultaneously. Find the probability of getting at least 7 heads. **[AU June '07]**

Solution Given: $n = 16$

and $$p = \text{probability of getting head} = \frac{1}{2}$$

$$q = 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore$

For a binomial distribution,

$$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, 3, ..., n$$

$$\Rightarrow P(X = x) = 16C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{16-x} = 16C_x \left(\frac{1}{2}\right)^{16}, \quad x = 0, 1, 2, ..., 16$$

$$\begin{aligned} P(X \geq 7) &= 1 - P(X < 7) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &\qquad + P(X = 4) + P(X = 5) + P(X = 6)] \\ &= 1 - \left(\frac{1}{2}\right)^{16} [16C_0 + 16C_1 + 16C_2 + 16C_3 + 16C_4 + 16C_5 + 16C_6] \\ &= 1 - \left(\frac{1}{2}\right)^{16} [1 + 16 + 120 + 560 + 1820 + 4368 + 8008] \\ &= 0.773 \end{aligned}$$

EXAMPLE 3.12 In 256 sets of 12 tosses of a fair coin, in how many cases may one expect 8 heads and 4 tails. **[AU December '03]**

Solution Let X be the random variable denoting the number of heads (remaining tails)

Given: $n = 12$

and p = probability of getting head $= \frac{1}{2}$

$\therefore$ $q = 1 - p = \frac{1}{2}$

The PMF of the binomial distribution is

$$\begin{aligned} P(X = x) &= nC_x p^x q^{n-x}, x = 0, 1, 2, ..., n \\ &= 12C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{12-x} = 12C_x \left(\frac{1}{2}\right)^{12}, x = 0, 1, 2, ..., 12 \end{aligned}$$

$$P(X = 8) = 12C_8 \left(\frac{1}{2}\right)^{12} = 0.12$$

Therefore, the number of times to get 8 heads and 4 tails appearing in 256 sets $= 256 \times 0.12 = 30.72 = 31$ times (approx.)

EXAMPLE 3.13 A die is thrown 8 times. What is the probability that 3 will show

(i) exactly 2 times,
(ii) at least 2 times, and
(iii) at most once.

Solution Given: $n = 8$

and p = probability of getting $3 = \frac{1}{6}$

$$\therefore \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Let X be a random variable denoting the number of times 3 is shown, then

$$P(X = x) = 8C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{8-x}, \ x = 0, 1, 2, \ldots, 8$$

(i) $P(X = 2) = 8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{8-2} = 0.26$

(ii) $P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$

$$= 1 - \left[8C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^8 + 8C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^7\right]$$

$$= 0.396$$

(iii) $P(X \leq 1) = P(X = 0) + P(X = 1)$

$$= 8C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^8 + 8C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^7$$

$$= 0.604$$

μ σ

EXAMPLE 3.14 If X follows B (3, 1/3) and Y follows B (5, 1/3), find $P(X + Y \geq 1)$. **[AU December '05]**

Solution Given: $p_1 = p_2 = \frac{1}{3}$

and
$$n_1 = 3,$$
$$n_2 = 5$$

Since $p_1 = p_2 = \frac{1}{3}$, $X + Y$ also follows binomial distribution, i.e.

$$X + Y \sim B\left(3 + 5, \frac{1}{3}\right)$$

$$= B\left(8, \frac{1}{3}\right)$$

$\therefore$

$$n = 8$$

$$p = \frac{1}{3}$$

$$q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(X+Y=x) = nC_x p^x q^{n-x} = 8C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{8-x}, \quad x = 0, 1, 2, ..., 8$$

$\therefore$
$$P(X+Y \geq 1) = 1 - P(X+Y=0)$$

$\therefore$
$$= 1 - \left(\frac{2}{3}\right)^8 = 0.961$$

EXAMPLE 3.15 If the MGF of a random variable X is of the form $(0.4e^t + 0.6)^8$, find the MGF of $3X + 2$.

Solution Given: The MGF of a binomial distribution is

$$M_X(t) = E(e^{tx}) = (q + pe^t)^n = (0.6 + 0.4e^t)^8$$

$\therefore$ X follows binomial distribution with $q = 0.6$, $p = 0.4$, $n = 8$
MGF of $3X + 2$ is given by

$$M_{3X+2}(t) = E[e^{(3X+2)t}] = E(e^{2t}e^{3Xt}) = e^{2t} \cdot E(e^{3Xt})$$
$$= e^{2t} E[e^{X(3t)}]$$
$$= e^{2t}(0.6 + 0.4e^{3t})^8$$

EXAMPLE 3.16 A discrete random variable X has moment generating function $\left(\frac{1}{4}+\frac{3}{4}e^t\right)^5$. Find $E(X)$, Var (X) and $P(X = 2)$.

Solution Given: $\left(\frac{1}{4}+\frac{3}{4}e^t\right)^5$.

Comparing it with $M_X(t) = (q + pe^t)^n$, we get

$$q = \frac{1}{4}$$

$$p = \frac{3}{4}$$

$$n = 5$$

$$E(X) = np = 5 \times \frac{3}{4} = \frac{15}{4}$$

$$\text{Var}(X) = npq = 5 \times \frac{3}{4} \times \frac{1}{4} = \frac{15}{16}$$

$$P(X = x) = nC_x p^x q^{n-x}, \; x = 0, 1, 2, ..., n$$

$$= 5C_2 \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{5-x}, x = 0, 1, 2, ..., 5$$

$$P(X = 2) = 5C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3$$

$$= 10 \times \frac{9}{16} \times \frac{1}{16 \times 4} = \frac{90}{1024} = \frac{45}{512} = 0.0879$$

EXAMPLE 3.17 A machine manufacturing bolts is known to produce 5% defective. In a random sample of 15 bolts, what is the probability that there are

(i) exactly 3 defective bolts and
(ii) not more than 3 defective bolts.

Solution Given: $p = 5\,\% = \frac{5}{100} = 0.05$

and $n = 15$

$\therefore$ $q = 1 - p = 1 - 0.05 = 0.95$

$\therefore$ $P(X = x) = nC_x p^x q^{n-x} = 15C_x (0.05)^x (0.95)^{15-x}$

(i) P(exactly 3 defective bolts)

$$P(X = 3) = 15\ C_3(0.05)^3\ (0.95)^{15-3}$$
$$= 455\ (0.05)^3(0.95)^{12}$$
$$= 0.0307$$

(ii) P(not more than 3 defective bolts)

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= 15C_0(0.05)^0\ (0.95)^{15} + 15C_1(0.05)^1\ (0.95)^{14} + 15C_2(0.05)^2\ (0.95)^{13} + 15C_3\ (0.05)^3\ (0.95)^{12}$$
$$= 0.4632 + 0.3657 + 0.1347 + 0.0307$$
$$= 0.994$$

EXAMPLE 3.18 The chance of an individual being a consumer of rice is 1/2. Assuming that there are 100 investigators, each takes 10 individuals to see whether they are consumers of rice, how many investigators would you expect to report that 3 people or less were consumer of rice?

Solution Given: $p = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 10$ and $N = 100$

Let X represents the number of consumers of rice then X follows binomial distribution, i.e.

$$X \sim B\left(10, \frac{1}{2}\right)$$

$\therefore$ $$P(X = x) = nC_x p^x q^{n-x} = 10C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} = 10C_x \left(\frac{1}{2}\right)^{10}$$

P(3 or less were consumers of rice)

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= 10C_0\left(\frac{1}{2}\right)^{10} + 10C_1\left(\frac{1}{2}\right)^{10} + 10C_2\left(\frac{1}{2}\right)^{10} + 10C_3\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10}(10C_0 + 10C_1 + 10C_2 + 10C_3) = \frac{176}{1024} = 0.1718$$

$\therefore$ For 100 investigators $= 100 \times 0.1718 = 17.18 = 17$ (approx.)

EXAMPLE 3.19 With the usual notation find p for a binomial random variate X if $n = 6$ and if $9P(X = 4) = P(X = 2)$.

Solution Given: X is a binomial random variate

and $n = 6$

$\therefore$
$$P(X = x) = nC_x\, p^x q^{n-x},\ x = 0, 1, 2, \ldots, n$$
$$= 6C_x\, p^x q^{6-x},\ x = 0, 1, 2, \ldots, 6$$

$$9P(X = 4) = P(X = 2) \Rightarrow 9(6C_4\, p^4 q^{6-4}) = 6C_2\, p^2 q^{6-2}$$

$$9\left[\frac{(6\times 5)\times(4\times 3)}{(1\times 2)\times(3\times 4)}\, p^4 q^2\right] = \left(\frac{6\times 5}{1\times 2}\right)p^2 q^4$$

$$9p^2 = q^2 = (1 - p)^2$$
$$9p^2 = (1 + p^2 - 2p)$$

$$8p^2 + 2p - 1 = 0 \Rightarrow p = \frac{-2 \pm \sqrt{4 + 32}}{16}$$

i.e. $p = -\dfrac{1}{2}$ or $p = \dfrac{1}{4}$

But, $p > 0$

$\therefore$
$$p = \frac{1}{4} = 0.25$$
$$q = 1 - 0.25 = 0.75$$

EXAMPLE 3.20 In a certain town, 20% samples of the population are literate. Assume that 200 investigators each take samples of 10 individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less are literates in the samples? **[AU December '03]**

Solution Given: P(a individual is literate) $= \dfrac{20}{100} = 0.2$

i.e. $p = 0.2$

$$q = 1 - p = 1 - 0.2 = 0.8$$

and $n = 10$ (sample size)

Let the random variable X denote the number of literates. Then

$$P(X = x) = nC_x p^x q^{n-x},\ x = 0, 1, 2, ..., n$$
$$= 10C_x(0.2)^x (0.8)^{10-x},\ x = 0, 1, ..., 10$$

P(an investigator reporting 3 or less as literates)

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= 10\ C_0\ (0.2)^0\ (0.8)^{10} + 10C_1\ (0.2)^1\ (0.8)^9 + 10C_2\ (0.2)^2\ (0.8)^8 + 10C_3\ (0.2)^3\ (0.8)^7$$
$$= (0.8)^7[(0.8)^3] + 10\ (0.2)^2\ (0.8)^2 + 45(0.2)^2\ (0.8) + 120(0.2)^3]$$
$$= 0.2097\ [0.512 + 1.28 + 1.44 + 0.96] = 0.8790$$

The number of investigators reporting 3 or less as literate out of 200 $= 200 \times 0.8790 = 175.8 = 176.$

EXAMPLE 3.21 It is known that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace? **[AU November '03]**

Solution Given:
$$p = 0.01$$
$$q = 1 - p = 1 - 0.01 = 0.99$$
and
$$n = 10$$

Let X be the random variable denoting the number of defective in a package. Then,

$$P(X = x) = nC_x p^x q^{n-x},\ x = 0, 1, 2, ..., n$$
$$= 10C_x(0.01)^x (0.99)^{10-x},\ x = 0, 1, 2, ..., 10$$

The company must replace the packages only when it has more than 1 defective screw.

$$P(\text{at most 1 screw is defective}) = P(X \le 1) = P(X = 0) + P(X = 1)$$
$$= 10C_0(0.1)^0\ (0.99)^{10} + 10C_1(0.1)^1\ (0.99)^9$$
$$= (0.99)^{10} + 10(0.1)\ (0.99)^9 = 0.96$$
$$P(X > 1) = 1 - P(X \le 1) = 1 - 0.96 = 0.04$$

$\therefore$ The proportion of packages sold to be replaced is 4%.

EXAMPLE 3.22 In a binomial distribution consisting of 5 independent trials with probability of 1 and 2 successes as 0.4096 and 0.2048 respectively, find the parameter p of the distribution.

Solution Given: $n = 5$, $P(X = 1) = 0.4096$ and $P(X = 2) = 0.2048$

We know that

$$P(X = x) = nC_x p^x q^{n-x},\ x = 0, 1, 2, ..., n$$

$$P(X = 1) = 5C_1 pq^4 = 5pq^4 = 0.4096 \quad \text{(i)}$$
$$P(X = 2) = 5C_2 p^2q^3 = 10p^2q^3 = 0.2048 \quad \text{(ii)}$$

Dividing Eq. (ii) by Eq. (i), we get

$$\frac{10p}{5q} = \frac{1}{2} \Rightarrow 20p = 5q \quad \Rightarrow \quad 20p = 5 - 5p \qquad (\because q = 1 - p)$$

$\Rightarrow$
$$25p = 5 \quad \Rightarrow \quad p = \frac{1}{5}$$

and
$$q = 1 - \frac{1}{5} = \frac{4}{5}$$

EXAMPLE 3.23 A variate takes values 0, 1, 2, ..., n with frequencies proportional to the binomial coefficients 1, nC_1, nC_2, ..., nC_n, find the mean and the variance of the distribution and show that the variance is half of the mean. **[AU November '07]**

Solution The total frequency $= 1 + nC_1 + nC_2 + \ldots + nC_n$
$= (1 + 1)^n = 2^n$

Given the variate takes values 0, 1, 2, ..., n with frequencies proportional to the binomial coefficients 1, nC_1, nC_2, ..., nC_n.

$\therefore$ Probabilities for 0, 1, 2, ..., n values of the variates are $\frac{1}{2^n}, \frac{nC_1}{2^n}, \frac{nC_2}{2^n}, \ldots, \frac{nC_n}{2^n}$ respectively.

The above are the terms of a binomial distribution $\left(\frac{1}{2} + \frac{1}{2}\right)^n$ where

$$p = \frac{1}{2}, \; q = \frac{1}{2}$$

$$\text{Mean} = np = \frac{n}{2}$$

$$\text{Var} = npq = n \times \frac{1}{2} \times \frac{1}{2} = \frac{n}{4} = \frac{\text{mean}}{2}$$

EXAMPLE 3.24 A perfect cubic die is thrown a large number of times in sets of 8, and getting a 5 or 6 is treated as a success. In what proportion of the sets can we expect 3 successes?

Solution Let X be a random variable denoting the number of times getting 5 or 6, i.e. success.

Given: $\quad p$ = probability of getting a 5 or 6 with six-faced die

$\therefore$
$$p = \frac{2}{6} = \frac{1}{3}, \; q = 1 - \frac{1}{3} = \frac{2}{3} \text{ and } n = 8$$

We know that

P(getting x successes)

$$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, ..., n$$

$$P(X = 3) = 8C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^5 = 0.2731$$

$\therefore$ The proportion (%) of sets we can expect 3 successes

$$= 0.2731 \times 100 = 27.31\%$$

EXAMPLE 3.25 In a large consignment of electric bulbs 10% are known to be defective. A random sample of 20 is taken for inspection. Find the probability that

(i) all are good bulbs,
(ii) at most 3 are defective bulbs and
(iii) exactly 3 are defective bulbs.

[AU December '07]

Solution Given:

$$p = 10\% = \frac{10}{100} = 0.1$$

$$q = 1 - p = 0.9$$

and

$$n = 20$$

If X is a random variable denoting the number of defective bulbs, then

$$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, ..., n$$
$$= 20C_x p^x q^{20-x}, x = 0, 1, 2, ..., 20$$

(i) P(all are good bulbs) = P(0 defective) = $P(X = 0)$

$$= 20C_0 (0.1)^0 (0.9)^{20} = 0.1216$$

(ii) P(at most 3 are defective bulbs) = $P(X = 0$ or 1 or 2 or 3$)$

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= 20C_0 (0.1)^0 (0.9)^{20} + 20C_1 (0.1)^1 (0.9)^{19} + 20C_2 (0.1)^2 (0.9)^{18} + 20C_3 (0.1)^3 (0.9)^{17}$$
$$= 0.1215 + 0.27 + 0.285 + 0.19$$
$$= 0.8666$$

(iii) P(exactly 3 defective bulbs)

$$P(X = 3) = 20C_3(0.1)^3 (0.9)^{17} = 0.19$$

EXAMPLE 3.26 An irregular six-faced die is thrown and the expectation that in 10 throws it will give 5 even numbers is twice the expectation that it will give 4 even numbers. How many times in 10000 sets of 10 throws each would you expect it to give no even number? **[AU June '03]**

Solution Let the random variable X denote the number of even numbers

P(getting x even number) $= P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, ..., n$

$$n = 10$$

Given: $$P(X = x) = 10C_x\, p^x\, q^{10-x},\ x = 0, 1, 2, \ldots, 10$$

$\therefore$
$$P(\text{getting 5 even numbers}) = 2P(\text{getting 4 even numbers})$$
$$P(X = 5) = 2\,P(X = 4)$$
$$10C_5\, p^5 q^5 = 2(10C_4\, p^4 q^6)$$
$$\frac{p}{5} = \frac{q}{3} \Rightarrow 3p = 5q = 5\,(1 - p)$$
$$8p = 5 \Rightarrow \frac{5}{8}$$

i.e.
$$q = 1 - p = 1 - \frac{5}{8} = \frac{3}{8}$$

$\therefore$

$\therefore$ $P(\text{getting } x \text{ even numbers}) = 10C_x\left(\frac{5}{8}\right)^x\left(\frac{3}{8}\right)^{10-x}, \qquad x = 0, 1, 2, \ldots, n$

$\therefore$ The required number of times that in 10000 sets of 10 throws each, we get no even number

$$= 10000 \times P(X = 0) = 10000 \times 10C_0\left(\frac{5}{8}\right)^0\left(\frac{3}{8}\right)^{10} = 0.5499 \text{ (approx.)}$$

EXAMPLE 3.27 A coin is tossed an infinite number of times. If the probability of a head in a single toss is p and $q = 1 - p$, find the probability that kth head is obtained at the nth tossing, but not earlier.

Solution Let X be a random variable denoting the number of heads.

Given: $$P(\text{getting a head}) = p$$
and $$q = 1 - p$$

If the kth head is obtained only at the nth toss, then $(k - 1)$ heads must be obtained in the first $(n - 1)$ tosses.

$\therefore$ The required probability $= P[(k - 1) \text{ heads in } (n - 1) \text{ tosses}] \times P[\text{head in the } k\text{th toss}]$

$$= (n - 1)C_{k-1}\, p^{k-1} q^{n-k} \times p$$
$$= (n-1)C_{k-1}\, p^k q^{n-k}$$

EXAMPLE 3.28 Each of two persons A_1 and A_2 tosses 3 fair coins. What is the probability that they obtain the same number of heads?

Solution $P(\text{getting a head in a single toss}) = \dfrac{1}{2} = p$

$\therefore$
$$q = 1 - p = \frac{1}{2}$$
$$n = 3$$

If the random variable X denotes the number of heads, then

$$P(X = x) = nC_x\, p^x q^{n-x},\ x = 0, 1, 2, \ldots, n$$

$$= 3C_x\left(\frac{1}{2}\right), \quad x = 0, 1, 2, 3$$

$P(A_1$ and A_2 get the same number of heads)

= P(both get no head) + P(both get 1 head each)

+ P(both get 2 heads each) + P(both get 3 heads each)

Since the events are mutually exclusive

= $P(A_1$ gets no head)$P(A_2$ gets no head)

+ $P(A_1$ gets 1 head) $P(A_2$ gets 1 head)

+ $P(A_1$ gets 2 heads) $P(A_2$ gets 2 heads)

+ $P(A_1$ gets 3 heads) $P(A_2$ gets 3 heads)

$$= \left[3C_0\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_1\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_2\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_3\left(\frac{1}{2}\right)^3\right]^2$$

$$= \frac{1}{64}(1+9+9+1) = \frac{5}{16}$$

EXAMPLE 3.29 If at least 1 child in a family with 2 children is a boy, what is the probability that both children are boys?

Solution The child may be a girl or boy.

$$P(\text{child is a boy}) = \frac{1}{2} = p$$

$$q = 1 - p = \frac{1}{2}$$

and

$$n = 2$$

If the random variable X denotes the number of boys, then

$$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

$$= 2C_x\left(\frac{1}{2}\right)^x, \quad x = 0, 1, 2$$

$$P(\text{at least 1 boy}) = P(X = 1 \text{ or } 2)$$

$$= 2C_1\left(\frac{1}{2}\right)^2 + 2C_2\left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$P(\text{both boys/at least 1 boy}) = P(X = 2/X = 1 \text{ or } 2)$$

$$= \frac{P(X = 2 \cap X = 1 \text{ or } 2)}{P(X = 1 \text{ or } 2)} = \frac{P(X = 2)}{P(X = 1) + P(X = 2)}$$

$$= \frac{2C_2\left(\frac{1}{2}\right)^2}{\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

EXAMPLE 3.30 A department has 10 working machines which may need adjustment from time to time during the day. Three of these machines are old, each having a probability of 1/11 of needing adjustment during the day and 7 are new, having corresponding probabilities of 1/21. Assuming that no machine needs adjustment twice on the same day, determine the probabilities that on a particular day

(i) just 2 old and no new machines need adjustment,

(ii) if just 2 machines need adjustment, they are of the same type.

Solution Let the random variable X_1 denote the number of old machines need adjustments and the random variable X_2 denote the number of new machines need adjustments.

Let p_1 = probability that an old machine needs adjustment.

$$\therefore \qquad p_1 = \frac{1}{11} \Rightarrow q_1 = 1 - p_1 = \frac{10}{11}$$

p_2 = probability that a new machine needs adjustment.

$$p_2 = \frac{1}{21} \Rightarrow q_2 = 1 - p_2 = \frac{20}{21}$$

There are 3 old machines, i.e. $n = 3$

$$\therefore \qquad P(X_1 = x) = nC_x\, p_1^x\, q_1^{n-x} = 3C_x \left(\frac{1}{11}\right)^x \left(\frac{10}{11}\right)^{3-x}, \quad x = 0, 1, 2, 3$$

There are 7 new machines, i.e. $n = 7$

$$\therefore \quad P(X_2 = x) = nC_x\, p_2^x\, q_2^{n-x} = 7C_x \left(\frac{1}{21}\right)^x \left(\frac{20}{21}\right)^{7-x}, \quad x = 0, 1, 2, 3, \ldots, 7$$

The random variables X_1 and X_2 are independent.

(i) The probability that just 2 old machines and no new machines need adjustment is given by

$$P(X_1 = 2 \cap X_2 = 0) = P(X_1 = 2)P(X_2 = 0)$$

$$= 3C_2 \left(\frac{1}{11}\right)^2 \left(\frac{10}{11}\right) \times 7C_0 \left(\frac{1}{21}\right)^0 \left(\frac{20}{21}\right)^7$$

$$= 0.016$$

(ii) If just 2 machines need adjustment and they are of the same type can happen in the following two mutually exclusive ways:

(a) 2 old and no new machine (or)

(b) 2 new and no old machines.

$\therefore$ The required probability

$$= P(X_1 = 2)\, P(X_2 = 0) + P(X_1 = 0)\, P(X_2 = 2)$$

$$= 0.016 + 3C_0\left(\frac{1}{11}\right)^0\left(\frac{10}{11}\right)^3 \times 7C_2\left(\frac{1}{21}\right)^2\left(\frac{20}{21}\right)^5$$

$$= 0.016 + 0.028 = 0.044$$

EXAMPLE 3.31 A and B play a game in which their chances of winning are in the ratio 3:2. Find A's chance of winning at least 3 games out of 5 games played.

Solution Let X be a random variable denoting the number of games A wins.

$$P(A \text{ winning}) = \frac{3}{5}$$

$$P(B \text{ winning}) = \frac{2}{5}$$

Here,

$$n = 5,\ p = \frac{3}{5},\ q = \frac{2}{5}$$

$$P(X = x) = nC_x\, p^x q^{n-x} = 5C_x\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{5-x} \quad x = 0, 1, 2, \ldots, 5$$

$$P(A \text{ winning at least 3 games out of 5}) = P(X \geq 3)$$

$$= P(X = 3) + P(X = 4) + P(X = 5)$$

$$= 5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 + 5C_4\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) + 5C_5\left(\frac{3}{5}\right)^5$$

$$= 0.6823$$

EXAMPLE 3.32 In a precision bombing attack, there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance of completely destroying the target?

Solution Let X be the random variable denoting the number of attacks. Since 2 direct hits are required to destroy the target, at least 2 should hit to destroy the target completely = $P(X \geq 2)$

$$P(X = x) = nC_x\, p^x q^{n-x},\ x = 0, 1, 2, \ldots, n$$

Given: $p = q = \dfrac{1}{2}$ (i.e. 50% chance)

$$P(X = x) = nC_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

$$= nC_x\left(\frac{1}{2}\right)^n \quad x = 0, 1, 2, \ldots, n$$

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - P(X = 0) - P(X = 1) \geq 0.99 \text{ (given)}$$

$$1-\left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n \geq 0.99$$

$$\Rightarrow \quad 1-\frac{n+1}{2^n} \geq 1-0.01 \quad \Rightarrow \quad \frac{n+1}{2^n} \leq 0.01 \Rightarrow 100(n+1) \leq 2^n$$

$\Rightarrow$

By trial we find the minimum value of n which satisfies the condition $100(n + 1) \geq 2^n$

When $n = 0$, $100 \not\leq 2$ (or) $100 > 2$

When $n = 1$, $200 \not\leq 2^2$, i.e. $200 \not\leq 4$ (or) $200 > 4$

$300 > 2^3$, $400 > 2^4$ ...

When $n = 11$, $1200 \leq 2^{11}$

By trial, we get the least value of n as 11.

$\therefore$ 11 bombs should be dropped.

EXAMPLE 3.33 The possibility of a man hitting a target is 1/4.

(i) If he fires 7 times, what is the probability of hitting the target twice?

(ii) How many times must he fire so that the probability of hitting the target at least once is greater than 2/3?

Solution Let X be a random variable which denotes the number of hits.

Given: $\quad p = \frac{1}{4}$

and $\quad q = 1 - p = \frac{3}{4}$

$$\therefore \quad P(X = x) = nC_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{n-x}, \quad x = 0, 1, 2, ..., n \qquad ...(i)$$

(i) Given: $n = 7$

$$\therefore \quad P(X = x) = 7C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{n-x}, \quad x = 0, 1, 2, ..., 7$$

$$\begin{aligned} P(\text{hitting at least twice}) &= P(X \geq 2) \\ &= 1 - P(X \leq 1) = 1 - P(X = 0) - P(X = 1) \\ &= 1-\left(\frac{3}{4}\right)^7 - 7\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6 = 0.5551 \end{aligned}$$

(ii)

$$\begin{aligned} P(\text{hitting at least once}) &= P(X \geq 1) \\ &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1-\left(\frac{3}{4}\right)^n \geq \frac{2}{3} \text{ (given)}, \quad \text{using (i)} \end{aligned}$$

$$\Rightarrow \qquad 1-\frac{2}{3} \geq \left(\frac{3}{4}\right)^n \quad \Rightarrow \quad \frac{1}{3} \geq \left(\frac{3}{4}\right)^n$$

$\therefore$ By trial, when $n = 4$, this condition is satisfied. Therefore, he must fire 4 times.

EXAMPLE 3.34 If the probability of success is 1/100, how many trials are necessary in order that probability of at least one success is greater than 1/2?

Solution Let the random variable X denotes the number of successes,

Given:
$$p = \frac{1}{100}$$

and
$$q = 1 - p = \frac{99}{100}$$

$$\therefore \qquad P(X = x) = nC_x \left(\frac{1}{100}\right)^x \left(\frac{99}{100}\right)^{n-x}, \; x = 0, 1, 2, \ldots, n$$

$$P(X = 0) = \left(\frac{99}{100}\right)^n = (0.99)^n$$

$$\begin{aligned} P(\text{hitting at least once}) &= P(X \geq 1) \\ &= 1 - P(X < 1) \\ &= 1 - P(X = 0) > \frac{1}{2} \text{ (given)} \\ &= 1 - (0.99)^n > \frac{1}{2} \end{aligned}$$

$$\Rightarrow \qquad 1 - \frac{1}{2} > (0.99)^n, \text{ i.e. } \frac{1}{2} > (0.99)^n$$

$$\Rightarrow \qquad \log(0.5) > n \log(0.99) \quad \Rightarrow \quad \frac{0.3010}{0.0044} < n$$

$$\Rightarrow \qquad 68.4 < n$$

$$\therefore \qquad n = 69$$

The least value of n is 69.

EXAMPLE 3.35 A system consists of n components, each of which will independently operate with probability p. The total system will be able to operate effectively if at least one-half of its components operates. For what values of p is a 5-component system more likely to operate effectively than a 3-component system.

Solution Let the random variable X denote the number of components operating effectively. If there are n components, then

$$P(X = x) = nC_x \, p^x q^{n-x}, \; x = 0, 1, 2, \ldots, n$$

The 5-component system operates effectively if one-half of its components operate, i.e. 3, 4, or 5 components operate.

$\therefore$ P(5-component system operates effectively)

$$= P(X \geq 3)$$
$$= P(X = 3) + P(X = 4) + P(X = 5)$$
$$= 5C_3 p^3 q^2 + 5C_4 p^4 q + 5C_5 p^5 \ (\because n = 5)$$

$\therefore$ P(3-component system operates effectively)

$$= P(X \geq 2)$$
$$= P(X = 2) + P(X = 3)$$
$$= 3C_2 p^2 q + 3C_3 p^3 \ (\because n = 3)$$

A 5-component system will function more effectively than a 3-component system only if the probability is more.

$\therefore$ P(5-component system operates effectively) $\geq P$(3-component system operates effectively)

$$5C_3 p^3 q^2 + 5C_4 p^4 q + 5C_5 p^5 \geq 3C_2 p^2 q + 3C_3 p^3$$
$$10p^3 q^2 + 5p^4 q + p^5 \geq 3p^2 q + p^3$$
$$10p^3(1-p)^2 + 5p^4(1-p) + p^5 \geq 3p^2(1-p) + p^3$$
$$10p(1 - 2p + p^2) + 5p^2 - 5p^3 + p^3 \geq 3 - 2p$$
$$3(2p^3 - 5p^2 + 4p - 1) \geq 0$$
$$\Rightarrow \quad 2p^3 - 5p^2 + 4p - 1 \geq 0$$
$$(p-1)^2 (2p-1) \geq 0 \ \Rightarrow \ 2p - 1 \geq 0$$

[Since $(p-1)^2 \geq 0$]

$$\Rightarrow \quad p \geq \frac{1}{2}$$

EXAMPLE 3.36 If the probability that a child is a boy is p, where $0 < p < 1$, find the expected number of boys in a family with n children, given that there is at least one boy.

Solution Let the random variable X denote the number of boys (success) out of n children.

$\therefore$ X follows binomial distribution.

$$P(X = x) = nC_x p^x q^{n-x}, \ x = 0, 1, 2, \ldots, n$$

We have to find

$$E(X/X \geq 1) = \sum_{x=1}^{n} xP(X = x / X \geq 1)$$
$$= \sum_{x=1}^{n} x \frac{P(X = x \cap X \geq 1)}{P(X \geq 1)}$$
$$= \sum_{x=1}^{n} x \frac{P(X = x)}{P(X \geq 1)}$$

$$= \sum_{x=1}^{n} \frac{x \cdot nC_x p^x q^{n-x}}{1 - P(X=0)}$$

$$= \sum_{x=1}^{n} \frac{x \times nC_x p^x q^{n-x}}{1 - q^n}$$

$$= \frac{1}{1-q^n} \sum_{x=1}^{n} x \times nC_x p^x q^{n-x}$$

$$= \frac{1}{1-q^n} E(X)$$

$$= \frac{1}{1-q^n}(np) = \frac{np}{1-q^n}$$

EXAMPLE 3.37 Two dice are thrown 100 times. Find the average number of times in which the number on the first die exceeds the number on the second die.

Solution The number of favourable cases are

(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)

The total number of possible outcomes when 2 dices are thrown = $6^2 = 36$

P(the number on the first die exceeds the number on the second die)

$$= \frac{15}{36} = \frac{5}{12}$$

$$\therefore \quad p = \frac{5}{12},\ q = 1 - p = \frac{7}{12}$$

$$n = 100$$

If the random variable X denotes the number of successes (the number on the first die is more than the second die), then X follows binomial distribution with mean np.

$\therefore$ The average (mean) number of times the success occurs

$$= E(X) = np = 100 \times \frac{5}{12} = 41.67 \approx 42 \text{ times}$$

EXAMPLE 3.38 A coin with probability of getting head $p = 1 - q$ is tossed n times. Find the probability that the number of heads obtained is even.

Solution Given: $P(\text{head}) = p$

$\therefore$ $P(\text{tail}) = 1 - p = q$ and the coin is tossed n times

P(even number of heads obtained)

$= P$(0 head or 2 heads or 4 heads or ...)

$$= P(0 \text{ head}) + P(2 \text{ heads}) + P(4 \text{ heads}) + \dots$$
$$= nC_0 p^0 q^n + nC_2 p^2 q^{n-2} + nC_4 p^4 q^{n-4} + \dots \quad \text{(i)}$$

We know that

$$(p+q)^n = (q+p)^n = nC_0 p^0 q^n + nC_1 pq^{n-1} + nC_2 p^2 q^{n-2} + nC_3 p^3 q^{n-3} + nC_4 p^4 q^{n-4} + \dots$$
$$(q-p)^n = nC_0 p^0 q^n - nC_1 pq^{n-1} + nC_2 p^2 q^{n-2} - nC_3 p^3 q^{n-3} + \dots$$
$$(q+p)^n + (q-p)^n = 2nC_0 p^0 q^n + 2nC_2 p^2 q^{n-2} + 2nC_4 p^4 q^{n-4} + \dots$$
$$= 2[nC_0 p^0 q^n + nC_2 p^2 q^{n-2} + nC_4 p^4 q^{n-4} + \dots]$$
$$= 2P(\text{even number of heads obtained}), \quad \text{[from Eq. (i)]}$$

$$\therefore \quad P(\text{even number of heads obtained}) = \frac{1}{2}[(q+p)^n + (q-p)^n]$$
$$= \frac{1}{2}[1 + (q-p)^n]$$

EXAMPLE 3.39 If X is binomially distributed random variable with parameters n and p, what is the distribution of $Y = n - X$?

Solution Since X follows binomial distribution

$$M_X(t) = E(e^{tX}) = (q + pe^t)^n$$
$$\therefore \quad M_Y(t) = E(e^{tY}) = E(e^{t(n-X)})$$
$$= e^{nt}E(e^{-Xt}) = e^{nt}E(e^{X(-t)})$$
$$= e^{nt}(q + pe^{-t})^n = (qe^t + p)^n$$
$$= (p + qe^t)^n$$

$\therefore$ By uniqueness theorem of MGF, $Y = n - X$ follows binomial distribution $B(n, q)$.

i.e., $\quad Y \sim B(n, q) \quad \Rightarrow \quad n - X \sim B(n, q)$

EXAMPLE 3.40 If the MGF of a random variable X is $\left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$ show that

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} 9C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}.$$

Solution Given: MGF $M_X(t) = (q + pe^t)^n = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$

$$\Rightarrow \quad q = \frac{2}{3},\ p = \frac{1}{3},\ n = 9$$

$\therefore$ X follows binomial distribution $\sim B\left(9, \frac{1}{3}\right)$

$$\therefore \quad E(X) = np = 9 \times \left(\frac{1}{3}\right) = 3 = \mu$$

$$\text{Var}(X) = npq = 9\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 2 = \sigma^2$$

$$\therefore \quad \sigma = \sqrt{\text{Var}(X)} = \sqrt{2}$$

$$\mu \pm 2\sigma = 3 \pm 2\sqrt{2} = 0.2 \text{ or } 5.8$$

$$\therefore \quad P(\mu - 2\sigma < X < \mu + 2\sigma) = P(0.2 < X < 5.8)$$

$$= P(1 \le X \le 5)$$

$$= \sum_{x-1}^{5} nC_x p^x q^{n-x}$$

$$= \sum_{x-1}^{5} 9C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

EXAMPLE 3.41 Five fair coins are flipped. If the outcomes are independent, find the probability mass function of the number of heads obtained.

Solution If random variable X denotes the number of heads (successes), then

$$P(X = x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

In tossing a coin,

$$p = \frac{1}{2},\ q = \frac{1}{2} \text{ and given that } n = 5$$

$$P(X = x) = 5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} = 5C_x \left(\frac{1}{2}\right)^5,\ x = 0, 1, 2, \ldots, 5$$

x	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

EXAMPLE 3.42 Ten coins are tossed 1024 times and the following frequencies are observed:

Number of heads	0	1	2	3	4	5	6	7	8	9	10
Frequencies	2	10	38	106	188	257	226	128	59	7	3

Compare these frequencies with the expected frequencies.

Solution Given:

$$n = 10$$

and $\quad N = \Sigma f = 1024$

p = probability of getting a head in one toss

$$p = \frac{1}{2}$$

$\therefore$
$$q = 1 - p = \frac{1}{2}$$

and

The expected frequency of x heads

$$= 1024 \times 10C_x \left(\frac{1}{2}\right)^{10-x} \left(\frac{1}{2}\right)^x, \quad x = 0, 1, 2, ..., 10$$

Hence we have the comparison

Observed frequencies	2	10	38	106	188	257	226	128	59	7	3
Expected frequencies	1	10	45	120	210	252	120	120	45	10	1

Generalization of Bernoulli Theorem/Multinomial Distribution

If $A_0, A_1, A_2, ..., A_k$ are exhaustive and mutually exclusive events associated with a random experiment such that $P(A_i) = p_i$, where $p_1 + p_2 + ... + p_k = 1$ and if the experiment is repeated n times, then the probability that A_1 occurs r_1 times, A_2 occurs r_2 times, ..., A_k occurs r_k times such that $r_1 + r_2 + ... + r_k = n$ is given by

$$P_n(r_1, r_2, r_3, ..., r_k) = \frac{n!}{r_1!r_2!\,r_3!...r_k!} p_1^{r_1} p_2^{r_2} p_3^{r_3}, ..., p_k^{r_k}$$

EXAMPLE 3.43 A fair die is rolled 5 times. Find the probability that 1 shown twice, 3 shown twice and 6 shown once.

Solution Given:

$$n = 5$$
$$A_1 = \text{getting } 1$$
$$A_2 = \text{getting } 3$$

and
$$A_3 = \text{getting } 6$$

$$P(A_1) = \frac{1}{6} = p_1,\ P(A_2) = \frac{1}{6} = p_2,\ P(A_3) = \frac{1}{6} = p_3$$

$r_1 = 2,\ r_2 = 2$ and $r_3 = 1,\quad r_1 + r_2 + r_3 = 5 = n$

Using the multinomial distribution, the required probability is

$$P_n(r_1, r_2, r_3) = \frac{n!}{r_1!r_2!\,r_3!} p_1^{r_1} p_2^{r_2} p_3^{r_3}$$

$$= \frac{5!}{2!2!1!}\left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right) = 0.0039$$

3.1.4 Poisson Distribution

If X is a discrete random variable that assumes only non-negative values such that its probability mass function is given by

$$P(X = x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, 3, \ldots; \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

then X is said to follow Poisson distribution, with the parameter λ.

Note:

(i) We can easily prove that the total probability is 1

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1$$

(ii) Poisson distribution occurs when there are events which do not occur as outcomes of a definite number of trials of an experiment but which occur at random points of time and space wherein our interest lies only in the number of occurrences of the event, not in its non-occurrences.

Poisson Distribution is a Limiting Case of Binomial Distribution

Poisson distribution is a limiting case of binomial distribution under the following conditions:

1. The number of trials n is indefinitely large, i.e. $n \to \infty$.
2. The probability of success p for each trial is very small, i.e. $p \to 0$.
3. $np = \lambda$ is finite or $p = \dfrac{\lambda}{n}$ and $q = 1 - \dfrac{\lambda}{n}$, where λ is a positive constant.
   In short, when $n \to \infty$ and p is very small.

Proof Let X be a binomially distributed random variable. Then probability mass function of a binomial distribution is

$$\begin{aligned} P(X = x) &= nc_x p^x q^{n-x},\ x = 0, 1, 2, \ldots, n \\ &= \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \\ &= \frac{1\cdot 2\cdot 3\ldots[n-(x+1)]\,(n-x)\,[n-(x-1)]\ldots(n-1)n}{1, 2, 3\ldots(n-x)x!} p^x (1-p)^{n-x} \end{aligned}$$

We know that, mean of the binomial distribution is np.

Let $np = \lambda \Rightarrow p = \dfrac{\lambda}{n}$ and $q = 1 - p = 1 - \dfrac{\lambda}{n}$

$$P(X=x) = \frac{n^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}}{x!}$$

Taking the limit as $n \to \infty$ in the above equation, we get

$$P(X=x) = \frac{\lambda^x \underset{n\to\infty}{\text{Lt}}\left(1-\frac{\lambda}{n}\right)^{n-x}}{x!} = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\left[\because \underset{n\to\infty}{\text{Lt}}\left(1-\frac{\lambda}{n}\right)^{n-x} = e^{-\lambda} \text{ and } \underset{n\to\infty}{\text{Lt}}\left(1-\frac{1}{n}\right) = 1 = \underset{n\to\infty}{\text{Lt}}\left(1-\frac{2}{n}\right) = \ldots\right]$$

Mean of Poisson Distribution

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} xP(X=x) = \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$\therefore$ Mean $\lambda e^{-\lambda} e^{\lambda} = \lambda$

Variance of Poisson Distribution

$$\sum_{x=0}^{\infty} x^2\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty}[x(x-1)+x]\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^2\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} + e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda^2 e^{-\lambda}e^{\lambda} + \lambda e^{-\lambda}e^{\lambda} = \lambda^2 + \lambda$$

$\therefore$ $\text{Var}(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Moment Generating Function of Poisson Distribution

EXAMPLE 3.44 Find the moment generating function of the Poisson distribution and hence, find the mean and variance.

[AU December '07, June '06; '07, April '08]

Solution The probability mass function of Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

The moment generating function of the Poisson distribution is

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

To find mean and varriance using MGF

$$\text{Mean} = E(X) = \mu_1' = \left[\frac{d}{dt}(M_X t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}(e^{-\lambda}e^{\lambda e^t})\right]_{t=0} = \left[e^{-\lambda}\cdot e^{\lambda e^t}\cdot \lambda e^t\right]_{t=0} = e^{-\lambda}\cdot e^{\lambda}\cdot\lambda = \lambda$$

$$\mu_2' = E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{d^2}{dt^2}(e^{-\lambda}e^{\lambda e^t})\right]_{t=0}$$

$$= [e^{-\lambda}e^{\lambda e^t}(\lambda e^t)^2 + e^{-\lambda}e^{\lambda e^t}\lambda e^t]_{t=0} = e^{-\lambda}e^{\lambda}\lambda^2 + e^{-\lambda}e^{\lambda}\lambda = \lambda^2 + \lambda$$

$$\text{Var} = E(X^2) - [E(X)]^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Note: For Poisson distribution,

Mean = variance = λ

Mean of the Poisson distribution = λ

Variance of the Poisson distribution = λ

Recurrence Formula for Central Moments of Poisson Distribution

Since $E(X) = \lambda$, we have $\mu_r = E[(X - \lambda)^r]$

$$= \sum_{x=0}^{\infty}(x-\lambda)^r\frac{\lambda^x e^{-\lambda}}{x!} \tag{3.4}$$

Differentiating Eq. (3.4) with respect to λ both sides,

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty}\frac{1}{x!}[x\lambda^{x-1}e^{-\lambda}(x-\lambda)^r + \lambda^x(-e^{-\lambda})(x-\lambda)^r - \lambda^x e^{-\lambda}r(x-\lambda)^{r-1}]$$

$$= \sum_{x=0}^{\infty}\frac{1}{x!}[\lambda^{x-1}e^{-\lambda}(x-\lambda)^r(x-\lambda) - \lambda^x e^{-\lambda}r(x-\lambda)^{r-1}]$$

$$= \sum_{x=0}^{\infty}\frac{1}{x!}\lambda^{x-1}e^{-\lambda}(x-\lambda)^r(x-\lambda) - \sum_{x=0}^{\infty}\frac{1}{x!}\lambda^x e^{-\lambda}r(x-\lambda)^{r-1}$$

$$\therefore \quad \lambda\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty}\frac{\lambda^x e^{-\lambda}}{x!}(x-\lambda)^{r+1} - \lambda r\sum_{x=0}^{\infty}\frac{1}{x!}\lambda^x e^{-\lambda}(x-\lambda)^{r-1}$$

$$\Rightarrow \quad \lambda\frac{d\mu_r}{d\lambda} = \mu_{r+1} - \lambda r\mu_{r-1}$$

Hence,

$$\mu_{r+1} = \lambda r\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda}$$

$$\therefore \quad \mu_{r+1} = \lambda\left(r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right) \tag{3.5}$$

Using Eq. (3.5), the central moments μ_2, μ_3 and μ_4 are obtained as follows:

$$\mu_0 = 1$$
$$\mu_1 = 0$$

for $r = 1$, $$\mu_2 = \lambda\left(\mu_0 + \frac{d\mu_1}{d\lambda}\right) = \lambda \qquad \left[\because \frac{d\mu_1}{d\lambda} = \frac{d}{d\lambda}(0) = 0\right]$$

for $r = 2$, $$\mu_3 = \lambda\left(2\mu_1 + \frac{d\mu_2}{d\lambda}\right) = \lambda \qquad \left[\because \frac{d\mu_2}{d\lambda} = \frac{d}{d\lambda}(\lambda) = 1\right]$$

for $r = 3$, $$\mu_4 = \lambda\left(3\mu_2 + \frac{d\mu_3}{d\lambda}\right) = \lambda(3\lambda + 1) = 3\lambda^2 + \lambda$$

$$\left[\because \frac{d\mu_3}{d\lambda} = \frac{d}{d\lambda}(\lambda) = 1\right]$$

Additive or Reproductive Property

Sum of independent Poisson variates is also a Poisson variate,
i.e. if $X_i (i = 1, 2, ..., n)$ are n independent Poisson variates with parameter $\lambda_i (i = 1, 2, ..., n)$, then $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n}\lambda_i$.

[AU December '03, June '06]

Proof We know that the MGF of the Poisson variate X_i is given by

$$M_{X_i}(t) = e^{\lambda_i(e^t - 1)}, \; i = 1, 2, ..., n$$

$$M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t)\cdot M_{X_2}(t)\dots M_{X_n}(t)$$
$$= e^{\lambda_1(e^t-1)}\, e^{\lambda_2(e^t-1)} \dots e^{\lambda_n(e^t-1)}$$
$$= e^{(\lambda_1+\lambda_2+\dots+\lambda_n)(e^t-1)}$$

which gives the MGF of a Poisson variate $X_1 + X_2 + X_3 + \dots + X_n$ with parameter $\lambda_1 + \lambda_2 + \dots + \lambda_n$.

Property

Sum of two independent Poisson random variables is also a Poisson random variable.

[AU June '06]

Proof Let X_1 and X_2 be two independent Poisson random variables with parameters λ_1 and λ_2 respectively.

Let $$X = X_1 + X_2$$

$\therefore$
$$\begin{aligned}
P(X = n) &= P(X_1 + X_2 = n) \\
&= \sum_{x=0}^{n} P(X_1 = x \cap X_2 = n - x) \\
&= \sum_{x=0}^{n} P(X_1 = x) P(X_2 = n - x) \\
&= \sum_{x=0}^{n} \frac{e^{-\lambda_1}\lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2}\lambda_2^{n-x}}{(n-x)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} \lambda_1^x \lambda_2^{n-x} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{x=0}^{n} nC_x \lambda_1^x \lambda_2^{n-x} \\
&= \frac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1 + \lambda_2)^n}{n!}
\end{aligned}$$

which is the PMF of a Poisson random variable $X_1 + X_2$ with parameter $(\lambda_1 + \lambda_2)$.

$\therefore$ Sum of two independent Poisson random variables is also a Poisson random variable.

EXAMPLE 3.45 Suppose the number of accidents occurring weekly on a particular stretch of a highway follow a Poisson distribution with mean 3. Calculate the probability that there is at least one accident this week.

Solution Given: $$\lambda = 3$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3}3^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$\begin{aligned}
P(X \geq 1) &= 1 - P(X < 1) = 1 - P(X = 0) \\
&= 1 - \frac{e^{-3}3^0}{0!} = 1 - e^{-3} = 0.9502
\end{aligned}$$

EXAMPLE 3.46 The PMF of a random variable X is given by

$$P(X = i) = \frac{C\lambda^i}{C!}, (i = 0, 1, 2, \ldots).$$

Find

(i) $P(X = 0)$ and

(ii) $P(X > 2)$. [AU November '0

$$P(X = i) = \frac{C\lambda^i}{C!}, (i = 0, 1, 2, \ldots)$$

Solution

We know that

$$\sum_{i=0}^{\infty} P(X = i) = 1 \Rightarrow \sum_{i=0}^{\infty} \frac{C\lambda^i}{i!} = 1$$

$$C\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right) = 1$$

$$Ce^{\lambda} = 1 \Rightarrow C = e^{-\lambda}$$

$$P(X = i) = P(i) = \frac{e^{-\lambda}\lambda^i}{i!}, i = 0, 1, 2, \ldots$$

$$P(X = 0) = e^{-\lambda}$$

$$\begin{aligned} P(X > 2) &= 1 - P(X \le 2) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\ &= 1 - \left(e^{-\lambda} + \frac{e^{-\lambda}\lambda}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}\right) \\ &= 1 - e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!}\right) \\ &= 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right) \end{aligned}$$

EXAMPLE 3.47 If X and Y are independent Poisson variables such that $P(X = 1) = P(X = 2)$, $P(Y = 2) = P(Y = 3)$, find the variance of $(X - 2Y)$.

[AU April/May '05]

Solution Given: X and Y are independent Poisson variables

$$\therefore \quad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2 \ldots$$

$$P(Y = y) = \frac{e^{-\lambda}\lambda^y}{y!}, y = 0, 1, 2 \ldots$$

Given: $P(X = 1) = P(X = 2)$

$$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!} \Rightarrow \frac{\lambda}{1} = \frac{\lambda^2}{2} \Rightarrow \lambda = 2$$

$\therefore$ The variance of the random variable X is $\lambda = 2$

$$\therefore \quad \text{Var}(X) = 2$$

Again, $P(Y = 2) = P(Y = 3)$

$$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}\lambda^3}{3!} \Rightarrow \frac{\lambda^2}{2} = \frac{\lambda^3}{6} \Rightarrow \lambda = 3$$

$\therefore$ The variance of the random variable Y is $\lambda = 3$

$$\text{Var}(Y) = 3$$

$\therefore$
$$\text{Var}(X - 2Y) = \text{Var}(X) + 4\,\text{Var}(Y)$$
$$= 2 + 4 \times 3 = 14$$

EXAMPLE 3.48 If X is a Poisson random variable with $P(X = 1) = P(X = 2)$, find $P(X \geq 3)$.

Solution Given: X is a Poisson random variable

$\therefore$ The PMF of X is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

$$P(X = 1) = \frac{e^{-\lambda}\lambda}{1!} \text{ and } P(X = 2) = \frac{e^{-\lambda}\lambda^2}{2!}$$

Given:
$$P\,(X = 1) = P(X = 2)$$

$$\frac{e^{-\lambda}\lambda}{1!} = \frac{e^{-\lambda}\lambda^2}{2!} \Rightarrow \lambda = 2$$

$\therefore$
$$P(X = x) = \frac{e^{-2}2^x}{x!}$$

$$P(X \geq 3) = 1 - P(X < 3) = 1[P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left(\frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!}\right)$$
$$= 1 - e^{-2}(1 + 2 + 2) = 0.323$$

EXAMPLE 3.49 • If the moment generating function of the random variable is $e^{4(e^t-1)}$ find $P(X = \mu + \sigma)$, where μ and σ^2 are the mean and variance of the Poisson random variable X. **[AU June '07]**

Solution We know that for a Poisson distribution, the moment generating function is

$$M_X(t) = e^{\lambda(e^t-1)} = e^{4(e^t-1)}$$

Therefore,

$$\text{Mean} = \text{variance} = \lambda = 4$$

$\therefore$
$$\sigma = \text{SD} = \sqrt{\text{Var}} = \sqrt{4} = 2$$

$$P(X=x)=e^{-\lambda}\frac{\lambda^x}{x!}=e^{-4}\frac{4^x}{x!},\quad x=0,1,2,\ldots$$

$$\therefore\quad P(X=\mu+\sigma)=P(X=6)=\frac{e^{-4}4^6}{6!}=0.1042$$

EXAMPLE 3.50 The number of monthly breakdown of a computer is a random variable having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month with

(i) only one breakdown,
(ii) at least one breakdown. **[AU June '07, November '09]**

Solution Mean of the Poisson distribution $=\lambda=1.8$
We know that the PMF of the Poisson distribution is

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}=\frac{e^{-1.8}(1.8)^x}{x!},\ x=0,1,2,\ldots$$

$$P(\text{only one breakdown})=P(X=1)=\frac{e^{-1.8}(1.8)^1}{1!}=0.2975$$

$$\begin{aligned}P(\text{at least one breakdown})&=P(X\geq 1)\\&=1-P(X=0)\\&=1-\frac{e^{-1.8}(1.8)^0}{0!}=1-e^{-1.8}=0.8347\end{aligned}$$

EXAMPLE 3.51 The atoms of a radioactive element are randomly disintegrating. If every gram of this element, on average, emits 3.9 alpha particles per second, what is the probability that during the next second the number of alpha particles emitted from 1 gram is

(i) at most 6,
(ii) at least 2,
(iii) at least 3 and at most 6. **[AU December '07]**

Solution Given: Mean $=\lambda=3.9$
We know that the PMF of the Poisson distribution is

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}=\frac{e^{-3.9}(3.9)^x}{x!},\ x=0,1,2,\ldots$$

(i) $$\begin{aligned}P(\text{at most }6)&=P(X\leq 6)\\&=P(X=0)+P(X=1)+P(X=2)+P(X=3)\\&\quad+P(X=4)+P(X=5)+P(X=6)\end{aligned}$$

$$= e^{-3.9}\left[\frac{(3.9)^0}{0!}+\frac{(3.9)^1}{1!}+\frac{(3.9)^2}{2!}+\frac{(3.9)^3}{3!}+\frac{(3.9)^4}{4!}+\frac{(3.9)^5}{5!}+\frac{(3.9)^6}{6!}\right]$$
$$=0.899$$

(ii) $P(\text{at least } 2) = P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$

$$= 1-e^{-3.9}\left[\frac{(3.9)^0}{0!}+\frac{(3.9)^1}{1!}\right]$$
$$= 0.901$$

(iii) $P(\text{at least 3 and at most } 6) = P(3 \leq X \leq 6)$

$$= P(X=3)+P(X=4)+P(X=5)+P(X=6)$$
$$= e^{-3.9}\left[\frac{(3.9)^3}{3!}+\frac{(3.9)^4}{4!}+\frac{(3.9)^5}{5!}+\frac{(3.9)^6}{6!}\right]= 0.646$$

EXAMPLE 3.52 A car hire firm has 2 cars. The number of demands for a car on each day is distributed as Poisson variate with mean 0.5. Calculate the proportion of days on which

(i) neither car is used,
(ii) some demand is refused.

Solution Given: Mean = λ = 0.5

Let the proportion of days on which there are x demands for a car is

$$P(X = x) = \frac{e^{-0.5}(0.5)^x}{x!}, x=0,1,2,...$$

(i) Proportion of days on which neither car is used:

$$P(\text{there is no demand}) = P(X = 0)= \frac{e^{-0.5}(0.5)^0}{0!}$$
$$= e^{-0.5} = 0.606, \text{ i.e. } 61\%$$

(ii) To find the proportion of days on which some demand is refused Since only 2 cars are there, the demand will be refused only when both cars are in use.

$$P(X > 2) = 1 - P(X \leq 2)$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1-e^{-0.5}\left(1+0.5+\frac{0.5^2}{2!}\right)$$
$$= 1 - (0.606 + 0.303 + 0.075) = 0.016, \text{ i.e. } 2\%$$

EXAMPLE 3.53 If X is a Poisson variate such that $P(X = 1) = 3/10$ and $P(X = 2) = 1/5$, find $P(X = 0)$ and $P(X = 3)$. **[AU November '98]**

Solution If X is a Poisson variate, then

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

Given:

$$P(X = 1) = \frac{3}{10} = e^{-\lambda}\lambda \quad \text{(i)}$$

$$P(X = 2) = \frac{1}{5} = \frac{e^{-\lambda}\lambda^2}{2} \quad \text{(ii)}$$

From Eqs. (i) and (ii),

$$\frac{3}{10} = e^{-\lambda}\lambda$$

$$= \frac{2}{5} = e^{-\lambda}\lambda^2$$

Dividing,

$$\frac{e^{-\lambda}\lambda^2}{e^{-\lambda}\lambda} = \frac{\frac{2}{5}}{\frac{3}{10}} \Rightarrow \lambda = \frac{4}{3}$$

$\therefore$

$$P(X = x) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^x}{x!}, x = 0, 1, 2, \ldots$$

$$P(X = 0) = e^{-\frac{4}{3}} = 0.2637$$

$$P(X = 3) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^3}{3!} = 0.1047$$

EXAMPLE 3.54 One percent of jobs arriving at a computer system need to wait until weekends for scheduling, owing to core-size limitations. Find the probability that among a sample of 200 jobs, there are no jobs that have to wait until weekends.

[AU June '07]

Solution Given:

and

$$p = 1\% = 0.01$$

$$n = 200$$

$$\text{Mean} = \lambda = n \times p = 200 \times 0.01 = 2$$

Let X denote the number of jobs that have to wait, then it follows Poisson distribution.

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

$\therefore$ The probability that no jobs to wait is

$$P(X = 0) = \frac{e^{-2}2^0}{0!} = e^{-2}$$

EXAMPLE 3.55 Write down the probability mass function of the Poisson distribution which is approximately equivalent to B (50, 0.03).

Solution Given: B(50, 0.03) is binomial distribution with $n = 50$, $p = 0.03$

$\therefore$ Mean of the Poisson distribution $= \lambda = np = 50 \times 0.03 = 1.5$

$\therefore$ The probability distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.5}(1.5)^x}{x!}, x = 0, 1, 2, \ldots$$

EXAMPLE 3.56 If 2% of the electric bulbs manufactured by a company are defective, find the probability that in a sample of 200 bulbs exactly 10 bulbs are defective ($e^{-4} = 0.01832$).

Solution Given: P(a bulb is defective) $= \dfrac{2}{100}$,

i.e. $p = 0.02$

and $n = 200$

$$\lambda = n \times p = 200 \times 0.02 = 4$$

We know that

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

P(exactly 10 bulbs are defective) $= P(X = 10)$

$$= \frac{e^{-4}4^{10}}{10!} = 0.005294$$

EXAMPLE 3.57 In turning out certain toys in a manufacturing process in a factory, the average number of defectives is 10%. What is the probability of getting exactly 3 defective toys in a sample of 10 toys chosen at random by using the Poisson distribution ($e^{-1} = 0.36788$). **[AU December '03]**

Solution Given: P(a toy is defective) $= p = \dfrac{10}{100} = \dfrac{1}{10} = 0.1$

and $n = 10$

$$\lambda = n \times p = 10 \times 0.1 = 1$$

Let the random variable X denote the number of defective toys.

We know that the PMF of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

$$= \frac{e^{-1}}{x!}, x = 0, 1, 2, \ldots$$

$P(\text{there are 3 defective toys}) = P(X = 3)$

$$= \frac{e^{-1}}{3!} = \frac{e^{-1}}{6} = 0.61313$$

EXAMPLE 3.58 If it is known from the past experience that in a certain plant there are on the average 4 industrial accidents, find the probability that in a given year there will be less than 4 accidents. **[AU June '03]**

Solution Given: Average = mean = $\lambda = 4$

Let X denote the number of accidents in a year.

We know that the PMF of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-4}4^x}{x!}, x = 0, 1, 2, \ldots$$

$P(\text{less than 4 accidents}) = P(X < 4)$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!}$$

$$= e^{-4}\left(1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!}\right) = 0.4335$$

EXAMPLE 3.59 The probability that a man aged 35 years will die before reaching the age of 40 years may be taken as 0.018. Out of a group of 400 men now aged 35 years, what is the probability that 2 men will die within next 5 years? **[AU December '03]**

Solution Given:

$$n = 400$$
$$p = 0.018$$
$$\text{Mean} = \lambda = n \times p = 400 \times 0.018 = 7.2$$

Let X be a random variable denoting the number of deaths within 5 years.

We know that the PMF of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-7.2}(7.2)^x}{x!}, x = 0, 1, 2, \ldots$$

$P(\text{2 men will die within next 5 years}) = P(X = 2)$

$$= \frac{e^{-7.2}(7.2)^2}{2!} = 0.01935$$

EXAMPLE 3.60 Assume that the chance of an individual coal miner being killed in an accident during a year is 1/1400. Using Poisson distribution, calculate the probability that in a mine employing 350 miners, there will be at least one fatal accident in a year ($e^{-0.25} = 0.7788$).

Solution Given:
$$p = \frac{1}{1400}$$
and
$$n = 350$$
$$\text{Mean } \lambda = n \times p = \frac{1}{1400} \times 350 = 0.25$$

Let the random variable X denote the number of accidents in a year. The PMF of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.25}(0.25)^x}{x!}, \; x = 0, 1, 2, \ldots$$

$$P(\text{at least one fatal accident in a year}) = P(X \geq 1)$$
$$= 1 - P(X < 1)$$
$$= 1 - P(X = 0)$$
$$= 1 - \frac{e^{-0.25}(0.25)^0}{0!}$$
$$= 1 - e^{-0.25}$$
$$= 1 - 0.7788$$
$$= 0.2212$$

EXAMPLE 3.61 At a busy traffic junction, the probability of an individual having an accident is $p = 0.01$. However, during a certain part of the day 1000 cars pass through the junction. What is the probability that two or more accidents occur during that period? ($e^{-0.1.} = 0.9048$) **[AU November '07]**

Solution Given:
$$p = 0.01$$
and
$$n = 1000$$
$$\text{Mean} = \lambda = n \times p = 1000 \times 0.01 = 0.1$$

Let the random variable X denote the number of accidents during a certain part of the day.

The PMF of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.1}(0.1)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$P(\text{2 or more accidents occur}) = P(X \geq 2)$$
$$= 1 - P(X < 2)$$
$$= 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - \left[\frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!}\right]$$
$$= 1 - 0.9048 = 0.0952$$

EXAMPLE 3.62 After correcting 50 pages, the proofreader finds that there are on the average of 2 errors per 5 pages. How many pages could one expect with (i) 0 error, (ii) 1 error and (iii) at least 3 errors in 1000 pages of the first print of the book?

Solution Let X be a random variable representing the number of errors per page.

The average number of errors per page $= \lambda = \dfrac{2}{5} = 0.4$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.4}(0.4)^x}{x!}, x = 0, 1, 2, \ldots$$

(i) $$P(0 \text{ error}) = P(X = 0) = \frac{e^{-0.4}(0.4)^0}{0!} = 0.67$$

$\therefore$ The average number of pages containing 0 error in 1000 pages

$$= 0.67 \times 1000 = 670$$

(ii) $$P(1 \text{ error}) = P(X = 1) = \frac{e^{-0.4}(0.4)^1}{1!} = 0.268$$

$\therefore$ The average number of pages containing 1 error in 1000 pages

$$= 0.268 \times 1000 = 268$$

(iii) Probability of at least 3 errors $= P(3 \text{ or more errors})$

$$P(X \geq 3) = 1 - P(X < 3\ )$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - (0.67 + 0.268 + 0.0536) = 0.0084$$

The number of pages which contains at least 3 errors in 1000 pages

$$= 0.0084 \times 1000 = 8$$

EXAMPLE 3.63 Find the probability of 5 or more telephone calls arriving in a 9-minute period in a college switchboard, if the telephone calls that are received at the rate of 2 every 3-minute following Poisson distribution.

Solution Let X_1, X_2 and X_3 be three independent Poisson random variables denoting the number of telephone calls received in 3 consecutive 3-minute periods, each with parameter 2.

Given: $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 2$

Let $X = X_1 + X_2 + X_3$, which follows Poisson distribution with parameters $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 6$

Let X represents the number of calls received in a 9-minute period

$$\therefore \quad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-6}6^x}{x_1}, x = 0, 1, 2, \ldots$$
$$P(X \geq 5) = 1 - P(X < 5)$$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)]$

$= 1 - e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right)$

$= 1 - 0.2851 = 0.7149$

EXAMPLE 3.64 A random variable has the MGF $e^{4(e^t-1)}$. Find its mean, SD, $P(\mu - 2\sigma < X < \mu + 2\sigma]$ and $P(\mu - 2\sigma < X < \mu)$.

Solution The MGF of a Poisson random variable X is $e^{\lambda(e^t - 1)}$

In this problem, it is given as $e^{4(e^t - 1)}$

Therefore, we get

$$e^{4(e^t - 1)} = e^{\lambda(e^t - 1)}$$

$\Rightarrow \quad \lambda = 4$

$$\therefore \quad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-4}4^x}{x!}, \quad x = 0, 1, 2, \ldots$$

But for a Poisson distribution

$$\text{Mean} = \text{variance} = \lambda = 4$$

$$\Rightarrow \quad \mu = 4 \text{ and } \sigma = 2$$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(0 < X < 8)$

$= P(1 \le X \le 7)$

$= e^{-4}\left(4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right) = 0.931$

Also, $P(\mu - 2\sigma < X < \mu) = P(0 < X < 4)$

$= P(X = 1) + P(X = 2) + P(X = 3)$

$= e^{-4}\left(4 + 8 + \frac{32}{3}\right) = 0.4131$

EXAMPLE 3.65 In a component manufacturing industry there is a small chance of 1/500 for any component to be defective. The components are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing

(i) no defective,
(ii) 1 defective,
(iii) 2 defective components respectively in a consignment of 10000 packets.

[AU June '04; '06]

Solution Given:

$$p = \frac{1}{500}$$

$$n = 10$$

and

$$N = 10000$$

$$\text{Mean} = \lambda = n \times p = 10 \times \frac{1}{500} = \frac{1}{50} = 0.02$$

i.e.

$$\lambda = 0.02$$

Let the random variable X denote the number of defective components. Using Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!.} = \frac{e^{-0.02}(0.02)^x}{x!}, x = 0, 1, 2, \ldots$$

(i) P(no defective component) $= P(X = 0)$

$$= \frac{e^{-0.02}(0.02)^0}{0!} = 0.980198$$

$\therefore$ The total number of packets containing no defective components in a consignment of 10000 packets

$$N \times P(X = 0) = 10000 \times 0.98019 = 9802 \text{ packets}$$

(ii) P(1 defective component) $= P(X = 1)$

$$= \frac{e^{-0.02}(0.02)^1}{1!} = 0.01960$$

The number of packets containing 1 defective components

$$N \times P(X = 1) = 10000 \times 0.01960 = 196 \text{ packets}$$

(iii) P(2 defective components) $= P(X = 2)$

$$= \frac{e^{-0.02}(0.02)^2}{2!} = 0.000196$$

The number of packets containing 2 defective components

$$N \times P(X = 2) = 10000 \times 0.0001960 = 2 \text{ packets}$$

EXAMPLE 3.66 Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well-shuffled cards at least once in 156 consecutive trials. **[AU June '07]**

Solution The probability of the ace of spades to be drawn $= p = \frac{1}{52}$

$$n = 156$$

$$\text{Mean} = \lambda = n \times p = 156 \times \frac{1}{52} = 3$$

Let the random variable X denote the number of draws of ace of spades in 156 consecutive trials.

Using Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3}(3)^x}{x!}, 0, 1, 2, \ldots$$

$$\begin{aligned} P(\text{at least once}) &= P(X \geq 1) \\ &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - \frac{e^{-3}(3)^0}{0!} = 1 - 0.0498 = 0.9502 \end{aligned}$$

EXAMPLE 3.67 An insurance company found that only 0.005% of the population is involved in a certain type of accident each year. If its 2000 policyholders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year?

Solution Given: $p = 0.005\% = \frac{0.005}{100} = \frac{1}{20000}$

and $n = 2000$

$$\text{Mean} = \lambda = n \times p = 2000 \times \frac{1}{20000} = 0.1$$

Let the random variable X denote the number of clients involved in certain type of accident each year.

Then using Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.1}(0.1)^x}{x!}, x = 0, 1, 2, \ldots$$

$$\begin{aligned} P(\text{not more than two clients}) &= P(X \leq 2) \\ &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!} \\ &= e^{-0.1}\left(1 + 0.1 + \frac{0.01}{2}\right) \\ &= 0.9048 \times 1.105 = 0.9998 \end{aligned}$$

EXAMPLE 3.68 If the probability of a bad reaction from a certain injection is 0.001, what is the chance that out of 2000 individuals, more than two will get a bad reaction? **[AU December '04]**

Solution Given: $p = 0.001$

and $n = 2000$

$$\text{Mean} = \lambda = n \times p = 2000 \times 0.001 = 2$$

Let the random variable X denote the number of individuals getting a bad reaction from a certain injection. Then using Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}(2)^x}{x!}, x = 0, 1, 2, ...$$

Now, P(more than two will get a bad reaction) $= P(X > 2)$

$$= 1 - P(X \leq 2)$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!} + \frac{e^{-2}(2)^2}{2!}\right]$$
$$= 1 - e^{-2}(1 + 2 + 2) = e^{-2} \times 5 = 1 - \frac{5}{e^2} = 0.323$$

EXAMPE 3.69 Wireless sets are manufactured with 25 soldered joints each on the average 1 joint in 500 defective. How many sets can be expected to be free from defective joints in a consignment of 10000 sets.

[AU November '03]

Solution Given: 1 joint in 500 is defective

$\therefore$ $$p = \frac{1}{500}$$

and $$n = 25$$

$$\text{Mean} = \lambda = n \times p = 25 \times \frac{1}{500} = 0.05$$

Let the random variable X denote the number of defective joints in a set.
$\therefore$ The probability that x joints are defective in a set using Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-(0.05)}(0.05)^x}{x!}, x = 0, 1, 2, ...$$

$$P(\text{no joint is defective}) = P(X = 0)$$
$$= \frac{e^{-(0.05)}(0.05)^0}{0!}$$
$$= 0.95122$$

Hence expected number of sets free from defects in 10000 sets = 10000 × 0.95122 = 9512.

EXAMPLE 3.70 Out of 2000 balls, 100 are red and the rest are white. If 60 balls are picked at random, what is the probability of picking up

(i) 3 red balls,
(ii) not more than 3 red balls in the sample?

Assume Poisson distribution for the number of red balls picked up in the sample, where $e^{-3} = 0.0498$.

Solution Since out of 2000 balls 100 are red, the probability of drawing a red ball is

$$p = \frac{100}{2000} = \frac{1}{20}$$

and $$n = 60$$

$\therefore$ $$\lambda = n \times p = 60 \times \frac{1}{20} = 3$$

Let the random variable X denote the number of red balls drawn.

Then using Poisson distribution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3}3^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) $P(3 \text{ red balls}) = P(X = 3)$

$$= \frac{e^{-3}3^3}{3!}$$

$$= \frac{0.0498 \times 27}{6} = 0.2241$$

(ii) $P(\text{not more than 3 red balls}) = P(X \leq 3)$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!}$$

$$= e^{-3}(1 + 3 + 4.5 + 4.5) = 0.0498\ (13) = 0.6474$$

EXAMPLE 3.71 VLSI chips, essential to the running of a computer system, fail in accordance with a Poisson distribution with the rate of one chip in about 5 weeks. If there are two spare chips on hand, and if a new supply will arrive in 8 weeks, what is the probability that during the next 8 weeks the system will be down for a week or more, owing to a lack of chips?

[AU December '2007]

Solution Given: Mean of Poisson distribution

$$= \lambda = 1 \text{ chip per 5 weeks}$$

$$= \frac{1}{5} \text{ chips per week}$$

$$= 0.2 \text{ chips per week}$$

$$= 1.4 \text{ for 7 weeks}$$

Let the random variable X denote the number of failures in 7 weeks.

The Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.4}(1.4)^x}{x!}, x = 0, 1, 2, ...$$

PC system down for at least one week before new supply in 8 weeks will happen only if there are 3 or more failures within 7 weeks (2 spare chips on hand)

$\therefore$ P(3 or more failures within 7 weeks)

$$= 1 - P(0, 1 \text{ or } 2 \text{ failures in 7 weeks})$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[\frac{e^{-1.4}(1.4)^0}{0!} + \frac{e^{-1.4}(1.4)^1}{1!} + \frac{e^{-1.4}(1.4)^2}{2!}\right]$$
$$= 1 - e^{-1.4}\left[1 + \frac{(1.4)}{1!} + \frac{(1.4)^2}{2!}\right] = 0.1665$$

EXAMPLE 3.72 If X is a Poisson variate such that $P(X = 2) = 9P(X = 4) + 90P(X = 6)$, find mean, $E(X^2)$ and $P(X \geq 2)$.

Solution For a Poisson distribution, the PMF is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, 3 ...$$

Given: $P(X = 2) = 9P(X = 4) + 90P(X = 6)$

$$\Rightarrow \quad \frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!} \Rightarrow 1 = \frac{3}{4}\lambda^2 + \frac{\lambda^4}{4}$$

$$\Rightarrow \quad \lambda^4 + 3\lambda^2 - 4 = 0 \Rightarrow (\lambda^2 + 4)(\lambda^2 - 1) = 0$$

$$\therefore \quad \lambda^2 = -4 \text{ or } \lambda^2 = 1$$

Hence, $\lambda = 1$ (since λ^2 and λ cannot be negative as $\lambda > 0$)

$$\therefore \quad \text{Mean} = \lambda = 1$$

For Poisson distribution,

$$E(X^2) = \lambda^2 + \lambda = 1 + 1 = 2$$

$$\therefore \quad P(X = x) = \frac{e^{-1}1^x}{x!} = \frac{e^{-1}}{x!}, \quad x = 0, 1, 2, ...$$

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - e^{-1}(1 + 1) = 1 - \frac{2}{e}$$
$$= 0.264$$

EXAMPLE 3.73 A manufacturer of cotton pins knows that 5% of his product are defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality?

[AU November '07]

Solution Let the random variable X denote the number of defectives in the box.

Given:
$$p = \frac{5}{100}$$

and
$$n = 100$$
$$\lambda = np = 5$$

$\therefore$
$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}5^x}{x!}, x = 0, 1, 2, \ldots$$

The probability that a box will fail to meet the guaranteed quality

$= P$ (it contains more than 4 pins defective)

$= P(X > 4) = 1 - P(X \leq 4)$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)]$

$$= 1 - e^{-5}\left(1 + 5 + \frac{25}{2} + \frac{125}{6} + \frac{625}{24}\right) = 0.5620$$

EXAMPLE 3.74 Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting 6 heads x times?

Solution The probability of getting 1 head with 1 coin $= \frac{1}{2}$

$\therefore$ The probability of getting 6 heads with 6 coins $= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$\therefore$ The average number of 6 heads with 6 coins in 6400 throws

$$\lambda = np = 6400 \times \frac{1}{64} = 100$$

$\therefore$ Mean of the Poisson distribution $= \lambda = 100$

By Poisson distribution, the approximate probability of getting 6 heads x times is given by

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(100)^x e^{-100}}{x!}, x = 0, 1, 2 \ldots$$

EXAMPLE 3.75 If X and Y are independent Poisson variates, show that the conditional distribution of X given $(X + Y)$ is binomial. **[AU May '06]**

Solution Let X be Poisson distributed with parameter λ and Y be Poisson distributed with parameter μ. Then $X + Y$ follows Poisson distribution with parameter $\lambda + \mu$. (since X and Y are independent).

$$P(X = r/X + Y = n) = \frac{P(X = r \cap Y = n - r)}{P(X + Y = n)}$$

$$= \frac{P(X=r)P(Y=n-r)}{P(X+Y=n)} = \frac{\dfrac{e^{-\lambda}\lambda^r e^{-\mu}\mu^{n-r}}{r!(n-r)!}}{\dfrac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!}}$$

$$= \frac{n!}{r!(n-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{n-r}$$

$\therefore$ $$P(X=r/X+Y=n) = nC_r p^r q^{n-r}$$

where $$p = \frac{\lambda}{\lambda+\mu}, q = 1-p = \frac{\mu}{\lambda+\mu}$$

Hence the conditional distribution is a binomial distribution with parameters n and p, where $p = \dfrac{\lambda}{\lambda+\mu}$.

EXAMPLE 3.76 Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2% of such fuses are defective. **[AU April '05]**

Solution Given:
$$n = 200$$
and
$$p = 2\% = 0.02$$
$\therefore$
$$\text{Mean } \lambda = n \times p = 200 \times 0.02 = 4, \text{ i.e. } \lambda = 4$$

The probability mass function of the random variable X which follows Poisson distribution is

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-4}4^x}{x!}, \; x = 0, 1, 2, \ldots$$

i.e. $$P(x \text{ defective bulbs}) = \frac{e^{-4}4^x}{x!}, \; x = 0, 1, 2, \ldots$$

Now

$$\begin{aligned}
P(\text{at most 5 defective fuses}) &= P(5 \text{ or less defective fuses}) \\
&= P(X \le 5) \\
&= P(X=0) + P(X=1) + P(X=2) + P(X=3) \\
&\quad + P(X=4) + P(X=5) \\
&= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} \\
&\quad + \frac{e^{-4}4^4}{4!} + \frac{e^{-4}4^5}{5!} \\
&= e^{-4}\left[1 + \frac{4}{1} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right] \\
&= e^{-4}(42.866) = 0.785
\end{aligned}$$

EXAMPLE 3.77 It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the number of packets containing exactly, at least and at most 2 defective items in a consignment of 1000 packets using

(i) binomial distribution and

(ii) Poisson approximation to binomial distribution.

[by binomial distribution AU November '09]

Solution

(a) Binomial distribution:

Given: p = probability that an item is defective = 0.05

$$\therefore \quad q = 1 - p = 1 - 0.05 = 0.95,$$
$$n = 20$$
and $$N = 1000$$

(i) If the random variable X represents the number of defective items, then

$$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$
$$= 20C_x (0.05)^x (0.95)^{20-x}$$
$$P(X = 2) = 20C_2 (0.05)^2 (0.95)^{18} = 0.1887$$

In a consignment of $N = 1000$ packets, the number of packets having exactly 2 defective items $= N \times P(X = 2)$

$$= 1000 \times 0.1887 = 189 \text{ (approx.)}$$

(ii) P(at least 2 defective items) $= P(X \geq 2)$

$$= 1 - P(X < 2)$$
$$= 1 - P(X = 0) - P(X = 1)$$
$$= 1 - (0.95)^{20} - 20C_1(0.05)(0.95)^{19}$$
$$= 0.2641$$

In a consignment of 1000 packets, the number of packets having at least 2 defective items $= N \times P(X \geq 2)$

$$= 1000 \times 0.2641 = 264 \text{ (approx.)}$$

(iii) P(at most 2 defective items) $= P(X \leq 2)$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$
$$= (0.95)^{20} + 20C_1(0.05)^1(0.95)^{19} + 20C_2(0.05)^2(0.95)^{18}$$
$$= 0.9246$$

In a consignment of 1000 packets, the number of packets having at most 2 defectives items $= N \times P(X \leq 2)$

$$= 1000 \times 0.9246 = 925 \text{ (approx.)}$$

(b) Poisson distribution: Since $p = 0.05$ is very small and $n = 20$ is sufficiently large, Binomial distribution may be approximated by Poisson distribution with parameter $\lambda = np = 0.05 \times 20 = 1$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1}}{x!}, \quad x = 0, 1, 2, \ldots$$

(i)
$$P(X = 2) = \frac{e^{-1}}{2!} = 0.1839$$

The number of packets having exactly 2 defective items out of 1000

$$= N \times P(X = 2)$$
$$= 1000 \times 0.1839 = 184 \text{ (approx.)}$$

(ii) P(at least 2 defective items) $= P(X \geq 2)$

$$= 1 - P(X < 2)$$
$$= 1 - P(X = 0) - P(X = 1)$$
$$= 1 - \frac{e^{-1}}{0!} - \frac{e^{-1}}{1!} = 1 - 2e^{-1} = 0.2642$$

The number of packets having at least 2 defective items out of 1000

$$= N \times P(X \geq 2)$$
$$= 1000 \times 0.2642 = 264 \text{ (approx.)}$$

(iii) P(at most 2 defective items) $= P(X \leq 2)$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$
$$= \frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-2}}{2!} = 0.9197$$

In a consignment of 1000 packets, the number of packets having at most 2 defective items $= N \times P(X \leq 2)$

$$= 1000 \times 0.9197 = 920 \text{ (approx.)}$$

EXAMPLE 3.78 Show that for a Poisson distribution with unit mean, mean deviation about mean is $2/e$ times the standard deviation.

Solution Given: Mean $= \lambda = 1$

$\therefore$ Variance $= 1$

$\therefore$
$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1}}{x!}, x = 0, 1, 2, \ldots$$

Mean deviation about mean 1 is

$$E(|X - 1|) = \sum_{x=0}^{\infty} |x - 1| p(x) = \sum_{x=0}^{\infty} |x - 1| \frac{e^{-1}}{x!}$$

$$= e^{-1}\sum_{x=0}^{\infty}\frac{|x-1|}{x!} = e^{-1}\left[1+\sum_{n=1}^{\infty}\frac{n}{(n+1)!}\right] = e^{-1}\left[1+\sum_{n=1}^{\infty}\frac{(n+1)-1}{(n+1)!}\right] \text{[Taking } (x-1=n)]$$

$$= e^{-1}\left[1+\sum_{n=1}^{\infty}\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)\right] = e^{-1}\left[1+\left(\frac{1}{1!}-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{3!}-\frac{1}{4!}\right)+\ldots+\infty\right]$$

$$= e^{-1}\left[\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\infty\right)-\left(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\infty\right)\right]$$

$$= e^{-1}\left[e^{1}-\left(e^{1}-1-\frac{1}{1!}\right)\right] = e^{-1}(1+1) = \frac{2}{e}$$

EXAMPLE 3.79 If X is a Poisson variate with mean λ, show that $\frac{X-\lambda}{\sqrt{\lambda}}$ is a variable with mean zero and variance unity. Find the MGF of this variate.

Solution Given: X is a Poisson variate

$\therefore$ Mean = $E(X) = \lambda$, Variance = $\text{Var}(X) = \lambda$

Now, $$E\left(\frac{X-\lambda}{\sqrt{\lambda}}\right) = \frac{1}{\sqrt{\lambda}}E(X-\lambda) = \frac{1}{\sqrt{\lambda}}[E(X)-\lambda] = \frac{1}{\sqrt{\lambda}}(\lambda-\lambda) = 0$$

and $$\text{Var}\left(\frac{X-\lambda}{\sqrt{\lambda}}\right) = \frac{1}{\lambda}\text{Var}(X-\lambda) = \frac{\text{Var}(X)}{\lambda} = \frac{\lambda}{\lambda} = 1$$

MGF of $\frac{X-\lambda}{\sqrt{\lambda}}$ is given by

$$M_X\left(\frac{X-\lambda}{\sqrt{\lambda}}\right) = E\left[e^{\left(\frac{X-\lambda}{\sqrt{\lambda}}\right)t}\right]$$

$$= E\left(e^{\frac{X}{\sqrt{\lambda}}t}\cdot e^{-\sqrt{\lambda}t}\right) \Rightarrow e^{-\sqrt{\lambda}\cdot t}\cdot E\left(e^{\frac{tX}{\sqrt{\lambda}}}\right)$$

$$= e^{-t\sqrt{\lambda}}\sum_{x=0}^{\infty}e^{\frac{tx}{\sqrt{\lambda}}}\cdot e^{-\lambda}\cdot\frac{\lambda^x}{x!} = e^{-t\sqrt{\lambda}}e^{-\lambda}e^{\lambda e^{\frac{t}{\sqrt{\lambda}}}}$$

$$= e^{-t\sqrt{\lambda}-\lambda+\lambda e^{\frac{t}{\sqrt{\lambda}}}}$$

EXAMPLE 3.80 Let X be a Poisson variate with parameter λ and Y be another discrete random variable whose conditional distribution for a given X is given by $P(Y = r/X = x) = xC_r p^r(1-p)^{x-r}$, $0 < p < 1$, $r = 0, 1, 2, \ldots$, then show that the unconditional distribution of Y is a Poisson distribution with parameter λp.

Solution If X is a Poisson variate, then

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!},\ x=0,1,2,\ldots$$

and also given $P(Y=r/X=x)=xC_rp^r(1-p)^{x-r},\quad r\le x$

$$P(X=x/Y=r)=P(X=x)\ P(Y=r/X=x)$$

$$=\frac{e^{-\lambda}\lambda^x}{x!}xC_rp^r(1-p)^{x-r},\quad r=0,1,2,\ldots,x$$

This gives the joint distribution of X and Y.

The distribution of Y is given by

$$P(Y=r)=\sum_{x=r}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}xC_rp^r(1-p)^{x-r}$$

$$=e^{-\lambda}\sum_{x=r}^{\infty}\frac{\lambda^x}{x!}\frac{x!}{r!(x-r)!}p^r(1-p)^{x-r}$$

$$=\frac{e^{-\lambda}(\lambda p)^r}{r!}\sum_{x=r}^{\infty}\frac{\lambda^{x-r}(1-p)^{x-r}}{(x-r)!}$$

$$=\frac{e^{-\lambda}(\lambda p)^r}{r!}\sum_{x=r}^{\infty}\frac{[\lambda(1-p)]^{x-r}}{(x-r)!}$$

$$=\frac{e^{-\lambda}(\lambda p)^r}{r!}\left[1+\lambda(1-p)+\frac{\lambda^2(1-p)^2}{2!}+\cdots\right]$$

$$=\frac{e^{-\lambda}(\lambda p)^r}{r!}\cdot e^{\lambda(1-p)}$$

$$=\frac{e^{-\lambda p}(\lambda p)^r}{r!}$$

EXAMPLE 3.81 Fit a Poisson distribution to the following data and calculate the theoretical frequencies.

Deaths	0	1	2	3	4
Frequency	122	60	15	2	1

Solution

$$N=\Sigma f=122+60+15+2+1$$
$$=200$$
$$\Sigma fx=0\times 122+1\times 60+2\times 15+3\times 2+4\times 1$$
$$=100$$

$$\text{Mean} = \frac{\Sigma fx}{N} = \frac{100}{200} = 0.5 = \lambda$$

$\therefore$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.5}(0.5)^x}{x!},\ x = 0, 1, 2, \ldots$$

$\therefore$

Hence the theoretical frequencies are given by

$$f(x) = N \cdot P(X = x) = 200 \times \frac{e^{-0.5}(0.5)^x}{x!}$$

Deaths	0	1	2	3	4
Observed frequency	122	60	15	2	1
Expected frequency	121	61	15	3	0

3.1.5 Geometric Distribution

Suppose we have a series of independent trials or repetitions and on each repetition or trial the probability of success p remains the same, then the probability that there are x failures preceding the first success is given by $q^x p$.

A random variable X is said to have a geometric distribution, if it assumes only non-negative values and its probability mass function is given by

$$P(X = x) = q^x p,\ x = 0, 1, 2, \ldots$$

where p is the probability of success and $q = 1 - p$.

Here $P(X = x) = q^x p$ gives the probability that the first success occurs only at the $(x + 1)$th trial, the previous x trials are failure.

Note:

(i) Since the various probabilities for $x = 0, 1, 2, \ldots$ are the various terms of geometric progression, hence the name geometric distribtuion.

(ii) The PMF of geometric distribution can also be defined as $P(X = x) = q^{x-1}, p,\ x = 1, 2, 3, \ldots$ which gives the probability that the first success occurs only at the xth trial and the preceding $(x - 1)$ are failures.

Moments of Geometric Distribution

$$\text{Mean} = \mu_1' = \sum_{x=0}^{\infty} x P(X = x) = \sum_{x=0}^{\infty} x pq^x$$

$$= pq \sum_{x=1}^{\infty} x q^{x-1} = pq(1-q)^2 = \frac{q}{p}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(X = x) = \sum_{x=0}^{\infty} [x(x-1) + x]\, pq^x$$

$$= \sum_{x=0}^{\infty} x(x-1)pq^x + \sum_{x=0}^{\infty} xpq^x$$

$$= pq^2 \sum_{x=2}^{\infty} [x(x-1)q^{x-2}] + \frac{q}{p} = 2pq^2(1-q)^{-3} + \frac{q}{p} = \frac{2q^2}{p^2} + \frac{q}{p}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \mu^2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p} = \frac{q(q+p)}{p^2} = \frac{q}{p^2}$$

Moment Generating Function of Geometric Distribution

EXAMPLE 3.82 Find the moment generating function of geometric distribution and hence find its mean and variance.

[AU May '06, June '07, April '08]

Solution We know that,

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} q^x p = \sum_{x=0}^{\infty} (qe^t)^x\, p = p\sum_{x=0}^{\infty} (qe^t)^x$$

$$= p[1 + qe^t + (qe^t)^2 + \cdots]$$

$$= p(1 - qe^t)^{-1}$$

$$= \frac{p}{(1 - qe^t)}$$

To find mean and variance using MGF

$$\text{Mean} = \mu_1'$$

$$= \left[\frac{d}{dt}\left(\frac{p}{1-qe^t}\right)\right]_{t=0} = \left\{\frac{d}{dt}[p(1-qe^t)^{-1}]\right\}_{t=0}$$

$$= [-p(1 - qe^t)^{-2}(-qe^t)]_{t=0} = [p(1-q)^{-2}(q)]$$

$$= \frac{pq}{(1-q)^2} = \frac{pq}{(p)^2} = \frac{q}{p}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}\left(\frac{p}{1-qe^t}\right)\right]_{t=0} = \left\{\frac{d}{dt}\left[\frac{pqe^t}{(1-qe^t)^2}\right]\right\}_{t=0}$$

$$= [e^t pq(1 - qe^t)^{-2} + pqe^t(-2)(1 - qe^t)^{-3}(-qe^t)]_{t=0}$$

$$= [pq(1-q)^{-2} + 2pq^2(1-q)^{-3}] = \left[\frac{pq}{p^2} + \frac{2pq^2}{p^3}\right]$$

$$= \left[\frac{q}{p} + \frac{2q^2}{p^2}\right]$$

$$\text{Variance} = \mu_2 = \frac{q}{p} + \frac{2q^2}{p^2} - \left(\frac{q}{p}\right)^2 = \frac{q}{p} + \frac{q^2}{p^2} = \frac{qp+q^2}{p^2} = \frac{q(p+q)}{p^2} = \frac{q}{p^2}$$

Therefore,

$$\text{Variance} = \mu_2 = \frac{q}{p^2}$$

Note: The MGF of geometric distribtuion can also be rewritten as

$$M_X(t) = \left(\frac{1}{p} - \frac{qe^t}{p}\right)^{-1}$$

Memoryless Property of Geometric Distribution

EXAMPLE 3.83 Prove that geometric distribution possesses memoryless property. **[AU April '08]**

Solution We have to show that

$$P(X > s + t/X > t) = P(X > s)$$

First, we find

$$P(X > k) = \sum_{x=k}^{\infty} q^x p = p\sum_{x=k}^{\infty} q^x = p(q^k + q^{k+1} + q^{k+2} + \cdots)$$

$$= pq^k(1 + q + q^2 + \cdots) = pq^k(1-q)^{-1} = pq^k p^{-1} = q^k \quad \text{...(i)}$$

$$\therefore \quad P(X > s + t/X > t) = \frac{P(X > s + t \cap X > t)}{P(X > t)} = \frac{P(X > s+t)}{P(X > t)}$$

$$= \frac{q^{s+t}}{q^t} = q^s = P(X > s), \quad \text{using (i)}$$

Thus geometric distribution possesses memoryless property.

EXAMPLE 3.84 If the probability is 0.05 that a certain kind of measuring device will show excessive drift, what is the probability that the sixth of these measuring devices tested will be the first to show excessive drift? **[AU December '07**

Solution Given: $p = 0.05$, $q = 1 - p = 0.95$ and $x = 6$

We know that the probability that the first success occurs on the xth hit is

$$P(X = x) = q^{x-1} p, \quad x = 1, 2, 3, \ldots$$

$$\therefore \quad P(X = 6) = (0.95)^5(0.05) = 0.0387$$

EXAMPLE 3.85 If the probability that the target is destroyed on any one shot is 0.5, what is the probability that it will be destroyed on the 6th attempt? [AU June '04]

Solution The probability that it will be destroyed on sixth attempt is equal to q^5p (first five attempts are failures and sixth one is a success)

$$p = \text{probability of hitting the target} = 0.5$$

$$q = 1 - p = 1 - 0.5 = 0.5$$

$$q^5p = (0.5)^6 = 0.0156$$

EXAMPLE 3.86 If the probability that an applicant for a driver's licence will pass road test on any given trial is 0.8, what is the probability that he will finally pass the test

(i) on the fourth trial,

(ii) fewer than 4 trials?

Solution Let X denote the number of trials required to get the first success. Then X follows geometric distribution.

$$P(X = x) = q^{x-1} p, \ x = 1, 2, 3, \ldots$$

Given: $p = 0.8, \therefore q = 1 - p = 0.2$

(i) The probability that the driver will finally pass the test on the fourth trial is (first three are failures and fourth test is a success)

$$P(X = 4) = q^3p = (0.2)^3\,(0.8) = 0.0064$$

(ii) The probability that the driver will pass the test in fewer than 4 trials is

$$\begin{aligned} P(X < 4) &= P(\text{pass in the first test or second test or third test}) \\ &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= p + qp + q^2p \\ &= p(1 + q + q^2) = (0.8)\,[1 + 0.2 + (0.2)^2] = 0.992 \end{aligned}$$

EXAMPLE 3.87 A die is thrown repeatedly until 6 appears. What is the probability that it must be thrown more than 5 times?

Solution p = probability of success

= probability of getting number 6 in throwing a die

$$\therefore \quad p = \frac{1}{6}$$

and

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

The probability that the die should be thrown more than 5 times (first five are failure)

$$= q^5p + q^6p + q^7p + \ldots$$
$$= q^5p(1 + q + q^2 + \ldots)$$
$$= q^5p(1 - q)^{-1}$$
$$= q^5p(p)^{-1} = q^5$$
$$= \left(\frac{5}{6}\right)^5 = 0.4019$$

Aliter

The probability of getting 6 in a throw $= \dfrac{1}{6}$

$\therefore$ $$p = \frac{1}{6}$$

and $$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Let X denote the number of throws for getting the number 6.
By geometric distribution

$$P(X = x) = q^{x-1}\, p,\ x = 1, 2, 3, \ldots$$

Since 6 can be got either in first, second, ... throws

$$P(X > 5) = 1 - P(X \le 5)$$
$$= 1 - \sum_{x=1}^{5}\left(\frac{5}{6}\right)^{x-1}\cdot\frac{1}{6}$$
$$= 1 - \left[\frac{1}{6} + \left(\frac{1}{5}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)\right]$$
$$= 1 - \frac{\frac{1}{6}\left[1 - \left(\frac{5}{6}\right)^5\right]}{1 - \frac{5}{6}} = \left(\frac{5}{6}\right)^5 = 0.4019$$

EXAMPLE 3.88 A coin is tossed until the first head occurs. Assuming the tosses are independent and the probability of a head occurring is p, find the value of p so that the probability that an odd number of tosses is required is equal to 0.6. Can you find a value of p so that the probability is 0.5 that an odd number of tosses is required?

Solution Let X denote the number of tosses required to get the first head (success). Then X follows geometric distribution given by

$$P(X = x) = pq^{x-1},\ x = 1, 2, 3, \ldots$$

$$\begin{aligned}
P(X = \text{an odd number}) &= P(X = 1 \text{ or } X = 3 \text{ or } X = 5, \ldots) \\
&= P(X = 1) + P(X = 3) + P(X = 5) + \ldots \\
&= p + pq^2 + pq^4 + pq^6 + \cdots \\
&= p(1 + q^2 + q^4 + q^6 + \cdots) \\
&= p(1-q^2)^{-1} = \frac{p}{1-q^2} \\
&= \frac{1-q}{(1-q)(1+q)} = \frac{1}{1+q}, (p = 1-q)
\end{aligned}$$

Given: $P(X = \text{an odd number}) = 0.6 = \dfrac{1}{1+q}$

$$1 = 0.6\ (1 + q) = 0.6 + 0.6q$$

$\Rightarrow$ $$0.6q = 1 - 0.6 = 0.4$$

$\Rightarrow$ $$q = \frac{0.4}{0.6} = \frac{2}{3}$$

and $$p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

Next, if $P(X = \text{an odd number}) = 0.5 = \dfrac{1}{1+q}$

$$0.5(1 + q) = 1 \quad \Rightarrow \quad 0.5q = 0.5$$
$$q = 1, \qquad p = 1 - q = 0$$

Note: When $P(X = x)$ = constant, the discrete random variable X is said to follow a discrete uniform distribution.

EXAMPLE 3.89 If the probability of success on each trial is 0.25, after how many trials can we expect first success?

Solution Given: $p = 0.25,\ q = 1 - p = 0.75$

If X denotes the number of trials required to get first success, then by geometric distribution, expected number of trials for first success is

$$E(X) = \frac{q}{p} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$$

EXAMPLE 3.90 Let one copy of a magazine out of 10 copies bears a special prize following geometric random distribution. Determine its mean and variance. **[AU June '07]**

Solution Given: $$p = \frac{1}{10}$$

$$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Mean of the geometric distribution is $\frac{q}{p} = 9$

$$\text{Variance} = \frac{q}{p^2} = \frac{9}{10} \times 10^2 = 90$$

EXAMPLE 3.91 Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.8.

(i) What is the probability that the target would be hit on 6th attempt?
(ii) What is the probability that it takes him less than 5 shots?
(iii) What is the probability that it takes him an even number of shots?
(iv) What is the average number of shots needed to hit on target?

[AU December '05]

Solution Given:

$$p = 0.8$$

and

$$q = 1 - p = 1 - 0.8 = 0.2$$

If X is a random variable denoting the number of shots required for the first success, then using geometric distribution

$$P(X = x) = q^{x-1}p, \qquad x = 1, 2, 3, \ldots$$

$$\Rightarrow \quad P(X = x) = (0.2)^{x-1}(0.8), \qquad x = 1, 2, 3, \ldots$$

(i) P(the target would be hit on the 6th attempt)

$$= P(X = 6)$$
$$= (0.2)^{6-1}(0.8) = (0.2)^5(0.8) = 0.000256$$

(ii) P(it takes him less 5 shots)

$$= P(X < 5)$$
$$= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$
$$= (0.8)[(0.2)^{1-1} + (0.2)^{2-1} + (0.2)^{3-1} + (0.2)^{4-1}]$$
$$= 0.8[1 + (0.2) + (0.2)^2 + (0.2)^3]$$
$$= 0.9984$$

(iii) P(it takes him an even number of shots)

$$= P(X = 2) + P(X = 4) + P(X = 6) + \cdots$$
$$= (0.2)^{2-1}(0.8) + (0.2)^{4-1}(0.8) + \cdots$$
$$= (0.2)(0.8)[1 + (0.2)^2 + (0.2)^4 + \cdots]$$
$$= 0.16[1 + (0.04) + (0.04)^2 + \cdots]$$
$$= 0.16[1 - 0.04]^{-1} \quad [(1 - x)^{-1} = 1 + x + x^2 + \cdots]$$
$$= 0.16(0.96) = 0.1536$$

(iv) Average (expected) number of shots needed to hit the target

$$= \frac{q}{p} = \frac{0.2}{0.8} \cong 1$$

EXAMPLE 3.92 A and B shoot independently until each has his own target. The probability of their hitting the target at each shot is 3/5 and 5/7 respectively. Find the probability that B will require more shots than A.

Solution Suppose A requires X number of trials to get his first success, with $p = \frac{3}{5}, q = 1 - \frac{3}{5} = \frac{2}{5}$, then X follows geometric distribution given by

$$P(X = r) = p_1 q_1^{\,r-1} = \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^{r-1} \quad r = 1, 2, 3, \ldots$$

Suppose B requires Y number of trials to get his first success, $p = \frac{5}{7}$, $q = 1 - \frac{5}{7} = \frac{2}{7}$, then Y follows geometric distribution given by

$$P\,(Y = r) = p_2 q_2^{\,r-1} = \left(\frac{5}{7}\right)\left(\frac{2}{7}\right)^{r-1} \quad r = 1, 2, 3, \ldots$$

The probability that B requires more trials to get his first success than A requires to get his first success is that if A requires r trials to get his first success that B requires $r + 1$, $r + 2$, ... trials to get his first success.

$$= \sum_{r=0}^{\infty} P(X = r \text{ and } Y = r + 1, r + 2, \ldots, \infty)$$

$$= \sum_{r=1}^{\infty} P(X = r)\, P[(Y = r + 1) \text{ or } (Y = r + 2) \ldots] \text{ by independence}$$

$$= \sum_{r=1}^{\infty}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^{r-1} \sum_{k=1}^{\infty}\left(\frac{5}{7}\right)\left(\frac{2}{7}\right)^{r+k-1}$$

$$= \sum_{r=1}^{\infty}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^{r-1}\left(\frac{2}{7}\right)^{r-1} \cdot \left(\frac{5}{7}\right)\sum_{k=1}^{\infty}\left(\frac{2}{7}\right)^{k}$$

$$= \sum_{r=1}^{\infty}\left(\frac{3}{7}\right)\left(\frac{4}{35}\right)^{r-1} \frac{2}{7}\left(1 - \frac{2}{7}\right)^{-1} \left[\because \sum_{r=1}^{\infty} x^n = x(1 - x)^{-1}\right]$$

$$= \sum_{r=1}^{\infty}\left(\frac{3}{7}\right)\left(\frac{4}{35}\right)^{r-1} \frac{2}{7}\left(\frac{7}{5}\right) = \frac{6}{35}\sum_{r=1}^{\infty}\left(\frac{4}{35}\right)^{r-1}$$

$$= \frac{6}{35}\left[1 + \frac{4}{35} + \left(\frac{4}{35}\right)^2 + \ldots\right] = \frac{6}{35}\left[1 - \frac{4}{35}\right]^{-1}$$

$$= \frac{6}{35}\left[\frac{31}{35}\right]^{-1} = \frac{6}{31}$$

EXAMPLE 3.93 Let X and Y be independent random variables such that $P(X = r) = P(Y = r) = q^r p$, $r = 0, 1, 2, \ldots$, where $p + q = 1$.

Find

(i) the distribution of $X + Y$, and
(ii) the conditional distribution of X given $X + Y = 3$.

Solution Given: $P(X = r) = P(Y = r) = q^r p$, $r = 0, 1, 2, \ldots$

$$P(X + Y = n) = \sum_{r=0}^{n} P[(X = r) \cap (Y = n - r)]$$

$$= \sum_{r=0}^{n} P(X = r) \cdot P(Y = n - r)$$

($\because$ X and Y are independent)

$$= \sum_{r=0}^{n} q^r p \cdot pq^{n-r} = p^2 \sum_{r=0}^{n} q^r \cdot q^{n-r}$$

$$= p^2(q^n + q^n + \cdots + q^n)\ (n + 1) \text{ times}$$

$$\therefore \quad P(X + Y = n) = p^2 q^n (n + 1)$$

$$P(X = r/X + Y = n) = \frac{P(X = r \cap X + Y = n)}{P(X + Y = n)}$$

$$= \frac{P(X = r \cap Y = n - r)}{P(X + Y = n)}$$

$$= \frac{P(X = r)\, P(Y = n - r)}{P(X + Y = n)}$$

$$= \frac{q^r p \cdot q^{n-r} p}{p^2 q^n (n+1)} = \frac{1}{(n+1)}$$

which is a discrete uniform distribution.

Hence, $P(X/X + Y = 3) = \dfrac{1}{4}$

EXAMPLE 3.94 If the MGF of X is $(5 - 4e^t)^{-1}$, find the distribution of X and $P(X = 5 \text{ or } 6)$. **[AU December '04]**

Solution The MGF of Geometric distribution is

$$\left(\frac{1 - qe^t}{p}\right)^{-1} = (5 - 4e^t)^{-1} \text{ (given)}$$

$$\Rightarrow \qquad \frac{1}{p} = 5, \ \frac{q}{p} = 4$$

$$\Rightarrow \qquad p = \frac{1}{5}$$

and
$$q = \frac{4}{5}$$

$\therefore$ The PMF of geometric distribution is

$$P(X = x) = q^{x-1} p, \qquad x = 0, 1, 2, \ldots$$

$$\begin{aligned} P(X = 5 \text{ or } 6) &= P(X = 5) + P(X = 6) \\ &= q^4 p + q^5 p = pq^4(1 + q) \\ &= \frac{1}{5}\left(\frac{4}{5}\right)^4\left(1 + \frac{4}{5}\right) \\ &= 0.147456 \end{aligned}$$

3.2 CONTINUOUS DISTRIBUTIONS

If X is a continuous random variable, then we have the following distributions:

1. Uniform distribution or rectangular distribution
2. Exponential distribution
3. Gamma distribution or Erlang distribution
4. Weibull distribution
5. Normal distribution or Gaussian distribution

3.2.1 Uniform Distribution or Rectangular Distribution

A random variable X is said to follow uniform distribution over an interval (a, b) if its probability density function is constant $= k$ (say), over the entire range of X

$$f(x) = \begin{cases} k, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Since the total probability is always unity, we have

$$\int_a^b f(x)dx = 1 \Rightarrow k\int_a^b dx = 1 \Rightarrow k[x]_a^b = 1$$

$$\Rightarrow \qquad k(b - a) = 1 \Rightarrow k = \frac{1}{b-a}$$

$$\therefore \qquad f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Note:

(i) $\int_{-\infty}^{\infty} f(x)dx = \int_a^b f(x)dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}[x]_a^b, = \frac{b-a}{b-a} = 1$

(ii) a and b ($a < b$) are the two parameters of the uniform distribution on (a, b).

(iii) The distribution is also known as rectangular distribution since curve $y = f(x)$ describes a rectangle over the X-axis and between the coordinates at $x = a$ and $x = b$.

(iv) The distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0, & \text{if } -\infty < x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b < x < \infty \end{cases}$$

Since $F(x)$ is not continuous at $x = a$ and $x = b$, it is not differentiable at these points. Thus, $\frac{d}{dx}F(x) = f(x) = \frac{1}{b-a} \neq 0$, exists everywhere except at the points $x = a$ and $x = b$.

(v) For a rectangular or uniform variate X in $(-a, a)$, PDF is given by

$$f(x) = \begin{cases} \dfrac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

Moments of Uniform Distribution

$$\mu_r' = \int_a^b x^r f(x)dx = \frac{1}{(b-a)}\int_a^b x^r\,dx = \frac{1}{(b-a)}\left(\frac{b^{r+1}-a^{r+1}}{r+1}\right)$$

In particular,

$$\text{Mean} = \mu_1' = \frac{1}{(b-a)}\left(\frac{b^2-a^2}{2}\right) = \frac{b+a}{2}$$

$$\mu_2' = \frac{1}{(b-a)}\left(\frac{b^3-a^3}{3}\right) = \frac{1}{3}(b^2+ab+a^2)$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{1}{3}(b^2+ab+a^2) - \left(\frac{b+a}{2}\right)^2 = \frac{1}{12}(b-a)^2$$

Mean Deviation about Mean, η

$$\eta = E(|X - \text{mean}|) = \int_a^b |x - \text{mean}|\, f(x)dx$$

$$= \frac{1}{(b-a)} \int_a^b \left| x - \frac{a+b}{2} \right| dx$$

$$= \frac{1}{(b-a)} \int_{-(b-a)/2}^{(b-a)/2} |t|\, dt$$

$$= \frac{2}{(b-a)} \int_0^{\frac{(b-a)}{2}} t\,dt = \frac{b-a}{4} \quad \text{[since } |t| \text{ is an even function]}$$

Mean and Variance of Uniform Distribtuion

We know that for a continuous random variable X which is defined in t interval (a, b) the rth moment about origin is

$$\mu_r' = \int_a^b x^r f(x) dx \qquad (3.$$

where $f(x)$ is a PDF of the random variable X

$$\therefore \qquad \mu_1' = \int_a^b x f(x) dx = \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{1}{2(b-a)} (b^2 - a^2) = \frac{b+a}{2}$$

$$\therefore \qquad \text{Mean} = \mu_1' = \frac{a+b}{2}$$

Putting $r = 2$ in Eq. (3.6), we get

$$\therefore \qquad \mu_2' = \int_a^b x^2 f(x) dx = \int_a^b \frac{x^2}{b-a} dx$$

$$x^2 f(x) dx = \int_a^b \frac{x^2}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b = \frac{1}{3(b-a)} (b^3 - a^3)$$

$$= \frac{1}{3(b-a)}(b-a)(b^2+ab+a^2)$$

$$\text{Var} = \mu_2' - \mu_1'^2$$

$$\therefore \quad = \frac{(b^2+ab+a^2)}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{(b-a)^2}{12}$$

Moment Generating Function of Uniform Distribtuion

EXAMPLE 3.95 Find the moment generating function, mean and variance of the random variable X with PDF $f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

[AU December '04, November '06]

Solution Given: $f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

$$\therefore \quad M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tX} f(x)dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a} dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{e^{tb}}{t} - \frac{e^{ta}}{t}\right) = \frac{1}{t(b-a)}(e^{bt} - e^{at})$$

$$\therefore \text{MGF} = \frac{e^{bt} - e^{at}}{t(b-a)}$$

$$M_X(t) = \frac{1}{t(b-a)}\left[1 + \frac{bt}{1!} + \frac{(bt)^2}{2!} + \frac{(bt)^3}{3!} + \cdots - \left(1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots\right)\right]$$

$$= \frac{1}{t(b-a)}\left[\frac{(b-a)t}{1!} + \frac{(b^2-a^2)t^2}{2!} + \frac{(b^3-a^3)t^3}{3!} + \cdots\right]$$

To find the mean and variance:

$$\mu_1' = \text{mean} = \text{coefficient of } \frac{t}{1!} = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

$$\mu_2' = \text{coefficient of } \frac{t^2}{2!} = \frac{b^3 - a^3}{3(b-a)}$$

$$\text{Var} = \mu_2' - (\mu_1')^2 = \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

EXAMPLE 3.96 If X is uniformly distributed over (0, 5), find

(i) $P(X < 2)$, (ii) $P(X > 3)$, (iii) $P(2 < X < 9)$.

Solution The random variable X is uniformly distributed over (0, 5). Therefore, its PDF is

$$f(x) = \begin{cases} \frac{1}{5}, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(X < 2) = \int_0^2 f(x)dx = \frac{1}{5}\int_0^2 dx = \frac{1}{5}[x]_0^2 = \frac{2}{5}$

(ii) $P(X > 3) = \int_3^5 f(x)dx = \frac{1}{5}\int_3^5 dx = \frac{1}{5}[x]_3^5 = \frac{2}{5}$

(iii) $P(2 < X < 9) = \int_2^5 f(x)dx = \frac{1}{5}\int_2^5 dx = \frac{1}{5}[x]_2^5 = \frac{3}{5}$

EXAMPLE 3.97 Subway trains on a certain line run every half hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes.

Solution Let X be the random variable which denote the waiting time for the next train. Assume that a man arrives at the station at random. The random variable X is distributed uniformly in (0, 30) and (30, 60) with PDF (half hour)

$$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$$

Probability that a man entering the station will have to wait at least twenty minutes if he arrives between 0 to 10 or 30 to 40 minutes

$$P(X \geq 20) = \int_0^{10} f(x)\,dx = \frac{1}{30}\int_0^{10} dx = \frac{1}{3}$$

Aliter

As there is a train every half hour in the station, a man has to wait at least 20 minutes (20 or more) even if he arrives the station during the period 30 to 40 minutes

$$f(x) = \begin{cases} \frac{1}{30}, & 30 < x < 60 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X \geq 20) = \int_{30}^{40} f(x)\,dx = \frac{1}{30}\int_{30}^{40} dx = \frac{1}{3}$$

EXAMPLE 3.98 If X is uniformly distributed random variable with mean 1 and variance 4/3, find $P(X > 0)$.

Solution For uniform distribution, we know that

$$\text{Mean} = \frac{a+b}{2} = 1 \Rightarrow a+b=2 \quad \text{(i)}$$

$$\text{Var} = \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 16$$

$\Rightarrow$ $\quad b - a = \pm 4 \quad$ (ii)

i.e. $\quad b - a = 4$ (since $a < b$)

Solving Eqs. (i) and (ii), we get

$$a = -1$$
$$b = 3 (a < b)$$

$$f(x) = \frac{1}{4} \text{ in } -1 < x < 3$$
$$= 0, \text{ otherwise}$$

$$\therefore \quad P(X > 0) = \int_0^3 f(x)\,dx = \int_0^3 \frac{1}{4}dx = \frac{1}{4}[x]_0^3 = \frac{3}{4}$$

EXAMPLE 3.99 Show that for the uniform distribution $f(x) = \frac{1}{2a}$, $-a < x < a$, the MGF about origin is $\frac{\sinh at}{at}$. Prove also that the moments of odd order are zero and $\mu'_{2r} = \frac{a^{2r}}{2r+1}$.

Solution We know that the MGF of uniform distribution in the interval (a, b) is

$$M_X(t) = E(e^{tX}) = \int_a^b e^{tx} f(x)dx$$

Here, $f(x) = \frac{1}{2a}$ in $-a < x < a$

$$\therefore \quad M_X(t) = \int_{-a}^{a} e^{tx} \frac{1}{2a} dx = \frac{1}{2a}\left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$= \frac{1}{2at}(e^{at} - e^{-at})$$

$$= \frac{1}{at}\sinh at$$

$$\left(\sinh x = \frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{at}\left[at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \frac{(at)^7}{7!} + \cdots + \frac{(at)^{2r+1}}{(2r+1)!} + \cdots\right]$$

$$= 1 + \frac{a^2t^2}{3!} + \frac{a^4t^4}{5!} + \frac{a^6t^6}{7!} + \cdots + \frac{a^{2r}t^{2r}}{(2r+1)(2r)!} + \cdots$$

$$\mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

As there are only even powers of t, we have

$$\mu_1' = \mu_3' = \mu_5' = \cdots = \mu_{2r+1}' = \cdots = 0$$

$$\mu_{2r}' = \text{coefficient of } \frac{t^{2r}}{2r!} = \frac{a^{2r}}{(2r+1)}$$

EXAMPLE 3.100 If X is uniformly distributed over $(-\alpha, \alpha)$, $\alpha > 0$, find α so that

(i) $P(X > 1) = 1/3$,

(ii) $P(|X| < 1) = P(|X| > 1)$.

Solution If X is uniformly distributed over $(-\alpha, \alpha)$, then its PDF is

$$f(x) = \begin{cases} \frac{1}{2\alpha}, & -\alpha < x < \alpha \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(X > 1) = \frac{1}{3}$

$$\Rightarrow \quad \int_1^{\alpha} f(x)\,dx = \frac{1}{3} \quad \Rightarrow \quad \int_1^{\alpha} \frac{1}{2\alpha}\,dx = \frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{2\alpha}[x]_1^{\alpha} = \frac{1}{3} \quad \Rightarrow \quad \frac{1}{2\alpha}(\alpha - 1) = \frac{1}{3}$$

$$\Rightarrow \quad 3(\alpha - 1) = 2\alpha \quad \Rightarrow \quad \alpha = 3$$

(ii) $P(|X| < 1) = P(|X| > 1)$

i.e. $P(-1 < X < 1) = 1 - P(-1 < X < 1)$

$$[\because P(|X| > 1) = 1 - P(|X| < 1)]$$

$$\Rightarrow \quad 2P(-1 < X < 1) = 1$$

$$2\int_{-1}^{1} \frac{1}{2\alpha}\,dx = 1 \quad \Rightarrow \quad \frac{1}{\alpha}\int_{-1}^{1} dx = 1 \quad \Rightarrow \quad \frac{1}{\alpha}[x]_{-1}^{1} = 1 \quad \Rightarrow \quad \frac{2}{\alpha} = 1 \quad \Rightarrow \quad \alpha = 2$$

EXAMPLE 3.101 A random variable Y is defined as $\cos \pi x$, where X has a uniform PDF over $\left(\frac{-1}{2}, \frac{1}{2}\right)$. Find mean and SD.

Solution The PDF of the random variable X over $\left(\frac{-1}{2}, \frac{1}{2}\right)$ is

$$f(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Now,
$$E(Y) = \int_{-\infty}^{\infty} y f(x)\,dx$$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos \pi x \cdot 1\,dx \qquad (\because y = \cos \pi x)$$

$$= \left(\frac{\sin \pi x}{\pi}\right)_{\frac{-1}{2}}^{\frac{1}{2}} = \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi} 2 \sin\left(\frac{\pi}{2}\right) = \frac{2}{\pi} = 0.636$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(x)dx = \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos^2 \pi x \cdot 1\, dx$$

$$= 2\int_0^{\frac{1}{2}} \cos^2 \pi x\, dx \qquad (\because \cos^2 x \text{ is an even function})$$

$$= 2\int_0^{\frac{1}{2}} \left(\frac{1+\cos 2\pi x}{2}\right) dx \qquad \left(\text{using } \cos^2 x = \frac{1+\cos 2x}{2}\right)$$

$$= \left[x + 2\frac{\sin 2\pi x}{4\pi}\right]_0^{\frac{1}{2}} = \left(\frac{1}{2} + \frac{\sin \pi}{2\pi}\right) - \left(0 + \frac{\sin 0}{2\pi}\right)$$

$$= \frac{1}{2} = 0.5 \qquad (\because \sin \pi = 0)$$

$\therefore \quad \text{Var}(Y) = E(Y^2) - [E(Y)]^2$

$= 0.5 - (0.636)^2 = 0.096$

EXAMPLE 3.102 A random variable X has a uniform distribution over $(-3, 3)$ compute

(i) $P(X < 2)$, $P(|X|) < 2)$, $P|X - 2| < 2)$, and

(ii) Find k for which $P(X > k) = \frac{1}{3}$.

Solution We know that the PDF of a random variable X which is distributed uniformly in $(-3, 3)$ is

$$f(x) = \begin{cases} \frac{1}{6}, & -3 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

By definition,

$$P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

(i) $$P(X < 2) = \int_{-3}^{2} f(x)dx = \int_{-3}^{2} \frac{1}{6} dx$$

$$= \frac{1}{6}[x]_{-3}^{2} = \frac{1}{6}(2+3) = \frac{5}{6}$$

We know that

$$P(|X| < 2) = P(-2 < X < 2)$$
$$= \int_{-2}^{2} f(x)dx = \frac{1}{6}\int_{-2}^{2} dx = \frac{1}{6}[x]_{-2}^{2} = \frac{4}{6} = \frac{2}{3}$$

Again,

$$P(|X - 2| < 2) = P(-2 < X - 2 < 2)$$
$$= P(0 < X < 4) = \int_{0}^{3} f(x)dx = \int_{0}^{3}\frac{1}{6}dx = \frac{1}{6}[x]_{0}^{3} = \frac{1}{2}$$

(ii) $P(X > k) = \dfrac{1}{3} \Rightarrow \displaystyle\int_{k}^{3} f(x)dx = \frac{1}{3}$

$$\int_{k}^{3}\frac{1}{6}dx = \frac{1}{3} \Rightarrow \frac{1}{6}(3 - k) = \frac{1}{3}$$

i.e. $3 - k = 2 \Rightarrow k = 1$

Note: If X is a continuous random variable, then $P(X = a) = 0 \;\forall\; a$.

EXAMPLE 3.103 Buses arrive at a specified stop at 15 minutes interval starting at 7 a.m., i.e. they arrive at 7, 7.15, 7.30, 7.45 .... If a passenger arrives at the stop at a random time that is uniformly distributed between 7 a.m. and 7.30 a.m., find the probability that he waits for

(i) less than 5 minutes, and
(ii) at least 12 minutes for a bus.

Solution X is a random variable denoting the minutes past 7.00 a.m.

$$\text{PDF } f(x) = \frac{1}{7.30 - 7} = \frac{1}{30} \text{ minutes.}$$

(i) Passenger waits for less than 5 minutes
$\Rightarrow$ he arrives between 7.10 a.m. and 7.15 a.m. or 7.25 a.m. and 7.30 a.m.

The required probability $= P(10 < X < 15) + P(25 < X < 30)$

$$= \int_{10}^{15}\frac{dx}{30} + \int_{25}^{30}\frac{dx}{30} = \frac{1}{3}$$

(ii) Passenger waits for at least 12 minutes
$\Rightarrow$ he arrives between 7.00 a.m. and 7.03 a.m. or 7.15 a.m. and 7.18 a.m.

The required probability $= P(0 < X < 3) + P(15 < X < 18)$

$$= \int_0^3 \frac{1}{30}\,dx + \int_{15}^{18} \frac{1}{30}\,dx = \frac{3}{30} + \frac{3}{30} = \frac{1}{5}$$

EXAMPLE 3.104 Starting at 5.00 a.m. every half hour there is a flight from San Francisco airport to Los Angeles International airport. Suppose that none of these planes is completely sold out and that they always have room for passengers. A person who wants to fly to Los Angeles arrives at the airport at a random time between 8.45 a.m. and 9.45 a.m. Find the probability that she waits

(i) at most 10 minutes, and
(ii) at least 15 minutes.

[AU December '07]

Solution Starting at 5.00 a.m. every half hour there is a flight from the airport. That is at 5, 5.30, 6, 6.30, 7 ,7.30, 8, 8.30, 9, 9.30, 10 a.m. and so on.

X be a random variable denoting the minutes past hours.

$$f(x) = \begin{cases} \dfrac{1}{60}, & 0 < x < 60 \\ 0, & \text{otherwise} \end{cases}$$

(i) If the person arrives either between 8.50 a.m. and 9.00 a.m. or 9.20 a.m. and 9.30 a.m., waits at most 10 minutes.

P(she waits at most 10 minutes)

$$= P(50 < X < 60) + P(20 < X < 30)$$

$$= \int_{50}^{60} \frac{1}{60}\,dx + \int_{20}^{30} \frac{1}{60}\,dx = \frac{1}{60}[x]_{50}^{60} + \frac{1}{60}[x]_{20}^{30}$$

$$= \frac{1}{60}(60-50) + \frac{1}{60}(30-20) = \frac{10}{60} + \frac{10}{60} = \frac{20}{60} = \frac{1}{3}$$

(ii) If the person arrives between 8.45 a.m. and 9.45 a.m. Then she waits for at least 15 minutes, if she arrives between 9.00 a.m. and 9.15 a.m. or 9.30 a.m. and 9.45 a.m.

P(she waits at least 15 minutes)

$$= P(0 < X < 15) + P(30 < X < 45)$$

$$= \int_{0}^{15} \frac{1}{60}\,dx + \int_{30}^{45} \frac{1}{60}\,dx = \frac{1}{60}[x]_{0}^{15} + \frac{1}{60}[x]_{30}^{45}$$

$$= \frac{1}{60}(15-0) + \frac{1}{60}(45-30) = \frac{15}{60} + \frac{15}{60} = \frac{30}{60} = \frac{1}{2}$$

EXAMPLE 3.105 *The variates a and b are independently and uniformly distributed in the interval (0, 3) and (0, 6) respectively. Find the probability that the equation $x^2 - ax + b = 0$ has two real roots.

Solution Since a and b are uniformly distributed in (0, 3), the PDF of a is

$$f(a) = \begin{cases} \frac{1}{3}, & 0 < a < 3 \\ 0, & \text{otherwise} \end{cases}$$

and PDF of b is

$$f(b) = \begin{cases} \frac{1}{6}, & 0 < b < 6 \\ 0, & \text{otherwise} \end{cases}$$

The equation $x^2 - ax + b = 0$ will have real roots only when

$$B^2 - 4AC \geq 0$$

i.e.
$$a^2 - 4b \geq 0$$

$\Rightarrow$
$$a^2 \geq 4b$$

i.e. the condition is $b \leq \dfrac{a^2}{4}$

$a^2 - 4b \geq 0$

$a^2 \geq 4b$

$b \leq \dfrac{a^2}{4}$

$\therefore$ The required probability $= \displaystyle\int_0^3 \int_0^{\frac{a^2}{4}} f(a, b)\, db\, da$

Since a and b are independent,

$$f(a, b) = f(a)\, f(b) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$

$\therefore$ The required probability $= \dfrac{1}{18} \displaystyle\int_0^3 \int_0^{\frac{a^2}{4}} db\, da$

$$= \frac{1}{18} \int_0^3 \frac{a^2}{4}\, da = \frac{1}{18 \times 4} \left[\frac{a^3}{3}\right]_0^3 = \frac{1}{8}$$

EXAMPLE 3.106 Trains arrive at a station at 15 minutes intervals starting at 4 a.m. If a passenger arrives at a station at a time that is uniformly distributed between 9.00 and 9.30, find the probability that he has to wait for the train for

(i) less than 6 minutes,

(ii) more than 10 minutes. **[AU December '09]**

Solution Let X be a random variable denoting the minutes past hours. Then we know that the PDF of a random variable X which is distributed uniformly in (0, 30) is

$$f(x)=\begin{cases}\dfrac{1}{30}, & 0<x<30\\ 0, & \text{otherwise}\end{cases}$$

(i) $P(\text{less than 6 minutes}) = P(9 < X < 15) + P(24 < X < 30)$

$$= \int_{9}^{15} \frac{1}{30}\,dx + \int_{24}^{30} \frac{1}{30}\,dx = \frac{2}{5}$$

(ii) $P(\text{more than 10 minutes}) = P(0 < X < 5) + P(15 < X < 20)$

$$= \int_{0}^{5} \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx = \frac{1}{3}$$

3.2.2 Exponential Distribution

A continuous random variable X is said to follow an exponential distribution with parameter $\lambda > 0$, if its PDF is given by

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\ge 0\\ 0, & \text{otherwise}\end{cases}$$

The general form of the exponential distribution is

$$f(x)=\frac{1}{a}e^{\frac{-x}{a}},\ a>0,\ x\ge 0 \text{ with parameter a}$$

Note: $\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} f(x)dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty} = 1$

Moment Generating Function of Exponential Distribution

EXAMPLE 3.107 Find the moment generating function of the exponential distribution and hence find its mean and variance.

[AU November '04; '06; '07; '08, June '06]

Solution The probability density function of exponential distribution is

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\ge 0\\ 0, & \text{otherwise}\end{cases}$$

By definition, the moment generating function $M_X(t) = E(e^{tx})$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_{0}^{\infty} e^{-x(\lambda - t)} dx = \lambda \left[\frac{e^{-x(\lambda-t)}}{-(\lambda - t)} \right]_0^{\infty} = \lambda \left[\frac{1}{(\lambda - t)} \right]$$

i.e. $$\text{MGF} = \frac{\lambda}{\lambda - t}$$

To find the mean and variance:

$$M_X(t) = \frac{\lambda}{\lambda - t} = \left(\frac{\lambda - t}{\lambda} \right)^{-1} = \left(1 - \frac{t}{\lambda} \right)^{-1} = 1 + \frac{t}{\lambda} + \left(\frac{t}{\lambda} \right)^2 + \cdots$$

$$\text{Mean} = \mu_1' = \text{coefficient of } \frac{t}{1!} = \frac{1}{\lambda}$$

$$\mu_2' = \text{coefficient of } \frac{t^2}{2!} = \frac{2!}{\lambda^2}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Moments, Mean and Variance

The nth moment about origin is given by

$$E(X^n) = \lambda \int_0^{\infty} x^n e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x^{(n+1)-1} e^{-\lambda x} dx$$

Using Gamma integral,

$$\frac{\overline{|(n)}}{a^n} = \int_0^{\infty} e^{-ax} x^{n-1} dx, \text{ we have}$$

$$E(X^n) = \mu_n' = \lambda \frac{\overline{|n+1}}{\lambda^{n+1}} = \frac{n!}{\lambda^n}$$

[$\overline{|(n+1)} = n!$, if n is an integer]

Hence, $$E(X^n) = \mu_n' = \frac{n!}{\lambda^n}, \text{ where } n \text{ is an integer.}$$

From this,

$$\text{Mean} = E(X) = \frac{1}{\lambda}$$

$$E(X^2) = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{\lambda^2}$$

$\therefore$

$$\text{Mean} = \frac{1}{\lambda}$$

$\Rightarrow$

$$\text{Variance} = \frac{1}{\lambda^2}$$

and

$$\text{SD} = \frac{1}{\lambda}$$

$\therefore$

Hence mean and standard deviation are equal.

Central Moments

$$\mu_2 = \frac{1}{\lambda^2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3$$

$$= \frac{6}{\lambda^3} - \frac{6}{\lambda^3} + \frac{2}{\lambda^3} = \frac{2}{\lambda^3}$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= \frac{24}{\lambda^4} - \frac{24}{\lambda^4} + \frac{12}{\lambda^4} - \frac{3}{\lambda^4} = \frac{9}{\lambda^4}$$

Distribution Function

$$F(x) = P(X \le x) = \int_0^x f(x)dx = \int_0^x \lambda e^{-\lambda x}\,dx = \lambda \int_0^x e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^x = 1 - e^{-\lambda x}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Exponential Distribution Possesses Memoryless Property

[AU May '06, '07 December '09]

Proof To prove this property, we have to show that

$$P(X > s + t/X > t) = P(X > s), \text{ for any } s, t > 0.$$

The PDF of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\therefore \qquad P(X > k) = \int_k^\infty \lambda e^{-\lambda x}\, dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty = \lambda\left(0 + \frac{e^{-\lambda k}}{\lambda}\right)$$

$$P(X > k) = e^{-\lambda k}$$

$$P(X > s + t/X > t) = \frac{P(X > s + t \cap X > t)}{P(X > t)} = \frac{P(X > s + t)}{P(X > t)} \qquad (3.7)$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$

[using Eq. (3.7)]

i.e. $\qquad P(X > s + t/X > t) = P(X > s)$

$\therefore$ Exponential distribution possesses memoryless property.

Note:

(i) If X is a continuous random variable with probability function,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

then X is said to follow exponential distribution with parameter λ.

$$\text{Mean} = \frac{1}{\lambda}, \ \text{Var} = \frac{1}{\lambda^2}$$

(ii) If $X_1, X_2, \ldots, X_n$ are independent random variables, each $X_i (i = 1, 2, 3, \ldots, n)$ having an exponential distribution with parameter $\lambda_i (i = 1, 2, 3, \ldots, n)$, then $Z = \min(X_1, X_2, \ldots, X_n)$ has exponential distribution with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$.

EXAMPLE 3.108 If X has an exponential distribution with mean = 2, find $P(X < 1/X < 2)$.

Solution We know that mean of the exponential distribution is $\frac{1}{\lambda}$.

$$\therefore \qquad \text{Mean} = 2 \Rightarrow \frac{1}{\lambda} = 2$$

$\lambda \quad \frac{1}{\lambda} = 2$

i.e.,
$$\lambda = \frac{1}{2} = 0.5$$

$$\text{PDF} = f(x) = 0.5\, e^{-(0.5)x}, \quad x \geq 0$$

$\lambda = \frac{1}{2}$

$$P(X < 1/X < 2) = \frac{P(X < 1 \cap X < 2)}{P(X < 2)} = \frac{P(X < 1)}{P(X < 2)}$$

$$P(X < 1) = \int_{-\infty}^{1} f(x)dx = \int_{0}^{1} 0.5e^{-0.5x}dx$$

$$= 0.5\left[\frac{e^{-0.5x}}{-(0.5)}\right]_0^1 = 0.5\left[\frac{e^{-0.5}-1}{-(0.5)}\right] = 0.393$$

$$P(X < 2) = \int_{0}^{2} f(x)dx = \int_{0}^{2} 0.5e^{-0.5x}dx$$

$$= 0.5\left[\frac{e^{-0.5x}}{-(0.5)}\right]_0^2 = 0.5\left[\frac{e^{-1}-1}{-(0.5)}\right] = 0.632$$

$$P(X < 1/X < 2) = \frac{P(X < 1)}{P(X < 2)} = \frac{0.393}{0.692} = 0.5679$$

EXAMPLE 3.109 If X follows an exponential distribution with $P(X \leq 1) = P(X > 1)$ find mean and variance.

Solution The probability density function of the exponential distribution is $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$

Given: $$P(X \leq 1) = P(X > 1) \quad \text{(i)}$$

$$P(X \leq 1) = \int_{-\infty}^{1} f(x)dx = \int_{0}^{1} \lambda e^{-\lambda x}dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^1$$

$$= -(e^{-\lambda}-1) = 1 - e^{-\lambda}$$

$$P(X > 1) = \int_{1}^{\infty} f(x)dx = \int_{1}^{\infty} \lambda e^{-\lambda x}dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_1^{\infty} = e^{-\lambda}$$

Substituting in Eq. (i), we get

$$1 - e^{-\lambda} = e^{-\lambda} \Rightarrow 2e^{-\lambda} = 1 \Rightarrow e^{-\lambda} = \frac{1}{2}$$

i.e., $$-\lambda = \log\left(\frac{1}{2}\right) = \log 1 - \log 2 = -\log 2$$

$\Rightarrow$ $$\lambda = \log 2 = 0.6931$$

$$\therefore \quad \text{Mean} = \frac{1}{\lambda} = \frac{1}{0.6931} = 1.4428$$

$$\text{Var} = \frac{1}{\lambda^2} = \frac{1}{(0.6931)^2} = 2.0817$$

EXAMPLE 3.110 The time (in hours) required to repair a watch is exponentially distributed with parameter $\lambda = 1/2$.

(i) What is the probability that the repair time exceeds 2 hours?
(ii) What is the probability that a repair takes 11 hours given that its duration exceeds 8 hours?

[AU May '03, June '06; '07, November '06]

With mean 120 days, find the probability that such a watch will

(a) have to set in less than 24 days, and
(b) not have to be reset in at least 180 days. **[AU June '03]**

Solution Let X be the random variable which denotes the time to repair the watch. The PDF of the exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Given: $\lambda = \frac{1}{2}$, and $f(x) = \frac{1}{2}e^{-\frac{1}{2}x}, \quad x \geq 0$

(i) $P(X > 2) = \int_2^{\infty} \frac{1}{2}e^{-\frac{1}{2}x}\,dx = \left[-e^{-\frac{1}{2}x}\right]_2^{\infty} = e^{-1}$

(ii) Using the memoryless property, we have

$$P(X \geq 11/X > 8) = P(X > 3) = \int_3^{\infty} \frac{1}{2}e^{-\frac{x}{2}}\,dx = e^{-1.5}$$

In the second case given

$$\text{Mean} = 120$$

$$\Rightarrow \quad \frac{1}{\lambda} = 120 \text{ i.e. } \lambda = \frac{1}{120}$$

The PDF of X is given by

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{120}e^{\frac{-x}{120}}$$

(a) $P(X < 24) = \int_0^{24} \frac{1}{120} e^{\frac{-x}{120}} dx = -\left[e^{\frac{x}{120}}\right]_0^{24}$

$= -(e^{-0.2} - 1) = 1 - e^{-0.2} = 1 - 0.8187 = 0.1813$

(b) $P(X \geq 180) = \int_{180}^{\infty} \frac{1}{120} e^{\frac{-x}{120}} dx = \frac{1}{120}\left[\frac{e^{\frac{-x}{120}}}{\frac{-1}{120}}\right]_{180}^{\infty}$

$= -e^{-\infty} + e^{-1.5} = 0.2231$

EXAMPLE 3.111 The daily consumption of milk in a city in excess of 10000 gallons is approximately exponentially distributed with mean $\theta = 1000$. The city has a daily stock of 20000 gallons, what is the probability that the stock is insufficient for both days if two days are selected at random?

Solution If the random variable X denotes the daily consumption of milk (in gallons) in a city, then $Y = X - 10000$ has exponential distribution with mean $\theta = 1000$.

$$\text{Mean} = \frac{1}{\lambda} = 1000 \Rightarrow \lambda = \frac{1}{1000}$$

The PDF of Y is

$$g(Y) = \frac{1}{1000} e^{\frac{-y}{1000}}, 0 < y < \infty$$

Since the daily stock of the city is 20000 gallons, the probability that the stock is insufficient on a pariticular day

$$= P(X > 20000)$$

$$= P(Y > 10000) = \int_{10000}^{\infty} \frac{1}{1000} e^{\frac{-y}{1000}} dy$$

$$= \left[-e^{\frac{-y}{1000}}\right]_{10000}^{\infty} = -(0 - e^{-10}) = e^{-10}$$

$\therefore$ The probability that the milk is insufficient for both the days if two days are selected at random $= (e^{-10})^2 = e^{-20}$.

EXAMPLE 3.112 If X is a continuous random variable with PDF $f(x) = e^{-x}$, $0 \leq x \leq \infty$, find the probability that the roots of the equation $t^2 - x(4t - 5) + 21 = 0$ may be real.

Solution The equation $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac \geq 0$.

$\therefore$ For the roots of the equation $t^2 - 4xt + 5x + 21 = 0$ to be real ($a = 1$, $b = 4x$, $c = 5x + 21$)

$$16x^2 - 4(5x + 21) \geq 0$$

i.e.,
$$4x^2 - 5x - 21 \geq 0$$

Solving $4x^2 - 5x - 21 = 0$, we get

$$x = \frac{5 \pm \sqrt{361}}{8}$$

i.e.
$$x = \frac{5 \pm 19}{8} \Rightarrow x = 3 \text{ or } -\frac{7}{4}$$

But $x > 0$. $\therefore$ $x = 3$

The roots of $t^2 - 4xt + 5x + 21 = 0$ may be real for $x = 3$

The required probability $= P(X \geq 3)$

$$= \int_3^\infty e^{-x} dx = \left[e^{-x}\right]_3^\infty = e^{-3}$$

EXAMPLE 3.113 A component has an exponential time to failure distribution with mean of 10000 hours.

The component has already been in operation for its mean life.

(i) What is the probability that it will fail by 15000 hours given that the component is already seen in operation for its mean life. condition

(ii) What is the probability that it operates for another 5000 hours. reliability.

Given that it is in operation at 15000 hours.

Solution Let X denote the time to failure of the component. Then X has exponential distribution with mean = 10000 hours.

$$\therefore \quad \frac{1}{\lambda} = 10000 \Rightarrow \lambda = \frac{1}{10000}$$

The PDF of X is

$$f(x) = \begin{cases} \frac{1}{10000} e^{\frac{-x}{10000}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) The probability that the component will fail by 15000 hours given it has already been in operation for its mean life

$$= P(X < 15000/X > 10000) \text{ (Mean life = 10000 hours)}$$

$$= \frac{P(10000 < X < 15000)}{P(X > 10000)}$$

$$= \frac{\int_{10000}^{15000} f(x)dx}{\int_{10000}^{\infty} f(x)dx} = \frac{\int_{10000}^{15000} e^{\frac{-1}{10000}x} dx}{\int_{10000}^{\infty} e^{\frac{-1}{10000}x} dx}$$

$$= \frac{e^{-1} - e^{-1.5}}{e^{-1}} = \frac{0.3679 - 0.2231}{0.3679} = 0.3936$$

(ii) The probability that the component will operate for another 5000 hours given that it is in operation at 15000 hours $(15000 + 5000 = 20000)$

$$= P(X > 20000/X > 15000)$$
$$= P(X > 5000) \quad \text{(using memoryless property)}$$

$$= \int_{5000}^{\infty} f(x)\, dx = e^{-0.5} = 0.6065$$

EXAMPLE 3.114 The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$, what is the probability that the required time

(i) exceeds 2 hours, and
(ii) exceeds 5 hours. **[AU May '06; '07, November '06]**

Solution Let X represent the time to repair the machine. Then the density function of X is given by

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{x}{2}}, \quad x > 0$$

(i) Now, $P(X > 2) = \int_{2}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_{2}^{\infty} = e^{-\frac{2}{2}} = e^{-1}$

(ii) $P(X > 5) = \int_{5}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_{5}^{\infty} = e^{-\frac{5}{2}} = 0.082$

EXAMPLE 3.115 If X is exponentially distributed with parameter λ, find the value of k such that $\frac{P(X > k)}{P(X \le k)} = a$. **[AU December '07]**

Solution Given: $\dfrac{P(X>k)}{P(X\le k)}=a \quad \Rightarrow \quad \dfrac{P(X>k)}{1-P(X>k)}=a$

i.e., $$P(X > k) = a[1 - P(X > k)]$$

i.e., $$P(X > k)\ (1 + a) = a$$

$$P(X > k) = \frac{a}{1+a} \tag{i}$$

Since X is exponentially distributed with parameter λ, we get

$$P(X > k) = \int_k^{\infty} f(x)dx \tag{ii}$$

From Eqs. (i) and (ii)

$$\int_k^{\infty} f(x)dx = \frac{a}{1+a} \quad \Rightarrow \quad \int_k^{\infty} \lambda e^{-\lambda x}\, dx = \frac{a}{1+a}$$

$$\lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^{\infty} = \frac{a}{1+a} \quad \Rightarrow \quad (-e^{-\infty} + e^{-\lambda k}) = \frac{a}{1+a}$$

$$\Rightarrow \qquad e^{-\lambda k} = \frac{a}{1+a} \quad \Rightarrow \quad e^{\lambda k} = \frac{1+a}{a}$$

Taking log on both sides,

$$\lambda k = \log\left(\frac{1+a}{a}\right) \quad \Rightarrow \quad k = \frac{1}{\lambda}\log\left(\frac{1+a}{a}\right)$$

EXAMPLE 3.116 If the number of kilometres that a car can run before its battery wears out is exponentially distributed with an average value of 10000 km and if the owner desires to take a 5000 km trip, what is the probability that he will be able to complete his trip without having to replace the car battery. Assume that the car has been used for some time. **[AU May '03]**

Solution Let X denote the number of kilometres that a car can run before its battery wears out.

Given: X follows an exponential distribution.

$$\therefore \qquad \text{Mean} = \frac{1}{\lambda} = 10000 \quad \Rightarrow \quad \lambda = \frac{1}{10000}$$

The PDF of X is

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{10000}\cdot e^{-\frac{x}{10000}}, \quad x \ge 0$$

$$P(X > 5000) = \int_{5000}^{\infty} f(x)dx = \frac{1}{10,000}\int_{5000}^{\infty} e^{-\frac{x}{10000}}dx$$

$$= \frac{1}{10000}\left[-10000 \cdot e^{-\frac{x}{10000}}\right]_{5000}^{\infty} = e^{-0.5} = 0.6065$$

EXAMPLE 3.117 The mileage which car owners get with certain kind of radial tyre is a random variable having an exponential distribution with mean 40000 km. Find the probabilities that one of these tyres will last

(i) at least 20000 km, and
(ii) at most 30000 km.

[AU December '09]

Solution Let X denote the mileage obtained with the tyre. Then X follows exponential distribution with mean

$$\frac{1}{\lambda} = 40000 \Rightarrow \lambda = \frac{1}{40000}$$

$$\therefore \qquad f(x) = \lambda e^{-\lambda x} = \frac{1}{40000} \cdot e^{-\frac{x}{40000}}, \quad x > 0$$

(i) The probability that the tyre will last for at least 20000 km

$$P(X > 20000) = \int_{20000}^{\infty} f(x)dx$$

$$= \frac{1}{40000}\int_{20000}^{\infty} e^{-\frac{x}{40000}}dx$$

$$= \frac{1}{40000}\left[-40000 \cdot e^{-\frac{x}{40000}}\right]_{20000}^{\infty}$$

$$= -\left[e^{-\frac{x}{40000}}\right]_{20000}^{\infty} = e^{-0.5} = 0.6065$$

(ii) The probability that the tyre will last for at most 30000 km

$$P(X \le 30000) = \int_{0}^{30000} f(x)dx = \frac{1}{40000}\int_{0}^{30000} e^{-\frac{x}{40000}}dx$$

$$= \frac{1}{40000}\left[-40000 \cdot e^{-\frac{x}{40000}}\right]_0^{30000}$$

$$= -\left[e^{-\frac{x}{40000}}\right]_0^{30000} = 1 - e^{-0.75} = 0.5270$$

EXAMPLE 3.118 Suppose X has an exponential distribution with mean equal to 10. Determine the value of x such that $P(X < x) = 0.95$.

Solution Given: $\text{Mean} = \frac{1}{\lambda} = 10 \Rightarrow \lambda = \frac{1}{10}$

$$P(X < x) = \int_{-\infty}^{x} \lambda e^{-\lambda x} dx = \int_{0}^{x} \lambda e^{-\lambda x} dx$$

$$= \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^x = 1 - e^{-\lambda x}$$

$$1 - e^{-\frac{1}{10}x} = 0.95 \qquad \text{(given)}$$

$$\frac{1}{-10}x = \log (0.05)$$

$$x = -10 \log (0.05)$$

$$x = 13.010$$

3.2.3 Gamma Distribution or Erlang Distribution

A continuous random variable X is said to follow general Gamma distribution or Erlang distribution with two parameters $\lambda > 0$ and $k > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Note:

(i) When $\lambda = 1$, the Erlang distribution is called simple Gamma distribution

$$f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}, \quad x \geq 0, k > 0$$

with one parameter $k > 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = \frac{\lambda^k}{\overline{|(k)}}\int_0^{\infty} x^{k-1}e^{-\lambda x}\,dx = \frac{\lambda^k}{\overline{|(k)}} \times \frac{\overline{|(k)}}{\lambda^k} = 1$

$$\left[\text{Since } \int_0^{\infty} x^{k-1}e^{-ax}dx = \frac{\overline{|(k)}}{a^k}\right]$$

(iii) When $k = 1$, the Erlang distribution reduces to exponential distribution

Moment Generating Function of Gamma Distribution

EXAMPLE 3.119

Solution The probability density function of the general gamma random variable X is

$$f(x) = \begin{cases} \dfrac{\lambda^k x^{k-1}e^{-\lambda x}}{\overline{|(k)}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where λ and k are the parameters.
The moment generating function,

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_0^{\infty} \frac{e^{tx}\lambda^k x^{k-1}e^{-\lambda x}}{\overline{|(k)}}\,dx = \frac{\lambda^k}{\overline{|(k)}}\int_0^{\infty} e^{tx}x^{k-1}e^{-\lambda x}\,dx$$

$$= \frac{\lambda^k}{\overline{|(k)}}\int_0^{\infty} x^{k-1}e^{-x(\lambda - t)}\,dx$$

$$= \frac{\lambda^k}{\overline{|(k)}(\lambda - t)^k}\overline{|(k)}\left[\text{using the Gamma function, } \int_0^{\infty} x^{n-1}e^{-ax}dx = \frac{\overline{|(n)}}{a^n}\right]$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^k$$

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right)^k\right]_{t=0} = \lambda^k\left[\frac{d}{dt}(\lambda - t)^{-k}\right]_{t=0}$$

$$= \lambda^k \left[(-k)(\lambda - t)^{-k-1}(-1)\right]_{t=0} = \frac{k}{\lambda}$$

$$\mu_2' = E(X^2) = \left[\frac{d^2}{dt^2}\left(\frac{\lambda}{\lambda - t}\right)^k\right]_{t=0} = \lambda^k \left\{\frac{d}{dt}\left[k(\lambda - t)^{-k-1}\right]\right\}_{t=0}$$

$$= k\lambda^k \left[(-k-1)(\lambda - t)^{-k-2} \cdot (-1)\right]\Big|_{t=0}$$

$$= k\lambda^k \left[(k+1)(\lambda - t)^{-k-2}\right]\Big|_{t=0} = k\lambda^k\, (k+1)\, \lambda^{-k-2}$$

$$= \frac{k(k+1)}{\lambda^2} = \frac{k^2 + k}{\lambda^2}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{k^2 + k}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k}{\lambda^2}$$

Note:

(i) Erlang distribution or general Gamma distribution both mean the same when k is an integer.

(ii) When $k = 1$, Gamma distribution/Erlang distribution tends to exponential distribution.

(iii) When $\lambda = 1$, the Erlang/general Gamma distribution reduces to simple Gamma distribution.

$$f(x) = \frac{x^{k-1} e^{-x}}{(k-1)!} = \frac{x^{k-1} e^{-x}}{\Gamma(k)}$$

Reproductive Property of Gamma Distribution

If $X_1, X_2, X_3, \ldots, X_n$ are n independent Gamma or Erlang random variables with parameters $(\lambda, k_1), (\lambda, k_2), \ldots, (\lambda, k_n)$ respectively, then $X_1 + X_2 + X_3 + \cdots + X_n$ is also a Gamma or Erlang random variable with parameter $(\lambda, k_1 + k_2 + \cdots + k_n)$.

Proof To prove this property, we use the MGF of Gamma distribution

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^k$$

Since $X_1, X_2, X_3, \ldots, X_n$ are n independent random variables, we have

$$M_{X_1 + X_2 + \cdots + X_n}(t) = E[e^{t(X_1 + X_2 + \cdots + X_n)}]$$

$$E = E\left(e^{tX_1}\, e^{tX_2} \cdots e^{tX_n}\right)$$

$$= E\left(e^{tX_1}\right) E\left(e^{tX_2}\right) \cdots E\left(e^{tX_n}\right)$$

($\because X_1, X_2 \cdots X_n$ are independent)

$$= \left(\frac{\lambda}{\lambda - t}\right)^{k_1} \left(\frac{\lambda}{\lambda - t}\right)^{k_2} \cdots \left(\frac{\lambda}{\lambda - t}\right)^{k_n}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{k_1 + k_2 + \cdots + k_n}$$

Therefore, $X_1 + X_2 + X_3 + \cdots + X_n$ is also a Gamma random variable with parameter $(\lambda, k_1 + k_2 + \cdots + k_n)$.

EXAMPLE 3.120 Find the MGF of a Gamma distribution (with one parameter k) and hence find its mean and variance.

[AU May '05, November '05; '08]

Solution If X is a Gamma random variable, then the PDF

$$f(x) = \frac{x^{k-1} e^{-x}}{\Gamma(k)}, \quad x \geq 0, k \geq 0$$

$$\text{MGF} = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} \frac{e^{tx} e^{-x} x^{k-1}}{\Gamma(k)} dx = \frac{1}{\Gamma(k)} \int_0^{\infty} x^{k-1} e^{-x(1-t)} dx$$

Using $\int_0^{\infty} x^{k-1} e^{-ax} dx = \frac{\Gamma(k)}{a^k}$, we get

$$\text{MGF} = \frac{1}{\Gamma(k)} \times \frac{\Gamma(k)}{(1-t)^k} = \left(\frac{1}{1-t}\right)^k$$

To find the mean and variance:

$$\text{Mean} = \mu_1' = E(X) = \left[\frac{d}{dt}\left(\frac{1}{1-t}\right)^k\right]_{t=0} = \left[\frac{d}{dt}(1-t)^{-k}\right]_{t=0}$$

$$= [(-k)(1-t)^{-k-1}(-1)]_{t=0} = k$$

$$\mu_2' = E(X^2) = \left[\frac{d^2}{dt^2}\left(\frac{1}{1-t}\right)^k\right]_{t=0} = \left[\frac{d^2}{dt^2}(1-t)^{-k}\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[k(1-t)^{-k-1}\right]\right\}_{t=0}$$

$$= [k(-k-1)(1-t)^{-k-2}(-1)]_{t=0} = k(k+1)$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = k(k+1) - k^2 = k$$

EXAMPLE 3.121 The daily consumption of milk in a city in excesss of 20000 gallons is approximately distributed as a Gamma variate with parameters $k = 2$ and $\lambda = 1/10000$. The city has a daily stock of 30000 gallons. What is the probability that the stock is insufficient on a particular day?

Solution Let X be the random variable denoting the daily consumption of milk (in gallons) in a city. Then $Y = X - 20000$ has Gamma distribution with PDF

$$g(y) = \frac{1}{(10000)^2\Gamma(2)} y^{2-1} e^{-\frac{y}{10000}}, \; y \geq 0$$

$$\left[\text{PDF of } X: f(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x} \therefore f(y) = \frac{\lambda^2 y^{2-1} e^{-\lambda y}}{\Gamma(2)}, \; y \geq 0\right]$$

$$g(y) = \frac{y e^{\frac{-y}{10000}}}{(10000)^2}, \; y \geq 0$$

Since the daily stock of the city is 30000 gallons, the stock is insufficient only if the demand on a particular day is more than 30000 gallons. The required probability that the stock is insufficient on a particular day is given by

$$P(X > 30000) = P(Y > 10000) = \int_{10000}^{\infty} g(y)dy = \int_{10000}^{\infty} \frac{y e^{\frac{-y}{10000}}}{(10000)^2} dy$$

$$= \frac{1}{(10000)^2}\left[y \frac{e^{\frac{-y}{10000}}}{\left(-\frac{1}{10000}\right)} - \frac{1.e^{\frac{-y}{10000}}}{\left(-\frac{1}{10000}\right)^2} \right]_{10000}^{\infty}$$

$$= \frac{1}{(10000)^2}\left[0 + \frac{10000 e^{-1}}{\frac{1}{10000}} + \frac{e^{-1}}{\left(\frac{1}{10000}\right)^2} \right]$$

$$= (e^{-1} + e^{-1}) = 2e^{-1} = \frac{2}{e}$$

EXAMPLE 3.122 Consumer demand for milk in a certain locality per month is known to be a general Gamma (Erlang) random variable. If the average demand is a litres and the most likely demand is b litres ($b < a$), what is the variance of the demand?

Solution Let X represent the monthly consumer demand of milk. Average demand is the value of $E(X)$. Most likely demand value of X for which its density function $f(x)$ is maximum is

$$f'(x) = 0 \text{ and } f''(x) < 0$$

If $f(x)$ is the density function of X, then

$$f(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}, \quad x > 0, E(X) = \frac{k}{\lambda} = a \qquad \text{(i) (given)}$$

$$f'(x) = \frac{\lambda^k}{\Gamma(k)} \left[(k-1)x^{k-2} e^{-\lambda x} - \lambda x^{k-1} e^{-\lambda x} \right]$$

$$= \frac{\lambda^k}{\Gamma(k)} x^{k-2} e^{-\lambda x} \left[(k-1) - \lambda x \right] = 0 \Rightarrow x = 0$$

or $$x = \frac{k-1}{\lambda}$$

$$f''(x) = \frac{\lambda^k}{\Gamma(k)} \left\{ -\lambda x^{k-2} e^{-\lambda x} + [(k-1) - \lambda x] \frac{d}{dx} (x^{k-2} e^{-\lambda x}) \right\}$$

$f''(x) < 0$, when $x = \dfrac{k-1}{\lambda}$

Therefore, $f(x)$ is maximum, when $x = \dfrac{k-1}{\lambda}$

i.e. most likely demand $= \dfrac{k-1}{\lambda} - b$

$$E(X) = \frac{k}{\lambda} = a, \text{ using it} \qquad \text{(i)}$$

$$\frac{k}{\lambda} - \frac{1}{\lambda} = b \quad \Rightarrow \quad a - \frac{1}{\lambda} = b$$

i.e. $$a - b = \frac{1}{\lambda} \qquad \text{(ii)}$$

Now, $$\text{Var}(X) = \frac{k}{\lambda^2} = \frac{k}{\lambda} \cdot \frac{1}{\lambda} = a(a-b) \qquad \text{[From Eqs. (i) and (ii)]}$$

EXAMPLE 3.123 In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be treated as a random variable having Gamma or Erlang distribution with parameters $\lambda = 1/2$ and $k = 3$. If the power plant of this city has a daily capacity of 12 millions kilowatt-hours, what is the probability that this power supply will be inadequate on any day?

Solution Let X denote the daily consumption of electric power (in million of kilowatt-hours) and given

$$\lambda = \frac{1}{2} \text{ and } k = 3$$

Then the PDF of X is given by

$$f(x)=\frac{\left(\frac{1}{2}\right)^3}{\Gamma(3)}x^2e^{\frac{-x}{2}},\ x\geq 0$$

The daily capacity of the power plant is 12 millions kilowatt-hours. The power supply is inadequate only if the demand for power supply is more than 12 millions on any day.

P(the power supply is inadequate)

$$=P(X>12)=\int_{12}^{\infty}f(x)dx=\int_{12}^{\infty}\frac{1}{\Gamma(3)}\left(\frac{1}{8}\right)x^2e^{\frac{-x}{2}}\,dx$$

$$=\frac{1}{16}\left[x^2\left(\frac{e^{\frac{-x}{2}}}{-\frac{1}{2}}\right)-(2x)\left(\frac{e^{\frac{-x}{2}}}{\frac{1}{4}}\right)+2\left(\frac{e^{\frac{-x}{2}}}{-\frac{1}{8}}\right)\right]_{12}^{\infty}$$

$$=\frac{e^{-6}}{16}(288+96+16)=25e^{-6}=0.0625$$

EXAMPLE 3.124 The lifetime (in hours) of a certain piece of equipment is a continuous random variable having range $0<x<\infty$ and the PDF is

$$f(x)=\begin{cases}xe^{-kx}, & 0<x<\infty\\ 0, & \text{otherwise}\end{cases}$$

Determine the constant k and evaluate the probability that the life-time exceeds 2 hours.

Solution Let X denote the life-time of a certain piece of equipment with the PDF

$$f(x)=\begin{cases}xe^{-kx}, & 0<x<\infty\\ 0, & \text{otherwise}\end{cases}$$

To find k:

$$\int_{-\infty}^{\infty}f(x)dx=1\ \Rightarrow\ \int_{0}^{\infty}e^{-kx}x\,dx=1\ \Rightarrow\ \int_{0}^{\infty}e^{-kx}x^{2-1}dx=1$$

Using $\int_0^{\infty}x^{n-1}e^{-ax}\,dx=\frac{\Gamma(n)}{a^n}$, we get

$$\frac{\Gamma(2)}{k^2}=1\ \Rightarrow\ k^2=1$$

$$k = 1$$

$\therefore$
$$f(x) = \begin{cases} xe^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

$\therefore$

P(life-time exceeds 2 hours) $= P(X > 2)$

$$= \int_2^\infty f(x)dx = \int_2^\infty xe^{-x}\,dx = \left[x(-e^{-x}) - (e^{-x})\right]_2^\infty$$

$$= 2e^{-2} + e^{-2} = 3e^{-2} = 0.4060$$

EXAMPLE 3.125 A random sample of size n is taken from a general Gamma (Erlang) distribution with parameters λ and k. Show that the mean $\bar{X}$ of the sample also follows a Gamma distribution with parameters λ and nk.

Solution If X follows Gamma distribution with parameters λ and k, then the MGF of X is

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^k$$

If $X_1, X_2, X_3, \ldots X_n$ are the members of the sample drawn, then each X_i, $i = 1, 2, \ldots, n$ follows Erlang distribution with MGF $\left(\dfrac{\lambda}{\lambda - t}\right)^k$.

Since $X_1, X_2, X_3, \ldots, X_n$ are independent by the reproductive property of Gamma distribution.

Now,

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$M_{\bar{X}}(t) = \frac{M_{X_1 + X_2 + \cdots + X_n}(t)}{n}$$

$$= M_{X_1 + X_2 + \cdots + X_n}\left(\frac{t}{n}\right)$$

$$= M_{X_1}\left(\frac{t}{n}\right) M_{X_2}\left(\frac{t}{n}\right) \cdots M_{X_n}\left(\frac{t}{n}\right)$$

$$= \left(\frac{\lambda}{\lambda - \frac{t}{n}}\right)^k \left(\frac{\lambda}{\lambda - \frac{t}{n}}\right)^k \cdots \left(\frac{\lambda}{\lambda - \frac{t}{n}}\right)^k \quad (n \text{ times})$$

$$M_{\bar{X}}(t) = \left(\frac{\lambda}{\lambda - \frac{t}{n}}\right)^{nk}$$

$\therefore$ $\bar{X}$ follows Gamma distribution with parameter λ and nk.

3.2.4 Weibull Distribution

The random variable X is said to follow Weibull distribution if its probability distribution is given by

$$f(x)=\begin{cases}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, & x>0\\ 0, & \text{otherwise}\end{cases}$$

where $\alpha>0$ and $\beta>0$ are the two parameters of the Weibull distribution.

Note: When $\beta=1$, Weibull distribution reduces to the exponential distribution with parameter α.

Moments of Weibull Distribution

EXAMPLE 3.126 Find the mean and variance of the Weibull distribution.

Solution The probability density function of Weibull distribution is given by

$$f(x)=\begin{cases}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, & x>0\\ 0, & \text{otherwise}\end{cases}$$

where α and β are the parameters.

$$\text{Mean} = E(X)=\mu_1'=\int_0^{\infty}x\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx=\alpha\beta\int_0^{\infty}x^{\beta}e^{-\alpha x^{\beta}}\,dx$$

Put $\qquad \alpha x^{\beta}=y \Rightarrow x=\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}$

$$dx=\frac{1}{\beta}\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}-1}\cdot\frac{1}{\alpha}\cdot dy$$

when $x=0$, $y=0$ and $x=\infty$, $y=\infty$

$$E(X)=\alpha\beta\int_0^{\infty}\left(\frac{y}{\alpha}\right)e^{-y}\cdot\frac{1}{\alpha\beta}\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}-1}\cdot dy$$

$$=\left(\frac{1}{\alpha}\right)^{\frac{1}{\beta}}\int_0^{\infty}e^{-y}\cdot y^{\frac{1}{\beta}+1-1}\,dy$$

$$\mu_1'=\frac{1}{\alpha^{\frac{1}{\beta}}}\Gamma\left(\frac{1}{\beta}+1\right) \qquad \left[\because \Gamma(n)=\int_0^{\infty}x^{n-1}e^{-x}dx\right]$$

$$E(X^2)=\mu_2'=\int_0^{\infty}x^2\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx=\alpha\beta\int_0^{\infty}x^{\beta+1}e^{-\alpha x^{\beta}}\,dx$$

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Making the subsitution as $\alpha x^{\beta} = y$, we get

$$\mu_2' = \int_0^\infty e^{-y}\left(\frac{y}{\alpha}\right)^{\frac{2}{\beta}} dy = \frac{1}{a^{2/\beta}}\int_0^\infty e^{-y} y^{\frac{2}{\beta}} dy$$

$$= \frac{1}{\alpha^{\frac{2}{\beta}}}\int_0^\infty e^{-y} y^{\left(\frac{2}{\beta}+1-1\right)} dy = \frac{1}{\alpha^{\frac{2}{\beta}}}\Gamma\left(\frac{2}{\beta}+1\right)$$

$$\text{Var} = \mu_2' - (\mu_1')^2.$$

$$= \frac{1}{\alpha^{\frac{2}{\beta}}}\Gamma\left(\frac{2}{\beta}+1\right) - \frac{1}{\alpha^{\frac{2}{\beta}}}\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2$$

$$= \frac{1}{\alpha^{\frac{2}{\beta}}}\left\{\Gamma\left(\frac{2}{\beta}+1\right) - \left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2\right\}$$

In general, the rth moment about the origin is

$$\mu_r' = E(X^r) = \frac{1}{\alpha^{(r/\beta)}}\Gamma\left(\frac{r}{\beta}+1\right)$$

Note: For a Weibull distribution

$$p(X \le a) = 1 - e^{-\alpha x^{\beta}},\ p(X > a) = e^{-\alpha x^{\beta}}.$$

EXAMPLE 3.127 Each of the 6 tubes of a radio set has a life length (in years) which may be considered as a random variable that follows a Weibull distribution with parameters $\alpha = 25$ and $\beta = 2$. If these tubes function independently of one another, what is the probability that no tube will have to be replaced during the first 2 months of service? **[AU April '04]**

Solution If X represents the life length of each tube, in years then its density function $f(x)$ is given by

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}},\quad x > 0$$

Given: $\alpha = 25,\ \beta = 2$

i.e., $f(x) = 50xe^{-25x^2},\ x > 0$

$$2 \text{ months} = \frac{2}{12} = \frac{1}{6} \text{ year}$$

P(a tube is not to be replaced during the first 2 months)

$$= P\left(X > \frac{1}{6}\right) = \int_{\frac{1}{6}}^{\infty} 50xe^{-25x^2} dx$$

Put $X = 25x^2$

$$P\left(X > \frac{1}{6}\right) = \int_{\frac{1}{6}}^{\infty} e^{-25x^2} d(25x^2) = \int_{\frac{1}{6}}^{\infty} e^{-X} d(X) = \left[\frac{e^{-X}}{-1}\right]_{\frac{1}{6}}^{\infty}$$

$\Rightarrow$

$$= \left[-e^{-25x^2}\right]_{\frac{1}{6}}^{\infty} = e^{-25/36} = 0.4993$$

$\therefore$ P(all the 6 tubes are not to be replaced during the first 2 months)
$= (e^{-25/36})^6 = e^{-25/6} = 0.0155$ (by independence)

EXAMPLE 3.128 If the life X (in years) of a certain type of car has a Weibull distribution with the parameter $\beta = 2$, find the value of the parameter α given that probability that the life of the car exceeds 5 years is $e^{-0.25}$. For these values of α and β, find the mean and variance of X.

[AU November '07, December '04]

Solution The density function of X is given by

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, \quad x > 0$$

Given: $\beta = 2$

$$f(x) = 2\alpha x e^{-\alpha x^2}, \quad x > 0 \qquad (\because \beta = 2)$$

$$P(X > 5) = \int_5^{\infty} 2\alpha x e^{-\alpha x^2} dx = \int_5^{\infty} e^{-\alpha x^2} d(\alpha x^2) = \left[-e^{-\alpha x^2}\right]_5^{\infty} = e^{-25\alpha}$$

Given that $P(X > 5) = e^{-0.25}$

$\therefore$

$$e^{-25\alpha} = e^{-0.25} \Rightarrow 25\alpha = 0.25 \Rightarrow \alpha = \frac{1}{100}$$

For the Weibull distribution with parameters α and β,

$$E(X) = \alpha^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right)$$

$\therefore$ Required mean $= \left(\frac{1}{100}\right)^{-\frac{1}{2}} \cdot \Gamma\left(\frac{3}{2}\right) = 10 \times \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = 5\sqrt{\pi} \qquad [\Gamma(n+1) = n\Gamma(n)]$

$$\text{Var}(X) = \alpha^{-\frac{2}{\beta}} \left\{\Gamma\left(\frac{2}{\beta} + 1\right) - \left[\Gamma\left(\frac{1}{\beta} + 1\right)\right]^2\right\}$$

$$= \left(\frac{1}{100}\right)^{-1} \left\{\Gamma(2) - \left[\Gamma\left(\frac{3}{2}\right)\right]^2\right\}$$

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$$= 100\left[1-\left(\frac{1}{2}\sqrt{\pi}\right)^2\right] = 100\left(1-\frac{\pi}{4}\right) = 21.46$$

EXAMPLE 3.129 If Y is the smallest item of 3 independent observations X_1, X_2, X_3 from a Weibull distribution with parameters α and β, show that Y also has a Weibull distribution. What are its parameters?

Solution Each of X_1, X_2, X_3 follows the Weibull distribution whose density function is given by

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, \quad x > 0$$

Now, $$P(Y > y) = P[\min(X_1, X_2, X_3) > y)]$$
$$= P(X_1 > y) \times P(X_2 > y) \times P(X_3 > y),$$
(since X_1, X_2, X_3 are independent)
$$= [P(X_i > y)]^3, \quad i = 1, 2, 3$$

Now, $$P(X_i > y) = \int_y^{\infty} \alpha\beta x^{\beta-1} e^{-\alpha x^{-\beta}} dx \qquad \text{(i)}$$
$$= \int_y^{\infty} \alpha x e^{-\alpha x^{\beta}} dx = \int_y^{\infty} e^{-\alpha x^{\beta}} d(\alpha x^{\beta})$$
$$= \left[-e^{-\alpha x^{\beta}}\right]_y^{\infty} = e^{-\alpha y^{\beta}} \qquad \text{(ii)}$$

Using Eqs. (ii) in (i), we have

$$P(Y > y) = \left(e^{-\alpha y^{\beta}}\right)^3 = e^{-3\alpha y^{\beta}}$$

Therefore, Y has a Weibull distribution with parameter 3α and β.

EXAMPLE 3.130 The life-time of a component measured in hours follows Weibull distribution with parameters $\alpha = 0.2$, $\beta = 0.5$. Find the mean life-time of the component.

[AU April '03, June '05]

Solution Mean of the Weibull distribution is

$$E(X) = \alpha^{\frac{-1}{\beta}}\, \Gamma\left(1+\frac{1}{\beta}\right)$$

Given: $$\alpha = 0.2, \ \beta = 0.5$$

$$E(X) = (0.2)^{\frac{-1}{0.5}}\, \Gamma\left(1+\frac{1}{0.5}\right) = 25\Gamma(1+2)$$

$$= 25\Gamma(3) = 25 \times 2! = 50 \text{ hours}$$

3.2.5 Normal Distribution or Gaussian Distribution

A random variable X is said to follow a normal distribution with mean μ and variance σ^2, denoted by $N(\mu, \sigma)$, if its density function is given by the probability law

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0 \qquad (3.8)$$

Special Cases

When $\mu = 0$ and $\sigma^2 = 1$, we get the standard normal distribution denoted by $N(0, 1)$ with PDF given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad -\infty < x < \infty.$$

Note: To prove total probability equal to 1

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let $\dfrac{x-\mu}{\sigma\sqrt{2}} = t, \quad dx = \sigma\sqrt{2}dt$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sigma\sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$$

Moment Generating Function of Normal Distribution

By definition,

$$M_X(t) = E(e^{tx}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Put $Z = \dfrac{(x-\mu)}{\sigma}$ so that $dx = \sigma dz$, $-\infty < Z < \infty$, $x = \sigma z + \mu$

$$\therefore \quad M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma Z + \mu)} e^{-\frac{Z^2}{2}} dz = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{Z^2}{2} - t\sigma Z\right)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[-\frac{1}{2}(Z - t\sigma)^2 + \frac{\sigma^2 t^2}{2}\right]} dz = \frac{e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z - t\sigma)^2} dz$$

Since the total area under normal curve is unity, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\,dz = 1$$

Hence,

$$M_X(t) = e^{\mu t+\frac{\sigma^2 t^2}{2}}$$

$$M_X(t) = e^{t\left(\mu+\frac{\sigma^2 t}{2}\right)}$$

$$= 1+\frac{t\left(\mu+\frac{\sigma^2 t}{2}\right)}{1!}+\frac{t^2\left(\mu+\frac{\sigma^2 t}{2}\right)^2}{2!}$$

$$+\frac{t^3\left(\mu+\frac{\sigma^2 t}{2}\right)^3}{3!}+\frac{t^4\left(\mu+\frac{\sigma^2 t}{2}\right)^4}{4!}+\ldots$$

$E(X)$ = equating the coefficients of $\frac{t}{1!} = \mu$

$E(X^2)$ = equating the coefficients of $\frac{t^2}{2!} = \sigma^2+\mu^2$

$\therefore$ $\quad \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \sigma^2+\mu^2-\mu^2 = \sigma^2.$

Note: For standard normal distribution $N(0, 1)$, $M_X(t) = e^{\frac{t^2}{2}}$.

Central Moments of Normal Distribution

$$\mu_r = E[(X-\mu)^r]$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^r e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

Let $\quad \frac{(x-\mu)}{\sigma\sqrt{2}} = t \Rightarrow dx = \sqrt{2}\sigma\,dt$

$$\mu_r = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\sqrt{2}\sigma t)^r e^{-t^2}\,dt$$

$$\mu_r = \frac{2^{\frac{r}{2}}\sigma^r}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^r e^{-t^2}\,dt$$

Case 1: Odd Moments about Mean

Let $r = 2n + 1$

$$\mu_{2n+1} = E[(X-\mu)^{2n+1}] = \frac{2^{\frac{2n+2}{2}}\sigma^{2n+1}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^{2n+1} e^{-t^2}\, dt = 0$$

Since the integrand $t^{2n+1}e^{-t^2}$ is an odd function

Hence all odd moments about the mean are zero for normal distribution.

Case 2: Even Moments

Let $r = 2n$

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^{2n} e^{-t^2}\, dt = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \cdot 2\int_{0}^{\infty} t^{2n} e^{-t^2}\, dt,$$

($\because$ the integrad is an even function)

Put $t^2 = u \Rightarrow 2t\, dt = du \Rightarrow dt = \dfrac{du}{2t}, dt = \dfrac{du}{2\sqrt{u}}$

$$\therefore \quad \mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{1\pi}} \int_0^{\infty} e^{-u} u^{n-\frac{1}{2}} du = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{n-\frac{1}{2}} du$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\left(n+\frac{1}{2}\right)-1} du \Rightarrow \mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right) \qquad \text{...(i)}$$

We know that

$$\Gamma(n+1) = n\Gamma(n)$$

$$\therefore \quad \Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\left(n-\frac{1}{2}\right)+1\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(\left(n-\frac{3}{2}\right)+1\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\therefore \quad \Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

Substituting in (i) we get,

$$\mu_{2n} = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\sigma^{2n}2^n}{\sqrt{\pi}}\cdot\sqrt{\pi}$$

$$= \frac{(2n-1)}{2}\frac{(2n-3)}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sigma^{2n}2^n$$

$$\mu_{2n} = (2n-1)(2n-3) \ldots 3 \cdot 1 \cdot \sigma^{2n}$$

$\therefore$ when $n = 1$, $\mu_2 = \sigma^2$, and when $n = 2$, $\mu_4 = 3\sigma^4$ and so on.

Median of Normal Distribution $N(\mu, \sigma)$

If X is a continuous random variable with density function $f(x)$, then M is called the median value of X, provided that

$$\int_{-\infty}^{M} f(x)dx = \int_{M}^{\infty} f(x)dx = \frac{1}{2}$$

For the normal distribution $N(\mu, \sigma)$, the median M is given by

$$\int_{M}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{2}$$

i.e.,
$$\int_{M}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \int_{\mu}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{2}$$

since $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = 1$ and the normal curve is symmetrical about $x = \mu$

$$\int_{M}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \frac{1}{2} = \frac{1}{2}$$

i.e.,
$$\int_{M}^{\mu} f(x)dx = 0 \quad \Rightarrow \quad M = \mu$$

Mode of Normal Distribution $N(\mu, \sigma)$

Mode of a continuous random variable X is defined as the value of x for which the density function $f(x)$ is maximum.

For the normal distribution $N(\mu, \sigma)$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$$\therefore \quad \log f(x) = k - \frac{1}{2\sigma^2}(x-\mu)^2 \left[k = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right)\right]$$

Differentiating with respect to x,

$$\frac{f'(x)}{f(x)} = -\frac{1}{\sigma^2}(x-\mu)$$

i.e.

$$f'(x) = -\frac{1}{\sigma^2}(x-\mu)f(x) = 0, \text{ when } x = \mu$$

$$f''(x) = -\frac{1}{\sigma^2}[(x-\mu)f'(x) + f(x)]$$

$\therefore$

$$[f''(x)]_{x=\mu} = -\frac{1}{\sigma^2}f(\mu) < 0$$

Therefore, $f(x)$ is maximum at $x = \mu$. That is, Mode of the distribution $N(\mu, \sigma) = \mu$.

Note: For the normal distribution, mean, median and mode are equal.

$$\text{Mean} = \text{Median} = \text{Mode} = \mu$$

Mean Deviation about Mean of Normal Distribution $N(\mu, \sigma)$

The absolute (central) moment of the first order of a random variable X is called the mean deviation (MD) about the mean of X.

$$\text{MD about the mean} = E\{|X - E(X)|\} = E\{|X - \mu|\}$$

For the normal distribution $N(\mu, \sigma)$,

$$\text{MD about the mean} = \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sqrt{2}\sigma t| e^{-t^2} \sqrt{2}\sigma \, dt, \text{ by facing } \frac{x-\mu}{\sqrt{2}\sigma} = t$$

$$= \sqrt{\frac{2}{\pi}}\sigma \int_{-\infty}^{\infty} |t| e^{-t^2} dt$$

$$= 2\sqrt{\frac{2}{\pi}}\sigma \int_{0}^{\infty} t e^{-t^2} dt, \text{ } (\because \text{ the integrand is an even function of } t)$$

$$= \sqrt{\frac{2}{\pi}}\sigma \left[\frac{e^{-t^2}}{-1}\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\sigma = \frac{4}{5}\sigma \text{ (approx.)}$$

Quartile Deviation of Normal Distribution $N(\mu, \sigma)$

The first quartile Q_1 and the third quartile Q_3 of $N(\mu, \sigma)$ (or of any continuous random variable) are defined by the equations

$$\int_{-\infty}^{Q_1} f(x)dx = \frac{1}{4} \quad \text{and} \quad \int_{-\infty}^{Q_3} f(x)dx = \frac{3}{4}$$

or equivalently

$$\int_{Q_1}^{\mu} f(x)dx = \frac{1}{4} \quad \text{and} \quad \int_{\mu}^{Q_3} f(x)dx = \frac{1}{4}$$

Proof If the curve $y = f(x)$ is symmetrical about $x = \mu$, then the quartile deviation (QD) is defined as

$$\text{QD} = \frac{1}{2}(Q_3 - Q_1)$$

For the normal distribution $N(\mu, \sigma)$, Q_1 is given by

$$\int_{Q_1}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{4}$$

$$\int_{\frac{(Q_1-\mu)}{\sigma}}^{0} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{4} \qquad \left(\text{putting } z = \frac{x-\mu}{\sigma}\right)$$

By symmetry of the normal curve and since $\frac{Q_1 - \mu}{\sigma} < 0$

$$\int_{0}^{\frac{(\mu - Q_1)}{\sigma}} \phi(z)\, dz = 0.25$$

From the table of normal areas (areas under standard normal curve), we get

$\therefore$
$$\int_0^{0.674} \phi(z)dz = 0.25$$

$\Rightarrow$
$$\frac{\mu - Q_1}{\sigma} = 0.674$$

i.e.
$$Q_1 = \mu - 0.674\sigma$$

By symmetry,

$$Q_3 = \mu + 0.674\sigma$$

$$QD = \frac{1}{2}(Q_3 - Q_1) = 0.674\sigma = \frac{2}{3}\sigma \text{ (approx.)}$$

Note: If X has the distribution $N(\mu, \sigma)$, then $Y = aX + b$ has the distribution $N(a\mu + b, a\sigma)$.

Proof $\quad M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$M_Y(t) = M_{aX+b}(t) = e^{bt} M_X(at)$$

$$\therefore \quad = e^{bt} e^{a\mu t + \frac{a^2\sigma^2 t^2}{2}} = e^{(a\mu + b)t + \frac{a^2\sigma^2 t^2}{2}}$$

which is the MGF of $N(a\mu + b, a\sigma)$.

In particular, if X has the distribution $N(\mu, \sigma)$, then $Z = \dfrac{X - \mu}{\sigma}$ has the distribution $N\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma} \cdot \sigma\right)$, i.e. $N(0, 1)$.

Additive Property or Reproductive Property

If X_i $(i = 1, 2, \ldots, n)$ are n independent normal random variables with mean μ_i and variance σ_i^2, then $\sum_{i=1}^{n} a_i X_i$ is also a normal random variable with mean $\sum_{i=1}^{n} a_i \mu_i$ and variance $\sum_{i=1}^{n} a_i^2 \sigma_i^2$.

Proof If X_i $(i = 1, 2, \ldots, n)$ are n independent normal random variables then

$$M_{\sum_{i=1}^{n} a_i X_i}(t) = M_{a_1X_1 + a_2X_2 + a_3X_3 + \cdots + a_nX_n}(t)$$

$$= M_{a_1X_1}(t) M_{a_2X_2}(t), \ldots, M_{a_nX_n}(t)$$

$$= M_{X_1}(a_1 t) M_{X_2}(a_2 t), \ldots, M_{X_n}(a_n t)$$

But $\quad M_{X_i}(t) = e^{\mu_i t + \sigma_i^2 \frac{t^2}{2}}, \quad i = 1, 2, \ldots, n$

$$M_{\sum a_i X_i}(t) = e^{a_1\mu_1 t + a_1^2 \sigma_1^2 \frac{t^2}{2}} + e^{a_2\mu_2 t + a_2^2\sigma_2^2 \frac{t^2}{2}} + \cdots + e^{a_n\mu_n t + a_n^2\sigma_n^2 \frac{t^2}{2}}$$

$$= \exp\left[(\Sigma a_i \mu_i)t + (\Sigma a_i^2 \sigma_i^2)\frac{t^2}{2}\right]$$

which is the MGF of the normal random variable with mean

$$\mu = \sum_{i=1}^{n} a_i \mu_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2 .$$

Hence the proof.

Deductions

1. If X_1 is $N(\mu_1, \sigma_1)$ and X_2 is $N(\mu_2, \sigma_2)$, then $X_1 + X_2$ is $N\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)$

   Similarly, $X_1 - X_2$ is $N\left(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)$

   Putting $a_1 = a_2 = 1$ and $a_3 = a_4 = \cdots = a_n = 0$, we get the above result.
2. If X_i ($i = 1, 2, ..., n$) are independent and identically distributed normal random variable with mean μ and standard deviation σ, then their mean $\bar{X}$ is $N(\mu, \sigma/\sqrt{n})$.

   Putting $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$ and assuming that each X_i is $N(\mu, \sigma)$, we get

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$\therefore$ $\bar{X}$ has a normal distribution $N\left(\frac{1}{n}\sum_{i=1}^{n}\mu, \sqrt{\sum_{i=1}^{n}\frac{1}{n^2}\sigma^2}\right) = N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

Normal Distribution as Limiting Form of Binomial Distribution

When n is very large and neither p nor q is very small, the standard normal distribution can be regarded as the limiting form of the standardized binomial distribution.

Proof When X follows the binomial distribution $B(n, p)$, the standardized binomial variable Z is given by $Z = \frac{X - np}{\sqrt{npq}}$. As X varies from 0 to n with step size 1, Z varies from $\frac{-np}{\sqrt{npq}}$ to $\frac{np}{\sqrt{npq}}$ with step size $\frac{1}{\sqrt{npq}}$. When neither

p nor q is very small and n is very large, Z varies from $-\infty$ to ∞ with infinitesimally small step size. Hence, in the limit, the distribution of Z may be expected to be a continuous distribution extending from $-\infty$ to ∞ and having mean 0 and standard deviation 1. In fact, the limiting form of the distribution of Z is standard normal distribution as seen below.

If X follows $B(n, p)$, then the MGF of X is given by

$$M_X(t) = (q + pe^t)^n$$

If $\quad Z = \dfrac{X - np}{\sqrt{npq}}$, then

$$M_Z(t) = M_{\frac{X}{\sqrt{npq}} - \frac{np}{\sqrt{npq}}}(t) = e^{\frac{-npt}{\sqrt{npq}}}\left(q + pe^{\frac{t}{\sqrt{npq}}}\right)^n$$

$$\log M_Z(t) = \frac{-npt}{\sqrt{npq}} + n\log\left(q + pe^{\frac{t}{\sqrt{npq}}}\right)$$

$$= \frac{-npt}{\sqrt{npq}} + n\log\left[q + p\left(1 + \frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + \frac{t^3}{6(npq)^{\frac{3}{2}}} + \cdots\right)\right]$$

$$= \frac{-npt}{\sqrt{npq}} + n\log\left[1 + \left(\frac{pt}{\sqrt{npq}} + \frac{pt^2}{2npq} + \frac{pt^3}{6(npq)^{\frac{3}{2}}} + \cdots\right)\right] \quad [\because p + q = 1]$$

$$= \frac{-npt}{\sqrt{npq}} + n\left[\frac{pt}{\sqrt{npq}}\left(1 + \frac{t}{2\sqrt{npq}} + \frac{t^2}{6npq} + \cdots\right)\right.$$

$$\left. - \frac{1}{2}\frac{p^2t^2}{npq}\left(1 + \frac{t}{2\sqrt{npq}} + \frac{t^2}{6npq} + \cdots\right)^2 + \cdots\right]$$

$$\left[\because \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$$

$$= \left(\frac{t^2}{2q} - \frac{pt^2}{2q}\right) + \text{terms containing } \frac{1}{\sqrt{n}} \text{ and lower powers of } n$$

$$= \frac{t^2}{2} + \text{terms containing } \frac{1}{\sqrt{n}} \text{ and lower powers of } n$$

$\therefore$
$$\lim_{n \to \infty} \log M_Z(t) = \frac{t^2}{2}$$

i.e.
$$\log_e\left(\lim_{n \to \infty} M_Z(t)\right) = \frac{t^2}{2}$$

$$\lim_{n \to \infty} M_Z(t) = e^{\frac{t^2}{2}}$$

$\therefore$

which is the MGF of the standard normal distribution. Hence the limit of the standardized binomial distribution, as n tends to ∞, is the standard normal distribution.

Importance of Normal Distribution

Normal distribution plays a very important role in statistical theory because of the following reasons:

1. A large number of random variables such as binomial and Poisson, occurring in many applications have a distribution closely resembling the normal distribution.
2. Many of the distributions of sample statistics, such as sample mean and sample variance, tend to normality for samples of large size. In particular, the sampling distributions like Student's t, Snedecor's F and Chi-square distributions tend to normality when the size of the sample is large.
3. Tests of significance for small samples are based on the assumption that samples have been drawn from normal populations.
4. Even if a variable is not normally distributed, it can sometimes be converted into a normal variable by simple transformation of the variable.
5. Normal distribution is applied to a large extent in statistical Quality Control in industry.

Area Under the Normal Curve

A random variable X is said to follow normal distribution with mean μ and variance σ^2 if its density function is given by the probability law

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0 \qquad (3.9)$$

$$\text{Area} = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Put $\quad \frac{x-\mu}{\sigma} = z;\ \sigma dz = dx$, when $x = \infty, z = \infty, x = -\infty, z = -\infty$

$\therefore$

$$\text{Area} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-z^2}{2}} \sigma dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\frac{-z^2}{2}} dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\left(\frac{z^2}{2}\right)} dz$$

Put
$$\frac{z}{\sqrt{2}} = u, dz = \sqrt{2}du$$

$\therefore$
$$\text{Area} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-u^2} \cdot \sqrt{2}du = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1 \qquad \left(\because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}\right)$$

$\therefore$ Area bounded by the normal curve is 1.

The equation of the normal curve gives the ordinate of the curve corresponding to any given value of x. We are usually interested in areas under the normal curve instead of its ordinate. The area under the curve gives us the proportion of the cases falling between two numbers or the probability of getting a value between two numbers.

The equation of the normal curve depends on μ and σ and for different values of μ and σ we will obtain different curves. Fortunately, we will be able to determine normal curve areas regardless of μ and σ by tabulating only the areas under the normal curve having $\mu = 0$ and $\sigma = 1$. Such a normal curve with 0 mean and unit standard deviation is known as the standard normal curve (Figure 3.1).

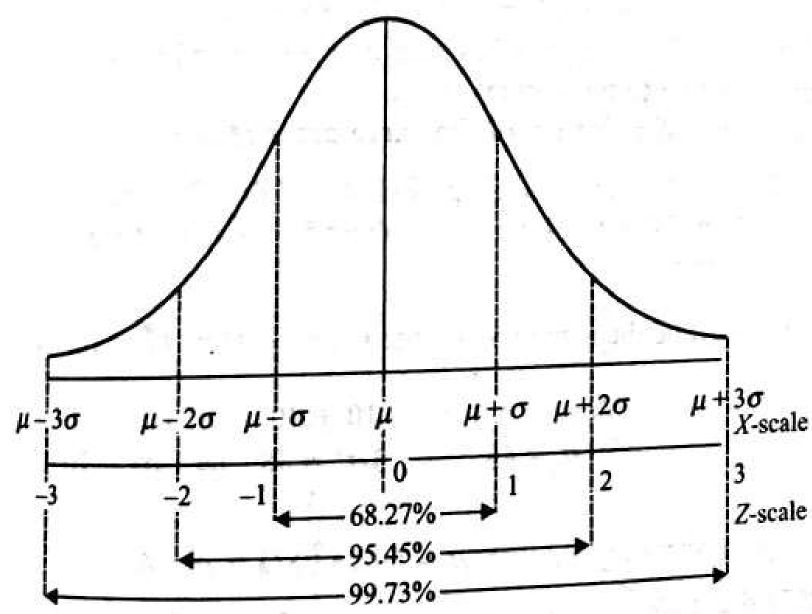

Figure 3.1 Standard normal curve.

A normal curve with mean μ and SD σ can be converted into a standard normal distribution by performing the change of the scale and origin as indicated above. In the original scale (x-scale) the mean and SD are μ and σ, in the new scale (the z-scale) they are 0 and 1. The formula that enables us to change from the x-scale to z-scale and vice versa is $Z = \frac{X - \mu}{\sigma}$. This transformation from X to Z is named as Z-transformation.

Note: Entries corresponding to negative values of z are unnecessary because the normal curve is symmetrical, i.e. the probability that Z is between (say) -1 to 0 is equal to the probability that Z lies between 0 and 1, i.e. $p(-1 \leq z \leq 0) = p(0 \leq z \leq 1)$.

Applications of Normal Distribution

The normal distribution is mostly used for the following purposes:

1. To approximate or "fit" a distribution of measurement is simple under certain conditions.
2. To approximate the binomial distribution and other discrete or continuous probability distributions under suitable conditions.
3. To approximate the distribution of means and certain other quantities calculated from samples, especially large samples.

Properties of Normal Curve

1. The normal curve is symmetrical about the mean (skewness = 0).
2. The height of the normal curve is at its maximum at the mean. For normal distribution mean, median and mode are equal.
3. The value of $f(x)$ approaches zero as $x \to -\infty$ and $x \to \infty$, i.e. X-axis is an asymptote to the normal curve.
4. The points of inflection of the curve occur at $\mu \pm \sigma$.

EXAMPLE 3.131 For a certain period normal distribution, the first moment about 10 is 40 and that the 4th moment about 50 is 48, what are the parameters of the distribution?

Solution Let μ be the mean and σ^2 the variance, then $\mu_1' = 40$ about $A = 10$.

$$\text{Mean} = A + \mu_1' = 10 + 40 \Rightarrow \mu = 50$$

Also

$$\mu_4 = 48 \Rightarrow 3\sigma^2 = 48 \Rightarrow \sigma^2 = 16$$
$$\Rightarrow \sigma = 4$$

$\therefore$ The parameters are mean $= \mu = 50$ and SD $= \sigma = 4$.

EXAMPLE 3.132 X is a normal variate with mean 2 and variance 4. Y is another normal variate independent of X with mean 2 and variance 3. What is the distribution of $X + 2Y$.

[AU December '09]

Solution Given X and Y are independent normal variates and $E(X) = 2$, $\text{Var}(X) = 4$, $E(Y) = 2$, $\text{Var}(Y) = 3$.

$\therefore$

$$\text{Mean of } (X + 2Y) = E(X + 2Y) = E(X) + E(2Y)$$

$$= E(X) + 2E(Y) = 1 + (2)\,(2) = 5$$

$$\text{Var}(X + 2Y) = \text{Var}(X) + 4\ \text{Var}(Y)$$

$$= (1 \times 4) + (4 \times 3) = 16$$

$\therefore$ $X + 2Y$ follows normal distribution with mean 5 and variance 16.

EXAMPLE 3.133 If X is a normal variate with mean 2 and SD 3, describe the distribution of $Y = \frac{1}{2}X - 1$. Also find $P\left(Y \geq \frac{3}{2}\right)$.

Solution Given: X is a normal variate with mean 2 and variance 9, i.e. $E(X) = 2$, $\text{Var}(X) = 9$

We know that if X is a normal variate, then $Y = aX + b$ is also a normal variate.

$$\therefore \qquad E(Y) = E\left(\frac{1}{2}X - 1\right) = \frac{1}{2}E(X) - 1 = \frac{1}{2} \times (2) - 1 = 0$$

$$\text{Var}(Y) = \text{Var}\left(\frac{1}{2}X - 1\right) = \left(\frac{1}{2}\right)^2 \text{Var}(X) = \frac{1}{4} \times 9 = \frac{9}{4}$$

$$\therefore \qquad \text{SD of } Y = \sqrt{\frac{9}{4}} = \frac{3}{2}, \text{ Mean of } Y = 0$$

$$\therefore \qquad Z = \frac{X - \mu}{\sigma} = \frac{Y - 0}{\frac{3}{2}} = \frac{Y}{\frac{3}{2}}$$

When $Y = \frac{3}{2}$, $Z = 1$

$$\therefore \qquad P\left(Y \geq \frac{3}{2}\right) = P(Z \geq 1) = 0.5 - P(0 < Z \leq 1)$$

$$= 0.5 - 0.3413 = 0.1587$$

EXAMPLE 3.134 A normal curve has $\mu = 20$ and $\sigma = 10$. Find the area between $x_1 = 15$ and $x_2 = 40$. **[AU November '07, May '03]**

Solution We know that the normal variate $Z = \frac{X - \mu}{\sigma}$,

when $x_1 = 15$, $\qquad Z = \frac{x_1 - \mu}{\sigma} = \frac{15 - 20}{10} = -0.50$

and when $\qquad x_2 = 40$, $Z = \frac{40 - 20}{10} = 2.0$

$$\therefore \quad P(15 < X < 40) = P(-0.50 < Z < 2.0)$$

$$= P(-0.50 < Z < 0) + P(0 < Z < 2.0)$$

$$= P(0 < Z < 0.5) + P(0 < Z < 2.0)$$

$$= (0.1915 + 0.4772) = 0.6687, \text{ from the Figure 3.2}$$

From the normla table, we find the areas corresponding to the Zs are 0.1915 and 0.4772 and, thus, the desired area between $x_1 = 15$ and $x_2 = 40$ is (0.1915 + 0.4772) = 0.6687 as shown in Figure 3.2.

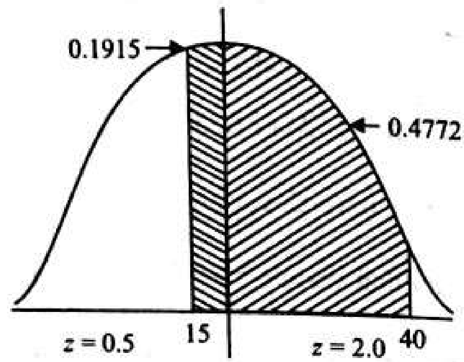

Figure 3.2

EXAMPLE 3.135 If X is a random variable normally distributed with mean zero and variance σ^2, find $E(|X|)$. **[AU December '07]**

Solution We know that

$$E(|X|) = \text{Mean deviation about origin}$$
$$= \text{Mean deviation about mean } 0$$

$$\text{Mean deviation about origin} = \frac{4}{5}\sigma$$

$$E(|X|) = \frac{4}{5}\sigma$$

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EXAMPLE 3.1376 In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution? 91

Solution Since 7% of items are under 35, 43% (50 – 7 = 43%) are between μ and 35. Similarly, the percentage of items between μ and 63 is 39% (89 – 50 = 39%) as shown in Figure 3.3.

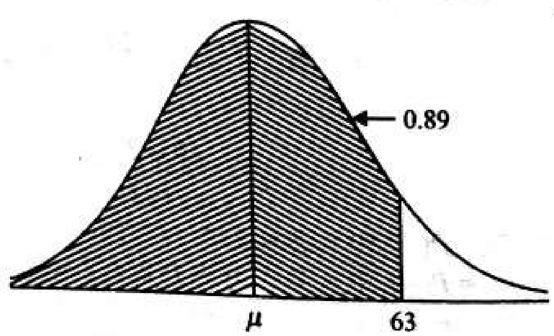


Figure 3.3

The standard normal variate corresponding to 0.43 (43%) is 1.48.

$$\therefore \qquad \frac{35-\mu}{\sigma} = -1.48 \qquad \text{(i)}$$

The standard normal variate corresponding to 0.39 (39%) is 1.23.

$$\therefore \qquad \frac{63-\mu}{\sigma} = 1.23 \qquad \text{(ii)}$$

From Eqs. (i) and (ii)

$$1.48\sigma - \mu = -35$$
$$1.23\sigma + \mu = 63$$

On addition of these equations, we get

$$2.71\sigma = 28 \Rightarrow \sigma = \frac{28}{2.71} = 10.33$$

$$1.48 \times 10.33 - \mu = -35 \Rightarrow -\mu = -35 - 15.3, \text{ i.e. } \mu = 50.3$$

Hence the mean of the distribution is 50.3 and $\sigma = 10.33$.

EXAMPLE 3.137 Assume the mean height of soldiers to be 68.22 inches with a variance of 10.8 inches. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall?

Solution Assume that the distribution of height is normal.

Standard normal variate $Z = \dfrac{x-\mu}{\sigma}$

Given: $X = 6$ feet $= 72$ inches

$\mu = 68.22$ inches

and $\sigma = \sqrt{10.8} = 3.286$

$\therefore$ When $X = 72$

$$Z = \frac{72-68.22}{3.286} = 1.15$$

Soldiers over 6 feet tall $P(X > 72) = P(Z > 1.15)$

$$= P(0 < Z < \infty) - P(0 < Z < 1.15)$$
$$= (0.5000 - 0.3749) = 0.1251$$

Area to the right of the ordinate at 1.15 from the normal table is $(0.5000 - 0.3749) = 0.1251$ as shown in Figure 3.4.

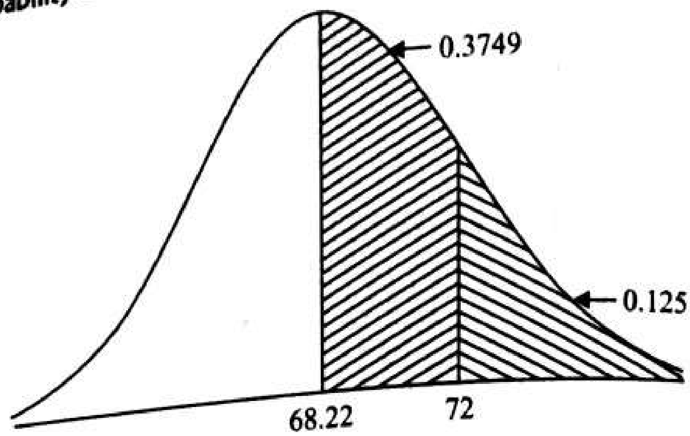


Figure 3.4

Hence the probability of getting soldier above 6 feet is 0.1251. Out of 1000 soldiers, the expectation is $0.1251 \times 1000 = 125.1$. Thus the expected number of soldiers over 6 feet tall = 125 soldiers.

EXAMPLE 3.138 The income of a group of 10000 persons was found to be normally distributed with mean = ₹ 750 per month and standard deviation = ₹ 50. Show that, of this group about 95% had income exceeding ₹ 668 and only 5% had income exceeding ₹ 832. What was the lowest income among the richest 100?

Solution Standard normal variate $Z = \dfrac{x - \mu}{\sigma}$

Given: $X = 668$, $\mu = 750$, $\sigma = 50$

$$Z = \frac{668 - 750}{50} = \frac{-82}{50} = -1.64$$

$$\begin{aligned} P(\text{Income exceeding ₹ } 668) &= P(X > 668) = P(Z > 1.64) \\ &= P(0 < Z < \infty) + P(-1.64 < Z < 0) \\ &= P(0 < Z < \infty) + P(0 < Z < 1.64) \\ &= (0.5000 + 0.4495) = 0.9495. \end{aligned}$$

$\therefore$ The expected number of persons getting above ₹ 668 $= 10000 \times 0.94950$
$= 9495$.

This is about 95% of the total 10000

Standard normal variate corresponding to 832 is

$$Z = \frac{832 - 750}{50} = \frac{82}{50} = 1.64$$

$$\begin{aligned} P(\text{Income exceeding ₹ } 832) &= P(X > 832) = P(Z > 1.64) \\ &= P(0 < Z < \infty) - P(0 < Z < 1.64) \\ &= (0.5000 - 0.4495) = 0.0505 \end{aligned}$$

[Area to the right of ordinate at 1.64 is $0.5000 - 0.4495 = 0.0505$]

The number of persons getting above ₹ 832 = 10,000 × 0.05050 = 505
This is 5% (approx.)

Now, probability of getting richest 100 = $\frac{100}{10000} = 0.01$
(from the normal table, (0.5 − 0.4904) = 0.0096 = 0.01)
Standard normal variate having 0.01 area to its right = 2.33

$$2.33 = \frac{X - 750}{50}$$

$$X = 2.33 \times 50 + 750 = ₹\ 866 \text{ (approx.)}$$

EXAMPLE 3.139 In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.
[AU April '05]

Solution Let mean be μ and standard deviation be σ. Given 31% of the items are under 45. They are lying to the left of the ordinate at $\mu = 45$ is 0.31 and, therefore, area lying to the right of the ordinate up to mean is (0.5 − 0.31) = 0.19.
The value of Z corresponding to this area is 0.5 (from the normal table).
Hence,

$$\frac{45 - \mu}{\sigma} = -0.5 \qquad \text{(i)}$$

8% of the item are above 64. Therefore, area to the right of the ordinate at 64 is 0.08. Area to the left of the ordinate at $x = 64$ up to mean ordinate is (0.5 − 0.08) = 0.42 and the value of Z corresponding to this area is 1.4 (from the normal table).

Hence,

$$\frac{64 - \mu}{\sigma} = 1.4 \qquad \text{(ii)}$$

From Eqs. (i) and (ii)

$$-\mu + 0.5\sigma = -45$$
$$\mu + 14\sigma = 64$$

i.e.,
$$1.9\sigma = 19 \Rightarrow \sigma = 10$$

and
$$\mu - 0.5 \times 10 = 45 \Rightarrow \mu = 50$$

∴
$$\text{Mean} = \mu = 50 \text{ and SD} = \sigma = 10$$

The mean of the distribution is 50 and standard deviation is 10.

EXAMPLE 3.140 The mean weight of 500 male students in a certain college is 151 lb, and the standard deviation is 15 lb. Assuming the weights are normally distributed, find how many students weigh

(i) between 120 and 155 lb, and
(ii) more than 185 lb.

Solution Given: $\mu = 151$ lb, $\sigma = 15$ lb

(i) Weights recorded as being between 120 lb and 155 lb can actually have any value from 119.5 lb to 155.5 lb, assuming they are recorded to the nearest pound.
Standard normal variate corresponding to $X = 119.5$ lb is

$$Z = \frac{X-\mu}{\sigma} \Rightarrow Z = \frac{119.5-151}{15} = -2.1$$

Standard normal variate corresponding to $X = 155.5$ lb

$$Z = \frac{155.5-151}{15} = 0.3$$

$$\begin{aligned} P(119.5 < X < 155.5) &= P(-2.1 < Z < 0.3) \\ &= P(-2.1 < Z < 0) + P(0 < Z < 0.3) \\ &= P(0 < Z < 2.1) + P(0 < Z < 0.3) \\ &= 0.4821 + 0.1179 = 0.6000 \end{aligned}$$

$\therefore$ The number of students weighing between 120 lb and 155 lb $= 500 \times 0.6000 = 300$

(ii) Students weighing more than 185 lb, must weigh at least 185.5 lb.
Standard normal variate corresponding to

$$X = 185.5$$

$$Z = \frac{185.5-151}{15} = 2.3$$

$$\begin{aligned} P(X > 185.5) &= P(Z > 2.3) = P(0 < Z < \infty) - P(0 < Z < 2.3) \\ &= 0.5000 - 0.4893 = 0.0107 \end{aligned}$$

$\therefore$ The number of students weighing more than 185 lb $= 500 \times 0.0107$
$= 5.35.$

EXAMPLE 3.141 In an intelligence test administered on 1000 students, the average was 42 and standard deviation 24. Find

(i) the number of students exceeding a score 50,
(ii) the number of children lying between 30 and 54, and
(iii) the value of score exceeded by the top 100 students.

Solution Given: $\mu = 42, X = 50, \sigma = 24$

$$Z = \frac{X-\mu}{\sigma} = \frac{50-42}{24} = 0.333$$

$$\begin{aligned} P(X > 50) &= P(Z > 0.333) \\ &= P(0 < Z < \infty) - P(0 < Z < 0.333) \\ &= 0.5 - 0.1304 = 0.3696 \end{aligned}$$

(i) The expected number of children exceeding a score of 50 out of 1000 $= 0.3696 \times 1000 = 369.6$ or 370

(ii) Standard normal variate for score $X = 30$

$$Z = \frac{X-\mu}{\sigma} = \frac{30-42}{24} = -0.5$$

Standard normal variate for score 54

$$Z = \frac{X-\mu}{\sigma} = \frac{54-42}{24} = 0.5$$

$$\therefore \quad P(30 < X < 54) = P(-0.5 < Z < 0.5)$$
$$= P(-0.5 < Z < 0) + P(0 < Z < 0.5)$$
$$= 2P(0 < Z < 0.5) = 2(0.1915) = 0.383$$

(Area to the right at $Z = 0.5 = 0.1915$)

The probability of having children between score 30 and 54 is 0.383. Thus, the number of children having scores between 30 and 54 out of $1000 = 0.383 \times 1000 = 383$.

(iii) The probability of getting top 100 students $= \frac{100}{1000} = 0.1$

Standard normal variate having 0.1 area to the right = 1.281
(Area to the right $Z = 1.281 = 0.5 - 0.3997 = 0.1$)
Standard normal variate for score X

$$Z = \frac{X-\mu}{\sigma} \Rightarrow 1.281 = \frac{X-42}{24} \Rightarrow 1.281 \times 24 = X - 42$$

i.e., $X = (1.281 \times 24) + 42 = 72.7$ or 73

EXAMPLE 3.142 The Coimbatore Corporation installed 2000 bulbs in the streets of Coimbatore. If these bulbs have an average life of 1000 burning hours, with a standard deviation of 200 hours, what number of bulbs might be expected to fail in the first 700 burning hours?

Solution Given: average life of bulbs $\mu = 1000$ hours, $\sigma = 200$ hours, X = burning hours = 700

$$Z = \frac{X-\mu}{\sigma} = \frac{700-1000}{200} = -1.5$$

$$\therefore \quad P(X < 700) = P(Z < -1.5) = P(-\infty < Z < 0) - P(-1.5 < Z < 0)$$
$$= 0.5 - P(0 < Z < 1.5) = 0.5 - 0.4332 = 0.668$$

$\therefore$ The number of bulbs expected to fail in the first 700 hours $= 0.67 \times 2000 = 134$.

EXAMPLE 3.143 In a certain examination, the percentages of passes and distinctions were 46 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively (assume the distribution of marks to be normal). Also determine what would have been the minimum qualifying marks for admission to re-examination of a failed candidate, had it been desired that the best 25% of them should be given another opportunity of being examined.

Solution Let μ be the mean and σ be the standard deviation of the normal distribution. The area to the right of the ordinate at $x = 40$ is 0.46 and, hence, the area between the mean and the ordinate at $x = 40$ is 0.04 as shown in Figure 3.5.

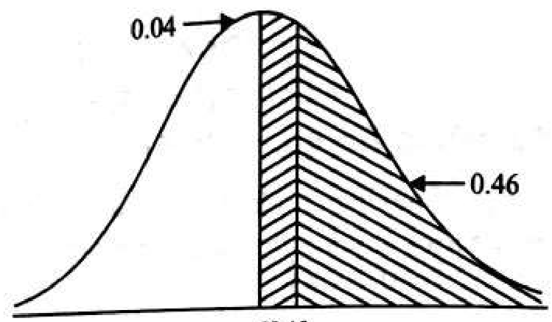


Figure 3.5

Now, from the normal table corresponding to 0.04, standard normal variate is 0.1.

$$Z = \frac{40-\mu}{\sigma} = 0.1 \Rightarrow 40 - \mu = 0.1\sigma \qquad \text{(i)}$$

Similarly,

$$Z = \frac{75-\mu}{\sigma} = 1.34 \Rightarrow 75 - \mu = 1.34\sigma \qquad \text{(ii)}$$

See Figure 3.6

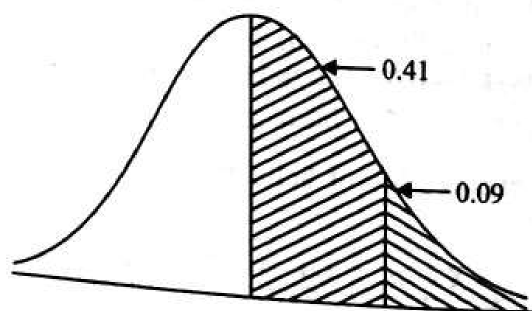


Figure 3.6

Subtracting Eq. (i) from Eq. (ii), we get

$$35 = 1.24\sigma \Rightarrow \sigma = \frac{35}{1.24} = 28.2$$

Putting the value of σ in Eq. (i) gives

$$\mu = 37.2 \text{ or } 37$$

Therefore, the average marks obtained by the candidates is 37.

Let us assume that X_1 is the minimum qualifying mark for admission to a re-examination and hence, the area to the right of $X = 40$ is 46%.

$\therefore$ % of students failing = 54 and this is the area to the left of 40. We want that the best 25% of these failed candidates should be given a chance to re-appear. Suppose this area is equal to the shaded area as shown in Figure 3.7.

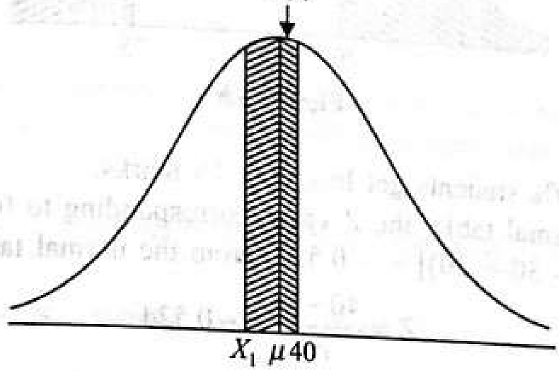


Figure 3.7

This area is 25% of 54 = 13.5

$\therefore$ Area between mean and ordinate at $X_1 = -(0.135 - 0.04) = -0.095$

(Negative sign is included because the area lies to the left of the mean co-ordinate)

Corresponding to this area, standard normal variate from the normal table is $Z = -0.0378$

$$\frac{X_1 - \mu}{\sigma} = -0.0378 \Rightarrow X_1 = \mu - 0.0378\sigma$$

$$X_1 = 37.2 - 0.0378 \times 28.2$$
$$= 37.2 - 1.065 = 36.0 \text{ (approx.)}$$

EXAMPLE 3.144 The results of a particular examination are given below in a summary form

Result	*% of candidates*
(i) Passed with distinction	10
(ii) Passed without distinction	60
(iii) Failed	30

It is known that a candidate gets plucked if he obtains less than 40 marks while he must obtain at least 75 marks in order to pass with distinction. Determine the mean and standard deviation of the distribution of marks assuming this to be normal.

Solution We have to calculate the mean and standard deviation from the given information. Figure 3.8 will help in understanding the question and finding its solution:

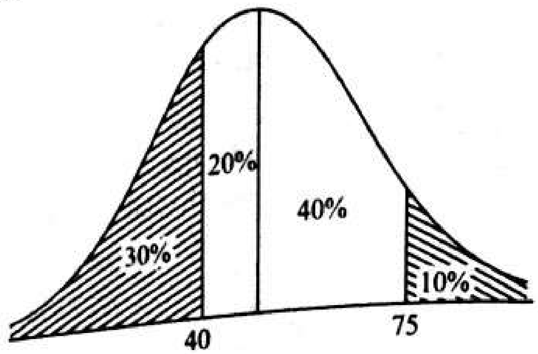


Figure 3.8

We know that 30% students get less than 40 marks.

$\therefore$ From the normal table, the Z value corresponding to 0.2 [(20% area (50 – 30 = 20)] = – 0.524 (from the normal table)

$$Z = \frac{40 - \mu}{\sigma} = -0.524 \tag{i}$$

10% students get distinction marks, i.e. 75 marks or more.

$\therefore$ From the normal table, the Z value corresponding to 0.4 [40% area (50 – 10 = 40)] is 1.28

$$Z = \frac{75 - \mu}{\sigma} = 1.28 \tag{ii}$$

Hence, solving Eqs. (i) and (ii)

$$\mu - 40 = 0.524\sigma$$
$$-\mu + 75 = 1.280\sigma$$

We get

$$35 = 1.804\sigma$$

$\therefore$
$$\sigma = \frac{35}{1.804} = 19.4$$

Therefore,
$$40 - \mu = 0.524 \times 19.4 \quad \Rightarrow \quad \mu = 50.17$$

Hence the mean and standard deviation of the distribution is

$$\mu = 50.17,\ \sigma = 19.4$$

EXAMPLE 3.145 If X is a normal variate with mean 50 and SD 10, find $P(Y \le 3137)$, where $Y = X^2 + 1$.

Solution Given: $P(Y \le 3137) = P(X^2 + 1 \le 3137)$

$$P(X^2 \le 3136) = P(-56 \le X \le 56)$$
$$= P(|X| \le 56)$$

But, $$Z = \frac{X-\mu}{\sigma}$$

When $X = -56$ $$Z = \frac{-56-50}{10} = -10.6$$

When $X = 56$ $$Z = \frac{56-50}{10} = 0.6$$

$$\begin{aligned} \therefore \quad P(Y \le 3137) &= P(-56 \le X \le 56) \\ &= P(-10.6 \le Z \le 0.6) \\ &= P(-10.6 \le Z \le 0) + P(0 \le Z \le 0.6) \\ &= 0.5 + 0.2257 = 0.7257 \end{aligned}$$

EXAMPLE 3.146 Given that X is normaly distributed with mean 10 and the probability $P(X > 12) = 0.1587$. What is the probability that X will fall in the interval (9, 11)?

Solution Given: X is normally distributed with mean $\mu = 10$.

Let $Z = \frac{X-\mu}{\sigma}$ be the standard normal variate.

For $X = 12$, $$Z_1 = \frac{12-10}{\sigma} \Rightarrow Z_1 = \frac{2}{\sigma}$$

Given: $$P(X > 12) = 0.1587 \Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{12-\mu}{\sigma}\right) = 0.1587$$

$$\Rightarrow \quad P(Z > Z_1) = 0.1587$$
$$\therefore \quad 0.5 - P(0 < Z < Z_1) = 0.1587$$
$$\Rightarrow \quad P(0 < Z < Z_1) = 0.3413$$
$P(0 < Z < 1) = 0.3413$ (from the normal table)

$$\therefore \quad Z_1 = 1 \Rightarrow \frac{2}{\sigma} = 1 \Rightarrow \sigma = 2$$

To find $P(9 < X < 11)$:

When $X = 9$, $Z = -\frac{1}{2}$

When $X = 11$, $Z = \frac{1}{2}$

$$\begin{aligned} \therefore \quad P(9 < X < 11) &= P(-0.5 < Z < 0.5) \\ &= 2P(0 < Z < 0.5) = 2 \times 0.1915 = 0.3830 \end{aligned}$$

EXAMPLE 3.147 If $\log_e x$ is normally distributed with mean 1 and variance 4, find $P(1/2 < X < 2)$ given that $\log_e 2 = 0.693$. **[AU May '07]**

Solution Given: Mean $\mu = 1$

and variance $\sigma^2 = 4 \Rightarrow \sigma = 2$

$$X = \log_e x$$

Let

$$Z = \frac{X - \mu}{\sigma} = \frac{\log_e x - 1}{2}$$

and

$$X = \frac{1}{2}$$

When

$$Z = \frac{\log_e \frac{1}{2} - 1}{2} = \frac{\log_e 1 - \log_e 2 - 1}{2}$$

$$= \frac{-\log_e 2 - 1}{2} = \frac{-0.693 - 1}{2} = -0.8465$$

When $x = 2$, $$Z = \frac{\log_e 2 - 1}{2} = \frac{0.693 - 1}{2} = -0.1535$$

$\therefore$ $P(1/2 < X < 2) = P(-0.8465 < Z < -0.1535)$

$= P(0.1535 < Z < 0.8465)$

(Area between $Z = 0$ and $Z = 0.8465$) – (Area between $Z = 0$ and $Z = 0.1535$)

$= 0.2996 - 0.0596 = 0.24$ (approx.)

EXAMPLE 3.148 An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a SD of 40 hours. Find

(i) the probability that a bulb burns more than 834 hours, and
(ii) the probability that a bulb burns between 778 and 834 hours.

Solution Given: $\mu = 800$ hours, $\sigma = 40$ hours

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 800}{40}$$

(i) P(a bulb burns more than 834 hours) $= P(X > 834)$

When $X = 834$, $Z = \frac{834 - 800}{40} = 0.85$

$\therefore$ $P(X > 834) = P(Z > 0.85)$

$= 1 - P(Z < 0.85) = 1 - 0.3023 = 0.6977$

(ii) To find $P(778 < X < 834)$

When $X = 778$, $Z = \frac{778 - 800}{40} = -0.55$

When $X = 834$, $Z = \frac{834 - 800}{40} = 0.85$

$\therefore$ $P(778 < X < 834) = P(-0.55 < Z < 0.85)$

$$= P(-0.55 < Z < 0) + P(0 < Z < 0.85)$$
$$= P(0 < Z < 0.55) + 0.3023$$
$$= 0.2088 + 0.3023 = 0.5111$$

EXAMPLE 3.149 The marks obtained by number of students in a certain subject are approximately normally distributed with mean 65 and SD 5. If 3 students are selected at random from this group, what is the probability that at least one of them would have scored above 75? (Given the area between $Z = 0$ and $Z = 2$ under the standard normal curve is 0.4772.)

[AU December '04]

Solution Given: $\mu = 65, \sigma = 5$

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 65}{5}$$

When $X = 75$, $Z = \dfrac{75 - 65}{5} = 2$

$\therefore$ $P(X > 75) = P(Z > 2)$
$$= 0.5 - P(0 < Z < 2) = 0.5 - 0.4772 = 0.0228$$

$\therefore$ $P(\text{a student scores} > 75) = 0.0228$

i.e. $p = 0.0228, q = 0.9772, n = 3$

Let Y be the number of students scoring more than 75.

$\therefore$ $P(Y = y) = nC_y p^y q^{n-y}$ (binomial)
$$= 3C_y (0.0228)^y (0.9772)^{3-y}$$

$\therefore$ $P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y = 0)$
$$= 1 - 3C_0 (0.0228)^0 (0.9772)^3$$
$$= 1 - (0.9772)^3 = 0.0667$$

EXAMPLE 3.150 The average seasonal rainfall in a place is 16 inches with a SD of 4 inches. What is the probability that in a year the rainfall in that place will be between 20 and 24 inches?

Solution Given: $\mu = 16, \sigma = 4$

The normal variate is

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 16}{4}$$

When $X = 20$, $Z = \dfrac{20 - 16}{4} = 1$

When $X = 24$, $Z = \dfrac{24 - 16}{4} = 2$

$\therefore$ $P(20 < X < 24) = P(1 < Z < 2)$
$$= P(0 < Z < 2) - P(0 < Z < 1)$$
$$= 0.4772 - 0.3413 = 0.1359$$

EXAMPLE 3.151 The weekly wages of 1000 workmen are normally distributed around a mean of ₹ 70 with a SD of ₹ 5. Estimate the number of workers whose weekly wages will be

(i) between ₹ 69 and ₹ 72,

(ii) less than ₹ 69, and

(iii) more than ₹ 72. **[AU January '04]**

Solution Given: $\mu = 70$, $\sigma = 5$

(i) To find $P(69 < X < 72)$:

When $X = 69, Z = \dfrac{X-\mu}{\sigma} = \dfrac{69-70}{5} = -0.2$

When $X = 72, Z = \dfrac{X-\mu}{\sigma} = \dfrac{72-70}{5} = 0.4$

$$\begin{aligned}\therefore \quad P(69 < X < 72) &= P(-0.2 < Z < 0.4)\\ &= P(-0.2 < Z < 0) + P(0 < Z < 0.4)\\ &= P(0 < Z < 0.2) + P(0 < Z < 0.4)\\ &= 0.0793 + 0.1554 \quad \text{(from the normal table)}\\ &= 0.2347\end{aligned}$$

Out of 1000 workmen, the number of workers whose wages lie between ₹ 69 and ₹ 72 = $1000 \times P(69 < X < 72)$

$$= 1000 \times 0.2347 = 234.7 \cong 235$$

(ii) To find $P(\text{less than ₹ } 69) = P(X < 69)$

When $X = 69$, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{69-70}{5} = -0.2$

$$\begin{aligned}\therefore \quad P(X < 69) &= P(Z < -0.2) = 0.5 - P(0 < Z < 0.2)\\ &= 0.5 - 0.0793 = 0.4207\end{aligned}$$

Out of 1000 workmen, the number of workers whose wages are less than ₹ 69 = $1000 \times P(X < 69)$

$$= 1000 \times 0.4207 = 420.7 \cong 421$$

(iii) $P(\text{more than ₹ } 72) = P(X > 72)$

When $X = 72$, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{72-70}{5} = 0.4$

$$\begin{aligned}\therefore \quad P(X > 72) &= P(Z > 0.4) = 0.5 - P(0 < Z < 0.4)\\ &= 0.5 - 0.1554 = 0.3446\end{aligned}$$

Out of 1000 workmen, the number of workers whose wages are grater than ₹ 72 = $1000 \times P(Z > 0.4)$

$$= 1000 \times 0.3446 = 344.6 \cong 345 \text{ (nearly)}$$

EXAMPLE 3.152 The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and SD is 0.005 cm. The purpose for which these washers are intended allows a maximum tolerance in diameter of 0.496 cm to 0.508 cm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed. **[AU December '07]**

Solution Given: $\mu = 0.502$, $\sigma = 0.005$

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 0.502}{0.005}$$

To find $P(0.496 < X < 0.508)$:

When $X = 0.496$, $Z = \dfrac{X - \mu}{\sigma} = \dfrac{0.496 - 0.502}{0.005} = -1.2$

When $X = 508$, $Z = \dfrac{X - \mu}{\sigma} = \dfrac{0.508 - 0.502}{0.005} = 1.2$

$$\begin{aligned} P(0.496 < X < 0.508) &= P(-1.2 < Z < 1.2) \\ &= P(-1.2 < Z < 0) + P(0 < Z < +1.2) \\ &= P(0 < Z < +1.2) + P(0 < Z < +1.2) \\ &= 2\,P(0 < Z < +1.2) \\ &= 2 \times 0.3849 = 0.7698 \end{aligned}$$

$\therefore$ Percentage of non-defective washers = 0.7698 × 100 = 76.98%

$\therefore$ Percentage of defective washers = 100 − 76.98 = 23.02%

EXAMPLE 3.153 A manufacturer produces covers where weight is normal with mean $\mu = 1.950$ g and SD $\sigma = 0.025$ g. The covers are sold in lots of 1000. How many covers in a lot may be heavier than 2 g?

Solution Given: $\mu = 1.950$, $\sigma = 0.025$

To find P(The number of covers heavier than 2 g) = $P(X > 2)$:

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 1.95}{0.025}$$

When $X = 2$, the normal variate $Z = \dfrac{2 - 1.95}{0.025} = 2$

$$\begin{aligned} \therefore \quad P(X > 2) &= P(Z > 2) \\ &= 0.5 - P(0 < Z < 2) = 0.5 - 0.4772 = 0.0228 \end{aligned}$$

$\therefore$ The number of covers heavier than 2 g = 1000 × $P(X > 2)$ = 1000 × 0.0228 = 23 (approx.)

EXAMPLE 3.154 Assuming that the diameters of 1000 brass plugs taken consecutively from a machine form a normal distribution with mean 0.7515 cm and SD 0.0020 cm, how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm? **[AU December '03]**

Solution The approved diameters ranges from 0.752 − 0.004 cm to 0.752 + 0.004 cm, i.e. 0.748 cm to 0.756 cm.

Given: $\mu = 0.7515$, $\sigma = 0.0020$
To find $P(0.748 < X < 0.756)$:
We know that

$$Z = \frac{X-\mu}{\sigma} = \frac{X-0.7515}{0.0020}$$

When $X = 0.748$, $Z = \frac{0.748-0.7515}{0.002} = -1.75$

When $X = 0.756$, $Z = \frac{0.756-0.7515}{0.002} = 2.25$

$$\begin{aligned} \therefore \quad P(0.748 < X < 0.756) &= P(-1.75 < Z < 2.25) \\ &= P(-1.75 < Z < 0) + P(0 < Z < 2.25) \\ &= P(0 < Z < 1.75) + P(0 < Z < 2.25) \\ &= 0.4599 + 0.4878 = 0.9477 \end{aligned}$$

$\therefore$ The number of plugs of approved diameters $= 1000 \times P(-1.75 < Z < 2.25)$

$$= 1000 \times 0.9477 = 947.7 = 948 \text{ (nearly.)}$$

EXAMPLE 3.155 The savings bank account of a customer showed an average balance of ₹ 150 and a SD of ₹ 50. Assuming that the account balances are normally distributed,

(i) what percentage of account is over ₹ 200?
(ii) what percentage of account is between ₹ 120 and ₹ 170? and
(iii) what percentage of account is less than ₹ 75?

[AU December '05]

Solution Given: Mean $= \mu = 150$, SD $= \sigma = 50$
We know that,

$$Z = \frac{X-\mu}{\sigma} = \frac{X-150}{50}$$

(i) To find $P(X \geq 200)$:

When $X = 200$, $Z = \frac{200-150}{50} = 1$

$$\begin{aligned} \therefore P(X \geq 200) &= P(Z > 1) \\ &= 0.5 - P(0 < Z < 1) = 0.5 - 0.3413 = 0.1587 \end{aligned}$$

$\therefore$ Percentage of account over ₹ 200 $= 100 \times 0.1587 = 15.87\%$

(ii) To find $P(120 \le X \le 200)$:

When $X = 120$, $Z = \dfrac{120-150}{50} = \dfrac{-30}{50} = -0.6$

When $X = 170$, $Z = \dfrac{170-150}{50} = \dfrac{20}{50} = 0.4$

$$\therefore \quad P(120 \le X \le 200) = P(-0.6 < Z < 0.4)$$
$$= P(0 < Z < 0.6) + P(0 < Z < 0.4)$$
$$= 0.2257 + 0.1554 = 0.3811$$

$\therefore$ Percentage of account between ₹ 120 and ₹ 170 $= 100 \times 0.3811$
$= 38.11\%$

(iii) To find $P(X < 75)$:

When $X = 75$, $Z = \dfrac{75-150}{50} = -1.5$

$$\therefore \quad P(X < 75) = P(Z < -1.5) = P(-\infty < Z < 0) - (-1.5 < Z < 0)$$
$$= 0.5 - P(0 < Z < 1.5) = 0.5 - 0.4332 = 0.0668$$

$\therefore$ Percentage of account less than ₹ 75 $= 100 \times 0.0668 = 6.68\%$

EXAMPLE 3.156 In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and SD of 60 hours. Estimate the number of bulbs likely to burn for

(i) more than 2150 hours,
(ii) less than 1950 hours, and
(iii) more than 1920 hours but less than 2160 hours.

[AU December '04]

Solution Given: $\mu = 2040$ hours, $\sigma = 60$ hours

(i) To find P(more than 2150 hours) $= P(X > 2150$ hours$)$:
We know that

$$Z = \frac{X-\mu}{\sigma} = \frac{X-2040}{60}$$

When $X = 2150$, $Z = \dfrac{2150-2040}{60} = 1.833$

$$\therefore \quad P(X > 2150) = P(Z_1 > 1.833)$$
$$= 0.5 - P(0 < Z < 1.833)$$
$$= 0.5 - 0.4664 = 0.0336$$

$\therefore$ The number of bulbs expected to burn for more than 2150 hours $= 2000 \times 0.0336 = 67$ (nearly)

(ii) To find P(less than 1950 hours) = $P(X < 1950$ hours):

When $X = 1950$, $Z = \dfrac{1950 - 2040}{60} = -1.5$

$$\therefore \quad P(X < 1950) = P(Z < -1.5)$$
$$= 0.5 - P(-1.5 < Z < 0)$$
$$= 0.5 - P(0 < Z < 1.5)$$
$$= 0.5 - 0.4332 = 0.0668$$

$\therefore$ The number of bulbs expected to burn for less than 1950 hours $= 2000 \times 0.0668 = 134$ (nearly)

(iii) To find P(more than 1920 hours but less than 2160 hours):

When $X = 1920$, $Z = \dfrac{1920 - 2040}{60} = -2$

When $X = 2160$, $Z = \dfrac{2160 - 2040}{60} = 2$

$$\therefore \quad P(1920 < X < 2160) = P(-2 < Z < 2)$$
$$= 2P(0 < Z < 2)$$
$$= 2 \times 0.4773 = 0.9546$$

$\therefore$ The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours $= 2000 \times 0.9546 = 1909$ (nearly)

EXERCISES

Bernoulli/Binomial Distributions

1. State Bernoulli theorem on independent trials.
2. A fair coin is tossed four times. What is the probability of getting more heads than tails? **[Ans.** 5/16]
3. When 12 coins are tossed 256 times, how many times may one expect 8 heads and 4 tails? **[Ans.** 31]
4. If war breaks out on the average once in 25 years, find the probability that in 50 years at a stretch, there will be no war.

$$\left[\textbf{Ans.} \quad \left(\frac{24}{25}\right)^{50}\right]$$

5. State the generalized form of Bernoulli theorem on independent trials.
6. Obtain the mean of the binomial distribution $B(n; p)$.
7. Find the binomial distribution whose mean is 6 and SD is $\sqrt{2}$.

$$\left[\textbf{Ans.} \quad 9C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{9-x}, \quad x = 0, 1, \cdots, 9\right]$$

8. If X is a binomial random variable with mean 2.4 and variance 1.44, find $P(X = 7)$.

$$\left[\textbf{Ans.}\quad nC_7\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right)^{n-7}\right]$$

9. If X is binomially distributed with $n = 5$ such that $P(X = 1) = 2P(X = 2)$, find $E(X)$ and $\text{Var}(X)$. **[Ans.** 1, 0.8]

10. If X is binomially distributed with $n = 6$ such that $P(X = 2) = 9P(X = 4)$, find $E(X)$ and $\text{Var}(X)$. **[Ans.** 1.5, 9/8]

11. Find the mean and variance of the binomial distribution.

[AU December '02; '03, November '06]

[Ans. np, npq]

12. The probability of a bomb hitting a target is 1/5. Two bombs are enough to destroy a bridge. If six bombs are aimed at the bridge, find the probability that the bridge is destroyed.

[Ans. $n = 6, p = 1/5, q = 4/5 \therefore P(X = 2) = 0.2458$]

13. A certain carton of eggs has 3 bad and 9 good eggs. If an omelette is made of 3 eggs randomly chosen from the carton,

(i) what is the probability that there are no bad eggs in the omelette?
(ii) what is the probability of having at least 1 bad egg in the omelette?
(iii) what is the probability of having exactly 2 bad eggs in the omelette?

[Ans. $X \sim B(3, 1/4)$ $n = 3, p = 1/4, q = 3/4$,
(i) $P(X = 0) = 0.4219$,
(ii) $P(X \geq 1) = 0.5781$,
(iii) $P(X = 2) = 0.1406$]

14. A bag contains 3 coins, one of which is coined with 2 heads, while the other 2 coins are normal and not biased. A coin is thrown at random from the bag and tossed 3 times in succession. If heads turn up each time, what is the probability that this is the two headed coin? **[Ans.** 4/15]

15. The following data due to weldon shows the results of throwing 12 dice 4096 times, a throw of 4, 5 or 6 being called a success (x):

x	0	1	2	3	4	5	6	7	8	9	10	11	12
f	7	60	198	430	731	948	847	536	257	71	11	40	96

Fit a binomial distribution and calculate the expected frequencies. Compare the actual mean and SD with those of the expected ones for the distribution.

[Ans. Expected frequencies: 1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, mean = 6, variance = 1.71]

16. If X is a random variable following binomial distribution with mean 2.4 and variance 1.44, find $P(X \geq 5)$.

[Ans. $P(X = x) = 6C_x(0.4)_x(0.6)^{6-x}$, $P(X \geq 5) = 1 - P(X < 5)$]

17. If the probability that a man aged 60 will live up to 70 is 0.65, what is the probability that out of 10 men, now 60, at least 7 will live up to 70? [Ans. 0.509]

18. If on the average, rain falls on 10 days in every 30 days, obtain the probability that
    (i) rain will fall on at least 3 days of a given week,
    (ii) first three days of a given week will be fine and the remaining 4 days wet. [Ans. (i) 0.4294, (ii) 0.0037]

19. A binary number (composed only of the digits 0 and 1) is made up of n digits. If the probability of an incorrect digit appearing is p and that errors in different digits are independent of one another, find the probability of forming an incorrect number. [Ans. $1 - (1 - p)^n$]

20. Suppose that twice as many items are produced (per day) by machine 1 as by machine 2. However, 4% of the items from machine 1 are defective while machine 2 produces only about 2% defectives. Suppose that the daily output of the 2 machines is combined and random sample of 10 items is taken from the combined output. What is the probability that this sample contains 2 defectives? [Ans. 0.0381]

21. Binary digits are transmitted over a noisy communication channel in blocks of 16 binary digits. The probability that a received binary digit is in error because of channel noise is 0.1. If errors occur in various digit positions within a block independently, find the probability that the number of errors per block is greater than or equal to 5. [Ans. 0.017]

22. A company is trying to market a digital transmission system (modem) that has a bit error probability of 10^{-4} and the bit errors are independent. The buyer will test the modem by sending a known message of 104 digits and checking the received message. If more than 2 errors occur, the modem will be rejected. Find the probability that the customer will buy the company's modem. [Ans. 0.9197]

23. A random experiment can terminate in one of 3 events A, B and C with probabilities 1/2, 1/4 and 1/4 respectively. The experiment is repeated 6 times. Find the probability that the events A, B and C occur once, twice and thrice respectively. [Ans. 0.0293]

24. A throws 3 fair coins and B throws 4 fair coins. Find the chance that A will throw more number of heads than would B. [Ans. 29/128]

25. A lot contains 1% defective items. What should be the number of items in a random sample so that the probability of finding at least 1 defective in it is at least 0.95? [Ans. 299]

26. What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success?

$$\left[\textbf{Ans.}\quad \frac{1-p}{p}\right]$$

27. What is the expectation of
   (i) the sum of the points on n dice, and
   (ii) the product of the points on n dice?
   [**Ans.** (i) $7n/2$, (ii) $(7/2)^n$]

28. Three tickets are chosen at random without replacement from 100 tickets, numbered 1, 2, 3, ... , 100. Find the expectation of the sum of the numbers.
   [**Ans.** 151.5]

29. From an urn containing 3 red and 2 black balls, a man is to draw 2 balls at random without replacement, being promised ₹ 20 for each red ball he draws and ₹ 10 for each black ball. Find his expectation.
   [**Ans.** ₹ 32]

Poisson Distribution

30. Write down the PMF of a Poisson distribution which is approximately equivalent to $B(100, 0.02)$.
   $\left[\textbf{Ans.}\ \ P(X=r)=e^{-2}\dfrac{2^r}{r!}\right]$

31. If X is a Poisson variate such that $2P(X = 0) + P(X = 2) = 2P(X = 1)$, find $E(X)$.
   [**Ans.** $\lambda = 2$]

32. If X is a Poisson variate such that $E(X^2) = 6$, find $E(X)$.
   [**Ans.** $\lambda = 6 - \lambda^2 \Rightarrow \lambda = 2$]

33. If X is a Poisson variate such that $P(X = 0) = 0.5$, find $\mathrm{Var}(X)$.
   [**Ans.** $\lambda = \log 2$]

34. If X is a Poisson variate with parameter $\lambda > 0$, prove that $E(X^2) = \lambda E(X + 1)$.

35. If X is a Poisson variate with parameter λ, prove that

$$E(X \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda}).$$

36. If the MGF of a discrete random variable X is $e^{9(e^t-1)}$, find $P(X = \lambda + \sigma)$ if λ and σ are the mean and SD of X.
   [**Ans.** $P(X = 13)$]

37. If X and Y are independent identical Poisson variates with mean 1, find $P(X + Y = 2)$.
   [**Ans.** $2/e$]

38. Wireless sets are manufactured with 20 soldered joints each, on the average 1 joint in 100 defectives. How many sets can be expected to be free from defective joints in a consignment of 20,000 sets?
   [**Ans.** 16375]

39. Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trails.
   [**Ans.** 0.865]

40. A large shipment of textbooks contains 2% with imperfect bindings. What is the probability that among 400 textbooks taken from this shipment exactly 5 have imperfect bindings? [Ans. 0.09]

41. The probability of getting no misprint in a page of a book is e^{-4}. What is the probability that a page contains

(i) 2 misprints,

(ii) more than 3 misprints? [Ans. (i) 0.1465, (ii) 0.5665]

42. A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the probability that a particular packet will violate the guarantee.

$$\left[\textbf{Ans.} \quad 1-\sum_{r=0}^{10}\left(e^{-10}\frac{10^r}{r!}\right)\right]$$

43. It is known from past experience that in a certain plant there are on the average 4 industrial accidents per month. Find the probability that in a given year there will be less than 4 accidents. [Ans. 0.4332]

44. In a certain factory producing cycle tyres there is a small chance of 1 in 500 tyres to be defective. The tyres are supplied in lots of 10. Using Poisson distribution, calculate the approximate number of lots containing no defective, 1 defective and 2 defective tyres respectively in a consignment of 10000 lots. [Ans. 9802, 196, 2]

45. If X is a Poisson variate with $\lambda = 1.5$, find the probability that

(i) $X = 3$, (ii) $X \le 3$.

[Ans. (i) 0.125, (ii) 0.934]

46. In a company, on an average, 3 workers are absent. Assuming Poisson distribution, find the probability that 5 are absent on a particular day.

[Ans. 0.101]

47. It is known that on an average 1 triplet is born in 10000 births. What is the probability of three triplets in a city in which there are 25000 births in a year? [Ans. 0.214]

48. An automatic machine makes paper clips from coils of a wire. On the average, 1 in 400 paper clips is defective. If the paper clips are packed in boxes of 100, what is the probability that any given box of clips will contain (i) no defective, (ii) 1 or more defectives, (iii) less than 2 defectives.

[Ans. (i) 0.7787, (ii) 0.2213, (iii) 0.97344]

49. The proofs of a 500-page book contains 500 misprints. Find the probability that there are at least 4 misprints in a randomly chosen page.

[Ans. 0.019]

50. If the average number of claims handled daily by an insurance company is 5, what proportion of days will have less than 3 claims? What is the

probability that there will be 4 claims in exactly 3 of the next 5 days? Assume that the number of claims on different days are independent.

[**Ans.** 0.1247, 0.0367]

51. An insurance company has discovered that only about 0.1% of the population is involved in a certain type of accident each year. If its 10000 policyholders were randomly selected from the population, what is the probability that not more than 5 of its clients are involved in such an accident next year?

[**Ans.** 0.067]

52. In a given city, 4% of all licenced drivers will be involved in at least 1 road accident in any given year. Determine the probability that among 150 licenced drivers randomly chosen in this city

(i) only 5 will be involved in at least 1 accident in any given year and

(ii) at most 3 will be involved in at least 1 accident in any given year.

[**Ans.** (i) 0.1606, (ii) 0.1512]

53. A radioactive source emits on the average 2.5 particles per second. Find the probability that 3 or more particles will be emitted in an interval of 4 seconds. [**Ans.** 0.0028]

54. It has been established that the number of defective stereos produced daily at a certain plant is Poisson distributed with mean 4. Over a 2-day span, what is the probability that the number of defective stereos does not exceed 3? [**Ans.** 0.0424]

55. In an industrial complex, the average number of fatal accidents per month is one-half. The number of accidents per month is adequately described by a Poisson distribution. What is the probability that 6 months will pass without a fatal accident? [**Ans.** 0.498]

56. If the number of telephone calls coming into a telephone exchange between 9 a.m. and 10 a.m. and between 10 a.m. and 11 a.m. are independent and follow Poisson distributions with parameters 2 and 6 respectively, what is the probability that more than 5 calls come between 9 a.m. and 11 a.m.?

[**Ans.** 0.8088]

57. Patients arrive randomly and independently at a doctor's consulting room from 5 p.m. at an average rate of 1 in 5 minutes. The waiting room can hold 12 persons. What is the probability that the room will be full, when the doctor arrives at 6 p.m.? [**Ans.** 0.1144]

58. The number of blackflies on a broad bean leaf follows a Poisson distribution with mean 2. A plant inspector, however, records the number of flies on a leaf only if at least 1 fly is present. What is the probability that he records 1 or 2 flies on a randomly chosen leaf? What is the expected number of flies recorded per leaf?

[**Hint:** If X is the number of flies on a leaf, we have to find $P(X = r/X \geq 1)$, $r = 1, 2, ..., 3$ and add them.] [**Ans.** 0.6244, 2.3]

59. Fit a Poisson distribution for the following distribution and hence find the expected frequencies.

x	0	1	2	3	4	5	6
f	314	335	204	86	29	9	3

[Ans. 301, 362, 217, 87, 26, 6, 1]

Geometric Distribution

60. Find the mean and variance of the discrete probability distribution given by

$$P(X = r) = e^{-t}(1 - e^{-t})^{r-1},\ r = 1, 2, \ldots, \infty.$$

[Ans. mean $= e^t - 1$, var $= e^t(e^t - 1)$]

61. If X is a geometric variate, taking values 1, 2, 3, ... , ∞, find $P(X$ is odd).

[Ans. $1/(1 + q)$]

62. Find the mean and variance of the distribution given by

$$P(X = r) = \frac{2}{3^r},\ r = 1, 2, \ldots, \infty.$$

[Ans. mean = 1/2, var = 3/4]

63. For the geometric distribution of X, which represents the number of Bernoulli trials required to get the first success Var(X) = 2E(X). Find the PMF of the distribution. **[Ans.** $P(X = r) = 2^r$, $r = 1, 2, 3, \ldots, \infty$]

64. Find the MGF of the geometric distribution given by

$$P(X = r) = q^{r-1}p, \text{ if } r = 1, 2, 3, \ldots, \infty.$$

[Ans. $M(t) = pe^t/(1 - qe^t)$]

65. If the MGF of a discrete random variable X taking values 1, 2, ... , ∞ is $e^t(5 - e^t)^{-1}$, find the mean and variance of X.

[Ans. mean = 4, var = 20]

66. If the probability that a certain test yields a positive reaction equals 0.4, what is the probability that fewer than 5 negative reactions occur before the first positive one? **[Ans.** 0.92]

67. Identify the distribution with the MGF $e^t(5 - 4e^t)^{-1}$.

$$\left[\textbf{Ans.}\ \text{Geometric distribution} = q^{x-1}p = \left(\frac{4}{5}\right)^{x-1}\left(\frac{1}{5}\right),\ x = 1, 2, 3, \ldots\right]$$

68. If the probability of a male child in a family is 1/3, how many children they are expected to have before the first male child is born?

[Ans. 2]

69. In a test, a light switch is turned on and off until it fails. If the probability that the switch will fail any time its turned on or off is 0.001, what is the probability that the switch will not fail during the first 800 times it is turned on or off? [Ans. 0.4529]

70. An item is inspected at the end of each day to see whether it is still functioning properly. If it is found to fail at the tenth inspection and not earlier, what is the maximum value of the probability of its failure on any day? [Ans. 1/10]

71. If X and Y are 2 independent random variables, each representing the number of failures preceding the first success in a sequence of Bernoulli trials with p as the probability of success in a single trial, show that

$$P(X = Y) = \frac{p}{1+q}, \text{ where } p + q = 1.$$

72. A throws 2 dice until he gets 6 and B throws independently 2 other dice until he gets 7. Find the probability that B will require more throws than A. [Ans. 25/61]

73. If 2 independent random variables X and Y have identical geometric distributions with parameter p, find the probability mass function of $(X + Y)$ and, hence, the expected value of $(X + Y)$.

$$\left[\textbf{Ans.} \quad P(X+Y=k) = (k-1)p^2q^{k-2}, k = 2, 3, \ldots \infty, \frac{2}{p}\right]$$

74. Suppose that a trainee soldier shoots a target in an independent fashion. The probability that the target is shot on anyone shot is 0.7.
   (i) What is the probability that the target would be hit on 10th attempt?
   (ii) What is the probability that it takes him less than 4 shots?
   (iii) What is the probability that it takes him an even number of shots.

[AU December '07]

[Ans. 0.000012, 0.973, 0.2307]

Uniform/Exponential Distribution

75. Define uniform distribution.

76. Write the PDF of the uniform random variable defined in the interval (a, b).

77. If X has a uniform distribution in $(-3, 3)$, find $P(|X - 2| < 2)$. [Ans. 1/2]

78. If X has a uniform distribution in $(-a, a)$, $a > 0$, find a such that $P(|X| < 1) = P(|X| > 1)$. [Ans. $a = 2$]

79. If the MGF of a continuous random variable X is $\frac{1}{t}(e^{5t} - e^{4t}), t \neq 0$, what is the distribution of X? What are its mean and variance?

[**Ans.** Uniform distribution, 9/2, 1/12]

80. A continuous random variable X has the density function $ce^{-x/5}$, $x > 0$. Find c, $E(X)$ and var(X). [**Ans.** 1/5, 25]

81. What do you mean by memoryless property of the exponential distribution?

82. If X and Y are independent identically distributed random variables, each with density function e^{-x}, $x > 0$, find the density function of $(X + Y)$.

83. If X has uniform distribution in (0, 2) and Y has exponential distribution with parameter λ, find λ such that $P(X < 1) = P(Y < 1)$.

[**Ans.** log 2]

84. If X has uniform distribution in $(-1, 3)$ and Y has exponential distribution with parameter λ, find λ such that Var(X) = Var(Y). [**Ans.** $\sqrt{3}/2$]

85. If X is a uniformly distributed random variable with mean 1 and variance 4/3 find $P(X < 0)$. [**Ans.** 1/2]

86. If X_1 and X_2 are independent rectangular variates in [0, 1], find the distribution of X_1/X_2. [**Ans.** 1]

87. A random variable X is uniformly distributed over $(-\alpha, \alpha)$, find α so that

(i) $P(X > 1) = 1/3$,
(ii) $P(|X| < 1) = P(|X| > 1)$. [**Ans.** (i) $\alpha = 3$, (ii) $\alpha = 2$]

88. A distribution is given by $f(x) = 1/2a \quad -a \leq x \leq a$. Find the first four central moments.

$$\left[\textbf{Ans.} \quad \mu_1 = \mu_3 = 0, \mu_2 = \frac{a^2}{3}, \mu_4 = \frac{a^4}{5}\right]$$

89. The density of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{100}, 100 < x < 200 \\ = 0, \text{ otherwise} \end{cases}.$$

Find $P(X \geq 150)$, $P(125 \leq X \leq 160)$. [**Ans.** 0.5, 0.35]

90. X is uniformly distributed with mean 1 and variance 4/3. If 3 independent observations of X are made, what is the probability that all of them are negative?

[**Ans.** 1/64]

91. A point D is chosen on the line AB whose length is 1 and whose mid-point is C. If the distance X from D to A is a random variable having a uniform distribution in $(0, a)$, what is the probability that AD, BD and AC will form a triangle?

[**Ans.** 9/16]

92. A passenger arrives at a bus stop at 10 a.m., knowing that the bus will arrive at some time uniformly distributed between 10 a.m. and 10.30 a.m. What is the probability that he will have to wait longer than 10 minutes? If at 10.15 a.m. the bus has not yet arrived, what is the probability that he will have to wait at least 10 additional minutes? **[Ans.** 2/3, 1/3]

93. A man and a woman agree to meet at a certain place between 10 a.m. and 11 a.m. They agree that the one arriving first will have to wait 15 minutes for the other to arrive. Assuming that the arrival times are independent and uniformly distributed, find the probability that they meet.

[Ans. 7/16]

94. The random variables a and b are independently and uniformly distributed in the intervals (0, 3) and (0, 6) respectively. Find the probability that the roots of the equation $x^2 - ax + b = 0$ are real. **[Ans.** 1/8]

95. If a, b, c are randomly chosen between 0 and 1, find the probability that the quadratic equation $ax^2 + bx + c = 0$ has real roots. **[Ans.** 1/9]

96. If the random variable a is uniformly distributed in the interval (1, 7), what is the probability that the roots of the equation $x^2 + 2ax + (2a + 3) = 0$ are real.

[Ans. 2/3]

97. A continuous random variable X has the PDF $f(x) = \alpha e^{\frac{-x}{9}}, x > 0$, find α, $E(X)$ and var (X). **[Ans.** $\alpha = 1/9$, $E(X) = 9$, var $(X) = 81$]

98. The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$.

   (i) What is the probability that the repair time exceeds 2 hours?

   (ii) What is the conditional probability that a repair takes at least 10 hours given that its duration exceeds 9 hours?

[Ans. (i) 0.3679, (ii) 0.6065]

99. If the number of kilometres that a car can run before its battery wears out is exponentially distributed with an average value of 10000 km and if the owner desires to take a 5000 km trip,

   (i) what is the probability that he will be able to complete his trip without having to replace the car battery? Assume that the car has been used for some time,

   (ii) what is the probability, when the distribution is not exponential?

$$\left[\textbf{Ans. } \text{(i) } 0.6065, \text{ (ii) } \frac{1 - F(t + 5000)}{1 - F(t)}, \text{ where } F(t) \text{ is the distribution function of the life of the car}\right]$$

100. The length of the shower on a tropical island during rainy season has an exponential distribution with parameter 2, time being measured in minutes. What is the probability that a shower will last more than 3 minutes? If a shower has already lasted for 2 minutes, what is the probability that it will last for at least one more minute? **[Ans.** (i) 0.0025, (ii) 0.1353]

101. If X is exponentially distributed with parameter λ, prove that the random variable $Y = e^{-\lambda X}$ is uniformly distributed in (0, 1).

102. If X_1, X_2, X_3 are independent random variables having exponential distributions with parameters $\lambda_1, \lambda_2, \lambda_3$ respectively, prove that $Y=$ minimum (X_1, X_2, X_3) follows an exponential distribution with parameter $(\lambda_1 + \lambda_2 + \lambda_3)$.

[**Hint:** Find the distribution function of $Y = F(y) = 1 - P[\text{minutes}(X_1, X_2, X_3) > y]$]

103. The mileage which a car owner gets with a certain kind of tyre is a random variable having an exponential distribution with mean 4000 km. Find the probabilities that one of these tyres will last

(i) at least 2000 km,

(ii) at most 3000 km.

[**Ans.** (i) 0.6065, (ii) 0.527]

104. The daily consumption of milk in excess of 20,000 gallons is approximately exponentially distributed with $\lambda = 3000$. The city has a daily stock of 35000 gallons. What is the probability that of 2 days selected at random, the stock is insufficient for both days.

[**Ans.** e^{-10}]

Gamma/Weibull/Normal Distributions

105. Write down the MGF of simple Gamma distribution and hence, find its mean and variance.

106. Define Weibull distribution and also give its mean and variance.

107. State the reproductive property of normal distribution.

108. Why is normal distribution considered an important distribution?

109. Find the value of k, mean and variance of the normal distribution whose density function is $k \cdot 2^{-x^2}, -\infty < x < \infty$.

$\left[\textbf{Ans.}\ \sqrt{\frac{\log 2}{\pi}}\right]$

110. If X follows $N(30, 5)$ and Y follows $N(15, 10)$, show that $P(26 \le X \le 40) = P(7 \le Y \le 35)$.

111. If X follows $N(3, 2)$, find the value of k such that $P(|X-3| > k) = 0.05$.

[**Ans.** 3.92]

112. For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. What are its mean and SD?

[**Ans.** 48, 2]

113. If X and Y are independent random variables having $N(1, 2)$ and $N(2, 2)$ respectively find the density function of $(X + 2Y)$.

114. If the random variable X follows an exponential distribution with parameter 2, prove that $Y = X^3$ follows a Weibull distribution with parameters 2 and 1/3.

115. If the service life, in hours, of a semiconductor is a random variable having a Weibull distribution with the parameters $\alpha = 0.025$ and $\beta = 0.5$,
   (i) how long can such a semiconductor be expected to last, and
   (ii) what is the probability that such a semiconductor will still be in operating condition after 4000 hours?.

[**Ans.** 3200, 0.2057]

116. In a sample of 1000 cases, the mean of a certain test is 14 and SD is 2.5. Assuming the distribution to be normal find
   (i) how many students score between 12 and 15,
   (ii) how many score above 18,
   (iii) how many score below 18, and
   (iv) how many score 16?

[**Ans.** (i) 443, (ii) 54, (iii) 8, (iv) 116]

117. The average test marks in a particular class is 79. The SD is 5. If the marks are distributed normally, how many students in a class of 200 did not receive marks between 75 and 82? [**Ans.** 97]

118. At a certain examination 10% of the students who appeared for the paper in statistics got less than 30 marks and 97% of the students got less than 62 marks. Assuming the distribution is normal, find the mean and the SD of the distribution. [**Ans.** $\mu = 43.04$, $\sigma = 10.03$]

119. If X is normally distributed with mean zero and variance unity, what is the expectation and variance of e^{aX}?

$$\left[\textbf{Ans.}\quad e^{\frac{a^2}{2}}, e^{a^2}\left(e^{a^2} - 1\right)\right]$$

120. Given X is distributed normally. If $P(X \leq 5) = 0.31$ and $P(X \geq 64) = 0.08$, find the mean and standard deviation of the distribution.

[**Ans.** $\mu = 50$, $\sigma^2 = 100$]

121. In a sample of 120 workers in a factory, the mean and standard deviation of wages were ₹ 11.35 and ₹ 3.03 respectively. Find the percentage of workers getting wages between ₹ 9 and ₹ 17 in the whole factory assuming that the wages are normally distributed? [**Ans.** 75.09%]

122. A random sample of size n is taken from a population which is exponentially distributed with parameter λ. If X is the sample mean, show that $n\lambda\bar{X}$ follows a simple Gamma distribution with parameter n.
[**Hint:** Use moment generating function.]

123. Find the probability of failure-free performance of roller-bearings over a period of 104 hours, if the life expectancy of the bearings is defined by Weibull distribution with parameters $\alpha = 10^{-7}$ and $\beta = 1.5$.

[Hint: P(failure-free performance over a period t) = P[the component does not fail in $(0, t)$] = $P(T \geq t)$, where T is the life expectancy or time to failure of the component.] [Ans. 0.9048]

124. The time when a country bus passes a certain point is distributed normally with a mean 9.25 a.m. and a SD of 3 minutes. What is the least time one could arrive at this point and still have a probability of 0.99 of catching the bus?

[Hint: If T is the time in minutes past 9 a.m., then T follows $N(25, 3)$.] [Ans. 9.18 a.m.]

125. The marks obtained by a number of students in a certain subject are assumed to be approximately normally distributed with mean 55 and a SD of 5. If 5 students are taken at random from this set, what is the probability that 3 of them would have scored marks above 60?

[Ans. 0.0283]

126. The life lengths in hours of 2 electronic devices A and B have distributions $N(40, 6)$ and $N(45, 3)$ respectively. If the electronic device is to be used for a 45 hours period, which device is to be preferred? If it is to be used for a 48 hours period, which device is to be preferred?

[Ans. (i) B, (ii) B]

127. The mean and SD of a certain group of 1000 high school grades, that are normally distributed are 78% and 11% respectively.
   (i) Find how many grades were above 90%?
   (ii) What was the highest grade of the lowest 10?
   (iii) Within what limits did the middle 900 lie?

[Ans. (i) 138, (ii) 52%, (iii) 60% and 96.1%]

128. The local authorities in a certain city install 10000 electric lamps in the streets of the city. If these lamps have an average life of 1000 burning hours with a SD of 200 h, how many lamps might be expected to fail
   (i) in the first 800 burning hours,
   (ii) between 800 and 1200 burning hours? After how many burning hours, would you expect
      (a) 10% of the lamps to fail,
      (b) 10% of the lamps to be still burning? Assume that the life of lamps is normally distributed. [Ans. (i) 1587, (ii) 6826, (a) 744 h, (b) 1256 h]

129. In a normal population with mean 15 and SD 3.5, it is found that 647 observations exceed 16.25. What is the total number of observations in the population? [Ans. 1800]

130. A random variable has a normal distribution with SD 10. If the probability that the random variable will take on a value less than 82.5 is 0.8212, what is the probability that it will take on a value greater than 58.3?

[Ans. 0.4332]

131. In a normal distribution, 7% of the items are under 35 and 89% are under 63. What are the mean and SD of the distribution? What percentage of items are under 49?

[**Ans.** 50.3, 10.33, 45%]

132. A normal population has coefficient of variation equal to 2% and 8% of the population lies above 120 cm. What percentage of the population lies below 115 cm? [**Ans.** 23%]

133. The breaking strength X of a certain kind of rope (in kg) has distribution $N(45, 1.8)$. Each 50 m coil of rope brings a profit of ₹ 1000, provided $X > 43$. If $X \leq 43$, the rope may be used for a different purpose and a profit of ₹ 400 per coil is realised. Find the expected profit per coil.

[**Ans.** ₹ 920 (nearly)]

4

Functions of a Random Variable

Let X be a continuous random variable (RV) with known PDF $f_X(x)$ and g be a real-valued function whose domain contains the range of the random variable X. Then $Y = g(X)$ is also a random variable. In general, the distribution of Y is uniquely determined, if g is a continuous and strictly monotonic function from $R \to R$. In this case, g^{-1} exists and continuous, such that for all x and y, $y = g(x)$ if $x = g^{-1}(y)$.

Let $Y = g(X)$, where $g(\cdot)$ be a function of another random variable. Then we can find the PDF of Y as follows.

Case 1

$g(X)$ is a strictly increasing function of X.

$$\begin{aligned} F_Y(y) &= P(Y \leq y), \text{ where } F_Y(y) \text{ is the CDF of } Y \\ &= P[g(X) \leq y] \\ &= P[X \leq g^{-1}(y)] \\ &= P[X \leq x] \\ &= F_X(x) \\ F_Y(y) &= F_X[g^{-1}(y)] \end{aligned}$$

Differentiating the above equation both sides with respect to y,

$$f_Y(y) = f_X(y)\frac{dx}{dy}, \quad \text{where } x = g^{-1}(y) \tag{4.1}$$

$f_X(x)$ is the PDF of the random variable X and $\frac{d}{dx}F_X(x) = f_X(x)$

Case 2

$g(X)$ is a strictly decreasing function of X.

$$F_Y(y) = P(Y \leq y)$$

$$= P[g(X) \le y]$$
$$= P[X \ge g^{-1}(y)]$$
$$= 1 - P[X \le g^{-1}(y)]$$
$$F_Y(y) = 1 - F_X[g^{-1}(y)]$$

Differentiating wtih respect to y, we get

$$f_Y(y) = -f_X(y)\frac{dx}{dy} \tag{4.2}$$

Combining Eqs. (4.1) and (4.2), we get

$$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$$

which is true for any continuous strictly monotonic real-valued function f whose domain includes the range of X, such that $Y = g(X)$.

Note:

(i) The above formula for $f_Y(y)$ can be used only when $x = g^{-1}(y)$ is single-valued.

(ii) For a discrete random variable X, the transformation $Y = g(X)$ only changes the range of X. The probability mass remaining the same.

$$P(X = x_i) = P(Y = y_i), \text{ where } y_i = g(x_i)$$

$$\therefore \quad f_Y(y_i) = f_X(y_i)$$

To find the PDF of the random variable Y given the PDF of X

(i) Write Y in terms of X and write the same relation in terms of y. For example, $Y = X^2 \Rightarrow X = \pm\sqrt{Y}$.

(ii) Find $\dfrac{dx}{dy}$.

(iii) Write $f_Y(y) = f_X(y)\left|\dfrac{dx}{dy}\right|$.

(iv) Using the given range of X, find the range of Y.
For example,
If $\quad f_X(x) = x^2, \; -1 < x < 2$
and $\quad Y = X^3$, then the range of Y is as follows:
When $x = -1$, $y = (-1)^3 = -1$, and $x = 2$, $y = (2)^3 = 8$
The range of Y is $-1 < y < 8$

EXAMPLE 4.1 If X is a normal random variable with mean 0 and variance σ^2, find the PDF of $Y = e^X$. **[AU December '07]**

Solution Given: $y = e^x$ which is a monotonic function.

$$\therefore \quad f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$$

Since X is a normal random variable with mean 0 and variance σ^2, the PDF is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left[\frac{(x-0)^2}{2\sigma^2}\right]} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, -\infty < x < \infty$$

$$y = e^x \Rightarrow x = \log y$$

and $$\frac{dx}{dy} = \frac{1}{y}$$

and also $$f_X(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(\log y)^2}{2\sigma^2}}, y \geq 0, \qquad [\because e^{-\infty} = 0, e^{\infty} = \infty]$$

$$\Rightarrow \quad f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(\log y)^2}{2\sigma^2}} \cdot \left|\frac{1}{y}\right| = \frac{1}{y\sigma\sqrt{2\pi}} e^{\frac{-(\log y)^2}{2\sigma^2}}, y \geq 0$$

which is the PDF of a log normal.

EXAMPLE 4.2 If X has an exponential distribution with parameter λ, find the PDF of $Y = \log X$. **[AU December '06; '07, June '07]**

Solution Given: X is an exponentially distributed random variable with parameter λ then PDF of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Given $$y = \log x \Rightarrow x = e^y$$

$$x = e^y \Rightarrow \frac{dx}{dy} = e^y$$

When $x = 0$, $e^y = 0 \Rightarrow y = -\infty$ and $x = \infty$, $e^y = \infty \Rightarrow y = \infty$

But $$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$$

Since $x = e^y$, we get $f_X(y) = \lambda e^{-\lambda e^y}$

$\therefore$ $$f_Y(y) = \lambda e^{-\lambda e^y} e^y, -\infty < y < \infty$$

EXAMPLE 4.3 If X is a continuous normal (Gaussian) random variable with zero mean and variance σ^2, find the PDF of the random variable Y, if $Y = X^2$.

Solution Given: X is a normal random variable with mean 0 and variance σ^2.

The PDF of X is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left[\frac{(x-0)^2}{2\sigma^2}\right]} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, -\infty < x < \infty \quad \text{(i)}$$

To find the PDF of Y

$$F_Y(y) = P(Y \le y) \quad \text{(By definition)}$$
$$= P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y}) \qquad [\because y = x^2 \Rightarrow x = \pm\sqrt{y}]$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \text{ by defnition of distribution function.}$$

Differentiating the above with respect to Y,

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \left(\frac{-1}{2\sqrt{y}}\right)$$
$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

$$f_X(\sqrt{y}) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-y}{2\sigma^2}} \text{ and } f_X(-\sqrt{y}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}}, \text{ from Eq. (i)}$$

$Y = X^2 \Rightarrow$ when $x = -\infty$, $y = \infty$

$$\therefore \quad f_Y(y) = \frac{1}{2\sqrt{y}} \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{y}{2\sigma^2}}, y > 0$$

EXAMPLE 4.4 The random variable Y is defined by $Y = \frac{1}{2}(X + |X|)$, where X is another random variable. Determine the density and distribution function of y interms of those of X.

Solution Given: $y = \frac{1}{2}(x + |x|)$

When $x > 0$, $y = \frac{1}{2}(x + |x|) = \frac{1}{2}(x + x) = \frac{2x}{2} = x, y > 0$

$$\Rightarrow y = x, y > 0 \qquad \begin{bmatrix} |x| = x, x > 0 \\ \quad = -x, x < 0 \end{bmatrix}$$

When $x < 0$, $y = \frac{1}{2}(x - x) = 0$

If $y < 0$, $F_Y(y) = P(Y \le y) = 0$

If $y \ge 0$, $F_Y(y) = P(Y \le y)$

$$F_Y(y) = P\,[X \le y / X \ge 0]$$

$$= \frac{P\,(X \le y,\, X \ge 0)}{P\,(X \ge 0)} = \frac{P\,(X \le y,\, X \ge 0)}{1 - P\,(X < 0)}$$

$$= \frac{P\,(0 \le X \le y)}{1 - P\,(X < 0)} = \frac{F_X\,(y) - F_X\,(0)}{1 - F_X\,(0)}$$

$\therefore$ $$f_Y(y) = 0, \text{ when } y < 0$$

and $$f_Y(y) = \frac{f_X\,(y)}{[1 - F_X\,(0)]}, \text{ when } y \ge 0$$

EXAMPLE 4.5 If the PDF of X is $f_X(x) = e^{-x}$, $x > 0$, find the PDF of $Y = 2X + 1$. **[AU April '03]**

Solution Given:

$$y = 2x + 1 \;\Rightarrow\; x = \frac{y-1}{2}$$

Now, $$f_Y(y) = f_X\,(y)\left|\frac{dx}{dy}\right| \tag{i}$$

Since, $$x = \frac{y-1}{2} \;\Rightarrow\; \frac{dx}{dy} = \frac{1}{2}$$

$$f_X(x) = e^{-x} \;\Rightarrow\; f_X(y) = e^{-\left(\frac{y-1}{2}\right)}$$

Since, $x > 0$, $\frac{y-1}{2} > 0 \;\Rightarrow\; y - 1 > 0 \;\Rightarrow\; y > 1$

Hence using (i) the PDF of Y is

$$f_Y(y) = \frac{1}{2}e^{-\left(\frac{y-1}{2}\right)}, \quad y > 1$$

EXAMPLE 4.6 If the PDF of a random variable X is $f_X(x) = 2x$, $0 < x < 1$, find the PDF of $Y = e^{-X}$. **[AU December '06]**

Solution Given:

$$y = e^{-x} \;\Rightarrow\; x = \log\left(\frac{1}{y}\right)$$

$\therefore$ $$\frac{dx}{dy} = \frac{1}{\frac{1}{y}} \cdot \left(\frac{-1}{y^2}\right) = \frac{-1}{y}$$

$$f_X(x) = 2x \;\Rightarrow\; f_X(y) = 2\log\left(\frac{1}{y}\right)$$

Now, $$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$$

$$= 2\log\left(\frac{1}{y}\right)\left|\frac{-1}{y}\right| = \frac{2}{y}\log\left(\frac{1}{y}\right) = \frac{-2\log y}{y}$$

Since $0 < x < 1$, $-1 < -x < 0$

$\Rightarrow$ $e^{-1} < e^{-x} < e^0$ as e^x is an increasing function of x.

$\Rightarrow$ $\frac{1}{e} < y < 1$

$\therefore$ $$f_Y(y) = \frac{-2\log y}{y}, \frac{1}{e} < y < 1$$

EXAMPLE 4.7 A random variable X has the PDF $f_X(x) = e^{-2|x|}$, $-\infty < x < \infty$ find the PDF of $Y = X^2$. **[AU December '07]**

Solution Given

$$y = x^2 \quad \Rightarrow \quad x = y^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$$

$$f_X(x) = e^{-2|x|}, \quad \Rightarrow \quad f_X(y) = e^{-2|\sqrt{y}|}$$

The PDF of Y is

$$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| = e^{-2|\sqrt{y}|}\cdot\frac{1}{2\sqrt{y}}, 0 < y < \infty$$

$[\because$ when $x = -\infty$, $y = x^2 = \infty]$

EXAMPLE 4.8 Find the distribution function of the random variable $Y = g(X)$ in terms of the distribution function of X if it is given that

$$g(x) = \begin{cases} x - c, & x > c \\ 0, & |x| \le c \\ x + c, & x < -c \end{cases}$$

Solution When $x > c$, $y = x - c \ge 0$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X - c \le y) \\ &= P(X \le y + c) \\ &= F_X(y + c) \end{aligned}$$

When $x < -c$, $y = x + c < 0$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X + c \le y) \\ &= P(X \le y - c) \\ &= F_X(y - c) \end{aligned}$$

When $$\begin{aligned} F_Y(y) &= F_X(y - c), y < 0 \\ &= F_X(y + c), y \ge 0 \end{aligned}$$

Differentiating with respect to y,

$$f_Y(y) = f_X(y - c),\ y < 0$$
$$= f_X(y + c),\ y \geq 0$$

EXAMPLE 4.9 Find the density function of $Y = aX + b$ in terms of the density function of X and, hence, find the PDF of $Y = 2X - 3$, if the PDF of X is $f(x) = \frac{x}{12}, 1 < x < 5$ and $= 0$ elsewhere.

Solution

$$F_Y(y) = P(Y \leq y)$$
$$= P(aX + b \leq y)$$
$$= P(aX \leq y - b)$$
$$= P\left(X \leq \frac{y-b}{a}\right),\ a > 0$$

$\therefore$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right), a > 0$$
$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$$
$$= P\left(X \geq \frac{y-b}{a}\right),\ a < 0$$
$$= 1 - P\left(X \leq \frac{y-b}{a}\right)$$
$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$\therefore$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right),\ \text{if } a > 0$$
$$= 1 - F_X\left(\frac{y-b}{a}\right),\ \text{if } a < 0 \qquad \text{(i)}$$

We have,

$$x = \frac{y-b}{a},\ \frac{dx}{dy} = \frac{1}{a} \qquad \text{(ii)}$$

Differentiating (i) with respect to x,

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}, a > 0,\ \text{using (ii)}$$
$$= -f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}, a < 0$$

$\therefore$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$y = 2x - 3 \Rightarrow a = 2, b = -3$ and $x = \frac{y+3}{2}$ is a single-valued function of y.

Given $f_x(x) = \dfrac{x}{12}, 1 < x < 5.$

When $x = 1$, $y = -1$ and $x = 5$, $y = 7$

$$\therefore \quad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{2} f_X\left(\frac{y+3}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{12}\right)\frac{y+3}{2} = \frac{y+3}{48}, \quad -1 < y < 7$$

Therefore, PDF of Y is

$$f_Y(y) = \begin{cases} \dfrac{y+3}{48} & -1 < y < 7 \\ 0, & \text{elsewhere} \end{cases}$$

EXAMPLE 4.10 If the continuous random variable X has the PDF

$$f_X(x) = \begin{cases} \dfrac{2}{9}(x+1), & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the PDF of $Y = X^2$.

Solution Given: $y = x^2$, and it is not monotonic in $(-1, 2)$. So, we divide the interval into two parts as $(-1, 1)$ and $(1, 2)$

$$\therefore \quad x = \pm\sqrt{y}$$

Given: $f_X(x) = \begin{cases} \dfrac{2}{9}(x+1), & \text{for } -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$

Since $(-1, 1)$ is a symmetric interval, $f_Y(y)$ in the interval $(-1, 1)$ is

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= P\left(-\sqrt{y} \le X \le \sqrt{y}\right) = F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) \text{ by definition} \qquad \text{(i)}$$

Differentiating (i) with respect to y,

$$f_Y(y) = f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

Given: $$f_X(x) = \frac{2}{9}(x+1), \qquad -1 < x < 2$$

$$f_X\left(\sqrt{y}\right) = \frac{2}{9}\left(\sqrt{y}+1\right)$$

and $$f_X\left(-\sqrt{y}\right) = \frac{2}{9}\left(1-\sqrt{y}\right)$$

$\therefore$ $$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}\left(\sqrt{y}+1\right)+\frac{2}{9}\left(1-\sqrt{y}\right)\right]$$

$$= \frac{2}{9}\cdot\frac{1}{2\sqrt{y}}\left(1+\sqrt{y}-\sqrt{y}+1\right) = \frac{2}{9\sqrt{y}}$$

$\therefore$ $$f_Y(y) = \frac{2}{9\sqrt{y}}, 0 < y < 1 \quad [\because\ y = x^2,\ y \text{ is always positive}]$$

When $1 < x < 2$, $y = x^2$ is monotonic (increasing),

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

Given $$f_X(x) = \frac{2}{9}(x+1)$$

$\therefore$ $$f_X(x) = \frac{2}{9}(\sqrt{y}+1)$$

$\therefore$ $$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| = \frac{2}{9}\left(\sqrt{y}+1\right)\times\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{9}\left(1+\frac{1}{\sqrt{y}}\right), \qquad 1 < y < 4$$

EXAMPLE 4.11 Given that the random variable X with density function

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the PDF of $Y = 8X^3$.

[AU June '07]

Solution $y = 8x^3$ is an increasing function in $(0, 1)$

Given: $$y = 8x^3 \quad \Rightarrow \quad x^3 = \frac{y}{8} \quad \Rightarrow \quad x = \left(\frac{y}{8}\right)^{\frac{1}{3}} = \frac{1}{2}y^{\frac{1}{3}}$$

and $$f_X(x) = 2x, \qquad 0 < x < 1$$

$$f_X(y) = \frac{2y^{\frac{1}{3}}}{2} = y^{\frac{1}{3}}$$

$$f_Y(y) = f_X(y)\frac{dx}{dy} = y^{\frac{1}{3}}\frac{dx}{dy} \qquad \text{(i)}$$

Given: $$x = \left(\frac{y}{8}\right)^{\frac{1}{3}} = \frac{1}{2}y^{\frac{1}{3}} \Rightarrow \frac{dx}{dy} = \frac{1}{6}y^{-\frac{2}{3}}$$

Using it in (i)

$$f_Y(y) = y^{\frac{1}{3}}\frac{1}{6}y^{-\frac{2}{3}} = \frac{1}{6}y^{-\frac{1}{3}} = \frac{1}{6}\frac{1}{y^{\frac{1}{3}}} = \frac{1}{6\sqrt[3]{y}}$$

The range for x is $0 < x < 1$

When $x = 0$, $y = 8 \times 0 = 0$ and $x = 1$, $y = 8 \times 1^3 = 8$

$$\therefore \qquad f_Y(y) = \frac{1}{6\sqrt[3]{y}}, \; 0 < y < 8$$

EXAMPLE 4.12 If X is a Gaussian random variable with zero mean and variance σ^2, find the PDF of $Y = |X|$.

Solution We know that

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(|X| \le y) \\ &= P(-y \le X \le y) \\ &= F_X(y) - F_X(-y) \end{aligned}$$

Differentiating with respect to y,

$$f_Y(y) = f_X(y) + f_X(-y) \qquad \text{(i)}$$

Given: $X \sim N(0, \sigma^2) \Rightarrow f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}}, \qquad -\infty < x < \infty,\ y = |x|$

$$f_X(y) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-y^2}{2\sigma^2}}, \qquad 0 < y < \infty$$

and $$f_X(-y) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-y^2}{2\sigma^2}}, \qquad 0 < y < \infty$$

Substituting in Eq. (i), we get

$$f_Y(y) = \frac{2}{\sigma\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}}, \qquad 0 < y < \infty$$

$[\because y = |x| \Rightarrow \text{ when } x = -\infty, y = \infty]$

EXAMPLE 4.13 If X is uniformly distributed in $(-1, 1)$, find the PDF of $y = \cos \pi x$.

Solution Since X is a uniformly distributed random variable in $(-1, 1)$, the PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Given: $y = \cos \pi x \Rightarrow \dfrac{dy}{dx} = -\pi \sin \pi x$

$$= -\pi\sqrt{1-y^2} \qquad \left[\because \sin \pi x = \sqrt{1-\cos^2 \pi x} = \sqrt{1-y^2}\right]$$

$$\therefore \quad \left|\frac{dx}{dy}\right| = \frac{1}{\pi\sqrt{1-y^2}}$$

The range of x is $-1 < x < 1 \Rightarrow -1 < \dfrac{1}{\pi}\cos^{-1} y < 1$

$$\Rightarrow -\pi < \cos^{-1} y < \pi \Rightarrow -1 < y < 1$$

But, $f_Y(y) = f_X(Y)\left|\dfrac{dx}{dy}\right| = \dfrac{1}{2}\dfrac{1}{\pi\sqrt{1-y^2}}$

$\therefore$ The PDF of y is $f_Y(y) = \begin{cases} \dfrac{1}{2\pi\sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$

EXAMPLE 4.14 A random variable X has an exponential distribution defined by the PDF $f(x) = e^{-x}$, $0 < x < \infty$, find the density function of

(i) $Y = 3X + 5$ and (ii) $Y = X^3$.

Solution Given:

(i) $y = 3x + 5 \Rightarrow \dfrac{y-5}{3} = x$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{3}$$

Using $f_X(x) = e^{-x}$ and $x = \dfrac{y-5}{3}$, we get

$$f_X(y) = e^{-\left(\frac{y-5}{3}\right)}$$

To find the PDF of Y, we have

$$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$$

$$\therefore \qquad f_Y(y) = \frac{1}{3}e^{-\left(\frac{y-5}{3}\right)}, \; y > 5$$

$$\left(\text{since } x > 0 \;\Rightarrow\; \frac{y-5}{3} > 0 \;\Rightarrow\; y > 5\right)$$

(ii) $y = x^3 \;\Rightarrow\; x = y^{\frac{1}{3}}$

$$\therefore \qquad \left|\frac{dx}{dy}\right| = \frac{1}{3}y^{-\frac{2}{3}} \quad \text{and} \quad x > 0 \;\Rightarrow\; y > 0$$

Therefore, the PDF of Y is

$$f_Y(y) = \frac{1}{3}e^{-(y)^{\frac{1}{3}}}\, y^{-\frac{2}{3}}, \qquad 0 < y < \infty$$

EXAMPLE 4.15 Let X be a random variable with PDF $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. If $Y = -2 \log X$, find the PDF of the random variable Y and $E(Y)$.

[AU June '07]

Solution Given:

$$y = -2 \log x$$

i.e.

$$\log x = -\frac{y}{2} \;\Rightarrow\; e^{-\frac{y}{2}} = x$$

$$\frac{dx}{dy} = e^{-\frac{y}{2}}\left(-\frac{1}{2}\right) = -\frac{e^{-\frac{y}{2}}}{2}$$

Given the PDF $f_X(x) = f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore \qquad f_X(y) = 1$$

Using $y = -2 \log x$, when $x = 0$, $y = \infty$ and $x = 1$, $y = 0$

$\therefore$ The PDF of $Y = -2 \log X$ is given by

$$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| = 1\left|-\frac{e^{-\frac{y}{2}}}{2}\right|$$

$$= \frac{1}{2}e^{-\frac{y}{2}}, \qquad 0 < y < \infty$$

Now,

$$E(Y) = E(-2 \log x) = \int_0^1 (-2 \log x)\, f(x)\, dx$$

$$= \int_0^1 -2 \log x\, dx = -2\int_0^1 \log x\, dx$$

$$= -2\left[[x\log x]_0^1 - \int_0^1 x\frac{1}{x}dx\right]$$

$$= -2[(0-0)-[x]_0^1] = 2$$

EXAMPLE 4.16 A random variable X is uniformly distributed over $(0, 2\pi)$. If $Y = \cos X$, find the PDF of Y and $E(Y)$. **[AU June '07]**

Solution Since X follows uniform distribution, its PDF is given by

$$f_X(x) = f(x) = \begin{cases} \frac{1}{2\pi}, & 0 < x < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Given: $y = \cos x \Rightarrow x = \cos^{-1}y$

$$\frac{dx}{dy} = -\frac{1}{\sqrt{1-y^2}}$$

and

$$f_X(y) = \frac{1}{2\pi}$$

Given: $0 < x < 2\pi \Rightarrow 0 < \cos^{-1}y < 2\pi$

$$\Rightarrow -1 < y < 1$$

The PDF $Y = \cos X$ is given by

$$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| = \frac{1}{2\pi}\left|\frac{-1}{\sqrt{1-y^2}}\right|$$

$$\therefore \quad f_Y(y) = \frac{1}{2\pi}\frac{1}{\sqrt{1-y^2}}, \quad -1 < y < 1$$

[$\because$ $y = \cos x$, and $0 < x < 2\pi$, $y = \cos\theta = \cos 2\pi = 1$, $y = \cos\pi = -1$]
[When $0 < \pi < 2\pi$, $y = \cos x \Rightarrow \cos 0 = 1$, $\cos\pi = -1$, $\cos 2\pi = 1$
$\therefore -1 < y < 1$]

Now,

$$E(y) = E(\cos X) = \int_0^{2\pi} \cos x\, f(x)dx = \int_0^{2\pi} \cos x \cdot \frac{1}{2\pi}dx$$

$$= \frac{1}{2\pi}(\sin x)_0^{2\pi} = 0$$

EXAMPLE 4.17 If X is uniformly distributed in $(-1, 1)$, find the PDF of $Y = \sin\left(\frac{\pi X}{2}\right)$. **[AU June '06]**

Solution Given:

$$y = \sin\left(\frac{\pi x}{2}\right) \Rightarrow \frac{\pi x}{2} = \sin^{-1}(y)$$

$$x = \frac{2}{\pi}\sin^{-1}(y)$$

$$\therefore \qquad \frac{dx}{dy} = \frac{2}{\pi\sqrt{1-y^2}}$$

Using $y = \sin\left(\frac{\pi x}{2}\right)$, we get

When $x = -1$, $y = \sin\left(-\frac{\pi}{2}\right) = -1$

When $x = 1$, $y = \sin\left(\frac{\pi}{2}\right) = 1.$

Since X is uniformly distributed in $(-1, 1)$, the PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow f_X(y) = \begin{cases} \frac{1}{2}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

$\therefore$ The PDF Y is given by

$$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$$

$$= \begin{cases} \frac{1}{\pi\sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

EXAMPLE 4.18 If X is uniformly distributed in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, find the PDF of $Y = \tan X$. **[AU December '03, May '03, June '06]**

Solution Given the PDF of X as

$$f_X(x) = \begin{cases} \frac{1}{\pi}, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

and $\qquad y = \tan x \Rightarrow x = \tan^{-1} y \Rightarrow \frac{dx}{dy} = \frac{1}{1+y^2}$

when $x = \frac{-\pi}{2}$, $y = \tan\left(\frac{-\pi}{2}\right) = -\infty$ and when $x = \frac{\pi}{2}$, $y = \tan\left(\frac{\pi}{2}\right) = \infty$

$$\therefore \qquad f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| = \frac{1}{\pi}\cdot\frac{1}{1+y^2}, \qquad -\infty < y < \infty$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+y^2}, \qquad -\infty < y < \infty$$

EXAMPLE 4.19 If the random variable X follows an exponential distribution with parameter 2, prove that $Y = X^3$ follows Weibull distribution with parameters 2 and $\frac{1}{3}$.

Solution Given: X follows exponential distribution with parameter $\lambda = 2$

$\therefore$ The PDF of X is

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Since $y = x^3$ is a monotonically increasing function and $x = y^{\frac{1}{3}}$, then

$$\frac{dx}{dy} = \frac{1}{3} y^{-\frac{2}{3}}$$

To find the PDF of y:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{3} y^{\frac{-2}{3}} \cdot 2e^{-2y^{\frac{1}{3}}}$$

$$= 2\left(\frac{1}{3}\right) y^{\frac{1}{3}-1} \cdot e^{-2y^{\frac{1}{3}}}, \qquad y > 0$$

Since $x > 0$, taking $\alpha = 2$ and $\beta = \left(\frac{1}{3}\right)$, the PDF of Y can be put in the form $f_Y(y) = \alpha\beta y^{\beta-1} e^{-\alpha y^{\beta}}$.

Therefore, the random variable Y follows Weibull distribution with parameters 2 and $\frac{1}{3}$.

EXAMPLE 4.20 If X has an exponential distribution with parameter 1, find the PDF of $Y = \sqrt{X}$. [AU April '0

Solution Given: $y = \sqrt{x}$ and $x = y^2$

Since X has an exponential distribution with parameter 1, the PDF of X is given by

$$f_X(x) = e^{-x}, x > 0 \qquad [\because f(x) = \lambda e^{-\lambda x}, \lambda =$$

$$x = y^2 \Rightarrow \frac{dx}{dy} = 2y, \quad f_X(y) = e^{-y^2}$$

$\therefore$

$$f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| = 2ye^{-y^2}$$

$\Rightarrow$

$$f_Y(y) = 2ye^{-y^2}, y > 0$$

EXAMPLE 4.21 If X is uniformly distributed in (0, 1) find the PDF of $Y = \dfrac{1}{2X+1}$.

Solution Given:

$$y = \frac{1}{2x+1}$$

$$\therefore \quad y = \frac{1}{2x+1} \quad \Rightarrow \quad 2x+1 = \frac{1}{y}$$

$$\Rightarrow \quad 2x = \frac{1}{y} - 1 \quad \Rightarrow \quad x = \frac{1-y}{2y}$$

$$\therefore \quad \frac{dx}{dy} = \frac{-1}{2y^2}$$

Since X is uniformly distributed in (0,1), the PDF of X is given by

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

The range of x is $0 < x < 1$,

When $x = 0$, $y = 1$ and when $x = 1$, $y = \dfrac{1}{3}$

The range of y is

$$\frac{1}{3} < y < 1 \qquad \left[\begin{array}{lll} 0 < x < 1 & \Rightarrow & 0 < 2x < 2 \\ 1 < 2x+1 < 3 & \Rightarrow & \dfrac{1}{3} < \dfrac{1}{2x+1} < 1 \\ & \Rightarrow & \dfrac{1}{3} < y < 1 \end{array}\right]$$

$$\therefore \quad f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| = \frac{1}{2y^2}$$

$\therefore$ The PDF of Y is given by

$$f_Y(y) = \frac{1}{2y^2}, \quad \frac{1}{3} < y < 1$$

EXERCISES

1. If X and Y are two random variables where $Y = g(X)$, how are the density functions of X and Y are related?

$$\left[\textbf{Ans.} \quad f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right| \text{ if } X \text{ is single-valued}\right]$$

2. If the PDF of X is $f_X(x) = 2x$, $0 < x < 1$, find the PDF of $Y = 3X + 1$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{2}{9}(y-1), 1 < y < 4\right]$$

3. If the PDF of X is $f_X(x) = e^{-x}$, $x > 0$, find the PDF of $Y = 2X + 1$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{1}{2}e^{-\left(\frac{y-1}{2}\right)}, y > 1\right]$$

4. If the random variable X is uniformly distributed in (1, 2) find the PDF of $Y = \dfrac{1}{X}$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{1}{y^2}, \frac{1}{2} < y < 1\right]$$

5. If the CDF of a random variable X is $F(x)$, show that the random variable $Y = F(x)$ follows a uniform distribution. [**Ans.** $f_Y(y) = 1, 0 < y < 1$]

6. A random variable X assumes three values -1, 0, 1 with probabilities 1/3, 1/2, 1/6 respectively, find the probability distribution of $Y = 3X + 1$.

$$\left[\textbf{Ans.}\quad \begin{array}{cccc} y: & -2 & 1 & 4 \\ p(y): & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

7. If X and Y are two random variables such that $Y = X^2$, how are the CDFs of X and Y related?

$$\left[\textbf{Ans.}\quad f_Y(y) = \begin{cases} F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right), & y \geq 0 \\ 0, & \text{otherwise} \end{cases}\right]$$

8. If X and Y are two random variables such that $Y = X^2$, how are the PDFs of X and Y related?

$$\left[\textbf{Ans.}\quad f_Y(y) = \begin{cases} f_X\left(\sqrt{y}\right) - f_X\left(-\sqrt{y}\right), & y \geq 0 \\ 0, & y < 0 \end{cases}\right]$$

9. If the PDF of X is $f(x) = e^{-x}$, $x > 0$, find the PDF of $Y = X^2$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{1}{2\sqrt{y}}e^{-\sqrt{y}}, y > 0\right]$$

10. If the random variables X and Y are related by $Y = |X|$, how are the CDFs of X and Y related? [**Ans.** $F_Y(y) = F_X(y) - F_X(-y)$]

11. If the random variables X and Y are related by $Y = |X|$, how are the PDFs of X and Y related? [**Ans.** $f_Y(y) = f_X(y) + f_X(-y), y > 0$]

12. If the random variable X is uniformly distributed in $(-1, 1)$, find the PDF of $Y = |X|$. [**Ans.** $f_Y(y) = 1, 0 < y < 1$]

13. If the random variables X and Y are related by $Y = \sqrt{X}$, how are their PDFs related? [**Ans.** $f_Y(y) = 2yf_X(y^2)$]

14. If X is uniformly distributed in (0, 1), find the PDF of $Y = \sqrt{X}$.

[**Ans.** $f_Y(y) = 2y,\ 0 < y < 1$]

15. If the PDF of a random variable X is $f_X(x) = 2x$ in (0, 1), find the PDF of $Y = \sqrt{X}$.

[**Ans.** $f_Y(y) = 4y^3,\ 0 < y < 1$]

16. If the PDF of X is $f_X(x) = e^{-x}$, $x > 0$ find the PDF of $Y = \dfrac{3}{(X+1)^2}$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{1}{6}\left(\frac{3}{y}\right)^{\frac{3}{2}} e^{1-\sqrt{\frac{3}{y}}},\ 0 \le y \le 3\right]$$

17. Find the PDF of $Y = |X|$ in terms of the PDF of X. Using this, find $f_Y(y)$, when $f_X(x) = 1/2$ in $-1 < x < 1$.

$$\left[\begin{array}{lll}\textbf{Ans.} & f_Y(y) = f_X(y) + f_X(-y), & y > 0 \\ & f_Y(y) = 1, & 0 < y < 1\end{array}\right]$$

18. If X is a continuous random variable with some unknown distribution defined over (0, 1) such that $P(X \le 0.29) = 0.75$, determine k so that $P(Y \le k) = 0.25$, where $Y = 1 - X$.

[**Ans.** $k = 0.71$]

19. If the random variable X is uniformly distributed in $\left(0, \dfrac{\pi}{2}\right)$ find the distribution of $Y = \sin X$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}},\ 0 < y < 1\right]$$

20. If X is a continuous random variable with PDF $f_X(x) = e^{-x}$, $x > 0$, find the PDF of $Y = 2X + 1$. Hence or otherwise find $P(Y \ge 5)$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{1}{2}e^{-(y-1)^2}\ y > 1,\ P(Y \ge 5) = e^{-2}\right]$$

21. (i) If the continuous random variable X is uniformly distributed in (−3, 3), find the density function of $Y = 2X^2 - 3$.

(ii) If the continuous random variable X is uniformly distributed in (−2, 2), find the density function of $Y = 6 - X^2$.

$$\left[\begin{array}{l}\textbf{Ans.}\quad \text{(i) } f_Y(y) = \dfrac{1}{6\sqrt{2}\sqrt{y+3}},\ -3 \le y \le 15 \\[2ex] \qquad\quad \text{(ii) } f_Y(y) = \dfrac{1}{4\sqrt{6-y}},\ 2 \le y \le 6\end{array}\right]$$

22. If the continuous random variable X has density function $f(x) = 2e^{-2x}$, $x > 0$, find the density function of $Y = X^2$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{1}{\sqrt{y}}e^{-2\sqrt{y}},\ y > 0\right]$$

23. If the density function of a continuous random variable X is given by $f_X(x) = e^{-x}, x > 0$, find the density function of $Y = 3/(X + 1)^2$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{1}{6}\left(\frac{3}{y}\right)^{\frac{3}{2}} e^{1-\sqrt{\frac{3}{y}}}, 0 \le y \le 3\right]$$

24. If the random variable X is uniform in $(-2\pi, 2\pi)$, find the density function of random variable

(i) $Y = X^3$, and (ii) $Y = X^4$.

$$\left[\begin{array}{l}\textbf{Ans.}\quad \text{(i)}\ f_Y(y) = \dfrac{1}{12\pi} y^{-\frac{2}{3}}, -8\pi^3 \le y \le 8\pi^3 \\ \qquad\quad \text{(ii)}\ f_Y(y) = \dfrac{1}{8\pi} y^{-\frac{3}{4}}, 0 \le y \le 16\pi^4\end{array}\right]$$

25. If the random variable X is uniformly distributed over $(-1, 1)$, find the density function of $Y = \cos\left(\dfrac{\pi X}{2}\right)$.

$$\left[\textbf{Ans.}\quad f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}, 0 \le y \le 1\right]$$

26. If X is an arbitrary random variable with continuous distribution function $F_X(X)$ and if $Y = F_X(X)$, show that Y is uniformly distributed in $(0, 1)$.

5

Two-dimensional Random Variables

In the previous chapters we have considered the random experiments, the outcome of which had only one characteristic and hence was assigned a single need value. But in many real life situations, we will be interest in the random experiments which has two or more characteristics (numerically) and hence to be assigned two or more real values. For example, both voltage and current might be of interest in certain electrical experiment. So, in this chapter we are going to deal with two-dimensional random variables.

Let S be the sample space associated with the random experiment E. Let $X = X(s)$, $Y = Y(s)$ be two functions, each assigning a real number to each outcome $s \in S$ of the random experiment, then (X, Y) is called the *two-dimensional random variable*.

If the possible values of (X, Y) are finite or countably infinite, then (X, Y) is called a *two-dimensional discrete random variable*. If (X, Y) can assume all values in a specified region R in the xy plane, then (X, Y) is called a *two-dimensional continuous random variable*.

Two random variables X and Y are said to be jointly distributed if they are defined on the same probability space. The sample points consists of two-tuples.

5.1 DISCRETE RANDOM VARIABLES X AND Y

5.1.1 Joint Probability Mass Function of (X, Y)

For two discrete random variables X and Y, the probability that X will take the value x_i and Y will take the value y_j is denoted by $P(X = x_i, Y = y_j) = P(x_i, y_j) = p_{ij}$, which is the intersection of the two events $X = x_i$ and $Y = y_j$.

Let (X, Y) be a two-dimensional discrete random variable such that $p(x_i, y_j) = P(X = x_i, Y = y_j) = p_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, then p_{ij} is called the joint probability mass function of (X, Y) if it satisfies the following conditions:

(i) $p_{ij} \geq 0, \forall\, i, j$ (ii) $\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} = 1$

The joint probability distribution is the set of triples (x_i, y_j, p_{ij}), where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

The function $p(x_i, y_j) = P(X = x_i, Y = y_j) = p_{ij}$ is called *joint probability function* or *joint probability mass function* for discrete random variables X and Y.

Note: To show $p(x_i, y_j) = P(X = x_i, Y = y_j) = p_{ij}$ to be a joint PMF, we have to prove

(i) $P_{ij} \geq 0 \;\forall\; i, j$

(ii) $\sum_j \sum_i P_{ij} = 1$

5.1.2 Joint Probability Distribution of (X, Y)

The set of triples (x_i, y_j, P_{ij}), $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ is called the *joint probability distribution* of (X, Y) and it can be represented in the form of table as follows:

X \ Y	y_1	y_2	y_3	…	y_m	$p_X(x)$
x_1	p_{11}	p_{12}	p_{13}	…	p_{1m}	P_{1*}
x_2	p_{21}	p_{22}	p_{23}	…	p_{2m}	P_{2*}
x_3	p_{31}	p_{32}	p_{33}	…	p_{3m}	P_{3*}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_1	p_{n1}	p_{n2}	p_{n3}	…	p_{nm}	P_{n*}
$p_Y(y_j)$	P_{*1}	P_{*2}	P_{*3}	…	P_{*m}	1

5.1.3 Marginal Probability Distribution

Let (X, Y) be a two-dimensional discrete random variable. Then the marginal probability function of the random variable X is defined as

$$P(X = x_i) = \sum_{j=1}^{m} p_{ij} = P_{i*}$$

The marginal probability function of the random variable Y is defined as

$$P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = P_{\bullet j}$$

The marginal distribution of X is the collection of pairs $(x_i, P_{i\bullet})$ and of Y is $(y_j, P_{\bullet j})$.

Note:

(i) The following table is used to calculate the marginal distributions when the random variable X takes horizontal values and Y takes vertical values.

$Y \backslash X$	x_1	x_2	x_3	$p_Y(y) = f(y)$
y_1	p_{11}	p_{21}	p_{31}	$p_{11} + p_{21} + p_{31} = P(Y = y_1)$
y_2	p_{12}	p_{22}	p_{32}	$p_{12} + p_{22} + p_{32} = P(Y = y_2)$
y_3	p_{13}	p_{23}	p_{33}	$p_{13} + p_{23} + p_{33} = P(Y = y_3)$
$p_x(x) = f(x)$	$p_{11} + p_{12} + p_{13} = P(X = x_1)$	$p_{21} + p_{22} + p_{23} = P(X = x_2)$	$p_{31} + p_{32} + p_{33} = P(X = x_3)$	

(ii) The following table is used to calculate the marginal distributions when the random variable Y takes horizontal values and X takes vertical values.

$X \backslash Y$	y_1	y_2	y_3	$P_x(y) = f(x)$
x_1	p_{11}	p_{12}	p_{13}	$p_{11} + p_{11} + p_{13} = P(X = x_1)$
x_2	p_{21}	p_{22}	p_{23}	$p_{21} + p_{22} + p_{23} = P(X = x_2)$
x_3	p_{31}	p_{32}	p_{33}	$p_{31} + p_{32} + p_{33} = P(X = x_3)$
$p_Y(x) = f(y)$	$p_{11} + p_{21} + p_{31} = P(Y = y_1)$	$p_{12} + p_{22} + p_{32} = P(Y = y_2)$	$p_{13} + p_{23} + p_{33} = P(Y = y_3)$	

5.1.4 Cumulative Distribution Function

The cumulative distribution function of a two-dimensional discrete random variable (X, Y) denoted by $F(x, y)$ is defined as

$$F(x, y) = P(X \le x_i, Y \le y_j)$$

$$= \sum_{y_j \le y} \sum_{x_i \le x} p_{ij}$$

5.1.5 Conditional Probability Distribution

Let (X, Y) be a two-dimensional discrete random variable, then

$$P(X = x_i / Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{P_{ij}}{p_{\bullet j}}$$

is called the conditional probability function of X given that $Y = y_j$

and
$$P(Y = y_j / X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)} = \frac{P_{ij}}{p_{i\bullet}}$$

is called the conditional probability function of Y given that $X = x_i$.

The collection of pairs $\left(y_j, \dfrac{p_{ij}}{p_{i\bullet}}\right)$, $j = 1, 2, 3, \ldots, m$ is called the conditional distribution of Y given that $X = x_i$ and $\left(x_i, \dfrac{p_{ij}}{p_{\bullet j}}\right)$, $i = 1, 2, 3, \ldots, n$ is called the conditional distribution of X given that $Y = y_j$.

5.2 CONTINUOUS RANDOM VARIABLES *X* AND *Y*

5.2.1 Joint Probability Density Function

Let (X, Y) be a two-dimensional continuous random variable such that

$$P\left(x - \frac{dx}{2} \le X \le x + \frac{dx}{2}, y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right) = f(x, y)dx\,dy$$

then $f(x, y)$ is called the joint probability density function of (X, Y) if it satisfies the following conditions:

(i) $f(x, y) \ge 0 \;\forall\; (x, y) \in R$, where R is the range space.

(ii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dx\,dy = 1$

Moreover, if $(a, b), (c, d) \in R$, then

$$P(a \le X \le b, c \le Y \le d) = \int_c^d \int_a^b f(x, y)dxdy$$

5.2.2 Cumulative Distribution Function

If (X, Y) is a continuous random variable, then the cumulative distribution function is defined as

$$F(x, y) = P(X \le x,\ Y \le y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)dxdy$$

Properties of Distribution Function

1. $F(-\infty, y) = F(x, -\infty) = 0,\ F(-\infty, \infty) = 1$
2. $F(X \le x,\ c \le Y \le d) = F(x, d) - F(x, c)$
3. $F(a \le X \le b,\ Y \le y) = F(b, y) - F(a, y)$
4. $F(a < X < b,\ c < Y < d) = F(b, d) - F(a, d) + F(a, c) - F(b, c)$
5. If $f(x, y)$ is continuous at (x, y), then $\dfrac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$
6. $0 \le F(x, y) \le 1$ and $F(x, y)$ is a non-decreasing function of x and y.

5.2.3 Marginal Probability Distribution

When (X, Y) is a two-dimensional continuous random variable, then the marginal density function of the random variable X is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

The marginal density function of the random variable Y is defined as

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

5.2.4 Conditional Probability Function

If (X, Y) is a two-dimensional continuous random variable, then

$$f(x/y) = \frac{f(x, y)}{f_Y(y)}$$

is called the conditional probability function of X given Y

and

$$f(y/x) = \frac{f(x, y)}{f_X(x)}$$

is called the conditional probability function of Y given X.

5.2.5 Independent Random Variables

The two-dimensional random variable (X, Y) is said to be independent if

$$p_{ij} = p_{*j} \cdot p_{i*}, \text{ when } (X, Y) \text{ is discrete}$$

and $$f(x, y) = f_X(x) \cdot f_Y(y), \text{ when } (X, Y) \text{ is continuous.}$$

Note: If X and Y are independent, then $f(x/y) = f(x)$, $f(y/x) = f(y)$.

EXAMPLE 5.1 For the bivariate probability distribution of (X, Y) given below, find $P(X \leq 1)$, $P(Y \leq 3)$, $P(X \leq 1, Y \leq 3)$, $P(X \leq 1/Y \leq 3)$, $P(Y \leq 3/X \leq 1)$, and $P(X + Y \leq 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution From the given table,

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= \sum_{j=1}^{6} P_{0j} + \sum_{j=1}^{6} P_{1j}$$

$$= \left(0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$$

$$P(Y \leq 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= \sum_{i=0}^{2} P_{i1} + \sum_{i=0}^{2} P_{i2} + \sum_{i=0}^{2} P_{i3}$$

$$= \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(\frac{1}{32} + \frac{1}{8} + \frac{1}{64}\right)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

$$P(X \le 1,\ Y \le 3) = P(X = 0,\ Y \le 3) + P(X = 1,\ Y \le 3)$$

$$= \sum_{j=1}^{3} P_{0j} + \sum_{j=1}^{3} P_{1j}$$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{9}{32}$$

$$P(X \le 1/Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)} = \frac{\frac{9}{32}}{\frac{23}{64}} = \frac{18}{23}$$

$$P(Y \le 3/X \le 1) = \frac{P(X \le 1, Y \le 3)}{P(X \le 1)} = \frac{\frac{9}{32}}{\frac{7}{8}} = \frac{9}{28}$$

$$P(X + Y \le 4) = \sum_{j=1}^{4} P_{0j} + \sum_{j=1}^{3} P_{1j} + \sum_{j=1}^{2} P_{2j} \qquad (\because i + j \le 4)$$

$$= \frac{3}{32} + \frac{2}{16} + \frac{1}{8} + \frac{2}{32} = \frac{13}{32}$$

EXAMPLE 5.2 The joint probability mass function of (X, Y) is given by $P(x, y) = K(2x + 3y)$, $x = 0, 1, 2$, $y = 1, 2, 3$. Find the marginal and conditional distributions for

(i) $P(X = 2,\ Y \le 2)$, (ii) $P(X \le 1,\ Y = 3)$, (iii) $P(X = 2)$, (iv) $P(X \le 2)$, (v) $P(X \le 1/Y \le 2)$, and (vi) $P(X = 0/Y = 3)$. **[AU April '04, December '07]**

Solution Given:

$X \backslash Y$	1	2	3	Total
0	$3K$	$6K$	$9K$	$18K$
1	$5K$	$8K$	$11K$	$24K$
2	$7K$	$10K$	$13K$	$30K$
Total	$15K$	$24K$	$33K$	$72K$

To find K:

We know that the total probability = 1

$$\therefore \qquad 72K = 1 \ \Rightarrow\ K = \frac{1}{72}$$

$X \backslash Y$	1	2	3	Total	
0	$\frac{3}{72}$	$\frac{6}{72}$	$\frac{9}{72}$	$\frac{18}{72}$	$P(X=0)$
1	$\frac{5}{72}$	$\frac{8}{72}$	$\frac{11}{72}$	$\frac{24}{72}$	$P(X=1)$
2	$\frac{7}{72}$	$\frac{10}{72}$	$\frac{13}{72}$	$\frac{30}{72}$	$P(X=2)$
Total	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	1	
	$P(Y=1)$	$P(Y=2)$	$P(Y=3)$		

(i) $P(X = 2, Y \leq 2) = P(X = 2, Y = 1) + P(X = 2, Y = 2)$

$$= \frac{7}{72} + \frac{10}{72} = \frac{17}{72}$$

(ii) $P(X \leq 1, Y = 3) = P(X = 0, Y = 3) + P(X = 1, Y = 3)$

$$= \frac{9}{72} + \frac{11}{72} = \frac{20}{72}$$

(iii) $P(X = 2) = \sum_{j=1}^{3} P_{2j} = P(X = 2, Y = 1) + P(X = 2, Y = 2)$

$+ P(X = 2, Y = 3)$

$$= \frac{7}{72} + \frac{10}{72} + \frac{13}{72} = \frac{30}{72}$$

(iv) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{18}{72} + \frac{24}{72} + \frac{30}{72} = \frac{72}{72} = 1$$

(v) $P(X \leq 1/Y \leq 2) = \dfrac{P(X \leq 1, Y \leq 2)}{P(Y \leq 2)}$

$$= \frac{P(X = 0, Y \leq 2) + P(X = 1, Y \leq 2)}{P(Y \leq 2)}$$

$$= \frac{\frac{9}{72} + \frac{13}{72}}{\frac{15}{72} + \frac{24}{72}} = \frac{\frac{22}{72}}{\frac{39}{72}} = \frac{22}{39}$$

(vi) $P(X = 0/Y = 3) = \dfrac{P(X = 0, Y = 3)}{P(Y = 3)}$

$$= \frac{\frac{9}{72}}{\frac{33}{72}} = \frac{9}{33}$$

EXAMPLE 5.3 Given the following bivariate probability distribution. Obtain

(i) marginal distributions of X and Y, and
(ii) the conditional distribution of X given $Y = 2$.

$Y \backslash X$	-1	0	1
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

Solution

$Y \backslash X$	-1	0	1	$\sum_x p(x, y)$	
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{4}{15}$	$P(Y = 0)$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{6}{15}$	$P(Y = 1)$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{5}{15}$	$P(Y = 2)$
$\sum_y p(x, y)$	$\frac{6}{15}$	$\frac{5}{15}$	$\frac{4}{15}$	1	
	$P(X = -1)$	$P(X = 0)$	$P(X = 1)$		

(i) The marginal distribution of X:

$$P(X = -1) = \frac{6}{15} = \frac{2}{5},\ P(X = 0) = \frac{5}{15} = \frac{1}{3} \text{ and } P(X = 1) = \frac{4}{15}$$

The marginal distribution of Y:

$$P(Y = 0) = \frac{4}{15},\ P(Y = 1) = \frac{6}{15} = \frac{2}{5} \text{ and } P(Y = 2) = \frac{5}{15} = \frac{1}{3}$$

(ii) The conditional distribution of X given $Y = 2$:

$$P(X = x/Y = 2) = \frac{P(X = x \cap Y = 2)}{P(Y = 2)}$$

$$\therefore \quad P(X = -1/Y = 2) = \frac{P(X = -1 \cap Y = 2)}{P(Y = 2)} = \frac{\frac{2}{5}}{\frac{1}{3}} = \frac{2}{5}$$

$$P(X = 0/Y = 2) = \frac{P(X = 0 \cap Y = 2)}{P(Y = 2)} = \frac{1}{5}$$

$$P(X = 1/Y = 2) = \frac{P(X = 1 \cap Y = 2)}{P(Y = 2)} = \frac{2}{5}$$

X	-1	0	1
$P(X/Y = 2)$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

EXAMPLE 5.4 From the following joint distribution of X and Y, find the marginal distribution.

$Y \diagdown X$	1	2	3
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0
2	$\frac{1}{28}$	0	0

Solution The marginal distributions are given in the following table:

$Y \diagdown X$	1	2	3	$P_Y(y)$
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{6}{14}$
2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$P_X(x)$	$\frac{10}{28}$	15/[illegible]	3/[illegible]	$\Sigma P(x) = 1$ $\Sigma P(y) = 1$

The marginal distribution of X:

$$P(X=1) = P(X=1,\ Y=0) + P(X=1,\ Y=1) + P(X=1,\ Y=2)$$

$$= \frac{3}{28} + \frac{3}{14} + \frac{1}{28} = \frac{10}{28}$$

$$P(X=2) = P(X=2,\ Y=0) + P(X=2,\ Y=1) + P(X=2,\ Y=2)$$

$$= \frac{9}{28} + \frac{3}{14} + 0 = \frac{15}{28}$$

$$P(X=3) = P(X=3,\ Y=0) + P(X=2,\ Y=1) + P(X=2,\ Y=2)$$

$$= \frac{3}{28} + 0 + 0 = \frac{3}{28}$$

The marginal distribution of Y:

$$P(Y=0) = P(X=1,\ Y=0) + P(X=2,\ Y=0) + P(X=3,\ Y=0)$$

$$= \frac{3}{28} + \frac{9}{28} + \frac{3}{28} = \frac{15}{28}$$

$$P(Y=1) = P(X=1,\ Y=1) + P(X=2,\ Y=1) + P(X=3,\ Y=1)$$

$$= \frac{3}{14} + \frac{3}{14} + 0 = \frac{6}{14}$$

$$P(Y=2) = P(X=1,\ Y=2) + P(X=2,\ Y=2) + P(X=3,\ Y=2)$$

$$= \frac{1}{28} + 0 + 0 = \frac{1}{28}$$

EXAMPLE 5.5 Let X and Y have the following joint probability distributions:

$Y \backslash X$	2	4
1	0.10	0.15
3	0.20	0.30
5	0.10	0.15

Show that X and Y are independent.

Solution

$Y \backslash X$	2	4	$P(Y = y)$
1	0.10	0.15	0.25
3	0.20	0.30	0.50
5	0.10	0.15	0.25
$P(X = x)$	0.40	0.60	1

From the table, we have $p_{ij} = p_{i*} \times p_{*j}$

For example,

$$p_{14} = 0.15$$
$$p_{1\bullet} = 0.25$$
$$p_{\bullet 4} = 0.60$$

$\therefore$
$$p_{1\bullet} \times p_{\bullet 4} = 0.25 \times 0.60 = 0.15 = p_{14}$$

Similarly,

$$p_{1\bullet} \times p_{\bullet 2} = 0.25 \times 0.40 = 0.10 = p_{12}$$
$$p_{3\bullet} \times p_{\bullet 2} = 0.50 \times 0.40 = 0.20 = p_{32}$$
$$p_{3\bullet} \times p_{\bullet 4} = 0.50 \times 0.60 = 0.30 = p_{34}$$
$$p_{5\bullet} \times p_{\bullet 2} = 0.25 \times 0.40 = 0.10 = p_{52}$$
$$p_{5\bullet} \times p_{\bullet 4} = 0.25 \times 0.60 = 0.15 = p_{54}$$

$\therefore$ The two random variables X and Y are independent.

EXAMPLE 5.6 Let X and Y are two random variables having the joint density function, $P(x, y) = 1/27(x + 2y)$, where x and y can assume only the integer values 0, 1 and 2. Find the conditional distribution of Y for $X = x$.

Solution The joint probability function gives the following table of joint probability distribution of X and Y.

Given: $P(x, y) = \dfrac{1}{27}(x + 2y), \quad x, y = 0, 1, 2$

$X \backslash Y$	0	1	2	$\sum_x p(x, y)$	
0	0	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{6}{27}$	$P(X = 0)$
1	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{5}{27}$	$\frac{9}{27}$	$P(X = 1)$
2	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$P(X = 2)$
$\sum_y p(x, y)$	$\frac{3}{27}$	$\frac{9}{27}$	$\frac{15}{27}$	1	
	$P(Y = 0)$	$P(Y = 1)$	$P(Y = 2)$		

$$P(0, 0) = \frac{1}{27}(0 + 2 \times 0) = 0, \; P(0, 1) = \frac{1}{27}(0 + 2 \times 1) = \frac{2}{27}$$

$$P(0, 2) = \frac{1}{27}(0 + 2 \times 2) = \frac{4}{27} \text{ and so on.}$$

The marginal probability distribution of X is given by

$$P(X = x_i) = \sum_j P_{ij} = P_{1*}$$

X	0	1	2
$P(x)$	$\frac{6}{27}$	$\frac{9}{27}$	$\frac{12}{27}$

The conditional distribution of Y for $X = x$ is given by

$$P(Y = y_j/X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)} = \frac{P_{ij}}{p_{i\bullet}}$$

$X \backslash Y$	0	1	2
0	0	$\frac{2}{6}$	$\frac{4}{6}$
1	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$
2	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{6}{12}$

EXAMPLE 5.7 Given is the joint distribution of X and Y

$Y \backslash X$	0	1	2
0	0.02	0.08	0.10
1	0.05	0.20	0.30
2	0.03	0.12	0.15

Obtain

(i) the marginal distribution, and
(ii) the conditional distribution of X given $Y = 0$.

Solution:

$Y \backslash X$	0	1	2	$P(Y = y)$
0	0.02	0.08	0.10	0.20
1	0.05	0.20	0.30	0.55
2	0.03	0.12	0.15	0.30
$P(X = x)$	0.10	0.40	0.55	1

(i) From the table, the marginal distribution of X is

$$P(X = 0) = 0.10, \quad P(X = 1) = 0.40, \quad P(X = 2) = 0.55$$

The marginal distribution of Y:

$$P(Y = 0) = 0.20, \quad P\ (Y = 1) = 0.50, \quad P\ (Y = 2) = 0.30$$

(ii) The conditional distribution of X for $Y = 0$ is

$X = x$	0	1	2
$P(X = x/Y = y)$	0.1	0.4	0.5

EXAMPLE 5.8 Let X and Y be integer-valued random variables with $P(X = m, Y = n) = q^2p^{m+n-2}$, $n, m = 1, 2, \ldots$, and $p + q = 1$. Are X and Y independent?

Solution

$$P(X = m) = \sum_{n=1}^{\infty} P(X = m, Y = n) = \sum_{n=1}^{\infty} q^2 p^{m+n-2} = \sum_{n=1}^{\infty} q^2 p^{m-2} p^n$$

$$= q^2 p^{m-2} \sum_{n=1}^{\infty} p^n = q^2 p^{m-2} \frac{p}{1-p}$$

$$= q^2 p^{m-2} \frac{p}{q}$$

$$P(X = m) = qp^{m-1}$$

$$P(Y = n) = \sum_{m=1}^{\infty} q^2 p^{m+n-2}$$

$$= q^2 p^{n-2} \sum_{m=1}^{\infty} p^m$$

$$= q^2 p^{n-2} \frac{p}{1-p}$$

$$= qp^{n-1}$$

$$P(X = m)\, P(Y = n) = qp^{m-1}\, qp^{n-1} = q^2p^{m+n-2}$$

$$= P\,(X = m, Y = n)$$

$\therefore$ X and Y are independent.

EXAMPLE 5.9 Two discrete random variables X and Y have the joint probability density function

$$P(x, y) = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!}, \; y = 0, 1, 2, \ldots, x;\; x = 0, 1, 2, \ldots$$

where λ, p are constants with $\lambda > 0$ and $0 < p < 1$. Find:

(i) The marginal probability density functions of X and Y.
(ii) The conditional distribution of Y for a given X and of X for a given Y.

[AU December '05]

Solution

(i) The marginal distribution of X is

$$P_X(x) = \sum_{y=0}^{x} p(x, y) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!}.$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} \frac{x! p^y (1-p)^{x-y}}{y!(x-y)!} = \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} xC_y p^y (1-p)^{x-y}$$

$$\therefore P_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}(1-p+p)^y = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0, 1, 2, \ldots$$

which is the probability function of a Poisson distribution with parameter λ.

The marginal distribution of Y is

$$P_Y(y) = \sum_{x=0}^{\infty} p(x, y) = \sum_{x=y}^{\infty} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!}$$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!} = \frac{(\lambda p)^y e^{-\lambda}}{y!} \times e^{\lambda(1-p)}$$

$$p_Y(y) = \frac{e^{-\lambda p}(\lambda p)^y}{y!}, y = 0, 1, 2, \ldots$$

which is the probability function of a Poisson distribution with parameter λp.

(ii) The conditional probability distribution of Y given X is

$$P(Y = y/X = x) = \frac{p(x, y)}{p_X(x)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y} x!}{y!(x-y)! \lambda^x e^{-\lambda}}$$

$$= \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y} = xC_y p^y (1-p)^{x-y}, x > y$$

The conditional probability distribution of X given Y is

$$P(X = x/Y = y) = \frac{p(x, y)}{p_Y(y)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y} y!}{y!(x-y)! e^{-\lambda p} (\lambda p)^y}$$

$$= \frac{e^{-\lambda(1-p)} (\lambda p)^y [\lambda(1-p)]^{x-y}}{(x-y)! (\lambda p)^y} = \frac{e^{-\lambda q} (\lambda q)^{x-y}}{(x-y)!},$$

where $q = 1 - p$, $x > y$

EXAMPLE 5.10 If the joint PDF of $f(x, y) = k(1 - x - y), 0 < x, y < \frac{1}{2}$, find k.

Solution We know that

$$\int_0^{\infty}\int_0^{\infty} f(x, y)dxdy = 1$$

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} k(1 - x - y)dxdy = 1$$

$$k\int_0^{\frac{1}{2}}\left[x - \frac{x^2}{2} - yx\right]_0^{\frac{1}{2}} dy = 1$$

$$k\int_0^{\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{8} - \frac{y}{2}\right)dy = 1$$

$$k\int_0^{\frac{1}{2}}\left(\frac{3}{8} - \frac{y}{2}\right)dy = 1$$

$$k\left[\frac{3}{8}y - \frac{y^2}{4}\right]_0^{\frac{1}{2}} = 1 \Rightarrow k\left(\frac{3}{16} - \frac{1}{16}\right) = 1 \Rightarrow k\left(\frac{1}{8}\right) = 1$$

$\therefore$ $$k = 8$$

EXAMPLE 5.11 Find K if the joint PDF of a bivariate random variable (X, Y) is given by

$$f(x, y) = \begin{cases} K(1 - x)(1 - y), & \text{if } 0 < x < 4; 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$$

[AU December '08]

Solution We know that if $f(x, y)$ is a joint PDF, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1$$

$\therefore$ $$\int_0^4\int_1^5 K(1 - x)(1 - y)dxdy = 1$$

$$K\int_1^5 \left[\frac{(1-x)^2}{-2}\right]_0^4 (1-y)dy = 1$$

$$-4K\int_1^5 (1-y)dy = 1$$

$$\Rightarrow \qquad -4K\left[\frac{(1-y)^2}{-2}\right]_1^5 = 1$$

i.e. $$-4K\left(\frac{16}{-2}+0\right) = 1 \Rightarrow 32K = 1 \Rightarrow K = \frac{1}{32}$$

EXAMPLE 5.12 For $\lambda > 0$, let

$$F(x, y) = \begin{cases} 1 - \lambda e^{-\lambda(x+y)}, & \text{if } x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Check whether F can be the joint probability distribution function of two random variables X and Y. **[AU December '09]**

Solution From the given problem, we have

$$F(-\infty, y) = F(x, -\infty) = 0$$
$$F(-\infty, \infty) = 1 - \lambda e^{-\lambda(\infty)} = 1 - 0 = 1 \qquad (\because e^{-\infty} = 0)$$

Also, $0 \le F(x, y) \le 1$ and $F(x, y)$ is a non-decreasing function.

$\therefore$ $F(x, y)$ can be the joint probability distribution function of the random variables X and Y.

EXAMPLE 5.13 If $f(x, y) = kye^{-x}$, $x > 0$, $0 < y < 2$ is the joint PDF of two random variables X and Y, what is the value of k?

Solution If $f(x, y)$ is a joint PDF, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1 \Rightarrow \int_0^2\int_0^{\infty} kye^{-x}dxdy = 1$$

$$\Rightarrow \qquad k\int_0^2 y\left[\frac{e^{-x}}{-1}\right]_0^{\infty} dy = 1 \Rightarrow k\int_0^2 y[0+1]dy = 1$$

$$\Rightarrow \qquad k\left[\frac{y^2}{2}\right]_0^2 = 1 \Rightarrow k\left(\frac{4}{2}-0\right) = 1$$

$$\Rightarrow \qquad 2k = 1 \Rightarrow k = \frac{1}{2}$$

EXAMPLE 5.14 If $f(x, y) = e^{-(x+y)}$, $x \geq 0$, $y \geq 0$ is the joint PDF of X and Y, find $P(X < 1)$. **[AU December '09]**

Solution Since $f(x, y)$ is the joint PDF

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$= \int_{0}^{\infty} e^{-(x+y)}\, dy$$

$$= e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = e^{-x}$$

$$\therefore \quad P(X < 1) = \int_0^1 f(x)dx$$

$$= \int_0^1 e^{-x}dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^1 = (1 - e^{-1}) = 0.6321$$

EXAMPLE 5.15 Let X and Y be continuous random variables with joint PDF

$$f(x, y) = \begin{cases} 2xy + \dfrac{3y^2}{2}, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(X + Y < 1)$. **[AU December '09, April '08]**

Solution Given: $f(x, y)$ is a joint PDF.

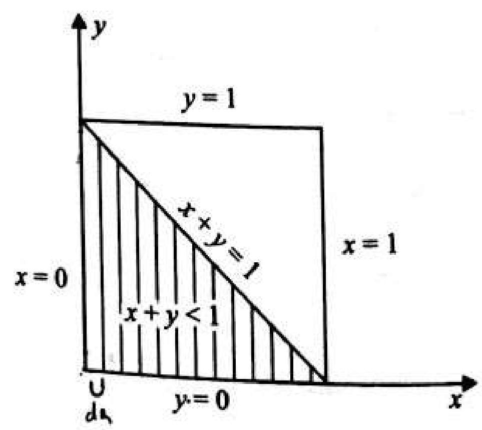


Figure 5.1

$$\therefore \quad P(X+Y<1) = \int_0^1 \int_0^{1-y} f(x, y)dxdy \text{, from the Figure 5.1}$$

$$= \int_0^1 \int_0^{1-y} \left(2xy + \frac{3y^2}{2}\right) dxdy$$

$$= \int_0^1 \left[yx^2 + \frac{3y^2}{2}x \right]_0^{1-y} dy$$

$$= \int_0^1 \left[y(1-y)^2 + \frac{3}{2}y^2(1-y) \right] dy$$

$$= \int_0^1 \left(y - \frac{y^2}{2} - \frac{y^3}{2} \right) dy = \left[\frac{y^2}{2} - \frac{y^3}{6} - \frac{y^4}{8} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{8}$$

$$= \frac{12-4-3}{24} = \frac{5}{24}$$

EXAMPLE 5.16 The joint PDF of a random variable (X, Y) is given by $f_{XY}(x, y) = Cxy$, $0 < x < 2$, $0 < y < 2$, where C is a constant. Find C.

[AU December '08]

Solution Given: $f(x, y) = Cxy$ is a joint PDF.

$$\therefore \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1 \Rightarrow \int_0^2 \int_0^2 Cxy\, dxdy = 1$$

$$\Rightarrow \quad C\int_0^2 y\left[\frac{x^2}{2}\right]_0^2 dy = 1 \Rightarrow 2C\int_0^2 y\, dy = 1$$

$$\Rightarrow \quad 2C\left[\frac{y^2}{2}\right]_0^2 = 1 \Rightarrow 4C = 1 \Rightarrow C = \frac{1}{4}$$

EXAMPLE 5.17 Let X and Y be continuous random variables with joint PDF

$$f(x, y) = \frac{3}{2}(x^2 + y^2),\ 0 < x < 1,\ 0 < y < 1.$$

Find $f(x/y)$.

[AU December '08]

Solution We know that

$$f(x/y) = \frac{f(x, y)}{f(y)}$$

To find $f(y)$:

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 \frac{3}{2}(x^2 + y^2)dx = \frac{3}{2}\left[\frac{x^3}{3} + y^2 x\right]_0^1$$

$\therefore$

$$f(y) = \frac{3}{2}\left(\frac{1}{3} + y^2\right), \qquad 0 < y < 1$$

$$f(x/y) = \frac{f(x, y)}{f(y)} = \frac{\left(\frac{3}{2}\right)(x^2 + y^2)}{\left(\frac{3}{2}\right)\left(\frac{1}{3} + y^2\right)}$$

$$f(x/y) = \frac{x^2 + y^2}{\left(\frac{1}{3}\right) + y^2}, \qquad 0 < y < 1$$

EXAMPLE 5.18 If two random variables X and Y have PDF $f(x, y) = Ke^{-(2x + y)}$ for $x, y > 0$, evaluate K.

Solution If $f(x, y)$ is a joint PDF, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

$$\Rightarrow \qquad \int_0^{\infty}\int_0^{\infty} Ke^{-(2x + y)}dxdy = 1 \quad \Rightarrow \quad K\int_0^{\infty} e^{-y}\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} dy = 1$$

$$\Rightarrow \qquad K\int_0^{\infty} e^{-y}\left(\frac{1}{2}\right)dy = 1 \quad \Rightarrow \quad \frac{K}{2}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow \qquad \frac{K}{2}(0 + 1) = 1 \quad \Rightarrow \quad K = 2$$

EXAMPLE 5.19 Find the marginal density functions of X and Y if

$$f(x, y) = \frac{2}{5}(2x + 3y), 0 \le x \le 1, 0 \le y \le 1.$$

Solution The marginal density function of X is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$= \int_0^1 f(x, y)dy, \quad (0 \le y \le 1)$$

$$= \int_0^1 \frac{2}{5}(2x + 3y)dy = \frac{2}{5}\left[2xy + \frac{3y^2}{2}\right]_0^1 = \frac{4x + 3}{5}, 0 \le x \le 1$$

The marginal density functions of Y is given by

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 f(x, y)dx$$

$$= \int_0^1 \frac{2}{5}(2x + 3y)dx = \frac{2}{5}\left[\frac{2x^2}{2} + 3yx\right]_0^1 = \frac{2 + 6y}{5}, \quad 0 \le y \le 1$$

EXAMPLE 5.20 The joint PDF of the two-dimensional random variables is

$$f(x, y) = \begin{cases} \frac{8}{9}xy, & 1 < x < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the marginal density functions of X and Y.
(ii) Find the conditional distribution of Y given $X = x$.

[AU April '03; '04]

Solution

(i) The marginal density function of X is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_x^2 \frac{8xy}{9}dy, (x \le y \le 2),$$

from the Figure 5.2

$$f(x) = \frac{8x}{9}\left[\frac{y^2}{2}\right]_x^2 = \frac{4x}{9}(4 - x^2), 1 \le x \le 2$$

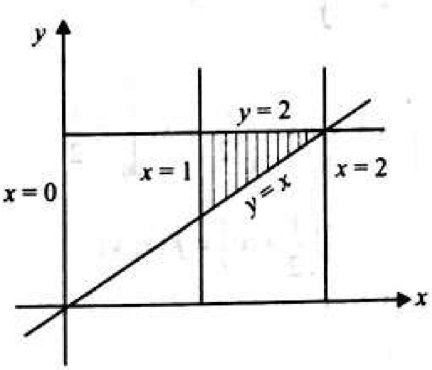


Figure 5.2

The marginal density function of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{1}^{y} \frac{8xy}{9} dx, \qquad (1 \le x \le y)$$

$$f(y) = \frac{8y}{9}\left[\frac{x^2}{2}\right]_1^y = \frac{4y}{9}(y^2 - 1), \qquad 1 \le y \le 2$$

(ii) The conditional density function of Y given $X = x$ is

$$f(y/x) = \frac{f(x, y)}{f(x)} = \frac{\frac{8xy}{9}}{\frac{4x(4 - x^2)}{9}} = \frac{2y}{4 - x^2}, \qquad x \le y \le 2$$

EXAMPLE 5.21 If X and Y have the joint PDF

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Check whether X and Y are independent or not. **[AU May '06]**

Solution The marginal density function of X is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$= \int_0^1 (x + y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}, \qquad 0 < x < 1$$

The marginal density function of Y is given by

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

$$= \int_0^1 (x + y)dx = \left[\frac{x^2}{2} + xy\right]_0^1 = \frac{1}{2} + y, \qquad 0 < y < 1$$

Now, $$f(x)\, f(y) = \left(x + \frac{1}{2}\right)\left(\frac{1}{2} + y\right) \ne f(x, y)$$

$$f(x, y) \ne f(x)\, f(y)$$

$\therefore$ X and Y are not independent.

EXAMPLE 5.22 The bivariate random variables X and Y have the PDF $f(x, y) = kx^2(8 - y)$, $x < y < 2x$, $0 \le x \le 2$. Find k. **[AU December '06]**

Solution We know that if $f(x, y)$ is a PDF, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1$$

$$\therefore \quad \int_{0}^{2}\int_{x}^{2x} kx^2(8-y)dydx = 1$$

i.e.

$$k\int_0^2 x^2\left[8y - \frac{y^2}{2}\right]_x^{2x} dx = 1$$

$$k\int_0^2 x^2\left(16x - \frac{4x^2}{2} - 8x + \frac{x^2}{2}\right)dx = 1$$

$$k\int_0^2 \left(8x^3 - \frac{3x^4}{2}\right)dx = 1 \Rightarrow k\left[8\frac{x^4}{4} - \frac{3}{2}\frac{x^5}{5}\right]_0^2 = 1$$

$$k\left(32 - \frac{48}{5}\right) = 1$$

$$k = \frac{5}{112}$$

EXAMPLE 5.23 The joint PDF of (X, Y) is given by $e^{-(x+y)}$, $0 < x, y < \infty$. Are X and Y independent? Why? **[AU December '03, April '08]**

Solution If X and Y are independent, then

$$f(x, y) = f_X(x)\, f_Y(y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^{\infty} e^{-x}e^{-y}dy = e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^{\infty} e^{-x}e^{-y}dx = e^{-y}\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = e^{-y}$$

$$f_X(x)\, f_Y(y) = e^{-x}\, e^{-y} = e^{-(x+y)} = f(x, y)$$

$\therefore$ They are independent.

EXAMPLE 5.24 If the joint PDF of X and Y is given by

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y), & 0 < x < 2,\ 2 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

Find

(i) $P(X < 1 \cap Y < 3)$, and

(ii) $P(X < 1/Y < 3)$.

[AU April '03, December '07; '09]

Solution We know that

$$P[(a_1 \le X \le b_1) \cap (a_2 \le Y \le b_2)] = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x, y)dydx$$

(i) $$P(X < 1 \cap Y < 3) = \int_0^1\int_2^3 f(x, y)dydx$$

$$= \int_0^1\int_2^3 \frac{1}{8}(6 - x - y)dydx$$

$$= \frac{1}{8}\int_0^1 \left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^1 \left[\left(18 - 3x - \frac{9}{2}\right) - \left(12 - 2x - \frac{4}{2}\right)\right]dx$$

$$= \frac{1}{8}\int_0^1 \left(6 - x - \frac{5}{2}\right)dx = \frac{1}{8}\int_0^1 \left(\frac{7}{2} - x\right)dx$$

$$= \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = \frac{3}{8} \qquad \text{(i)}$$

(ii) $$P(X < 1/Y < 3) = \frac{P(X < 1, Y < 3)}{P(Y < 3)} \qquad \text{(ii)}$$

$$P(Y < 3) = \int_0^2\int_2^3 \frac{1}{8}(6 - x - y)dydx$$

$$= \frac{1}{8}\int_0^2 \left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^2 \left[\left(18 - 3x - \frac{9}{2}\right) - (12 - 2x - 2)\right]dx$$

$$= \frac{1}{8}\int_0^2 \left(\frac{7}{2} - x\right)dx$$

$$= \frac{1}{8}\left[\frac{7x}{2} - \frac{x^2}{2}\right]_0^2 = \frac{1}{8}(7-2) = \frac{5}{8}$$

$$\therefore \qquad P(Y < 3) = \frac{5}{8} \qquad \text{(iii)}$$

Substituting Eqs. (i) and (iii) in Eq. (ii), we get

$$P(X < 1/Y < 3) = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$$

EXAMPLE 5.25 Show that the function

$$f(x, y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

is a joint PDF of X and Y. **[AU April '03]**

Solution $f(x, y)$ is a joint PDF if it satisfies the following conditions:

(i) $f(x, y) \ge 0$,

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1$

Given:

$$f(x, y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) $f(x, y) \ge 0$ in the given interval $0 \le (x, y) \le 1$

(ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)dxdy$$

$$= \frac{2}{5}\int_0^1 \left[x^2 + 3xy\right]_0^1 dy = \frac{2}{5}\int_0^1 (1+3y)dy$$

$$= \frac{2}{5}\left(y + \frac{3y^2}{2}\right)_0^1 = \frac{2}{5}\left(1 + \frac{3}{2}\right) = \frac{2}{5}\left(\frac{5}{2}\right) = 1$$

Since $f(x, y)$ satisfies the necessary conditions, it is a joint PDF.

EXAMPLE 5.26 If the joint PDF of (X, Y) is 1/4, $0 \le x, y \le 1$, find $P(X + Y \le 1)$. **[AU December '05]**

Solution Given: $f(x, y) = \frac{1}{4}$, $0 \le x, y \le 1$,

$$P(X + Y \le 1) = \int_0^1 \int_0^{1-y} f(x, y)dxdy,$$

[from Figure 5.3]

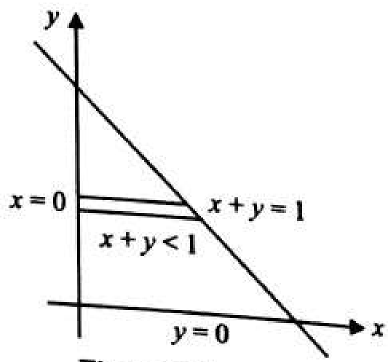


Figure 5.3

$$P(X + Y \le 1) = \int_0^1 \int_0^{1-y} \frac{1}{4}dxdy$$

$$= \frac{1}{4}\int_0^1 [x]_0^{1-y} dy = \frac{1}{4}\int_0^1 (1-y)dy$$

$$= \frac{1}{4}\left[y - \frac{y^2}{2}\right]_0^1 = \frac{1}{4}\left(1 - \frac{1}{2}\right) - 0 = \frac{1}{8}$$

EXAMPLE 5.27 The joint PDF of the random variable (X, Y) is given by $f(x, y) = Kxye^{-(x^2 + y^2)}$, $x > 0, y > 0$. Find the value of K and prove also that X and Y are independent. **[AU November '04; '06; '07, December '08]**

Solution We know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

$$\therefore \quad \int_0^{\infty}\int_0^{\infty} Kxy \cdot e^{-(x^2 + y^2)}dxdy = 1$$

$$K\int_0^{\infty}\int_0^{\infty} xe^{-x^2} \cdot ye^{-y^2} dxdy = 1$$

$$K\left(\int_0^\infty xe^{-x^2}dx \cdot \int_0^\infty ye^{-y^2}dy\right) = 1$$

$$K\left[\frac{1}{2}\int_0^\infty e^{-x^2}d(x^2)\cdot\frac{1}{2}\int_0^\infty e^{-y^2}d(y^2)\right] = 1$$

$$\frac{K}{4}\left\{\left[\frac{e^{-x^2}}{-1}\right]_0^\infty \cdot \left[\frac{e^{-y^2}}{-1}\right]_0^\infty\right\} = 1$$

$$\frac{K}{4}(1\times 1) = 1 \quad \Rightarrow \quad K = 4$$

$\therefore$ $$f(x, y) = 4xy\, e^{-(x^2+y^2)}, \quad x \ge 0, y \ge 0$$

To show that X and Y are independent, we have to show that

$$f(x, y) = f_X(x)\cdot f_Y(y).$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = 4\int_0^\infty xye^{-x^2}\cdot e^{-y^2}dy = 4xe^{-x^2}\int_0^\infty y\cdot e^{-y^2}dy$$

$$= 4xe^{-x^2}\frac{1}{2}\int_0^\infty e^{-y^2}d(y^2) = 2xe^{-x^2}\left[\frac{e^{-y^2}}{-1}\right]_0^\infty$$

$$= 2xe^{-x^2}, x > 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = 4\int_0^\infty xye^{-x^2}\cdot e^{-y^2}dx = 4ye^{-y^2}\int_0^\infty x\cdot e^{-x^2}dx$$

$$= 4ye^{-y^2}\frac{1}{2}\int_0^\infty e^{-x^2}d(x^2) = 2ye^{-y^2}\left[\frac{e^{-x^2}}{-1}\right]_0^\infty$$

$$f_Y(y) = 2ye^{-y^2}, y > 0$$

$$f_X(x)\cdot f_Y(y) = 2xe^{-x^2}\, 2ye^{-y^2} = 4xye^{-(x^2+y^2)} = f(x, y)$$

$\therefore$ X and Y are independent.
Hence proved.

EXAMPLE 5.28 Let X_1 and X_2 be two random variables with joint PDF given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & x_1, x_2 \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the marginal densities of X_1 and X_2.

(ii) Are they independent?
(iii) Find $P(X_1 \le 1, X_2 \le 1)$ and $P(X_1 + X_2 \le 1)$.

[AU December '04, April '08]

Solution

(i) $$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 = \int_0^{\infty} e^{-(x_1+x_2)}dx_2$$

$$= \int_0^{\infty} e^{-x_1}e^{-x_2}dx_2 = e^{-x_1}\left[\frac{e^{-x_2}}{-1}\right]_0^{\infty} = e^{-x_1},\ x_1 > 0$$

(ii) $$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_1 = \int_0^{\infty} e^{-(x_1+x_2)}dx_1$$

$$= \int_0^{\infty} e^{-x_1}e^{-x_2}dx_1 = e^{-x_2}\left[\frac{e^{-x_1}}{-1}\right]_0^{\infty} = e^{-x_2},\ x_2 > 0$$

For independent random variables,

$$f(x_1, x_2) = f(x_1)\, f(x_2)$$

$$f(x_1, x_2) = e^{-(x_1+x_2)} = e^{-x_1} \cdot e^{-x_2} = f(x_1)\, f(x_2)$$

$\therefore$ The random variables are independent.

(iii) $$P(X_1 \le 1, X_2 \le 1) = \int_0^1\int_0^1 e^{-(x_1+x_2)}dx_1dx_2 = \int_0^1\int_0^1 e^{-x_1} \cdot e^{-x_2}dx_1dx_2$$

$$= \int_0^1 \left[\frac{e^{-x_1}}{-1}\right]_0^1 e^{-x_2}dx_2 = \left[\frac{e^{-1}}{-1} + 1\right]\int_0^1 e^{-x_2}dx_2$$

$$= (1 - e^{-1})\,(1 - e^{-1}) = (1 - e^{-1})^2$$

$$P(X_1 + X_2 \le 1) = \int_0^1 \int_0^{1-x_2} e^{-(x_1+x_2)}dx_1dx_2 = \int_0^1 \int_0^{1-x_2} e^{-x_1} \cdot e^{-x_2}dx_1dx_2$$

[from Figure 5.4]

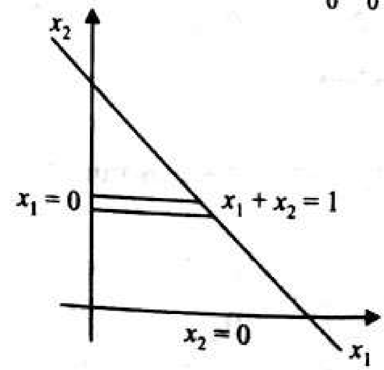


Figure 5.4

$$= \int_0^1 e^{-x_2} \left[\frac{e^{-x_1}}{-1} \right]_0^{1-x_2} dx_2$$

$$= \int_0^1 e^{-x_2} \left[\frac{e^{-(1-x_2)} - 1}{-1} \right] dx_2 = \int_0^1 e^{-x_2} (1 - e^{-1} e^{x_2}) dx_2$$

$$= \int_0^1 (e^{-x_2} - e^{-1}) dx_2 = \left[\frac{e^{-x_2}}{-1} - e^{-1}(x_2) \right]_0^1$$

$$= \frac{e^{-1}}{-1} - e^{-1} + 1 = 1 - 2e^{-1}$$

EXAMPLE 5.29 If the joint PDF of two random variables X and Y is

$$f(x, y) = \begin{cases} \frac{1}{3}(3x^2 + xy), & 0 < x < 1, 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

find $P(X + Y \geq 1)$. **[AU November '07]**

Solution We know that

$$P(X + Y \geq 1) = 1 - P(X + Y < 1)$$

$$P(X + Y < 1) = \int_0^1 \int_0^{1-y} f(x, y) dx dy \text{, [from Figure 5.5]}$$

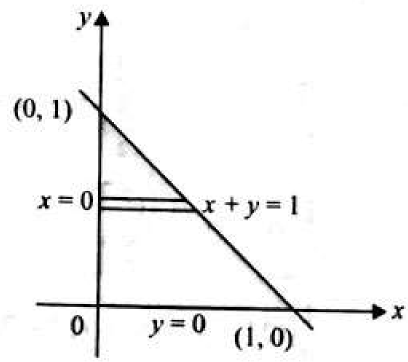


Figure 5.5

$$= \int_0^1 \int_0^{1-y} \frac{1}{3}(3x^2 + xy) \, dx dy$$

$$= \frac{1}{3} \int_0^1 \left[x^3 + \frac{x^2 y}{2} \right]_0^{1-y} dy$$

$$= \frac{1}{3}\int_0^1 \left[(1-y)^3 + \frac{(1-y)^2 y}{2}\right] dy$$

$$= \frac{1}{3}\int_0^1 \left(\frac{-y^3}{2} + 2y^2 - \frac{5y}{2} + 1\right) dy$$

$$= \frac{1}{3}\left[\frac{-y^4}{8} + 2\frac{y^3}{3} - \frac{5y^2}{4} + y\right]_0^1$$

$$= \frac{1}{3}\left(-\frac{1}{8} + \frac{2}{3} - \frac{5}{4} + 1\right) = \frac{7}{72}$$

$\therefore$
$$P(X + Y \geq 1) = 1 - P(X + Y < 1)$$

$$= 1 - \frac{7}{72} = \frac{65}{72}$$

EXAMPLE 5.30 If the joint PDF of the random variables X and Y is given by

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y), & 0 < x < 2, 2 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

Find $f(y/x = 2)$.

[AU May '03]

Solution We know that the conditional density function of Y given X is

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_2^4 \frac{1}{8}(6 - x - y)dy$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4$$

$$= \frac{1}{8}[(24 - 4x - 8) - (12 - 2x - 2)]$$

$$= \frac{1}{8}(6 - 2x), \quad 0 < x < 2$$

$\therefore$
$$f(y/x) = \frac{f(x, y)}{f(x)}$$

$$= \frac{\frac{1}{8}(6 - x - y)}{\frac{1}{8}(6 - 2x)} = \frac{(6 - x - y)}{(6 - 2x)}$$

$$\therefore \quad f(y/x=2) = \frac{6-2-y}{6-4} = \frac{(4-y)}{2}, \quad 2<y<4$$

EXAMPLE 5.31 Find the marginal density functions of X and Y, if

$$f(x, y) = \frac{2}{5}(2x+5y), \quad 0 \le x \le 1, 0 \le y \le 1.$$

[AU December '06, June '06]

Solution By definition, the marginal density function of X is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 \frac{2}{5}(2x+5y)dy$$

$$= \frac{2}{5}\left[2xy + \frac{5y^2}{2}\right]_0^1 = \frac{2}{5}\left(2x+\frac{5}{2}\right) = \frac{4x+5}{5}, \quad 0 \le x \le 1$$

The marginal density function of Y is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 \frac{2}{5}(2x+5y)dx$$

$$= \frac{2}{5}\left[\frac{2x^2}{2} + 5xy\right]_0^1 = \frac{2}{5}(1+5y) = \frac{2+10y}{5}, \quad 0 \le y \le 1$$

EXAMPLE 5.32 Given

$$f_{XY}(x, y) = \begin{cases} cx(x-y), & 0<x<2, -x<y<x \\ 0, & \text{elsewhere} \end{cases}$$

Evaluate

(i) c, (ii) $f_X(x)$,

(iii) $f_{(Y/X)}(y/x)$, and (iv) $f_Y(y)$. **[AU June '05; '06, April '08]**

Solution

(i) To find the value of c, we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

$$\therefore \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = \int_0^2\int_{-x}^{x} cx(x-y)dydx = 1 \qquad \text{(i)}$$

$$c\int_0^2\int_{-x}^{x}(x^2-xy)dydx = c\int_0^2\left[\left(x^2y - \frac{xy^2}{2}\right)\right]_{-x}^{x} dx = 1$$

$$\Rightarrow \quad c\int_0^2\left[\left(x^3-\frac{x^3}{2}+x^3+\frac{x^3}{2}\right)\right]dx = c\int_0^2 2x^3dx = 1$$

$$\Rightarrow \quad 2c\left[\frac{x^4}{4}\right]_0^2 = 2c\times\frac{16}{4} = 1$$

$$\Rightarrow \quad 8c = 1 \quad \Rightarrow \quad c = \frac{1}{8}$$

(ii) $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$

$$f_X(x) = \frac{1}{8}\int_{-x}^{x}(x^2-xy)dy = \frac{1}{8}\left[\left(x^2y-\frac{xy^2}{2}\right)\right]_{-x}^{x}, \quad \text{[from Figure 5.6]}$$

$$= \frac{1}{8}\left(x^3-\frac{x^3}{2}+x^3+\frac{x^3}{2}\right) = \frac{2x^3}{8} = \frac{x^3}{4}, \qquad 0<x<2$$

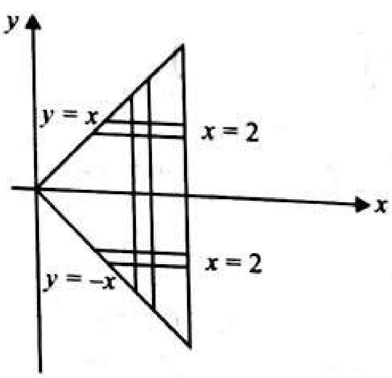


Figure 5.6

(iii) $$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{\dfrac{x(x-y)}{8}}{\dfrac{x^3}{4}} = \frac{x-y}{2x^2}, \qquad -x<y<x$$

(iv) $$f_Y(y)\int_{-\infty}^{\infty} f(x, y)dx = \int_{-y}^{2}\frac{1}{8}x(x-y)dx, \qquad -2\le y\le 0$$

$$\frac{1}{8}\left[\frac{x^3}{3}-\frac{x^2y}{2}\right]_{-y}^{2} = \frac{1}{8}\left[\frac{8}{3}-2y-\left(\frac{-y^3}{3}-\frac{y^3}{2}\right)\right]$$

$$= \frac{1}{3}-\frac{y}{4}+\frac{5y^3}{48}$$

and $$f_Y(y) = \int_y^2 \frac{1}{8} x(x-y)dx, \quad 0 \le y \le 2$$

$$= \frac{1}{8}\left(\frac{x^3}{3} - \frac{x^2 y}{2}\right)_y^2 = \frac{1}{8}\left(\frac{8}{3} - \frac{4y}{2} - \frac{y^3}{3} + \frac{y^3}{2}\right)$$

$$= \frac{1}{3} - \frac{y}{4} + \frac{y^3}{48}$$

$$\therefore \quad f_Y(y) = \begin{cases} \dfrac{1}{3} - \dfrac{y}{4} + \dfrac{5y^3}{48}, & -2 < y < 0 \\ \dfrac{1}{3} - \dfrac{y}{4} + \dfrac{y^3}{48}, & 0 < y < 2 \end{cases}$$

EXAMPLE 5.33 If the joint PDF of a two-dimensional random variable is given by

$$f(x, y) = \begin{cases} 2, & 0 < x < 1, 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

find the marginal density functions of X and Y. **[AU December '03; '07]**

Solution The marginal distribution of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^x 2dy = 2[y]_0^x, \text{ [from Figure 5.7]}$$

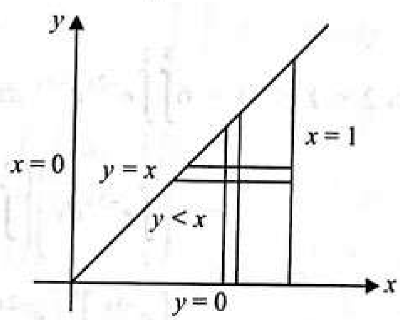


Figure 5.7

$$f_X(x) = 2x, \quad 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_y^1 2dx = 2[x]_y^1$$

$$= 2(1-y), \quad 0 < y < 1$$

EXAMPLE 5.34 The joint PDF of the random variables (X, Y) is $f(x, y) = 8xy$, $0 < x < 1, 0 < y < x$. Find the conditional density function

(i) $f_{Y/X}(y/x)$, and

(ii) $f_Y(y)$. **[AU December '05, June '06, November '07]**

Solution Given: $f(x, y) = 8xy,\ 0 < x < 1,\ 0 < y < x$

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^x 8xy\, dy, \text{ (from the Figure 5.7)}$$

$$= 8x\left[\frac{y^2}{2}\right]_0^x = 8x \times \frac{x^2}{2}$$

$$f_X(x) = 4x^3, \qquad 0 < x < 1$$

(i) $f_{Y/X}(y/x) = \dfrac{f(x, y)}{f_X(x)} = \dfrac{8xy}{4x^3} = \dfrac{2y}{x^2}$

(ii) $f_Y(y) = \displaystyle\int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 8xy\, dx$

$$= 8y\left[\frac{x^2}{2}\right]_0^1 = 8y\left(\frac{1}{2}\right) = 4y, \qquad 0 < y < 1$$

EXAMPLE 5.35 If the joint PDF of the two random variables is given by

$$f(x, y) = \begin{cases} 6e^{-2x-3y}, & \text{for } x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

find

(i) $P(1 < X < 2 \cap 2 < Y < 3)$, and (ii) $P(X < 2 \cap Y > 2)$.

Solution

(i) $$P(1 < X < 2 \cap 2 < Y < 3) = 6\int_2^3\int_1^2 e^{-2x}e^{-3y}\, dxdy$$

$$= 6\left(\int_2^3 e^{-3y}dy\right)\left(\int_1^2 e^{-2x}dx\right)$$

$$= 6\left[\frac{e^{-3y}}{-3}\right]_2^3 \left[\frac{e^{-2x}}{-2}\right]_1^2$$

$$= [e^{-3y}]_2^3\, [e^{-2x}]_1^2$$

$$= (e^{-9} - e^{-6})(e^{-4} - e^{-2})$$

(ii) $$P(X < 2 \cap Y > 2) = 6\int_2^{\infty}\int_0^2 e^{-2x}e^{-3y}dxdy$$

$$= 6\left[\frac{e^{-2x}}{-2}\right]_0^2 \left[\frac{e^{-3y}}{-3}\right]_2^{\infty}$$

$$= [-e^{-2x}]_0^2\, [-e^{-3y}]_2^{\infty} = (1 - e^{-4})e^{-6} = 0.0025$$

EXAMPLE 5.36 The joint PDF of a bivariate random variable (X, Y) is given by

$$f(x, y) = \begin{cases} Kxy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find K.
(ii) Find $P(X + Y < 1)$.
(iii) Are X and Y are independent random variables?

[AU December '05; '07; '09]

Solution

(i) We know that if $f(x, y)$ is a PDF, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1$$

$$\therefore \qquad \int_0^1\int_0^1 K\,xy\,dydx = 1 \quad \Rightarrow \quad K\int_0^1 y\left[\frac{x^2}{2}\right]_0^1 dy = 1$$

i.e.
$$K\int_0^1 \frac{y}{2}dy = 1 \quad \Rightarrow \quad \frac{K}{2}\left(\frac{y^2}{2}\right)_0^1 = 1$$

$$\Rightarrow \qquad K\left(\frac{1}{4}\right) = 1 \quad \Rightarrow \quad K = 4$$

(ii) Now, $\quad P(X + Y < 1) = \int_0^1\int_0^{1-y} f(x, y)dxdy$, from the Figure 5.3

$$\therefore \qquad = \int_0^1\int_0^{1-y} 4xy\,dxdy = 4\int_0^1 y\left[\frac{x^2}{2}\right]_0^{1-y} dy = 4\int_0^1 \frac{y(1-y)^2}{2}dy$$

$$= 2\int_0^1 (y^3 + y - 2y^2)dy = 2\left[\frac{y^4}{4} + \frac{y^2}{2} - \frac{2y^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{4} + \frac{1}{2} - \frac{2}{3}\right) = 2\left(\frac{1}{12}\right) = \frac{1}{6}$$

(iii) The marginal density function of X is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 4xydy = 4x\left[\frac{y^2}{2}\right]_0^1 = 2x$$

The marginal density function of Y is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 4xydx = 4y\left[\frac{x^2}{2}\right]_0^1 = 2y$$

$$f(x)\,f(y) = 2x \cdot 2y = 4xy = f(x, y)$$

$\therefore$ X and Y are independent.

EXAMPLE 5.37 If the joint PDF of a random variable (X, Y) is given by

$$f(x, y) = \begin{cases} x^2 + \dfrac{xy}{3}, & 0 \le x \le 1; 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

find the conditional densities of X given Y and Y given X.

[AU November '06]

Solution The conditional density function of Y on X is

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right)dy$$

$$= \left[x^2 y + \frac{xy^2}{6}\right]_0^2 = 2x^2 + \frac{2x}{3}, \qquad 0 \le x \le 1$$

$\therefore$

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

$$= \frac{x^2 + \dfrac{xy}{3}}{2x^2 + \dfrac{2x}{3}} = \frac{3x^2 + xy}{2(3x^2 + x)} = \frac{3x + y}{2(3x + 1)}$$

Similarly, it can be shown that

$$f(y) = \frac{y + 2}{6}, \qquad 0 \le y \le 2$$

$$f(x/y) = \frac{f(x, y)}{f(y)} = 2\left(\frac{3x^2 + xy}{y + 2}\right)$$

EXAMPLE 5.38 If $f(x, y) = \begin{cases} xe^{-x(y+1)}, & x \ge 0,\ y \ge 0 \\ 0, & \text{otherwise} \end{cases}$ is the joint PDF of a two-dimensional random variable (X, Y), find the marginal and conditional density functions.

[AU December '05]

Solution The marginal densities of X and Y are given by

$$f_X(x) = \int_0^{\infty} xe^{-x(y+1)}dy = e^{-x}[-e^{-xy}]_0^{\infty} = e^{-x}, \qquad x \ge 0$$

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)}dx = \left[x\left(\frac{-e^{-x(y+1)}}{y+1}\right) - (1)\left(\frac{e^{-(y+1)x}}{(y+1)^2}\right)\right]_0^\infty$$

$\therefore$
$$f_Y(y) = \frac{1}{(y+1)^2}, \qquad y \geq 0$$

The conditional density function is given by

$$f(x/y) = \frac{f(x,y)}{f_Y(y)} = (y+1)^2 xe^{-x(y+1)}, \qquad x \geq 0, y \geq 0$$

and
$$= f(y/x) = \frac{f(x,y)}{f_X(x)} = xe^{-xy}, \qquad x \geq 0, y \geq 0$$

EXAMPLE 5.39 The joint PDF of two random variables X and Y is

$$f(x,y) = \begin{cases} xye^{\frac{-(x^2+y^2)}{2}}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Find the marginal PDFs of X and Y. Are X and Y independent?

Solution The marginal density of X is

$$f_X(x) = \int_0^\infty xye^{\frac{-(x^2+y^2)}{2}} dy = xe^{\frac{-x^2}{2}} \int_0^\infty ye^{\frac{-y^2}{2}} dy$$

$$= xe^{\frac{-x^2}{2}} \int_0^\infty e^{\frac{-y^2}{2}} d\left(\frac{y^2}{2}\right)$$

$\therefore$
$$f_X(x) = xe^{\frac{-x^2}{2}} \left[-e^{\frac{-y^2}{2}}\right]_0^\infty = xe^{\frac{-x^2}{2}} (0+1)$$

$\Rightarrow$
$$f_X(x) = xe^{\frac{-x^2}{2}}, \qquad x \geq 0$$

Similarly, the marginal density of Y is

$$f_Y(y) = \int_0^\infty f(x,y)dx \;\Rightarrow\; f_Y(y) = ye^{\frac{-y^2}{2}} \int_0^\infty xe^{\frac{-x^2}{2}} dx$$

$$= ye^{\frac{-y^2}{2}} \int_0^\infty e^{\frac{-x^2}{2}} d\left(\frac{x^2}{2}\right) = ye^{\frac{-y^2}{2}} \left[\frac{e^{\frac{-x^2}{2}}}{-1}\right]_0^\infty$$

$$= ye^{\frac{-y^2}{2}}, \qquad y \geq 0$$

$$f_X(x)\, f_Y(y) = xye^{\frac{-(x^2+y^2)}{2}} = f(x, y)$$

$\therefore$ X and Y are independent.

EXAMPLE 5.40 The joint PDF of two random variables X and Y is given by $f(x, y) = \dfrac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}}$, where $|\rho| < 1$ and $-\infty < x, y < \infty$.

Find the marginal density functions of X and Y. Are X and Y independent, if $\rho = 0$?

Solution The marginal PDF of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{\frac{-(x^2-2\rho xy+y^2)}{2(1-\rho^2)}} dy$$

To find the marginal density function of X:

$$x^2 - 2\rho xy + y^2 = (y^2 - 2\rho xy + \rho^2 x^2) + (1 - \rho^2)x^2$$
$$= (y - \rho x)^2 + (1 - \rho^2)x^2$$

$$\therefore \quad e^{\frac{-1}{2}\left[\frac{(x^2-2\rho xy+y^2)}{(1-\rho^2)}\right]} = e^{\frac{-1}{2}\left[\frac{(y-\rho x)^2+(1-\rho^2)x^2}{(1-\rho^2)}\right]} = e^{\frac{-x^2}{2}} e^{\frac{-1}{2}\left[\frac{(y-\rho x)^2}{(1-\rho^2)}\right]}$$

$$\therefore \qquad f_X(x) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-x^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[\frac{(y-\rho x)}{\sqrt{(1-\rho^2)}}\right]^2} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-\rho^2}} e^{\frac{-1}{2}\left[\frac{(y-\rho x)}{\sqrt{(1-\rho^2)}}\right]^2} dy$$

Put
$$z = \frac{(y-\rho x)}{\sqrt{(1-\rho^2)}} \Rightarrow dz = \frac{dy}{\sqrt{(1-\rho^2)}}$$

$$\therefore \qquad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz\right)$$

But, we know that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz = 1$$

Therefore, the marginal density of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad -\infty < x < \infty$$

To find the marginal density function of Y:
Writing $x^2 - 2\rho xy + y^2 = (x - \rho y)^2 + (1 - \rho^2)y^2$

$$\therefore \quad f_Y(y) = \frac{e^{\frac{-y^2}{2}}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[\frac{(x-\rho y)}{\sqrt{(1-\rho^2)}}\right]^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1-\rho^2}} e^{\frac{-1}{2}\left[\frac{(x-\rho y)}{\sqrt{(1-\rho^2)}}\right]^2} dx$$

Using $\quad z = \dfrac{(x-\rho y)}{\sqrt{(1-\rho^2)}}$, we get

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz\right]$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \text{ as } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz = 1$$

Therefore, the marginal density of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}, \quad -\infty < y < \infty$$

When $\rho = 0, f(x, y) = \dfrac{1}{2\pi} e^{\frac{-(x^2+y^2)}{2}}$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$$

Also, $\quad f_X(x)\, f_Y(y) = \dfrac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)} = f(x, y)$

$\therefore$ If $\rho = 0$, X and Y are independent.

EXAMPLE 5.41 The joint distribution function of two random variables X and Y is given by

$$F(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal and conditional densities of X and Y. Are X and Y independent? Also compute $P(X \le 1, Y \le 1)$.

Solution We know that the joint PDF $f(x, y) = \dfrac{\partial^2 F(x, y)}{\partial x \partial y}$

$$\therefore \quad f(x, y) = \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-x} - e^{-y} + e^{-(x+y)}\right] = \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)}\right] = e^{-(x+y)}$$

The joint PDF $f(x, y) = \begin{cases} e^{-(x+y)}, & x \ge 0, y \ge 0 \\ 0, & \text{otherwise} \end{cases}$

The marginal densities of X and Y are given as follows:

$$f_X(x) = \int_0^\infty f(x, y)dy = \int_0^\infty e^{-(x+y)}dy = e^{-x}[-e^{-y}]_0^\infty = e^{-x}, \quad x > 0$$

$$f_Y(y) = \int_0^\infty f(x, y)dx = \int_0^\infty e^{-(x+y)}dx = e^{-y}[-e^{-x}]_0^\infty = e^{-y}, \quad y > 0$$

The conditional density functions are:

$$f(x/y) = \frac{f(x, y)}{f_Y(y)} = e^{-x}, \quad x > 0$$

$$f(y/x) = \frac{f(x, y)}{f_X(x)} = e^{-y}, \quad y > 0$$

Since $f_X(x)\, f_Y(y) = e^{-(x+y)} = f(x, y)$, X and Y independent.

$$P(X \le 1, Y \le 1) = \int_0^1\int_0^1 e^{-(x+y)}dydx = \left(\int_0^1 e^{-x}dx\right)\left(\int_0^1 e^{-y}dy\right)$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^1 \left[\frac{e^{-y}}{-1}\right]_0^1$$

$$= (1 - e^{-1})\,(1 - e^{-1}) = (1 - e^{-1})^2$$

EXAMPLE 5.42 The joint density function of a random variable (X, Y) is given by

$$f(x, y) = \begin{cases} axy, & 1 \le x \le 3, 2 \le y \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

Find

(i) the value of a,

(ii) the marginal densities of X and Y,

(iii) the conditional densities of X given Y and Y given X.

Solution Since $f(x, y)$ is a joint PDF,

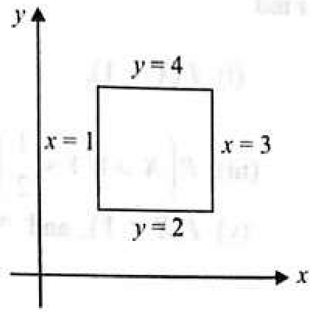


Figure 5.8

(i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx \int_{1}^{3}\int_{2}^{4} axy \cdot dydx = 1$

See Figure 5.8.

$$\Rightarrow \int_{1}^{3} ax\left[\frac{y^2}{2}\right]_2^4 dx = \int_{1}^{3} ax\left(\frac{16}{2}-\frac{4}{2}\right)dx = 1$$

$$\Rightarrow \int_{1}^{3} 6ax \cdot dx = 6a\left[\frac{x^2}{2}\right]_1^3 = 6a\left(\frac{9}{2}-\frac{1}{2}\right) = 1$$

i.e., $6a \times \frac{8}{2} = 24a = 1 \Rightarrow a = \frac{1}{24}$

(ii) $$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{2}^{4} \frac{xy}{24} \cdot dy = \frac{x}{24}\left[\frac{y^2}{2}\right]_2^4$$

$$= \frac{x}{24}\left(\frac{16}{2}-\frac{4}{2}\right) = \frac{x}{24} \times \frac{12}{2} = \frac{x}{4}, \quad 1 \le x \le 3$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{1}^{3} \frac{xy}{24} \cdot dy = \frac{y}{24}\left[\frac{x^2}{2}\right]_1^3$$

$$= \frac{y}{24}\left(\frac{9}{2}-\frac{1}{2}\right) = \frac{y}{24} \times \frac{8}{2} = \frac{y}{6}, \quad 2 \le y \le 4$$

(iii) $$f_{(Y/X)}(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{\frac{xy}{24}}{\frac{x}{4}} = \frac{xy}{24} \times \frac{4}{x} = \frac{y}{6}, \quad 2 \le y \le 4$$

(iv) $$f_{(X/Y)}(x/y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{xy}{24}}{\frac{y}{6}} = \frac{x}{4}, \quad 1 \le x \le 3$$

Aliter

Since $f(x, y) = f(x)\, f(y)$, X and Y are independent.

$$\therefore \quad f(y/x) = f(y) = \frac{y}{6}, \quad 2 \le y \le 4$$

$$f(x/y) = f(x) = \frac{x}{4}, \quad 1 \le x \le 3$$

EXAMPLE 5.43 The joint PDF of two-dimensional random variables (X, Y) is given by

$$f(x, y) = xy^2 + \frac{x^2}{8}, \quad 0 \le x \le 2, 0 \le y \le 1.$$

Find

(i) $P(X > 1)$,

(ii) $P\left(Y < \frac{1}{2}\right)$,

(iii) $P\left(X > 1 \big/ Y < \frac{1}{2}\right)$,

(iv) $P\left(Y < \frac{1}{2} \big/ X > 1\right)$,

(v) $P(X < Y)$, and

(vi) $PX + Y) \le 1$.

[AU November '07]

Solution Given: $f(x, y) = xy^2 + \frac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1.$

(i) $P(X > 1) = \int\limits_0^1 \int\limits_1^2 \left(xy^2 + \frac{x^2}{8}\right) dxdy$, [from Figure 5.9]

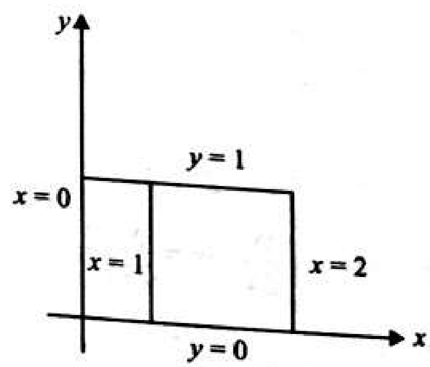


Figure 5.9

$$= \int\limits_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{3 \times 8}\right]_1^2 dy$$

$$= \int\limits_0^1 \left(\frac{4y^2}{2} + \frac{8}{3 \times 8} - \frac{y^2}{2} - \frac{1}{3 \times 8}\right) dy$$

$$= \int\limits_0^1 \left(2y^2 + \frac{7}{24} - \frac{y^2}{2}\right) dy = \left[\frac{2y^3}{3} + \frac{7y}{24} - \frac{y^3}{6}\right]_0^1$$

$$= \frac{2}{3} + \frac{7}{24} - \frac{1}{6} = \frac{19}{24} \qquad \text{(i)}$$

(ii) $P\left(Y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_0^2\left(xy^2+\frac{x^2}{8}\right)dxdy$, [from Figure 5.10]

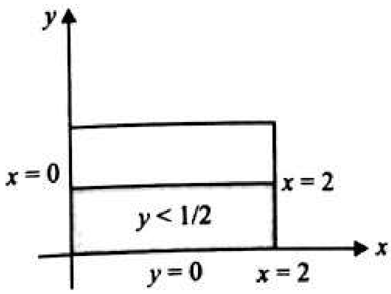


Figure 5.10

$$=\int_0^{\frac{1}{2}}\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_0^2 dy=\int_0^{\frac{1}{2}}\left[\frac{4y^2}{2}+\frac{8}{24}\right]dy$$

$$=\left[\frac{4y^3}{6}+\frac{8y}{24}\right]_0^{\frac{1}{2}}=\frac{4\left(\frac{1}{2}\right)^3}{6}+\frac{8\left(\frac{1}{2}\right)}{24}=\frac{1}{4} \qquad \text{(ii)}$$

(iii) $$P\left(X>1\Big/Y<\frac{1}{2}\right)=\frac{P\left(X>1\cap Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)}$$

$$=\frac{P\left(X>1,Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)} \qquad \text{(iii)}$$

$P\left(X>1,Y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_1^2\left(xy^2+\frac{x^2}{8}\right)dxdy$ [from Figure 5.11]

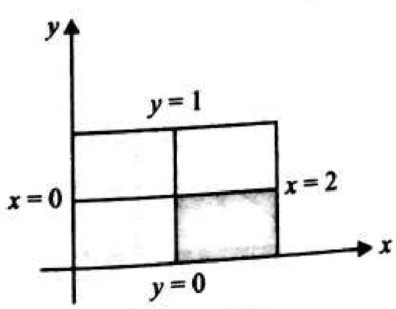


Figure 5.11

$$= \int_0^{\frac{1}{2}} \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2 dy$$

$$= \int_0^{\frac{1}{2}} \left(\frac{4y^2}{2} + \frac{8}{24} - \frac{y^2}{2} - \frac{1^3}{24}\right) dy$$

$$= \left[\frac{4y^3}{6} + \frac{8y}{24} - \frac{y^3}{6} - \frac{1^3 y}{24}\right]_0^{\frac{1}{2}}$$

$$= \frac{4\left(\frac{1}{2}\right)^3}{6} + \frac{8\left(\frac{1}{2}\right)}{24} - \frac{\left(\frac{1}{2}\right)^3}{6} - \frac{\left(\frac{1}{2}\right)}{24} = \frac{5}{24} \qquad \text{(iv)}$$

Substituting in Eq. (iii), we get,

$$P\left(X > 1/Y < \frac{1}{2}\right) = \frac{\frac{5}{24}}{\frac{1}{4}} = \frac{5}{6}$$

(iv) $P\left(Y < \frac{1}{2}/X > 1\right) = \dfrac{P\left(Y < \frac{1}{2}, X > 1\right)}{P(X > 1)}$

$$= \frac{\frac{5}{24}}{\frac{19}{24}} = \frac{5}{19} \qquad \text{[using Eqs. (i) and (iv)]}$$

(v) $P(X < Y) = \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dy dx$, [from Figure 5.12]

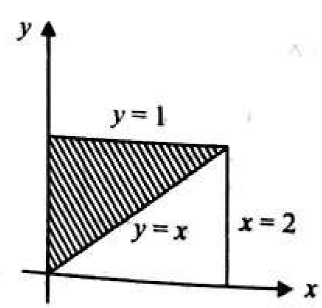


Figure 5.12

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^y dy$$

$$= \int_0^1 \left[\frac{y^4}{2} + \frac{y^3}{24}\right] dy = \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{53}{480}$$

(vi) $P[(X + Y) \le 1] = \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dxdy,$ [from Figure 5.13]

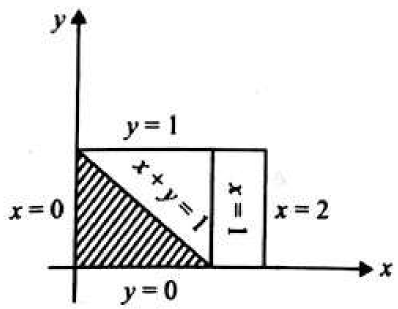


Figure 5.13

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^{1-y} dy = \int_0^1 \left[\frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24}\right] dy$$

$$= \left[\frac{1}{2}\left(\frac{y^3}{3} - \frac{2y^4}{4} + \frac{y^5}{5}\right) + \frac{1}{24}\left(y - \frac{3y^2}{2} + \frac{3y^3}{3} - \frac{y^4}{4}\right)\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \frac{1}{24}\left(1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{60} + \frac{1}{96} = \frac{13}{480}$$

EXAMPLE 5.44 Determine the value of C that makes the function $f(x, y) = C(x + y)$ a joint PDF over the range $0 < x < 3$ and $x < y < x + 2$. Also determine the following:

(i) $P(X < 1, Y < 2)$
(ii) $P(Y > 2)$
(iii) $E(X)$ **[AU December '09]**

Solution If $f(x, y)$ is a joint PDF, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1$$

$$\int_0^3 \int_x^{x+2} C(x+y)dydx = 1 \quad \Rightarrow \quad C\int_0^3 \left[xy + \frac{y^2}{2}\right]_x^{x+2} dx = 1$$

$\Rightarrow$
$$C\int_0^3 (4x+2)dx = 1 \quad \Rightarrow \quad C[2x^2 + 2x]_0^3 = 1$$

$$24C = 1$$

$\Rightarrow$
$$C = \frac{1}{24}$$

i.e.

(i) $P(X < 1, Y < 2) = \dfrac{1}{24}\displaystyle\int_0^1 \int_x^2 (x+y)dydx$, [from Figure 5.14]

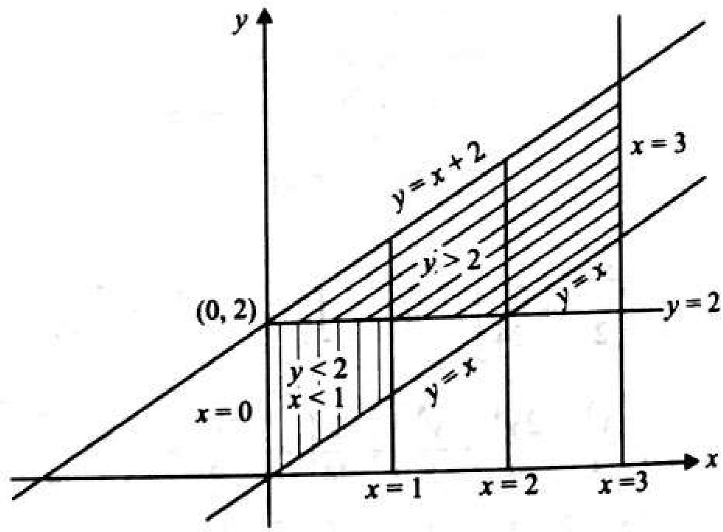


Figure 5.14

$$= \frac{1}{24}\int_0^1 \left[xy + \frac{y^2}{2}\right]_x^2 dx$$

$$= \frac{1}{24}\int_0^1 \left[2x + 2 - \frac{3x^2}{2}\right] dx$$

$$= \frac{1}{24}\left[x^2 + 2x - \frac{x^3}{2}\right]_0^1$$

$$P(X < 1, Y < 2) = \frac{5}{48}$$

(ii) $$P(Y>2)=\frac{1}{24}\int_0^2\int_2^{x+2}(x+y)dydx+\frac{1}{24}\int_2^3\int_x^{x+2}(x+y)dydx$$

$$=\frac{1}{24}\int_0^2\left[xy+\frac{y^2}{2}\right]_2^{x+2}dx+\frac{1}{24}\int_2^3\left[xy+\frac{y^2}{2}\right]_x^{x+2}dx$$

$$=\frac{1}{24}\int_0^2\left[\frac{3x^2}{2}+2x\right]dx+\frac{1}{24}\int_2^3(4x+2)\,dx$$

$$=\frac{1}{24}\left\{\left[\frac{x^3}{2}+x^2\right]_0^2+[2x^2+2x]_2^3\right\}$$

$$=\frac{5}{6}$$

(iii) To find $E(X)$, first we find $f_X(x)$.

$$f_X(x)=\int_x^{x+2}f(x,y)dy$$

$$=\frac{1}{24}\int_x^{x+2}(x+y)dy=\frac{1}{24}\left[xy+\frac{y^2}{2}\right]_x^{x+2}$$

$$=\frac{1}{24}(4x+2)=\frac{1}{12}(2x+1),\qquad 0\le x\le 3$$

$$\therefore\qquad E(X)=\int_0^3 xf_X(x)dx=\frac{1}{12}\int_0^3 x(2x+1)dx$$

$$=\frac{1}{12}\left[\frac{2x^3}{3}+\frac{x^2}{2}\right]_0^3=\frac{1}{12}\left(18+\frac{9}{2}\right)$$

$$=\frac{45}{24}$$

EXAMPLE 5.45 If X is the proportion of persons who will respond to one kind of mail-order solicitation, Y is the proportion of persons who will respond to another kind of mail-order solicitation and the joint PDF of X and Y is given by

$$f(x,y)=\frac{2}{5}(x+4y),\qquad 0<x<1, 0<y<1$$

find the probabilities that

(i) at least 30% will respond to the first kind of mail-order solicitation.

(ii) at most 50% will respond to the second kind of mail-order solicitation given that there has been 20% response to the first kind of mail-order solicitation. [AU June '06]

Solution Given: X is the proportion of persons who will respond to first kind of mail-order solicitation and Y is the proportion of persons who will respond to second kind of mail-order solicitation and the joint PDF of the random variable (X, Y) is

$$f(x, y) = \frac{2}{5}(x + 4y), \qquad 0 < x < 1,\ 0 < y < 1$$

(i) To find

$$P(X \geq 30\%) = P\left(X \geq \frac{30}{100}\right)$$

$$= P\left(X \geq \frac{3}{10}\right) = 1 - P\left(X < \frac{3}{10}\right)$$

The marginal density functions of X and Y:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{0}^{1} \frac{2}{5}(x + 4y)dy$$

$$= \frac{2}{5}\left[xy + \frac{4y^2}{2}\right]_0^1$$

$$= \frac{2}{5}(x + 2), \qquad 0 < x < 1$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{0}^{1} \frac{2}{5}(x + 4y)dx$$

$$= \frac{2}{5}\left[\frac{x^2}{2} + 4yx\right]_0^1$$

$$= \frac{2}{5}\left(\frac{1}{2} + 4y\right) = \frac{1}{5}(1 + 8y)$$

$$\therefore \qquad f(y) = \frac{1}{5}(1 + 8y), \qquad 0 < y < 1$$

(i) The probability that at least 30% will respond to the first kind of mail-order solicitation

$$P(X \geq 30\%) = P\left(X \geq \frac{3}{10}\right)$$

$$= \int_{\frac{3}{10}}^{\infty} f(x)dx = \int_{\frac{3}{10}}^{1} \frac{2}{5}(x+2)dx$$

$$= \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{\frac{3}{10}}^{1} = \frac{2}{5}\left[\frac{1}{2} + 2 - \left(\frac{9}{200} + \frac{6}{10}\right)\right]$$

$$= \frac{171}{500} = 0.342$$

(ii) To find the probability that at most 50% will respond to the second kind of mail-order solicitation given that there has been 20% response to the first kind of mail-order solicitation
We know that

$$f(y/x) = \frac{f(x, y)}{f(x)} = \frac{\frac{2}{5}(x+4y)}{\frac{2}{5}(x+2)}$$

$$f(y/x) = \frac{x+4y}{x+2}$$

$$f(y/x = 0.2) = \frac{0.2+4y}{2.2}$$

$$\therefore \quad P\left(Y \leq \frac{1}{2}\Big/X = 0.2\right) = \int_{0}^{\frac{1}{2}} f(y/x = 0.2)dy$$

$$= \frac{1}{2.2}\int_{0}^{\frac{1}{2}} (0.2+4y)dy = \frac{1}{2.2}\left[0.2y + \frac{4y^2}{2}\right]_{0}^{\frac{1}{2}}$$

$$= \frac{1}{2.2}\left[0.2 \times \frac{1}{2} + 2\left(\frac{1}{4}\right)\right] = \frac{1}{2.2}(0.1+0.5)$$

$$= \frac{0.6}{2.2} = 0.2727$$

5.2.6 Expectation of Two-dimensional Random Variables

If (X, Y) is a two-dimensional random variable, then the mean or expectation of (X, Y) is defined as follows

Case 1

When X and Y are discrete random variables, then

$$E(X) = \sum_{x_i} x_i P(X = x_i) = \sum_{x_i} x_i p(x_i)$$

$$E(Y) = \sum_{y_j} y_j P(Y = y_j) = \sum_{y_j} y_j p(y_j)$$

$$E(X/Y) = \sum_{x_i} x_i P(X = x_i / Y = y_j)$$

$$E(Y/X) = \sum_{y_j} y_j P(Y = y_j / X = x_i)$$

and
$$E(XY) = \sum_{x_i}\sum_{y_j} x_i y_j \, P(X = x_i, Y = y_j) = \sum_{x_i}\sum_{y_j} x_i y_j p(x_i, y_j)$$

Case 2

If X and Y are continuous random variables, then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

and
$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy$$

Conditional Expected Values

$$E(Y/X) = \int_{-\infty}^{\infty} y f(y/x)\, dy$$

and
$$E(X/Y) = \int_{-\infty}^{\infty} x f(x/y)\, dx$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x, y)\, dx\, dy$$

Note:

(i) If X and Y are independent random variables, then

$$E(X/Y) = E(X)$$

and $\quad E(Y/X) = E(Y)$

(ii) $\quad E[E(X/Y)] = E(X)$

and $E[E(Y/X)] = E(Y)$

EXAMPLE 5.46 The joint distribution of X and Y is given by

$$f(x, y) = \frac{x+y}{21}, x = 1, 2, 3, y = 1, 2$$

Find the marginal distributions of X and Y. Find the mean of X and Y also.

Solution From the given problem,

$X \backslash Y$	1	2	$P(X = x_i)$	
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{5}{21}$	$P(X = 1)$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{7}{21}$	$P(X = 2)$
3	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{9}{21}$	$P(X = 3)$
$P(Y = y_j)$	$\frac{9}{21}$	$\frac{12}{21}$	1	
	$P(Y = 1)$	$P(Y = 2)$		

The marginal distributions of X are

$$P(X = 1) = \frac{5}{21}, P(X = 2) = \frac{7}{21}, P(X = 3) = \frac{9}{21}$$

The marginal distributions of Y are

$$P(Y = 1) = \frac{9}{21}, P(Y = 2) = \frac{12}{21}$$

$$E(X) = \sum_{x=1}^{3} xP(X = x) = 1P(X = 1) + 2P(X = 2) + 3P(X = 3)$$

$$= \frac{5}{21} + \frac{14}{21} + \frac{27}{21} = \frac{46}{21}$$

$$E(Y) = \sum_{y=1}^{2} yP(Y = y) = 1P(Y = 1) + 2P(Y = 2)$$

$$= \frac{9}{21} + \frac{24}{21} = \frac{33}{21} = \frac{11}{7}$$

EXAMPLE 5.47 Let X and Y are two random variables each having three values -1, 0, 1 and having the following joint probability distribution:

Y \ X	-1	0	1	Total
-1	0	0.1	0.1	0.2
0	0.2	0.2	0.2	0.6
1	0	0.1	0.1	0.2
Total	0.2	0.4	0.4	1.0

Prove that X and Y have different expectations. Also prove that X and Y are uncorrelated and find Var(X) and Var(Y). **[AU December '05]**

Solution From the table given

$$P(X = -1) = 0.2,\ P(X = 0) = 0.4,\ P(X = 1) = 0.4,\ P(Y = -1) = 0.2,$$
$$P(Y = 0) = 0.6,\ P(Y = 1) = 0.2,$$

$$\begin{aligned} E(X) &= \Sigma xp(x) = \Sigma xP(X = x) \\ &= -1 \times 0.2 + 0 \times 0.4 + 1 \times 0.4 \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} E(Y) &= \Sigma yp(y) = \Sigma yP(Y = y) \\ &= -1 \times 0.2 + 0 \times 0.6 + 1 \times 0.2 \\ &= 0 \end{aligned}$$

$\therefore$ X and Y have different expectations.

$$\begin{aligned} E(XY) &= \Sigma xyP(X = x, Y = y) \\ &= (-1 \times -1) \times 0 + (-1 \times 0) \times 0.1 + (-1 \times 1) \times 0.1 \\ &\quad + (0 \times -1) \times 0.2 + (0 \times 0) \times 0.2 + (0 \times 1) \times 0.2 \\ &\quad + (1 \times -1) \times 0 + (1 \times 0) \times 0.1 + (1 \times 1) \times 0.1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X)\,E(Y) \\ &= 0 - 0.2 \times 0 = 0 \end{aligned}$$

$\therefore$ X and Y are uncorrelated.

$$\begin{aligned} E(X^2) &= \Sigma x^2P(X = x) \\ &= (-1)^2 \times 0.2 + 0 \times 0.4 + 1^2 \times 0.4 \\ &= 0.2 + 0.4 = 0.6 \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \Sigma y^2P(Y = y) \\ &= (-1)^2 \times 0.2 + 0 \times 0.6 + 1^2 \times 0.2 \\ &= 0.4 \end{aligned}$$

$\therefore$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 0.6 - (0.2)^2 \end{aligned}$$

$$= \frac{14}{25} = 0.56$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$$
$$= 0.4 - 0 = 0.4$$

EXAMPLE 5.48 If the joint PDF is given by $f(x, y) = x + y$, $0 \le x, y \le 1$, find $E(XY)$. **[AU December '09]**

Solution By definition,

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dxdy$$

$$= \int_0^1\int_0^1 xy(x + y)\, dxdy$$

$$= \int_0^1\int_0^1 (x^2 y + xy^2)\, dxdy$$

$$= \int_0^1 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{3}$$

EXAMPLE 5.49 If $f(x, y) = 2x$, $0 \le x \le y \le 1$, find $E(XY)$ and $E(Y)$.

Solution We know that

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)\, dxdy$$

$$E(XY) = \int_0^1\int_0^y xy \cdot 2x\, dxdy = \int_0^1 \left[\frac{2yx^3}{3}\right]_0^y dy, \quad \text{[from Figure 5.15]}$$

$$= \int_0^1 \frac{2y^4}{3}\, dy = \left[\frac{2y^5}{15}\right]_0^1 = \frac{2}{15}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^y 2x\, dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^y = y^2, \qquad 0 < y < 1$$

$$E(Y) = \int_R y f_Y(y)dy = \int_0^1 y^3 dy$$

$$= \left[\frac{y^4}{4}\right]_0^1 = \frac{1}{4}$$

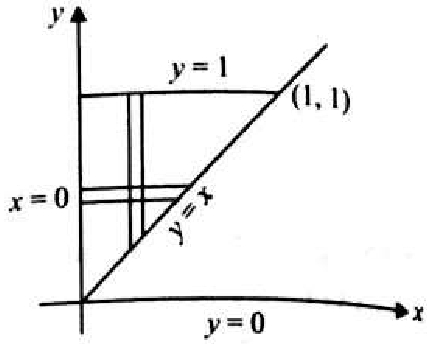


Figure 5.15

EXAMPLE 5.50 The joint PDF of (X, Y) is given by

$$f(x, y) = \begin{cases} 24xy, & x > 0,\ y > 0,\ x + y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the conditional mean and variance of Y given X.

Solution Given: $f(x, y) = 24xy, \quad x > 0, y > 0, x + y \le 1$

$$\therefore \quad f_X(x) = \int_0^{1-x} 24xy\, dy = 24x \int_0^{1-x} y\, dy,$$

[from Figure 5.16]

$$= 24x\left[\frac{y^2}{2}\right]_0^{1-x} = 24x\frac{(1-x)^2}{2}$$

$$= 12x\,(1-x)^2, \qquad 0 < x < 1$$

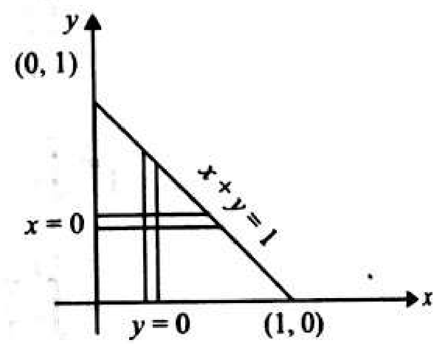


Figure 5.16

$$f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{2y}{(1-x)^2}, \qquad 0 < y < 1 - x$$

$$E(Y/X) = \int_0^{1-x} y f(y/x) dy$$

$$= \int_0^{1-x} \frac{2y^2}{(1-x)^2} dy = \frac{2}{(1-x)^2}\left[\frac{y^3}{3}\right]_0^{1-x} = \frac{2}{3}(1-x), \qquad x > 0$$

$$E(Y^2/X = x) = \int_0^{1-x} y^2 f(y/x) dy$$

$$= \int_0^{1-x} y^2 \frac{2y}{(1-x)^2} dy = \frac{2}{(1-x)^2}\left[\frac{y^4}{4}\right]_0^{1-x} = \frac{1}{2}(1-x)^2, \qquad x > 0$$

$$\text{Var}(Y/X) = E(Y^2/X) - [E(Y/X)]^2$$

$$= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2 = \frac{1}{18}(1-x)^2, \qquad x > 0$$

EXAMPLE 5.51 If (X, Y) is uniformly distributed over the semicircle bounded by $y = \sqrt{1-x^2}$ and $y = 0$, find $E(X/Y)$ and $E(Y/X)$. Also verify $E[E(X/Y)] = E(X)$ and $E[E(Y/X)] = E(Y)$.

Solution We know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dy\,dx = 1$$

Since (X, Y) is uniformly distributed, we assume that $f(x, y) = K$, a constant

when $y = 0$, $\sqrt{1-x^2} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

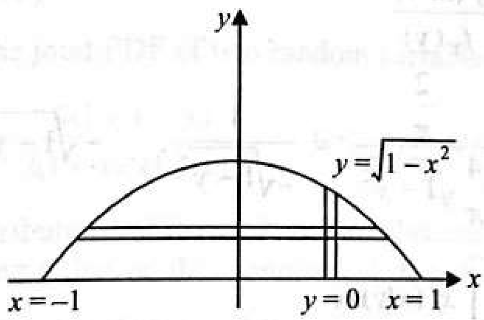


Figure 5.17

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} K\,dy\,dx = 1 \quad \Rightarrow \quad K\int_{-1}^{1}[y]_0^{\sqrt{1-x^2}}\,dx = 1,$$

[from Figure 5.17]

$$K\int_{-1}^{1}\sqrt{1-x^2}\,dx = 2K\int_{0}^{1}\sqrt{1-x^2}\,dx = 1$$

Using $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right)$, we get

$$2K\int_{0}^{1}\sqrt{1-x^2}\,dx = K\left(\frac{\pi}{2}\right) = 1 \Rightarrow K = \frac{2}{\pi}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

$$= \int_{0}^{\sqrt{1-x^2}} \frac{2}{\pi}\,dy = \frac{2}{\pi}\sqrt{1-x^2}; \quad -1 \le x \le 1$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}\, dx = \frac{2}{\pi}\left(\sqrt{1-y^2} + \sqrt{1-y^2}\right)$$

$$= \frac{4}{\pi}\sqrt{1-y^2}, \qquad 0 \le y \le 1$$

$$E(X) = \int_{-1}^{1} x f_x(x) dx = \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}\, dx = 0 \text{ [since the integrand is odd]}$$

We have

$$f(x/y) = \frac{f(x, y)}{f_Y(y)}$$

$$= \frac{\frac{2}{\pi}}{\frac{4}{\pi}\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}}, \qquad -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$

$$E(X/Y) = \int_{-\infty}^{\infty} x f(x/y) dx$$

$$E(X/Y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{x}{2\sqrt{1-y^2}}\, dx = \frac{1}{2\sqrt{1-y^2}}\left[\frac{x^2}{2}\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 0$$

$$\therefore \quad E[E(X/Y)] = E(0) = 0 = E(X)$$

$$f(y/x) = \frac{f(x, y)}{f_X(x)}$$

$$= \frac{\frac{2}{\pi}}{\frac{2}{\pi}\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}, \qquad 0 \le y \le \sqrt{1-x^2}$$

$$E(Y/X) = \int_{-\infty}^{\infty} y f(y/x) dy$$

$$E(Y/X) = \int_{0}^{\sqrt{1-x^2}} y \frac{1}{\sqrt{1-x^2}}\, dy = \frac{1}{\sqrt{1-x^2}}\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} = \frac{1}{2}\sqrt{1-x^2}$$

$$E(Y) = \int_0^1 y f_y(y) dy = \frac{4}{\pi}\int_0^1 y\sqrt{1-y^2}\, dy$$

$$= \frac{-2}{\pi}\int_0^1 \sqrt{1-y^2}\, d(1-y^2) = \frac{-2}{\pi}\left[\frac{(1-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{4}{3\pi}$$

$$E[E(Y/X)] = E\left(\frac{1}{2}\sqrt{1-x^2}\right) = \frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}\, f_X(x)dx$$

$$= \frac{2}{\pi}\int_0^1 (1-x^2)\,dx = \frac{2}{\pi}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{\pi}\left(1-\frac{1}{3}\right) = \frac{4}{3\pi}$$

$\therefore \quad E[E(Y/X)] = E(Y)$

EXAMPLE 5.52 The joint PDF of two random variables X and Y is given by

$$f(y/x) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}, \qquad 0 \le x < \infty, 0 \le y < \infty$$

Find the marginal distributions of X and Y, the conditional distribution of Y for $X = x$ and the expected value of this conditional distribution.

Solution The marginal distribution of X is

$$f_X(x) = \int_0^\infty \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\,dy$$

$$= \frac{9}{2(1+x)^4}\int_0^\infty [(1+y)^{-3} + x(1+y)^{-4}]\,dy$$

$$= \frac{9}{2(1+x)^4}\left[-\frac{(1+y)^{-2}}{2} - \frac{x(1+y)^{-3}}{3}\right]_0^\infty$$

$$= \frac{9}{2(1+x)^4}\left(\frac{1}{2}+\frac{x}{3}\right) = \frac{3}{4}\frac{2x+3}{(1+x)^4}, \qquad 0 \le x < \infty$$

Similarly, it can be shown that the marginal distribution of Y is

$$f_Y(y) = \frac{3}{4}\frac{2y+3}{(1+y)^4}, \qquad 0 \le y < \infty$$

The conditional PDF of Y for $X = x$ is

$$f(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\cdot\frac{4(1+x)^4}{3(2x+3)}$$

$$= \frac{6(1+x+y)}{(2x+3)(1+y)^4}, \quad \begin{cases} 0 \le x \le \infty \\ 0 \le y \le \infty \end{cases}$$

The conditional expectation

$$E(Y/X = x) = \int_0^\infty yf(y/x)dy$$

$$= \int_0^\infty y \frac{6(1+x+y)}{(2x+3)(1+y)^4} dy$$

$$= \frac{6}{(2x+3)} \int_0^\infty \frac{(1+y-1)(1+x+y)}{(1+y)^4} dy$$

$$= \frac{6}{(2x+3)} \int_0^\infty \frac{[(1+y)^2 + (x-1)(1+y) - x]}{(1+y)^4} dy$$

$$= \frac{6}{(2x+3)} \int_0^\infty [(1+y)^{-2} + (x-1)(1+y)^{-3} - x(1+y)^{-4}] dy$$

$$= \frac{6}{(2x+3)} \left[-\frac{1}{1+y} + (x-1)\left(\frac{-1}{2(1+y)^2}\right) + \frac{x}{3(1+y)^3} \right]_0^\infty$$

$$= \frac{6}{(2x+3)} \left(1 + \frac{x-1}{2} - \frac{x}{3}\right)$$

$$\therefore \quad E(Y/X = x) = \frac{6}{(2x+3)} \left(\frac{6+3x-3-2x}{6}\right) = \frac{x+3}{2x+3}$$

5.3 COVARIANCE

If (X, Y) is a two-dimensional random variable with $E(X)$ and $E(Y)$ finite, then the covariance of (X, Y) denoted by $\text{Cov}(X,Y)$ is defined as

$$\text{Cov}(X, Y) = \{E[X - E(X)][Y - E(Y)]\} = E(XY) - E(X)E(Y)$$

Note:

(i) If X and Y are independent, then $\text{Cov}(X, Y) = 0$ but the converse need not be true.

(ii) $\text{Cov}(X + Y, X - Y) = \text{Var}(X) - \text{Var}(Y)$

(iii) $\text{Cov}(aX + bY, cX + dY) = ac\ \text{Var}(X) + bd\ \text{Var}(Y) + (ad + bc)\ \text{Cov}(X, Y)$

5.4 CORRELATION AND REGRESSION

In a bivariate distribution, we may be interested to find out if there is any correlation or covariance between the two variables under study. If the change in one variable affects a change in the other variable, the variables are said to be correlated. If the two variables deviate in the same direction, i.e. if the increase (or decrease) in one variable results in corresponding increase (or decrease) in the other, correlation is said to be direct or positive. But if they constantly deviate in the opposite directions, i.e. if the increase (or decrease) in one variable results in corresponding decrease (or increase) in the other, correlation is said to be inverse or negative.

It is to be noted that covariance is positive for positive correlation and negative for negative correlation.

Note: The closeness of relationship between two variables is not proportional to the correlation coefficient.

5.4.1 Karl Pearson Coefficient of Correlation

It is a numerical measure of intensity or degree of linear relationship between the two variables. Correlation coefficient between two random variables X and Y, denoted by ρ_{XY} (or r_{XY}) and is defined as

$$r(X,Y) = r_{XY} = \rho(X,Y) = \rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$$= \frac{E(XY) - E(X)E(Y)}{\sigma_X\ \sigma_Y}$$

If X and Y are discrete random variables taking values $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ respectively, then

$$\text{Cov}(X, Y) = \sum (x-\bar{x})(y-\bar{y}) - \bar{x}\,\bar{y} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y}$$

$$\text{Var}(X) = E[(x-\bar{x})^2] = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$$\text{Var}(Y) = E[(y-\bar{y})^2] = \frac{1}{n}\sum_{j=1}^{n} y_j^2 - \bar{y}^2$$

Properties

1. The correlation coefficient lies between -1 and 1, i.e.

$$\rho^2 \le 1 \quad \Rightarrow \quad -1 \le \rho \le 1$$

Proof We know that

$$E\left\{\left[\frac{X-E(X)}{\sigma_X}\right]\pm\left[\frac{Y-E(Y)}{\sigma_Y}\right]\right\}^2 \geq 0$$

$$\Rightarrow \quad E\left[\frac{X-E(X)}{\sigma_X}\right]^2 + E\left[\frac{Y-E(Y)}{\sigma_Y}\right]^2 \pm 2E\left\{\frac{[X-E(X)][Y-E(Y)]}{\sigma_X\sigma_Y}\right\} \geq 0$$

$$\Rightarrow \quad \frac{E(X^2)-[E(X)]^2}{\sigma_X{}^2} + \frac{E(Y^2)-[E(Y)]^2}{\sigma_Y{}^2} \pm 2\rho(X,Y) \geq 0$$

$$\Rightarrow \quad 1 + 1 \pm 2\rho(X, Y) \geq 0$$

$$\Rightarrow \quad -1 \leq \rho(X, Y) \leq 1$$

$$\Rightarrow$$

Hence the proof.

2. The correlation coefficient is independent of change of scale and origin.

Proof Let $U = \dfrac{X-a}{h}, V = \dfrac{Y-b}{k}$, where $a, b, h, k > 0$

Then,
$$X = hU + a, \; Y = kV + b$$

$$E(X) = hE(U) + a, \; E(Y) = kE(V) + b$$

$$\sigma_X^2 = E\{[X - E(X)]^2\} = E\{h^2[U - E(U)]^2 = h^2\sigma_U^2$$

$$\Rightarrow \quad \sigma_X = h\sigma_U$$

$$\sigma_Y^2 = E\{[Y - E(Y)]^2\} = E\{k^2[V - E(V)]^2\} = k^2\sigma_V^2$$

$$\Rightarrow \quad \sigma_Y = h\sigma_V$$

$$\begin{aligned}\text{Cov}(X, Y) &= E\{[X - E(X)]\,[Y - E(Y)]\}\\ &= E\{h[U - E(U)]\,k[V - E(V)]\}\\ &= hkE\{[U - E(U)]\,[V - E(V)]\}\\ &= hk\,\text{Cov}\,(U, V)\end{aligned}$$

$$\rho(X, Y) = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{hk\,\text{Cov}(U,V)}{h\sigma_U k\sigma_V} = \frac{\text{Cov}(U,V)}{\sigma_U\sigma_V} = \rho(U,V)$$

Hence the proof.

3. Two independent random variables are uncorrelated if $\text{Cov}(X, Y) = 0$. But the converse need not be true.

4. $E(X/Y = y)$ is known as the regression curve X on Y.

5. $E(Y/X = x)$ is known as the regression curve Y on X.

6. The correlation coefficient is also denoted by $r\,(X, Y) = r_{XY}$.

EXAMPLE 5.53 Let X and Y be any two random variables and a, b be constants. Prove that $\text{Cov}(aX, bY) = ab\,\text{Cov}(X, Y)$. **[AU December '08]**

Solution By definition,

$$\begin{aligned}\text{Cov}(X, Y) &= E(XY) - E(X)\,E(Y)\\ \text{Cov}(aX, bY) &= E(aX\,bY) - E(aX)E(bY)\\ &= abE(XY) - aE(X)bE(Y)\\ &= ab[E(XY) - E(X)E(Y)]\\ &= ab\ \text{Cov}(X, Y)\end{aligned}$$

Hence the proof.

EXAMPLE 5.54 If $Y = -2X + 3$, find the $\text{Cov}(X, Y)$. **[AU May '08]**

Solution By definition,

$$\begin{aligned}\text{Cov}(X, Y) &= E(XY) - E(X)E(Y)\\ &= E[X(-2X + 3)] - E(X)\,E(-2X + 3)\\ &= E(-2X^2 + 3X) - E(X)\,[(-2)\,E(X) + 3]\\ &= -2E(X^2) + 3E(X) + 2[E(X)]^2 - 3E(X)\\ &= -2E(X^2) + 2[E(X)]^2\\ &= -2\{E(X^2) - [E(X)]^2\}\\ &= -2\ \text{Var}(X)\end{aligned}$$

EXAMPLE 5.55 Two random variables X and Y have joint PDF

$$f(x, y) = \begin{cases}\dfrac{xy}{96}, & 0 < x < 4, 1 < y < 5\\ 0, & \text{otherwise}\end{cases}$$

Find $E(X)$, $E(Y)$, $E(XY)$, $E(2X + 3Y)$, $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Cov}(X, Y)$. What can you infer from $\text{Cov}(X, Y)$? **[AU May '06]**

Solution Given: $f(x, y) = \dfrac{xy}{96}, 0 < x < 4, 1 < y < 5$

The marginal density function of X is

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} f(x, y)dy = \frac{1}{96}\int_1^5 xy\,dy\\ &= \frac{x}{96}\left[\frac{y^2}{2}\right]_1^5 = \frac{x}{96}\left(\frac{25-1}{2}\right) = \frac{x}{8}\end{aligned}$$

$\therefore$
$$f(x) = \frac{x}{8}, \quad 0 < x < 4$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \frac{1}{96}\int_0^4 xy\,dx = \frac{y}{96}\left[\frac{x^2}{2}\right]_0^4 = \frac{y}{12}$$

$$f(y) = \frac{y}{12}, \qquad 1 < y < 5$$

$\therefore$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^4 x \cdot \frac{x}{8} dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy = \int_1^5 y \cdot \frac{y}{12} dy = \frac{1}{12}\left[\frac{y^3}{3}\right]_1^5 = \frac{31}{9}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dxdy$$

$$= \int_1^5\int_0^4 xy\, \frac{xy}{96} dxdy$$

$$= \frac{1}{96}\int_1^5 y^2 \left[\frac{x^3}{3}\right]_0^4 dy = \frac{1}{96} \times \frac{64}{3}\int_1^5 y^2\, dy$$

$$= \frac{64}{3 \times 96}\left[\frac{y^3}{3}\right]_1^5 = \frac{64}{3 \times 96}\left(\frac{125 - 1}{3}\right)$$

$$= \frac{124}{3} \times \frac{64}{288} = \frac{248}{27}$$

$$E(2X + 3Y) = 2E(X) + 3E(Y)$$

$$= 2 \times \frac{8}{3} + 3 \times \frac{31}{9} = \frac{47}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^4 x^2 \frac{x}{8} dx$$

$$= \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4 = 8$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y) dy = \int_1^5 y^2 \frac{y}{12} dy$$

$$= \frac{1}{12}\left[\frac{y^4}{4}\right]_1^5 = \frac{1}{48}(5^4 - 1) = 13$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 8 - \left(\frac{8}{3}\right)^2 = \frac{8}{9}$$

$$\sigma_X^2 = \frac{8}{9} \Rightarrow \sigma_X = \frac{2\sqrt{2}}{3}$$

$$\therefore \quad \text{Var}(Y) = \sigma_Y^2 = 13 - \left(\frac{31}{9}\right)^2 = \frac{92}{81} \Rightarrow \sigma_Y^2 = \frac{2\sqrt{23}}{9}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= \frac{248}{27} - \frac{8}{3} \times \frac{31}{9} = 0$$

Since $\text{Cov}(X, Y) = 0$, X and Y are uncorrelated.

EXAMPLE 5.56 Two random variables X and Y have the following joint PDF:

$$f(x, y) = \begin{cases} 2 - x - y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find Var(X) and Var(Y) and also find the covariance between X and Y.

[AU December '04, '05; June '06]

Solution The marginal PDF of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 (2 - x - y)dy$$

$$= \left[2y - xy - \frac{y^2}{2}\right]_0^1 = 2 - x - \frac{1}{2} = \frac{3}{2} - x$$

Similarly, the marginal PDF of Y is given by

$$f_Y(y) = \int_0^1 f(x, y)dx = \int_0^1 (2 - x - y)\,dx = \left[2x - \frac{x^2}{2} - xy\right]_0^1$$

$$= \frac{3}{2} - y$$

Now, $$E(X) = \int_0^1 x f_X(x)dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \int_0^1 \left(\frac{3x}{2} - x^2\right)dx$$

$$= \left[\frac{3}{2}\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$$

Similarly,

$$E(Y) = \int_0^1 y f_Y(y)dy = \int_0^1 y\left(\frac{3}{2} - y\right)dy = \left(\frac{3}{2}\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = \frac{5}{12}$$

and

$$E(X^2) = \int_0^1 x^2 f_X(x)dx = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx = \int_0^1\left(\frac{3x^2}{2} - x^3\right)dx$$

$$= \left[\frac{3}{2}\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_0^1 = \frac{3}{6} - \frac{1}{4} = \frac{6-3}{12} = \frac{1}{4}$$

Similarly,

$$E(Y^2) = \int_0^1 y^2 f_Y(y)dy = \int_0^1 y^2\left(\frac{3}{2} - y\right)dy = \left(\frac{3}{2}\frac{y^3}{3} - \frac{y^4}{4}\right)_0^1 = \frac{1}{4}$$

Now, $\quad \mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144} = \frac{11}{144}$$

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$E(XY) = \int_0^1\int_0^1 xy\, f(x, y)\,dxdy = \int_0^1\int_0^1 xy(2 - x - y)\,dy$$

$$= \int_0^1\int_0^1 (2xy - x^2y - xy^2)\,dxdy$$

$$= \int_0^1\left[\frac{2x^2}{2}y - \frac{x^3}{3}y - \frac{x^2}{2}y^2\right]_0^1 dy$$

$$= \int_0^1\left(y - \frac{1}{3}y - \frac{1}{2}y^2\right)dy = \int_0^1\left(\frac{2}{3}y - \frac{1}{2}y^2\right)dy$$

$$= \left[\frac{2}{3}\left(\frac{y^2}{2}\right) - \frac{1}{2}\frac{y^3}{3}\right]_0^1 = \frac{2}{6} - \frac{1}{6} = \frac{1}{6}$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)\,E(Y)$$

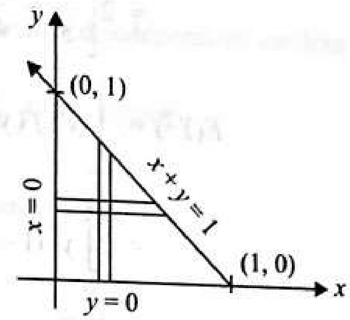
$$= \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12} = \frac{1}{6} - \frac{25}{144} = \frac{24-25}{144} = -\frac{1}{144}$$

EXAMPLE 5.57 Find the correlation between X and Y if the joint PDF of X and Y is

$$f(x, y) = \begin{cases} 2, & x > 0,\ y > 0,\ x + y < 1 \\ 0, & \text{elsewhere} \end{cases}$$ **[AU April '08]**

Solution By definition,

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$



As we are integrating first with respect to y, take a strip parallel to y-axis. Along the strip, y varies from $y = 0$ to $y = 1 - x$. The strip moves from $x = 0$ to $x = 1$ to cover the entire region.

$$f(x) = \int_{0}^{1-x} 2\, dy = 2[y]_0^{1-x} = 2(1-x), \text{ [refer Figure 5.3]}$$

$$\therefore \quad f(x) = 2(1 - x), \quad 0 < x < 1$$

Now, $$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

As we are integrating first with respect to x, take a strip parallel to x-axis. Along the strip, x varies from $x = 0$ to $x = 1 - y$ and the strip moves from $y = 0$ to $y = 1$ to cover the entire region.

$$f(y) = \int_{0}^{1-y} 2\, dx = 2\,[x]_0^{1-y} = 2(1-y), \quad 0 < y < 1$$

$$\therefore \quad f(y) = 2(1 - y), \quad 0 < y < 1$$

As $$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{1} x\, 2(1-x)\, dx$$

$$= 2\int_0^1 (x - x^2)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy = 2\int_0^1 y\,(1-y)\, dy$$

$$= 2\int_0^1 (y - y^2)\, dy = 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x, y) dx = 2\int_0^1 x^2 (1-x)\, dx$$

$$= 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\, dy$$

$$= 2\int_0^1 y^2(1-y)\, dy = 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{6}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dx\, dy$$

$$= \int_0^1\int_0^{1-x} 2xy\, dy\, dx = 2\int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 x[y^2]_0^{1-x} dx = \int_0^1 x(1-x)^2\, dx = \int_0^1 (x - 2x^2 + x^3)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$$

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \frac{1}{6} - \frac{1}{9} = \frac{3}{54} = \frac{1}{18}$$

Similarly, it can be shown that

$$\sigma_Y^2 = \frac{1}{18}$$

∴ The correlation coefficient

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{12} - \frac{1}{3} \times \frac{1}{3}}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}} = \frac{\frac{1}{12} - \frac{1}{9}}{\frac{1}{18}} = \frac{-3}{108} \times 18 = \frac{-1}{2}$$

$$\therefore \quad \rho_{XY} = \frac{-1}{2}$$

EXAMPLE 5.58 If X and Y are two uniformly distributed independent random variables over the triangular region R bounded by $y = 0$, $x = 3$ and $y = \frac{4}{3}x$, find $f_X(x)$, $f_Y(y)$, $E(X)$, $E(Y)$, Var(X), Var(Y) and ρ_{XY}.

Solution Given: X and Y are two uniformly distributed independent random variables. So, we assume $f(x, y) = k$.

We know that the total probability is 1 in the given region.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dxdy = 1 \quad\Rightarrow\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\,dxdy = 1$$

$$\int_{0}^{4}\int_{\frac{3}{4}y}^{3} k\,dx\,dy = 1 \quad\Rightarrow\quad k\int_{0}^{4}[x]_{\frac{3}{4}y}^{3}\,dy = 1, \text{ [from Figure 5.18]}$$

$$k\int_{0}^{4}\left(3-\frac{3}{4}y\right)dy = 1 \quad\Rightarrow\quad k\left[3y-\frac{3y^2}{8}\right]_0^4 = 1$$

$$k\left(12-\frac{48}{8}\right) = 1 \Rightarrow k(6) = 1$$

i.e., $$k = \frac{1}{6}$$

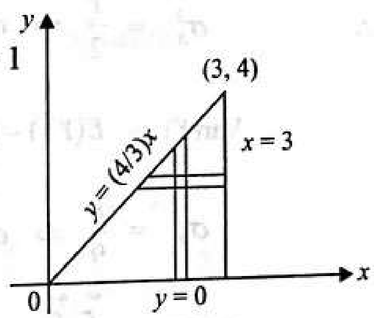


Figure 5.18

$$f_X(x) = \int_{0}^{\frac{4}{3}x}\frac{1}{6}\,dy = \frac{1}{6}[y]_0^{\frac{4}{3}x} = \frac{1}{6}\cdot\frac{4}{3}x,$$

[refer Figure 5.18]

$$= \frac{2}{9}x, \quad 0 \le x \le 3$$

$$f_Y(y) = \int_{\frac{3}{4}y}^{3}\frac{1}{6}\,dx = \frac{1}{6}[x]_{\frac{3}{4}y}^{3} = \frac{1}{6}\left(3-\frac{3}{4}y\right)$$

$$= \frac{1}{2}\left(1-\frac{y}{4}\right) = \left(\frac{1}{2}-\frac{y}{8}\right), \quad 0 \le y \le 4$$

$$E(X) = \int_{0}^{3} x f_X(x)\,dx = \int_{0}^{3} x\cdot\frac{2}{9}x\,dx = \frac{2}{9}\int_{0}^{3} x^2\,dx$$

$$= \frac{2}{9}\left[\frac{x^3}{3}\right]_0^3 = \frac{2}{9}\left(\frac{3^3}{3}\right) = 2$$

$$E(Y) = \int_0^4 y f_Y(y)dy = \int_0^4 y\cdot\left(\frac{1}{2}-\frac{y}{8}\right)dy = \int_0^4\left(\frac{y}{2}-\frac{y^2}{8}\right)dy$$

$$= \left(\frac{y^2}{4}-\frac{y^3}{24}\right)_0^4 = \left(\frac{4^2}{4}-\frac{4^3}{24}\right) = 4-\frac{4\times4\times4}{24} = \frac{4}{3}$$

$$\therefore \quad E(Y) = \frac{4}{3}$$

$$E(X^2) = \int_0^3 x^2\cdot\frac{2}{9}x\,dx = \frac{2}{9}\left[\frac{x^4}{4}\right]_0^3 = \frac{2}{9}\cdot\frac{3^4}{4} = \frac{9}{2}$$

$$E(Y^2) = \int_0^4\left(\frac{y^2}{2}-\frac{y^3}{8}\right)dy = \left[\frac{y^3}{6}-\frac{y^4}{32}\right]_0^4 = \frac{4^3}{6}-\frac{4^4}{32} = \frac{8}{3}$$

$$\text{Var}(X) = E(X^2)-[E(X)]^2 = \frac{9}{2}-(2)^2 = \frac{9}{2}-4 = \frac{1}{2}$$

$$\therefore \quad \sigma_X^2 = \frac{1}{2} \Rightarrow \sigma_X = \frac{1}{\sqrt{2}}$$

$$\text{Var}(Y) = E(Y^2)-[E(Y)]^2 = \frac{8}{3}-\left(\frac{4}{3}\right)^2 = \frac{8}{9}$$

$$\sigma_Y^2 = \frac{8}{9} \Rightarrow \sigma_Y = \frac{2\sqrt{2}}{3}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dxdy$$

$$= \int_0^4\int_{\frac{3}{4}y}^{3}\frac{1}{6}xy\,dxdy = \int_0^4\left[\frac{x^2}{2}\right]_{\frac{3}{4}y}^{3}\frac{y}{6}dy$$

$$= \frac{1}{6}\int_0^4 y\left[\frac{3^2}{2}-\frac{\left(\frac{3}{4}y\right)^2}{2}\right]dy$$

$$= \frac{1}{6}\left(\frac{9y^2}{4}-\frac{\left(\frac{3}{4}\right)^2 y^4}{8}\right)_0^4 = \frac{1}{6}\left[\frac{9\times4^2}{4}-\frac{\left(\frac{3}{4}\right)^2 4^4}{8}\right]$$

$$= \frac{1}{6}(36-18) = \frac{1}{6}\times18 = 3$$

We know that

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

$$\therefore \qquad \rho_{XY} = \frac{3 - 2 \times \frac{4}{3}}{\frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3}} = \frac{3 - \frac{8}{3}}{\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

EXAMPLE 5.59 Let X and Y be the random variables having joint PDF

$$f(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the correlation coefficient ρ_{XY}. **[AU November '06]**

Solution We know that

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)\,dxdy = \int_0^1\int_0^1 xy\frac{3}{2}(x^2 + y^2)\,dxdy$$

$$= \frac{3}{2}\int_0^1\int_0^1 (x^3 y + xy^3)\,dxdy = \frac{3}{2}\int_0^1 \left[\frac{x^4}{4}y + \frac{x^2}{2}y^3\right]_0^1 dy$$

$$= \frac{3}{2}\int_0^1 \left(\frac{y}{4} + \frac{y^3}{2}\right)dy = \frac{3}{2}\left[\frac{y^2}{8} + \frac{y^4}{8}\right]_0^1 = \frac{3}{16}(1+1) = \frac{3}{8}$$

The marginal PDF of X is

$$f(x) = \int_0^1 f(x, y)\,dy = \frac{3}{2}\int_0^1 (x^2 + y^2)\,dy$$

$$= \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left(x^2 + \frac{1}{3}\right)$$

$$= \frac{3}{6}(3x^2 + 1) = \frac{1}{2}(3x^2 + 1)$$

Similarly, we can find the PDF of Y as

$$f(y) = \frac{1}{2}(3y^2 + 1)$$

Now,

$$E(X) = \int_0^1 xf(x)\,dx = \int_0^1 x\frac{1}{2}(3x^2 + 1)\,dx$$

$$= \frac{1}{2}\int_0^1 (3x^3 + x)\,dx = \frac{1}{2}\left[\frac{3x^4}{4} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{3}{4} + \frac{1}{2}\right) = \frac{5}{8}$$

Similarly,
$$E(Y) = \frac{5}{8}$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \frac{1}{2}(3x^2 + 1)\,dx$$

$$= \frac{1}{2}\int_0^1 (3x^4 + x^2)dx = \frac{1}{2}\left[\frac{3x^5}{5} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{3}{5} + \frac{1}{3}\right) = \frac{14}{30} = \frac{7}{15}$$

Similarly,
$$E(Y^2) = \frac{7}{15}$$

$$\text{Var}(X) = E(X^2) - [E(X)^2]$$

$$= \frac{7}{15} - \left(\frac{5}{8}\right)^2 = \frac{7}{15} - \frac{25}{64} = \frac{73}{960}$$

$\therefore$
$$\sigma_X = \sqrt{\frac{73}{960}}$$

Similarly,

$$\sigma_Y = \sqrt{\frac{73}{960}}$$

$$\rho(X, Y) = \frac{E(XY) - E(X)E(Y)}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{\frac{3}{8} - \left(\frac{5}{8} \times \frac{5}{8}\right)}{\sqrt{\frac{73}{960}} \times \sqrt{\frac{73}{960}}} = \frac{\frac{3}{8} - \frac{25}{64}}{\frac{73}{960}} = \frac{24 - 25}{64} \times \frac{960}{73} = -0.2055$$

EXAMPLE 5.60 If the joint PDF of (X, Y) is given by $f(x, y) = x + y$, $0 \le x, y \le 1$, find ρ_{XY}. **[AU May '03, December '03, November '03; '04]**

Solution By definition,

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} \quad \text{(i)}$$

Now, $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)\,dxdy = \int_0^1\int_0^1 xy(x+y)\,dxdy$$

$$= \int_0^1\int_0^1 (x^2y + xy^2)\,dxdy = \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{y}{3} + \frac{y^2}{2}\right] dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Also, the joint PDF of X and Y is given by

$$f_X(x) = \int_0^1 f(x, y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$$

$$f_Y(y) = \int_0^1 f(x, y)\,dx = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2} + xy\right]_0^1 = y + \frac{1}{2}$$

Now, $$E(X) = \int_0^1 xf_X(x)\,dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx = \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(Y) = \int_0^1 yf_Y(y)\,dy = \int_0^1 y\left(y + \frac{1}{2}\right)dy = \int_0^1 \left(y^2 + \frac{y}{2}\right)dy = \frac{7}{12}$$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2\left(x + \frac{1}{2}\right)dx = \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Similarly,

$$E(Y^2) = \frac{5}{12}$$

$$\text{Var}(X) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

Also $$\text{Var}(Y) = \frac{11}{144}$$

and so $$\sigma_X = \sigma_Y = \frac{\sqrt{11}}{12}$$

Hence Eq. (i)

$$\Rightarrow \qquad \rho_{XY} = \frac{\frac{1}{3} - \frac{7}{12} \times \frac{7}{12}}{\frac{\sqrt{11}}{12} \times \frac{\sqrt{11}}{12}} = \frac{-1}{144} \times \frac{144}{11} = \frac{-1}{11}$$

EXAMPLE 5.61 Find the correlation coefficient ρ_{XY} for the bivariate random variable (X, Y) having the joint probability function

$$f(x, y) = \begin{cases} 2xy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

[AU December '09]

Solution The correlation coefficient

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} \tag{i}$$

Given: $f(x, y) = \begin{cases} 2xy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 2xy\, dy$$

$$= 2x\left[\frac{y^2}{2}\right]_0^1 = x$$

$\therefore$ $$f(x) = x, \quad 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 2xy\, dx$$

$$= 2y\left[\frac{x^2}{2}\right]_0^1 = y$$

$\therefore$ $$f(y) = y, \quad 0 < y < 1$$

$\therefore$ $$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 xf(x)\,dx$$

$$= \int_0^1 x \cdot x\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E(Y) = \int_{-\infty}^{\infty} y\,f(y)\,dy = \int_0^1 y \cdot y\, dy = \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)\,dxdy$$

$$= \int_0^1\int_0^1 xy\, 2xy\, dxdy$$

$$= 2\int_0^1\int_0^1 x^2 y^2 dx dy$$

$$= 2\int_0^1 y^2 \left[\frac{x^3}{3}\right]_0^1 dy = \frac{2}{3}\int_0^1 y^2 dy = \frac{2}{3}\int_0^1 \left[\frac{y^3}{3}\right]_0^1 = \frac{2}{9}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 . x\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 y^2 \cdot y\,dy = \left[\frac{y^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\sigma_X^2 = E(X^2) - [E(X)]^2$$

$$= \frac{1}{4} - \left(\frac{1}{3}\right)^2 = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$$

$$\therefore \quad \sigma_X = \frac{\sqrt{5}}{6}$$

$$\sigma_Y^2 = E(Y^2) - [E(Y)]^2$$

$$= \frac{1}{4} - \left(\frac{1}{3}\right)^2 = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$$

$$\therefore \quad \sigma_Y = \frac{\sqrt{5}}{6}$$

Substituting these values in Eq. (i)

$$\rho_{XY} = \frac{\frac{2}{9} - \frac{1}{9}}{\frac{\sqrt{5}}{6} \times \frac{\sqrt{5}}{6}} = \frac{1}{9} \times \frac{36}{5} = \frac{4}{5}$$

$$\therefore \quad \rho_{XY} = 0.8$$

EXAMPLE 5.62 Let the joint probability density function of (X, Y) be $f(x, y) = e^{-y}$, $0 < x < y < \infty$. Find the correlation coefficient $r(X, Y)$.

[AU December '08]

Solution The correlation coefficient

$$r_{XY} = \frac{E(XY) - E(X)\,E(Y)}{\sigma_X \sigma_Y}$$

First we find the marginal density functions $f(x)$ and $f(y)$.

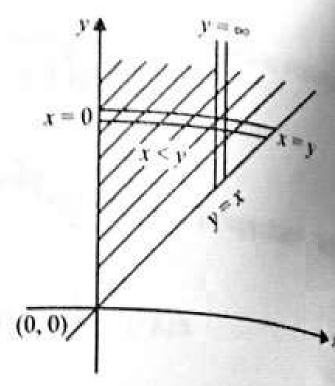


Figure 5.19

To find $f(x)$, we integrate with respect to y. So, take a strip parallel to y axis. Along the strip, y varies from $y = x$ to $y = \infty$ and the strip moves from $x = 0$ to $x = \infty$ to cover the entire region.

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_{x}^{\infty} e^{-y}\,dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_x^{\infty} = (-e^{-\infty} + e^{-x})$$

$$= e^{-x},\ 0 < x < \infty \quad (e^{-\infty} = 0)$$

To find $f(y)$ we integrate with respect to x.

So, take a strip parallel to x axis. Along the strip x varies from 0 to $x = y$ and the strip moves from $y = 0$ to $y = \infty$ to cover the entire region.

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_{0}^{y} e^{-y}\,dx = e^{-y}[x]_0^y = ye^{-y}$$

$$f(y) = ye^{-y},\ 0 < y < \infty$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{0}^{\infty} xe^{-x}\,dx = \left[x\left(\frac{e^{-x}}{-1}\right) - 1(e^{-x})\right]_0^{\infty} = 1$$

$$E(Y) = \int_{-\infty}^{\infty} yf(y)\,dy = \int_{0}^{\infty} y \cdot y \cdot e^{-y}\,dy$$

$$= \left[y^2\left(\frac{e^{-y}}{-1}\right) - 2y\left(\frac{e^{-y}}{(-1)^2}\right) + 2\left(\frac{e^{-y}}{(-1)^3}\right)\right]_0^{\infty}$$

$$E(Y) = 2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{\infty} x^2 e^{-x}\,dx$$

$$= \left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x(e^{-x}) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$$

$$\therefore \quad E(X^2) = 2$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_{0}^{\infty} y^3 e^{-y}\,dy$$

$$= \left[y^3\left(\frac{e^{-y}}{-1}\right) - 3y^2(e^{-y}) + 6y\left(\frac{e^{-y}}{-1}\right) - 6(e^{-y})\right]_0^{\infty}$$

$$= 6$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy$$

As we are integrating first with respect to x, take a strip parallel to x axis. Along the strip, x varies from 0 to y. As the strip moves from $y = 0$ to $y = \infty$, the required region will be covered.

$$E(XY) = \int_0^{\infty}\int_0^{y} e^{-y}\,dx\,dy = \int_0^{\infty} e^{-y}[x]_0^y\,dy = \int_0^{\infty} ye^{-y}\,dy$$

$$= \left[y\left(\frac{e^{-y}}{-1}\right) - 1\,e^{-y}\right]_0^{\infty} = 1$$

$$\text{Var}(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

$$= 2 - 1 = 1$$

$$\sigma_X = 1$$

$$\text{Var}(Y) = \sigma_Y^2 = E(Y^2) - [E(Y)]^2$$

$$= 6 - 2^2 = 2$$

$$\sigma_Y = \sqrt{2}$$

$$r(X, Y) = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

$$= \frac{1 - 1 \times 2}{1 \times \sqrt{2}} = \frac{-1}{\sqrt{2}} = -0.7071$$

$$|r(X, Y)| = 0.7071$$

EXAMPLE 5.63 The independent variables X and Y have the probability density functions given by

$$f_X(x) = \begin{cases} 4ax, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 4by, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the correlation coefficient between $X + Y$ and $X - Y$.

Solution If $f(x)$ and $f(y)$ are the PDF of X and Y, then we have

$$\int_0^1 f(x)dx = 1 \quad \Rightarrow \quad \int_0^1 4ax\,dx = 1 \Rightarrow 4a\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$\Rightarrow \qquad 4\frac{a}{2} = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$

$$\int_0^1 f(y)dy = 1 \quad \Rightarrow \quad \int_0^1 4by\,dy = 1$$

$$\Rightarrow \quad 4b\left[\frac{y^2}{2}\right]_0^1 = 1 \quad \Rightarrow \quad b = \frac{1}{2}$$

$$\therefore \quad f(x) = 2x, \qquad 0 \le x \le 1$$
$$f(y) = 2y, \qquad 0 \le y \le 1$$

As X and Y are independent variables, $\text{Cov}(X, Y) = 0$. Let $U = X + Y$, $V = X - Y$ then

$$\begin{aligned}
\text{Cov}(U, V) &= \text{Cov}(X + Y, X - Y) \\
&= \text{Cov}(X, X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Cov}(Y, Y) \\
&= E(X^2) - [E(X)]^2 - 0 + 0 - E(Y^2) + [E(Y)]^2 \\
&= \sigma_X^2 - \sigma_Y^2
\end{aligned}$$

$$\begin{aligned}
\text{Var}(U) &= E(U^2) - [E(U)]^2 = E(X + Y)^2 - [E(X + Y)]^2 \\
&= E(X^2 + Y^2 + 2XY) - [E(X) + E(Y)]^2 \\
&= E(X^2) + E(Y^2) + 2E(X)\,E(Y) - [E(X)]^2 - [E(Y)]^2 - 2E(X)E(Y)
\end{aligned}$$

$$\text{Var}(U) = \sigma_X^2 + \sigma_Y^2$$

Similarly,

$$\begin{aligned}
\text{Var}(V) &= E(X - Y)^2 - [E(X - Y)]^2 \\
&= E(X^2) + E(Y^2) - 2E(X)\,E(Y) - [E(X)]^2 - [E(Y)]^2 + 2E(X)\,E(Y) \\
&= \sigma_X^2 + \sigma_Y^2
\end{aligned}$$

$$\therefore \quad \rho(U, V) = \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U)\text{Var}(V)}}$$

$$\rho_{UV} = \frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} \tag{i}$$

$$E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 2x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$E(Y) = \int_0^1 y\,f(y)\,dy = \int_0^1 2y^2\,dy = 2\left[\frac{y^3}{3}\right]_0^1 = \frac{2}{3}$$

$$E(X^2) = \int_0^1 x^2\,f(x)\,dx = \int_0^1 2x^3 dx = 2\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

$$E(Y^2) = \int_0^1 y^2\,f(y)\,dy = \int_0^1 2y^3\,dy = 2\left[\frac{y^4}{4}\right]_0^1 = \frac{1}{2}$$

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$\therefore$

$$= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Substituting in Eq. (i), we get

$\therefore$

$$\rho_{UV} = \frac{\frac{1}{18} - \frac{1}{18}}{\frac{1}{18} + \frac{1}{18}} = 0$$

$\therefore$ The correlation coefficient between U and V is 0.

EXAMPLE 5.64 If X, Y, Z are uncorrelated random variables having same variance, find the correlation coefficient between $(X + Y)$ and $(Y + Z)$.

Solution Let

$$U = X + Y$$

Given:

$$\text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = \sigma^2$$

$$V = Y + Z$$

$$\rho = \frac{E(UV) - E(U)E(V)}{\sigma_U \, \sigma_V}$$

$$E(U) = E(X + Y) = E(X) + E(Y)$$

$$E(V) = E(Y + Z) = E(Y) + E(Z)$$

$$\begin{aligned} E(UV) &= E[(X + Y)(Y + Z)] \\ &= E(XY + XZ + Y^2 + YZ) \\ &= E(XY) + E(XZ) + E(Y^2) + E(YZ) \\ &= E(X)E(Y) + E(X)E(Z) + E(Y^2) + E(Y)E(Z) \end{aligned}$$

($\because$ X, Y and Z are independent)

$$\begin{aligned} \text{Var}(U) &= E(U^2) - [E(U)]^2 \\ &= E[(X + Y)^2] - [E(X + Y)]^2 \\ &= E(X^2) + E(Y^2) + 2E(XY) - [E(X)]^2 - [E(Y)]^2 - 2E(X)E(Y) \\ &= E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2 \\ &= \text{Var}(X) + \text{Var}(Y) = \sigma^2 + \sigma^2 = 2\sigma^2 \end{aligned}$$

Simillarly,

$$\text{Var}(V) = 2\sigma^2$$

$$\rho_{UV} = \frac{E(UV) - E(U)E(V)}{\sigma_U \, \sigma_V}$$

$$\rho = \frac{E(X)E(Y) + E(X)E(Z) + E(Y^2) + E(Y)E(Z) - [E(X) + E(Y)][E(Y) + E(Z)]}{2\sigma^2}$$

$$= \frac{E(Y^2) - [E(Y)]^2}{2\sigma^2} = \frac{\text{Var}(Y)}{2\sigma^2} = \frac{\sigma^2}{2\sigma^2} = \frac{1}{2}$$

EXAMPLE 5.65 If the independent random variables X and Y have variance 36 and 16 respectively, find the correlation coefficient between $(X + Y)$ and $(X - Y)$.

[AU December '06, April '08]

Solution Given: $\text{Var}(X) = 36$

and $\text{Var}(Y) = 16$

Let $U = X + Y$

and $V = X - Y$

$$\rho = \frac{E(UV) - E(U)E(V)}{\sigma_U\, \sigma_V}$$

$$E(U) = E(X + Y) = E(X) + E(Y)$$

$$E(V) = E(X - Y) = E(X) - E(Y)$$

$$E(UV) = E[(X + Y)(X - Y)] = E(X^2 - Y^2)$$

$$= E(X^2) - E(Y^2)$$

$$\sigma_U^2 = E(U^2) - [E(U)]^2$$

$$= E[(X + Y)^2] - [E(X + Y)]^2$$

$$= E(X^2) + E(Y^2) + 2E(XY) - \{[E(X)]^2 - [E(Y)]^2 + 2E(X)\,E(Y)]\}$$

$$= E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2$$

$$= \text{Var}(X) + \text{Var}(Y)$$

$$= 36 + 16 = 52$$

$$\sigma_U = \sqrt{52}$$

$$\sigma_V^2 = E(V^2) - [E(V)]^2$$

$$= E[(X - Y)^2] - [E(X - Y)]^2$$

$$= E(X^2) + E(Y^2) - 2E(XY) - [E(X)]^2 - [E(Y)]^2 + 2E(X)\,E(Y)$$

$$= E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2$$

$$= \text{Var}(X) + \text{Var}(Y)$$

$$\sigma_V^2 = 52 \;\Rightarrow\; \sigma_V = \sqrt{52}$$

$$\rho_{UV} = \frac{E(UV) - E(U)E(V)}{\sigma_U\, \sigma_V}$$

$$= \frac{E(X^2) - E(Y^2) - [E(X) + E(Y)][E(X) - E(Y)]}{52}$$

$$= \frac{E(X^2) - [E(X)]^2 - E(Y^2) - [E(Y)]^2}{52}$$

$$= \frac{\text{Var}(X) - \text{Var}(Y)}{52} = \frac{36 - 16}{52} = \frac{20}{52} = \frac{5}{13}$$

$$\rho_{UV} = \frac{5}{13}$$

EXAMPLE 5.66 If X and Y are two independent random variables with zero mean, find when $U = X\cos\alpha + Y\sin\alpha$ and $V = X\sin\alpha - Y\cos\alpha$ are also independent.

Solution $E(U) = E(X)\cos\alpha + E(Y)\sin\alpha$

$E(V) = E(X)\sin\alpha - E(Y)\cos\alpha$

Given: $E(X) = 0$

$E(Y) = 0 \quad \Rightarrow \quad E(U) = 0$

and $E(V) = 0$

$E(XY) = E(X)\,E(Y) = 0$ $\qquad$ ($\because$ X and Y are independent)

To show that

$$E(UV) = E(U)\,E(V)$$

$$\begin{aligned}
E(UV) &= E[(X\cos\alpha + Y\sin\alpha)(X\sin\alpha - Y\cos\alpha)] \\
&= E(X^2)\cos\alpha\sin\alpha - E(XY)\cos^2\alpha \\
&\qquad + E(XY)\sin^2\alpha - E(Y^2)\sin\alpha\cos\alpha \\
&= E(X^2)\cos\alpha\sin\alpha - E(Y^2)\sin\alpha\cos\alpha \qquad [\because E(XY) = 0] \\
&= \cos\alpha\sin\alpha\,[E(X^2) - E(Y^2)] \\
E(U)E(V) &= E(X\cos\alpha + Y\sin\alpha)\,E(X\sin\alpha - Y\cos\alpha) \\
&= [E(X)\cos\alpha + E(Y)\sin\alpha]\,[E(X)\sin\alpha - E(Y)\cos\alpha] \\
&= 0 \qquad [\because E(X) = E(Y) = 0]
\end{aligned}$$

$E(UV) = E(U)\,E(V) = 0$ only when $E(X^2) - E(Y^2) = 0$

$\Rightarrow \quad E(X^2) = E(Y^2)$

$\therefore$ U and V are independent only when X and Y have equal variance.

EXAMPLE 5.67 Two random variables X and Y are related as $Y = 4X + 9$. Find the correlation coefficient between X and Y. **[AU 2007]**

Solution Given:

$$\begin{aligned}
Y &= 4X + 9 \\
E(Y) &= E(4X + 9) = 4E(X) + 9 \\
E(XY) &= E[X(4X + 9)] = E(4X^2 + 9X) = 4E(X^2) + 9E(X) \\
\therefore \quad \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= [4E(X^2) + 9E(X)] - E(X)\,[4E(X) + 9] \\
&= 4E(X^2) + 9E(X) - 4[E(X)]^2 - 9E(X) \\
&= 4\{E(X^2) - [E(X)]^2\} \\
&= 4\sigma_X^2
\end{aligned}$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{4\sigma_X^2}{\sigma_X\sigma_Y} = 4\frac{\sigma_X}{\sigma_Y} \qquad \text{(i)}$$

But, $\sigma_Y^2 = E(Y^2) - [E(Y)]^2$

$= E[(4X+9)^2] - [E(4X+9)]^2$

$= E(16X^2 + 81 + 72X) - [4E(X) + 9]^2$

$= 16E(X^2) + 81 + 72E(X) - 16[E(X)]^2 - 81 - 72E(X)$

$= 16\{E(X^2) - [E(X)]^2\} = 16\sigma_X^2$

$\Rightarrow \quad \sigma_Y = 4\sigma_X$

Substituting in Eq. (i), we get

$$\rho_{XY} = 4\frac{\sigma_X}{4\sigma_X} = 1$$

EXAMPLE 5.68 X, Y and Z are uncorrelated random variables with 0 mean and SD 5, 12 and 9 respectively. If $U = X + Y$ and $V = Y + Z$, find the correlation coefficient between U and V. **[AU April '03]**

Solution Given: $E(X) = E(Y) = E(Z) = 0$

and $E(U) = E(X) + E(Y) = 0$

$E(V) = E(Y) + E(Z) = 0$

$E(UV) = E[(X+Y)(Y+Z)]$

$= E(XY + XZ + Y^2 + YZ)$

$= E(XY) + E(XZ) + E(Y^2) + E(YZ)$

$\because$ X, Y and Z are uncorrelated

$\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, Z) = \mathrm{Cov}(X, Z) = 0$

So, $E(XY) = E(X)\,E(Y) = 0$

and $E(XZ) = E(YZ) = 0$

$\because$ $E(UV) = E(Y^2)$

Given: $\mathrm{Var}(X) = \mathrm{SD}^2 = 5^2 = 25$, $\mathrm{Var}(Y) = 12^2 = 144$, (i)

$\mathrm{Var}(Z) = 9^2 = 81$

$$\rho = \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}(U)\mathrm{Var}(V)}} = \frac{E(UV) - E(U)E(V)}{\sigma_U\,\sigma_V} \qquad \text{(ii)}$$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 \Rightarrow 5^2 = E(X^2) - 0 \Rightarrow E(X^2) = 25$

Similarly,

and $E(Y^2) = 144$

$E(Z^2) = 81$

$\therefore$ From Eq. (i), $E(UV) = 144$

From Eq. (ii)
$$\rho = \frac{144 - 0}{\sigma_U \cdot \sigma_V} = \frac{144}{\sigma_U \cdot \sigma_V}$$

$$\sigma_U^2 = E(U^2) - [E(U)]^2$$
$$= E[(X + Y)^2] - 0$$
$$= E(X^2) + E(Y^2) + 2E(XY)$$
$$= 25 + 144 = 169$$

$\therefore$
$$\sigma_U = 13$$

$$\sigma_V^2 = E[(Y + Z)^2]$$
$$= E(Y^2) + E(Z^2) + 2E(YZ)$$
$$= 144 + 81 = 225$$

$\therefore$
$$\sigma_V = 15$$

Substituting in Eq. (ii), we get
$$\rho = \frac{144}{13 \times 15} = \frac{144}{195} = \frac{48}{65}$$

5.5 REGRESSION

The term 'regression' literally means 'stepping back towards the average'.

Regression analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of the data.

5.5.1 Line of Regression

The line of regression is the line which gives the best estimate to the value of one variable for any specific value of the other variable. Thus the line of regression is the line of best fit and is obtained by the principle of least squares. The regression equation of Y on X is obtained on minimizing the sum of the squares of the errors parallel to Y axis while the regression equation of X on Y is obtained on minimizing the sum of the squares of the errors parallel to X axis.

If the variables in a bivariate distribution are related, then the points in the scatter diagram (which is a diagrammatic representation of bivariate data) will cluster round some curve called the *curve of regression*. If the curve is a straight line, it is called the *line of regression* and said to be linear regression between the variables, otherwise regression is said to be curvilinear.

5.5.2 Equations of Lines of Regression

The regression line of y on x is

$$y - \bar{y} = \rho \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

which is used to predict or estimate the value of y for any given x.

The coefficient of x in the regression line of y on x is the regression coefficient of y on x and is denoted by

$$b_{yx} = \rho \frac{\sigma_y}{\sigma_x}$$

The regression line of x on y is

$$x - \bar{x} = \rho \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

which is used to predict or estimate the value of x for any given y.

The coefficient of y in the regression line of x on y is the regression coefficient of x on y and is denoted by

$$b_{xy} = \rho \frac{\sigma_x}{\sigma_y}$$

$\therefore$ The correlation coefficient ρ in terms of the regression coefficients is

$$\rho = \pm \sqrt{b_{xy}\, b_{yx}}$$

5.5.3 Properties of Correlation and Regression Coefficients

1. Correlation coefficient is the geometric mean between regression coefficients, i.e.

$$\rho = \pm \sqrt{b_{xy}\, b_{yx}} .$$

2. The correlation coefficient cannot numerically exceed the arithmetic mean between regression coefficient.
3. If one of the regression coefficients is greater than unity, the other must be less than unity numerically.
4. Regression coefficients are independent of the origin, but not scale.
5. The correlation coefficient and the two regression coefficients have the same sign.

The regression coefficients are obtained by the following expressions in case of discrete values of X and Y:

$$b_{yx} = \rho\frac{\sigma_y}{\sigma_x} = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{n\Sigma x^2 - (\Sigma x)^2} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

and

$$b_{xy} = \rho\frac{\sigma_x}{\sigma_y} = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{n\Sigma y^2 - (\Sigma y)^2} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

5.5.4 Angle between the Regression Lines

The angle between the regression lines is given by

$$\tan\theta = \frac{1-\rho^2}{\rho}\cdot\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

Note:

(i) When $r = \pm 1$, $\tan\theta = 0$, i.e. $\theta = 0$ or π. In this case, the two lines of regression coincide. When $\theta = \frac{\pi}{2}$, the lines of regression are perpendicular.

(ii) Whenever the two lines intersect, there are two angles between them, one acute angle and the other obtuse angle.

(iii) Both regression lines pass through the point $(\bar{x}, \bar{y})$ where $\bar{x}$ = mean of x and $\bar{y}$ = mean of y.

(iv) Regression curve of Y on X is $y = E(Y/X = x)$ and regression curve of X on Y is $x = E(X/Y = y)$.

EXAMPLE 5.69 Calculate the correlation coefficient for the following heights (in inches) of fathers X and their sons Y. **[AU June '06, November '07]**

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

Solution **Method 1:**

X	Y	XY	X^2	Y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\Sigma(X) = 544$	$\Sigma(Y) = 552$	$\Sigma XY = 37560$	$\Sigma X^2 = 37028$	$\Sigma Y^2 = 38132$

Now, $$\overline{X} = \frac{544}{8} = 68$$

$$\overline{Y} = \frac{552}{8} = 69$$

$$\overline{X}\overline{Y} = 68 \times 69 = 4692$$

$$\sigma_X = \sqrt{\frac{1}{n}\Sigma X^2 - \overline{X}^2} = \sqrt{\frac{37028}{8} - 4624} = 2.121$$

$$\sigma_Y = \sqrt{\frac{1}{n}\Sigma Y^2 - \overline{Y}^2} = \sqrt{\frac{38132}{8} - 4761} = 2.345$$

$$r(X, Y) = \frac{\frac{1}{n}\Sigma XY - \overline{X}\overline{Y}}{\sigma_X \sigma_Y} = \frac{\frac{1}{8} \times 37560 - 4692}{2.121 \times 2.345} = 0.6030$$

Note: Correlation coefficient is independent of change of origin and scale, i.e.

$$r(X, Y) = r(U, V)$$

where

$$U = \frac{X - a}{h}$$

and

$$V = \frac{Y - b}{K}$$

Here a and b are some arbitrary constants and usually the mid-values of the given data X and Y respectively.

Method 2:

X	Y	$U = X - 68$	$V = Y - 68$	UV	U^2	V^2
65	67	−3	−1	3	9	1
66	68	−2	0	0	4	0
67	65	−1	−3	3	1	9
67	68	−1	0	0	1	0
68	72	0	4	0	0	16
69	72	1	4	4	1	16
70	69	2	1	2	4	1
72	71	4	3	12	16	9
		$\Sigma U = 0$	$\Sigma V = 8$	$\Sigma UV = 24$	$\Sigma U^2 = 36$	$\Sigma V^2 = 52$

Now,

$$\overline{U} = \frac{\Sigma U}{n} = \frac{0}{8} = 0$$

$$\bar{V} = \frac{\Sigma V}{n} = \frac{8}{8} = 1$$

$$\text{Cov}(X, Y) = \text{Cov}(U, V)$$

$$= \frac{\Sigma UV}{n} - \bar{U}\bar{V} = \frac{24}{8} - 0 = 3$$

$$\sigma_U = \sqrt{\frac{1}{n}\Sigma U^2 - \bar{U}^2} = \sqrt{\frac{36}{8} - 0} = 2.121$$

$$\sigma_V = \sqrt{\frac{1}{n}\Sigma V^2 - \bar{V}^2} = \sqrt{\frac{52}{8} - 1} = 2.345$$

$$r(X, Y) = r(U, V)$$

$$= \frac{\text{Cov}(U, V)}{\sigma_U \cdot \sigma_V} = \frac{3}{2.121 \times 2.345}$$

$$= 0.6031$$

EXAMPLE 5.70 Find the coefficient of correlation between industrial production and export using the following data: **[AU May '06]**

Production (x)	55	56	58	59	60	60	62
Export (y)	35	38	37	39	44	43	44

Solution

x	y	$X = x - 58$	$Y = y - 40$	XY	X^2	Y^2
55	35	−3	−5	15	9	25
56	38	−2	−2	4	4	4
58	37	0	−3	0	0	9
59	39	1	−1	−1	1	1
60	44	2	4	8	4	16
60	43	2	3	6	4	9
62	44	4	4	16	16	16
		$\Sigma X = 4$	$\Sigma Y = 0$	$\Sigma XY = 48$	$\Sigma X^2 = 38$	$\Sigma Y^2 = 80$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{4}{7} = 0.5714$$

$$\bar{Y} = \frac{\Sigma Y}{n} = \frac{0}{7} = 0$$

$$\text{Cov}(X, Y) = \frac{\Sigma XY}{n} - \bar{X}\bar{Y} = \frac{48}{7} - 0 = 6.857$$

$$\sigma_X = \sqrt{\frac{\Sigma X^2}{n} - \bar{X}^2} = \sqrt{\frac{38}{7} - (0.5714)^2} = 2.2588$$

$$\sigma_Y = \sqrt{\frac{\Sigma Y^2}{n} - \overline{XY}} = \sqrt{\frac{80}{7} - 0} = 3.38$$

$$r(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y} = \frac{6.857}{2.258 \times 3.38} = 0.898$$

EXAMPLE 5.71 Find the correlation coefficient for the following data:

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Solution

x	y	$X = \frac{x-22}{4}$	$Y = \frac{y-24}{6}$	XY	X^2	Y^2
10	18	−3	−1	3	9	1
14	12	−2	−2	4	4	4
18	24	−1	0	0	1	0
22	6	0	−3	0	0	9
26	30	1	1	1	1	1
30	36	2	2	4	4	4
		$\Sigma X = -3$	$\Sigma Y = -3$	$\Sigma XY = 12$	$\Sigma X^2 = 19$	$\Sigma Y^2 = 19$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{-3}{6} = -0.5$$

$$\bar{Y} = \frac{\Sigma Y}{n} = \frac{-3}{6} = -0.5$$

$$\text{Cov}(X, Y) = \frac{\Sigma XY}{n} - \overline{XY}$$

$$= \frac{12}{6} - (-0.5)(-0.5) = \frac{7}{4}$$

$$\sigma_X = \sqrt{\frac{\Sigma X^2}{n} - \bar{X}^2} = \sqrt{\frac{19}{6} - (-0.5)^2} = 1.708$$

$$\sigma_Y = \sqrt{\frac{\Sigma Y^2}{n} - \overline{XY}} = \sqrt{\frac{19}{6} - (-0.5)^2} = 1.708$$

$$r(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y} = \frac{7/4}{1.708 \times 1.708} = 0.5999$$

EXAMPLE 5.72 Calculate the correlation coefficient and the lines of regression from the following data: **[AU December '04, June '06]**

x	62	64	65	69	70	71	72	74
y	126	125	139	145	165	152	180	208

Solution

x	y	$X = x - 70$	$Y = y - 165$	XY	X^2	Y^2
62	126	−8	−39	312	64	1521
64	125	−6	−40	240	36	1600
65	139	−5	−26	130	25	676
69	145	−1	−20	20	1	400
70	165	0	0	0	0	0
71	152	1	−13	−13	1	169
72	180	2	15	30	4	225
74	208	4	43	172	16	1849
		$\Sigma X = -13$	$\Sigma Y = -80$	$\Sigma XY = 891$	$\Sigma X^2 = 147$	$\Sigma Y^2 = 6440$

$$\bar{X} = \frac{1}{n}\Sigma X = \frac{1}{8}\times(-13) = \frac{-13}{8}$$

$$\bar{Y} = \frac{1}{n}\Sigma Y = \frac{1}{8}\times(-80) = \frac{-80}{8} = -10$$

$$\frac{1}{n}\Sigma XY = \frac{1}{8}\times(891) = 111.375$$

$$\sigma_X^2 = \frac{1}{n}\Sigma X^2 - \bar{X}^2 = \frac{1}{8}\times(147) - \left(\frac{-13}{8}\right)^2 = 15.73$$

$$\sigma_Y^2 = \frac{1}{n}\Sigma Y^2 - \bar{Y}^2 = \frac{1}{8}\times(6440) - (-10)^2 = 705$$

$$\rho_{XY} = \frac{\frac{1}{n}\Sigma XY - \bar{X}\bar{Y}}{\sigma_X\sigma_Y} = \frac{111.375 - \left(\frac{-13}{8}\right)(-10)}{3.96\times 26.55} \Rightarrow \rho_{XY} = 0.9$$

To obtain the regression lines:
The regression line of x on y is

$$x - \bar{x} = \rho\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

which is used to predict or estimate the value of x for any given y.

$$\bar{X} = \frac{-13}{8}$$

$$\bar{Y} = -10$$

$$\bar{x} = \bar{X} + 70 = \frac{-13}{8} + 70 = 68.375$$

$$\bar{y} = \bar{Y} + 165 = -10 + 165 = 155$$

$$x - 68.375 = 0.9 \times \frac{3.96}{26.55}(y - 155)$$

$\Rightarrow$ $$x - 68.375 = 0.134(y - 155)$$

$$x = 0.134y - 20.8 + 68.375$$

$\Rightarrow$ $$x = 0.134y + 47.5$$

The regression line of y on x is

$$y - \bar{y} = \rho\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$\Rightarrow$ $$y - 155 = 0.9 \times \frac{26.55}{3.96}(x - 68.375)$$

$$y = 6.034x - 412.58 + 155$$

$\Rightarrow$ $$y = 6.034x - 257.58$$

EXAMPLE 5.73 Calculate the correlation coefficient between the variables x and y from the following data:

x	1	2	3	4	5	6	7
y	9	8	10	12	11	13	14

Solution We know that the correlation coefficient of X and Y is

$$\rho_{xy} = \frac{\frac{1}{n}\Sigma xy - \bar{x}\cdot\bar{y}}{\sigma_x \cdot \sigma_y}$$

where $$\sigma_x^2 = \frac{1}{n}\Sigma x^2 - \bar{x}^2$$

$$\sigma_y^2 = \frac{1}{n}\Sigma y^2 - \bar{y}^2$$

x	y	xy	x^2	y^2
1	9	9	1	81
2	8	16	4	64
3	10	30	9	100
4	12	48	16	144
5	11	55	25	121
6	13	78	36	169
7	14	98	49	196
$\Sigma x = 28$	$\Sigma y = 77$	$\Sigma xy = 334$	$\Sigma x^2 = 140$	$\Sigma y^2 = 875$

$$\bar{x} = \frac{1}{n}\Sigma x = \frac{1}{7}\times 28 = 4$$

$$\bar{y} = \frac{1}{n}\Sigma y = \frac{1}{7}\times 77 = 11$$

$$\frac{1}{n}\Sigma xy = \frac{1}{7}\times 334 = \frac{334}{7}$$

$$\sigma_x^2 = \frac{1}{n}\Sigma x^2 - \bar{x}^2 = \frac{1}{7}\times 140 - 4^2 = 4$$

$$\sigma_y^2 = \frac{1}{n}\Sigma y^2 - \bar{y}^2 = \frac{1}{7}\times 875 - 11^2 = 4$$

$$\rho_{xy} = \frac{\frac{1}{n}\Sigma xy - \bar{x}\,\bar{y}}{\sigma_x\sigma_y} = \frac{\frac{334}{7} - (11\times 4)}{\sqrt{4}\,\sqrt{4}} = \frac{13}{14} = 0.928$$

EXAMPLE 5.74 Calculate the correlation coefficient and the lines of regression from the following data:

x	22	26	29	30	31	31	34	35
y	20	20	21	29	27	24	27	31

Solution Using the given data,

x	y	xy	x^2	y^2
22	20	440	484	400
26	20	520	676	400
29	21	609	841	441
30	29	870	900	841
31	27	837	961	729
31	24	744	961	576
34	27	918	1156	729
35	31	1085	1225	961
$\Sigma x = 238$	$\Sigma y = 199$	$\Sigma xy = 6023$	$\Sigma x^2 = 7204$	$\Sigma y^2 = 5077$

$$\bar{x} = \frac{1}{n}\Sigma x = \frac{1}{8}\times 238 = 29.75$$

$$\bar{y} = \frac{1}{n}\Sigma y = \frac{1}{8}\times 199 = 24.875$$

$$\frac{1}{n}\Sigma xy = \frac{1}{8}\times 6023 = 752.875$$

$$\sigma_x^2 = \frac{1}{n}\Sigma x^2 - \bar{x}^2 = \frac{1}{8}\times 7204 - 29.75^2 = 15.4375$$

$$\sigma_y^2 = \frac{1}{n}\Sigma y^2 - \bar{y}^2 = \frac{1}{8}\times 5077 - 24.875^2 = 15.859$$

$$\rho_{xy} = \frac{\frac{1}{n}\Sigma xy - \bar{x}\,\bar{y}}{\sigma_x \sigma_y} = \frac{752.875 - 740.00}{\sqrt{15.4375 \times 15.859}} = 0.82$$

To obtain the regression lines of x on y:

$$x - \bar{x} = \rho \frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$(x - 29.75) = 0.82 \times \sqrt{\frac{15.4375}{15.859}}(y - 24.875)$$

$$x - 29.75 = 0.81y - 20.125$$

$$x = 0.81y + 9.63$$

The regression line of y on x:

$$y - \bar{y} = \rho \frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$(y - 24.875) = 0.82 \times \sqrt{\frac{15.859}{15.4375}}(x - 29.75)$$

$$y - 24.875 = 0.83x - 24.73$$

$$y = 0.83x + 0.15$$

EXAMPLE 5.75 Calculate the correlation coefficient, lines of regression and angle between them from the following data:

x	4	7	5	6	8	5	6	6	4	9
y	2.5	6	4.5	5	4.5	2	3	4.5	3	5.5

Solution Using the given data,

x	y	xy	x^2	y^2
4	2.5	10	16	6.25
7	6	42	49	36
5	4.5	22.5	25	20.25
6	5	30	36	25
8	4.5	36	64	20.25
5	2	10	25	4
6	3	18	36	9
6	4.5	27	36	20.25
4	3	12	16	9
9	5.5	49.5	81	30.25
$\Sigma x = 60$	$\Sigma y = 40.5$	$\Sigma xy = 257$	$\Sigma x^2 = 384$	$\Sigma y^2 = 180.25$

$$\bar{x} = \frac{1}{n}\Sigma x = \frac{60}{10} = 6$$

$$\bar{y} = \frac{1}{n}\Sigma y = \frac{40.5}{10} = 4.05$$

$$\frac{\Sigma xy}{n} = \frac{257}{10} = 25.7$$

$$\sigma_x^2 = \frac{1}{n}\Sigma x^2 - \bar{x}^2 = \frac{384}{10} - 6^2 = 2.4$$

$$\sigma_y^2 = \frac{1}{n}\Sigma y^2 - \bar{y}^2 = \frac{180.25}{10} - 4.05^2 = 1.6225$$

Regression lines are

$$x - \bar{x} = \rho\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 6 = 0.709 \times \sqrt{\frac{2.4}{1.6225}}(y - 4.05)$$

$$x = 2.505 + 0.86y$$

$$y - \bar{y} = \rho\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 4.05 = 0.709 \times \sqrt{\frac{1.6}{2.4}}(x - 6)$$

$$y = 0.58x + 0.55$$

We know that

$$\rho_{xy} = \frac{\frac{1}{n}\Sigma xy - \bar{x}\,\bar{y}}{\sigma_x\sigma_y} = \frac{25.7 - 6 \times 4.05}{\sqrt{2.4 \times 1.6225}} = 0.709$$

$\therefore$ The correlation coefficient $\rho_{xy} = 0.709$

The angle between the regression lines is

$$\tan\theta = \frac{1-\rho^2}{\rho}\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2} = \frac{1-(0.709)^2}{0.709} \times \frac{2.4 \times 1.6225}{(2.4)^2 + (1.6225)^2} = 0.32545$$

$$\therefore \quad \theta = 0.31464$$

EXAMPLE 5.76 From the following data, find

(i) the two regression equations,

(ii) the coefficient of correlation between the marks in economics and statistics,

(iii) the most likely marks in statistics when marks in economics are 30.

[AU June '06, May '07, December '09]

Marks in economics	25	28	35	32	31	36	29	38	34	32
Marks in statistics	43	46	49	41	36	32	31	30	33	39

Solution

x	y	$x-\overline{x}$ $= x - 32$	$y-\overline{y}$ $= y - 38$	$(x-\overline{x})^2$	$(y-\overline{y})^2$	$(x-\overline{x})(y-\overline{y})$
25	43	−7	5	49	25	−35
28	46	−4	8	16	64	−32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	−1	−2	1	4	2
36	32	4	−6	16	36	−24
29	31	−3	−7	9	49	21
38	30	6	−8	36	64	−48
34	33	2	−5	4	25	−10
32	39	0	1	0	1	0
$\Sigma x = 320$	$\Sigma y = 380$	$\Sigma(x-\overline{x}) = 0$	$\Sigma(y-\overline{y}) = 0$	$\Sigma(x-\overline{x})^2 = 140$	$\Sigma(y-\overline{y})^2 = 398$	$\Sigma(x-\overline{x})(y-\overline{y}) = -93$

$$\overline{x} = \frac{\Sigma x}{n} = \frac{320}{10} = 32$$

$$\overline{y} = \frac{\Sigma y}{n} = \frac{380}{10} = 38$$

Coefficient of regression of y on x is

$$b_{yx} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(x-\overline{x})^2} = \frac{-93}{140} = -0.6643$$

Coefficient of regression of x on y is

$$b_{xy} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(y-\overline{y})^2} = \frac{-93}{398} = -0.2337$$

Equation of the line of regression of x on y is

$$x-\overline{x} = b_{xy}(y-\overline{y})$$
$$x - 32 = -0.2337(y - 38)$$
$$x = -0.2337y + 40.8806$$

Equation of the line of regression of y on x is

$$y-\overline{y} = b_{yx}(x-\overline{x})$$

$$y - 38 = -0.6643(x - 32)$$
$$y = -0.6643x + 59.2576$$

Coefficient of correlation,

$$\rho^2 = b_{xy} \times b_{yx}$$
$$= -0.6643 \times (-0.2337) = 0.1552$$
$$\rho = \pm 0.394$$

Now, to find the most likely marks in statistics (y) when marks in economics (x) is 30. We use the line of regression of y on x, i.e.

$$y = -0.6643x + 59.2576$$

Put $x = 30$, we get

$$y = -0.6643 \times 30 + 59.2576 = 39.3286$$

EXAMPLE 5.77 Heights of fathers and sons are given in centimetres.

Heights of fathers (x)	150	152	155	157	160	161	164	166
Heights of sons (y)	154	156	158	159	160	162	161	164

Find the two lines of regression and calculate the expected average height of the son when the height of the father is 154 cm.

[AU December '07]

Solution Let 160 and 159 be assumed means of x and y.
Using the given data, we get the following table:

x	y	$X = x - 160$	$Y = y - 159$	X^2	Y^2	XY
150	154	−10	−5	100	25	50
152	156	−8	−3	64	9	24
155	158	−5	−1	25	1	5
157	159	−3	0	9	0	0
160	160	0	1	0	1	0
161	162	1	3	1	9	3
164	161	4	2	16	4	8
166	164	6	5	36	25	30
		$\Sigma X = -15$	$\Sigma Y = 2$	$\Sigma X^2 = 251$	$\Sigma Y^2 = 74$	$\Sigma XY = 120$

$$\bar{x} = 160 + \frac{\Sigma X}{n} = 160 - \frac{15}{8} = 158.13$$

$$\bar{y} = 159 + \frac{\Sigma Y}{n} = 159 + \frac{2}{8} = 159.25$$

Since regression coefficients are independent of change of origin, we have regression coefficient of y on x.

$$b_{yx} = b_{YX} = \frac{n\Sigma XY - \Sigma X\, \Sigma Y}{n\Sigma X^2 - (\Sigma X)^2} = \frac{8\times 120 - (-15)\times 2}{8\times 251 - (-15)^2} = 0.555$$

Regression coefficient of x on y is

$$b_{xy} = b_{XY} = \frac{n\Sigma XY - \Sigma X\, \Sigma Y}{n\Sigma Y^2 - (\Sigma Y)^2} = \frac{990}{8\times 74 - 4} = 1.68$$

Equation of the line of regression of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$
$$x - 158.13 = 1.68(y - 159.25)$$
$$x = 1.68y - 109.41$$

Equation of the line regression of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$
$$y - 159.25 = 0.56(x - 158.13)$$
$$y = 0.56x + 70.697$$

When $x = 154$

$$y = 0.56(154) + 70.697$$
$$= 156.937$$

Note:

$$b_{xy} = \frac{\frac{1}{n}\Sigma xy - \bar{X}\bar{Y}}{\frac{1}{n}\Sigma x^2 - \bar{X}^2} = \frac{\frac{1}{n}\Sigma xy - \frac{\Sigma X}{n}\frac{\Sigma Y}{n}}{\frac{1}{n}\Sigma x^2 - \left(\frac{\Sigma X}{n}\right)^2} = \frac{n\Sigma XY - \Sigma X \Sigma Y}{n\Sigma X^2 - (\Sigma X)^2}$$

EXAMPLE 5.78 If $\bar{x} = 970, \bar{y} = 18, \sigma_x = 38, \sigma_y = 2$ and $\rho = 0.6$, find the line of regression of x on y. **[AU December '09]**

Solution Given: $\bar{x} = 970$, $\bar{y} = 18$, $\sigma_x = 38$, $\sigma_y = 2$, $\rho = 0.6$

The regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y}) = \rho\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$\therefore \quad (x - 970) = 0.6\left(\frac{38}{2}\right)(y - 18)$$
$$x = 970 + (0.6)(19)(y - 18)$$
$$= 764.8 + 11.4y$$

$\therefore$ The regression line of x on y is

$$x = 11.4y + 764.8$$

EXAMPLE 5.79 A statistical investigation obtains the following regression equation in a survey:

$$x - y = 6$$

$$0.64x + 4.08 = 0$$

and

Here x denotes the age of husband and y denotes the age of wife. Find

(i) the means of x and y,
(ii) the correlation coefficient between x and y and σ_y = SD of y, if σ_x = SD of $x = 4$.

Solution By one of the properties of regression lines, all the regression lines pass through their means ($\bar{x}$ and $\bar{y}$).

$\therefore$ $(\bar{x}, \bar{y})$ satisfies the equation.

(i) $$\bar{x} - \bar{y} = 6$$

and $$0.64\bar{x} + 4.08 = 0 \Rightarrow \bar{x} = \frac{-4.08}{0.64} = -6.37$$

$$-6.37 - \bar{y} = 6 \Rightarrow \bar{y} = -12.37$$

(ii) Given: $$x - y = 6 \Rightarrow y = x - 6$$

and $$x = 0y - 6.37$$

The regression coefficient of x on y is the coefficient of y in the regression line of x on y, i.e.

$$\frac{\rho\sigma_x}{\sigma_y} = \text{coefficient of } y = b_{xy} = 0$$

$$\frac{\rho\sigma_y}{\sigma_x} = \text{coefficient of } x = b_{yx} = 1$$

$$\rho^2 = b_{xy}b_{yx} = 0(1) = 0 \Rightarrow \rho = 0$$

Given: $\sigma_x = 4$

But, $$\rho\frac{\sigma_x}{\sigma_y} = 0 \Rightarrow \rho\sigma_x = 0 \times \sigma_y$$

$$0 \times 4 = 0 \times \sigma_y$$

$\Rightarrow$ σ_y can assume any value.

EXAMPLE 5.80 The two regression lines are $4x - 5y + 33 = 0$ and $20x - 9y = 107$, and $\text{Var}(X) = 25$. Find

(i) the means of x and y,
(ii) the values of ρ and σ_y, and
(iii) the angle between the regression lines.

[AU June '06, December '07]

Solution

(i) Given: $4x - 5y = -33$

$20x - 9y = 107$

Since all the regression lines pass through the means,

$$4\bar{x} - 5\bar{y} = -33 \quad \text{(i)}$$

$$20\bar{x} - 9\bar{y} = 107 \quad \text{(ii)}$$

Multiplying Eq. (i) by 5,

$$20\bar{x} - 25\bar{y} = -165 \quad \text{(iii)}$$

Equation (iii)–Eq. (ii) gives

$$-16\,\bar{y} = -272$$

$$\bar{y} = 17$$

Substituting in Eq. (i), we get

$$4\bar{x} = -33 + 5(17)$$

$$\bar{x} = \frac{85-33}{4} = 13$$

(ii) $20x - 9y = 107$

$$x = \frac{9}{20}y + \frac{107}{20}$$

$$x = 0.45y + 5.35$$

$$b_{xy} = \frac{\rho\sigma_x}{\sigma_y} = 0.45 = \text{coefficient of } y$$

$$5y = 33 + 4x$$

$$y = \frac{4}{5}x + \frac{33}{5}$$

$$= 0.8x + 6.6$$

$$b_{yx} = \frac{\rho\sigma_y}{\sigma_x} = 0.8 = \text{coefficient of } x$$

$$\rho^2 = b_{xy} \times b_{yx} = 0.36$$

$\therefore$ The correlation coefficient $\rho = 0.6$

(iii) Given: $\sigma_x^2 = 25$

$\therefore \quad \sigma_x = 5$

$$b_{xy} = \frac{\rho\sigma_x}{\sigma_y} = 0.45 \Rightarrow \sigma_y = \frac{0.6 \times 5}{0.45}$$

$$= 6.67$$

The angle between the regression lines are

$$\tan\theta = \frac{1-\rho^2}{\rho}\cdot\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

$$\Rightarrow \qquad \tan\theta = \frac{1-(0.6)^2}{0.6}\times\left(\frac{5\times 6.67}{25+44.49}\right)$$

$$\theta = \tan^{-1}(0.512) \Rightarrow \theta = 27.11$$

EXAMPLE 5.81 If X, Y denote the deviation of variance from the arithmetic mean and if $\rho = 0.5$, $\Sigma XY = 120$, $\sigma_y = 8$, $\Sigma X^2 = 90$. Find n, number of times.

Solution Given $X = x-\bar{x}, Y = y-\bar{y}$

$$\therefore \qquad \rho = \frac{\frac{1}{n}\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\frac{1}{n}\Sigma(x-\bar{x})^2}\sqrt{\frac{1}{n}\Sigma(y-\bar{y})^2}} = \frac{\frac{1}{n}\Sigma XY}{\sqrt{\frac{1}{n}\Sigma X^2}\sqrt{\sigma_y^2}}$$

$$0.5 = \frac{\frac{1}{n}(120)}{\frac{1}{\sqrt{n}}\times 9.48\times 8}$$

$$\sqrt{n} = \frac{120}{9.48\times 8\times 0.5} = 3.16$$

$$n = 10$$

EXAMPLE 5.82 Can $Y = 5 + 2.8X$ and $X = 3 - 0.5Y$ be the estimated regression equations of Y on X and X on Y respectively? Explain your answer.

[AU June '06, November '07]

Solution Given: $X = 3 - 0.5Y \Rightarrow b_{XY} = -0.5$

$Y = 5 + 2.8X \Rightarrow b_{YX} = 2.8$

$\therefore \qquad \rho^2 = b_{XY}\times b_{YX} = (-0.5)\times(2.8) = -1.4$

$\therefore \qquad \rho = \sqrt{-1.4}$ which is imaginary quantity.

ρ cannot be imaginary.

$\therefore$ The given lines are not estimated as regression equations.

EXAMPLE 5.83 If $y = 2x - 3$ and $y = 5x + 7$ are the two regression lines, find the mean values of x and y. Find the correlation coefficient between x and y. Find an estimate of x when $y = 1$.

Solution The two regression lines always pass through their means $(\bar{x}, \bar{y})$

$\therefore$ We have

$$\bar{y} = 2\bar{x} - 3$$

$$\bar{y} = 5\bar{x} + 7 \Rightarrow 2\bar{x} - \bar{y} = 3$$

$$5\bar{x} - \bar{y} = -7$$

Solving we get

$$\bar{x} = \frac{-10}{3}$$

$$\bar{y} = \frac{-29}{3}$$

From the given equations, we have

$$x = \frac{1}{5}y - \frac{7}{5} \Rightarrow b_{xy} = \frac{1}{5}$$

$$y = 2x - 3 \Rightarrow b_{yx} = 2$$

$\therefore$ The correlation coefficient $\rho_{xy} = b_{xy}\, b_{yx} = \frac{2}{5}$

When $y = 1$, $x = \frac{1}{5} - \frac{7}{5} = \frac{-6}{5}$

Aliter

If we choose the regression equation for x as $x = \frac{1}{2}y + \frac{3}{2}$ and for y as $y = 5x + 7$, then $b_{xy} = \frac{1}{2}, b_{yx} = 5$

$$\therefore \qquad \rho = \frac{1}{2} \times 5 = 2.5 > 1$$

$\therefore$ It is not possible.

EXAMPLE 5.84 Given that $x = 4y + 5$ and $y = kx + 4$ are the lines of regression of x on y and y on x respectively, show that $0 < 4k < 1$. If $k = 1/16$, find the means of two variables and coefficient of correlation between them.

Solution Given: $x = 4y + 5$, regression line of x on y

$\therefore \qquad b_{xy} = 4$

The regression line of y on x is

$$y = kx + 4$$

$$\therefore \qquad b_{yx} = k$$

$$\rho^2 = b_{xy}\, b_{yx} = 4k$$

But, $0 \le \rho^2 \le 1 \Rightarrow 0 \le 4k \le 1$

If $k = \frac{1}{16}$, then

$$\rho^2 = 4k = \frac{4}{16} = \frac{1}{4} \Rightarrow \rho = \pm\frac{1}{2}$$

Since b_{xy} and b_{yx} are positive, ρ is also positive.

$$\rho = \frac{1}{2}$$

$\therefore$ The two lines of regression are

$$x = 4y + 5$$

$$y = \frac{x}{16} + 4$$

Since the regression lines pass through their means

$$\bar{x} = 4\bar{y} + 5 \qquad \text{(i)}$$

$$\bar{y} = \frac{\bar{x}}{16} + 4 \qquad \text{(ii)}$$

Equation (i) + 4 × Eq. (ii) gives

$$\bar{x} = \frac{\bar{x}}{4} + 21 \Rightarrow 3\bar{x} = 84 \Rightarrow \bar{x} = 28$$

Substituting in Eq. (i)

$$28 = 4\bar{y} + 5 \Rightarrow \bar{y} = 5.75$$

$\therefore$

$$\rho = \frac{1}{2}$$

$$\bar{x} = 28$$

$$\bar{y} = 5.75$$

EXAMPLE 5.85 In a bivariate population $\sigma_x = \sigma_y = \sigma$ and the angle between the regression lines is $\tan^{-1}(6)$, obtain the value of correlation coefficient.

Solution We know that the angle between the regression lines is

$$\tan\theta = \frac{1-\rho^2}{\rho}\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

$$= \frac{1-\rho^2}{\rho}\frac{\sigma^2}{2\sigma^2}$$

$$\tan\theta = \frac{1-\rho^2}{2\rho}$$

$$\theta = \tan^{-1}\left(\frac{1-\rho^2}{2\rho}\right) = \tan^{-1}(6) \text{ (given)}$$

$\Rightarrow$

$$\frac{1-\rho^2}{2\rho} = 6$$

$$1 - \rho^2 = 12\rho$$

i.e.

$$\rho^2 + 12\rho - 1 = 0$$

$$\rho = \frac{-12 \pm \sqrt{12^2 - 4}}{2}$$

$$\rho = -0.0839 \text{ or } -11.9161$$

$\therefore$

But, $-1 \le \rho \le 1$

$\therefore$

$$\rho = -0.0839$$

EXAMPLE 5.86 The tangent of the angle between the two lines of regression is 0.5 and $\sigma_x = \frac{1}{4}\sigma_y$, find the value of correlation coefficient.

Solution Given the angle between the two regression lines is 0.5

$$\tan\theta = \frac{1-\rho^2}{\rho}\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_x^2}$$

and

$$\sigma_x = \frac{1}{4}\sigma_y \Rightarrow 4\sigma_x = \sigma_y$$

$\therefore$

$$0.5 = \frac{1-\rho^2}{\rho}\frac{\sigma_x 4\sigma_x}{\sigma_x^2+16\sigma_x^2}$$

$$\frac{1}{2} = \frac{1-\rho^2}{\rho}\left(\frac{4}{17}\right)$$

$$\frac{1-\rho^2}{\rho} = \frac{17}{8} \Rightarrow 8\rho^2 + 17\rho - 8 = 0$$

$$\rho = \frac{-17 \pm \sqrt{17^2 + 4\times 8\times 8}}{16} = \frac{-17 \pm \sqrt{545}}{16}$$

$$= 0.3966 \quad \text{or} \quad -2.5216$$

But, $-1 \le \rho \le 1$

$\therefore$

$$\rho = 0.3966$$

EXAMPLE 5.87 Let X be a random variable with mean 3 and variance 2. Find the second moment of X about the origin. Another random variable Y is defined by $y = -6x + 22$. Find the mean value of Y and correlation of X and Y.

Solution Given:

$$E(X) = 3$$

and

$$\text{Var}(X) = 2$$

$$y = -6x + 22$$

Second moment of $X = \mu_2'$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \text{Var}(X) + [E(X)]^2$$
$$= 2 + 3^2 = 11$$

$\Rightarrow$ $$\mu_2' = 11$$

Since the regression line passes through $(\bar{x}, \bar{y})$

$$\bar{y} = -6(\bar{x}) + 22 = 4$$

$$\bar{y} = E(Y) = 4$$

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} \quad \text{(i)}$$

$$E(XY) - E(X)\,E(Y) = E[X(-6X + 22)] - 12$$
$$= E(-6X^2 + 22X) - 12$$
$$= -6E(X^2) + 22\,E(X) - 12$$
$$= -6(11) + 22\,(3) - 12$$
$$= -66 + 66 - 12$$
$$= -12$$

Given: $\text{Var}(X) = 2 = \sigma_X^2$

$$\sigma_Y^2 = \text{Var}(Y) = E[(-6X + 22)^2] - [E(-6X + 22)]^2$$
$$= E(36X^2 + 484 - 264X) - \{36[E(X)]^2 + 484 - 264E(X)\}$$
$$= 36\{E(X^2) - [E(X)]^2\} + 484 - 484 - 264\,E(X) + 264E(X)$$
$$= 36\,\text{Var}(X)$$
$$= 36(2) = 72$$

$$\sigma_X = \sqrt{2}$$

and $$\sigma_Y = \sqrt{72} = 6\sqrt{2}$$

Substituting in (i), we get

$$\rho_{XY} = \frac{-12}{\sqrt{2} \times 6\sqrt{2}} = -1$$

EXAMPLE 5.88 If the lines of regression of x on y and y on x are respectively, $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Prove that $a_2b_1 \le a_1b_2$.

Solution Given the lines of regression as

$$a_1x + b_1y + c_1 = 0 \ (x \text{ on } y)$$
$$a_2x + b_2y + c_2 = 0 \ (y \text{ on } x)$$

$$x = \frac{-b_1}{a_1}y - \frac{c_1}{a_1} \quad (x \text{ on } y)$$

$$y = \frac{-a_2}{b_2}x - \frac{c_2}{b_2} \quad (y \text{ on } x)$$

$$b_{xy} = \frac{-b_1}{a_1}$$

$\therefore$

$$b_{yx} = \frac{-a_2}{b_2}$$

$$\rho^2 = b_{xy} \times b_{yx} = \frac{-b_1}{a_1} \times \frac{-a_2}{b_2}$$

$\therefore$

$$\rho^2 = \frac{a_2 b_1}{a_1 b_2}$$

$$\rho^2 \leq 1$$

But,

$$\frac{a_2 b_1}{a_1 b_2} \leq 1$$

$\therefore$

$$a_2 b_1 \leq a_1 b_2$$

$\Rightarrow$

EXAMPLE 5.89 Find the mean of which two variables are involved given that the correlation coefficient between the variables is $-\frac{1}{2}$. The line of regression passes through the points (4, 0) and (−14, 3) and the other line of the regression passes through the point (+1, −1).

Solution The correlation coefficient between the variables is

$$\rho = -\frac{1}{2} = -0.5$$

Equation of a line passing through any 2 points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Given: $(x_1, y_1) = (4, 0)$

and $(x_2, y_2) = (-14, 3)$

$$\frac{y - 0}{3 - 0} = \frac{x - 4}{-14 - 4}$$

$$\frac{y}{3} = \frac{x-4}{-18} \quad \Rightarrow \quad -18y = 3x - 12$$

$$3x + 18y = 12 \qquad \text{(i)}$$

$$18y = 12 - 3x$$

$$y = \frac{12 - 3x}{18}$$

$\therefore$ The regression line of y on x

$$y = -0.167x + 0.67$$

$$b_{yx} = -0.167, \text{ coefficient of } x$$

$$\rho = \pm\sqrt{b_{yx}\, b_{xy}}$$
$$\rho^2 = b_{yx} \cdot b_{xy}$$
$$(-0.5)^2 = -0.167 b_{xy}$$
$$b_{xy} = -1.497$$

The other line of regression, i.e. the regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$
$$x - \bar{x} = -1.497(y - \bar{y})$$

The line passes through the point (+1, −1)

$$x - 1 = -1.497(y + 1)$$
$$x = -1.497y - 1.497 + 1$$
$$x = -1.497y - 0.497$$

∴ The two lines of regression are

$$x = -1.497y - 0.497$$

and

$$y = -0.167x + 0.67$$

All the regression lines pass through their means

∴
$$\bar{x} + 1.497\bar{y} = -0.497 \qquad \text{(i)}$$
$$0.167\bar{x} + \bar{y} = 0.67 \qquad \text{(ii)}$$

Multiplying Eq. (ii), by 1.497 gives

$$0.249\,\bar{x} + 1.497\,\bar{y} = 1.002 \qquad \text{(iii)}$$

Equation (i) — Eq. (iii) gives

$$0.751\,\bar{x} = -1.499$$
$$\bar{x} = -1.99$$

Substituting $\bar{x} = -1.99$ in Eq. (i), we get

$$-1.99 + 1.497\,\bar{y} = -0.497$$
$$\bar{y} = 0.997$$

∴ Mean = $(\bar{x}, \bar{y})$ = (−1.99, 0.997)

EXAMPLE 5.90 If Var(X) = 208.89, Var(Y) = 288.76, Var($X - Y$) = 137.61, find $\rho(x, y)$.

Solution

$$\rho(x, y) = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{(x-y)}^2}{2\sigma_x\sigma_y}$$

$$= \frac{208.89^2 + 288.76^2 - 137.61^2}{2 \times 208.89 \times 288.76} = 0.895$$

EXAMPLE 5.91 The probability density function of two random variables X and Y is given by

$$f(x, y) = \frac{3}{2}(x^2 + y^2), \quad 0 \le x \le 1, 0 \le y \le 1$$

Find the lines of regression of X on Y and Y on X. **[AU December '09]**

Solution Given: $f(x, y) = \frac{3}{2}(x^2 + y^2), \quad 0 \le x \le 1, 0 \le y \le 1$

Marginal PDF of X and Y:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 \frac{3}{2}(x^2 + y^2)dy$$

$$= \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left(x^2 + \frac{1}{3}\right)$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 \frac{3}{2}(x^2 + y^2)dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3} + y^2 x\right]_0^1 = \frac{3}{2}\left(\frac{1}{3} + y^2\right)$$

$$\therefore \quad f(x) = \frac{3}{2}\left(x^2 + \frac{1}{3}\right), \quad 0 < x < 1$$

$$f(y) = \frac{3}{2}\left(y^2 + \frac{1}{3}\right), \quad 0 < y < 1$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x\frac{3}{2}\left(x^2 + \frac{1}{3}\right)dx$$

$$= \frac{3}{2}\left[\frac{x^4}{4} + \frac{x^2}{6}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{5}{8}$$

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy = \int_0^1 y\frac{3}{2}\left(y^2 + \frac{1}{3}\right)dy = \frac{5}{8}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2\frac{3}{2}\left(x^2 + \frac{1}{3}\right)dx$$

$$= \frac{3}{2}\int_0^1 \left(x^4 + \frac{x^2}{3}\right)dx$$

$$= \frac{3}{2}\left[\frac{x^5}{5} + \frac{x^3}{9}\right]_0^1 = \frac{3}{2}\left(\frac{1}{5} + \frac{1}{9}\right) = \frac{7}{15}$$

Similarly, it can be proved that

$$E(Y^2) = \frac{7}{15}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{7}{15} - \left(\frac{5}{8}\right)^2 = \frac{73}{960}$$

and

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{73}{960}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dx dy$$

$$= \frac{3}{2}\int_0^1\int_0^1 xy(x^2 + y^2)\, dx dy$$

$$= \frac{3}{2}\int_0^1 y\left[\frac{x^4}{4} + y^2\frac{x^2}{2}\right]_0^1 dy$$

$$= \frac{3}{2}\int_0^1 y\left(\frac{1}{4} + \frac{y^2}{2}\right) dy$$

$$= \frac{3}{2}\left[\frac{y^2}{8} + \frac{y^4}{8}\right]_0^1 = \frac{3}{2} \times \frac{2}{8} = \frac{3}{8}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)\, E(Y)$$

$$= \frac{3}{8} - \frac{5}{8} \times \frac{5}{8} = \frac{-1}{64}$$

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{-\frac{1}{64}}{\sqrt{\frac{73}{960}}\sqrt{\frac{73}{960}}}$$

$$= -\frac{1}{64} \times \frac{960}{73} = -\frac{960}{4762} = -\frac{15}{73}$$

$$\rho_{XY} = -\frac{15}{73}$$

The line of regression of X on Y is given by

$$X - E(X) = \frac{\text{Cov}(X, Y)}{\sigma_Y^2}[Y - E(Y)]$$

$$X-\frac{5}{8}=\frac{-\frac{1}{64}}{\frac{73}{960}}\left(Y-\frac{5}{8}\right)$$

$$X=\frac{5}{8}-\frac{15}{73}\left(Y-\frac{5}{8}\right)$$

$$X=\frac{-15}{73}Y+\frac{55}{73}$$

The regression line of Y on X is

$$Y-E(Y)=\frac{\text{Cov}(X,Y)}{\sigma_X^2}[X-E(X)]$$

$$Y-\frac{5}{8}=\frac{-\frac{1}{64}}{\frac{73}{960}}\left(X-\frac{5}{8}\right)$$

$\therefore$

$$Y=\frac{5}{8}-\frac{15}{73}\left(X-\frac{5}{8}\right)$$

$$=\frac{-15}{73}X+\frac{55}{73}$$

EXAMPLE 5.92 Let the random variable X have the marginal density

$$f(x)=1,\quad -\frac{1}{2}<x<\frac{1}{2}$$

and the conditional density of Y be

$$f(y/x)=1,\quad x<y<x+1,-\frac{1}{2}<x<0$$

$$=1,\quad -x<y<1-x,0<x<\frac{1}{2}$$

Show that the variables X and Y are uncorrelated.

Solution By definition,

$$E(X)=\int_{-\frac{1}{2}}^{\frac{1}{2}}x\,f(x)dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}x\cdot 1\cdot dx=\left[\frac{x^2}{2}\right]_{-\frac{1}{2}}^{\frac{1}{2}}=0$$

$$f(y/x) = \frac{f(x, y)}{f(x)} \Rightarrow f(x, y) = f(y/x)\, f(x)$$

$$f(x, y) = 1, \quad x < y < x + 1, -\frac{1}{2} < x < 0$$

$$= 1, \quad -x < y < 1 - x, 0 < x < \frac{1}{2}$$

$$\therefore \quad E(XY) = \int_{-\frac{1}{2}}^{0} \int_{x}^{x+1} xy\, dxdy + \int_{0}^{\frac{1}{2}} \int_{-x}^{1-x} xy\, dxdy$$

$$= \int_{-\frac{1}{2}}^{0} x \left[\frac{y^2}{2}\right]_{x}^{x+1} dx + \int_{0}^{\frac{1}{2}} x \left[\frac{y^2}{2}\right]_{-x}^{1-x} dy$$

$$= \frac{1}{2} \int_{-\frac{1}{2}}^{0} x(2x+1)dx + \frac{1}{2} \int_{0}^{\frac{1}{2}} x(1-2x)dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x^3 + \frac{x^2}{2}\right]_{-\frac{1}{2}}^{0} + \frac{1}{2}\left[\frac{x^2}{2} - \frac{2}{3}x^3\right]_{0}^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{1}{12} - \frac{1}{8} - \frac{1}{12} + \frac{1}{8}\right) = 0$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$
$$= 0 - 0 \times E(Y) = 0$$

$\therefore$ The random variable X and Y are uncorrelated.

EXAMPLE 5.93 Two random variables have the joint PDF

$$f(x, y) = \frac{1}{3}(x + y), \quad 0 \le x \le 1, 0 \le y \le 2$$

Find

(i) the correlation coefficient,
(ii) the two regression lines, and
(iii) the two regression curves for the means.

Solution

(i) The marginal PDF of X and Y are as follows:

$$f_X(x) = \frac{1}{3}\int_{0}^{2}(x + y)dy = \frac{1}{3}\left[xy + \frac{y^2}{2}\right]_0^2 = \frac{2}{3}(1 + x), \quad 0 \le x \le 1$$

$$f_Y(y) = \frac{1}{3}\int_0^1 (x+y)dx = \frac{1}{3}\left[\frac{x^2}{2} + xy\right]_0^1 = \frac{(1+2x)}{6}, \qquad 0 \le y \le 2$$

$$E(X) = \int_0^1 xf(x)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 = \frac{5}{9}$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{7}{18}$$

$$\therefore \quad \text{Var}(X) = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}$$

$$E(Y) = \frac{1}{6}\int_0^2 (y + 2y^2)dy = \frac{1}{6}\left[\frac{y^2}{2} + \frac{2y^3}{3}\right]_0^2 = \frac{1}{6}\left(2 + \frac{16}{3}\right) = \frac{11}{9}$$

$$E(Y^2) = \frac{1}{6}\int_0^2 (y^2 + 2y^3)dy = \frac{1}{6}\left[\frac{y^3}{3} + \frac{y^4}{2}\right]_0^2 = \frac{1}{6}\left(\frac{8}{3} + 8\right) = \frac{16}{9}$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}$$

$$\text{Also, } E(XY) = \int_0^1\int_0^2 xyf(x,y)dxdy = \frac{1}{3}\int_0^1\int_0^2 (x^2y + xy^2)dxdy$$

$$= \frac{1}{3}\left\{\left[\frac{x^3}{3}\right]_0^1\left[\frac{y^2}{2}\right]_0^2 + \left[\frac{x^2}{2}\right]_0^1\left[\frac{y^3}{3}\right]_0^2\right\} = \frac{1}{3}\left(\frac{2}{3} + \frac{4}{3}\right) = \frac{2}{3}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y) = \frac{2}{3} - \frac{55}{81} = -\frac{1}{81}$$

$$\therefore \quad \rho(X, Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \frac{-\frac{1}{81}}{\sqrt{\frac{13}{162} \times \frac{23}{81}}} = -\sqrt{\frac{2}{229}} = -0.0818$$

(ii) Two lines of regression are given by

$$Y - E(Y) = \frac{\text{Cov}(X,Y)}{\sigma_X^2}[X - E(X)]$$

$$\Rightarrow \qquad Y - \frac{11}{9} = -\frac{2}{13}\left(X - \frac{5}{9}\right)$$

$$X - E(X) = \frac{\text{Cov}(X, Y)}{\sigma_Y^2}[Y - E(Y)]$$

$$\Rightarrow \qquad X - \frac{5}{9} = -\frac{1}{23}\left(Y - \frac{11}{9}\right)$$

(iii) To find the regression curves for means

$$f(y/x) = \frac{f(x, y)}{f(x)} = \frac{x+y}{2+2x}$$

$$f(x/y) = \frac{f(x, y)}{f(y)} = \frac{2(x+y)}{1+2y}$$

The regression curve of Y on X is given by

$$y = E(Y/X = x) = \int_0^2 yf(y/x)dy = \int_0^2 \frac{y(x+y)}{2(1+x)}dy$$

$$= \frac{1}{2(1+x)}\int_0^2 (xy + y^2)dy = \frac{1}{2(1+x)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_0^2 = \frac{3x+4}{3(1+x)}$$

$$\therefore \quad y = \frac{3x+4}{3(1+x)}$$

The regression curve of X on Y is given by

$$x = E(X/Y = y) = \int_0^1 xf(x/y)dx = \int_0^1 x\frac{2(x+y)}{1+2y}dx$$

$$= \frac{2}{1+2y}\int_0^1 (x^2 + xy)dx = \frac{2}{1+2y}\left[\frac{x^3}{3} + \frac{x^2y}{2}\right]_0^1 = \frac{2+3y}{3(1+2y)}$$

$$\therefore \quad x = \frac{2+3y}{3(1+2y)}$$

EXAMPLE 5.94 Given $f(x, y) = xe^{-x(y+1)}$, $x \geq 0, y \geq 0$ is the joint PDF of a random variable (X, Y), find the regression of Y on X. Also, find whether it is linear or not.

Solution Given: $f(x, y) = xe^{-x(y+1)}, \qquad x \geq 0, y \geq 0$

$\therefore$ The marginal PDF of X is

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^{\infty} xe^{-x(y+1)}dy$$

$$= xe^{-x}\int_0^{\infty} xe^{-xy}\,dy = xe^{-x}\left[\frac{e^{-xy}}{-x}\right]_0^{\infty}$$

$$= xe^{-x}\left(0+\frac{1}{x}\right) = e^{-x}$$

$$f(x) = e^{-x}, \quad x \geq 0$$

The conditional PDF of Y given X is

$$f(y/x) = \frac{f(x, y)}{f(x)} = \frac{xe^{-xy}e^{-x}}{e^{-x}}$$

$\therefore$

$$f(y/x) = xe^{-xy}, \quad y \geq 0, x \geq 0$$

The regression curve of Y on X is

$$y = E(Y/X = x) = \int_0^{\infty} y\, f(y/x)\,dy$$

$$= \int_0^{\infty} yxe^{-xy}\,dy = x\int_0^{\infty} ye^{-xy}\,dy$$

$$= x\left[y\left(\frac{e^{-xy}}{-x}\right) - 1\cdot\frac{e^{-xy}}{(-x)^2}\right]_0^{\infty} = x\left[(0-0) - 0 + \frac{1}{x^2}\right] = \frac{1}{x}$$

$\therefore \quad y = \frac{1}{x} \Rightarrow xy = 1$ (a rectangular hyperbola)

$\therefore$ The regression of Y on X is not linear.

EXAMPLE 5.95 If the density function is defined by

$$f(x, y) = \frac{y}{(1+x)^4}\, e^{\frac{-y}{1+x}}, \quad x \geq 0, y \geq 0$$

then obtain the regression equation of Y on X for the distribution.

Solution Given: $f(x, y) = \frac{y}{(1+x)^4}\, e^{\frac{-y}{1+x}}, \quad x \geq 0, y \geq 0.$

The marginal PDF of X is

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

$$= \int_0^{\infty} \frac{y}{(1+x)^4}\, e^{\frac{-y}{1+x}}\,dy$$

$$= \frac{1}{(1+x)^4}\left[y\frac{e^{\frac{-y}{1+x}}}{\left(-\frac{1}{1+x}\right)} - 1\cdot\frac{e^{\frac{-y}{1+x}}}{\left(-\frac{1}{1+x}\right)^2}\right]_0^\infty$$

$$= \frac{1}{(1+x)^4}\left[(0-0) - 0 + \frac{1}{\left(\frac{1}{1+x}\right)^2}\right]$$

$$= \frac{1}{(1+x)^4}(1+x)^2 = \frac{1}{\left(\frac{1}{1+x}\right)^2}$$

The conditional PDF of y given x is

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{y}{(1+x)^4}e^{\frac{-y}{1+x}}}{\left(\frac{1}{1+x}\right)^2}$$

$$\therefore \qquad f(y/x) = \frac{y}{(1+x)^2}e^{\frac{-y}{1+x}}, \qquad y \geq 0$$

The regression equation of Y on X is

$$y = E(Y/X) = \int_0^\infty y\, f(y/x)\, dy$$

$$= \frac{1}{(1+x)^2}\int_0^\infty y^2 e^{-y(1+x)} dy$$

$$= \frac{1}{(1+x)^2}\left[y^2\frac{e^{\frac{-y}{1+x}}}{\left(-\frac{1}{1+x}\right)} - 2y\frac{e^{\frac{-y}{1+x}}}{\left(-\frac{1}{1+x}\right)^2} + 2\frac{e^{\frac{-y}{1+x}}}{\left(-\frac{1}{1+x}\right)^3}\right]_0^\infty$$

$$= \frac{1}{(1+x)^2}\left[(0-0) - (0-0) + 0 + \frac{2}{\left(\frac{1}{1+x}\right)^3}\right]$$

$$= \frac{2}{(1+x)^2}(1+x)^3$$

$$y = 2(1+x) \quad \Rightarrow \quad y = 2x + 2$$

$\therefore$ The regression of y on x is linear.

EXAMPLE 5.96 The joint PDF of the random variable (X, Y) is given by

$$f(x, y) = \begin{cases} 1, & |y| < x, 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the regression curve of Y on X and X on Y. Verify whether the curves are linear or not.

Solution Given: $f(x, y) = \begin{cases} 1, & |y| < x, 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

The marginal PDF of X is

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{-x}^{x} 1\, dy,$$

from the Figure 5.20 taking parallel strip to y-axis

$$= [y]_{-x}^{x} = 2x$$

$\therefore \quad f(x) = 2x, \quad 0 < x < 1$

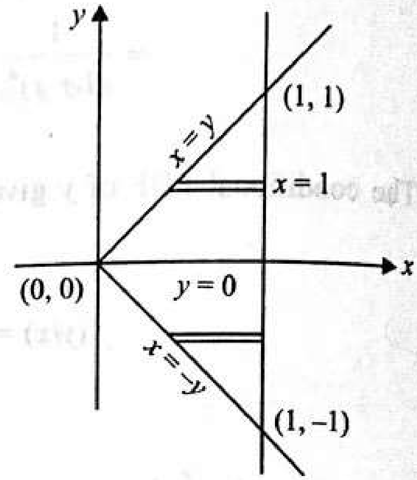


Figure 5.20

The marginal PDF of y is

$$f(y) = \int_{y}^{1} f(x, y)\, dx = \int_{y}^{1} 1dx$$

$$= [x]_{y}^{1} = 1 - y \qquad \text{(taking parallel strip to } x\text{-axis)}$$

$\therefore \quad f(x) = 1 - y, \quad 0 < y < 1$

$$f(x) = \int_{-y}^{1} f(x, y)\, dx = \int_{-y}^{1} 1dx = [x]_{-y}^{1} = 1 + y$$

$\therefore \quad f(x) = 1 + y, \quad -1 < y < 0$

$\qquad\quad = 1 - y, \quad 0 < y < 1$

The conditional PDFs of x on y and y on x:

$$f(y/x) = \frac{f(x, y)}{f(x)} = \frac{1}{2x}, \quad 0 < x < 1, -x < y < x$$

$$f(x/y) = \frac{f(x, y)}{f(y)}$$

$$= \frac{1}{1+y}, \quad -1 < y < 0, 0 < x < 1$$

$$= \frac{1}{1-y}, \quad 0 < y < 1, 0 < x < 1$$

The regression curve of Y on X is

$$y = E(Y/X = x) = \int_{-\infty}^{\infty} y\, f(y/x)\, dy = \int_{-x}^{x} y \frac{1}{2x} dy$$

$$= \frac{1}{2x}[y^2/2]_{-x}^{x} = 0$$

∴ The regression curve of y on x is $y = 0$, which is linear.
The regression curve of x on y is

$$x = E(X/Y = y) = \int_0^1 x \frac{1}{1-y} dx, \quad 0 < y < 1$$

$$= \frac{1}{1-y}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2(1-y)}, \quad 0 < y < 1$$

and

$$x = E(X/Y = y) = \int_0^1 x \frac{1}{1+y} dx, \quad -1 < y < 0$$

$$= \frac{1}{1+y}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2(1+y)}, \quad -1 < y < 0$$

∴ The regression curve of X on Y is

$$x = \frac{1}{2(1+y)}, \quad -1 < y < 0$$

$$= \frac{1}{2(1-y)}, \quad 0 < y < 1$$

which is not linear.

EXAMPLE 5.97 Find the regression curves for the means of the distribution with PDF given by

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}.$$

Solution The marginal PDF of X is

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_x^{\infty} e^{-y} dy$$

(parallel strip to y-axis in Figure 5.21)

$$= \left[\frac{e^{-y}}{-1}\right]_x^{\infty} = e^{-x}, \quad 0 < x < \infty$$

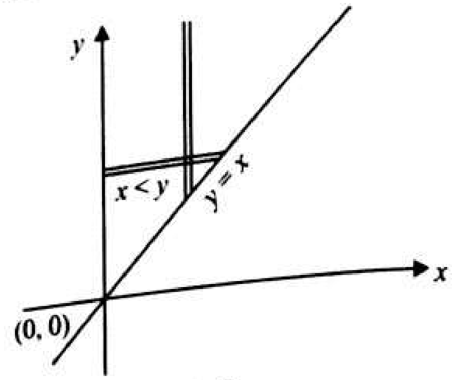


Figure 5.21

The marginal PDF of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{0}^{y} e^{-y}dx$$

(parallel strip to x-axis in Figure 5.21)

$$= e^{-y}[x]_0^y = ye^{-y}, \qquad 0 < y < \infty$$

The conditional densities are

$$f(x/y) = \frac{f(x, y)}{f(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}, \qquad x < y < \infty$$

$$f(y/x) = \frac{f(x, y)}{f(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}, \qquad 0 < x < y$$

The regression curve for means:

The regression curve for Y on X is

$$y = E(Y/X = x) = \int_{x}^{\infty} ye^{x-y}dy = e^{x}\int_{x}^{\infty} ye^{-y}dy$$

$$= e^{x}\left\{y\left(\frac{e^{-y}}{-1}\right) - 1\left[\frac{e^{-y}}{(-1)^2}\right]\right\}_x^{\infty}$$

$$= e^{x}(0 + xe^{-x} - 0 + e^{-x})$$

$\therefore$ $y = x + 1$, the regression curve of y on x

The regression curve of X on Y is

$$x = E(X/Y = x) = \int_{0}^{y} xf(x/y)\,dx$$

$$= \int_{0}^{y} x\frac{1}{y}dx = \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y$$

$$x = \frac{y}{2} \Rightarrow 2x - y = 0$$

∴ which is the regression curve of X on Y.

5.6 RANK CORRELATION

The rank correlation between the ranks of X and Y is given by

$$r_{XY} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} \text{ (if the rank or data is not repeated)}$$

where $d_i = x_i - y_i$, n = number of data given

and $$r_{XY} = 1 - \frac{6\left(\sum_{i=1}^{n} d_i^2 + \text{CF}\right)}{n(n^2-1)} \text{ (if the data or rank is repeated)}$$

where

$$\text{CF} = \text{correction factor} = \frac{m(m^2-1)}{12}$$

m = number of times the data is repeated.

EXAMPLE 5.98 Calculate the rank correlation coefficient from the following data, which give the ranks of 10 students in maths and computer science.

Maths (x)	1	5	3	4	7	6	10	2	9	8
Computer science (y)	6	9	1	3	5	4	8	2	10	7

Solution

x	y	$d_i = x_i - y_i$	d_i^2
1	6	–5	25
5	9	–4	16
3	1	2	4
4	3	1	1
7	5	2	4
6	4	2	4
10	8	2	4
2	2	0	0
9	10	–1	1
8	7	1	1
			$\Sigma d_i^2 = 60$

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$$\therefore \quad \sum_{i=1}^{n} d_i^2 = 60, n = 10$$

In this problem, the ranks are not repeated.

$$\therefore \quad r_{XY} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} = 1 - \frac{6 \times 60}{10 \times 99} = 0.63636$$

EXAMPLE 5.99 Find the rank correlation coefficient from the following data: **[AU January '06]**

Ranks in X	1	2	3	4	5	6	7
Ranks in Y	4	3	1	2	6	5	7

Solution In this problem, the ranks are not repeated.

x	y	$d_i = x_i - y_i$	d_i^2
1	4	−3	9
2	3	−1	1
3	1	2	4
4	2	2	4
5	6	−1	1
6	5	1	1
7	7	0	0
			$\Sigma d_i^2 = 20$

Rank correlation coefficient,

$$r(X,Y) = 1 - \frac{6\sum d_i^2}{n(n^2-1)} = 1 - \frac{6 \times 20}{7(49-1)} = 1 - \frac{120}{336} = 0.6429$$

EXAMPLE 5.100 The ranks of some 16 students in mathematics and physics are as follows. Calculate rank correlation coefficients for proficiency in mathematics and physics.

Rank in maths (X)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Ranks in physics (Y)	1	10	3	4	5	7	2	6	8	11	15	9	14	12	16	13

Solution In this problem, the ranks are not repeated.

Ranks in maths X	Ranks in physics Y	$d_i = x_i - y_i$	d_i^2
1	1	0	0
2	10	−8	64
3	3	0	0
4	4	0	0
5	5	0	0
6	7	−1	1
7	2	5	25
8	6	2	4
9	8	1	1
10	11	−1	1
11	15	−4	16
12	9	3	9
13	14	−1	1
14	12	2	4
15	16	−1	1
16	13	3	9
		$\Sigma d_i = 0$	$\Sigma d_i^2 = 136$

Rank correlation coefficient is

$$r(X,Y) = 1 - \frac{6\Sigma d_i^2}{n(n^2-1)} = 1 - \frac{6\times 136}{16(256-1)} = 1 - \frac{816}{4080} = 0.8$$

EXAMPLE 5.101 Obtain the rank correlation coefficient for the following data:

X	68	64	75	50	64	80	75	40	55	64
Y	62	58	68	45	81	60	68	48	50	70

X	Y	Rank X (x_i)	Rank Y (y_i)	$d_i = x_i - y_i$	d_i^2
68	62	4	5	−1	1
64	58	6	7	−1	1
75	68	2.5	3.5	−1	1
50	45	9	10	−1	1
64	81	6	1	5	25
80	60	1	6	−5	25
75	68	2.5	3.5	−1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	70	6	2	4	16
				$\Sigma d_i = 0$	$\Sigma d_i^2 = 72$

In X, 75 is repeated twice which are in the positions 2nd and 3rd ranks. Therefore, $(2 + 3)/2 = 2.5$ to be given for each 75.

Also in X, 64 is repeated thrice which are in the position 5th, 6th and 7th ranks. Therefore, common rank is 6, i.e. $(5 + 6 + 7)/3 = 6$ is to be given for each 64.

Similarly in Y, 68 is repeated twice which are in the positions 3rd and 4th ranks. Therefore, common rank is 3.5, i.e. $(3 + 4)/2 = 3.5$ is to be given for each 68.

Correction factors: In X, 75 is repeated twice

$$\therefore \quad CF = \frac{m(m^2-1)}{12} = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

In X, 64 is repeated thrice

$$\therefore \quad CF = \frac{m(m^2-1)}{12} = \frac{3(3^2-1)}{12} = \frac{24}{12} = 2$$

In Y, 68 is repeated twice

$$\therefore \quad CF = \frac{m(m^2-1)}{12} = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

$$\therefore \quad \text{Rank correlation } r = 1 - \frac{6\left(\sum_{i=1}^{n} d_i^2 + CF\right)}{n(n^2-1)}$$

$$= 1 - \frac{6\left(72 + \frac{1}{2} + 2 + \frac{1}{2}\right)}{10(10^2-1)} = 1 - \frac{6 \times 75}{10 \times 99}$$

$$= 1 - \frac{450}{990} = 0.5454$$

EXAMPLE 5.102 Find the rank correlation coefficient from the following data:

x	124 123	100 104	105 99	112 113	102 121	93 103	99 101	115
y	80 89	100 104	102 111	91 102	100 98	111 111	109 123	100

Solution Here the ranks are repeated. In x, only the value 99 is repeated twice, i.e. 13th and 14th ranks.

$$\therefore \quad \text{Rank of } 99 = \frac{13+14}{2} = 13.5$$

In y, the value 111 is repeated thrice, i.e. 2nd, 3rd and 4th ranks.

$$\therefore \quad \text{Rank of } 111 = \frac{2+3+4}{3} = 3$$

The value 100 is repeated thrice, i.e. 9th, 10th and 11th ranks.

$$\therefore \quad \text{Rank of } 100 = \frac{9+10+11}{3} = 10$$

The value 102 is repeated twice, i.e. 7th and 8th ranks.

$$\therefore \quad \text{Rank of } 102 = \frac{7+8}{2} = 7.5$$

x	y	X	Y	$d_i = x_i - y_i$	d_i^2
124	80	1	15	−14	196
100	100	12	10	2	4
105	102	7	7.5	−0.5	0.25
112	91	6	13	−7	49
102	100	10	10	0	0
93	111	15	3	12	144
99	109	13.5	5	8.5	72.25
115	100	4	10	−6	36
123	89	2	14	−12	144
104	104	8	6	2	4
99	111	13.5	3	10.5	110.25
113	102	5	7.5	−2.5	6.25
121	98	3	12	−9	81
103	111	9	3	6	36
101	123	11	1	10	100
					$\Sigma d_i^2 = 983$

$$\text{CF} = \text{correction factor} = \frac{m(m^2-1)}{12}$$

$$\text{CF for } 99 = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

$$\text{CF for } 111 = \frac{3(3^2-1)}{12} = 2$$

$$\text{CF for } 100 = \frac{3(3^2-1)}{12} = 2$$

$$\text{CF for } 102 = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

$$r = 1 - \frac{6\left(\sum_{i=1}^{n} d_i^2 + \text{CF}\right)}{n(n^2-1)}$$

$$= 1 - \frac{6\left(983 + \frac{1}{2} + 2 + 2 + \frac{1}{2}\right)}{15(15^2-1)} = -0.764$$

EXAMPLE 5.103 A sample of 12 fathers and their eldest sons have the following data about their heights in inches.

Fathers (X)	65	63	67	64	68	62	70	66	68	67	69	71
Sons (Y)	68	66	68	65	69	66	68	65	71	67	68	70

[AU April '03]

Calculate the rank correlation coefficient.

Solution

Fathers X	Sons Y	Rank of X	Rank of Y	$d_i = x_i - y_i$	d_i^2
65	68	9	5.5	3.5	12.25
63	66	11	9.5	1.5	2.25
67	68	6.5	5.5	1	1
64	65	10	11.5	−1.5	2.25
68	69	4.5	3	1.5	2.25
62	66	12	9.5	2.5	6.25
70	68	2	5.5	−3.5	12.25
66	65	8	11.5	−3.5	12.25
68	71	4.5	1	3.5	12.25
67	67	6.5	8	−1.5	2.25
69	68	3	5.5	−2.5	6.25
71	70	1	2	−1	1
				$\Sigma d_i = 0$	$\Sigma d_i^2 = 72.5$

Correction factors:

In X, 68 is repeated twice,

$$\therefore \quad \text{CF} = \frac{m(m^2-1)}{12} = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

In X, 67 is repeated twice,

$$\therefore \quad \text{CF} = \frac{m(m^2-1)}{12} = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

In Y, 68 is repeated 4 times,

$$\therefore \quad \text{CF} = \frac{m(m^2-1)}{12} = \frac{4(4^2-1)}{12} = 5$$

In Y, 66 is repeated twice,

$$\therefore \quad CF = \frac{m(m^2-1)}{12} = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

In Y, 65 is repeated twice,

$$\therefore \quad CF = \frac{m(m^2-1)}{12} = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

$$\therefore \quad \text{Rank correlation } r(X, Y) = 1 - \frac{6\left(\sum_{i=1}^{n} d_i^2 + CF\right)}{n(n^2-1)}$$

$$= 1 - \frac{6\left(72.5 + \frac{1}{2} + \frac{1}{2} + 5 + \frac{1}{2} + \frac{1}{2}\right)}{12(12^2-1)}$$

$$= 1 - \frac{6 \times 79.5}{12 \times 143} = 0.722$$

5.7 TRANSFORMATION OF RANDOM VARIABLES

In most of the electronic and electrical systems, we will be interested in finding the properties of a signal after it has been subjected to certain processing operations by the system. These signal processing operations may be viewed as transformations of a set of input variables to a set of output variables. If the input is a set of random variables, then the output will also be a set of random variables. In this section we deal with the techniques for obtaining the probability law for the set of output random variables when the probability law for the set of input random variables and the nature of transformation are known.

Let X and Y be any two random variables whose joint PDF is known and the random variables U and V be defined by the transformations $U = f_1(x, y)$, $V = f_2(x, y)$.

Then the joint PDF of U and V is given by

$$f(u, v) = f_{XY}(u, v)\ |J|$$

where J is the Jacobian of the transformation, i.e.

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

EXAMPLE 5.104 Let X and Y be independent uniform random variables over (0, 1), find the PDF of $Z = X + Y$.

[AU December '05]

Solution Given X and Y are independent uniform random variables over (0, 1).

The PDF of X and Y are given by

$$f(x) = 1, \quad 0 < x < 1,$$

$$f(y) = 1, \quad 0 < y < 1$$

$$\therefore \quad f(x, y) = f(x)\,f(y) = 1, \quad 0 < x < 1,\ 0 < y < 1$$

[$\because$ X and Y are independent]

Let $W = Y$

$$\therefore \quad Z = X + Y = W + X \Rightarrow X = Z - W$$

$w = y$ and $x = z - w$

The joint PDF of (z, w) is given by

$$f(z, w) = f_{XY}(z, w)\left|\frac{\partial(x, y)}{\partial(z, w)}\right|$$

$$J = \left|\frac{\partial(x, y)}{\partial(z, w)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$|J| = 1$$

The joint PDF of (z, w) is

$$g(z, w) = f_{XY}(z, w)\,|J| = 1 \times 1 = 1$$

$$\therefore \quad g(z, w) = 1$$

Since $0 < y < 1$, $0 < w < 1$ and $0 < x < 1 \Rightarrow 0 < z - w < 1$

$$\therefore \quad g(z, w) = 1, \quad 0 < w < z, \quad 0 < w < 1$$

$$\therefore \quad f(z) = \int_0^1 g(z, w)\,dw = \int_0^1 1\,dw = 1[w]_0^1 = 1$$

$$\therefore \quad f(z) = 1, \quad 0 < w < z$$

EXAMPLE 5.105 If X and Y are independently and exponentially distributed variables, obtain the distribution of $Z = X + Y$. **[AU May '05]**

Solution Given: X and Y are independently and exponentially distributed random variables. Let X be an exponential random variable with parameter λ_1, and Y be another exponential random variable with parameter λ_2. Then

$$f_X(x) = \lambda_1 e^{-\lambda_1 x}, \qquad x \geq 0$$

$$f_Y(y) = \lambda_2 e^{-\lambda_2 y}, \qquad y \geq 0$$

$$f(x, y) = f_X(x)\, f_Y(y) = \lambda_1 \lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}, \qquad x \geq 0,\ y \geq 0$$

Given: $Z = X + Y$

Let $W = Y$

$\therefore\ y = w$ and $x = z - w$

$$J = \left|\frac{\partial(x, y)}{\partial(z, w)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$|J| = 1$$

$$f(z, w) = f_{XY}(z, w)\,|J| = \lambda_1 \lambda_2\, e^{-[\lambda_1(z - w) + \lambda_2(w)]}$$

$$= \lambda_1 \lambda_2\, e^{-\lambda_1 z} e^{(\lambda_2 - \lambda_1) w}, \qquad w \geq 0,\ z \geq w$$

$$y \geq 0 \;\Rightarrow\; w \geq 0$$

$$x \geq 0 \;\Rightarrow\; z - w \geq 0 \;\Rightarrow\; z \geq w, \text{ i.e. } w \leq z$$

$\therefore\ 0 \leq w \leq z$

$$f(z) = \int_0^z f(z, w)\, dw = \int_0^z \lambda_1 \lambda_2 e^{-\lambda_1 z}\, e^{-(\lambda_2 - \lambda_1) w}\, dw$$

$$= \lambda_1 \lambda_2 e^{-\lambda_1 z} \left[\frac{e^{-(\lambda_2 - \lambda_1) w}}{-(\lambda_2 + \lambda_1)}\right]_0^z$$

$$= \lambda_1 \lambda_2\, e^{-\lambda_1 z} \left[\frac{e^{-\lambda_2 z} e^{\lambda_1 z} - 1}{-(\lambda_2 + \lambda_1)}\right]$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right), \qquad z \geq 0$$

EXAMPLE 5.106 Let X and Y be independent standard normal random variables. Find the PDF of $Z = X + Y$. **[AU December '05]**

Solution Given: $Z = X + Y$

Let $W = Y$

$$z = x + y \;\Rightarrow\; z = x + w \;\Rightarrow\; x = z - w,\ y = w$$

Given: X and Y are independent standard normal random variables

$\therefore$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2}} \qquad -\infty < y < \infty$$

$$\therefore \qquad f(x, y) = f_X(x)\, f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma^2} e^{-\frac{(x^2 + y^2)}{2}} \qquad -\infty < x, y < \infty$$

$$f_{ZW}(z, w) = f_{XY}(z, w)\ |J|$$

$$|J| = \frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{ZW}(z, w) = \frac{1}{2\pi} e^{\frac{-[(z-w)^2 + w^2]}{2}}$$

$$= \frac{1}{2\pi} e^{\frac{-(z^2 - 2zw + 2w^2)}{2}}, \qquad -\infty < z, w < \infty$$

$$f_Z(z) = \int_{-\infty}^{\infty} f(z, w)dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} e^{\frac{-(2w^2 - 2zw)}{2}} dw$$

$$= \frac{e^{-\frac{z^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{-(w^2 - zw)} dw$$

$$= \frac{e^{-\frac{z^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{-\left(w^2 - zw + \frac{z^2}{4} - \frac{z^2}{4}\right)} dw$$

$$= \frac{e^{-\frac{z^2}{2}}}{2\pi} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-\left(w - \frac{z}{2}\right)^2} dw$$

$$= \frac{e^{-\frac{z^2}{4}}}{2\pi} \sqrt{\pi}$$

$$f(z) = \frac{e^{-\frac{z^2}{4}}}{2\sqrt{\pi}}, \qquad -\infty < z < \infty$$

EXAMPLE 5.107 If the joint PDF of the random variables X and Y is given by

$$f(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

find the density function of the random variable $U = \frac{X}{Y}$. **[AU April '08]**

Solution Given: $f(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$

Let $V = Y$,

$$U = \frac{X}{Y}$$

$$\therefore \quad v = y \Rightarrow uv = x$$

$$|J| = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$$\therefore \quad |J| = v$$

$$\therefore \quad f(u, v) = f_{XY}(u, v)\,|J| \Rightarrow f(u, v) = 2v$$

Given: $0 < x < y < 1 \Rightarrow 0 < y < 1 \Rightarrow v < 1$

and $0 < uv < v \Rightarrow 0 < u < 1$

$\therefore \quad f(u, v) = 2v, \quad 0 < u < 1, \quad 0 < v < 1$

EXAMPLE 5.108 The random variables X and Y are statistically independent having a gamma distribution with parameters $(m, 1/2)$ and $(n, 1/2)$ respectively.

Derive the PDF of a random variable $U = \frac{X}{X+Y}$. **[AU December '08]**

Solution Given: X and Y are independent gamma random variables with paramater $(m, 1/2)$ and $(n, 1/2)$ respectively.

$$\therefore \quad f_X(x) = \frac{1}{\Gamma(m)}\left(\frac{1}{2}\right)^m x^{m-1} e^{-\frac{x}{2}}, \quad x > 0$$

$$f_Y(y) = \frac{1}{\Gamma(n)}\left(\frac{1}{2}\right)^n y^{n-1} e^{-\frac{y}{2}}, \quad y > 0$$

Since X and Y are independent $f(x, y) = f_X(x)\, f_Y(y)$

$$\therefore \quad f(x, y) = \frac{x^{m-1} y^{n-1}}{\Gamma(m)\Gamma(n)}\left(\frac{1}{2}\right)^{m+n} e^{-\left(\frac{x+y}{2}\right)}, \quad x > 0, y > 0$$

Given: $U = \frac{X}{X+Y}$

Let $V = X + Y$

$\therefore$ $x = uv$

and $y = v - uv = v(1 - u)$

$$J = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & 1-u \end{vmatrix} = v$$

$$|J| = v$$

$\therefore$
$$f(u, v) = f_{XY}(u, v)|J| = v\frac{(uv)^{m-1}[v(1-u)]^{n-1}}{\Gamma(m)\,\Gamma(n)\,2^{m+n}}\;e^{-\frac{v}{2}}$$

$$y > 0 \;\Rightarrow\; v(1-u) > 0 \;\Rightarrow\; v > 0$$

and
$$(1-u) > 0 \;\Rightarrow\; 1 > u \quad \text{or} \quad u < 1$$

$\therefore$
$$x > 0 \;\Rightarrow\; \text{as } v > 0,\ u > 0$$

$\therefore$
$$f(u, v) = \frac{u^{m-1}(1-u)^{m+n-1}}{\Gamma(m)\Gamma(n)\;2^{m+n}}\;e^{-\frac{v}{2}}, \qquad 0 \le u \le 1, v > 0$$

To find $f(u)$:

$$f(u) = \int_0^\infty u^{m-1}(1-u)^{n-1}\frac{v^{m+n-1}}{2^{m+n}}\,e^{-\frac{v}{2}}\,dv$$

$$= \frac{u^{m-1}\,(1-u)^{n-1}}{2^{m+n}\,\Gamma(m)\,\Gamma(n)}\int_0^\infty v^{m+n-1}\,e^{-\frac{v}{2}}\,dv$$

Put
$$\frac{v}{2} = t \;\Rightarrow\; dv = 2dt$$

$$f(u) = \frac{u^{m-1}\,(1-u)^{n-1}}{\Gamma(m)\Gamma(n)\,2^{m+n}}\int_0^\infty (2t)^{m+n-1}\,e^{-t}\,2\,dt$$

$$= \frac{u^{m-1}\,(1-u)^{n-1}}{\Gamma(m)\,\Gamma(n)}\int_0^\infty t^{m+n-1}\,e^{-t}\,dt$$

$$= \frac{u^{m-1}\,(1-u)^{n-1}}{\Gamma(m)\,\Gamma(n)}\,\Gamma(m+n)$$

$\therefore$
$$f(u) = \frac{u^{m-1}\,(1-u)^{n-1}}{\beta(m, n)}, \qquad 0 \le u \le 1$$

EXAMPLE 5.109 The joint PDF of a two-dimensional random variables (X, Y) is given by

$$f(x, y) = \begin{cases} 4xye^{-(x^2+y^2)}, & x \ge 0,\ y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the density function of $U = \sqrt{X^2 + Y^2}$. **[AU June '06, December '09]**

Solution Given: $u = \sqrt{x^2 + y^2}$ and let $v = x$

$x \geq 0, y \geq 0 \;\Rightarrow\; v \geq 0$ and $u \geq v$

$\Rightarrow \quad u \geq 0$ and $0 \leq v \leq u$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{J} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{\sqrt{x^2 + y^2}} & \dfrac{y}{\sqrt{x^2 + y^2}} \\ 1 & 0 \end{vmatrix} = \frac{-y}{\sqrt{x^2 + y^2}}$$

$$\therefore \quad |J| = \frac{\sqrt{x^2 + y^2}}{y}$$

The joint PDF of U and V is given by

$$f_{UV}(u, v) = f_{XY}(u, v)\,|J| = 4xye^{-(x^2 + y^2)}\,\frac{\sqrt{x^2 + y^2}}{y} = 4ve^{-u^2}u$$

$$\therefore \quad f(u, v) = \begin{cases} 4uve^{-u^2}, & u \geq 0, 0 \leq v \leq u \\ 0, & \text{otherwise} \end{cases}$$

Hence the density function of $U = \sqrt{X^2 + Y^2}$ is

$$f_U(u) = \int_0^u f(u, v)\,dv = 4ue^{-u^2}\int_0^u v\,dv = 4ue^{-u^2}\left[\frac{v^2}{2}\right]_0^u$$

$$\therefore \quad f_U(u) = \begin{cases} 2u^3e^{-u^2}, & u \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

EXAMPLE 5.110 Given the joint density function of X and Y as

$$f(x, y) = \begin{cases} \dfrac{1}{2}xe^{-y}, & 0 < x < 2, y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the distribution of $X + Y$.

Solution Given: $u = x + y$

Let $v = y$

$\therefore \quad x = u - v, \; y = v$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

The regions $0 < x < 2$ and $y > 0$ transform into $0 < u - v < 2$ and $v > 0$, as shown in Figure 5.22.

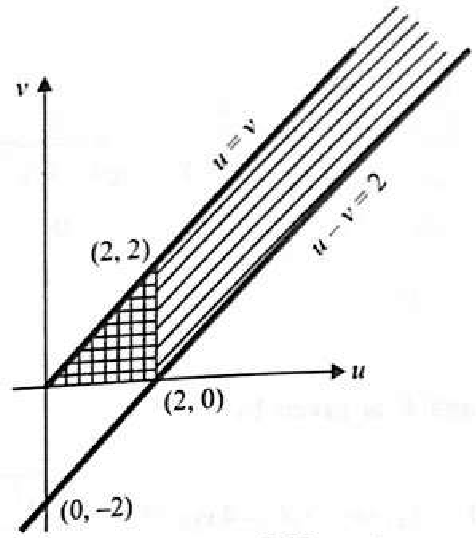

Figure 5.22

The joint density function of U and V is given by

$$f_{UV}(u, v) = f_{XY}(u, v)|J|$$

$$f(u, v) = \frac{1}{2}(u - v)e^{-v}, \quad 0 < v < u, u > 0 \quad (\because 0 < u - v \Rightarrow v < u)$$

The density function of $U = X + Y$ is obtained by splitting the range of U into two parts.

From the figure,

(i) $0 < u < 2$ (region I)

(ii) $u > 2$ (region II) (parallel strip to v axis)

For $0 < u < 2$ (in region I), the density of U is

$$f_U(u) = \int_0^u g(u, v)dv = \frac{1}{2}\int_0^u (u - v)e^{-v}\, dv$$

$$= \frac{1}{2}[(u - v)(-e^{-v}) - (-1)(e^{-v})]_0^u = \frac{1}{2}(u + e^{-u} - 1)$$

and for $u > 2$ (region II),

$$f_U(u) = \frac{1}{2}\int_{u-2}^{u} (u - v)e^{-v}dv = \frac{1}{2}[(u - v)(-e^{-v}) - (-1)(e^{-v})]_{u-2}^{u}$$

$$= \frac{1}{2}[e^{-v}(1 + v - u)]_{u-2}^{u} = \frac{1}{2}e^{-u}(1 + e^2) = \frac{1}{2}(e^{-u} + e^{2-u})$$

Hence $$f_U(u) = \begin{cases} \frac{1}{2}(e^{-u} + u - 1), 0 < u \le 2 \\ \frac{1}{2}(e^{-u} + e^{2-u}), 2 < u < \infty \\ 0, \qquad \text{otherwise} \end{cases}$$

EXAMPLE 5.111 If the PDF of a two-dimensional random variables (X, Y) is given by $f(x, y) = x + y$, $0 \le x, y \le 1$, find the PDF of $U = XY$.

Solution Given: $U = XY$

Let $v = y$

$\therefore$ $x = \frac{u}{v}$

and $y = v$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & 0 \\ \frac{-u}{v^2} & 1 \end{vmatrix} = \frac{1}{v}$$

The joint PDF of u and v is given by

$$f_{UV}(u, v) = f_{XY}(u, v)|J| = (x + y)\frac{1}{v} = \left(\frac{u}{v} + v\right)\frac{1}{v} = 1 + \frac{u}{v^2}$$

$0 \le y \le 1 \Rightarrow 0 \le v \le 1$

and $0 \le x \le 1 \Rightarrow 0 \le u \le v$

Range of u and v is shown in Figure 5.23.

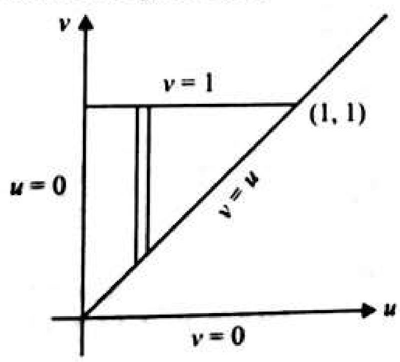

Figure 5.23

Limits of v : u to 1 (parallel strip to v axis)

The PDF of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f(u, v)dv = \int_u^1 \left(1 + \frac{u}{v^2}\right)dv = [v]_u^1 - \left[\frac{u}{v}\right]_u^1 = 1 - u - u + 1$$

$$f_U(u) = 2(1 - u), \qquad 0 \le u \le 1$$

EXAMPLE 5.112 If X and Y are two independent random variables which are exponentially distributed with parameter 1, find the joint PDF of $U = X + Y$ and $V = X - Y$. **[AU April '08]**

Solution Since X and Y are independent, $f(x, y) = f_X(x) \cdot f_Y(y)$

Given: X and Y are exponentially distributed random variables with parameter 1, i.e. $\lambda = 1$

$$\therefore \quad f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = e^{-y}, \quad y \geq 0 \qquad [\because f(x) = \lambda e^{-\lambda x}, \lambda = 1]$$

$$f(x, y) = e^{-(x+y)}, \quad x \geq 0, y \geq 0$$

Given:
$$u = x + y$$
$$v = x - y$$

i.e.
$$u + v = 2x \Rightarrow x = \frac{u+v}{2}$$

$$x \geq 0, y \geq 0 \Rightarrow u = x + y \geq 0$$

$$u - v = 2y \Rightarrow y = \frac{u-v}{2}$$

$$\therefore \quad y \geq 0 \Rightarrow u \geq v \Rightarrow v \leq u$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$|J| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

The joint PDF of (u, v) is

$$f(u, v) = f_{XY}(u, v)\,|J| = \frac{1}{2} e^{-\left(\frac{u+v}{2} + \frac{u-v}{2}\right)} = \frac{1}{2} e^{-(u)}, \qquad u \geq 0, 0 \leq v \leq u$$

EXAMPLE 5.113 If X and Y are independent random variables with $f_X(x) = e^{-x}\, U(x)$ and $f_Y(y) = 3e^{-3y} U(y)$, find $f_Z(z)$, if $Z = X/Y$.

Solution Let $w = y$

$$\therefore \quad z = \frac{x}{y} = \frac{x}{w} \Rightarrow x = zw$$

$$y = w$$

Since X and Y are independent,

$$f(x, y) = f_X(x) \cdot f_Y(y)$$

$$= e^{-x}\, U(x)\, 3e^{-3y}\, U(y) = 3e^{-(x+3y)}, \qquad x, y \geq 0$$

$$J = \frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} \qquad \left[\because U(x) = \begin{cases} 1, \text{if } x \geq 0 \\ 0, \text{otherwise} \end{cases} \text{ and } U(y) = \begin{cases} 1, \text{if } y \geq 0 \\ 0, \text{otherwise} \end{cases} \right]$$

$$J = w \Rightarrow |J| = w$$

$\therefore$

$\therefore$ $$f(z, w) = f_{XY}(z, w)\,|J|$$
$$= w \cdot 3e^{-(zw + 3w)}$$
$$= 3\,we^{-w(z + 3)}$$

Since $$y \geq 0,\ w \geq 0$$
$$x \geq 0 \Rightarrow z \geq 0 \text{ since } w \geq 0$$

$\therefore$ $$f(z, w) = 3we^{-w(z + 3)}, \quad z \geq 0,\ w \geq 0$$

$$f_Z(z) = \int_R f(z, w)\,dw = \int_0^\infty 3we^{-w(z+3)}\,dw$$

$$= 3\left[w \cdot \frac{e^{-w(z+3)}}{-(z+3)} - 1 \cdot \frac{e^{-w(z+3)}}{-(z+3)^2} \right]_0^\infty$$

$\therefore$ $$f_Z(z) = \frac{3}{(z+3)^2}, \quad z \geq 0$$

EXAMPLE 5.114 If X and Y each following exponential distribution with mean 1 and are independent, find the PDF of $U = X - Y$.

[AU December '03, June '06, November '07]

Solution The mean of the exponential distribution is $\frac{1}{\lambda}$

Given: mean $= \frac{1}{\lambda} = 1 \Rightarrow \lambda = 1$

$\therefore$ The PDFs of X and Y are

$$f_X(x) = e^{-x}, \quad x \geq 0$$
$$f_Y(y) = e^{-y}, \quad y \geq 0$$

Since X and Y are independent,

$$f(x, y) = f_X(x)\,f_Y(y)$$

$\Rightarrow$ $$f(x, y) = e^{-(x + y)},\ x \geq 0,\ y \geq 0$$

Let $$V = Y$$

i.e. $$v = y$$

Given: $$U = X - Y$$

$\therefore$ $$u = x - y \Rightarrow x = u + v$$

$\therefore$ $$x = u + v$$

and $\qquad y = v$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$|J| = 1$$

$\therefore \qquad f(u, v) = e^{-(u+v+v)}$

i.e. $\qquad f(u, v) = e^{-(u+2v)}, \qquad u + v \geq 0$

Since $y \geq 0$, $v \geq 0$ and $u + v \geq 0 \Rightarrow v \geq -u$, when $u \leq 0$, $v \geq 0$ and $u \geq 0$, $v \leq 0$

$\therefore \qquad f_U(u) = \int_R f(u, v)dv$

$$= \int_{-u}^{\infty} e^{-(u+2v)}dv, \qquad \text{when } u < 0$$

$$= \int_{0}^{\infty} e^{-(u+2v)}dv, \qquad \text{when } u > 0$$

i.e. $\qquad f_U(u) = e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty} = \frac{e^{-u}e^{2u}}{2} = \frac{e^{u}}{2}, \qquad \text{when } u < 0$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{0}^{\infty} = \frac{e^{-u}}{2}, \qquad \text{when } u > 0$$

$\therefore \qquad f_U(u) = \frac{e^{u}}{2}, \quad u < 0$

$$= \frac{e^{-u}}{2}, \qquad u > 0$$

EXAMPLE 5.115 If X and Y are independent random variables each following normal distribution with mean 0 and variance 4, find the PDF of $Z = 2X + 3Y$.

Solution Given: $Z = 2X + 3Y$

Let $\qquad W = Y$

$\therefore \qquad z = 2x + 3y \Rightarrow x = \frac{z - 3w}{2}$

$$y = w$$

$$J = \frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{-3}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

For normal distribution,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Given:

$$\mu = 0$$

and

$$\sigma^2 = 4$$

Since X and Y are independent,

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

$$\Rightarrow \quad f_{XY}(x, y) = \frac{1}{8\pi} e^{-\frac{(x^2+y^2)}{8}}, \quad -\infty < x, y < \infty$$

$$f_{ZW}(z, w) = |J|\, f_{XY}(u, v) = \frac{1}{2} \times \frac{1}{8\pi} e^{-\left[\frac{(z-3w)^2 + 4w^2}{32}\right]}, \quad -\infty < z, w < \infty$$

$$f_Z(z) = \frac{1}{16\pi} \int_{-\infty}^{\infty} e^{-\left(\frac{13w^2 - 6zw + z^2}{32}\right)} dw$$

$$= \frac{1}{16\pi} e^{-\left(\frac{z^2}{8\times 13}\right)} \int_{-\infty}^{\infty} e^{\frac{-13\left(w - \frac{3z}{13}\right)^2}{32}} dw$$

$$= \frac{1}{2\sqrt{13}\sqrt{2\pi}} e^{\frac{-z^2}{2(2\sqrt{13})^2}}, \quad -\infty < z < \infty$$

which follows normal distribution with mean zero and variance $(2\sqrt{13})^2$, i.e. $N(0,\ 2\sqrt{13}\,)$.

EXAMPLE 5.116 If X and Y are independent random variables having density functions

$$f(x) = \begin{cases} 2e^{-2x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

and

$$g(y) = \begin{cases} 3e^{-3y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$

find the density function of their sum $U = X + Y$. **[AU April '03]**

Solution Given:

$$f(x) = 2e^{-2x}, \quad x \ge 0$$

$$g(y) = 3e^{-3y}, \quad y \ge 0$$

and

$$u = x + y \Rightarrow x = u - y$$

We know that

$$f_U(u) = \int_0^u f_x(u-y) g_y(y)\,dy$$

$$= \int_0^u 2e^{-2(u-y)} 3e^{-3y}\,dy = 6\int_0^u e^{-2u} e^{2y} e^{-3y}\,dy$$

$$= 6e^{-2u}\int_0^u e^{-y}\,dy = 6e^{-2u}[-e^{-y}]_0^u = 6e^{-2u}(1-e^{-u}), \quad u>0$$

$$(\because\ x \geq 0, y \geq 0 \ \Rightarrow\ u > 0)$$

EXAMPLE 5.117 If the joint density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 6e^{-3x_1 - 2x_2}, & \text{for } x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

find the probability density of $V = X_1 + X_2$. **[AU November '06]**

Solution $\therefore$ Given: $f_{XY}(x_1, x_2) = \begin{cases} 6e^{-3x_1-2x_2}, & \text{for } x_1 > 0,\ x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$

We have to find the PDF of $V = X_1 + X_2$

Let us assume that $U = X_1$

$\therefore$ $$x_1 = u,\ x_2 = v - u$$

The PDF of (U, V) is given by

$$f_{UV}(u,v) = f_{X_1X_2}(u,v)\left|\frac{\partial(x_1, x_2)}{\partial(u,v)}\right| \qquad \text{(i)}$$

Now

$$J = \left|\frac{\partial(x_1, x_2)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \qquad \text{(ii)}$$

Substituting Eq. (ii) in (i), we get

$$f_{UV}(u, v) = 1 \times 6e^{-3x_1 - 2x_2} = 6e^{-3u - 2(v-u)}$$

$$= 6e^{-3u - 2v + 2u} = 6e^{-u - 2v}$$

The range of U and V is

$$x_1 = u > 0 \ \Rightarrow\ u > 0$$

$$x_2 = v - u > 0 \ \Rightarrow\ v > u > 0$$

$\therefore$ The PDF of V is

$$f_V(v) = \int_0^v 6e^{-u-2v}\,du = \int_0^v 6e^{-u}e^{-2v}\,du$$

$$= 6e^{-2v}\int_0^v e^{-u}\,du = 6e^{-2v}\left[\frac{e^{-u}}{-1}\right]_0^v$$

$$= 6e^{-2v}\left[\frac{e^{-v}}{-1}-\left(\frac{1}{-1}\right)\right] = 6e^{-2v}(-e^{-v}+1)$$

$$= 6e^{-2v}(1-e^{-v}) = 6(e^{-2v}-e^{-3v}), \qquad v > u > 0$$

EXAMPLE 5.118 If X and Y are independent random variables with PDF $e^{-x}, x \geq 0$ and $e^{-y}, y \geq 0$ respectively, find the density functions of $U = \dfrac{X}{X+Y}$ and $V = X + Y$. Are U and V independent?

[AU June '06, November '06, December '09]

Solution Given: $u = \dfrac{x}{x+y}$

$$v = x + y$$

$$\therefore \quad uv = x, \qquad v = y + uv$$

$$y = v - uv \;\Rightarrow\; y = v(1-u)$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= v(1-u) + uv = v$$

$$|J| = v$$

Since X and Y are independent,

$$f(x, y) = f_X(x)\cdot f_Y(y) = e^{-(x+y)}, \qquad x \geq 0, \qquad y \geq 0$$

$$f(u, v) = f_{XY}(u, v)\cdot |J|$$

$$= v\cdot e^{-(uv+v-uv)}$$

$$f(u, v) = ve^{-v}, \qquad uv \geq 0, \qquad v(1-u) \geq 0$$

Since $x \geq 0$ and $y \geq 0$, $\quad v = x + y \geq 0$

$$\therefore \quad y \geq 0 \Rightarrow v(1-u) \geq 0 \;\Rightarrow\; v \geq 0 \text{ and } (1-u) \geq 0$$

$$1 - u \geq 0 \;\Rightarrow\; u \leq 1$$

But, $x = uv \geq 0 \;\Rightarrow\; u \geq 0$ $\qquad [\because v > 0]$

$\therefore \quad 0 \leq u \leq 1.$

[$\because$ Let $v \leq 0$ and $1 - u \leq 0$, then $1 \leq u$, i.e. $u \geq 1$, $v \leq 0$.

But if $v \leq 0$, then

$$x = uv \geq 0 \;\Rightarrow\; u \leq 0$$

which is not possible.]

$$\therefore \qquad 0 \leq u \leq 1 \text{ and } v \geq 0.$$

To show that U and V are independent, we have to show that

$$f(u, v) = f_U(u) \cdot f_V(v)$$

$$f_U(u) = \int_R f(u, v)dv = \int_0^\infty ve^{-v}dv = \left[v\frac{e^{-v}}{-1} - (1)\frac{e^{-v}}{(-1)^2} \right]_0^\infty = 1$$

$$f_V(v) = \int_R f(u, v)du = \int_0^1 ve^{-v}du = ve^{-v}\int_0^1 du = ve^{-v}[u]_0^1 = ve^{-v}$$

$$f(u, v) = f_U(u)\, f_V(v) = ve^{-v}$$

$\therefore$ U and V are independent.

EXAMPLE 5.119 If X and Y are independent random variables each following $N(0, \sigma^2)$, find the density function of $R = \sqrt{X^2 + Y^2}$ and $\phi = \tan^{-1}\left(\frac{Y}{X}\right)$.

[AU December '03, '04; May '04, 09]

Solution Given: X and Y are independent random variables with PDF,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2\sigma^2}}$$

$\therefore$

$$f(x, y) = f_X(x)\, f_Y(y) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2 + y^2)}{2\sigma^2}}$$

Here we consider the transformation

$$x = r\cos\theta$$
$$y = r\sin\theta$$

that is, transformation from cartesian to polar coordinates, we get

$$r = \sqrt{x^2 + y^2}$$

and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

The joint PDF of (r, θ) is

$$f_{r\theta}(r,\theta) = |J|\, f_{xy}(r,\theta) = \frac{|r|}{2\pi\sigma^2} e^{\frac{-r^2}{2\sigma^2}}, \qquad r \geq 0, 0 \leq \theta \leq 2\pi$$

EXAMPLE 5.120 If X and Y are independent variates uniformly distributed in (0, 1), find the distributions of XY. [AU May '06]

Solution Given: X and Y are uniformly distributed in (0, 1)

∴ The PDF of X and Y are

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Since X and Y are independent, the joint PDF of X and Y is

$$f(x,y) = \begin{cases} 1, & 0 \leq x, y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $\quad U = XY$

and $\quad V = X$

∴ The transformation is

$$u = xy \text{ and } v = x$$

$$\Rightarrow \qquad x = v,\ y = \frac{u}{v}$$

The joint PDF of (U, V) is given by

$$f_{UV}(u,v) = f_{XY}(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \qquad \text{(i)}$$

Now,

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}$$

$$|J| = \frac{1}{v}$$

The joint PDF of (U, V) is

$$f_{UV}(u,v) = f_{XY}(u,v)\,|J|$$

$$= \frac{1}{v} f_{XY}(u,v)$$

$$f(u, v) = \frac{1}{v}(1) = \frac{1}{v}$$

$\therefore$

Since

$$0 \le x \le 1 \Rightarrow 0 \le v \le 1 \qquad [\because v = x]$$

$$0 \le y \le 1 \Rightarrow 0 \le \frac{u}{v} \le 1$$

$$\Rightarrow 0 \le u \le v \le 1 \qquad (\because 0 \le v \le 1)$$

$\therefore$ The marginal PDF of $U = XY$ is

$$f_U(u) = \int_{-\infty}^{\infty} f(u, v)dv = \int_u^1 \frac{1}{v} dv = [\log v]_u^1$$

$$= \log 1 - \log u = -\log u$$

$\therefore$

$$f(u) = -\log u, \qquad 0 \le u \le 1$$

EXAMPLE 5.121 The joint PDF of X and Y is given by

$$f(x, y) = e^{-(x+y)}, \qquad x > 0, y > 0$$

find the PDF of $U = \dfrac{X+Y}{2}$. **[AU December '07]**

Solution Given: $f(x, y) = e^{-(x+y)}, \qquad x > 0, y > 0$

and

$$U = \frac{X+Y}{2}$$

i.e.

$$u = \frac{x+y}{2}$$

Let us make the transformation

$$u = \frac{1}{2}(x+y)$$

and

$$v = y \qquad \text{(i)}$$

$\therefore$

$$2u = x + y \Rightarrow 2u - v = x$$

$$x = 2u - v, \; y = v$$

The Jacobian J of the transformation is given by

$$J = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

The PDF of (u, v) is given by

$$f(u, v) = f_{XY}(u, v)\,|J|$$

$$= e^{-(x+y)} \cdot 2 = 2e^{-2u}$$

$$x > 0 \Rightarrow 2u - v > 0$$

$$\Rightarrow 2u > v$$

$$y > 0 \Rightarrow v > 0$$

$\therefore$ $$2u > 0$$

i.e. $$u > 0$$

$\therefore$ $$u > 0,\ 0 < v < 2u$$

$\therefore$ $$f(u, v) = 2e^{-2u}, \quad u > 0,\ 0 < v < 2u$$

The PDF of u is given by

$$f_U(u) = \int_0^{2u} f(u, v)\,dv$$

$$= 2e^{-2u}\int_0^{2u} dv = 4ue^{-2u}, \quad u \geq 0$$

5.8 CENTRAL LIMIT THEOREM (CLT)

5.8.1 Liapounoff's Form

If $X_1, X_2, X_3, \ldots, X_n$ is a sequence of independent identically distributed random variables with $E(X_i) = \mu$ and Var $(X_i) = \sigma_i^2$ and if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain conditions S_n follows normal distribution with mean $n\mu$ and variance $n\sigma^2$ as $n \to \infty$.

Note:

If $\bar{X} = \dfrac{S_n}{n} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$, then $\bar{X}$ follows normal distribution with mean

$E(\bar{X}) = \dfrac{n\mu}{n} = \mu$, variance $\dfrac{\bar{X}}{n} = \dfrac{\sigma^2}{n}$

i.e. $\bar{X}$ follows $N\left(\mu, \dfrac{\sigma^2}{n}\right)$.

5.8.2 Lindberg–Levy's Form

If $X_1, X_2, X_3, \ldots, X_n$, is a sequence of independent identically distributed random variables with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma_i^2$, $i = 1, 2, \ldots$ and if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain conditions S_n follows normal distribution with mean $n\mu$ and variance $n\sigma^2$ as $n \to \infty$.

5.8.3 Applications of Central Limit Theorem

1. It provides a simple method for computing approximate probabilities of sums of independent random variables.
2. It gives us the fact that the empirical frequencies of so many natural "populations" exhibit a bell-shaped curve (normal curve).

Central Limit Theorem

THEOREM 1 If $X_1, X_2, X_3, \ldots, X_n$ are n independent identically distributed random variables with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$ and if $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the variate $Z = \frac{X - \mu}{\sigma/\sqrt{n}}$ has a distribution that approaches the standard normal distribution as $n \to \infty$ provided the moment generating function exists.

Proof Given: $\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$

Using the definition of moment generating function, we have the moment generating function of the random variate Z about the origin is given by,

$$M_Z(t) = E(e^{tX}) = E\left[e^{\frac{t\sqrt{n}}{\sigma}(\bar{x} - \mu)}\right]$$

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E\left(e^{\frac{t\sqrt{n}}{\sigma}\bar{x}}\right)$$

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E\left[e^{\frac{t}{\sigma\sqrt{n}}(x_1 + x_2 + x_3 + \cdots + x_n)}\right]$$

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E\left(e^{\frac{tx_1}{\sigma\sqrt{n}}} \cdot e^{\frac{tx_2}{\sigma\sqrt{n}}} e^{\frac{tx_3}{\sigma\sqrt{n}}} \cdots e^{\frac{tx_n}{\sigma\sqrt{n}}}\right)$$

Since $X_1, X_2, X_3, \ldots, X_n$ are independent, we have

$$M_Z(t) = e^{\frac{-\mu t\sqrt{n}}{\sigma}} E\left(e^{\frac{tx_1}{\sigma\sqrt{n}}}\right) E\left(e^{\frac{tx_2}{\sigma\sqrt{n}}}\right) E\left(e^{\frac{tx_3}{\sigma\sqrt{n}}}\right) \cdots E\left(e^{\frac{tx_n}{\sigma\sqrt{n}}}\right)$$

All variables $X_1, X_2, X_3, \ldots, X_n$ have the same moment generating function,

$$M_Z(t) = e^{\frac{-\mu t\sqrt{n}}{\sigma}} M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right) M_{X_2}\left(\frac{t}{\sigma\sqrt{n}}\right) M_{X_3}\left(\frac{t}{\sigma\sqrt{n}}\right) \cdots M_{X_n}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} \left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

where

$$M_{X_i}\left(\frac{t}{\sigma\sqrt{n}}\right) = M_X\left(\frac{t}{\sigma\sqrt{n}}\right), \quad i = 1, 2, \ldots, n$$

$$\therefore \quad \log M_Z(t) = -\frac{\mu t\sqrt{n}}{\sigma} + n\log\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]$$

$$= -\frac{\mu t\sqrt{n}}{\sigma} + n\log\left[1 + \mu_1' \frac{t}{\sigma\sqrt{n}} + \frac{\mu_2'}{2!}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \frac{\mu_3'}{3!}\left(\frac{t}{\sigma\sqrt{n}}\right)^3 + \cdots\right]$$

$$\therefore \quad \log M_Z(t) = -\frac{\mu t\sqrt{n}}{\sigma} + n\left[\left(\mu_1' \frac{t}{\sigma\sqrt{n}} + \frac{\mu_2'}{2!}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \cdots\right) - \frac{1}{2}\left(\mu_1' \frac{t}{\sigma\sqrt{n}} + \frac{\mu_2'}{2!}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \cdots\right)^2 \cdots\right]$$

Using $\mu_1' = \mu =$ mean, we get

$$\log M_Z(t) = -\frac{\mu t\sqrt{n}}{\sigma} + \frac{\mu t\sqrt{n}}{\sigma} + \frac{t^2}{2\sigma^2}\left[\mu_2' - (\mu_1')^2\right] + \text{terms containing } n \text{ in the denominator}$$

$$\Rightarrow \quad \log M_Z(t) = \frac{t^2}{2\sigma^2}(\sigma^2) + \text{terms containing } n \text{ in the denominator}$$

$$\therefore \quad \lim_{n\to\infty} \log M_Z(t) = \frac{t^2}{2} \Rightarrow M_Z(t) = e^{\frac{t^2}{2}} \text{ as } n \to \infty$$

which is the moment generating function of $N(0, 1)$.

Therefore, as $n \to \infty$, the distribution of Z tends to the standard normal distribution.

Hence the proof.

EXAMPLE 5.122 A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using CLT, find with what probability can we assert that the mean of the sample will not differ from $\mu = 60$ by more than 4. **[AU April '03, December '04]**

Solution Given:

$$n = 100$$
$$\mu = 60$$
$$\sigma^2 = 400$$

Given that sample mean $\bar{X}$ will not differ from $\mu = 60$ by more than 4

$$\therefore \quad P(|\bar{X} - 60| \le 4) = P\left(\frac{|\bar{X} - 60|}{\sigma/\sqrt{n}} \le \frac{4}{\sigma/\sqrt{n}}\right)$$

$$= P\left(|Z| \le \frac{4\sqrt{n}}{20}\right) \qquad \left(\because Z = \frac{X - 60}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= P\left(|Z| \le \frac{4\sqrt{100}}{20}\right) = P(|Z| \le 2)$$
$$= P(-2 \le Z \le 2)$$
$$= 2\,P(0 \le Z \le 2)$$
$$= 2 \times 0.4773$$

$$\therefore \quad P(|\bar{X} - 60| \le 4) = 0.9546$$

EXAMPLE 5.123 If X_i, $i = 1, 2, \ldots, 50$ are independent random variables each having a Poisson distribution with parameter $\lambda = 0.03$ and $S_n = X_1 + X_2 + X_3 + \cdots + X_{50}$. Evaluate $P(S_n \ge 3)$ using CLT. Compare your answer with the exact value of the probability.

Solution Given: $n = 50$
Here $X_1, X_2, X_3, \ldots, X_{50}$ are Poisson random variables.
For Poisson distribution:

$$\text{Mean} = \text{Var} = \lambda$$

$$\therefore \quad \mu = \text{Mean} = \lambda = 0.03$$
$$\sigma^2 = \text{Var} = \lambda = 0.03$$

$$\text{Mean of } S_n = E(S_n) = n\mu = 50 \times 0.03 = 1.5$$

$$\text{Standard deviation} = \sqrt{n}\,\sigma = \sqrt{50} \times \sqrt{0.03} = \sqrt{1.5}$$

$\therefore$ $S_n = X_1 + X_2 + \cdots + X_{50}$ follows a normal distribution with mean $n\mu = 1.5$.

$$P(S_n \ge 3) = P\left(\frac{S_n - n\mu}{\sqrt{n}\cdot\sigma} \ge \frac{3 - n\mu}{\sqrt{n}\cdot\sigma}\right)$$

$$= P\left(\frac{S_n - 1.5}{\sqrt{1.5}} \geq \frac{3 - 1.5}{\sqrt{1.5}}\right) \quad \left(Z = \frac{S_n - 1.5}{\sqrt{1.5}}\right)$$
$$= P(Z \geq 1.23)$$
$$= P(0 \leq Z \leq \infty) - P(0 \leq Z \leq 1.23)$$
$$= 0.5 - 0.3888 = 0.1112$$

To find the exact probability:

Here, $S_n = X_1 + X_2 + \cdots + X_{50}$ follows a Poisson distribution. By the reproductive property of Poisson distribution, S_n follows Poisson distribution with mean

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{50} = \lambda = 50 \times 0.03 = 1.5$$

$$P(S_n = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$P(S_n \geq 3) = 1 - P(S_n < 3)$$
$$= 1 - [P(S_n = 0) + P(S_n = 1) + P(S_n = 2)]$$
$$= 1 - \frac{e^{-1.5}(1.5)^0}{0!} - \frac{e^{-1.5}(1.5)^1}{1!} - \frac{e^{-1.5}(1.5)^2}{2!}$$
$$= 0.1912$$

EXAMPLE 5.124 A distribution with unknown mean μ has variance 1.5. Using CLT, find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 that the probability sample mean will be within 0.5 of the population mean. **[AU April '03, December '04]**

Solution Let $\bar{X}$ denote the sample mean of a sample of size n taken from the distribution.

Given: $\quad E(X_i) = \mu$

and $\quad \mathrm{Var}(X_i) = 1.5 \quad (i = 1, 2, \ldots, n)$

By CLT, $\bar{X}$ follows normal distribution with mean μ and variance $\frac{\sigma^2}{n}$, i.e.

$\bar{X}$ follows $N\left(\mu, \frac{\sigma^2}{n}\right)$

Given that the probability will be at least 0.95 that the probability sample mean will be within 0.5 of the population mean.

We have to find n, the size of the sample, such that

$$P(\mu - 0.5 < \bar{X} < \mu + 0.5) \geq 0.95 \quad \Rightarrow \quad P[|\bar{X} - \mu| < 0.5] \geq 0.95$$

i.e. $$P\left(\frac{|\bar{X} - \mu|}{\sigma/\sqrt{n}} < \frac{0.5}{\sigma/\sqrt{n}}\right) \geq 0.95 \quad \Rightarrow \quad P\left(|Z| < \frac{0.5\sqrt{n}}{\sqrt{1.5}}\right) \geq 0.95$$

$\Rightarrow$ $$P(|Z| < 0.4082\sqrt{n}) \geq 0.95$$

From the normal table,

$$P(|Z| < 1.96) = 0.95$$

$\therefore$ The least value of n such that $P(|Z| < 0.4082\sqrt{n}) = 0.95$ is

$$0.4082\sqrt{n} = 1.96 \Rightarrow n = \left(\frac{1.96}{0.4082}\right)^2 = 24$$

$\therefore$ Required sample size $n = 24$.

EXAMPLE 5.125 Twenty dice are thrown. Find approximately the probability that the sum obtained is between 65 and 75 using CLT. **[AU December '09]**

Solution Let X_i be the random variable representing the number shown on the ith dice, $i = 1, 2, ..., 20$, then

$$\mu = E(X_i) = \sum_{i=1}^{6} i\left(\frac{1}{6}\right) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

$$E(X_i^2) = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}$$

Then,

$$\text{Var}(X_i) = E(X_i^2) - [E(X_i)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Let S_n be the total score. Then

$$S_n = X_1 + X_2 + \cdots + X_{20}$$

$$E(S_n) = 20 \times \frac{7}{2} = 70 = (n\mu)$$

$$\text{Var}(S_n) = 20 \times \frac{35}{12} = \frac{175}{3} = (n\sigma^2)$$

By CLT,

$$S_n \sim N(n\mu, n\sigma^2)$$

i.e.

$$S_n \sim N\left(70, \frac{175}{3}\right)$$

Let $Z = \dfrac{S_n - 70}{\sqrt{\dfrac{175}{3}}}$ be the Standard normal variate when $S_n = 65$, $Z = -0.65$ and $S_n = 75$, $S_n = 0.65$

$$\therefore \quad P(65 < S_n < 75) = P(-0.65 < Z < 0.65)$$
$$= 2P(0 < Z < 0.65) = 2 \times 0.2422 = 0.4844$$

$\therefore$ The required probability = 0.4844

EXAMPLE 5.126 A coin is tossed 10 times. What is the probability of getting 3 or 4 or 5 heads? Use central limit theorem. **[AU December '09]**

Solution Let X denote the number of heads in 10 tosses and it follows binomial distribution.

$$\text{Mean} = np$$

and $\quad \text{Var} = npq$

Here $\quad n = 10$

$$p = \text{probability of getting head in a trial} = \frac{1}{2}$$

$$q = 1 - p = \frac{1}{2}$$

$\therefore$

$$\text{Mean of } X = np = 10 \times \frac{1}{2} = 5$$

$$\text{Var } X = nqp = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$$

$\therefore \quad \mu = 5$

and $\quad \sigma = \sqrt{2.5} = 1.58$

To find $P(3 \le X \le 5)$:

Since X is a discrete random variable, we find $P(2.5 \le X \le 5.5)$.

$$\therefore \quad P(2.5 \le X \le 5.5) = P\left(\frac{2.5-5}{1.58} < Z < \frac{5.5-5}{1.58}\right), \text{ where } Z = \frac{X-\mu}{\sigma}$$

$$= P(-1.58 < Z < 0.32)$$

$$= P(-1.58 < Z < 0) + P(0 < Z < 0.32)$$

$$= P(0 < Z < 1.58) + P(0 < Z < 0.32)$$

$$\therefore \quad P(2.5 \le X \le 5.5) = 0.4429 + 0.1255$$

Hence, the required probability = 0.5684.

EXAMPLE 5.127 If V_i, $i = 1, 2, \ldots, 20$ are independent noise voltages received in an adder and V is the sum of the voltages received, find the probability that the total incoming voltage exceeds 105 using CLT. Assume that each of the random variables V_i is uniformly distributed over (0, 10).

Solution Since $V_1, V_2, V_3, \ldots, V_{20}$ follows a uniform distribution

$$\text{Mean} = \mu = \frac{b+a}{2} = \frac{10}{2} = 5$$

$$\text{Var}\,\sigma^2 = \frac{(b-a)^2}{12} = \frac{100}{12} = 8.33$$

Given: V is the sum of the voltages received,

i.e.

$$V = V_1 + V_2 + V_3 + \cdots + V_{20}$$

To find $P(V > 105)$:
By CLT, V follows a normal a distribution

$$V \sim N(n\mu, \sqrt{n}\sigma) = N(100, 12.9)$$

To find the probability that the total incoming voltage exceeds 105 is

$$P(V > 105) = P\left(\frac{V - n\mu}{\sigma\sqrt{n}} > \frac{105 - n\mu}{\sigma\sqrt{n}}\right)$$

$$= P\left(Z > \frac{105 - 100}{12.9}\right) = P\left(Z > \frac{5}{12.9}\right)$$

$$= P(Z > 0.387) = P\ (0 < Z < \infty) - P(0 < Z < 0.387)$$

$$\therefore \quad P(V > 105) = 0.5 - 0.1517 = 0.3483$$

EXAMPLE 5.128 The life-time of a certain brand of an electric bulb may be considered as a random variable with mean 1200 and SD 250. Find the probability using CLT that the average lifetime of 60 bulbs exceeds 1250 hours.

[AU April '04, December '08]

Solution Let $\bar{X}$ denote the mean life-time of 60 bulbs.
Given: $n = 60$

$$\mu = 1200$$

and

$$\sigma = 250$$

$\bar{X}$ follows normal distribution with mean μ and variance $\frac{\sigma^2}{n}$

To find the probability that average lifetime of 60 bulbs exceeds 1250 hours

$$P(\bar{X} \geq 1250) = P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \geq \frac{1250 - \mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= P\left(Z \geq \frac{1250 - 1200}{\frac{250}{\sqrt{60}}}\right)$$

$$= P(Z \geq 1.55)$$

$$= P(0 \leq Z \leq \infty) - P(\ 0 \leq Z \leq 1.55)$$

$$= 0.5 - 0.4394 = 0.0606$$

EXAMPLE 5.129 If $X_1, X_2, X_3, \ldots, X_n$ are Poisson variates with parameter $\lambda = 2$, use CLT to estimate $P(120 \leq S_n \leq 160)$ where $S_n = X_1 + X_2 + \cdots + X_n$ and $n = 75$.

[AU April '03, December '07]

Solution For Poisson distribution,

$$\text{mean} = \text{variance} = \lambda$$

$\therefore$ Mean of each random variable $X_1, X_2, X_3, \ldots, X_n$ is

$$\mu = \lambda = 2$$
$$\sigma^2 = \text{Var} = \lambda = 2$$

By CLT,
$S_n = X_1 + X_2 + \cdots + X_{75}$ follows a normal distribution with mean $n\lambda$ and SD $\sqrt{n\lambda}$.

$$S_n \sim N(n\lambda, \sqrt{n\lambda}) = N(150, \sqrt{150})$$

$$\begin{aligned} P(120 \le S_n \le 160) &= P\left(\frac{120-150}{\sqrt{150}} \le \frac{S_n - 150}{\sqrt{150}} \le \frac{160-150}{\sqrt{150}}\right) \\ &= P(-2.45 \le Z \le 0.816) \\ &= P(-2.45 \le Z \le 0) + P(0 \le Z \le 0.816) \\ &= P(0 \le Z \le 2.45) + P(0 \le Z \le 0.816) \\ &= 0.4929 + 0.2939 \\ &= 0.7868 \end{aligned}$$

EXAMPLE 5.130 Suppose that orders at a restaurent are identically independent random variables with mean μ = ₹ 8 and standard deviation σ = ₹ 2. Estimate

(i) the probability that first 100 customers spend a total of more than ₹ 840, i.e. $P(X_1 + X_2 + X_3 + \cdots + X_{100} > 840)$,

(ii) $P(780 < X_1 + X_2 + X_3 + \cdots + X_{100} < 820)$. **[AU April '08]**

Solution Given: $X_1, X_2, \ldots, X_{100}$ are identically independent random variables with mean μ = ₹ 8.

$$\sigma = 2$$
$$n = 100$$

Let
$$\bar{X} = X_1 + X_2 + X_3 + \cdots + X_{100}$$

Then $\bar{X}$ follows normal distribution with mean $n\mu = 800$ and variance $n\sigma^2 = 400$

$\therefore$
$$\text{SD} = \sqrt{n\sigma^2} = \sigma\sqrt{n} = 20$$

(i) To estimate the probability that first 100 customers spend a total of more than ₹ 840

$$P(\bar{X} > 840) = P\left(\frac{\bar{X} - n\mu}{\sigma\sqrt{n}} > \frac{840 - n\mu}{\sigma\sqrt{n}}\right)$$

$$= P\left(Z > \frac{840 - 800}{20}\right)$$
$$= P(Z > 2)$$
$$= P(Z > 0) - P(0 < Z < 2)$$
$$= 0.5 - 0.4772 = 0.0228$$

$$P(780 < \bar{X} < 820) = P\left(\frac{780 - n\mu}{\sigma\sqrt{n}} < \frac{\bar{X} - n\mu}{\sigma\sqrt{n}} < \frac{820 - n\mu}{\sigma\sqrt{n}}\right)$$
$$= P\left(\frac{780 - 800}{20} < Z < \frac{820 - 800}{20}\right)$$
$$= P(-1 < Z < 1) = 2P(0 < Z < 1)$$
$$= 2 \times 0.3413 = 0.6826$$

EXAMPLE 5.131 Using CLT, show that for large n,

$$\frac{c^n}{(n-1)!} x^{n-1} e^{-cx} = \frac{c}{\sqrt{2\pi n}} e^{\frac{-(cx-n)^2}{2n}}, \qquad x > 0.$$

Solution Let $X_1, X_2, X_3, \ldots, X_n$ be independently identically distributed exponential random variables with PDF $f(x) = ce^{-cx}$, $x > 0$ and mean $\frac{1}{c}$, variance $\frac{1}{c^2}$ whose moment generating function is $\frac{c}{c-t}$.

Let $S_n = X_1 + X_2 + \cdots + X_n$

Moment generating function of S_n

$$M_{X_1 + X_2 + \cdots + X_n}(t) = M_{X_1}(t) \times M_{X_2}(t) \times M_{X_3}(t) \times \cdots \times M_{X_n}(t)$$
$$= \frac{c}{c-t} \times \frac{c}{c-t} \times \cdots \times \frac{c}{c-t} = \left(\frac{c}{c-t}\right)^n$$

which is the moment generating function of Gamma distribution. Therefore, S_n follows Gamma distribution when n is finite with PDF

$$f(x) = \frac{c^n}{(n-1)!} x^{n-1} e^{-cx} \tag{i}$$

As $n \to \infty$ by CLT, S_n follows normal distribution with mean $n\mu$, variance $n\sigma^2$ and SD $\sqrt{n}\sigma$, i.e. it follows normal distribution with mean $\frac{n}{c}$ and SD $\frac{\sqrt{n}}{c}$. Using the PDF of normal distribution, we get

$$f(x) = \frac{1}{\sqrt{2\pi}\,\frac{\sqrt{n}}{c}} e^{-\frac{1}{2}\left(\frac{x-\frac{n}{c}}{\frac{\sqrt{n}}{c}}\right)^2} \qquad \left[\because f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right]$$

$$= \frac{c}{\sqrt{2\pi n}} e^{-\frac{1}{2}\left(\frac{cx-n}{\sqrt{n}}\right)^2} = \frac{c}{\sqrt{2\pi n}} e^{-\frac{1}{2n}(cx-n)^2}$$

From Eqs. (i) and (ii), we get (ii)

$$\frac{c^n}{(n-1)!} x^{n-1} e^{-cx} = \frac{c}{\sqrt{2\pi n}} e^{-\frac{(cx-n)^2}{2n}}, \quad x > 0$$

THEOREM 2 Liapounoff's form of the CLT holds good for the sequence $\{X_k\}$ if

$$\underset{n \to \infty}{\mathrm{Lt}} \frac{\sum_{k=1}^{n} E(|X_k - \mu_k|^3)}{\left[\sum_{k=1}^{n} \mathrm{Var}(X_k)\right]^{3/2}} = 0$$

EXAMPLE 5.132 Show that the CLT holds good for the sequence X_k if $P(X_k = \pm k^{\alpha}) = \frac{1}{2} k^{-2\alpha}$ and $P(X_k = 0) = 1 - k^{-2\alpha}$, $\alpha < \frac{1}{2}$.

Solution

x_k :	$-k^{\alpha}$	k^{α}	0
$P(x_k)$:	$\frac{1}{2} k^{-2\alpha}$	$\frac{1}{2} k^{-2\alpha}$	$1 - k^{-2\alpha}$

$$E(x_k) = \Sigma x_k\, p(x_k) = -k^{\alpha} \frac{1}{2} k^{-2\alpha} + k^{\alpha} \frac{1}{2} k^{-2\alpha} + 0(1 - k^{-2\alpha})$$

$$= 0 \ (\mu_k)$$

$$E[|X_k - \mu_k|^3] = E[|X_k|^3] = \Sigma |X_k|^3 P(x_k)$$

$$= |-k^{\alpha}|^3 \frac{1}{2} k^{-2\alpha} + |k^{\alpha}|^3 \frac{1}{2} k^{-2\alpha} + 0$$

$$= k^{3\alpha - 2\alpha} \frac{1}{2} + k^{3\alpha - 2\alpha} \frac{1}{2}$$

$$= k^{\alpha}$$

$$\sum_{k=1}^{n} E[|X_k - \mu_k|^3] = \sum_{k=1}^{n} k^{\alpha} = 1^{\alpha} + 2^{\alpha} + 3^{\alpha} + \cdots + n^{\alpha} \le n \cdot n^{\alpha}$$

$$\begin{aligned} \text{Var}(X_k) &= E(X_k^2) - [E(X_k)]^2 \\ &= E(X_k^2) - 0 \\ &= (-k^{\alpha})^2 \frac{1}{2} k^{-2\alpha} + (k^{\alpha})^2 \frac{1}{2} k^{-2\alpha} + 0 = 1 \end{aligned}$$

$$\sum_{k=1}^{n} [\text{Var}(X_k)]^{\frac{3}{2}} = \sum_{k=1}^{n} (1)^{3/2} = 1 + 1 + \cdots + n \text{ times} = n$$

$$\underset{n\to\infty}{\text{Lt}} \frac{\sum_{k=1}^{n} E[|X_k - \mu_k|^3]}{\left[\sum_{k=1}^{n} \text{Var}(X_k)\right]^{\frac{3}{2}}} = \underset{n\to\infty}{\text{Lt}} \frac{n \cdot n^{\alpha}}{n^{\frac{3}{2}}} \le \underset{n\to\infty}{\text{Lt}} \frac{n^{\alpha}}{n^{\frac{1}{2}}} \le \underset{n\to\infty}{\text{Lt}} \frac{1}{n^{\frac{1}{2}-\alpha}}$$

$$= 0 \qquad \left(\text{Since } \alpha < \frac{1}{2}\right)$$

Hence CLT holds good.

EXERCISES

Discrete and Continuous Random Variables

1. Define two-dimensional random variable.
2. Define two-dimensional discrete and continuous random variables.
3. Define joint PDF of two discrete random variables X and Y.
4. What is the joint probability distribution of X and Y and represent it in the form of a table.
5. Define marginal probability function of the random variables X and Y.
6. Define marginal distribution of the discrete random variable X and represent it in the form of a table.
7. Define marginal distribution of the discrete random variable Y and represent it in the form of a table.
8. Define conditional probability distribution of the two random variables X and Y. [Both discrete and continuous cases.]
9. When we say the two random variables X and Y are independent. [Both discrete and continuous cases.]
10. Define joint PDF for continuous random variables X and Y.

11. Define marginal density function for the continuous random variables X and Y.
12. Write any two properties of joint distribution function.
13. Find the value of k if $f(x, y) = k(1 - x)(1 - y)$, $0 < x, y < 1$ is joint density function. **[Ans.** $k = 4$**]**
14. If $f(x, y) = k(1 - x - y)$, $0 < x, y < 1/2$ is a joint PDF, find k. **[Ans.** $k = 8$**]**
15. Given the joint PDF of (x, y) is $f(x, y) = 6e^{-2x-3y}$, $x \geq 0$, $y \geq 0$. Are X and Y independent? **[Ans.** Yes**]**
16. If the joint PDF of (s, y) is $f(x, y) = e^{-(x+y)}$, find the marginal density function of X. **[Ans.** e^{-X}**]**
17. Two discrete random variables X and Y have $P(X = 0, Y = 0) = 2/9$; $P(X = 0, Y = 1) = 5/9$. Examine whether X and Y are independent.

[Hint:

Y \ X	0	1
0	$\frac{2}{9}$	$\frac{1}{9}$
1	$\frac{5}{9}$	$\frac{1}{9}$

18. The joint probability distribution of X and Y is given by the following table:

X \ Y	1	3	9
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$

(i) Find the probability distribution of Y.
(ii) Find the conditional distribution of Y given $X = 2$.
(iii) Are X and Y independent?

19. Let X and Y have joint density function $f(x, y) = 2$, $0 < x < y < 1$. Find the marginal density function and the conditional density function of Y given $X = x$.

$$\left[\textbf{Ans.}\quad f_X(x) = 2(1-x), 0 < x < 1, f_Y(y) = 2y, 0 < y < 1, f(Y/X) = \frac{1}{1-x}\right]$$

20. Find the marginal density functions of X and Y if

$$f(x, y) = \frac{2}{5}(2x + 3y), \qquad 0 \le x \le 1,\ 0 \le y \le 1.$$

$$\left[\textbf{Ans.}\quad f_X(x) = \frac{4x+3}{5},\ f_Y(y) = \frac{2+6y}{5}\right]$$

21. The joint probability density function of the two-dimensional random variable is

$$f(x, y) = \begin{cases} \frac{8}{9}xy, 1 < x < y < 2 \\ 0, \quad \text{otherwise} \end{cases}$$

Find whether X and Y are independent.

$$\left[\begin{array}{l}\textbf{Ans. No.}\quad f_X(x) = \frac{4x}{9}(4 - x^2), 1 \le x \le 2,\ f_Y(y) = \frac{4y}{9}(y^2 - 1), 1 \le y \le 2 \\ f(Y/X) = \frac{2y}{4 - x^2}, x \le y \le 2\end{array}\right]$$

22. The joint density function of X and Y is

$$f(x, y) = \begin{cases} e^{-(x+y)}, 0 \le x < y \le \infty \\ 0, \quad \text{otherwise} \end{cases}$$

Are X and Y independent?

[**Ans.** Yes. $f_X(x) = e^{-x}$, $f_Y(y) = e^{-y}$]

23. The joint density function of two random variables X and Y is

$$f(x, y) = \begin{cases} \frac{1}{3}(3x^2 + xy), 0 < x < 1, 0 < y < 2 \\ 0, \quad \text{otherwise} \end{cases}$$

Find $P(X + Y \ge 1)$. $\left[\textbf{Ans.}\ \frac{125}{144}\right]$

24. Examine whether the variables X and Y are independent whose joint density is $xe^{-x(y+1)}$, $0 < x, y < \infty$.

$$\left[\textbf{Ans. No.}\quad f_X(x) = e^{-x},\ f_Y(y) = \frac{1}{(y+1)^2}\right]$$

25. Two random variables X and Y have the joint PDF

$$f(x, y) = \begin{cases} Ae^{-(2x+y)}, x, y \ge 0 \\ 0, \text{ otherwise} \end{cases}$$

Find (i) A, (ii) the marginal PDF, and (iii) the joint CDF.

$$\left[\begin{array}{l}\textbf{Ans.}\quad \text{(i) } A = 1/2 \\ \text{(ii) } f(x) = \begin{cases} 2e^{-2x}, & x, y \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad f(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases} \\ \text{(iii) Joint CDF } f(x, y) = \begin{cases} (1 - e^{-2x})(1 - e^{-y}), & x \geq 0,\ y \geq 0 \\ 0, & x < 0,\ y < 0 \end{cases}\end{array}\right]$$

26. Verify whether X and Y are statistically independent or not. Given:

$$f(x, y) = \begin{cases} kxy, & 0 \leq x \leq y,\ 0 \leq y \leq 4 \\ 0, & \text{otherwise} \end{cases}.$$

[Ans. X and Y are not independent**]**

27. If $f(x, y) = \begin{cases} 2 - x - y, & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

find (i) the marginal probability functions, and (ii) the conditional probability functions.

$$\left[\begin{array}{l}\textbf{Ans.}\quad \text{(i) } f(x) = \frac{3}{2} - x,\ f(y) = \frac{3}{2} - y, \\ \text{(ii) } f(y/x) = \dfrac{2 - x - y}{\frac{3}{2} - x},\ f(x/y) = \dfrac{2 - x - y}{\frac{3}{2} - y}\end{array}\right]$$

28. The joint PDF of two random variables is given by

$$f(x, y) = \begin{cases} C(1 + xy), & 0 \leq x \leq 6,\ 0 \leq y \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) the constant c, (ii) $f(x, 3)$, and (iii) $f(x/3)$.

$$\left[\begin{array}{l}\textbf{Ans.}\quad \text{(i) } c = 1/255, \\ \text{(ii) } f(x, 3) = \begin{cases} \frac{1}{255}(1 + 3x), & 0 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases} \\ \text{(iii) } f\left(\frac{x}{3}\right) = \begin{cases} \frac{1}{6}(1 + 3x), & 0 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}\end{array}\right]$$

29. The joint probability density function of the two random variables X and Y is given by

$$f(x, y) = \begin{cases} \dfrac{x^3 y^3}{16}, & 0 < x < 2, 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal densities of X and Y.

$$\left[\textbf{Ans.} \quad f(x) = \frac{x^3}{4}, \ 0 \le x \le 2 \qquad f(y) = \frac{y^3}{4}, \ 0 \le y \le 2\right]$$

30. The joint PDF of the random variables X and Y is

$$f(x, y) = \frac{1}{4} e^{|x| - |y|}, \qquad -\infty < x < \infty, -\infty < y < \infty$$

Are X and Y statistically independent variables? Calculate the probability that $X \le 1$ and $Y \le 0$.

$$\left[\textbf{Ans.} \quad \text{Yes}, \frac{1}{4}(2 - e^{-1})\right]$$

31. The density function of a two-dimensional continuous random variables is given as

$$f(x, y) = \begin{cases} e^{-(x+y)}, & \text{for } x \ge 0, y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find $P(1/2 < X < 2;\ 0 < Y < 4)$.

$$\left[\textbf{Ans.} \quad \left(e^{-\frac{1}{2}} - e^{-2}\right)\left(1 - e^{-4}\right)\right]$$

32. Consider the two-dimensional density function

$$f(x, y) = \begin{cases} 2, & \text{for } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) the marginal density function, and (ii) the conditional density function.

$$\left[\begin{aligned} &\textbf{Ans.} \quad \text{(i) } f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \\ &\qquad f(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \\ &\qquad \text{(ii) } f(y/x) = \frac{1}{x}, 0 < x < 1 \\ &\qquad f(x/y) = \frac{1}{1-y}, 0 < y < 1 \end{aligned}\right]$$

33. If X and Y have the JDF

$$f(x, y) = \begin{cases} e^{-(x+y)}, & \text{for } x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find (i) $P(0 < x < 1/y = 2)$, (ii) $P(X > Y)$, and (iii) $P(X + Y < 1)$.

$$\left[\textbf{Ans.} \quad \text{(i) } 1 - \frac{1}{e}, \quad \text{(ii) } \frac{1}{2}, \text{ and (iii) } 1 - \frac{2}{e}\right]$$

34. Let X_1 and X_2 be two random variables with joint density function given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1 + x_2)}, & \text{for } x_1 \geq 0, x_2 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal densities of X_1 and X_2. Also find $P(X_1 \leq 1, X_2 \leq 1)$, and $P(X_1 + X_2 \leq 1)$.

$$\left[\begin{array}{l} \textbf{Ans.} \quad \text{Marginal density of } X_1 \text{ is } e^{x_1}, \\ \text{marginal density of } X_2 \text{ is } e^{-x_2}, (1 - e^{r})^2, \left(1 - \frac{2}{e}\right) \end{array}\right]$$

35. Test whether X and Y are independent or not.

$$f(x, y) = Ae^{-|x| - 2|y|}. \qquad \textbf{[Ans.} \quad \text{Independent]}$$

36. If X and Y are two random variables, having joint density function

$$f(x, y) = \begin{cases} x + y, & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the marginal distributions of X and Y.

$$\left[\begin{array}{l} \textbf{Ans.} \quad f(x) = \left(x + \frac{1}{2}\right), 0 \leq x \leq 1 \\ \qquad\quad f(y) = \left(y + \frac{1}{2}\right), 0 \leq y \leq 1 \end{array}\right]$$

37. The two random variables X and Y have the joint PDF

$$f(x, y) = \begin{cases} 6(1 - x - y), & \text{for } x, y > 0, x + y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal distributions of X and Y. Test whether X and Y are independent. **[Ans.** $f_X(x) = 3(1 - x)^2$, $0 < x < 1$; $f_Y(y) = 3(1 - y)^2$, $0 < y < 1$, not independent**]**

38. The random variables X and Y have the joint PDF.

$$f(x, y) = \begin{cases} 2, & x + y < 1, x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the conditional distribution of Y given $X = x$.

39. Two random variables have the joint density function given by

$$f(x, y) = \begin{cases} 4xy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that X and Y are independent.

40. Let X and Y be jointly distributed with PDF

$$f(x, y) = \begin{cases} \frac{1}{4}(1 + xy), |x| < 1, & |y| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that X and Y are independent.

41. Let X and Y be two random variables with JDF

$$f(x, y) = \begin{cases} 8xy, 0 < x, y < 1 \\ 0, \quad \text{otherwise} \end{cases}$$

Find (i) the marginal distribution function of X and Y, (ii) the conditional distributions, and (iii) $P(X < 1/2, y < 1/4)$.

[**Ans.** (i) $f(x) = 4x(1 - x^2)$, $0 < x < 1$; $f(y) = 4y^3$, $0 < y < 1$,
(ii) $f(x/y) = 2x/y^2$, $0 < x < y < 1$, $f(y/x) = 2y/1 - x^2$, $0 < x < y < 1$,
(iii) 1/256]

42. If the joint PDF of the random variables X and Y is

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2), 0 \le x \le 1, 0 \le y \le 1 \\ 0, \quad \text{otherwise} \end{cases}$$

Obtain the marginal PDF of X and Y and $P\left(\frac{1}{4} \le y \le \frac{3}{4}\right)$.

$\left[\textbf{Ans.} \quad f(x) = 6/5\left(x + \frac{1}{3}\right), f(y) = 6/5\left(\frac{1}{2} + y^2\right), 0.4625\right]$

43. If X and Y are two random variables having joint PDF

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y), 0 < x < 2, 2 < y < 4 \\ 0, \quad \text{otherwise} \end{cases}$$

Find (i) $P(X < 1 \cap Y < 3)$, and (ii) $P(X + Y < 3)$.

[**Ans.** (i) 3/8, (ii) 5/24]

44. The joint PDF of the random variables X and Y is given by

$$f(x, y)\begin{cases} \frac{1}{y} e^{-\frac{x}{y}} e^{-y}, & x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(X > 1/Y = y)$. $\left[\text{Ans. } e^{-\frac{1}{y}}\right]$

Covariance, Correlation, Regression and Rank Correlation

45. If the PDF of a random variable (X, Y) is given by $f(x, y) = 2 - x - y$, in $0 \le x \le y < 1$, find $E(X)$ and $E(Y)$. **[Ans.** 5/12, 5/12]
46. If the PDF of (X, Y) is given by $f(x, y) = 2$ in $0 \le x \le y \le 1$, find $E(X)$. **[Ans.** 1/3]
47. A coin is tossed until a tail appears. What is the expectation of the number of tosses? **[Ans.** 2]
48. If the joint PDF of (X, Y) is $f(x, y) = 24y(1 - x)$, $0 \le y \le x \le 1$, find $E(XY)$. **[Ans.** 4/15]
49. If X and Y are independent random variable X with PDFs $f(x) = \frac{8}{x^3}, x > 2$ and $f(y) = 2y$, $0 < y < 1$, find $E(XY)$. **[Ans.** 8/3]
50. From an urn containing 3 red and 2 black balls, a man is to draw 2 balls without replacement. He gets ₹ 20 for each red ball and ₹ 10 for each black ball. Find his expectation. **[Ans.** ₹ 32]
51. If the continuous random variable X has the density function $f(x) = 2xe^{-x^2}$, $x \ge 0$ and if $Y = X^2$, find the mean and variance of Y. **[Ans.** 1, 11]
52. If X and Y are independent random variable X with density functions $f(x) = \frac{8}{x^3}, x > 2$ and $f(y) = 2y$, $0 < y < 1$ respectively, find $E(Z)$ if $Z = XY$. **[Ans.** 8/3]
53. If the two independent variables X and Y have the variance 36 and 16 respectively, find the correlation coefficient between $(X + Y)$ and $(X - Y)$. **[Ans.** 5/13]
54. If X and Y are random variables such that $Y = aX + b$ where a and b are real constants, show that the correlation coefficient $r(X,Y)$ between them has magnitude one.
55. Find correlation (X, Y) for the following discrete bivariate distribution:

Y \ X	5	15
10	0.2	0.4
20	0.3	0.1

[AU May '05]

56. Calculate the correlation coefficient for the following data:

X	56	42	72	36	63	47	55	49	38	42	68	60
Y	147	125	160	118	149	128	150	145	115	140	152	155

[**Ans.** 0.896]

57. Calculate the correlation coefficient for the following data:

X	15	13	17	14	18	12	20	16	19	16	18	15
Y	18	16	18	15	18	17	19	21	21	17	18	20

[**Ans.** 0.703]

58. For the following discrete bivariate distribution, determine the correlation coefficient.

Y \ X	0	1	2	3
1	$\frac{5}{48}$	$\frac{7}{48}$	–	–
2	$\frac{9}{48}$	$\frac{5}{48}$	$\frac{5}{24}$	–
3	–	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{5}{48}$

[**Ans.** 0.545]

59. The joint PDF of two random variables X and Y is given by $f(x, y) = 8xy$, $0 \le x \le y \le 1$ and $= 0$, otherwise. Obtain the regression curve of X on Y.

[**Ans.** $3x - 2y = 0$]

60. Suppose that a random variable (X, Y) has the joint PDF $f(x, y) = 2$, $0 < x < y < 1$ and $= 0$, otherwise. Find the regression curve of X on Y.

[**Ans.** $2x - y = 0$]

61. Obtain regression equation of y on x for the following distribution:

$$f(x, y) = \frac{9}{2}\frac{1+x+y}{(1+x)^4(1+y)^4}, \qquad x, y \ge 0.$$

$$\left[\textbf{Ans.} \quad y = \frac{x+3}{2x+3}\right]$$

62. If the joint PDF of (x, y) is given by $f(x, y) = 2$, $0 \le x \le y \le 1$, find the conditional mean and conditional variance of X given that $Y = y$.

[**Ans.** $y/2, y^2/12$]

63. The random variable (X, Y) has the following PDF:

$$f(x, y) = \begin{cases} \frac{1}{2}(x+y), 0 \le x \le 2, \; 0 \le y \le 2 \\ 0, \qquad \text{otherwise} \end{cases}$$

(i) Obtain the marginal distributions of X

(ii) $E(X)$ and $E(X^2)$.

(iii) Compute covariance of (X, Y). **[AU May '05]**

[Ans. (i) $f(x) = x + 1$, $f(y) = y + 1$, (ii) 14/3, 20/4, (iii) –148/9**]**

64. If the joint PDF of (x, y) is given by $f(x, y) = 21\, x^2y^3$, $0 \le x \le y \le 1$, find the conditional mean and variance of X given that $Y = y$, $0 < y < 1$.

$$\left[\textbf{Ans.}\quad \frac{3y}{4}, \frac{3y^2}{80}\right]$$

65. If the joint PDF of (x, y) is given by $f(x, y) = 3xy\,(x + y)$, $0 \le x, y \le 1$ verify that $E[E(Y/X)] = E(Y) = 11/24$.

Transformation of Random Variables

66. If X and Y are independent random variables with density functions $f(x) = e^{-x}$, $x \ge 0$ and $g(y) = 2e^{-2y}$, $y \ge 0$, find the probability distribution of $U = \dfrac{X}{Y}$.

$$\left[\textbf{Ans.}\quad \frac{1}{(u+2)^2}, u \ge 0\right]$$

67. If $f(x, y) = e^{-(x+y)}$ is the joint PDF of X and Y, find the PDF of $V = X - Y$.

$$\left[\textbf{Ans.}\quad g(v) = \frac{1}{2}e^{-|v|}\right]$$

68. Write down the formula to find the PDF of $Z = X + Y$, if X and Y are independent random variables with PDFs $f_X(x)$ and $f_Y(y)$ respectively.

$$\left[\textbf{Ans.}\quad f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy\right]$$

69. Write down the formula to find the PDF of $Z = XY$ in terms of the PDFs of X, Y if they are independent.

$$\left[\textbf{Ans.}\quad f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_X\left(\frac{z}{y}\right) f_Y(y)\, dy\right]$$

70. Write down the formula for the PDF of $Z = \dfrac{X}{Y}$ in terms of the PDFs of X, Y if they are independent.

$$\left[\textbf{Ans.}\quad f_Z(z) = \int_{-\infty}^{\infty} |y| f_X(yz)\, f_Y(y)\, dy\right]$$

71. If $Z = g(X, Y)$ and $W = h(X, Y)$, how are the joint PDFs of (X, Y) and (Z, W) related?

$$\left[\textbf{Ans.}\quad f_{ZW}(z, w) = |J|\, f_{XY}(x, y), \text{ where } J = \begin{vmatrix} x_z & x_w \\ y_z & y_w \end{vmatrix}\right]$$

72. If $Z = 2X + 3Y$ and $W = Y$, how are the joint PDFs of (X, Y) and (Z, W) related?

$$\left[\textbf{Ans.}\quad f_{ZW}(z, w) = \frac{1}{2} f_{XY}(x, y)\right]$$

73. If $U = XY$ and $V = Y$, how are the joint PDFs of (X, Y) and (U, V) related?

$$\left[\textbf{Ans.}\quad f_{UV}(u, v) = \frac{1}{|v|} f_{XY}(x, y)\right]$$

74. If $U = \dfrac{X}{Y}$ and $V = Y$, how are the joint PDFs of (X, Y) and (U, V) related?

[**Ans.** $f_{UV}(u, v) = |v|\, f_{XY}(x, y)$]

75. If $U = X + Y$ and $V = X - Y$, how are the joint PDFs of (X, Y) and (U, V) related?

$$\left[\textbf{Ans.}\quad F_{UV}(u, v) = \frac{1}{2} f_{XY}(x, y)\right]$$

76. If $x = R\cos\phi$ and $y = R\sin\phi$, how are the joint PDFs of (X, Y) and (R, ϕ) related?

[**Ans.** $f_{R\phi} = (r, \theta) = |r|\, f_{XY}(x, y)$]

77. If X and Y are independent random variables each exponentially distributed with parameter λ, find the PDF of $U = \dfrac{X}{X+Y}$.

$$\left[\textbf{Ans.}\quad g(u) = \frac{1}{2\lambda} e^{-\lambda|u|}, -\infty < u < \infty\right]$$

78. Let X and Y have common PDF $\alpha e^{-\alpha x}, 0 < x < \infty,\ \alpha > 0$, find the PDF of $X - Y$.

$$\left[\textbf{Ans.}\quad \frac{\alpha}{2} e^{-\alpha|x|}\right]$$

79. If X and Y are independent random variables with joint PDF $f(x, y)$, find the density functions of (i) $Z = XY$, and (ii) $Z = X/Y$.

$$\left[\textbf{Ans.}\quad \int_{-\infty}^{\infty} \frac{1}{|y|} f_X\left(\frac{z}{y}\right) f_Y(y)\,dy, \text{ (ii) } \int_{-\infty}^{\infty} |y|\, f_X(yz)\, f_Y(y)\,dy\right]$$

80. Let X and Y be two independent normal variables with mean 0 and variance σ^2. Find the joint density function of V and W, where $V =$

$X\cos\theta + Y\sin\theta$, $w = X\sin\theta - Y\cos\theta$, where θ is a constant angle.

$$\left[\textbf{Ans.}\quad f(v, w) = \frac{1}{2\pi\sigma^2}\, e^{-(x^2+y^2)/2\sigma^2}\right]$$

81. If X and Y are random variables with joint density $f(x, y) = 4xy$, $0 < x < 1$, $0 < y < 1$ and $= 0$, otherwise, find the joint PDF of $V = X^2$ and $W = XY$.

$$\left[\textbf{Ans.}\quad f(v, w) = \frac{2w}{v},\ w^2 < v < 1, 0 < w < 1\right]$$

82. Consider two random variables $V = X + Y$ and $W = X - Y$, where X and Y are random variables with joint PDF $f(x, y)$. Find PDF of V and W.

$$\left[\textbf{Ans.}\quad f(v, w) = \frac{1}{2}\cdot f\left(\frac{v+w}{2}, \frac{v-w}{2}\right)\right]$$

83. The density function of the independent random variables X and Y are

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0, \lambda > 0 \\ 0, \quad x < 0 \end{cases}$$

and

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, y > 0, \lambda > 0 \\ 0, \quad y < 0 \end{cases}$$

Find the density function of $Z = X/Y$.

$$\left[\textbf{Ans.}\quad \frac{1}{(z+1)^2}, z \geq 0\right]$$

84. If $U = X/Y$ and $V = X + Y$, where X and Y are independent and identically distributed random variables, having the density function $f(x) = xe^{-x}$, $x \geq 0$, prove that U and V are independent.

85. Prove that $Z = \dfrac{X}{Y}$ follows a Cauchy distribution, where X and Y are independent random variables each following $N(0, 2)$.

$$\left[\textbf{Ans.}\quad \frac{1}{\pi}\cdot\frac{1}{1+z^2}\right]$$

86. If X and Y are independent random variables with density functions $f_X(x) = e^{-x}\,U(x)$ and $f_Y(y) = 2e^{-2y}\,U(y)$, find the density function of $Z = X + Y$.

[**Ans.** $f_Z(z) = 2\,(e^{-z} - e^{-2z})\,U(z)$]

87. If X and Y are independent random variables with identical uniform distributions in the interval $(-1, 1)$, find the density function $Z = X + Y$.

$$\left[\textbf{Ans.}\quad f_Z(z) = \begin{cases} \dfrac{1}{4}(2+z), \text{if } z < 0 \\ \dfrac{1}{4}(2-z), \text{if } z > 0 \end{cases}\right]$$

Central Limit Theorem

88. State the Liapounoff's form of central limit theorem.
89. State the Lindberg–Levy's form of central limit theorem.
90. Write the applications of central limit theorem.
91. Suppose that $X_i (i = 1, 2, ..., 50)$ are independent identically distributed random variables with common mean 10 and variance 4. Using CLT, find $P(S_n \geq 110)$, where $S_n = X_1 + X_2 + \cdots + X_{10}$. **[Ans.** 0.057]
92. Verify central limit theorem for the independent random variables X_k, where for each k, $P(X_k \pm 1) = \frac{1}{2}$.
93. A distribution with unknown mean μ has variance 1.5. Using CLT, find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 that the probability sample mean will be within 0.5 of the population mean. **[Ans.** $n = 24$]
94. Suppose that in a certain circuit, 20 resistors are connected in series. The mean and variance of each resistor are 5 and 0.20 respectively. What is the probability that the total resistance of the circuit will exceed 98 Ohms, assuming independence. **[Ans.** 0.8413]
95. Twenty dice are thrown. Find approximately the probability that the sum obtained is between 65 and 75 using CLT.

Hints:
$$E(X_i) = \sum_{i=1}^{6} i\left(\frac{1}{6}\right) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$
$$E(X_i^2) = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}$$
$$\text{Var}(X_i) = E(X_i^2) - [E(x_i)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

[Ans. 0.4844]

96. What is the probability that the average of 150 random points from the interval (0, 1) is within 0.02 of the mid-point of the interval?

Hints: $\text{Var}(X_i) = \frac{1}{12}$ and $E(X_i) = \frac{1}{2}$
$$P\left(\left|\overline{X} - \frac{1+D}{2}\right| \leq 0.02\right) = P(|\overline{X} - 0.51| \leq 0.02)$$

[Ans. $P(-0.8486 \leq Z \leq 0.8486) = 0.6046$]

97. A coin is tossed 10 times. What is the probability of getting 3 or 4 or 5 heads? Use central limit theorem.

$$\left[\begin{array}{l} \textbf{Hints: } n=10, P=1/2, q=1-p=1/2 \\ \text{Mean of } x = np = 10\times\frac{1}{2}=5 \\ \text{Var of } x = npq = 10\times\frac{1}{2}\times\frac{1}{2}=2.5 \\ \mu=5, \sigma=\sqrt{2.5}=1.58 \end{array}\right]$$

[Ans. 0.5684]

98. The burning time of a certain type of lamp is an experimental random variable with mean 30 hours. What is the probability that 144 of these lamps will provide a total of more than 4500 hours of burning time?

[Ans. 0.3085]

99. The length of a continuous nylon filament that can be drawn without a break occurring is an exponential random variable with mean 5000 feet. What is the approximate probability that the average length of 100 filaments lies between 4750 feet and 5500 feet?

[Ans. 0.5538]

100. A random sample of size 25 is taken from a normal population with mean 49.5 and variance 1.69. Using CLT, find the probability that the mean of this sample falls between 48.85 and 50.20.

[Ans. 0.9902]

101. The resisters r_1, r_2, r_3 and r_4 are independent random variables and are uniform in the interval (450, 550). Using the central limit theorem, find $P(1900 = r_1 + r_2 + r_3 + r_4 \leq 2100)$.

[Ans. 0.9164]

6

T-3

Random Processes

In this chapter, the concept of a random (or stochastic) process is introduced. The theory of random processes was first developed in connection with the study of fluctuation and noise in physical systems. A random process is the mathematical model of an empirical process whose developement is governed by probability laws. Random processes provide useful models for the studies of such diverse fields as statistical physics, communication and control, time series analysis, population growth and management sciences.

The signals like voltage or current waveforms used in electrical systems for collecting, controlling and providing power to a variety of devices are functions of time and are of two classes:

(i) determinstic, and
(ii) random.

A random signal always has some element of uncertainty associated with it and, hence, it is not possible to determine its value exactly at any given point of time. But we may be able to describe the random signal in terms of its average properties such as average power in the random signal, its spectral distribution and the probability that the signal amplitude exceeds a given value. The probabilistic model used for characterizing a random signal is called a *random process* or *stochastic process*.

A variable which assigns a real number to every outcome of a random experiment is called a *random variable*, whereas a function or rule which assigns a time function to every outcome of a random experiment is called a *random process*.

6.1 BASICS OF RANDOM PROCESS

6.1.1 Random Process Concept

The concept of a random process is based on enlarging the random variable concept to include time. Recall that a random variable X is a mapping between the sample space S and the real line R. That is, $X:S \to R$.

A random process (stochastic process) is a mapping from the sample space into an ensemble of time functions (known as sample functions). To every $s \in S$, there corresponds a function of time (a sample function) $X(t, s)$. Often, from the notation, we drop the s variable and write just $X(t)$. However, the sample space s variable is always there, even if it is not shown explicitly.

Interpretations

1. If s and t both are fixed, then $X(t, s)$ is a real number.
2. For a fixed $t = t_0$, $X(t_0; s)$ is a random variable mapping S into the real line.
3. For a fixed $s_0 \in S$, $X(t; s_0)$ is a well-defined, non-random, function of time.

6.1.2 Continuous and Discrete Random Processes

For a continuous random process, probabilistic variable s takes a continuum of values and for every fixed value of time $t = t_0$, $X(t_0; s)$ is a continuous random variable.

For a discrete random process, probabilistic variable s takes on only discrete values and for every fixed value of time $t = t_0$, $X(t_0; s)$ is a discrete random variable.

For example: Let the random variable A be uniform in (0, 1). Define the continuous random process $X(t; s) = A(s) S(t)$, where $S(t)$ is a unit-amplitude, T-periodic square wave. Notice that sample functions contain periodically spaced (in time) jump discontinuities, however, the random process is continuous.

Consider the coin tossing experiment with $S = \{H, T\}$. Then $X(t; H) = \sin t$, $X(t; T) = \cos t$ defines a discrete random process. Notice that the sample functions are continuous functions of time. However, the process is discrete.

6.1.3 Statistics of Random Process

Distribution and Density Functions

The first-order distribution function is defined as

$$F(x; t) = P[X(t) \le x]$$

The first-order density function is defined as $f(x;t) = \frac{d}{dx}F(x;t)$

These definitions generalize to the nth-order case. For any given positive integer n, let $x_1, x_2, \ldots, x_n$ denote n realization variables, and let $t_1, t_2, \ldots, t_n$ denote n time variables. Then, define the nth-order distribution function as

$F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P[X(t_1) \le x_1, X(t_2) \le x_2, X(t_3) \le x_3, \ldots, X(t_n) \le x_n]$

Similarly, define the nth-order density function as

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{\partial^n F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)}{\partial x_1 \partial x_2 \partial x_3 \ldots \partial x_n}$$

Mean Function

The mean of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_X(x;t)dx$$

where $X(t)$ is treated as a random variable for a fixed value of t and $f_X(x;t)$ is the density function. In general, $\mu_X(t)$ is a function of time, and it is often called the ensemble average of $X(t)$.

Autocorrelation Function $R_X(t_1, t_2)$

The autocorrelation function (ACF) is defined by

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

Note:

$$R_X(t_1, t_2) = R_X(t_2, t_1)$$

and

$$R_X(t, t) = E[X^2(t)]$$

Autocovariance Function

The autocovariance function of $X(t)$ is defined by

$$\begin{aligned} \text{Cov}[X(t_1), X(t_2)] &= E\{[X(t_1) - \mu_X(t_1)]\,[X(t_2) - \mu_X(t_2)]\} \\ &= R_X(t_1, t_2) - \mu_X(t_1)\,\mu_X(t_2) \\ &= R_{XX}(t_1, t_2) - E[X(t_1)]\,E[X(t_2)] \end{aligned}$$

6.1.4 Definition of Random Process

A random process is a collection of random variables $\{X(t, s)\}$ or a family of random variables $\{X(t, s)\}$ which are functions of real variables say time t where $s \in S$ (sample space) and $t \in T$ (T is a parameter set or index set).

6.1.5 Classification of Random Process

Depending on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified as follows:

The term 'sequence' or 'process' refers to the nature of T and the term 'discrete' or 'continuous' refers to the nature of S.

1. If both S and T are discrete, then the random process $\{X(t)\}$ is said to be discrete random sequence.
   For example: The outcome of the nth toss of a fair dice $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, ...\}$ $S = \{1, 2, 3, ...\}$.
2. If S is continuous and T is discrete, then the random process $\{X(t)\}$ is said to be continuous random sequence.
   For example: The temperature at the end of the nth hour of the day.
3. If T is continuous and S is discrete, then the random process $\{X(t)\}$ is said to be discrete random process.
   For example: If $X(t)$ represents the number of telephone calls received in the interval of $(0, t)$, then $\{X(t)\}$ is a discrete random process as $S = \{1, 2, 3, ...\}$.
4. If both S and T are continuous, then the random process $\{X(t)\}$ is said to be continuous random process.
   For example: If $X(t)$ represents the maximum temperature at a place in the interval of $(0, t)$, then $\{X(t)\}$ is a continuous random process.

As the dependence of a random process on S is obvious, S will be omitted hereafter in the notation of a random process.

If the random process is discrete, then it is denoted as $\{X(n)\}$ or $\{X_n\}$. -sd

If the random process is continuous, then it is denoted as $\{X(t)\}$. -so

Any individual member of a random process is called a sample function or realization of a random process.

The set of all possible values of a sample function of the random process is called state space S.

A random process $\{X(t)\}$ is said to be a stationary process if some of its probability distributions or averages are constant. Or, if certain probability distribution or averages do not depend on 't', then the random process is called as a stationary process.

A random process is said to be first-order stationary if its first-order distribution $F(x, t) = P[X(t) \leq x]$ and its first-order density function $f(x, t) = \frac{d}{dx}F(x,t)$ are constants.

6.1.6 Stationary Random Processes

Strict Sense Stationary (SSS) Process

It is also called strongly stationary process. A random process $\{X(t)\}$ is said to be a SSS process, if all its finite dimensional distributions are invariant under time translation, i.e. if the joint distribution (or density) of $X(t_1)$, $X(t_2)$, $X(t_3)$, ..., $X(t_n)$ and $X(t_1 + h)$, $X(t_2 + h)$, $X(t_3 + h)$, ..., $X(t_n + h)$ for arbitrary t_1, t_2, t_3, ..., t_n are the same for all $h > 0$ and for all $n \geq 1$, then $\{X(t)\}$ is said to be stationary of order n.

For example:

(i) The Bernoulli process $\{X_n, n \geq 1\}$, where $P(X_n = 1) = p$ and $P(X_n = 0) = 1 - p = q$, is a SSS process.
(ii) Sine wave process $\{X(t)\}$, where $X(t) = Y \cos \omega t$ and Y is uniformly distributed in (0, 1) is a SSS process.

Jointly strict sense stationary process: The two real valued random processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly SSS process if their joint distribution is invariant under time translation.

The complex random process $Z(t) = X(t) + iY(t)$ is said to be a SSS process if the random processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly stationary in the strict sense.

Wide Sense Stationary (WSS) Process

It is also called weakly stationary process. A random process with finite first- and second-order moments is said to be a WSS process if

(i) its expectation is a constant and
(ii) its autocorrelation is a function of time difference (τ).

Jointly wide sense stationary process: The random processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly WSS process if $\{X(t)\}$ and $\{Y(t)\}$ are individually WSS process and its cross-correlation is a function of time difference (τ).

Example: $\{X(t)\} = A \cos(\omega_0 t + \theta)$, where A and ω_0 are constants and θ is uniformly distributed random variable in $(0, 2\pi)$.

Note: A SSS process is a WSS process, but a WSS process need not be a SSS process.

6.1.7 Evolutionary Random Process

A random process which is not stationary in any sense is called *evolutionary random process*.

Example: Poisson process.

6.1.8 Averages of Random Processes

Mean

The mean of a random process $X(t)$ is $E[X(t)] = \mu_X(t) = \mu(t)$.

Autocorrelation

Let $X(t_1)$ and $X(t_2)$ are any two members of the random process $\{X(t)\}$. Then, the autocorrelation of the random process $\{X(t)\}$ is the expectation of product of $X(t_1)$ and $X(t_2)$.

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

Note: $R_{XX}(t_1, t_2)$ is denoted by $R(t_1, t_2)$ or $R_X(t_1, t_2)$.

Autocovariance

The autocovariance of any two members $X(t_1)$ and $X(t_2)$ of the random process $\{X(t)\}$ is defined as

$$\begin{aligned} C_{XX}(t_1, t_2) &= E\{[X(t_1) - \mu(t_1)]\,[X(t_2) - \mu(t_2)]\} \\ &= E[X(t_1)\, X(t_2) - \mu(t_1)\, \mu(t_2)] \\ &= R_{XX}(t_1, t_2) - \mu(t_1)\, \mu(t_2) \end{aligned}$$

Autocorrelation Coefficient

The autocorrelation coefficient of a random process is given by

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)\, C_{XX}(t_2, t_2)}}$$

6.1.9 Cross-correlation

If $X(t_1)$ and $Y(t_2)$ are two random variables of the random processes $\{X(t)\}$ and $\{Y(t)\}$ respectively, then the cross-correlation of $\{X(t)\}$ and $\{Y(t)\}$ is defined as $R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$.

Cross-covariance of Two Processes

Cross-covariance of $\{X(t_1)\}$ and $\{Y(t_2)\}$ is defined as

$$\begin{aligned} C_{XY}(t_1, t_2) &= E\{[X(t_1) - \mu(t_1)]\,[Y(t_2) - \mu(t_2)]\} \\ &= E\{X(t_1)\, Y(t_2) - E[X(t_1)]\, E[Y(t_2)]\} \\ &= E[X(t_1)\, Y(t_2) - \mu_X(t_1)\, \mu_Y(t_2)] \\ &= E[X(t_1)\, Y(t_2)] - \mu_X(t_1)\, \mu_Y(t_2) \end{aligned}$$

Cross-correlation Coefficient

Cross-correlation coefficient is defined by

$$\rho_{XY}(t_1, t_2) = \frac{C_{XY}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)\, C_{YY}(t_2, t_2)}}$$

Note: To show that the given random process is a WSS process, we have to show that

(i) $E\{X(t)\}$ is a constant.
(ii) $R_{XX}(t_1, t_2)$ is a function of time difference (τ).

EXAMPLE 6.1 Show that $X(t) = A \cos\lambda t + B \sin\lambda t$ where A and B are random variables is a WSS process, if

(i) $E(A) = E(B) = 0$, (ii) $E(A^2) = E(B^2)$, and (iii) $E(AB) = 0$.

[AU November '04, June '05, May '08]

Solution Given: $E(A) = E(B) = 0$, $E(AB) = 0$
and $E(A^2) = E(B^2)$

To show that $\{X(t)\}$ is a WSS process, we have to show that

(i) $E[X(t)]$ = a constant.
(ii) Autocorrelation = $R(t_1, t_2)$ is a function of time difference $(t_1 - t_2)$

Given: $X(t) = A \cos\lambda t + B \sin\lambda t$

(i)
$$\begin{aligned} E[X(t)] &= E(A \cos\lambda t) + E(B \sin\lambda t) \\ &= \cos\lambda t\, E(A) + \sin\lambda t\, E(B) \\ &= \cos\lambda t(0) + \sin\lambda t(0) = 0 \end{aligned}$$

(ii)
$$\begin{aligned} R(t_1, t_2) &= E[X(t_1)\, X(t_2)] \\ &= E[(A \cos\lambda t_1 + B \sin\lambda t_1)(A \cos\lambda t_2 + B \sin\lambda t_2)] \\ &= E(A^2 \cos\lambda t_1 \cos\lambda t_2 + AB \sin\lambda t_1 \cos\lambda t_2 \\ &\qquad + AB \cos\lambda t_1 \sin\lambda t_2 + B^2 \sin\lambda t_1 \sin\lambda t_2) \\ &= \cos\lambda t_1 \cos\lambda t_2\, E(A^2) + \sin\lambda t_1 \cos\lambda t_2 E(AB) \\ &\qquad + \cos\lambda t_1 \sin\lambda t_2\, E(AB) + \sin\lambda t_1 \sin\lambda t_2\, E(B^2) \\ &= \cos\lambda t_1 \cos\lambda t_2\, E(A^2) + \sin\lambda t_1 \sin\lambda t_2 E(B^2) \qquad [\because E(AB) = 0] \\ &= E(A^2)(\cos\lambda t_1 \cos\lambda t_2 + \sin\lambda t_1 \sin\lambda t_2) \qquad [\because E(A^2) = E(B^2)] \\ &= E(A^2)\,[\cos\lambda(t_1 - t_2)] = \text{a function of time difference} \end{aligned}$$

Therefore, it is a WSS process.

EXAMPLE 6.2 If $X(t) = Y\cos\omega t + Z\sin\omega t$, where Y and Z are independent $N(0, \sigma^2)$ random variables and ω is a constant, then prove that $\{X(t)\}$ is a WSS process. **[AU April '04]**

Solution Given: X and Y are independent $N(0, \sigma^2)$, i.e. X and Y are normal random variables with mean 0 and variance σ^2

$$E(Y) = E(Z) = 0,\ E(Y^2) = E(Z^2) = \sigma^2$$

$\Rightarrow$

To prove that $\{X(t)\}$ is a WSS process, we have to show that

(i) $E[X(t)]$ = a constant.

(ii) Autocorrelation $R(t_1, t_2)$ is a function of time difference $(t_1 - t_2)$.

Given: $X(t) = Y\cos\omega t + Z\sin\omega t$,

$E(Y) = E(Z) = 0, \qquad E(Y^2) = E(Z^2) = \sigma^2$

Since Y and Z are independent, $E(YZ) = E(Y)E(Z) = 0$.

$$\begin{aligned}
\text{(i)}\quad E[X(t)] &= E(Y\cos\omega t) + E(Z\sin\omega t)\\
&= \cos\omega t\, E(Y) + \sin\omega t\, E(Z)\\
&= \cos\omega t\,(0) + \sin\omega t\,(0) \qquad [\because E(Y) = E(Z) = 0]\\
&= 0, \text{ a constant}
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad R(t_1, t_2) &= E[X(t_1)\cdot X(t_2)]\\
&= E[(Y\cos\omega t_1 + Z\sin\omega t_1)(Y\cos\omega t_2 + Z\sin\omega t_2)]\\
&= E\,(Y^2\cos\omega t_1\cos\omega t_2 + YZ\cos\omega t_1\sin\omega t_2\\
&\qquad + YZ\sin\omega t_1\cos\omega t_2 + Z^2\sin\omega t_1\sin\omega t_2)\\
&= \cos\omega t_1\cos\omega t_2\, E(Y^2) + \cos\omega t_1\sin\omega t_2\, E(YZ)\\
&\qquad + \sin\omega t_1\cos\omega t_2\, E(YZ) + \sin\omega t_1\sin\omega t_2\, E(Z^2)\\
&= \sigma^2[\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2], \quad [\because E(YZ) = 0]\\
&= \sigma^2[\cos\omega(t_1 - t_2)] = \sigma^2\cos\omega\tau, \text{ a function of time difference}
\end{aligned}$$

Therefore, it is a WSS process.

EXAMPLE 6.3 If $X(t) = P + Qt$ where P and Q are independent random variables with $E(P) = p$, $E(Q) = q$, $\text{Var}(P) = \sigma_1^2$, $\text{Var}(Q) = \sigma_2^2$, find $E[X(t)]$, $R(t_1, t_2)$ and $C(t_1, t_2)$. Is the process $\{X(t)\}$ stationary?

Solution Given:

$$X(t) = P + Qt$$

and

$$E(P) = p,\ E(Q) = q$$

$$\text{Var}(P) = \sigma_1^2,\ \text{Var}(Q) = \sigma_2^2$$

As P and Q are independent random variables

$$\begin{aligned}
E(PQ) &= E(P)E(Q) = pq\\
E[X(t)] &= E(P + Qt) = E(P) + tE(Q)\\
E[X(t)] &= p + qt \text{ (not a constant and depends on 't')}\\
R(t_1, t_2) &= E[X(t_1)\cdot X(t_2)]\\
&= E[(P + Qt_1)(P + Qt_2)]
\end{aligned}$$

$$= E(P^2 + PQt_2 + PQt_1 + Q^2t_1t_2)$$
$$= E(P^2) + t_2E(PQ) + t_1E(PQ) + t_1t_2\, E(Q^2)$$

$$\text{Var}(P) = \sigma_1^2 \Rightarrow E(P^2) - [E(P)]^2 = \sigma_1^2 \quad \text{(i)}$$

i.e. $\quad E(P^2) - p^2 = \sigma_1^2$

$\therefore \quad E(P^2) = \sigma_1^2 + p^2$

Similarly, $\quad E(Q^2) = \sigma_2^2 + q^2$

Substituting in Eq. (i), we get

$R(t_1, t_2) = \sigma_1^2 + p^2 + pq(t_1 + t_2) + (\sigma_2^2 + q^2)t_1t_2$, a function of time

$$C(t_1, t_2) = R(t_1, t_2) - E[X(t_1)]\, E[X(t_2)]$$
$$= \sigma_1^2 + p^2 + pq(t_1 + t_2) + (\sigma_2^2 + q^2)t_1t_2 - (p + qt_1)(p + qt_2)$$

$C(t_1, t_2) = \sigma_1^2 + \sigma_2^2\, t_1t_2$, a function of time.

Since $E[X(t)]$ is a function of t, the process is not stationary.

EXAMPLE 6.4 Show that $X(t) = A\cos\lambda t + B\sin\lambda t$ is stationary in the wide sense if and only if A and B are uncorrelated random variables with zero mean and equal variance. **[AU April '04]**

Solution If A and B are uncorrelated random variables with zero mean and equal variances, i.e. $E(A) = E(B) = 0$, $E(AB) = 0$, $E(A^2) = E(B^2)$, then $X(t)$ is a WSS process. [Proved in Example (6.1)]

Now, to prove the converse:

Let $\{X(t)\}$ be a WSS process, then

(i) $E[X(t)]$ = a constant.

(ii) $R(t_1, t_2)$ = a function of time difference.

$$\therefore \quad E[X(t)] = E(A\cos\lambda t + B\sin\lambda t)$$
$$= \cos\lambda t\, E(A) + \sin\lambda t\, E(B)$$
$$= \text{a constant, only when } E(A) = E(B) = 0$$

$$\therefore \quad E(A) = E(B) = 0 \quad \text{(i)}$$

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$
$$= E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)]$$
$$= (\cos\lambda t_1 \cos\lambda t_2)\, E(A^2) + (\sin\lambda t_1 \sin\lambda t_2)\, E(B^2)$$
$$+ E(AB)\,(\sin\lambda t_1 \cos\lambda t_2 + \cos\lambda t_1 \sin\lambda t_2)$$

Autocorrelation will be a function of time difference only if

$$E(AB) = 0$$

From (i)

$$E(A) = E(B) = 0$$

$$\therefore \quad E(AB) = E(A)E(B) = 0$$

i.e. only if A and B are uncorrelated or independent with $E(A) = E(B) = 0$ and $E(A^2) = E(B^2)$, the autocorrelation will be a function of time difference.

Hence the proof.

EXAMPLE 6.5 Given a random variable Ω with density $f(\omega)$ and another random variable ϕ uniformly distributed in $(-\pi, \pi)$ and independent of Ω. If $X(t) = a\cos(\Omega t + \phi)$, prove that $X(t)$ is a WSS process.

Solution To show that $X(t)$ is a WSS process, we have to show that

(i) $E[X(t)]$ is a constant and

(ii) $R_{XX}(t_1, t_2)$ is a function of time difference.

Given: θ is uniformly distributed in $(-\pi, \pi)$.

$$\therefore \qquad f(\phi) = \begin{cases} \dfrac{1}{2\pi}, & -\pi < \phi < \pi \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X(t)] &= E[a\cos(\Omega t + \phi)] \\ &= aE(\cos\Omega t \cos\phi - \sin\Omega t \sin\phi] \\ &= a[E(\cos\Omega t \cos\phi) - E(\sin\Omega t \sin\phi)] \\ &= a[E(\cos\Omega t)\, E(\cos\phi) - E(\sin\Omega t)\, E(\sin\phi)] \end{aligned} \qquad \text{(i)}$$

$$E(\cos\phi) = \int_{-\pi}^{\pi} \cos\phi \frac{1}{2\pi} d\phi$$

$$= \frac{1}{2\pi}[\sin\phi]_{-\pi}^{\pi} = \frac{1}{2\pi}(0) = 0$$

$$E(\sin\phi) = \int_{-\pi}^{\pi} \sin\phi \frac{1}{2\pi} d\phi$$

$$= \frac{1}{2\pi}[-\cos\phi]_{-\pi}^{\pi} = \frac{-1}{2\pi}[(-1)-(-1)] = 0$$

Sustituting in Eq. (i), we get $E[X(t)] = 0$, a constant.

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = a^2E[\cos(\Omega t_1 + \phi)\cos(\Omega t_2 + \phi)]$$

$$= \frac{a^2}{2}E[\cos(\Omega t_1 + \phi + \Omega t_2 + \phi) + \cos(\Omega t_1 - \Omega t_2)]$$

$$= \frac{a^2}{2}(E\{\cos[\Omega(t_1 + t_2) + 2\phi]\} + E[\cos\Omega(t_1 - t_2)])$$

$$= \frac{a^2}{2}\{E[\cos\Omega(t_1 + t_2)\cos 2\phi - \sin\Omega(t_1 + t_2)\sin 2\phi] + E[\cos\Omega(t_1 - t_2)]\}$$

Since Ω and ϕ are independent.

$$R_{XX}(t_1, t_2) = \frac{a^2}{2}\{E[\cos\Omega(t_1+t_2)]\,E(\cos 2\phi) - E[\sin\Omega(t_1+t_2)]\,E(\sin 2\phi)\}$$
$$+\frac{a^2}{2}E[\cos\Omega(t_1-t_2)] \quad \text{(ii)}$$

$$E(\cos 2\phi) = \int_{-\pi}^{\pi}\cos 2\phi\frac{1}{2\pi}d\phi = \frac{1}{2\pi}\left[\frac{\sin 2\phi}{2}\right]_{-\pi}^{\pi} = 0$$

$$E(\sin 2\phi) = \int_{-\pi}^{\pi}\sin 2\phi\frac{1}{2\pi}d\phi$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2\phi}{2}\right]_{-\pi}^{\pi} = \frac{-1}{2\pi}(1-1) = 0$$

Substituting in Eq. (ii), we get

$$R_{XX}(t_1, t_2) = \frac{a^2}{2}E[\cos\Omega(t_1-t_2)] \text{ = a function of time difference.}$$

$\therefore$ The given random process is a WSS process.

Note:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
$$(1-x)^{-3} = \frac{1}{2}[1\times 2 + 2\times 3x + 3\times 4x^2 + \cdots]$$

EXAMPLE 6.6 The process $X(t)$ whose probability distribution under certain condition is given by

$$P[X(t)=n] = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, 3, \ldots \\ \dfrac{at}{1+at}, & n = 0 \end{cases}$$

Show that it is not a stationary process. **[AU June '07, April '08]**

Solution Given: $P(n) = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, 3, \ldots \\ \dfrac{at}{1+at}, & n = 0 \end{cases}$

$$E[X(t)] = \sum_i x_i\, p(x_i) = \sum_{n=1}^{\infty} np(n)$$

$$= 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + \cdots$$

$$= 1 \times \frac{1}{(1+at)^2} + 2 \times \left[\frac{(at)}{(1+at)^3}\right] + 3 \times \left[\frac{(at)^2}{(1+at)^4}\right] + \cdots$$

$$= \frac{1}{(1+at)^2}\left\{1 + 2 \times \frac{(at)}{(1+at)} + 3 \times \left[\frac{(at)}{(1+at)}\right]^2 + \cdots\right\}$$

$$= \frac{1}{(1+at)^2}\left\{1 - \left[\frac{(at)}{(1+at)}\right]\right\}^{-2} = \frac{1}{(1+at)^2}\left[\frac{1}{(1+at)}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2} = 1, \text{ a constant} \qquad \text{(i)}$$

$$E[X^2(t)] = \sum_{n=1}^{\infty} n^2 p(n) = \sum_{n=1}^{\infty} n(n+1)\, p(n) - \sum_{n=1}^{\infty} np(n)$$

$$= [1 \times 2 \times p(1) + 2 \times 3 \times p(2) + 3 \times 4 \times p(3) + \cdots] - 1,$$

from Eq. (i)

$$= 2 \times \frac{1}{2}\left[1 \times 2 \times \frac{1}{(1+at)^2} + 2 \times 3 \times \frac{(at)}{(1+at)^3} + 3 \times 4 \times \frac{(at)^2}{(1+at)^4} + \cdots\right] - 1$$

$$= \left\{\frac{2}{(1+at)^2}\left[1 - \frac{at}{(1+at)}\right]^{-3}\right\} - 1$$

$$= \left\{\frac{2}{(1+at)^2}\left[\frac{1}{(1+at)}\right]^{-3}\right\} - 1$$

$$= \left[\frac{2}{(1+at)^2}(1+at)^3\right] - 1 \Rightarrow 2(1+at) - 1$$

$$= 1 + 2at, \text{ a function of time}$$

$\therefore$

$$\text{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2$$

$$= 1 + 2at - 1^2 = 2at, \text{ a function of time.}$$

$\therefore$ The given process is not a stationary process.

EXAMPLE 6.7 If the random process $\{X(t)\}$ takes the value -1 with probability 1/3 and takes the value $+1$ with probability 2/3, find whether $\{X(t)\}$ is a stationary process or not.

Solution Given: $P(X=-1)=\frac{1}{3}$ and $P(X=1)=\frac{2}{3}$

$$E[X(t)] = \sum_i x_i p(x_i) = \sum_{n=-1,1} np(n)$$

$$= (-1)\times p(-1) + 1\times p(1) = (-1)\times\frac{1}{3} + 1\times\frac{2}{3} = \frac{1}{3}, \text{ a constant}$$

$$E[X^2(t)] = \sum_i x_i^2 p(x_i) = \sum_{n=-1,1} n^2 p(n)$$

$$= (-1)^2\times p(-1) + 1^2\times p(1) = \frac{1}{3}+\frac{2}{3} = 1$$

$$\text{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2$$

$$= 1-\left(\frac{1}{3}\right)^2 = 1-\frac{1}{9} = \frac{8}{9}, \text{ a constant}$$

$E[X(t)]$ and $\text{Var}[X(t)]$ are independent of time.

$\therefore$ $\{X(t)\}$ is a stationary process.

EXAMPLE 6.8 Assume a random process $X(t)$ with four sample functions:

$$X(t, s_1) = \cos t \qquad X(t, s_2) = -\cos t$$
$$X(t, s_3) = \sin t \qquad X(t, s_4) = -\sin t$$

which are equally likely. Show that it is a wide sense stationary process.

[AU December '05]

Solution To prove $X(t)$ is a WSS process, we have to show that

(i) $E[X(t)]$ = a constant

(ii) Autocorrelation is a function of time difference.

Since the given sample functions are equally likely, their probability $p(x) = \frac{1}{4}$.

$$\therefore \quad \text{Mean} = E[X(t)] = \sum_{i=1}^{4} p(x)\, X(t, s_i) = \sum_{i=1}^{4}\frac{1}{4} X(t, s_i)$$

$$= \frac{1}{4}[X(t,s_1) + X(t,s_2) + X(t,s_3) + X(t,s_4)]$$

$$= \frac{1}{4}[\cos t - \cos t + \sin t - \sin t]$$

$$= 0, \text{ which is a constant}$$

Autocorrelation function:

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$= \frac{1}{4}\sum_{i=1}^{4} X(t_1, s_i)\, X(t_2, s_i)$$

$$= \frac{1}{4}[X(t_1, s_1)\, X(t_2, s_1) + X(t_1, s_2)\, X(t_2, s_2) + X(t_1, s_3)\, X(t_2, s_3) + X(t_1, s_4)\, X(t_2, s_4)]$$

$$= \frac{1}{4}[\cos t_1 \cos t_2 + \cos t_1 \cos t_2 + \sin t_1 \sin t_2 + \sin t_1 \sin t_2]$$

$$= \frac{1}{4}[2\cos t_1 \cos t_2 + 2\sin t_1 \sin t_2]$$

$$= \frac{2}{4}[\cos t_1 \cos t_2 + \sin t_1 \sin t_2]$$

$$= \frac{1}{2}\cos(t_1 - t_2),$$ which is a function of time difference.

$\therefore$ The process $X(t)$ is a WSS process.

EXAMPLE 6.9 A random process $X(t)$ is characterized by four sample functions:

$X(t, s_1) = -1$, $X(t, s_2) = -2$, $X(t, s_3) = 3$, $X(t, s_4) = t$

Assume that the sample functions are equally likely. Find the autocorrelation function and check whether it is a WSS process. **[AU April '04]**

Solution Since the given sample functions are equally likely, their probability $p(x) = \frac{1}{4}$.

$\therefore$ Autocorrelation function

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= \frac{1}{4}\sum_{i=1}^{4} x_i(t_1)\, x_i(t_2)$$

$$= \frac{1}{4}[x_1(t_1)\, x_1(t_2) + x_2(t_1)\, x_2(t_2) + x_3(t_1)\, x_3(t_2) + x_4(t_1)\, x_4(t_2)]$$

$$= \frac{1}{4}[(-1)(-1) + (-2)(-2) + (3)(3) + t_1 t_2]$$

$$= \frac{1}{4}[14 + t_1 t_2]$$

Since $R_{XX}(t_1, t_2)$ is not a function of time difference, $\{X(t)\}$ is not a WSS process.

EXAMPLE 6.10 If $X(t) = A \sin(\omega_0 t + \phi)$ where A and ω_0 are constants and ϕ is a uniformly distributed random variable in $(0, 2\pi)$, calculate the autocorrelation function of the process and see whether the process is a WSS process. **[AU December '04; '09]**

Solution Given: $X(t) = A \sin(\omega_0 t + \phi)$

The PDF of the random variable ϕ which is uniformly distributed in $(0, 2\pi)$ is

$$f(\phi) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \phi \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X(t)] &= E[A \sin(\omega_0 t + \phi)] \\ &= AE[\sin \omega_0 t \cos \phi + \cos \omega_0 t \sin \phi] \\ &= A \sin \omega_0 t \, E(\cos \phi) + A \cos \omega_0 t \, E(\sin \phi) \end{aligned} \qquad \text{(i)}$$

$$E(\cos \phi) = \int_0^{2\pi} \frac{1}{2\pi} \cos \phi \, d\phi = \frac{1}{2\pi}[\sin \phi]_0^{2\pi} = \frac{1}{2\pi} \times 0 = 0$$

$$E(\sin \phi) = \int_0^{2\pi} \frac{1}{2\pi} \sin \phi \, d\phi = \frac{1}{2\pi}[-\cos \phi]_0^{2\pi} = -\frac{1}{2\pi} \times (1-1) = 0$$

Substituting in Eq. (i), we get

$$E[X(t)] = 0, \quad \text{a constant.}$$

To find the autocorrelation:

$$\begin{aligned} R(t_1, t_2) = E[X(t_1)X(t_2)] &= E[A \sin(\omega_0 t_1 + \phi) \, A \sin(\omega_0 t_2 + \phi)] \\ &= A^2 E[\sin(\omega_0 t_1 + \phi) \sin(\omega_0 t_2 + \phi)] \\ &= A^2 E\left\{\frac{1}{2}[\cos(\omega_0 t_1 + \phi - \omega_0 t_2 - \phi) - \cos(\omega_0 t_1 + \phi + \omega_0 t_2 + \phi)]\right\} \end{aligned}$$

$$\left\{\because \sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]\right\}$$

$$\begin{aligned} &= \frac{1}{2} A^2 E[\cos \omega_0 (t_1 - t_2) - \cos(\omega_0 (t_1 + t_2) + 2\phi)] \\ &= \frac{1}{2} A^2 \{E[\cos \omega_0 (t_1 - t_2)] - E[\cos(\omega_0 (t_1 + t_2))] + 2\phi\} \\ &= \frac{A^2}{2} E[\cos \omega_0 (t_1 - t_2)] - \frac{A^2}{2} \int_0^{2\pi} \frac{1}{2\pi} \cos[\omega_0 (t_1 + t_2) + 2\phi] d\phi \\ &= \frac{A^2}{2} \cos \omega_0 (t_1 - t_2) - \frac{A^2}{2} \frac{1}{2\pi} \left[\frac{\sin[\omega_0 (t_1 + t_2) + 2\phi]}{2}\right]_0^{2\pi} \end{aligned}$$

$$= \frac{A^2}{2}\cos\omega_0(t_1 - t_2) - \frac{A^2}{4\pi}\cdot\frac{1}{2}\{\sin[\omega_0(t_1 - t_2) + 4\pi] - \sin[\omega_0(t_1 + t_2) + 0]\}$$

$$= \frac{A^2}{2}\cos\omega_0(t_1 - t_2) - \frac{A^2}{8\pi}\{\sin[\omega_0(t_1 + t_2)] - \sin[\omega_0(t_1 + t_2)]\}$$

$$= \frac{A^2}{2}\cos\omega_0(t_1 - t_2) \qquad [\because \sin(4\pi + x) = \sin x]$$

$\therefore R(t_1, t_2) = \dfrac{A^2}{2}\cos\omega_0(t_1 - t_2)$, a function of time difference.

$\therefore$ The process is a WSS process.

EXAMPLE 6.11 Consider the random process $V(t) = \cos(\omega t + \theta)$ where θ is a random variable with probability density

$$v(\theta) = \begin{cases} \dfrac{1}{2\pi}, & -\pi \le \theta < \pi \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that first and second moments of $V(t)$ are independent of time.

(ii) If θ = a constant, will the ensemble mean of $V(t)$ be time independent?

[AU December '08]

Solution: Given: $V(t) = \cos(\omega t + \theta)$

and
$$v(\theta) = \begin{cases} \dfrac{1}{2\pi}, & -\pi \le \theta < \pi \\ 0, & \text{otherwise} \end{cases}$$

(i) The first moment:

$$E[V(t)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t + \theta)d\theta$$

$$= \frac{1}{2\pi}[\sin(\omega t + \theta)]_{-\pi}^{\pi} = \frac{1}{2\pi}[-\sin\omega t - (-\sin\omega t)] = 0$$

The second moment:

$$E[V^2(t)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2(\omega t + \theta)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1 + \cos 2(\omega t + \theta)}{2}\right]d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos 2(\omega t+\theta)}{2}d\theta$$

$$= \frac{1}{2\pi}\times\frac{1}{2}(2\pi) + \frac{1}{2\pi}\left[\frac{\sin 2(\omega t+\theta)}{2}\right]_{\pi}^{-\pi}$$

$$= \left(\frac{1}{2\pi}\times\frac{2\pi}{2}\right) + \frac{1}{2\pi}(0) = \frac{1}{2}$$

Thus, the first and second moments of $V(t)$ are independent of time.

(ii) When θ is a constant, ensemble mean
$E[X(t)] = E[\cos(\omega t + \theta)] = \cos(\omega t + \theta)$, which is time dependent.
$\therefore$ $V(t)$ is not time independent.

EXAMPLE 6.12 Let the random process be $X(t) = \cos(t + \phi)$, where ϕ is a random variable with density function $f(\phi) = \frac{1}{\pi}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Check whether the process is stationary or not. **[AU December '09]**

Solution If a random process is stationary, then

$$E[X(t)] = \text{constant}$$

Given: $X(t) = \cos(t + \phi)$

and $f(\phi) = \frac{1}{\pi}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$$\therefore \quad E[X(t)] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} X(t)\, f(\phi)d\phi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\phi)\cdot\frac{1}{\pi}\cdot d\phi$$

$$= \frac{1}{\pi}[\sin(t+\phi)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\sin\left(t+\frac{\pi}{2}\right) - \sin\left(t-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}[\cos t + \cos t] = \frac{2\cos t}{\pi} \quad \text{(a function of } t\text{)}$$

Since $E[X(t)]$ is not a constant, $X(t)$ is not a stationary process.

EXAMPLE 6.13 A random process has sample functions of the form $X(t) = A\cos(\omega t + \theta)$ in which A and ω are constants and θ is a random variable. Prove that this process is not stationary if θ is not uniformly distributed over a range of 2π.

[AU December '05, June '07]

Solution Since θ is not a uniformly distributed random variable, $f(\theta)$ is not a constant.

$\therefore$ $E[X(t)] = E[A\cos(\omega t + \theta)] = AE[\cos\omega t \cos\theta - \sin\omega t - \sin\theta]$

$\therefore$ $E[X(t)]$ is not a constant, as it contains θ.

Since $E[X(t)] \neq$ constant, the random process $X(t)$ is not a stationary random process.

EXAMPLE 6.14 Examine whether the Poisson Process $X(t)$ given by the probability law $P[X(t) = n] = \dfrac{e^{-\lambda t}(\lambda t)^n}{\underline{|n}}$, $n = 0, 1, 2, \ldots$ is covariance stationary.

[AU December '07, May '05]

Solution The mean is given by

$$E[X(t)] = \sum_{n=0}^{\infty} nP_n(t)$$

$$= \sum_{n=0}^{\infty} n\frac{e^{-\lambda t}(\lambda t)^n}{\underline{|n}} = \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{\underline{|n-1}} = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{\underline{|n-1}}$$

$$= (\lambda t)e^{-\lambda t}\left[1 + \frac{(\lambda t)}{\underline{|1}} + \frac{(\lambda t)^2}{\underline{|2}} + \cdots\right]$$

$$= (\lambda t)e^{-\lambda t}e^{\lambda t} = \lambda t, \text{ depends on } t.$$

Hence Poisson process is not a stationary process. That is, it is not covariance stationary.

EXAMPLE 6.15 Show that the random process $X(t) = A\cos(\omega_0 t + \theta)$ is not stationary if A and ω_0 are constants and θ is uniformly distributed random variable in $(0, \pi)$.

[AU December '05; '09, April '07]

Solution Since θ is uniformly distributed random variable in $(0, \pi)$, its PDF is given by

$$f(\theta) = \begin{cases} \dfrac{1}{\pi - 0}, & 0 < \theta < \pi \\ \dfrac{1}{\pi}, & 0 < \theta < \pi \end{cases}$$

Now,

$$E[X(t)] = E[A\cos(\omega_0 t + \theta)]$$

$$= \int_0^{\pi} A\cos(\omega_0 t + \theta)\, f(\theta) d\theta = \int_0^{\pi} A\cos(\omega_0 t + \theta)\, \frac{1}{\pi} d\theta$$

$$= \frac{A}{\pi}\int_0^{\pi} \cos(\omega_0 t + \theta)\, d\theta = \frac{A}{\pi}[\sin(\omega_0 t + \theta)]_0^{\pi}$$

$$= \frac{A}{\pi}[\sin(\omega_0 t + \pi) - \sin(\omega_0 t + 0)] \qquad [\because \sin(\pi + \theta) = -\sin\theta]$$

$$= \frac{-2A\sin\omega_0 t}{\pi}, \text{ which depends on } t.$$

$\therefore$ $E[X(t)]$ is not a constant.

Since $E[X(t)]$ is not a constant, the random process $\{X(t)\}$ is not a stationary random process.

EXAMPLE 6.16 If $Y(t) = X(t)\cos(\omega t + \theta)$ where R and ϕ are independent random variables and θ is uniformly distributed in $(-\pi, \pi)$, prove that $R(t_1, t_2) = \frac{1}{2}E(R^2)\cos\omega(t_1 - t_2)$.

Solution Given: $X(t) = R\cos(\omega t + \theta)$

$$R(t_1, t_2) = E[X(t_1), X(t_2)]$$
$$= E[R\cos(\omega t_1 + \theta)\, R\cos(\omega t_2 + \theta)]$$
$$= E[R^2\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)]$$
$$= \frac{E(R^2)}{2}E[\cos(\omega t_1 + \omega t_2 + \theta + \theta)] + [\cos(\omega t_1 - \omega t_2)]$$
$$= \frac{1}{2}E(R^2)\, E[\cos\omega(t_1 - t_2)] + \frac{1}{2}E(R^2)\, E[\cos\omega(t_1 + t_2 + 2\theta)]$$
$$= \frac{1}{2}E(R^2)\cos\omega(t_1 - t_2) + \frac{1}{2}E(R^2)\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos\omega[(t_1 + t_2) + 2\theta]d\theta$$
$$= \frac{1}{2}E(R^2)\cos\omega(t_1 - t_2) + \frac{1}{2}E(R^2)\frac{1}{2\pi}\left\{\frac{\sin[\omega(t_1 + t_2) + 2\theta]}{2}\right\}_{-\pi}^{\pi}$$
$$= \frac{1}{2}E(R^2)\cos\omega(t_1 - t_2) + \frac{1}{2}E(R^2)\cdot 0$$
$$[\because \sin(2\pi + x) = \sin x,\ \sin(2\pi - x) = -\sin x]$$

i.e. $R(t_1, t_2) = \frac{1}{2}E(R^2)\cos\omega(t_1 - t_2)$

Hence proved.

EXAMPLE 6.17 A stochastic (random) process is described by $X(t) = A \sin t + B \cos t$ where A and B are independent random variables with zero means and equal standard deviation. Show that the process is stationary of second order.
[AU April '04]

Solution To show that it is stationary of second order, we have to show that $E[X(t)]$ and $E[X^2(t)]$ are constants.

Given: $$E(A) = E(B) = 0$$
and $$E(A^2) = E(B^2) = \sigma^2$$

Since A and B are independent,

$$E(AB) = E(A)E(B) = 0$$

Given:
$$X(t) = A \sin t + B \cos t$$
$$\begin{aligned}E[X(t)] &= E(A \sin t + B \cos t)\\ &= \sin t\, E(A) + \cos t\, E(B)\\ &= 0, \text{ a constant}\end{aligned}$$
$$\begin{aligned}E[X^2(t)] &= E[X(t) \cdot X(t)]\\ &= E[(A \sin t + B \cos t)(A \sin t + B \cos t)]\\ &= E(A^2 \sin^2 t + AB \sin t \cos t + AB \sin t \cos t + B^2 \cos^2 t)\\ &= \sin^2 t\, E(A^2) + \cos^2 t\, E(B^2) \qquad [\because E(AB) = 0]\\ &= \sigma^2(\sin^2 t + \cos^2 t)\\ &= \sigma^2, \text{ a constant.}\end{aligned}$$

∴ The process is stationary of second order.

EXAMPLE 6.18 Given a random variable Y with characteristic function $\phi(\omega) = E(e^{i\omega y})$ and a random process defined by $X(t) = \cos(\lambda t + y)$. Show that $X(t)$ is stationary in the wide sense if $\phi(1) = \phi(2) = 0$.
[AU May '04, December '05]

Solution Given: $X(t) = \cos(\lambda t + y)$

$$\phi(\omega) = E(e^{i\omega y}) \Rightarrow \phi(1) = E(e^{iy}) = E(\cos y + i \sin y)$$
$$\phi(1) = 0 \Rightarrow \phi(1) = E(\cos y) + iE(\sin y) = 0 \qquad \text{(i)}$$
$$\Rightarrow \quad E(\cos y) = 0 \text{ and } E(\sin y) = 0 \qquad \text{(given)}$$
$$\phi(2) = E(e^{2iy}) = 0,$$
$$\phi(2) = E(\cos 2y + i \sin 2y) = 0$$
$$\Rightarrow \quad E(\cos 2y) + iE(\sin 2y) = 0 \qquad \text{(ii)}$$
$$\Rightarrow \quad E(\cos 2y) = 0 \text{ and } E(\sin 2y) = 0$$
$$\begin{aligned}E[X(t)] &= E[\cos(\lambda t + y)] = E(\cos \lambda t \cos y - \sin \lambda t \sin y)\\ &= \cos \lambda t\, E(\cos y) - \sin \lambda t\, E(\sin y)\\ &= 0, \text{ a constant, using Eq. (i)}\end{aligned}$$
$$\begin{aligned}R(t_1, t_2) &= E[X(t_1)\, X(t_2)]\\ &= E[\cos(\lambda t_1 + y) \cos(\lambda t_2 + y)]\end{aligned}$$

$$= \frac{1}{2} E[\cos(\lambda t_1 + y - \lambda t_2 - y) + \cos(\lambda t_1 + y + \lambda t_2 + y)]$$

$$= \frac{1}{2} \{\cos(\lambda t_1 - \lambda t_2) + \cos[\lambda(t_1 + t_2) + 2y]\}$$

$$= \frac{1}{2} E[\cos(\lambda t_1 - \lambda t_2)] + \frac{1}{2} E[\cos \lambda(t_1 + t_2) \cos 2y - \sin \lambda(t_1 + t_2) \sin 2y]$$

$$= \frac{1}{2} E[\cos(\lambda t_1 - \lambda t_2)] + \frac{1}{2} [\cos \lambda(t_1 + t_2) E(\cos 2y) - \sin \lambda(t_1 + t_2) E(\sin 2y)]$$

$$= \frac{1}{2} \cos \lambda(t_1 - t_2) + \frac{1}{2} [\cos \lambda(t_1 + t_2) \times 0 - \sin \lambda(t_1 + t_2) \times 0]$$

using Eq. (ii)

$$= \frac{1}{2} [\cos \lambda(t_1 - t_2)], \text{ a function of time difference.}$$

and

$\therefore$ The process is a WSS process.

EXAMPLE 6.19 Let $X(t) = A \cos \lambda t + B \sin \lambda t$ where A and B are independent normally distributed random variables $N(0, \sigma^2)$. Obtain the covariance function of the process $\{X(t): -\infty < t < \infty\}$. Is $\{X(t)\}$ covariance stationary?

Solution Given $N(0, \sigma^2) \Rightarrow$ mean = 0, variance = σ^2

$$E(A) = E(B) = 0$$
$$E(A^2) = E(B^2) = \sigma^2$$

Since A and B are independent

$$E(AB) = E(A)\, E(B) = 0$$
$$E[X(t)] = E(A\cos \lambda t + B \sin \lambda t)$$
$$= E(A)\cos \lambda t + E(B)\sin \lambda t$$
$$= 0, \text{ a constant}$$

The ACF of $\{X(t)\}$ is

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$
$$= E(A \cos \lambda t_1 + B \sin \lambda t_1)(A \cos \lambda t_2 + B \sin \lambda t_2)]$$
$$= E[A^2 \cos \lambda t_1 \cos \lambda t_2 + AB(\cos \lambda t_1 \sin \lambda t_2 + \sin \lambda t_1 \cos \lambda t_2) + B^2 \sin \lambda t_1 \sin \lambda t_2]$$
$$= E(A^2) \cos \lambda t_1 \cos \lambda t_2 + E(B^2) \sin \lambda t_1 \sin \lambda t_2$$
$$= \sigma^2(\cos \lambda t_1 \cos \lambda t_1 + \sin \lambda t_1 \sin \lambda t_2)$$
$$= \sigma^2 \cos \lambda(t_1 - t_2)$$

$$\text{Covariance} = C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)]\, E[X(t_2)]$$
$$= \sigma^2 \cos \lambda(t_1 - t_2) - 0$$

$\therefore$ $E[X(t)]$ is a constant and covariance is a function of time difference.

$\therefore$ $\{X(t)\}$ is covariance stationary.

EXAMPLE 6.20 If the $2n$ random variables A_r and B_r are uncorrelated with zero mean and $E(A_r^2) = E(B_r^2) = \sigma_r^2$, show that the process $X(t) = \sum_{r=1}^{n}(A_r \cos \omega_r t + B_r \sin \omega_r t)$ is WSS. What are the mean and autocorrelation of $\{X(t)\}$?

Solution To show that $\{X(t)\}$ is a WSS process, we have to prove that

(i) $E[X(t)]$ = a constant.

(ii) $R(t_1, t_2)$ is a function of time difference.

Given: $E(A_r) = E(B_r) = 0$, $E(A_r^2) = E(B_r^2) = \sigma_r^2$, $r = 0, 1, 2, \ldots, n$

$$E(A_1) = E(A_2) = \cdots = E(A_r) = \cdots = E(A_n) = 0$$
$$E(B_1) = E(B_2) = \cdots = E(B_r) = \cdots = E(B_n) = 0.$$

Also, $E(A_r B_s) = E(A_r)\, E(B_s) = 0 \;\; \forall\; r, s = 1, 2, \ldots, n$

$$X(t) = \sum_{r=1}^{n}(A_r \cos \omega_r t + B_r \sin \omega_r t)$$

$$E[X(t)] = E\left[\sum_{r=1}^{n}(A_r \cos \omega_r t + B_r \sin \omega_r t)\right]$$
$$= E[(A_1 \cos \omega_1 t + A_2 \cos \omega_2 t + \cdots + A_n \cos \omega_n t) + (B_1 \sin \omega_1 t + B_2 \sin \omega_2 t + \cdots + B_n \sin \omega_n t)]$$
$$= E(A_1 \cos \omega_1 t) + E(A_2 \cos \omega_2 t) + \cdots + E(A_n \cos \omega_n t) + E(B_1 \sin \omega_1 t) + E(B_2 \sin \omega_2 t) + \cdots + E(B_n \sin \omega_n t)$$
$$= \cos \omega_1 t E(A_1) + \cos \omega_2 t E(A_2) + \cdots + \cos \omega_n t E(A_n) + \sin \omega_1 t E(B_1) + \sin \omega_2 t E(B_2) + \cdots + \sin \omega_n t E(B_n)$$
$$= 0, \text{ a constant}$$

$$R(t_1, t_2) = E[X(t_1) X(t_2)]$$
$$= E\left\{\left[\sum_{r=1}^{n}(A_r \cos \omega_r t_1 + B_r \sin \omega_r t_1)\right]\left[\sum_{r=1}^{n}(A_r \cos \omega_r t_2 + B_r \sin \omega_r t_2)\right]\right\}$$
$$= E\left\{\begin{array}{l}[(A_1 \cos \omega_1 t_1 + B_1 \sin \omega_1 t_1) + (A_2 \cos \omega_2 t_1 + B_2 \sin \omega_2 t_1) + \cdots \\ \qquad + (A_n \cos \omega_n t_1 + B_n \sin \omega_n t_1)] \times \\ {[(A_1 \cos \omega_1 t_2 + B_1 \sin \omega_1 t_2) + (A_2 \cos \omega_2 t_2 + B_2 \sin \omega_2 t_2) + \cdots} \\ \qquad + (A_n \cos \omega_n t_2 + B_n \sin \omega_n t_2)]\end{array}\right\}$$

$$= E\begin{bmatrix} (A_1A_1 \cos\omega_1 t_1 \cos\omega_1 t_2) + (B_1B_1 \sin\omega_1 t_1 \sin\omega_1 t_2) + \cdots \\ + (A_nA_n \cos\omega_n t_1 \cos\omega_n t_2) + (B_nB_n \sin\omega_n t_1 \sin\omega_n t_2) \\ + (A_1A_2 \cos\omega_1 t_1 \cos\omega_2 t_2) + (B_1B_2 \sin\omega_1 t_1 \sin\omega_2 t_2) \\ + (A_1B_2 \cos\omega_1 t_1 \sin\omega_1 t_2) + (A_nA_{n-1} \cos\omega_n t_1 \cos\omega_{n-1} t_2) + \cdots \\ + (B_nB_{n-1} \sin\omega_n t_1 \sin\omega_{n-1} t_2) + (A_nB_{n-1} \cos\omega_n t_1 \cos\omega_{n-1} t_2) \\ + (B_nA_{n-1} \sin\omega_n t_1 \sin\omega_{n-1} t_2) \end{bmatrix}$$

Using $E(A_rB_s) = 0$ for all r and s, $E(A_rA_s) = 0$ for $r \neq s$, $E(B_rB_s) = 0$ for $r \neq s$ and $E(A_r^2) = E(B_r^2) = \sigma_r^2$, $r = 1, 2, \ldots, n$

$$= \begin{bmatrix} (\cos\omega_1 t_1 \cos\omega_1 t_2)\, E(A_1^2) + (\sin\omega_1 t_1 \sin\omega_1 t_2)\, E(B_1^2) + \cdots \\ + (\cos\omega_n t_1 \cos\omega_n t_2)\, E(A_n^2) + (\sin\omega_n t_1 \sin\omega_n t_2)\, E(B_n^2) \end{bmatrix}$$

$$= \sum_{r=1}^{n} \sigma_r^2 (\cos\omega_r t_1 \cos\omega_r t_2 + \sin\omega_r t_1 \sin\omega_r t_2)$$

$$= \sum_{r=1}^{n} \sigma_r^2 \cos\omega_r (t_1 - t_2), \text{ a function of time difference.}$$

$\therefore$ It is a WSS process.

EXAMPLE 6.21 Two random processes $X(t)$ and $Y(t)$ are defined by $X(t) = A \cos\omega_0 t + B \sin\omega_0 t$ and $Y(t) = B \cos\omega_0 t - A \sin\omega_0 t$. Show that $X(t)$ and $Y(t)$ are jointly WSS process if A and B are uncorrelated random variables with 0 means and the same variances with ω_0 as a constant.

[AU April '04, December '09]

Solution To show that the two random processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS process, we have to show that they are

(i) individually WSS process, and
(ii) their cross-correlation is a function of time difference.

Given: $X(t) = A \cos\omega_0 t + B \sin\omega_0 t$ (i)
where A and B are uncorrelated random variables.
Given: means of random variables are zero and A and B are uncorrelated

$$\therefore \quad E(A) = E(B) = 0$$
$$E(AB) = E(A)E(B) = 0$$

As A and B have the same variance, we get

$$\text{Var}(A) = E(A^2) - [E(A)]^2 = \sigma^2 \Rightarrow E(A^2) = \sigma^2$$
$$\text{Var}(B) = E(B^2) - [E(B)]^2 = \sigma^2 \Rightarrow E(B^2) = \sigma^2$$

$$\therefore\ E[X(t)] = E(A\cos\omega_0 t + B\sin\omega_0 t)$$
$$= \cos\omega_0 t\, E(A) + \sin\omega_0 t\, E(B)$$
$$= 0, \text{ a constant} \qquad [\because E(A) = E(B) = 0]$$

$$R(t_1, t_2) = E[X(t_1)\cdot X(t_2)]$$
$$= E[(A\cos\omega_0 t_1 + B\sin\omega_0 t_1)(A\cos\omega_0 t_2 + B\sin\omega_0 t_2)]$$
$$= E[A^2\cos\omega_0 t_1\cos\omega_0 t_2 + AB\cos\omega_0 t_1\sin\omega_0 t_2 + AB\sin\omega_0 t_1\cos\omega_0 t_2 + B^2\sin\omega_0 t_1\sin\omega_0 t_2]$$
$$= \cos\omega_0 t_1\cos\omega_0 t_2\, E(A^2) + \cos\omega_0 t_1\sin\omega_0 t_2\, E(AB) + \sin\omega_0 t_1\cos\omega_0 t_2\, E(AB) + \sin\omega_0 t_1\sin\omega_0 t_2\, E(B^2)$$
$$= \sigma^2(\cos\omega_0 t_1\cos\omega_0 t_2 + \sin\omega_0 t_1\sin\omega_0 t_2)$$
$$= \sigma^2[\cos\omega_0(t_1 - t_2)], \text{ a function of time difference}$$

$\therefore$ $X(t)$ is a WSS process.

$$E[Y(t)] = E(B\cos\omega_0 t - A\sin\omega_0 t)$$
$$= \cos\omega_0 t\, E(B) - \sin\omega_0 t\, E(A)$$
$$= 0, \text{ a constant} \qquad [\because E(A) = E(B) = 0]$$

$$R(t_1, t_2) = E[X(t_1)\cdot X(t_2)]$$
$$= E[(B\cos\omega_0 t_1 - A\sin\omega_0 t_1)(B\cos\omega_0 t_2 - A\sin\omega_0 t_2)]$$
$$= E(B^2\cos\omega_0 t_1\cos\omega_0 t_2 - AB\cos\omega_0 t_1\sin\omega_0 t_2 - AB\sin\omega_0 t_1\cos\omega_0 t_2 + A^2\sin\omega_0 t_1\sin\omega_0 t_2)$$
$$= \cos\omega_0 t_1\cos\omega_0 t_2\, E(B^2) - \cos\omega_0 t_1\sin\omega_0 t_2\, E(AB) - \cos\omega_0 t_2\sin\omega_0 t_1\, E(AB) + \sin\omega_0 t_1\sin\omega_0 t_2 E(A^2)$$
$$= \sigma^2(\cos\omega_0 t_1\cos\omega_0 t_2 + \sin\omega_0 t_1\sin\omega_0 t_2)$$
$$= \sigma^2[\cos(t_1 - t_2)], \text{ a function of time difference.}$$

$\therefore$ $Y(t)$ is a WSS process.

$$R_{XY}(t_1, t_2) = E[X(t_1)\cdot Y(t_2)]$$
$$= E[(A\cos\omega_0 t_1 + B\sin\omega_0 t_1)(B\cos\omega_0 t_2 - A\sin\omega_0 t_2)]$$
$$= E(AB\cos\omega_0 t_1\cos\omega_0 t_2 - A^2\cos\omega_0 t_1\sin\omega_0 t_2 + B^2\sin\omega_0 t_1\cos\omega_0 t_2 - AB\sin\omega_0 t_1\sin\omega_0 t_2]$$
$$= \cos\omega_0 t_1\cos\omega_0 t_2\, E(AB) - \cos\omega_0 t_1\sin\omega_0 t_2\, E(A^2) + \sin\omega_0 t_1\cos\omega_0 t_2\, E(B^2) - \sin\omega_0 t_1\sin\omega_0 t_2\, E(AB)$$
$$= \sigma^2(\sin\omega_0 t_1\cos\omega_0 t_2 - \cos\omega_0 t_1\sin\omega_0 t_2)$$
$$= \sigma^2\sin\omega_0(t_1 - t_2), \text{ a function of time difference.}$$

$\therefore$ The cross-correlation is a function of time difference and, hence, $X(t)$ and $Y(t)$ are jointly WSS process.

EXAMPLE 6.22 If $U(t) = (X\cos t + Y\sin t)$ and $V(t) = (Y\cos t + X\sin t)$ where X and Y are independent random variables such that $E(X) = 0$, $E(Y) = 0$, $E(X^2) = E(Y^2) = 1$, show that $\{U(t)\}$ and $\{V(t)\}$ are individually stationary in the wide sense, but not jointly stationary.

Solution Given: $E(X) = 0$, $E(Y) = 0$ and $E(X^2) = E(Y^2) = 1$. Also X and Y are independent random variables, i.e. $E(XY) = E(X)\ E(Y) = 0$

To show $\{U(t)\}$ and $\{V(t)\}$ are stationary in the wide sense:

$$\begin{aligned}
E[U(t)] &= E(X \cos t + Y \sin t)\\
&= \cos t\ E(X) + \sin t\ E(Y) = 0\\
R_{UU}(t_1, t_2) &= E[U(t_1)\, U(t_2)]\\
&= E[(X \cos t_1 + Y \sin t_1) \times X \cos t_2 + Y \sin t_2)]\\
&= E(X^2 \cos t_1 \cos t_2 + XY \cos t_1 \sin t_2 + XY \sin t_1 \cos t_2 + Y^2 \sin t_1 \sin t_2)\\
&= \cos t_1 \cos t_2\ E(X^2) + \cos t_1 \sin t_2\ E(XY) + \sin t_1 \cos t_2\ E(XY)\\
&\qquad + \sin t_1 \sin t_2\ E(Y^2)\\
&= \cos t_1 \cos t_2 + \sin t_1 \sin t_2 = \cos(t_1 - t_2)\\
&= \cos \tau, \text{ where } \tau = t_1 - t_2\\
&= \text{a function of time difference.}
\end{aligned}$$

$\therefore$ $U(t)$ is a WSS process.

Again,

$$\begin{aligned}
E[V(t)] &= E(Y \cos t + X \sin t)\\
&= \cos t\ E(Y) + \sin t\ E(X) = 0\\
R_{VV}(t_1, t_2) &= E[V(t_1)\, V(t_2)]\\
&= E[(Y \cos t_1 + X \sin t_1) \times (Y \cos t_2 + X \sin t_2)]\\
&= E(Y^2 \cos t_1 \cos t_2 + XY \cos t_1 \sin t_2 + XY \sin t_1 \cos t_2 + X^2 \sin t_1 \sin t_2)\\
&= \cos t_1 \cos t_2\ E(Y^2) + \cos t_1 \sin t_2\ E(XY) + \sin t_1 \cos t_2\ E(XY)\\
&\qquad + \sin t_1 \sin t_2\ E(X^2)\\
&= \cos t_1 \cos t_2 + \sin t_1 \sin t_2 = \cos(t_1 - t_2)\\
&= \cos \tau, \text{ where } \tau = t_1 - t_2\\
&= \text{a function of time difference.}
\end{aligned}$$

$\therefore$ $V(t)$ is a WSS process.

To find whether they are jointly WSS or not.

$$\begin{aligned}
R_{UV}(t_1, t_2) &= E[U(t_1)\, V(t_2)]\\
&= E[(X \cos t_1 + Y \sin t_1) \times (Y \cos t_2 + X \sin t_2)]\\
&= E(XY \cos t_1 \cos t_2 + X^2 \cos t_1 \sin t_2 + Y^2 \sin t_1 \cos t_2 + XY \sin t_1 \sin t_2)\\
&= \cos t_1 \cos t_2\ E(XY) + \cos t_1 \sin t_2\ E(X^2) + \sin t_1 \cos t_2\ E(Y^2)\\
&\qquad + \sin t_1 \sin t_2\ E(XY)\\
&= \cos t_1 \sin t_2 + \sin t_1 \cos t_2 = \sin(t_1 + t_2)
\end{aligned}$$

which is not a function of time difference, i.e. the cross-correlation is not a function of time difference.

$\therefore$ $U(t)$ and $V(t)$ are individually WSS processes but not jointly WSS.

6.2 MARKOV PROCESS AND MARKOV CHAIN

6.2.1 Markov Process

A random process $\{X(t)\}$ is called a Markov process if
$P[X(t_n) = a_n/X(t_{n-1}) = a_{n-1}, X(t_{n-2}) = a_{n-2}, ..., X(t_2) = a_2, X(t_1) = a_1]$
$= P[X(t_n) = a_n/X(t_{n-1}) = a_{n-1}]$ for all $t_1 < t_2 < ... < t_n$.

In other words, if the future behaviour of a process depends on the present but not on the past, then the process is called a *Markov process*.

A Markov process can be classified into four types depending upon the values taken by time t and the state space $\{X_n\}$:

(i) A continuous random process satisfying Markov property is known as *continuous parameter Markov process*.
(ii) A continuous random sequence satisfying Markov property is known as *discrete parameter Markov process*.
(iii) A discrete random sequence satisfying Markov property is known as *discrete parameter Markov chain*.
(iv) A discrete random process satisfying Markov property is known as *continuous parameter Markov chain*.

6.2.2 Markov Chain

If $P(X_n = a_n/X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ..., X_0 = a_0) = P(X_n = a_n/X_{n-1} = a_{n-1})\ \forall\ n$, then the process $\{X_n\}, n = 0, 1, 2, ...$ is called a *Markov chain* and the constants $\{a_0, a_1, ..., a_n\}$ are called the *states of the Markov chain*. In other words, a discrete parameter Markov process is called a *Markov chain*.

One-step Transition Probability

The conditional transition probability $P(X_n = a_j/X_{n-1} = a_i)$ is called the *one-step transition probability* from state a_i to state a_j at the nth step and is denoted by $P_{ij}(n-1, n)$.

Homogeneous Markov Chain

If the one-step transition probability does not depend on the step, i.e., $P_{ij}(n-1, n) = P_{ij}(m-1, m)$, the Markov chain is called a *homogeneous Markov chain*. That is, the chain is said to be stationary.

Transition Probability Matrix (TPM)

When the Markov chain is homogeneous, the one-step transition probability is denoted by P_{ij}. The matrix $P = (P_{ij})$ is called the *transition probability matrix* satisfying the conditions

(i) $P_{ij} \geq 0, \ \forall i, j$ and

(ii) $\sum_i P_{ij} = 1$ for all j

That is, the sum of the elements of any row of the TPM is 1.

n-Step Transition Probability

The conditional probability that the process is in state a_j at step n, given that it was in state a_i at step 0, $P(X_n = a_j/X_0 = a_i)$ is called the *n-step transition probability* and is denoted by

$$P_{ij}^{(n)} = P(X_n = a_j/X_0 = a_i)$$

Note: $P_{ij}^{(1)} = P_{ij}$.

Probability Distribution of the Process

If the probability that the process in state a_i is $p_i (i = 1, 2, \ldots, n)$ at any arbitrary step, then the row vector $P = (p_1, p_2, \ldots, p_n)$ is called the *probability distribution of the process* at that time.

In particular, $P^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots, p_n^{(0)})$ is the initial probability distribution, where $p_1^{(0)} + p_2^{(0)} + \cdots + p_n^{(0)} = 1$.

The nth step probability distribution of the Markov Chain is given by $P^{(n)}$ and it is computed from the intial probability distribution $P^{(0)}$ and the TPM P. i.e. $P^{(1)} = P^{(0)}P$; $P^{(2)} = P^{(1)}P, \ldots, P^{(n)} = P^{(n-1)}P$

Chapman–Kolmogorov Theorem

If P is the TPM of a homogeneous Markov chain, then n-step TPM $P^{(n)}$ is equal to P^n, i.e.

$$[p_{ij}^{(n)}] = [p_{ij}]^n$$

Regular Markov Chain

A stochastic matrix P is said to be a regular matrix if all the entries of P^m, for some positive integer m are positive. A homogeneous Markov chain is said to be regular if its TPM is regular.

Steady-state Distribution

If a homogeneous Markov chain is regular, then every sequence of state probability distributions approaches a unique fixed distribution, called the *steady-state distribution of the Markov chain.*

$$\lim_{n \to \infty} [P^{(n)}] = \pi$$

where $P^{(n)} = [p_1^{(n)}, p_2^{(n)}, \ldots, p_k^{(n)}]$ and $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$

If P is the TPM of the regular Markov chain and $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is the steady-state distribution, then $\pi P = \pi$ and $\pi_1 + \pi_2 + \cdots + \pi_k = 1$.

Classification of States of a Markov Chain†

1. Accessibility: Suppose that the state j has the property that it can be reached from any state i, then it is said to be accessible from i, i.e. there exists a probability p_{ij}^m in m steps such that

(i) $p_{ij}^m > 0$ for $m \geq 0$

(ii) $i \to j$ if $p_{ij}^m > 0$, $m \geq 0$

It is shown in Figure 6.1.

Figure 6.1 Accessibility.

2. Communication: If two states are accessible from each other, then they are said to communicate with each other, i.e. the states i and j are communicating states, i.e. $i \leftrightarrow j$ if $j \to i$ and $i \to j$. It is shown in Figure 6.2.

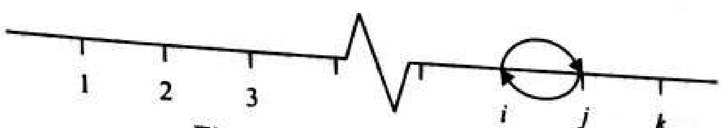

Figure 6.2 Communication.

3. Essentiality: A state which has the property that it will communicate with any state from which it is accessible is said to be an essential state.

Properties of Communicating States

The communicating states satisfy the following properties:

1. Reflexivity: A state will communicate with itself, i.e. $i \leftrightarrow i$. It is shown in Figure 6.3.

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†AU May '05, December '04

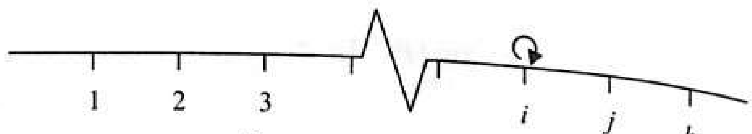

Figure 6.3 Reflexivity.

2. Symmetry: If the states i and j are communicating with each other, then j and i also communicate with each other.

3. Transitivity: If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$, i.e. if i communicates with j and j communicates with k, then i communicates with k. It is shown in Figure 6.4.

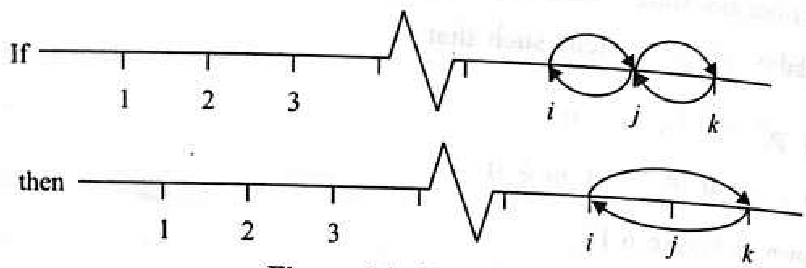

Figure 6.4 Transitivity.

Irreducible Markov Chain

If $p_{ij}^{(n)} > 0$ for some n for all i and j, then every state can be reached from every other state. When this condition is satisfied, the Markov chain is said to be irreducible. The TPM of irreducible Markov chain is an irreducible matrix. Otherwise, the chain is said to be reducible.

Return State

State i of a Markov chain is called a return state if $p_{ii}^{(n)} > 0$ for some $n \geq 1$. It is shown in Figure 6.5.

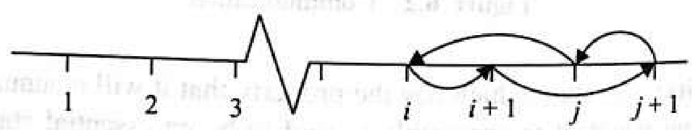

Figure 6.5 Return state.

Periodicity

The period d_i of a return state i is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$, i.e.

$$d_i = \text{GCD}\{m : p_{ii}^{(m)} > 0\}$$

State i is said to be periodic with period d_i if $d_i > 1$ and aperiodic if $d_i = 1$.

Obviously, state i is aperiodic if $p_{ii} \neq 0$. The probability that the chain returns to state i, having started from state i, for the first time at the nth step (i.e. after n transitions) is denoted by $f_{ii}^{(n)}$ and called the *first return time probability* or the *recurrence time probability*.

$\{n, f_{ii}^{(n)}\}, n = 1, 2, 3, \ldots$ is the distribution of recurrence times of the state i.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain and $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ is called the mean recurrence time of the state i.

A state i is said to be *persistent or recurrrent* if the return to state i is certain, i.e. if $F_{ii} = 1$.

The state i is said to be *transient* if the return to state i is uncertain, i.e. if $F_{ii} < 1$.

The state i is said to be *non-null persistent* if its mean reccurence time μ_{ii} is finite and *null persistent* if $\mu_{ii} = \infty$.

Ergodic State

A non-null persistent and aperiodic state is called an *ergodic state*.

Note: If the state i is non-null persistent, then $\lim_{n \to \infty} p_{ii}^{(n)} > 0$ and if the state i is null persistent or transient, then $\lim_{n \to \infty} \{p_{ii}^{(n)}\} \to 0$.

Absorbing State

A state i is called an absorbing state if $p_{ij} = 1$, for $i = j$ and $p_{ij} = 0$ for $i \neq j$.

Note: The following two results are very helpful to classify the states of a Markov chain:

(i) If a Markov chain is irreducible, all its states are of the same type. They are all transient, all null persistent or all non-null persistent. All its states are either aperiodic or periodic with same period.

(ii) If a Markov chain is finite irreducible, then all its states are non-null persistent.

Calculation of Joint Probability

$P(X_n = a_n, X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, X_{n-3} = a_{n-3}, \ldots, X_1 = a_1, X_0 = a_0)$
$= P(X_n = a_n / X_{n-1} = a_{n-1}) \ldots P(X_1 = a_1 / X_0 = a_0)\, P(X_0 = a_0) = p_{n-1,n}\, p_{n-2,n-1} \ldots p_{01} P(X_0 = a_0)$

EXAMPLE 6.23 Check whether $\left(\frac{1}{2} \quad \frac{1}{4} \quad 0 \quad \frac{1}{4}\right)$ is a probability vector.

Solution Since

(i) all the elements are ≥ 0, and

(ii) the row sum is equal to 1, the given vector is a probability vector.

EXAMPLE 6.24 Let $A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ be a stochastic matrix, check whether it is regular.

[AU June '06]

Solution A matrix A is regular if all entries of some power are > 0, i.e. all entries of A^n are positive for some n.

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Since all the entries in A^2 are positive (> 0), A is regular.

EXAMPLE 6.25 If the TPM of a Markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, find the steady-state distribution of the chain. **[AU November '05]**

Solution If $\pi = (\pi_1, \pi_2)$ is the steady-state distribution of the chain, then $\pi P = \pi$ and $\pi_1 + \pi_2 = 1$ (i)

$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$$\frac{1}{2}\pi_2 = \pi_1 \quad \Rightarrow \quad \pi_2 = 2\pi_1$$

Substituting in Eq. (i)

$$3\pi_1 = 1 \quad \Rightarrow \quad \pi_1 = \frac{1}{3}$$

$$\therefore \qquad \pi_2 = \frac{2}{3}$$

The steady-state distribution $\pi = \left(\frac{1}{3}, \frac{2}{3}\right)$

Note: When all the entries of a TPM are positive, i.e. if the TPM is regular and of the form

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}, \quad 0 < a < 1, 0 < b < 1$$

then it can be easily shown that

$$P^n = \frac{1}{a+b}\left[\begin{pmatrix} b & a \\ b & a \end{pmatrix} + (1-a-b)^n \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}\right]$$

and the steady-state probability is given by

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

EXAMPLE 6.26 In a hypothetical market, there are only two brands A and B. A customer buys brand A with probability 0.7 if his last purchase was A and buys brand B with probability 0.4 if his last purchase was B. Assuming MC model, obtain

(i) one-step TPM P say,
(ii) n-step TPM P^n, and
(iii) the stationary distribution.

Hence highlight the proportion of customers who would buy brand A and brand B in the long run. **[AU December '05]**

Solution Since there are only two brands A and B in the market, we have a 2×2 TPM.

A customer buys

(a) brand A with probability 0.7 if his last purchase was A
$A \to A$ with probability 0.7
(b) brand B with probability 0.4 if his last purchase was B
$B \to B$ with probability 0.4

(i) $$\text{TPM} = P = \begin{array}{c} \\ A \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} \end{array} = \begin{array}{c} \\ A \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 1-0.3 & 0.3 \\ 0.6 & 1-0.6 \end{pmatrix} \end{array}$$

Since all the entries of the TPM P are positive, it is regular.

(ii) Therefore, using the above note, we get

$$P^n = \frac{1}{a+b}\left[\begin{pmatrix} b & a \\ b & a \end{pmatrix} + (1-a-b)^n \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}\right]$$

where $a = 0.3$ and $b = 0.6$

$$P^n = \frac{1}{0.3+0.6}\left[\begin{pmatrix} 0.6 & 0.3 \\ 0.6 & 0.3 \end{pmatrix} + (1-0.3-0.6)^n \begin{pmatrix} 0.3 & -0.3 \\ -0.6 & 0.6 \end{pmatrix}\right]$$

$$P^n = \frac{1}{0.9}\left[\begin{pmatrix} 0.6 & 0.3 \\ 0.6 & 0.3 \end{pmatrix} + (0.1)^n \begin{pmatrix} 0.3 & -0.3 \\ -0.6 & 0.6 \end{pmatrix}\right]$$

$$= \frac{1}{0.9}\left[\begin{pmatrix} 0.6 + (0.1)^n \times 0.3 & 0.3 - (0.1)^n \times 0.3 \\ 0.6 - (0.1)^n \times 0.6 & 0.3 + (0.1)^n \times 0.6 \end{pmatrix}\right]$$

(iii) To find the proportion of customers who would buy brand A and brand B in the long run, that is, the steady-state distribution of the chain, then

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right) = \left(\frac{0.6}{0.3+0.6}, \frac{0.3}{0.3+0.6}\right)$$

$$= \left(\frac{2}{3}, \frac{1}{3}\right) = (0.667, 0.333)$$

Therefore, 66.7% customers would buy brand A and 33.3% customers would buy brand B.

Aliter

Solving as before using $\pi P = \pi$ and $\pi_1 + \pi_2 = 1$, we get

$$(\pi_1, \pi_2)\begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} = (\pi_1 \quad \pi_2)$$

$$\begin{aligned}
0.7\pi_1 + 0.6\pi_2 = \pi_1 \quad &\Rightarrow \quad 0.7\pi_1 + 0.6 - 0.6\pi_1 = \pi_1 \text{ using } 1 - \pi_1 = \pi_2 \\
0.9\pi_1 = 0.6 \quad &\Rightarrow \quad \pi_1 = 0.667 \\
0.6\pi_1 + 0.4\pi_2 = \pi_2 \quad &\Rightarrow \quad 0.6 \times 0.667 + 0.4\pi_2 = \pi_2 \\
&\Rightarrow \quad \pi_2 = 0.333
\end{aligned}$$

Therefore, the steady-state distribution is

$$\pi = \left(\frac{0.6}{0.3+0.6}, \frac{0.3}{0.3+0.6}\right) = \left(\frac{2}{3}, \frac{1}{3}\right) = (0.667, 0.333)$$

EXAMPLE 6.27 At an intersection, a working traffic light will be out of order the next day with probability 0.07, and an out-of-order traffic light will be working the next day with probability 0.88. Let $X_n = 1$ if on day n the traffic light will work, $X_n = 0$ if on day n the traffic light will not work. Is $\{X_n : n = 0, 1, 2, \ldots\}$ a Markov chain? If so, write the transition probability matrix.

[AU December '07]

Solution The traffic light will work on the next day depends on whether it works or not today.

Since the states of X_n depend only on X_{n-1} but not on X_{n-2}, X_{n-3}, X_{n-4}, ... or earlier states $\{X_n : n = 0, 1, 2, \ldots\}$ is a Markov chain.

The required transition probability matrix (TPM)

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & \;\;1 \end{array} \\ \begin{pmatrix} 0.12 & 0.88 \\ 0.07 & 0.93 \end{pmatrix} \end{array}$$

EXAMPLE 6.28 Using limiting behaviour of homogeneous Markov chain, find steady-state probability of the chain given by the TPM

$$P = \begin{pmatrix} 0.1 & 0.6 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}.$$

Solution In the steady-state, the probability will be the same, i.e. there will be no change in the probability with respect to time.

Let $\pi = (\pi_1, \pi_2, \pi_3)$ be the probability in the steady-state, then $\pi \cdot P = \pi$ and $\pi_1 + \pi_2 + \pi_3 = 1$.

$$\therefore \qquad (\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0.1 & 0.6 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3)$$

$$0.1\pi_1 + 0.5\pi_2 + 0.1\pi_3 = \pi_1$$
$$0.6\pi_1 + 0.1\pi_2 + 0.2\pi_3 = \pi_2$$
$$0.3\pi_1 + 0.4\pi_2 + 0.7\pi_3 = \pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

Solving the above equations, we get

$$\pi_1 = 0.2021$$
$$\pi_2 = 0.2553$$
$$\pi_3 = 0.5425$$

Imp

EXAMPLE 6.29 There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the two marbles selected are interchanged. The state of the related Markov chain is the number of red marbles in urn A, after the interchange.

(i) Find the transition probability matrix of the system.

(ii) What is the probability that there are two red marbles in urn A after three steps? second state

(iii) In the long run, what is the probability that there are 2 red marbles in urn A? **[AU, June '07]**

Solution The urn A contains only two marbles. Therefore, as the states denote the number of red marbles in A, the states are $0(R)$, $1(R)$, or $2(R)$.

In state 0, $\quad A \to 0R, 2W \qquad B \to 3R, 0W$

In state 1, $A \to 1W, 1R$ $\quad B \to 2R, 1W$

In state 2, $A \to 2R, 0W$ $\quad B \to 1R, 2W$

Suppose the state is in 0. After one interchange A will have 1 red marble. Therefore, from state 0 it will go only to state 1. As there is no other possibility, the probability is 1. Suppose the state is in 1, then after one interchange either it will go to state 0 or state 2, or it will remain in 1.

So, after one interchange state 1 will go to 0, only if 1 red from A and 1 white from B are interchanged with probability

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \left[\text{i.e. } P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{3} \right]$$

The state is in 1, then after one interchange it will remain in state 1, only if 1 white from A and 1 white from B are interchanged or 1 red from A and 1 red from B are interchanged with probability

$$= \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{1}{2}$$

The state is in 1, and then after one interchange it will go to state 2, only if 1 white from A and 1 red from B are interchanged with probability

$$= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \text{ and so on.}$$

Similarly, $\quad 2 \to 1 \Rightarrow 1 \times \frac{2}{3} = \frac{2}{3}$

$\quad 2 \to 2 \Rightarrow 1 \times \frac{1}{3} = \frac{1}{3}$

$\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3}$

Therefore, the TPM $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 2 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}$

Now, $P^0 = (1, 0, 0)$ as there is no red marble in A in the beginning.

$$P^1 = P^0 \cdot P = (0\ 1\ 0)$$

$$P^2 = P^1 \cdot P = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3} \right)$$

$$P^3 = P^2 \cdot P = \left(\frac{1}{12}, \frac{23}{26}, \frac{5}{18} \right)$$

$P(2 \text{ red marbles in } A \text{ after 3 steps}) = \dfrac{5}{18}$

In the steady-state, we know that $\pi P = \pi$ and $\pi_1 + \pi_2 + \pi_3 = 1$ (i)

i.e.
$$(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3)$$

$$\frac{1}{6}\pi_2 = \pi_1$$

$$\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{3}\pi_3 = \pi_2$$

$$0\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{3}\pi_3 = \pi_3 \quad \Rightarrow \quad \pi_3 = \frac{1}{2}\pi_2$$

Substituting in Eq. (i) gives

$$\frac{1}{6}\pi_2 + \pi_2 + \frac{1}{2}\pi_2 = 1 \quad \Rightarrow \quad \pi_2\left(\frac{1}{6} + 1 + \frac{1}{2}\right) = 1$$

i.e.
$$\frac{10}{6}\pi_2 = 1 \quad \Rightarrow \quad \pi_2 = \frac{6}{10}$$

So, $\pi_1 = \dfrac{1}{10}$ and $\pi_3 = \dfrac{3}{10}$

EXAMPLE 6.30 Three girls G_1, G_2, G_3 are throwing a ball to each other. G_1 always throws the ball to G_2 and G_2 always throws the ball to G_3. But G_3 is just as likely to throw the ball to G_2 as to G_1. Prove that the process is Markovian. Find the TPM and classify the states.

[AU December '05]

Solution

$$\text{TPM} = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} G_1 & G_2 & G_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

The process $\{X_n\}$ depends only on X_{n-1} but not on $X_{n-2}, X_{n-3}, \ldots$.
The process $\{X_n\}$ is Markov process

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Since $\quad p_{12} > 0,\ p_{23} > 0,\ p_{31} > 0,\ p_{32} > 0$ in P

$$p_{13}^{(2)} > 0,\ p_{21}^{(2)} > 0,\ p_{22}^{(2)} > 0,\ p_{33}^{(2)} > 0 \text{ and } p_{11}^{(3)} > 0.$$

The Markov chain is irreducible.

$$P^4 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix},\ P^5 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{pmatrix} \text{ and so on.}$$

Since for $m = 1, 2, 3$, all the states are greater than 0,
The Markov chain is irreducible.
Since the Markov chain is finite, the states are non-null persistent.
Again $P_{ii}^{(2)}, P_{ii}^{(3)}, P_{ii}^{(4)}, \ldots$ are greater than 0 for $i = 2, 3$
and Period = GCD(2, 3, ...) = 1.
$\therefore$ The states 2 and 3 are periodic with period 1, that is they are aperiodic.
Also, $P_{11}^{(3)}, P_{11}^{(5)}, P_{11}^{(7)}, \ldots$ are > 0

$$\text{Period} = \text{GCD}\{3, 5, \ldots\} = 1$$

The state 1 is also aperiodic.
$\therefore$ The Markov chain is irreducible, non-null persistent and aperiodic and hence, all the states are ergodic.

doubt
Given

EXAMPLE 6.31 The TPM of the Markov chain $\{X_n\}$, with $n = 1, 2, 3, \ldots$ having 3 states 1, 2, 3 is $P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$ and initial distribution is $P^{(0)} = (0.7,\ 0.2,\ 0.1)$. Find

(i) $P(X_2 = 3)$, and
(ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$. [AU December '03, April '07]

Solution Given $\quad P^{(0)} = (0.7,\ 0.2,\ 0.1)$

i.e.

$$P(X_0 = 1) = 0.7$$
$$P(X_0 = 2) = 0.2$$

$$P(X_0 = 3) = 0.1$$

and

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \end{array} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$P^{(2)} = P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(i) $P(X_0 = 2) = 0.2$

$$P(X_2 = 3) = P_{13}^{(2)} P(X_0 = 1) + P_{23}^{(2)} P(X_0 = 2) + P_{33}^{(2)} P(X_0 = 3)$$
$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 = 0.279$$

$$P(X_1 = 3, X_0 = 2) = P(X_1 = 3/X_0 = 2) \cdot P(X_0 = 2)$$
$$= P_{23} \cdot P(X_0 = 2)$$
$$= (0.2)\ (0.2) = 0.04$$

$$P(X_2 = 3, X_1 = 3, X_0 = 2) = P(X_2 = 3/X_1 = 3, X_0 = 2)$$
$$\times P(X_1 = 3, X_0 = 2)$$
$$= P(X_2 = 3/X_1 = 3) \times P(X_1 = 3, X_0 = 2)$$

Using Markov property,

$$P(X_2 = 3/X_1 = 3, X_0 = 2) = P(X_2 = 3/X_1 = 3)$$
$$= P_{33} \times P(X_1 = 3)$$
$$= 0.3 \times 0.04 = 0.012 \qquad \text{(i)}$$

(ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

$$= P(X_3 = 2/X_2 = 3)\ P(X_2 = 3, X_1 = 3, X_0 = 2)$$
$$= P_{32} \times 0.012 \quad \text{from Eq. (i)}$$
$$= 0.4 \times 0.012 = 0.0048$$

EXAMPLE 6.32 **Find the nature of the states of the Markov chain with**

$$\text{TPM} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \end{array}.$$

[AU December '03, June '05, April '08]

Solution Given:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix}$$

Therefore, $P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$, $P^3 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$, $P^4 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

We note that $P^2 = P^4 = \ldots = P^{2n}$ and $P = P^3 = P^5 = \ldots = P^{2n+1}$ for $n = 1, 2, 3, \ldots$

Also, $p_{00}^{(2)} > 0,\ p_{01}^{(1)} > 0,\ p_{02}^{(2)} > 0$

$p_{10}^{(1)} > 0,\ p_{11}^{(2)} > 0,\ p_{12}^{(1)} > 0$

$p_{20}^{(2)} > 0,\ p_{21}^{(1)} > 0,\ p_{22}^{(2)} > 0$

Therefore, the chain is irreducible. Again,

$$p_{ii}^{(2)} > 0 \quad p_{ii}^{(4)} > 0 \quad p_{ii}^{(6)} > 0, \ldots, i = 1, 2, 3$$

$\therefore$ Period = GCD (2, 4, 6, ...) = 2, for all 3 states.

Since the chain is finite and irreducible, all its states are non-null persistent. But period of each state is not 1. Therefore all states are not ergodic.

ord **EXAMPLE 6.33** A fair die is tossed repeatedly. If $X(n)$ denote the maximum of the numbers occurring in the first n tosses, find the probability matrix P of the Markov chain $\{X_n\}$, $P(X_2 = 6)$, and P^2. **[AU June '07]**

Solution When a fair die is tossed, the states are $\{1, 2, 3, 4, 5, 6\}$.

$\therefore$ Initial probability $P^{(0)} = \left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$

$P(X_0 = 1) = P(X_0 = 2) = P(X_0 = 3) = P(X_0 = 4) = P(X_0 = 5) = P(X_0 = 6) = \frac{1}{6}$

$$\text{TPM} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

When 2 is the maximum number in the previous throw, then in the next throw if the number 1 or 2 which ever occurs, 2 will be the maximum number.

$$\therefore \quad P(X \le 2) = P(X = 1) + P(X = 2)$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} \text{ and so on}$$

$$P^2 = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{36} \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

We know that

$$P^{(1)} = P^{(0)} \cdot P, \; P^{(2)} = P^{(1)} \cdot P, \; P^{(3)} = P^{(2)} \cdot P \; \ldots$$

$$\therefore \quad P^{(1)} = P^{(0)} \cdot P = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{(1)} = \left(\frac{1}{36}, \frac{3}{36}, \frac{5}{36}, \frac{7}{36}, \frac{9}{36}, \frac{11}{36}\right)$$

$$P^{(2)} = P^{(1)} \cdot P = \left(\frac{1}{216}, \frac{7}{216}, \frac{19}{216}, \frac{37}{216}, \frac{61}{216}, \frac{91}{216}\right)$$

$$P(X_2 = 6) = \frac{91}{216} = 0.4213$$

EXAMPLE 6.34 A man either drives a car or catches a train to go to office each day. He never goes 2 days in row by train; but if he drives one day, then the next day he is just as likely to drive again as he travel by train. On the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appear. Find

(i) the probability that he takes a train on the third day, and
(ii) the probability that he drives to work in the long run?

[AU December '04, April '07]

Solution Travel pattern denotes the state space = (T, C)
Here T-train and C-car
In throwing a die,

$$P(\text{getting number } 6) = \frac{1}{6}$$

$$P(\text{getting the numbers other than } 6) = \frac{5}{6}$$

$$P^{(1)} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix} \text{ (first day)}$$

$$\text{TPM} = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

$$P^{(2)} = P^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

(i) $$P^{(3)} = P^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$\therefore \quad P(\text{taking train on third day}) = P^{(3)} = \frac{11}{24}$$

(ii) In the long run, $\pi P = \pi$ and $\pi_1 + \pi_2 = 1$ (i)

$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

i.e. $$0 + \frac{1}{2}\pi_2 = \pi_1 \Rightarrow \pi_1 = \frac{1}{2}\pi_2$$

Substituting in Eq. (i), we get

$$\frac{1}{2}\pi_2 + \pi_2 = 1 \Rightarrow \frac{3}{2}\pi_2 = 1 \Rightarrow \pi_1 = \frac{1}{3} \text{ and } \pi_2 = \frac{2}{3}$$

$$\therefore \quad P(\text{taking car in the long run}) = \frac{2}{3} = 0.667$$

EXAMPLE 6.35 The territory of a salesman consists 3 cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in B. However, if he sells either in B or C, then the next day he is twice as likely to sell in city A as in the other city. How often does he sell in each of the cities in steady state?

Solution
$$\text{TPM} = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \end{array}$$

In the steady-state or in the long run, we know that

$$\pi P = \pi \quad \text{and} \quad \pi_1 + \pi_2 + \pi_3 = 1 \qquad \text{(i)}$$

$$(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3)$$

$$\frac{2}{3}\pi_2 + \frac{2}{3}\pi_3 = \pi_1 \qquad \text{(ii)}$$

$$\pi_1 + \frac{1}{3}\pi_3 = \pi_2 \qquad \text{(iii)}$$

$$\frac{1}{3}\pi_2 = \pi_3 \qquad \text{(iv)}$$

From Eq. (iii), $$\pi_1 = \pi_2 - \frac{\pi_3}{3} = \pi_2 - \frac{\pi_2}{9} = \frac{8\pi_2}{9}$$

Substituting Eq. (i), we get

$$\pi_1 = \frac{8\pi_2}{9} + \pi_2 + \frac{\pi_2}{3} = 1$$

$$\frac{20\pi_2}{9} = 1 \Rightarrow \pi_2 = \frac{9}{20}$$

Hence $$\pi_1 = \frac{8}{20}$$

∴ $$\pi_3 = \frac{3}{20}$$

$$\pi = \left(\frac{8}{20}, \frac{9}{20}, \frac{3}{20}\right)$$

i.e. 40%, 45%, 15% of probability that the salesman sells in city A, B and C respectively.

EXAMPLE 6.36 Write the transition probability matrix for the following problem. A travelling sales representative is at an integral point of the x axis between the origin and at the point $x = 3$. He can take a unit step either to left or right. If he goes to the right, the probability is 0.7 and if he goes to the left, the probability is 0.3.

Solution $$\text{TPM} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

If he is in state 0, then he can take a unit step to his right and that is the only possibility. Therefore, the probability is 1.

If he is in the state 1 or state 2, then either he can take a unit step to his right with probability 0.7 or to his left with probability 0.3.

If he is at the state 3, then he can take a unit step only to his left with probability 1.

EXAMPLE 6.37 On a given day, a retired English Professor, Dr. Charles Fish, amuses himself with only one of the following activities: reading (activity 1), gardening (activity 2), or working on his book about a river valley (activity 3). For $1 \le i \le 3$, let $X_n = i$ if Dr. Fish devotes day n to activity i. Suppose that $\{X_n : n = 1, 2, ...\}$ is a Markov chain, and depending on which of these activities on the next day is given by the TPM

$$P = \begin{pmatrix} 0.30 & 0.25 & 0.45 \\ 0.40 & 0.10 & 0.50 \\ 0.25 & 0.40 & 0.35 \end{pmatrix}$$

Find the proportion of days Dr. Fish devotes to each activity.

[AU November '07]

Solution To find the proportion of days Dr. Fish devotes to each activity, we have to the find steady-state probability.

In the steady-state, we know that

$$\pi P = \pi \text{ and } \pi_1 + \pi_2 + \pi_3 = 1 \quad \text{(i)}$$

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0.30 & 0.25 & 0.45 \\ 0.40 & 0.10 & 0.50 \\ 0.25 & 0.40 & 0.35 \end{pmatrix} = (\pi_1 \ \pi_2 \ \pi_3)$$

$$0.30\pi_1 + 0.4\pi_2 + 0.25\pi_3 = \pi_1 \quad \text{(ii)}$$

$$0.25\pi_1 + 0.10\pi_2 + 0.40\pi_3 = \pi_2 \quad \text{(iii)}$$

$$0.45\pi_1 + 0.50\pi_2 + 0.35\pi_3 = \pi_3 \quad \text{(iv)}$$

Solving Eqs. (i)–(iv) gives

$$\pi_1 = 0.306, \quad \pi_2 = 0.267, \quad \pi_3 = 0.427$$

$$\pi = (0.306, \quad 0.267, \quad 0.427)$$

i.e. Dr. Fish devotes 30.6% of the day for reading, 26.7% of the day for gardening and 42.7% of the day for working on his book.

EXAMPLE 6.~~38~~ 37 A raining process is considered as a two-state Markov chain. If it rains, it is considered to be in state 0 and if does not rain, the chain is in state 1. The transition probability of the Markov chain is defined as $P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$. Find the probability that it will rain for three days from today assuming that it is raining today. Find also the unconditional probability that it will rain after three days with the initial probabilities of the state 0 and 1 as 0.4 and 0.6 respectively.

Solution Given: The initial probabilities $P(0) = (0.4, 0.6)$

$$\text{and the TPM } P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

Given: state 0—raining

and state 1—not raining

To find the probability that it will rain for three days from today assuming that it is raining today:

$$P^2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{pmatrix}$$

$$P^3 = P^2P = \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{pmatrix}\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{pmatrix}$$

The probability that it will rain for three days from today assuming that it is raining today is the state

$$0 \to 0 \text{ in } P^3 = 0.376$$

To find the unconditional probability that it will rain after three days with the initial probabilities:

$$P^{(0)} = (0.4,\ 0.6)$$

$$P(3) = P^{(0)}P^3 = (0.4 \quad 0.6)\begin{pmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{pmatrix} = (0.3376 \quad 0.6624)$$

Therefore, the unconditional probability that it will rain after three days is given by

$$P(X_3 = 0) = 0.3376$$

EXAMPLE 6.39 Two boys B_1, B_2 and 2 girls G_1, G_2 are throwing a ball from one to another. Each boy throws the ball to the other boy with probability 1/2 and to each girl with probability 1/4. On the other hand, each girl throws the ball to each boy with probability 1/2 and never to the other girl. In the long run, how does each receive the ball?

Solution The TPM of the Markov chain is

$$\begin{array}{c} \\ B_1 \\ B_2 \\ G_1 \\ G_2 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & G_1 & G_2 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \end{array}$$

In the steady-state or in the long run, we know that

$$\pi P = \pi$$

and

$$\pi = \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \qquad \text{(i)}$$

$$(\pi_1, \pi_2, \pi_3, \pi_4)\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3\ \pi_4)$$

$$\pi_2 + \pi_3 + \pi_4 = 2\pi_1$$
$$\pi_1 + \pi_3 + \pi_4 = 2\pi_2$$
$$\pi_1 + \pi_2 = 4\pi_3$$

and
$$\pi_1 + \pi_2 = 4\pi_4$$

So we get,
$$\pi_3 = \pi_4$$

and
$$6\pi_4 = 1 \Rightarrow \pi_4 = \frac{1}{6} \Rightarrow \pi_3 = \frac{1}{6}$$

Also,
$$\pi_1 + \pi_2 = \frac{2}{3} \quad \text{(ii)}$$

and
$$2\pi_1 - \pi_2 = \frac{1}{3} \quad \text{(iii)}$$

Solving Eq. (ii) and Eq. (iii), we get $\pi_1 = \frac{2}{6}$, $\pi_2 = \frac{2}{6}$

∴ The steady-state probability distribution is $\pi = \left(\frac{2}{6}, \frac{2}{6}, \frac{1}{6}, \frac{1}{6}\right)$

Thus in the long run, each boy receives the ball $\frac{1}{3}$ of the time and each girl $\frac{1}{6}$ of the time.

EXAMPLE 6.40 A gambler has ₹ 2. He bets ₹ 1 at a time and wins ₹ 1 with probability 1/2. He stops playing if he loses ₹ 2 or wins ₹ 4.

(i) What is the TPM of the related Markov chain?
(ii) What is the probability that he has lost his money at the end of 5 plays?
(iii) What is the probability that the game lasts more than 7 plays.

[AU April '04, December '06]

Solution Let X_n denote the amount with the gambler at the end of nth round of the play.

The states of the Markov chain are

$X_n = \{0, 1, 2, 3, 4, 5, 6\}$

$p^{(0)} = \{0, 0, 1, 0, 0, 0, 0\}$ (initially he has only ₹ 2)

(i) The TPM of the Markov chain is

$$\text{TPM} = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 3 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 5 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

The states 0 and 6 are said to be absorbing barriers because the chain cannot come out of those 2 states. Because, if he losses ₹ 2 then he will have ₹ 0 and if he wins ₹ 4 then he will have ₹ 6 with him. In these two cases he stops playing.

$$P^{(1)} = P^{(0)} \cdot P = \left(0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0\right)$$

$$P^{(2)} = P^{(1)} \cdot P = \left(\frac{1}{4}, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, 0\right)$$

$$P^{(3)} = P^{(2)} \cdot P = \left(\frac{1}{4}, \frac{1}{4}, 0, \frac{3}{8}, 0, \frac{1}{8}, 0\right)$$

$\frac{1}{8}+\frac{3}{16}$

$$P^{(4)} = P^{(3)} \cdot P = \left(\frac{3}{8}, 0, \frac{5}{16}, 0, \frac{1}{4}, 0, \frac{1}{16}\right)$$

$$P^{(5)} = P^{(4)} \cdot P = \left(\frac{3}{8}, \frac{5}{32}, 0, \frac{9}{32}, 0, \frac{1}{8}, \frac{1}{16}\right)$$

(ii) After the 5th play the probability of having ₹ 0 $= \frac{3}{8}$ (i.e. he losts ₹ 2)

$$P^{(6)} = P^{(5)} \cdot P = \left(\frac{29}{64}, 0, \frac{14}{64}, 0, \frac{13}{64}, 0, \frac{1}{8}\right)$$

(iii) $P^{(7)} = P^{(6)} \cdot P = \left(\frac{29}{64}, 0, \frac{14}{128}, 0, \frac{27}{128}, 0, \frac{13}{128}, \frac{1}{8}\right)$

$\therefore$ The game lasts more than 7 plays is (he stops playing only when he has ₹ 0 or ₹ 6)

$$= P(X_7 = 1) + P(X_7 = 2) + P(X_7 = 3) + P(X_7 = 4) + P(X_7 = 5)$$

$$= \frac{7}{64} + 0 + \frac{27}{128} + 0 + \frac{13}{128} = \frac{54}{128} = 0.4218$$

6.3 POISSON RANDOM PROCESS

Let $X(t)$ represent the number of occurrences of a certain event in the interval $(0, t)$, then $\{X(t)\}$ is said to be a discrete Poisson random process if it satisfies the following postulates:

1. P[1 occurrence in the interval $(t, t + \Delta t)$] $= \lambda\Delta t + 0(\Delta t)$.
2. P[0 occurrence in the interval $(t, t + \Delta t)$] $= 1 - \lambda\Delta t + 0(\Delta t)$.
3. P[2 or more occurrences in $(t, t + \Delta t)$] $= 0(\Delta t)$.
4. $X(t)$ is independent of the number of occurrences of the event in the interval prior and after $(0, t)$.
5. Probability that the event occurs a specified number of times in the interval $(t_1, t_1 + \Delta t)$ depends only on Δt, but not on t_1.

6.3.1 Probability Law of Poisson Process

Let λ be the number of occurrences per unit time.

Let $P_n(t) = P[X(t) = n]$ = probability that there are n occurrences in $(0, t)$.

$$\therefore \qquad P_{n+1}(t) = P[X(t) = n + 1]$$

$P_n(t + \Delta t) = P[(n - 1)$ occurrence in $(0, t)$ and 1 occurrence in $(t + \Delta t)]$
$+ P[n$ occurrence in $(0, t)$ and 0 occurrence in $(t, t + \Delta t)]$

$$P_n(t + \Delta t) = P_{n-1}(t)(\lambda\Delta t) + P_n(t)(1 - \lambda\Delta t)$$

$$= P_{n-1}(t)(\lambda\Delta t) + P_n(t) - P_n(t)\lambda\Delta t$$

$$P_n(t + \Delta t) - P_n(t) = \lambda\Delta t[P_{n-1}(t) - P_n(t)]$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \lambda[P_{n-1}(t) - P_n(t)]$$

Taking the limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \lambda[P_{n-1}(t) - P_n(t)]$$

$$\frac{d}{dt}[P_n(t)] = \lambda[P_{n-1}(t) - P_n(t)]$$
$$P'_n(t) = \lambda[P_{n-1}(t) - P_n(t)] \tag{6.1}$$

Let the solution of Eq. (6.1) be

$$P_n(t) = \frac{(\lambda t)^n}{n!} f(t) \tag{6.2}$$

$$\frac{d}{dt}[P_n(t)] = \frac{\lambda^n t^{n-1}}{(n-1)!} f(t) + \frac{(\lambda t)^n}{n!} f'(t)$$

and $$P_{n-1}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} f(t)$$

Substituting in Eq. (6.1), we get

$$\frac{\lambda^n t^{n-1}}{(n-1)!} f(t) + \frac{(\lambda t)^n}{n!} f'(t) = \lambda\left[\frac{(\lambda t)^{n-1}}{(n-1)!} f(t) - \frac{(\lambda t)^n}{n!} f(t)\right]$$

Cancelling $\frac{\lambda^n t^{n-1}}{(n-1)!}$ throughout, we get

$$f(t) + \frac{t}{n} f'(t) = f(t) - \frac{\lambda t}{n} f(t)$$
$$\frac{t}{n} f'(t) = \frac{-\lambda t}{n} f(t) \Rightarrow f'(t) = -\lambda f(t)$$
$$\Rightarrow \frac{f'(t)}{f(t)} = -\lambda$$

Integrating with respect to t, we get

$$\log [f(t)] = -\lambda t + A$$
$$f(t) = e^{-\lambda t + A} = e^{-\lambda t} e^A$$
$$f(t) = ke^{-\lambda t}, \text{ where } k = e^A$$

Now, $f(0) = P$[number of occurrence in (0,0)] $= P_0(0) = 1$

$$f(t) = ke^{-\lambda t} \Rightarrow f(0) = k = 1 \Rightarrow k = 1$$
$$\Rightarrow f(t) = e^{-\lambda t},$$

Substituting in Eq. (6.2), we get

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

6.3.2 Mean of Poisson Process

Let $\{X(t)\}$ be a Poisson random process, then

$$E[X(t)] = \sum_{n=0}^{\infty} nP_n(t), \text{ where } P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0, 1, 2, \cdots$$

$$= \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}\lambda t}{(n-1)!} = \lambda t e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t}\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right] = \lambda t e^{-\lambda t} e^{\lambda t}$$

$$\therefore \quad E[X(t)] = \lambda t \qquad \left(e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$E[X^2(t)] = \sum_{n=1}^{\infty} n^2 P_n(t) = \sum_{n=1}^{\infty} n^2 \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=1}^{\infty} [n(n-1) + n] \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=1}^{\infty} \left[n(n-1) \frac{e^{-\lambda t}(\lambda t)^n}{n!} + n \frac{e^{-\lambda t}(\lambda t)^n}{n!} \right]$$

$$= e^{-\lambda t} \sum_{n=2}^{\infty} \left[\frac{(\lambda t)^2 (\lambda t)^{n-2}}{(n-2)!} \right] + \lambda t$$

$$= e^{-\lambda t} (\lambda t)^2 \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots \right] + \lambda t$$

$$= e^{-\lambda t} (\lambda t)^2 e^{\lambda t} + \lambda t$$

$$\therefore \quad E[X^2(t)] = (\lambda t)^2 + \lambda t$$

$$\mathrm{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2 = (\lambda t)^2 + \lambda t - (\lambda t)^2 \tag{6.3}$$

$$\therefore \quad \mathrm{Var}[X(t)] = \lambda t$$

6.3.3 Autocorrelation of Poisson Process

$$\begin{aligned}
R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E\{X(t_1)\,[X(t_2) - X(t_1) + X(t_1)]\} \\
&= E\{X(t_1)\,[X(t_2) - X(t_1)] + X^2(t_1)\} \\
&= E[X(t_1)]\,E[X(t_2) - X(t_1)] + E[X^2(t_1)] \\
&= E[X(t_1)]\,[EX(t_2) - EX(t_1)] + E[X^2(t_1)] \\
&= \lambda t_1(\lambda t_2 - \lambda t_1) + [(\lambda t_1)^2 + \lambda t_1] \\
&= \lambda^2 t_1 t_2 - \lambda^2 t_1^2 + \lambda^2 t_1^2 + \lambda t_1
\end{aligned}$$

from Eq. (6.3)

$$\therefore \quad R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda t_1, \qquad t_2 > t_1$$
$$= \lambda^2 t_1 t_2 + \lambda [\min(t_1, t_2)]$$

6.3.4 Autocovariance of Poisson Process

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)]\, E[X(t_2)]$$
$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda t_1 \lambda t_2, \qquad t_2 > t_1$$
$$= \lambda t_1, \qquad t_2 > t_1$$
$$\therefore \quad C_{XX}(t_1, t_2) = \lambda \min(t_1, t_2)$$

6.3.5 Correlation Coefficient of Poisson Process

$$C(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)\, C_{XX}(t_2, t_2)}}$$
$$= \frac{\lambda t_1}{\sqrt{\lambda t_1 \lambda t_2}} = \frac{t_1}{\sqrt{t_1 t_2}}, \qquad t_2 > t_1$$
$$\therefore \quad C(t_1, t_2) = \sqrt{\frac{t_1}{t_2}}, \qquad t_2 > t_1$$

6.3.6 Properties of Poisson Process

1. Poisson process is not a stationary process.

Let $\{X(t)\}$ be a Poisson random process, then

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad n = 0, 1, 2, \ldots$$
$$E[X(t)] = \sum_{n=0}^{\infty} n P_n(t)$$
$$= \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{(n-1)!}$$
$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1} \lambda t}{(n-1)!} = \lambda t e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$
$$= \lambda t e^{-\lambda t}\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right] = \lambda t e^{-\lambda t} e^{\lambda t}$$
$$\therefore \quad E[X(t)] = \lambda t \qquad \left(e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

Since $E[X(t)] = \lambda t$, which is a function of time, it is not a stationary process.

2. Sum of two independent Poisson processes is also a Poisson process.

Proof Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be two independent Poisson random processes with mean arrival rate λ_1 and λ_2 respectively.

Let $X(t) = X_1(t) + X_2(t)$

Since $\{X_1(t)\}$ and $\{X_2(t)\}$ are Poisson processes, we have

$$E[X_1(t)] = \lambda_1 t, \qquad E[X_2(t)] = \lambda_2 t$$

and

$$E[X_1^2(t)] = \lambda_1^2 t + \lambda_1 t, \qquad E[X_2^2(t)] = \lambda_2^2 t + \lambda_2 t$$

$$\begin{aligned}
E[X(t)] &= E[X_1(t) + X_2(t)] \\
&= E[X_1(t)] + E[X_2(t)] \\
&= \lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)t \\
E[X^2(t)] &= E[X_1(t) + X_2(t)]^2 \\
&= E[X_1^2(t) + X_2^2(t) + 2X_1(t)X_2(t)] \\
&= E[X_1^2(t)] + E[X_2^2(t)] + 2E[X_1(t)X_2(t)] \\
&= \lambda_1^2 t^2 + \lambda_1 t + \lambda_2^2 t^2 + \lambda_2 t + 2E[X_1(t)]\,E[X_2(t)] \\
&= \lambda_1^2 t^2 + \lambda_1 t + \lambda_2^2 t^2 + \lambda_2 t + 2\lambda_1 t \lambda_2 t \\
&= (\lambda_1 + \lambda_2)^2 t^2 + (\lambda_1 + \lambda_2)t
\end{aligned}$$

$\therefore$ $X_1(t) + X_2(t)$ is also a Poisson process with mean arrival rate $\lambda_1 + \lambda_2$.

3. Difference of two independent Poisson processes is not a Poisson process.

Proof Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be two independent Poisson random processes with mean arrival rate λ_1 and λ_2 respectively.

Let $X(t) = X_1(t) - X_2(t)$

We know that

$$E[X_1(t)] = \lambda_1 t \qquad E[X_2(t)] = \lambda_2 t$$

and

$$E[X_1^2(t)] = \lambda_1^2 t^2 + \lambda_1 t \qquad E[X_2^2(t)] = \lambda_2^2 t^2 + \lambda_2 t$$

$$\begin{aligned}
E[X(t)] &= E[X_1(t) - X_2(t)] \\
&= E[X_1(t)] - E[X_2(t)] = \lambda_1 t - \lambda_2 t = (\lambda_1 - \lambda_2)t \\
E[X^2(t)] &= E[X_1(t) - X_2(t)]^2 \\
&= E[X_1^2(t) + X_2^2(t) - 2X_1(t)\,X_2(t)] \\
&= E[X_1^2(t)] + E[X_2^2(t)] - 2E[X_1(t)]\,E[X_2(t)] \\
&= \lambda_1^2 t^2 + \lambda_1 t + \lambda_2^2 t^2 + \lambda_2 t - 2\lambda_1 t \lambda_2 t \\
&= (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 + \lambda_2)t \\
&\neq (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 - \lambda_2)t
\end{aligned}$$

Since the parameter is not $\lambda_1 - \lambda_2$, difference of two independent Poisson processes is not a Poisson process.

4. Poisson process is a Markov process (memoryless property).

Proof Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be two independent Poisson processes with same parameter λ. Then

$$P[X(t_1) = n_1] = \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!}$$

and $$P[X(t_2) = n_2] = \frac{e^{-\lambda t_2}(\lambda t_2)^{n_2}}{n_2!}, \qquad t_2 > t_1$$

The second-order probability function of a homogeneous Poisson process, is

$$P[X(t_1) = n_1, X(t_2) = n_2] = P[X(t_2) = n_2/X(t_1) = n_1] \cdot P[X(t_1) = n_1] \tag{6.4}$$

$$\left[\because P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(A)\, P(B/A)\right]$$

$$= P[X(t_1) = n_1]\, P[(n_2 - n_1) \text{ number of occurrences in } (t_2 - t_1)]$$

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!} \times \frac{e^{-\lambda(t_2 - t_1)}[\lambda(t_2 - t_1)]^{n_2 - n_1}}{(n_2 - n_1)!}$$

$$= \frac{e^{-\lambda t_1}\lambda^{n_1} t_1^{n_1}}{n_1!} \times \frac{e^{-\lambda t_2} e^{\lambda t_1}(\lambda)^{n_2 - n_1}(t_2 - t_1)^{n_2 - n_1}}{(n_2 - n_1)!}$$

$$= \frac{e^{-\lambda t_2}(\lambda)^{n_2} t_1^{n_1}(t_2 - t_1)^{n_2 - n_1}}{n_1!(n_2 - n_1)!}, \qquad n_2 \geq n_1$$

The third-order probability density function of a Poisson random process is

$$P[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3]$$

$$= P[X(t_1) = n_1, X(t_2) = n_2]\, P[X(t_3) = n_3/X(t_2) = n_2]$$

$$= \frac{e^{-\lambda t_2}(\lambda)^{n_2} t_1^{n_1}(t_2 - t_1)^{n_2 - n_1}}{n_1!(n_2 - n_1)!} \frac{e^{-\lambda(t_3 - t_2)}[\lambda(t_3 - t_2)]^{n_3 - n_2}}{(n_3 - n_2)!}$$

$$= \frac{t_1^{n_1}(t_2 - t_1)^{n_2 - n_1}}{n_1!(n_2 - n_1)!} \frac{e^{-\lambda t_3}(\lambda)^{n_3}(t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)!}$$

$$= \frac{t_1^{n_1}(t_2 - t_1)^{n_2 - n_1} e^{-\lambda t_3}(\lambda)^{n_3}(t_3 - t_2)^{n_3 - n_2}}{n_1!(n_2 - n_1)!(n_3 - n_2)!}$$

$$= \frac{e^{-\lambda t_3} t_1^{n_1}(t_2 - t_1)^{n_2 - n_1}(t_3 - t_2)^{n_3 - n_2}(\lambda)^{n_3}}{n_1!(n_2 - n_1)!(n_3 - n_2)!}, \qquad n_3 \geq n_2 \geq n_1 \tag{6.5}$$

Now to prove Poisson process is a Markov Process:

$$P[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1] = \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_1) = n_1, X(t_2) = n_2]}$$

By substituting Eqs. (6.4) and (6.5), we get

$$= \frac{\dfrac{e^{-\lambda t_3} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1} (t_3 - t_2)^{n_3 - n_2} (\lambda)^{n_3}}{n_1!(n_2 - n_1)!(n_3 - n_2)!}}{\dfrac{e^{-\lambda t_2} (\lambda)^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1!(n_2 - n_1)!}}$$

$$= \frac{\dfrac{e^{-\lambda t_3} (t_2)^{-n_1} (t_3 - t_2)^{n_3 - n_2} (\lambda)^{n_3}}{n_1!(n_2 - n_1)!(n_3 - n_2)!}}{\dfrac{e^{-\lambda t_2} (\lambda)^{n_2} (t_2)^{-n_1}}{n_1!(n_2 - n_1)!}}$$

$$= \frac{e^{-\lambda t_3} (t_3 - t_2)^{n_3 - n_2} \lambda^{n_3}}{(n_3 - n_2)! e^{-\lambda t_2} \lambda^{n_2}}$$

$$= \frac{e^{-\lambda (t_3 - t_2)} (\lambda)^{(n_3 - n_2)} (t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)!}$$

$$= \frac{e^{-\lambda (t_3 - t_2)} [\lambda (t_3 - t_2)]^{(n_3 - n_2)}}{(n_3 - n_2)!}$$

$= P[(n_3 - n_2)$ number of occurrences in $(t_3 - t_2)]$

$= P[X(t_3) = n_3 / X(t_2) = n_2]$

$\therefore$ The probability of the occurrence of the event $X(t_3) = n_3$ depends only on the previous value $X(t_2) = n_2$, but not on $X(t_1) = n_1$.

$\therefore$ Poisson process is a Markov process.

5. The interarrival (interval between two successive occurrences) time of a Poisson process with parameter λ follows an exponential distribution with mean $\frac{1}{\lambda}$.

Proof Let $X(t)$ denote the number of occurrences in a time interval t.

Let T be the interval between two successive occurrences E_i and E_{i+1}. T is a continuous random variable.

Let the event E_i occurs at the time instant t_i.

$$P(T > t) = P[E_{i+1} \text{ did not occur in } (t_i, t_i + t)]$$
$$= P[\text{no event occurs in } (t_i, t_i + t)]$$

$$= P[X(t) = 0] = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$\Rightarrow \quad P(T > t) = e^{-\lambda t}.$$

$\therefore$ Cumulative distribution function,

$$F(t) = P(T \le t) = 1 - P(T > t) = 1 - e^{-\lambda t}$$

Probability density function,

$$f(t) = F'(t) = 0 - (-\lambda)e^{-\lambda t} = \lambda e^{-\lambda t},$$

which is the PDF of exponential function.
Hence proved.

6. If the number of occurrences of an event E in an interval of length t is a Poisson process $\{X(t)\}$ with parameter λ and if each occurrences of E has a constant probability p of being recorded and the recordings are independent of each other, then the number $N(t)$ of the recorded occurrences in t is also a Poisson process with parameter λp.

Proof $P[N(t) = n] = \sum_{r=0}^{\infty} P$ (the event E occurs $(n + r)$ times in t and n of them being recorded)

$$= \sum_{r=0}^{\infty} P\,[\text{the event } E \text{ occurs } (n+r) \text{ times}] \times P[n \text{ of them being recorded out of } (n+r) \text{ occurrences}]$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!}(n+r)C_n p^n q^r, (q = 1-p)$$

$$= e^{-\lambda t} p^n (\lambda t)^n \sum_{r=0}^{\infty} \frac{(\lambda t)^r q^r}{(n+r)!} \frac{(n+r)!}{r!n!}$$

$$\left[\text{since } nC_r = \frac{n!}{(n-r)!r!}\right]$$

$$= \frac{e^{-\lambda t}(\lambda pt)^n}{n!} \sum_{r=0}^{\infty} \frac{(\lambda qt)^r}{r!}.$$

$$= \frac{(\lambda pt)^n}{n!} e^{-\lambda t}\left[1 + \frac{(\lambda qt)}{1!} + \frac{(\lambda qt)^2}{2!} + \cdots\right]$$

$$= \frac{(\lambda pt)^n}{n!} e^{-\lambda t} e^{\lambda qt} = \frac{(\lambda pt)^n}{n!} e^{-\lambda(1-q)t}$$

$$= \frac{(\lambda pt)^n}{n!} e^{-\lambda pt} \qquad (\because p + q = 1 \Rightarrow p = 1 - q)$$

which is the PDF of the Poisson process with parameter λp.

$\therefore$ $N(t)$ follows Poisson process with parameter λp.

EXAMPLE 6.41 Suppose that customers arrive at a bank according to a Poisson process with a mean rate 3 per minute. Find the probability that during a time interval of two minutes

(i) exactly 4 customers arrive,

(ii) more than 4 customers arrive.

$1 - e^{6}\left[1 + \frac{6}{1!} + \frac{6}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right]$

Solution Given: mean arrival rate $\lambda = 3$

and $t = 2$ minutes

We know that

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \frac{e^{-6}(6)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

(i) $P[X(2) = 4] = \dfrac{e^{-6}(6)^4}{4!} = 0.1338$

(ii) $P[X(2) > 4] = 1 - P[X(2) \le 4]$

$= 1 - \{P[X(2) = 0] + P[X(2) = 1] + P[X(2) = 2] + P[X(2) = 3] + P[X(2) = 4]\}$

$$= 1 - \left[\frac{e^{-6}(6)^0}{0!} - \frac{e^{-6}(6)^1}{1!} - \frac{e^{-6}(6)^2}{2!} - \frac{e^{-6}(6)^3}{3!} - \frac{e^{-6}(6)^4}{4!}\right]$$

$$= 1 - e^{-6}\left[1 + \frac{6}{1!} + \frac{(6)^2}{2!} + \frac{(6)^3}{3!} + \frac{(6)^4}{4!}\right] = 0.7298$$

EXAMPLE 6.42 If the customers arrive at a bank according to a Poisson process with mean rate 2 per minute, find the probability that during a 1-minute interval no customer arrives.

Solution Given: $t = 1$

and $\lambda = 2$

We know that

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$P[X(t) = n] = \frac{e^{-2}(2)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$P[X(1) = 0] = e^{-2(1)} = e^{-2} = 0.135$$

EXAMPLE 6.43 A radioactive source emits particles at a rate of 5 per minute in accordance with Poisson process. Each particle emitted has a probability of 0.6 of being recorded. Find the probability that 10 particles are recorded in a 4-minute period.

Solution We know that

$$P[X(t) = n] = \boxed{\frac{e^{-\lambda pt}(\lambda pt)^n}{n!}} \quad n = 0, 1, 2, \ldots$$

Given: $p = 0.6$, $n = 10$, $t = 4$

and $\lambda = 5$

$\lambda p = 3$

$\therefore$

$$P[X(t) = n] = \frac{e^{-3t}(3t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$P[X(4) = 10] = \frac{e^{-3(4)}(12)^{10}}{10!} = 0.104$$

EXAMPLE 6.44 The particles are emitted from a radioactive source at the rate of 20 per hour. Find the probability that exactly 5 particles are emitted during a 15-minute period.

Solution Given: $\lambda = 20$ per hour,

$$t = 15 \text{ minutes} = \frac{1}{4} \text{ hour}$$

and $n = 5$

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$P\left[X\left(\frac{1}{4}\right) = 5\right] = \frac{e^{-20\left(\frac{1}{4}\right)}\left[20\left(\frac{1}{4}\right)\right]^5}{5!} = \frac{e^{-5}(5)^5}{5!} = 0.175$$

EXAMPLE 6.45 A machine goes out of order whenever a component fails. The failure of this part follows a Poisson process with a mean rate 1 per week. Find the probability that 2 weeks have ellapsed since last failure. If there are 5 spare parts of this component in an inventory and that the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in next 10 weeks.

Solution Given: $t = 2$

$\lambda = 1 \Rightarrow \lambda t = 2$

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$P[X(2) = 0] = \frac{e^{-2}(2)^0}{0!} = 0.135$$

Since there are only 5 spare parts, within 10 weeks maximum 5 failures can occur such that the machine will not be out of order. For $t = 10$,

$$P[X(10) \le 5] = \begin{Bmatrix} P[X(10)=0] + P[X(10)=1] + P[X(10)=2] \\ + P[X(10)=3] + P[X(10)=4] + P[X(10)=5] \end{Bmatrix}$$

$$= \frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!} + \frac{e^{-10}(10)^4}{4!} + \frac{e^{-10}(10)^5}{5!}$$

$$= e^{-10}\left[1 + \frac{(10)}{1!} + \frac{(10)^2}{2!} + \frac{(10)^3}{3!} + \frac{(10)^4}{4!} + \frac{(10)^5}{5!}\right] = 0.067$$

EXAMPLE 6.46 Queries presented in a computer database are following a Poisson process of rate $\lambda = 6$ queries per minute. An experiment consists of monitoring the database for m minutes and recording $N(m)$ the number of queries presented.

(i) What is the probability that no queries arriving in a one-minute interval?
(ii) What is the probability that exactly 6 queries arriving in a one-minute interval?
(iii) What is the probability that less than 3 queries arriving in a half-minute interval? **[AU May '07]**

Solution Given: $N(m)$ = number of queries presented in m minutes

and $\lambda = 6$

$\therefore$ $\lambda t = 6t$

$$P[N(t) = x] = \frac{e^{-6t} \cdot (6t)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) $P[N(1) = 0] = e^{-6} = 0.00248$

(ii) $P[N(1) = 6] = \dfrac{e^{-6} \cdot 6^6}{6!} = 0.1607$

For $t = 1/2$, $x = 0, 1, 2$, we have

(iii) $P\left[N\left(\frac{1}{2}\right) < 3\right] = e^{-3}\left(1 + \frac{3}{1!} + \frac{3^2}{2!}\right) = 0.4231$

EXAMPLE 6.47 Customers arrive at the complaint department of a store at the rate of 5 per hour for male customers and 10 per hour for female customers. If arrivals in each case follow Poisson process, calculate the probabilities that

(i) at most 4 male customers,
(ii) at most 4 female customers will arrive in a 30-minute period.

Solution Given: mean arrival rate = $\lambda = 5$ per hour (male)
and $\lambda = 10$ per hour (female)

By Poisson process, probability that n customers arrive in a time interval of t hours is given by

$$P[X(t) = n] = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

(i) Mean arrival rate for male customers is $\lambda = 5$

$$P[X(t) = n] = \frac{e^{-5t} \cdot (5t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Since λ is given in hours $t = 30$ minutes $= 30 \times \frac{1}{60} = \frac{1}{2}$ hour

P[at most 4 male customers arrive in a 30-minute period]

$$= P\left[X\left(\frac{1}{2}\right) \leq 4\right]$$

$$= P\left[X\left(\frac{1}{2}\right) = 0\right] + P\left[X\left(\frac{1}{2}\right) = 1\right] + P\left[X\left(\frac{1}{2}\right) = 2\right] + P\left[X\left(\frac{1}{2}\right) = 3\right] + P\left[X\left(\frac{1}{2}\right) = 4\right]$$

$$= e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$$

$$= e^{-2.5}(3.5 + 3.125 + 2.6042 + 1.6279)$$

$$= 0.8912$$

(ii) Mean arrival rate for female customers is $\lambda = 10$

$\therefore$ P[at most 4 female customers arrive in a 30-minute period]

$$= P\left[X\left(\frac{1}{2}\right) \leq 4\right]$$

where $P[X(t) = n] = \dfrac{e^{-10t}(10t)^n}{n!}, \quad n = 0, 1, 2, \ldots$

$\therefore$ The required probability $= e^{-5}\left(1 + 5 + \dfrac{5^2}{2!} + \dfrac{5^3}{3!} + \dfrac{5^4}{4!}\right)$

$$= e^{-5}[1 + 5 + 12.5 + 20.8333 + 26.0417]$$

$$= 0.4405$$

EXAMPLE 6.48 Assume that the number of messages input to a communication channel in an interval of duration t seconds is a Poisson process with mean rate $\lambda = 0.3$. Compute

(i) the probability that exactly three messages will arrive during a 10-second interval,

(ii) the probability that the number of message arrivals in an interval of duration 5 seconds is between 3 and 7.

Solution Given: $\lambda = 0.3$

By Poisson process, probability of n messages arrive in t seconds

$$P[X(t) = n] = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

(i) P(exactly 3 messages arrive during a 10-second interval)

$$P[X(10) = 3] = \frac{e^{-(0.3)10}[(0.3)10]^3}{3!}$$

$$= \frac{e^{-3}3^3}{6}$$

$$= 0.2240$$

(ii) $$P[3 \le X(5) \le 7] = \sum_{n=3}^{7} \frac{e^{-1.5} \cdot (1.5)^n}{n!}$$

$$= e^{-1.5}\left(\frac{(1.5)^3}{3!} + \frac{(1.5)^4}{4!} + \frac{(1.5)^5}{5!} + \frac{(1.5)^6}{6!} + \frac{(1.5)^7}{7!}\right)$$

$$= 0.22313[0.5625 + 0.2109 + 0.0633 + 0.0158 + 0.0034]$$

$$= 0.1910$$

EXAMPLE 6.49 If customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between 2 consecutive arrivals is

(i) more than 1 minute

(ii) between 1 and 2 minutes, and

(iii) 4 minutes or less.

Solution The interval T between 2 consecutive arrivals follows an exponential distribution with parameter $\lambda = 2$ (given).

$$f(t) = \lambda e^{-\lambda t}, \quad \lambda > 0, t > 0$$
$$f(t) = 2e^{-2t}, \quad \lambda < 0, t > 0$$

(i) $$P(T > 1) = \int_1^{\infty} f(t)dt = \int_1^{\infty} 2e^{-2t}dt = e^{-2} = 0.135$$

(ii) $P(1 < T < 2) = \int_1^2 f(t)dt = \int_1^2 2e^{-2t} dt = e^{-2} - e^{-4} = 0.117$

(iii) $P(T \le 4) = \int_0^4 f(t)\, dt = 1 - e^{-8} = 0.999$

1 hr

EXAMPLE 6.50 A fisherman catches fish at a Poisson rate of 2 per hour from a large lake with lots of fish. If he starts fishing at 10:00 a.m., what is the probability that he catches one fish by 10:30 a.m. and three fishes by noon?

Solution Let $X(t)$ be the total number of fishes caught at or before time t

$$P[X(t) = n] = \frac{e^{-2t}(2t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Given: $\lambda = 2$

t = 10 a.m.–10.30 a.m. = 30 minutes = $\frac{1}{2}$ hour, $n = 1$ fish

t = 10 a.m.–12 noon = 2 hours, $n = 3$ fishes

$$P\left[X\left(\frac{1}{2}\right) = 1 \text{ and } X(2) = 3\right] = P\left[X\left(\frac{1}{2}\right) = 1\right] P\left[X\left(2 - \frac{1}{2}\right) = 3 - 1\right]$$

$$= P\left[X\left(\frac{1}{2}\right) = 1\right] P\left[X\left(\frac{3}{2}\right) = 2\right]$$

$$= \left\{\frac{1 \cdot e^{-1}}{1!}\right\}\left\{\frac{e^{-3} \cdot 3^2}{2!}\right\}$$

$$= 0.082$$

EXAMPLE 6.51 If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two independent Poisson processes with parameter λ_1 and λ_1 respectively, show that $P\{X_1(t) = k/[X_1(t) + X_2(t)] = n\} = nC_k p^k q^{n-k}$, where $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ and $q = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$.

[AU November '06]

Solution We know that

$$P(A \cap B) = P(A)P(B) \text{ (if } A \text{ and } B \text{ are independent)}$$

and $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P\{X_1(t) = k/[X_1(t) + X_2(t)] = n\} = \frac{P[X_1(t) = k \cap X_1(t) + X_2(t) = n]}{P[X_1(t) + X_2(t) = n]}$$

$$= \frac{P[X_1(t) = k \cap X_2(t) = n-k]}{P[X_1(t) + X_2(t) = n]}$$

$$= \frac{P[X_1(t) = k]\, P[X_2(t) = n-k]}{P[X_1(t) + X_2(t) = n]}$$

Since $\{X_1(t)\}$ and $\{X_2(t)\}$ are two independent Poisson processes, $\{X_1(t) + X_2(t)\}$ is also a Poisson process with parameter $\lambda_1 + \lambda_2$.

$$\Rightarrow \frac{\dfrac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \dfrac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)t}[(\lambda_1+\lambda_2)t]^n}{n!}}$$

$$\Rightarrow \frac{(\lambda_1)^k t^k (\lambda_2)^{n-k} t^{n-k} n!}{(\lambda_1+\lambda_2)^n t^n k!(n-k)!} = \frac{(\lambda_1)^k (\lambda_2)^{n-k} n!}{(\lambda_1+\lambda_2)^n k!(n-k)!}$$

$$= \frac{(\lambda_1)^k (\lambda_2)^{n-k}}{(\lambda_1+\lambda_2)^k (\lambda_1+\lambda_2)^{n-k}} \frac{n!}{k!(n-k)!}$$

$$= nC_k \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k} = nC_k p^k q^{n-k}$$

where $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$ and $q = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$

Hence proved.

EXAMPLE 6.52 If $\{X(t)\}$ is a Poisson process prove that

$$P[X(s) = r/X(t) = n] = nC_r \left(\frac{s}{t}\right)^r \left(1 - \frac{s}{t}\right)^{n-r} \quad \text{given } s < t$$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Solution $P[X(s) = r/X(t) = n] = \dfrac{P[X(s) = r \cap X(t) = n]}{P[X(t) = n]}$

$$= \frac{P[X(s) = r \cap X(t-s) = n-r]}{P[X(t) = n]}$$

Since $X(s)$ and $X(t-s)$ are independent

$$P[X(s) = r/X(t) = n] = \frac{P[X(s) = r]\, P[X(t-s) = n-r]}{P[X(t) = n]}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^r}{r!} \dfrac{e^{-\lambda(t-s)}[\lambda(t-s)]^{n-r}}{(n-r)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{e^{-\lambda s}(\lambda)^r s^r e^{-\lambda t} e^{\lambda s}(\lambda)^{n-r}(t-s)^{n-r} n!}{e^{-\lambda t}(\lambda)^n t^n r!(n-r)!}$$

$$= \frac{s^r (t-s)^{n-r} n!}{t^n r!(n-r)!} = nC_r \left(\frac{s}{t}\right)^r \left(\frac{t-s}{t}\right)^{n-r}$$

$$= nC_r \left(\frac{s}{t}\right)^r \left(1-\frac{s}{t}\right)^{n-r}$$

Hence proved.

EXAMPLE 6.53 The number of accidents in Coimbatore follows a Poisson process with a mean of 2 per day and the number X_i of the people involved in the ith accident has the distribution (independent) $P(X_i = k) = \frac{1}{2^k}, (k \geq 1)$. Find the mean and variance of the number of people involved in accidents per week.

Solution Given: a distribution $P(X_i = k) = \frac{1}{2^k}, \quad k = 1, 2, 3, \ldots$

The mean and variance are 2 and 2.

$$\text{Mean} = \sum_{x=1}^{\infty} x \frac{1}{2^x} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots\right] = 2$$

Easily, we can show that $E(X^2) = 6$.

Let us assume that the number of accidents on any day be n.

Let $X_1, X_2, \ldots, X_n$ be the number of people involved in the accident. Since $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with mean 2 and variance 2, $X_1 + X_2 + \cdots + X_n$ follows a normal distribution with mean $2n$ and variance $2n$ (by Central Limit Theorem). Hence the number of people involved with n accidents on a day is $2n$.

Let n denote the number of people involved in accidents on any day. Then

$$P[X(t) = n] = \frac{e^{-2t}(2t)^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} \frac{2xe^{-2t}(2t)^x}{x!} = 2E\{X(t)\} = 4t \qquad (\because \lambda = 2)$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \sum_{x=0}^{\infty} \frac{4x^2 e^{-2t}(2t)^x}{x!} - 16t^2$$

$= 4E[X^2(t)] - 16t^2 = 4(2t + 4t^2) - 16t^2 \qquad [\because E[X^2(t)] = \lambda t + \lambda^2 t^2]$

$= 8t$

Hence for the number of people involved in accidents per week with the given distribution,

$$\text{Mean} = 4t = 4 \times 7 = 28$$
$$\text{Var} = 8t = 8 \times 7 = 56$$

EXAMPLE 6.54 If $\{X(t)\}$ is a Poisson process, then prove that correlation coefficient between $X(t)$ and $X(t + s)$ is $\sqrt{\dfrac{t}{t+s}}$.

Solution $\{X(t)\}$ follows Poisson process.

$\therefore$
$$E[X(t)] = \lambda t$$
$$E[X(t + s)] = \lambda(t + s)$$

The autocorrelation function is

$$\begin{aligned}
R_{XX}(t, t + s) &= E[X(t)\, X(t + s)] \\
&= E\{X(t)\, [X(t + s) - X(t) + X(t)]\} \\
&= E\{X(t)\, [X(t + s) - X(t)] + E[X^2(t)]\} \\
&= E[X(t)]\, E[X(t + s) - X(t)] + E[X^2(t)] \\
&= \lambda t[\lambda(t + s) - \lambda t] + \lambda^2 t^2 + \lambda t \\
&= \lambda t(\lambda t + \lambda s - \lambda t) + \lambda^2 t^2 + \lambda t \\
&= \lambda^2 t^2 + \lambda^2 ts + \lambda t
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(t, t + s) &= R_{XX}(t, t + s) - E[X(t)]\, E[X(t + s)] \\
&= \lambda^2 ts + \lambda^2 t^2 + \lambda t - (\lambda t)(\lambda t + \lambda s) \\
&= \lambda^2 ts + \lambda^2 t^2 + \lambda t - \lambda^2 t^2 - \lambda^2 ts \\
&= \lambda t
\end{aligned}$$

Correlation coefficient between $X(t)$ and $X(t + s)$ is

$$r = \frac{\text{Cov}(t, t+s)}{\sqrt{\text{Var}[X(t)]}\,\sqrt{\text{Var}[X(t+s)]}} \qquad \text{[For Poisson process mean = variance]}$$

$$= \frac{\lambda t}{\sqrt{\lambda t}\,\sqrt{\lambda(t+s)}} = \sqrt{\frac{t}{t+s}}$$

EXAMPLE 6.55 If T_n is the random variable denoting the time of occurrence of the nth event in a Poisson process with parameter λ, show that the distribution

function $F_n(t) = \begin{cases} 1 - \displaystyle\sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}, & t \geq 0 \\ 0, & t < 0 \end{cases}$.

Find also the probability density function $f_n(t)$ of T_n.

Solution The probability distribution function,

$$F_n(t) = P(T_n \le t) = 1 - P(T_n > t) \quad \text{(i)}$$

Here T_n is a random variable which denotes the time of nth occurrence.

$$P(T_n > t) = P[(n-1) \text{ or less number of occurrences in } (0, t)]$$

$$= \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Substituting in Eq. (i), we get

$$F_n(t) = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

The probability density function,

$$f_n(t) = F_n'(t) = \frac{d}{dt}F_n(t) = \frac{d}{dt}\left[1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\right]$$

$$= 0 - \sum_{k=0}^{n-1} \frac{1}{k!}\frac{d}{dt}[e^{-\lambda t}(\lambda t)^k]$$

$$= -\sum_{k=0}^{n-1} \frac{1}{k!}[e^{-\lambda t}\lambda^k k t^{k-1} + e^{-\lambda t}\lambda^k t^k(-\lambda)]$$

$$= e^{-\lambda t}\lambda \sum_{k=0}^{n-1} \frac{(\lambda^k t^k)}{k!} - e^{-\lambda t}\sum_{k=0}^{n-1} \frac{(\lambda^k t^{k-1} k)}{k!}$$

$$= e^{-\lambda t}\lambda \left[\sum_{k=0}^{n-1} \frac{(\lambda^k t^k)}{k!} - \sum_{k=1}^{n-1} \frac{(\lambda^{k-1} t^{k-1})}{(k-1)!}\right]$$

$$= e^{-\lambda t}\lambda \left\{\left[1 + \frac{(\lambda t)}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-2}}{(n-2)!} + \frac{(\lambda t)^{n-1}}{(n-1)!}\right] - \left[1 + \frac{(\lambda t)}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-2}}{(n-2)!}\right]\right\}$$

$$= e^{-\lambda t}\lambda \frac{(\lambda t)^{n-1}}{(n-1)!} = \frac{e^{-\lambda t}\lambda^n t^{n-1}}{(n-1)!}, \qquad t \ge 0$$

6.3.7 Applications of Poisson Process

1. Number of customers arriving at a queueing system.
2. Number of telephone calls arriving at a switchboard.

3. Number of cars arriving at a service station.
4. Number of requests arriving at a server in a centre and many more.

6.4 BERNOULLI RANDOM PROCESS

Let X_n be a random variable which denotes the outcome of the nth trial such that $P(X_n = 1) = p$, if the trial is a success and $P(X_n = 0) = q$, if the trial is a failure. That is, X_n takes the value 1 if the outcome is a success and it takes the value 0 if the outcome is a failure. Then $\{X_n\}$, $n = 1, 2, 3, \ldots$ is a *Bernoulli Random Process*.

6.4.1 Properties of Bernoulli Random Process

1. It is a discrete random process.
2. It is a SSS process.

6.5 BINOMIAL RANDOM PROCESS

The discrete sequence of partial sum $\{S_n, n = 1, 2, 3, \ldots\}$ where $S_n = X_1 + X_2 + \cdots + X_n$ and $X_1, X_2, \ldots, X_n$ are Bernoulli random variables is called a *Binomial Random Process*.

1. The time is assumed to be divided into unit intervals. Therefore, it is a discrete time process.
2. At most one arrival can occur in any interval.
3. Arrivals can occur randomly and independently in each interval with probability p.

6.5.1 Properties of Binomial Random Process[††]

1. It is a Markov Process.

Proof Let

$$S_n = X_1 + X_2 + \cdots + X_n$$
$$S_n = S_{n-1} + X_n$$
$$P(S_n = m/S_{n-1} = m) = P(X_n = 0) = q = 1 - p$$
$$P(S_n = m/S_{n-1} = m - 1) = P(X_n = 1) = p$$

$\therefore$ S_n depends on S_{n-1} only.

Hence binomial process is a Markov process.

[††] AU December '09

2. If S_n is a binomial random variable, then

$$P(S_n = m) = nC_m p^m q^{n-m},\ E(S_n) = np \text{ and } \operatorname{Var}(S_n) = npq$$

3. The distribution of the number of slots m_i between ith and $(i-1)$th arrival is geometric with parameter p and starts from 0. The random variable m_i, $i = 1, 2, 3, \ldots$ are mutually independent. The geometric distribution is given by pq^{i-1}, $i = 1, 2, \ldots$.

If $Y_n = \sum_{i=1}^{n} X_i$ which is the total number of pulses from the time instant 1 to n, then the random process $\{Y_n, n \geq 1\}$ is a binomial random process but not a strict sense stationary process because $P(Y_n = r) = nC_r p^r q^{n-r}$ where $r = 1, 2, \ldots, n$ depends on n.

6.6 SINE WAVE RANDOM PROCESS

The sine wave random process is represented by $X(t) = A \sin(\omega t + \phi)$ where A is the amplitude, ω is the frequency and ϕ is the phase.

It is also represented by $X(t) = A \cos(\omega t + \phi)$.

EXAMPLE 6.56 Verify whether the sine wave process $X(t)$ where $X(t) = Y \cos \omega t$ and Y is uniformly distributed in (0, 1) is a SSS process and also find its variance.

Solution Given: $f(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$

and $\qquad X(t) = Y \cos \omega t$

$\therefore \qquad E[X(t)] = E(Y \cos \omega t) = \cos \omega t\ E(Y)$

$$E(Y) = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

$$E[X(t)] = \frac{1}{2}\cos \omega t = \text{a function of time.}$$

Therefore, it is not a SSS process.

To find the variance:

$$E[X^2(t)] = E(Y^2 \cos^2 \omega t) = \frac{1}{2} E[Y^2 (1 + \cos 2\omega t)]$$

$$= \left(\frac{1 + \cos 2\omega t}{2}\right) E(Y^2)$$

$$= \frac{1}{2}(\cos 2\omega t + 1)\, E(Y^2)$$

$$E(Y^2) = \int_0^1 y^2 dy = \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E[X^2(t)] = \frac{1}{2}\left(\frac{1}{3}\right)(\cos 2\omega t + 1) = \frac{1}{6}(\cos 2\omega t + 1)$$

$$\text{Var}[X(t)] = E[(X^2(t)] - \{E[X(t)]\}^2$$

$$= \frac{1}{6}(\cos 2\omega t + 1) - \frac{1}{4}\cos^2 \omega t$$

$$= \frac{1}{6}(\cos 2\omega t) + \frac{1}{6} - \frac{1}{4}(\cos^2 \omega t)$$

$$= \frac{1}{6}(\cos 2\omega t) + \frac{1}{6} - \frac{1}{8}(\cos 2\omega t + 1)$$

$$= \frac{1}{6}(\cos 2\omega t) + \frac{1}{6} - \frac{1}{8} - \frac{1}{8}(\cos 2\omega t)$$

$$= \frac{1}{24}(\cos 2\omega t) + \frac{1}{24} = \frac{1}{24}(1 + \cos 2\omega t)$$

Variance is also a function of time.

Therefore, it is not a SSS process.

EXAMPLE 6.57 For the sine wave process $X(t) = Y \cos 10t$, $-\infty < t < \infty$, the amplitude Y is a random variable with uniform distribution in the interval (0, 1). Determine whether the process is stationary or not.

Solution It is similar to the previous problem with $\omega = 10$

$$\therefore \quad E[X(t)] = E(Y)\cos 10t = \frac{1}{2}\cos 10t, \text{ a function of time}$$

$$E[X^2(t)] = \frac{1}{6}(\cos 20t + 1)$$

and $$\text{Var } X(t) = \frac{1}{24}(\cos 20t + 1)$$

Variance is also a function of time.

$\therefore$ It is not a stationary process.

EXAMPLE 6.58 Consider the process $X(t) = 10 \sin(200t + \phi)$ where ϕ is uniformly distibuted in the interval $(-\pi, \pi)$. Check whether the process is stationary or not.

Solution Given: $$f(\phi) = \begin{cases} \dfrac{1}{2\pi}, & -\pi < \phi < \pi \\ 0, & \text{otherwise} \end{cases}$$

$$X(t) = 10\sin(200t + \phi)$$

and $$E[X(t)] = E[10\sin(200t + \phi)] = 10E[\sin(200t + \phi)] \quad \text{(i)}$$

$$E[\sin(200t + \phi)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(200t + \phi)d\phi$$

$$= \frac{1}{2\pi}[-\cos(200t + \phi)]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[-\cos(200t + \pi) + \cos(200t - \pi)]$$

$$= \frac{1}{2\pi}(\cos 200t - \cos 200t) = 0$$

$$[\because \cos(200t + \pi) = -\cos 200t, \text{ and } \cos(200t - \pi) = \cos 200t]$$

Substituting in Eq. (i), we get

$$E[X(t)] = 0, \text{ a constant}$$

$$E[X^2(t)] = E[100\sin^2(200t + \phi)]$$

$$= \frac{100}{2}E[1 - \cos 2(200t + \phi)]$$

$$= 50\{E(1) - E[\cos(400t + 2\phi)]\} \quad \text{(ii)}$$

$$E[\cos(400t + 2\phi)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(400t + 2\phi)d\phi$$

$$= \frac{1}{2\pi}\left[\frac{\sin(400t + 2\phi)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}[\sin(400t + 2\pi) - \sin(400t - 2\pi)]$$

$$= \frac{1}{4\pi}[\sin 400t - \sin 400t] = 0$$

Substituting in Eq. (ii), we get

$$E[X^2(t)] = 50E(1) = 50$$

$$\text{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2 = 50 - 0 = 50$$

Variance is not a function of time, i.e. it is a constant.
Therefore, it is a stationary process.

EXAMPLE 6.59 Verify whether the sine wave random process $X(t) = Y\sin \omega t$, where Y is uniformly distributed in the interval -1 to 1 is WSS or not.

[AU April '06]

Solution Given: $X(t) = Y \sin \omega t$

$$\text{Mean} = E[X(t)] = E(Y \sin \omega t) = \int_{-\infty}^{\infty} y \sin \omega t\, f(y) dy$$

and

$$f(y) = \begin{cases} \dfrac{1}{2}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Mean} = \int_{-1}^{1} y \sin \omega t \frac{1}{2} dy = \frac{\sin \omega t}{2} \int_{-1}^{1} y\, dy = 0 \; [\because y \text{ is an odd function}]$$

By definition of autocorrelation,

$$\begin{aligned} R_{XX}(t, t + \tau) &= E[X(t)\, X(t + \tau)] \\ &= E[y^2 \sin \omega t \sin \omega(t + \tau)] \\ &= \int_{-\infty}^{\infty} y^2 \sin \omega t \sin \omega(t + \tau)\, f(y) dy \\ &= \frac{1}{2} \cdot \frac{1}{2} [\cos \omega \tau - \cos(2\omega t + \tau)] \int_{-1}^{1} y^2 dy \end{aligned}$$

$$[\because \cos(A - B) - \cos(A + B) = 2 \sin A \sin B]$$

$$\begin{aligned} &= \frac{1}{4}[\cos \omega \tau - \cos(2\omega t + \tau)] \left[\frac{y^3}{3}\right]_{-1}^{1} \\ &= \frac{1}{4}[\cos \omega \tau - \cos(2\omega t + \tau)] \left[\frac{1}{3} - \left(\frac{-1}{3}\right)\right] \\ &= \frac{1}{6}[\cos \omega \tau - \cos(2\omega t + \tau)] \end{aligned}$$

which depends on τ as well as t. Hence $X(t)$ is not a WSS process.

EXAMPLE 6.60 Show that the random process $X(t) = Ae^{j\omega t}$, is WSS if and only if $E(A) = 0$.

Solution Given: $X(t) = Ae^{j\omega t}$. Let $E(A) = 0$

$\therefore$

$$E[X(t)] = E(Ae^{j\omega t}) = e^{j\omega t} E(A) = 0$$

Since $X(t)$ is a complex variable,

$$\begin{aligned} R(t_1, t_2) &= E[X(t_1)\, X^*(t_2)] \\ &= E(Ae^{j\omega t_1} \bar{A} e^{-j\omega t_2}) = E(Ae^{j\omega t_1})\, E(\bar{A} e^{-j\omega t_2}) \\ &= E(A) e^{j\omega t_1} E(\bar{A}) e^{-j\omega t_2} = E(A\bar{A}) e^{j\omega t_1} e^{-j\omega t_2} \\ &= e^{j\omega(t_1 - t_2)} E(A\bar{A}) = \text{a function of time difference.} \end{aligned}$$

$\therefore$ $X(t)$ is a WSS process.

When $X(t)$ is a WSS process, $E[X(t)]$ is a constant.

$\therefore$ $E(Ae^{j\omega t}) = e^{j\omega t} E(A)$ = a constant, only when $E(A) = 0$.

Hence proved.

EXAMPLE 6.61 Verify whether the random process $X(t) = A\cos(\omega_0 t + \phi)$ is a WSS process where ϕ is a uniformly distributed random variable in $(0, \pi)$.

[AU April '07, December '05]

Solution Given: $f(\phi) = \begin{cases} \dfrac{1}{\pi}, & 0 < \phi < \pi \\ 0, & \text{otherwise} \end{cases}$

and $X(t) = A\cos(\omega_0 t + \phi)$

$$E[X(t)] = E[A\cos(\omega_0 t + \phi)] = AE[\cos(\omega_0 t + \phi)] \tag{i}$$

$$E[\cos(\omega_0 t + \phi)] = \frac{1}{\pi}\int_0^{\pi} \cos(\omega_0 t + \phi)d\phi = \frac{1}{\pi}[\sin(\omega_0 t + \phi)]_0^{\pi}$$

$$= \frac{1}{\pi}[\sin(\omega_0 t + \pi) - \sin(\omega_0 t + 0)]$$

$$= \frac{1}{\pi}(-\sin\omega_0 t - \sin\omega_0 t) = \frac{-2\sin\omega_0 t}{\pi}$$

Substituting in Eq. (i) we get

$$E[X(t)] = \frac{-2A\sin\omega_0 t}{\pi}$$

$\therefore$ $E[X(t)]$ is not a constant.

$\therefore$ It is not a WSS process.

6.7 ERGODIC RANDOM PROCESS

It is used to estimate various statistical quantities of a random process.

A random process $X(t)$ is said to be ergodic random process if its ensemble averages are equal to its time averages.

Time Averages

(i) Mean time average: It is denoted by $\overline{X}_T$ and defined as

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

(ii) Autocorrelation time average: It is denoted by $\langle R_{XX}(T)\rangle$ and is defined as

$$\langle R_{XX}(T)\rangle = \frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+\tau)dt$$

6.7.1 Mean Ergodic Random Process

A random process $\{X(t)\}$ is said to be mean ergodic if

$$\overline{X}_T = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)dt = \mu$$

6.7.2 Correlation Ergodic Random Process

A random processs $\{X(t)\}$ is said to be correlation ergodic if

$$\langle R_{XX}(\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+\tau)dt = R_{XX}(\tau)$$

EXAMPLE 6.62 Consider a random process $X(t) = \cos(\omega t + \theta)$ where ω is a constant and θ is a random variable with a probability density,

$$p(\theta) = \begin{cases} \dfrac{1}{2}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Prove that $X(t)$ is an ergodic process.

Solution Given:

$$X(t) = \cos(\omega t + \theta)$$

$$\begin{aligned}
E[X(t)] &= E[\cos(\omega t + \theta)] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t + \theta)d\theta \\
&= \frac{1}{2\pi}[\sin(\omega t + \theta)]_0^{2\pi} \\
&= \frac{1}{2\pi}[\sin(\omega t + 2\pi) - \sin \omega t] \\
&= 0, \text{ a constant} \qquad [\text{since } \sin(2\pi + x) = \sin x]
\end{aligned}$$

$$\begin{aligned}
R_{XX}(\tau) &= E[X(t)X(t+\tau)] \\
&= E\{\cos(\omega t + \theta)\cos[\omega(t+\tau) + \theta]\}
\end{aligned}$$

$$= \frac{1}{2} E[\cos(2\omega t + \omega\tau + 2\theta) + \cos(\omega\tau)]$$

$$= \frac{1}{2}\{E[\cos(2\omega t + \omega\tau + 2\theta)] + E[\cos(\omega\tau)]\}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{2\pi}\cos[2\omega t + \omega\tau + 2\theta]\, d\theta + \frac{\cos(\omega\tau)}{2}$$

$$= \frac{1}{4\pi}\left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2}\right]_0^{2\pi} + \frac{\cos(\omega\tau)}{2}$$

$$= \frac{1}{4\pi}\left[\frac{\sin(2\omega t + \omega\tau + 4\pi) - \sin(2\omega t + \omega\tau)}{2}\right] + \frac{\cos(\omega\tau)}{2}$$

$$= 0 + \frac{\cos(\omega\tau)}{2} = \frac{1}{2}\cos(\omega\tau)$$

Time average:

$$\bar{X}_T = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos(\omega t + \theta)dt = \lim_{T\to\infty} \frac{1}{2T}\left[\frac{\sin(\omega t + \theta)}{\omega}\right]_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{\sin(\omega T + \theta) - \sin(-\omega T + \theta)}{\omega}\right]$$

$$= 0 \qquad \left[\text{since } \lim_{T\to\infty}\frac{1}{2T} = \frac{1}{\infty} = 0\right]$$

$$\therefore \quad \bar{X}_T = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt = 0 = E[X(t)]$$

$$\langle R_{XX}(T)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos(\omega t + \theta)\cos[\omega(t+\tau) + \theta]dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[\frac{\cos[\omega t + \theta + \omega(t+\tau) + \theta] + \cos(\omega t + \theta - \omega t - \omega\tau - \theta)}{2}\right]dt$$

$$= \lim_{T \to \infty} \frac{1}{4T} \int_{-T}^{T} [\cos(2\omega t + 2\theta + \omega\tau) + \cos\omega\tau]dt$$

$$= \lim_{T \to \infty} \frac{1}{4T} \left\{ \left[\frac{\sin(2\omega t + 2\theta + \omega\tau)}{2\omega} \right]_{-T}^{T} + \cos\omega\tau[t]_{-T}^{T} \right\}$$

$$= \lim_{T \to \infty} \frac{1}{4T} \left\{ \left[\frac{\sin(2\omega T + 2\theta + \omega\tau)}{2\omega} - \frac{\sin(-2\omega T + 2\theta + \omega\tau)}{2\omega} \right] + [\cos\omega\tau[T - (-T)]] \right\}$$

$$= 0 + \lim_{T \to \infty} \frac{1}{4T} 2T \cos\omega\tau = \lim_{T \to \infty} \frac{\cos\omega\tau}{2} = \frac{\cos\omega\tau}{2}$$

$$\langle R_{XX}(T) \rangle = \underset{T \to \infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} X(t)\, X(t+\tau)dt = \frac{\cos\omega\tau}{2} = R_{XX}(\tau)$$

$\therefore$ The random process is an ergodic random proccess.

EXAMPLE 6.63 The WSS process $X(t)$ is given by $X(t) = 10\cos(100t + \theta)$ where θ is uniformly distributed over $(-\pi, \pi)$. Prove that $\{X(t)\}$ is correlation ergodic.

Solution
$$R_{XX}(t_1, t_2) = E[X(t)X(t + \tau)]$$
$$= E\{10\cos(100t + \theta)\, 10\cos[100(t + \tau) + \theta]\}$$
$$= \frac{100}{2} E\{\cos[100(2t + \tau) + 2\theta] + \cos(100\tau)\}$$
$$= 50 \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos[100(2t + \tau) + 2\theta]d\theta + 50E(\cos 100\tau)$$
$$= 50E[\cos 100(\tau)] \qquad \text{(i)}$$
$$R_{XX}(\tau) = 50\cos 100(\tau)$$
$$\langle R_{XX}(T) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, X(t+\tau)dt$$
$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} 10\cos(100t + \theta)\, 10\cos[100(t + \tau) + \theta]dt$$
$$= \lim_{T \to \infty} \frac{100}{4T} \int_{-T}^{T} [\cos(200t + 2\theta + 100\tau) + \cos(-100\tau)]dt$$

$$= \lim_{T\to\infty} \frac{100}{4T}\left\{\left[\frac{\sin(200t+2\theta+100\tau)}{200}\right]_{-T}^{T} + \cos 100\tau[t]_{-T}^{T}\right\}$$

$$= \lim_{T\to\infty} \frac{25}{T}\left\{\left[\frac{\sin(200T+2\theta+100\tau)-\sin(-200T+2\theta+100\tau)}{200}\right]+\cos 100\tau[2T]\right\}$$

$$= 0 + 50\cos 100\tau = 50\cos 100\tau \qquad \text{(ii)}$$

From Eqs. (i) and (ii), it is proved that

$$\langle R_{XX}(T)\rangle = R_{XX}(\tau) = 50\cos 100\tau$$

$\therefore$ $\{X(t)\}$ is correlation ergodic.

EXAMPLE 6.64 For the random process $X(t) = A\cos\omega t + B\sin\omega t$, where A and B are random variables with $E(A) = E(B) = 0$, $E(A^2) = E(B^2) > 0$, and $E(AB) = 0$. Prove that the process is a mean ergodic.

Solution Given:

$$X(t) = A\cos\omega t + B\sin\omega t$$

$$E(A) = E(B) = 0 = E(AB)$$

$$E(A^2) = E(B^2) > 0$$

$$E[X(t)] = E(A\cos\omega t + B\sin\omega t) = \cos\omega t E(A) + \sin\omega t E(B)$$

$$E[X(t)] = 0, \text{ a constant} \qquad \text{(i)}$$

$$\bar{X}_T = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(A\cos\omega t + B\sin\omega t)dt$$

$$= \lim_{T\to\infty}\frac{A}{2T}\int_{-T}^{T}(\cos\omega t)dt + \lim_{T\to\infty}\frac{B}{2T}\int_{-T}^{T}(\sin\omega t)dt$$

$$= \lim_{T\to\infty}\frac{A}{2T}\left[\frac{\sin\omega t}{\omega}\right]_{-T}^{T} + 0$$

[Since the second integrand is an odd function of t]

$$= \lim_{T\to\infty}\frac{A}{2T}\left(\frac{\sin\omega T + \sin\omega T}{\omega}\right)$$

$$= \lim_{T\to\infty}\frac{A}{2T}\left[\frac{2\sin\omega T}{\omega}\right] = 0 \qquad \text{(ii)}$$

From Eqs. (i) and (ii), it follows that $\bar{X}_T = E[X(t)] = 0$. Therefore, $\{X(t)\}$ is a mean ergodic process.

6.7.3 Distribution Ergodic Random Process

The random process $\{X(t)\}$ is said to be distribution erogodic random process if there is another random process $\{Y(t)\}$ such that $Y(t)=\begin{cases}1, & X(t)\le x\\ 0, & X(t)>x\end{cases}$ and also the process $\{Y(t)\}$ is mean ergodic random process.

Mean Ergodic Theorem

If $\{X(t)\}$ is a random process with constant mean μ and if $\bar{X}_T=\dfrac{1}{2T}\displaystyle\int_{-T}^{T}X(t)dt$, then $\{X(t)\}$ is said to be mean ergodic if $\lim\limits_{T\to\infty}\operatorname{Var}(\bar{X}_T)=0$.

Proof Given: $\lim\limits_{T\to\infty}\operatorname{Var}(\bar{X}_T)=0$

$$E(\bar{X}_T) = E\left[\frac{1}{2T}\int_{-T}^{T}X(t)dt\right]$$

$$= \frac{1}{2T}\int_{-T}^{T}E[X(t)]dt=\frac{1}{2T}\int_{-T}^{T}\mu dt \qquad [\text{since } E[X(t)]=\mu]$$

$$= \frac{1}{2T}\mu[t]_{-T}^{T}=\frac{1}{2T}\mu[T-(-T)]=\mu$$

By Tchebycheff's inequality,

$$P\left(\left|\bar{X}_T-E(\bar{X}_T)\right|\le\varepsilon\right)\ge 1-\frac{\operatorname{Var}(\bar{X}_T)}{\varepsilon^2}$$

Taking limit as $T\to\infty$ on both sides, we get

$$P\left\{\lim_{T\to\infty}|\bar{X}_T-\mu|\le\varepsilon\right\}\ge 1-\lim_{T\to\infty}\frac{\operatorname{Var}(\bar{X}_T)}{\varepsilon^2}$$

$$P\left\{\lim_{T\to\infty}|\bar{X}_T-\mu|\le\varepsilon\right\}\ge 1 \qquad \left[\text{since }\lim_{T\to\infty}\frac{\operatorname{Var}(\bar{X}_T)}{\varepsilon^2}=0\right]$$

$$\therefore \quad P\left\{\lim_{T\to\infty}|\bar{X}_T-\mu|\le\varepsilon\right\}=1 \qquad [\because \text{Probability} \not> 1]$$

Since ε is very small value nearly equal to zero.

$$\lim_{T\to\infty}(\bar{X}_T-\mu)=0 \Rightarrow \lim_{T\to\infty}\bar{X}_T=\mu$$

$\therefore$ It is a mean ergodic.

Note:

(i) The sufficient condition for the random process $\{X(t)\}$ to be mean ergodic is $\lim_{T\to\infty} \text{Var}(\bar{X}_T) = 0$.

(ii) If $\bar{X}_T$ is the time average of a stationary random process $\{X(t)\}$, then

$$\text{Var}(\bar{X}_T) = \frac{1}{T}\int_0^{2T}\left[1 - \frac{|\tau|}{2T}\right] C_{XX}(\tau)\,d\tau$$

(iii) When $E[X(t)] = 0 =$ mean, then $C_{XX}(\tau) = R_{XX}(\tau)$.

EXAMPLE 6.65 Express the autocorrelation function of the process $\{X'(t)\}$ in terms of the autocorrelation function of the process $\{X(t)\}$.

Solution We know that

$$R_{XX'}(t_1, t_2) = E\{X(t_1) \times X'(t_2)\}$$

$$= E\left\{X(t_1)\lim_{h\to 0}\left[\frac{X(t_2+h) - X(t_2)}{h}\right]\right\}$$

$$= \lim_{h\to 0}\left[\frac{R_{XX}(t_1, t_2+h) - R_{XX}(t_1, t_2)}{h}\right]$$

$$= \frac{\partial}{\partial t_2} R_{XX}(t_1, t_2) \qquad \text{(i)}$$

and
$$R_{X'X}(t_1, t_2) = \frac{\partial}{\partial t_1} R_{XX}(t_1, t_2) \qquad \text{(ii)}$$

Using Eqs. (i) and (ii),

$$R_{X'X'}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_{XX}(t_1, t_2) \qquad \text{(iii)}$$

If $\{X(t)\}$ is a stationary process, then $t_1 - t_2 = \tau$. From Eqs. (i), (ii) and (iii), we get

$$R_{XX'}(\tau) = -\frac{\partial}{\partial \tau} R_{XX}(\tau)$$

$$R_{X'X}(\tau) = \frac{\partial}{\partial \tau} R_{XX}(\tau)$$

and
$$R_{X'X'}(\tau) = -\frac{\partial^2}{\partial \tau^2} R_{XX}(\tau)$$

EXAMPLE 6.66 Prove that the random process $\{X(t)\}$ with constant mean is mean ergodic, if $\lim_{T\to\infty}\left[\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)\,dt_1 dt_2\right] = 0$. **[AU December '09]**

Solution By mean ergodic theorem, the condition for the mean ergodicity of the process $\{X(t)\}$ is

$$\lim_{T \to \infty} \{\text{Var}(\bar{X}_T)\} = 0$$

where
$$\bar{X}_T = \frac{1}{2T} \int_{-T}^{T} X(t)dt$$

and
$$E(\bar{X}_T) = E\{X(t)\} = \frac{1}{2T} \int_{-T}^{T} E[X(t)]dt$$

Then
$$\bar{X}_T^2 = \frac{1}{4T^2} \int_{-T}^{T}\int_{-T}^{T} X(t_1)\, X(t_2)dt_1 dt_2$$

$\therefore$
$$E\{\bar{X}_T^2\} = \frac{1}{4T^2} \int_{-T}^{T}\int_{-T}^{T} E[X(t_1) \times X(t_2)]dt_1 dt_2$$

$$= \frac{1}{4T^2} \int_{-T}^{T}\int_{-T}^{T} R(t_1, t_2)dt_1 dt_2$$

$\therefore$
$$\text{Var}(\bar{X}_T) = E\{\bar{X}_T^2\} - [E(\bar{X}_T)]^2$$

$$= \frac{1}{4T^2} \int_{-T}^{T}\int_{-T}^{T} \{R(t_1, t_2) - E[X(t_1)]\, E[X(t_2)]\}dt_1 dt_2$$

$$= \frac{1}{4T^2} \int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)dt_1 dt_2$$

Therefore, the condition $\lim_{T \to \infty} [\text{Var}(\bar{X}_T)] = 0$ is equivalent to the condition

$$\lim_{T \to \infty} \left[\frac{1}{4T^2} \int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)dt_1 dt_2\right] = 0$$

Hence the result.

EXAMPLE 6.67 A random binary transmission process $\{X(t)\}$ is a WSS process with zero mean and autocorrelation function $R(\tau) = 1 - \frac{|\tau|}{T}$, where T is a constant. Find the variance of the time average of $\{X(t)\}$ and also the mean over $(0, T)$. Is $\{X(t)\}$ mean ergodic?

Solution Given: $R_{XX}(\tau) = 1 - \frac{|\tau|}{T}$

$$E(\bar{X}_T) = E\left[\frac{1}{2T}\int_{-T}^{T} X(t)dt\right] = \frac{1}{T}\int_{0}^{T} E[X(t)]dt$$

$$= \frac{1}{T}\int_{0}^{T} 0dt = 0 \qquad \text{[since zero mean]}$$

In the interval (0, T),

$$\text{Var}(\bar{X}_T) = \frac{1}{T}\int_{-T}^{T}\left[1-\frac{|\tau|}{T}\right]C_{XX}(\tau)d\tau \qquad \text{(i)}$$

Since $E[X(t)] = 0$, we have

$$C_{XX}(\tau) = R_{XX}(\tau)$$

$$\therefore \qquad \text{Var}(\bar{X}_T) = \frac{2}{T}\int_{0}^{T}\left[1-\frac{|\tau|}{T}\right]R_{XX}(\tau)d\tau$$

because the integrand is an even function.

$$= \frac{2}{T}\int_{0}^{T}\left(1-\frac{\tau}{T}\right)\left(1-\frac{\tau}{T}\right)d\tau \qquad \text{[since } |\tau| = \tau \text{ in } (0, T)]$$

$$= \frac{2}{T}\int_{0}^{T}\left(1-\frac{\tau}{T}\right)^2 d\tau = \frac{2}{T}\int_{0}^{T}\left[1+\left(\frac{\tau}{T}\right)^2-\frac{2\tau}{T}\right]d\tau$$

$$= \frac{2}{T}\left[\tau+\frac{\tau^3}{3T^2}-\frac{2\tau^2}{2T}\right]_0^T = \frac{2}{T}\left(T+\frac{T}{3}-T\right) = \frac{2}{3}$$

$$\lim_{T\to\infty} \text{Var}(\bar{X}_T) = \frac{2}{3} \neq 0$$

$\therefore$ It is not mean ergodic.

EXAMPLE 6.68 If $\{X(t)\}$ is a WSS process with mean μ and autocovariance function $C_{XX}(\tau) = \begin{cases} \sigma_X^2\left(1-\dfrac{|\tau|}{\tau_0}\right), & \text{for } 0 \le |\tau| \le \tau_0 \\ 0, & \text{for } |\tau| \ge \tau_0 \end{cases}$, find the variance of $\{X(t)\}$ over (0, T). Also examine if the process $\{X(t)\}$ is a mean ergodic.

Solution $E(\bar{X}_T) = E\left[\frac{1}{2T}\int_{-T}^{T} X(t)dt\right]$

$$= \frac{1}{2T}\int_{-T}^{T} E[X(t)]dt = \frac{1}{2T}\int_{-T}^{T}\mu dt \qquad \{\text{since } E[X(t)] = \mu\}$$

$$= \frac{1}{2T}\mu[t]_{-T}^{T} = \frac{1}{2T}\mu[T-(-T)] = \mu$$

$$\text{Var}(\bar{X}_T) = \frac{1}{T}\int_{-T}^{T}\left[1-\frac{|\tau|}{T}\right]C_{XX}(\tau)d\tau$$

$$= \frac{2}{T}\int_{0}^{\tau_0}\left[\sigma_X^2\left(1-\frac{|\tau|}{T}\right)\left(1-\frac{|\tau|}{\tau_0}\right)\right]d\tau \qquad \text{[even function of } \tau]$$

$$= \frac{2\sigma_X^2}{T}\int_{0}^{\tau_0}\left[\left(1-\frac{\tau}{T}\right)\left(1-\frac{\tau}{\tau_0}\right)\right]d\tau \qquad [\text{since } |\tau| = \tau \text{ in } (0, \tau_0)]$$

$$= \frac{2\sigma_X^2}{T}\int_{0}^{\tau_0}\left[1-\frac{\tau}{\tau_0}-\frac{\tau}{T}+\frac{\tau^2}{\tau_0 T}\right]d\tau$$

$$= \frac{2\sigma_X^2}{T}\left[\tau-\frac{\tau^2}{2\tau_0}-\frac{\tau^2}{2T}+\frac{\tau^3}{3\tau_0 T}\right]_0^{\tau_0}$$

$$= \frac{2\sigma_X^2}{T}\left[\tau_0-\frac{\tau_0^2}{2\tau_0}-\frac{\tau_0^2}{2T}+\frac{\tau_0^3}{3\tau_0 T}\right]$$

$$= \frac{2\sigma_X^2}{T}\left(\tau_0-\frac{\tau_0}{2}-\frac{\tau_0^2}{2T}+\frac{\tau_0^2}{3T}\right) = \frac{2\sigma_X^2}{T}\left(\frac{\tau_0}{2}-\frac{\tau_0^2}{6T}\right)$$

$$\lim_{T\to\infty}\text{Var}(\bar{X}_T) = \lim_{T\to\infty}\frac{2\sigma_X^2}{T}\left(\frac{\tau_0}{2}-\frac{\tau_0^2}{6T}\right) = 0$$

∴ It is a mean ergodic.

EXAMPLE 6.69 If the autocorrelation function of a stationary process $\{X(t)\}$ is given by $C(\tau) = Ae^{-\alpha|\tau|}$, prove that $\{X(t)\}$ is mean ergodic. Also find $\text{Var}(\bar{X}_T)$ where $\bar{X}_T$ is the time average of $\{X(t)\}$ over $(-T, T)$.

Solution Given: $C(\tau) = Ae^{-\alpha|\tau|}$

To prove $X(t)$ is mean ergodic:

We have to show that

$$\lim_{T\to\infty}\text{Var}(\bar{X}_T) = 0$$

$$C(\tau) = Ae^{-\alpha|\tau|}$$

$$\text{Var}(\bar{X}_T) = \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)Ae^{-\alpha|\tau|}d\tau$$

$$= \frac{2}{T}\int_0^T \left(Ae^{-\alpha|\tau|} - \frac{\tau}{T} Ae^{-\alpha|\tau|} \right) d\tau \qquad [\because |\tau| \text{ is an even function}]$$

$$= \frac{2}{T}\int_0^T \left(Ae^{-\alpha\tau} - \frac{\tau}{T} Ae^{-\alpha\tau} \right) d\tau = \frac{2A}{T}\int_0^T \left(1 - \frac{\tau}{T}\right) e^{-\alpha\tau}\, d\tau$$

$$= \frac{2A}{T}\left[\left(1 - \frac{\tau}{T}\right)\frac{e^{-\alpha\tau}}{-\alpha} - \left(\frac{-1}{T}\right)\frac{e^{-\alpha\tau}}{\alpha^2} \right]_0^T$$

$$= \frac{2A}{T}\left[\frac{1}{\alpha} + \left(\frac{1}{T}\right)\frac{e^{-\alpha T}}{\alpha^2} - \frac{1}{T\alpha^2} \right]$$

$\frac{e^{-\alpha T}}{-\alpha} + \frac{Te^{-\alpha T}}{\alpha}$

$$\lim_{T \to \infty} \operatorname{Var}(\overline{X}_T) = 0$$

$\therefore$ The random process $\{X(t)\}$ is mean ergodic.

EXAMPLE 6.70 If the random process $X(t) = \sin(\omega t + Y)$ where Y is a random variable uniformly distributed in $(0, 2\pi)$, prove that for the process $\{X(t)\}$, $C(t_1, t_2) = R(t_1, t_2) = \dfrac{\cos \omega(t_1 - t_2)}{2}$.

Solution Given:

$$f(x) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le x < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

and

$$X(t) = \sin(\omega t + Y)$$

$$\begin{aligned} E[X(t)] &= E[\sin(\omega t + Y)] \\ &= E(\sin \omega t \cos Y + \cos \omega t \sin Y) \\ &= \sin \omega t\, E(\cos Y) + \cos \omega t\, E(\sin Y) \qquad \text{(i)} \end{aligned}$$

$$E(\cos Y) = \frac{1}{2\pi}\int_0^{2\pi} \cos y\, dy$$

$$= \frac{1}{2\pi}[\sin y]_0^{2\pi} = \frac{1}{2\pi}(\sin 2\pi - \sin 0) = 0$$

$$E(\sin Y) = \frac{1}{2\pi}\int_0^{2\pi} \sin y\, dy = \frac{1}{2\pi}[-\cos y]_0^{2\pi}$$

$$= \frac{1}{2\pi}(-\cos 2\pi + \cos 0) = \frac{1}{2\pi}(-1 + 1) = 0$$

Substituting in Eq. (i)

$$E[X(t)] = 0$$

$$\begin{aligned} R(t_1, t_2) &= E[X(t_1)\, X(t_2)] \\ &= E[\sin(\omega t_1 + Y) \sin(\omega t_2 + Y)] \end{aligned}$$

$$= \frac{1}{2} E\{\cos[\omega(t_1 - t_2)] - \cos[\omega(t_1 + t_2) + 2Y]\}$$

$$= \frac{1}{2}\{E\cos[\omega(t_1 - t_2)]\} - E\cos[\omega(t_1 + t_2) + 2Y]\} \qquad \text{(ii)}$$

Now, $$E\{\cos[\omega(t_1 + t_2) + 2Y]\} = \frac{1}{2\pi}\int_0^{2\pi} \cos[\omega(t_1 + t_2) + 2y]dY$$

$$= \frac{1}{2\pi}\{\sin[\omega(t_1 + t_2) + 2y]\}_0^{2\pi} = 0$$

Substituting in Eq. (ii)

$$R(t_1, t_2) = \frac{1}{2} E[\cos \omega(t_1 - t_2)] = \frac{\cos \omega(t_1 - t_2)}{2}$$

$$C(t_1, t_2) = R(t_1, t_2) - [E(X)]^2$$

$$C(t_1, t_2) = \frac{\cos \omega(t_1 - t_2)}{2} - 0 = R(t_1, t_2)$$

Hence proved.

6.8 NORMAL OR GAUSSIAN PROCESS

Gaussian process play an important role in the theory and analysis of random phenomena, because they are good approximations to the observations and multivariate Gaussian distributions are analytically simple. One of the most important use of the Gaussian process is to model and analyze the effects of thermal noise in electronic circuits used in communication systems. A number of processes such as Wiener process and the shot-noise process can be approximated, as per central limit theorm, by a Gaussian process.

If the random variables $X(t_1), X(t_2), ..., X(t_n)$ are jointly normal for any set of $t_1, t_2, ..., t_n$, $n = 1, 2, 3, ...$, then the real-valued random process $\{X(t)\}$ is called a Gaussian process or normal process.

Gaussian process is specified completely by the mean and covariances, i.e. by the first- and second-order moments.

The first-order density of a Gaussian process is given by

$$\Lambda = (\lambda_{11}) = [\text{cov}\{X(t_1), X(t_2)\}]$$
$$= [\text{Var }\{X(t_1)\}] = (\sigma_1^2)$$

$$|\Lambda| = \sigma_1^2$$

and $$|\Lambda|_{11} = 1$$

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$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{21}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\therefore \quad |\Lambda| = \sigma_1^2\sigma_2^2(1-\rho^2), \text{ where } \rho_{12} = \rho_{21} = \rho$$

$$|\Lambda|_{11} = \sigma_1^2, \quad |\Lambda|_{12} = -\rho\sigma_1\sigma_2, \quad |\Lambda|_{21} = -\rho\sigma_1\sigma_2, \quad |\Lambda|_{22} = \sigma_2^2$$

$$\therefore \quad f(x_1, x_2) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\left[\frac{(x-\mu_1)^2}{2\sigma_1^2}\right]}$$

The second-order density of a Gaussian process which we often use it is given by

$$\therefore \quad f(x_1, x_2; t_1, t_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right] - \left[\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right] + \left[\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}}$$

where ρ is the correlation coefficient between the variables X_1 and X_2.

The nth-order Density of Normal or Gaussian Process

Let $\lambda_{ij} = \text{Cov}[X(t_i), X(t_j)]$, $\Lambda = (\lambda_{ij})$ = a square matrix of order n and $|\Lambda|_{ij}$ = cofactor of λ_{ij} in $|\Lambda|$. Then the nth-order density of the Gaussian process is given by

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{1}{(\sqrt{2\pi})^n (\sqrt{|\Lambda|})} e^{-\left[\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda|_{ij}(x_i-\mu_i)(x_j-\mu_j)\right]}$$

where

$$\mu_i = E[X(t_i)]$$

6.8.1 Properties of Normal or Gaussian Process

1. If a Gaussian process $\{X(t)\}$ is wide sense stationary, then it is strictly stationary.

Proof The nth-order density of a Gaussian process $X(t)$ is given by

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{1}{(\sqrt{2\pi})^n (\sqrt{|\Lambda|})} e^{-\left[\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda|_{ij}(x_i-\mu_i)(x_j-\mu_j)\right]}$$

where

$$\mu_i = E[X(t_i)]$$

Λ is the nth-order square matrix (λ_{ij}), $i, j = 1, 2, 3, \ldots, n$

where

$$\lambda_{ij} = C\{X(t_i), X(t_j)\}, \; |\Lambda|_{ij} = \text{cofactor of } \lambda_{ij} \text{ in } |\Lambda|.$$

If the Gaussian process is WSS, then $\lambda_{ij} = \text{Cov}[X(t_i), X(t_j)]$ is a function of $(t_i - t_j)\ \forall\ i, j$.

Therefore, the nth-order densities of $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ and $\{X(t_1 + h), X(t_2 + h), \ldots, X(t_n + h)\}$ are identical.

Therefore, the Gaussian process is a SSS process.

2. If the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ of a Gaussian process $\{X(t)\}$, are uncorrelated, then they are independent.

Proof Let $X(t_1), X(t_2), \ldots, X(t_n)$ are n member functions of the Gaussian process $\{X(t)\}$.

Since the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are uncorrelated,

$$\lambda_{ij} = \text{Cov}[X(t_i), X(t_j)] = \begin{cases} 0, & i \neq j \\ \sigma_i^2, & i = j \end{cases}.$$

$\therefore$ $[\Lambda]$ is the diagonal matrix with the elements in the principal diagonal equal to σ_i^2.

i.e.
$$\Lambda = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

$\therefore$
$$|\Lambda| = \sigma_1^2 \sigma_2^2 \ldots \sigma_n^2$$

and
$$|\Lambda|_{ij} = \begin{cases} \sigma_1^2 \sigma_2^2 \cdots \sigma_{i-1}^2 \sigma_{i+1}^2 \cdots \sigma_n^2, & i = j \\ 0, & i \neq j \end{cases}$$

Therefore, the nth-order density of the Gaussian process becomes

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{1}{\left(\sqrt{2\pi}\right)^n \sigma_1 \sigma_2 \ldots \sigma_n} e^{-\left[\frac{1}{2}\sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}\right]}$$

$$= \frac{1}{\left(\sqrt{2\pi}\right)\sigma_1} e^{-\left[\frac{(x_1 - \mu_1)^2}{2\sigma_1^2}\right]} \cdot \frac{1}{\left(\sqrt{2\pi}\right)\sigma_2} e^{-\left[\frac{(x_2 - \mu_2)^2}{2\sigma_2^2}\right]}$$

$$\cdots \frac{1}{\left(\sqrt{2\pi}\right)\sigma_n} e^{-\left[\frac{(x_n - \mu_n)^2}{2\sigma_n^2}\right]}$$

$$= f(x_1, t_1) \cdot f(x_2, t_2) \ldots f(x_n, t_n)$$

$\therefore \quad X(t_1), X(t_2), \ldots, X(t_n)$ are independent.

3. If the input to a linear system is a Gaussian process, the output is also a Gaussian process.

Proof Let $\{X(t)\}$ be an input process and $\{Y(t)\}$ be its corresponding output process.

Let $X(t_1), X(t_2), \ldots, X(t_n)$ be n sample functions of the process $\{X(t)\}$ and $Y(t_1), Y(t_2), \ldots, Y(t_n)$ be the corresponding output sample functions of $\{Y(t)\}$

$$Y(t) = \int_{-\infty}^{\infty} X(u)\, h(t-u)\,du = \sum_{j=1}^{n} [h(t-u_j)\,\Delta u]\, X(u_j) \quad \text{as } n \to \infty$$

$$\therefore \quad Y(t_i) = \sum_{j=1}^{n} [h(t_i - u_j)\Delta u]\, X(u_j), \quad \text{as } n \to \infty$$

$$y_i = Y(t_i) = \sum_{j=1}^{n} h_{ij} x_j, \qquad i = 1, 2, \ldots, n \tag{6.6}$$

where $h_{ij} = h(t_i - u_j)$ and $X(u_j) = x_j$.

Writing it in the matrix form, we get

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_{n1} & h_{n2} & h_{n3} & \cdots & h_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{6.7}$$

i.e. $\quad \bar{Y} = H\bar{X}$ (say)

where $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$

$$\therefore \quad E(\bar{Y}) = E(H\bar{X}) = HE(\bar{X})$$

$$\text{i.e.} \quad \mu_{\bar{Y}} = H\mu_{\bar{X}} \tag{6.8}$$

Now, $\quad f_y(y_1, y_2, \ldots, y_n) = |J(x_1, x_2, \ldots, x_n)|\, f_X(x_1, x_2, \ldots, x_n)$

where $\quad |J(x_1, x_2, \ldots, x_n)| = |J(y_1, y_2, \ldots, y_n)|^{-1}$

Now,
$$|J(y_1, y_2, \ldots, y_n)| = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \cdots & \dfrac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial y_n}{\partial x_1} & \dfrac{\partial y_n}{\partial x_2} & \cdots & \dfrac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_{n1} & h_{n2} & h_{n3} & \cdots & h_{nn} \end{vmatrix} = |H|$$

Since $\frac{\partial y_i}{\partial x_i} = h_{ij}$, ($i = 1, 2, ..., n$ and $j = 1, 2, ..., n$), from (6.6).

Therefore, $f_Y(y_1, y_2, ..., y_n) = \frac{1}{|H|} f_X(x_1, x_2, ..., x_n)$

$$= \frac{1}{|H|\left(\sqrt{2\pi}\right)^n \sqrt{|\Lambda_x|}} e^{-\left[\left(\frac{1}{2|\Lambda_x|}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda_x|_{ij}(x_i - \mu_i)(x_j - \mu_j)\right]} \qquad (6.9)$$

where

$$\Lambda_x = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{pmatrix}$$

and $\qquad C_{ij} = \text{Cov}[X(t_i), X(t_j)]$

The matrix form of the RHS of Eq. (6.9) is

$$f_Y(y_1, y_2, ..., y_n) = \frac{1}{|H|\left(\sqrt{2\pi}\right)^n \sqrt{|\Lambda_x|}} e^{-\left[\frac{1}{2}(\bar{X} - \mu_{\bar{X}})^T \Lambda_x^{-1}(\bar{X} - \mu_{\bar{X}})\right]}$$

$$= \frac{1}{|H|\left(\sqrt{2\pi}\right)^n \sqrt{|\Lambda_x|}} e^{-\left[\frac{1}{2}(H^{-1}\bar{Y} - H^{-1}\mu_{\bar{Y}})^T \Lambda_x^{-1}\left(H^{-1}\bar{Y} - H^{-1}\mu_{\bar{Y}}\right)\right]}$$

$$= \frac{1}{|H|\left(\sqrt{2\pi}\right)^n \sqrt{|\Lambda_x|}} e^{-\left\{\frac{1}{2}\left(H^T\right)^{-1}(\bar{Y} - \mu_{\bar{Y}})^T \Lambda_x^{-1}\left[H^{-1}(\bar{Y} - \mu_{\bar{Y}})\right]\right\}}$$

$$= \frac{1}{|H|\left(\sqrt{2\pi}\right)^n \sqrt{|\Lambda_x|}} e^{-\left\{(\bar{Y} - \mu_{\bar{Y}})^T [(H^T)^{-1}\Lambda_x^{-1}H^{-1}](\bar{Y} - \mu_{\bar{Y}})\right\}} \qquad (6.10)$$

Let C_{ij} be the covariance matrix of the output process $Y(t)$, i.e.

$$C'_{ij} = \text{Cov}[Y(t_i), Y(t_j)] = E\left(\{Y_i - E[Y(t_i)]\}\{Y_j - E[Y(t_j)]\}\right)$$

$$= E\left\{\sum_{r=1}^{n} h_{ir}[X_r - E(X_r)] \sum_{s=1}^{n} h_{js}[X_s - E(X_s)]\right\}, \qquad \text{by (6.6)}$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} h_{ir}h_{js} E\{[X_r - E(X_r)][X_s - E(X_s)]\}$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} h_{ir}h_{js} \text{Cov}[X(t_r), X(t_s)]$$

$\therefore \quad \Lambda_y = H\Lambda_x H^T \Rightarrow \Lambda_y^{-1} = (H\Lambda_x H^T)^{-1} = (H^T)^{-1}\Lambda_x^{-1} H^{-1}$ (6.11)

and $\quad |\Lambda_y| = |H|^2 |\Lambda_x|$ (6.12)

Substituting Eqs. (6.11) and (6.12) in Eq. (6.10) gives

$$f(y_1, y_2, \ldots, y_n) = \frac{1}{(\sqrt{2\pi})^n \sqrt{|\Lambda_y|}} e^{-\left[\frac{1}{2}(\bar{Y}-\mu_{\bar{Y}})^T \Lambda_y^{-1}(\bar{Y}-\mu_{\bar{Y}})\right]}$$

which is the nth-order density function of a Gaussian process $\{Y(t)\}$. Hence the output process $\{Y(t)\}$ is a Gaussian process.

EXAMPLE 6.71 Let Z and θ be independent random variables such that Z has a density function $f(z) = \begin{cases} 0, & z < 0 \\ ze^{\frac{-z^2}{2}}, & z > 0 \end{cases}$ and θ is uniformly distributed in $(0, 2\pi)$. Show that $(X_t; -\infty < t < \infty)$ is a Gaussian process if $X_t = Z\cos(2\pi t + \theta)$.

Solution Given: $X_t = Z\cos(2\pi t + \theta)$

Let $\omega = \cos(2\pi t + \theta)$, then $X_t = Z\omega$. Let $X_t = X$, then $X = Z\omega$.

To find the density function of $\omega = \cos(2\pi t + \theta)$.

$\omega = \cos(2\pi t + \theta) \Rightarrow \theta = \cos^{-1}(\omega) - 2\pi t$. For a given value of ω, θ has only two values in $(0, 2\pi)$ and let them be θ_1 and θ_2.

Given θ is uniformly distributed in $(0, 2\pi)$.

$$\therefore \quad f(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \quad f(\omega) = f_\theta(\theta_1)\left|\frac{d\theta_1}{d\omega}\right| + f_\theta(\theta_2)\left|\frac{d\theta_2}{d\omega}\right|$$

$$= \frac{1}{2\pi}\left|-\frac{1}{\sqrt{1-\omega^2}}\right| + \frac{1}{2\pi}\left|\frac{1}{\sqrt{1-\omega^2}}\right| = \frac{1}{\pi\sqrt{1-\omega^2}}, \quad |\omega| < 1$$

To find the density function of $X = Z\omega$:

Let $y = \omega$, then $x = z\omega$ and $y = \omega$, i.e. $z = \frac{x}{y}$ and $\omega = y$.

Therefore, the joint PDF of (X, Y) is

$$f(x, y) = |J|\, f(z, \omega) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} \end{vmatrix} f(z, \omega) = \begin{vmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ 0 & 1 \end{vmatrix} f(z, \omega)$$

$$f(x, y) = \frac{1}{y} f(z)\, f(\omega), \text{ since } \omega \text{ and } z \text{ are independent.}$$

The marginal density function of X is

$$f(x) = \int_{-1}^{1} \frac{1}{|y|} f(z)\, f(\omega) dy = \int_{-1}^{1} \frac{1}{|y|} \frac{x}{y} e^{\frac{-x^2}{2y^2}} \frac{1}{\pi\sqrt{1-y^2}} dy, \qquad \frac{x}{y} > 0$$

when $x < 0$, we have $y < 0$

$$f(x) = \frac{1}{\pi} \int_{-1}^{0} \left(\frac{-x}{y^2}\right) e^{\frac{-x^2}{2y^2}} \frac{1}{\sqrt{1-y^2}} dy$$

Replacing y by $-y$ in the integral, we get

$$f(x) = \frac{1}{\pi} \int_{0}^{1} \frac{x}{y^2} e^{\frac{-x^2}{2y^2}} \frac{1}{\sqrt{1-y^2}} dy, \qquad -\infty < x < \infty$$

When $x > 0$, $y > 0$, we have

$$f(x) = \frac{1}{\pi} \int_{0}^{1} \frac{x}{y^2} e^{\frac{-x^2}{2y^2}} \frac{1}{\sqrt{1-y^2}} dy, \qquad -\infty < x < \infty$$

Let $t = \frac{x^2}{2y^2}$, $\sqrt{t} = \frac{x}{\sqrt{2}y}$, then

$$dt = \frac{x^2}{2} \cdot \frac{-2}{y^3} dy \Rightarrow dt = \frac{-x^2}{y^3} dy$$

When $y = 0$, $t = \infty$ and $y = 1$, $t = \frac{x^2}{2}$

$$\therefore \qquad f(x) = \frac{1}{\pi} \int_{\frac{x^2}{2}}^{\infty} \frac{1}{\sqrt{2t - x^2}} e^{-t} dt$$

Put $t - \frac{x^2}{2} = u \Rightarrow 2t - x^2 = 2u$, $dt = du$

when $t = \frac{x^2}{2}, u = 0$ and $t = \infty$, $u = \infty$.

$$\therefore \qquad f(x) = \frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{1}{\sqrt{u}} e^{-u} e^{\frac{-x^2}{2}} du = \frac{e^{\frac{-x^2}{2}}}{\pi\sqrt{2}}\int_0^\infty e^{-u} u^{-\frac{1}{2}} du$$

$$= \frac{1}{\pi\sqrt{2}} e^{\frac{-x^2}{2}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\pi\sqrt{2}} e^{\frac{-x^2}{2}} \sqrt{\pi}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \qquad -\infty < x < \infty$$

Therefore, each member of the process $\{X_t\}$ follows a normal distribution with mean 0 and variance 1.

Again, for any $a_1, a_2, \ldots, a_n$, $a_1X_{t_1} + a_2X_{t_2} + \cdots + a_nX_{t_n}$ follows a normal distribution.

$\therefore$ $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ are jointly normal for any n.

Hence, the process $\{X_t\}$ is a Gaussian process.

6.8.2 Processes Depending on Stationary Gaussian Process

Square Law Detector Process

If $Y(t) = X^2(t)$, where $\{X(t)\}$ is a zero mean stationary Gaussian process, then $\{Y(t)\}$ which is called a square law detector process is wide sense stationary.

Proof To prove $\{Y(t)\}$ is wide sense stationary, we have to show that

(i) $E\{Y(t)\}$ is constant.
(ii) $R_{YY}(t_1, t_2)$ is a function of time difference.

Given: $Y(t) = X^2(t)$, the square law detector process where $\{X(t)\}$ is a zero mean Gaussian process.

(i) $E[X(t)] = 0$ and
(ii) $R_{XX}(t_1, t_2)$ is a function of time difference.

$$\therefore \qquad E\{Y(t)\} = E\{X^2(t)\} = E\{X(t)\,X(t)\} = R_{XX}(0) = \text{constant}$$

$$R_{YY}(t_1, t_2) = E[Y(t_1)\,Y(t_2)] = E[X^2(t_1)\,X^2(t_2)]$$

$$= E[(X^2(t_1)]\,E[X^2(t_2)] + 2\{E[X(t_1)\,X(t_2)]\}^2$$

[$\because$ $X(t_1)$ and $X(t_2)$ are jointly normal,
$E(X^2Y^2) = E(X^2)\,E(Y^2) + 2E^2(XY)$]

$$= R_{XX}(0)\, R_{XX}(0) + 2R_{XX}^2(t_1, t_2)$$
$$= [R_{XX}(0)]^2 + 2 \text{ [a function of time difference]}$$

Therefore, $R_{YY}(t_1, t_2)$ is also a function of time difference. Hence $\{Y(t)\}$, the square law detector process is a wide sense stationary process.

Two important results:

1. If X and Y are two normal random variables with zero means, variances σ_1^2 and σ_2^2 and correlation coefficient ρ, then the probability that they are of the same sign is $\frac{1}{2} + \frac{1}{\pi}\sin^{-1}(\rho)$ and the probability that they are of opposite sign is $\frac{1}{2} - \frac{1}{\pi}\sin^{-1}(\rho)$.
2. If X and Y are two normal random variables with zero means, variances σ_1^2 and σ_2^2 and correlation coefficient ρ, then $E(|XY|) = \frac{2}{\pi}\sigma_1\sigma_2$ $(\cos\alpha + \alpha \sin\alpha)$ where $\sin\alpha = \rho$.

Full Wave Linear Detector Process

If $\{X(t)\}$ is a zero mean stationary Gaussian process and if $Y(t) = |X(t)|$, $\{Y(t)\}$ is called a full wave linear detector process which is a WSS process.

(i) $$E[Y(t)] = E[|X(t)|] = \int_{-\infty}^{\infty} |x| \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} dx \qquad (\because \mu = 0)$$

$$= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\int_0^{\infty} x e^{\frac{-x^2}{2\sigma^2}} dx \qquad \left[\because |x| \text{ and } e^{\frac{-x^2}{2\sigma^2}} \text{ are even function of } x\right]$$

Put, $\frac{x^2}{2\sigma^2} = t \Rightarrow \frac{2xdx}{2\sigma^2} = dt \Rightarrow xdx = \sigma^2 dt$

$$E[Y(t)] = \sqrt{\frac{2}{\pi}}\sigma\int_0^{\infty} e^{-t} dt$$

$$= \sqrt{\frac{2}{\pi}}\sigma\left[\frac{e^{-t}}{-1}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\sigma$$

$\because$ $\text{Var}[X(t)] = R_{XX}(0)$, when $E[X(t)] = 0$, $\sigma = \sqrt{R_{XX}(0)}$

$$\therefore \quad E[|X(t)|] = E[Y(t)] = \sqrt{\frac{2}{\pi}}\sqrt{R_{XX}(0)}, \text{ a constant} \qquad (6.13)$$

(ii) $R_{YY}(\tau) = E[|X(t)\,X(t+\tau)|] = \frac{2}{\pi}\sigma^2(\cos\alpha + \alpha\sin\alpha)$

[using result (2)]

where $\sin\alpha = \rho_{xx} = \frac{C_{XX}(\tau)}{\sigma^2} = \frac{R_{XX}(\tau)}{\sigma^2}$.

Since $\{X(t)\}$ is a WSS process, $R_{XX}(\tau)$ is a function of time difference, $R_{YY}(\tau)$ is also a function of time difference.

$\therefore$ $\{Y(t)\}$ is a WSS random process.

Also, $$E[Y^2(t)] = R_{YY}(0) = \frac{2}{\pi}R_{XX}(0)\left\{0 + \frac{\pi}{2}\cdot 1\right\} = R_{XX}(0)$$

and $$\text{Var}[Y(t)] = \left(1 - \frac{2}{\pi}\right)R_{XX}(0).$$

Half-wave Linear Detector Process

If $\{X(t)\}$ is a zero mean stationary Gaussian process and if

$$Z(t) = \begin{cases} X(t), & \text{for } X(t) \ge 0 \\ 0, & \text{for } X(t) < 0 \end{cases}$$

then $\{Z(t)\}$ is called a half-wave linear detector process which is also a WSS process.

(i) $Z(t)$ can be written as

$$Z(t) = \frac{1}{2}[X(t) + |X(t)|] \qquad \left[\because |X(t)| = \begin{cases} X(t), & X(t) \ge 0 \\ -X(t), & X(t) < 0 \end{cases}\right]$$

$$\therefore E[Z(t)] = \frac{1}{2}\{E[X(t)] + E(|X(t)|)\} = \frac{1}{2}\left[0 + \sqrt{\frac{2}{\pi}}R_{XX}(0)\right], \text{ using (6.13)}$$

$$\therefore E[Z(\tau)] = \sqrt{\frac{R_{XX}(0)}{2\pi}}$$

$$R_{ZZ}(\tau) = E[Z(t)\,Z(t+\tau)]$$

$$= \frac{1}{4}E\{[X(t) + |X(t)|][X(t+\tau) + |X(t+\tau)|]\}$$

$$= \frac{1}{4}E\begin{bmatrix} X(t)\,X(t+\tau) + X(t)|X(t+\tau)| \\ +|X(t)|\,X(t+\tau) + |X(t)|\,|X(t+\tau)| \end{bmatrix}$$

$$= \frac{1}{4}[R_{XX}(\tau) + R_{YY}(\tau)]$$

where $\{Y(t)\}$ is the full-wave linear detector process.

Since $R_{XX}(\tau)$ and $R_{YY}(\tau)$ are functions of time difference, $R_{ZZ}(\tau)$ is also a function of time difference.

$\therefore$ $\{Z(t)\}$ is a WSS random process. Also, $E[Z^2(t)] = \frac{1}{2}R_{XX}(0)$ and

$\text{Var}[Z(t)] = \frac{1}{2}\left(1 - \frac{1}{\pi}\right)R_{XX}(0)$.

Hard Limiter Process

If $\{X(t)\}$ is a zero mean stationary Gaussian process and if

$$Y(t) = \begin{cases} 1, & \text{for } X(t) \geq 0 \\ -1, & \text{for } X(t) < 0 \end{cases}$$

then $\{Y(t)\}$ is called a hard limiter process or ideal limiter process which is also a WSS process.

(i) $E[Y(t)] = 1 \times P[X(t) \geq 0] - 1 \times P[X(t) < 0]$

$\therefore E[Y(t)] = 0$

$$Y(t)\,Y(t+\tau) = \begin{cases} 1, & \text{if } X(t)\,X(t+\tau) \geq 0 \\ -1, & \text{if } X(t)\,X(t+\tau) < 0 \end{cases}$$

$\therefore$ $P[Y(t)\;Y(t+\tau) = 1] = P[X(t)X(t+\tau) \geq 0]$

Using the result that if X and Y are two normal random variables with zero mean, variance σ_1^2 and σ_2^2 and correlation coefficient ρ, then the probability that they are of the same sign is $\frac{1}{2} + \frac{1}{\pi}\sin^{-1}(\rho)$ and the probability that they are of opposite sign is $\frac{1}{2} - \frac{1}{\pi}\sin^{-1}(\rho)$, we get

$$P[Y(t)Y(t+\tau) = 1] = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}(\rho)$$

and $$P[Y(t)Y(t+\tau) = -1] = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}(\rho)$$

$$\therefore\; E[Y(t)Y(t+\tau)] = 1 \times \left[\frac{1}{2} + \frac{1}{\pi}\sin^{-1}(\rho)\right] - 1 \times \left[\frac{1}{2} - \frac{1}{\pi}\sin^{-1}(\rho)\right]$$

$$R_{YY}(\tau) = E[Y(t)\,Y(t+\tau)] = \frac{2}{\pi}\sin^{-1}(\rho) = \frac{2}{\pi}\sin^{-1}\left[\frac{R_{XX}(\tau)}{R_{XX}(0)}\right]$$

which is a function of time difference.

$\therefore$ $\{Y(t)\}$ is a WSS random process.

Also, $E[Y^2(t)] = 1$

and $\text{Var}[Y(t)] = 1$

EXAMPLE 6.72 Suppose that $\{X(t)\}$ is a normal process with mean $\mu(t) = 3$ and $C(t_1, t_2) = 4e^{-0.2|t_1 - t_2|}$, find the probability that

(i) $X(5) \le 2$,
(ii) $|X(8) - X(5)| \le 1$.

Solution Given: $X(t)$ is a normal process with $\mu(t) = 3$

and $$C(t_1, t_2) = 4e^{-0.2|t_1 - t_2|}$$

$\therefore$ $$C(5, 5) = 4e^{-0.2|0|} = 4 \Rightarrow \text{Var}[X(5)] = 4$$

(i) To find $P[X(5) \le 2]$:
We have $E[X(5)] = \mu = 3$, $\text{Var}[X(5)] = 4 \Rightarrow \sigma = 2$

$$\therefore \quad Z = \frac{X - \mu}{\sigma} = \frac{X(5) - 3}{2}$$

$$P[X(5) \le 2] = P\left[\frac{X(5) - 3}{2} \le \frac{2 - 3}{2}\right]$$

$$= P\left(Z \le \frac{2-3}{2}\right) = P(Z \le -0.5) = 0.5 - P(0 < Z < -0.5)$$

$$= 0.5 - 0.1915 = 0.3085$$

(ii) To find $P[|X(8) - X(5)|] \le 1$:

Let $X = X(8) - X(5)$

$$E(X) = E[X(8) - X(5)] = E[X(8)] - E[X(5)] = 3 - 3 = 0.$$

$$\text{Var}(X) = \text{Var}[X(8) - X(5)] = C(8, 8) + C[(5, 5) - 2C(8, 5)]$$

$$= 4 + 4 - 8e^{-0.2|3|} = 8 - 8e^{-0.6} = 3.6095$$

$$\text{SD} = \sqrt{\text{Var}(X)} = \sigma = \sqrt{3.6095} = 1.8998$$

Hence,

$$P[|X(8) - X(5)| \le 1] = P(X \le 1) = P\left(\frac{X}{1.8998} \le \frac{1}{1.8998}\right)$$

$$= P\left[|Z| \le \frac{1}{1.8998}\right] \text{ where } Z = \frac{X - \mu}{\sigma} = \frac{X - 0}{1.8998}$$

$$= P(|Z| \le 0.526) = 2P(0 < Z < 0.53)$$

$$\therefore \quad = P[X(8) - X(5) \le 1] = 2 \times 0.2019 = 0.4038$$

EXAMPLE 6.73 Suppose that $X(t)$ is a Gaussian process with $\mu_X = 2$, $R_{XX}(\tau) = 5e^{-0.2|\tau|}$, find the probability that $X(4) \le 1$.

Solution Given: $X(t)$ is a Gaussian process with mean $\mu_X = 2$ and ACF

$$R_{XX}(\tau) = 5e^{-0.2|\tau|}$$

$$R_{XX}(t_1, t_2) = 5e^{-0.2|t_2 - t_1|} \qquad \text{[since } \tau = t_2 - t_1\text{]}$$

$$\therefore \quad \text{Var } X(t) = R_{XX}(t, t) - E[X(t)]\,[E[X(t)]]$$

$$\text{Var}[X(4)] = 5e^{-0.2|0|} - 4 = 1$$

Since $E[X(t)] = 2 \Rightarrow E[X(4)] = 2$ $\qquad \therefore \sigma = SD = \text{Var}[X(4)] = 1$

To find $P[X(4) \leq 1]$:

$$P[X(4) \leq 1] = P\left[\frac{X(4)-2}{1} \leq \frac{1-2}{1}\right] = P[Z \leq -1]$$

$$= 0.5 - P[0 \leq Z \leq 1] = 0.5 - 0.3413$$

$$= 0.1587$$

EXAMPLE 6.74 The process $\{X(t)\}$ is normal with $\mu_t = 0$ and $R_{XX}(\tau) = 4e^{-3|\tau|}$. Find a memoryless system $g(X)$ such that the first-order density $f_Y(y)$ of the resulting output $Y(t) = g[X(t)]$ is uniform in the interval (6, 9).

Solution Since $\{X(t)\}$ is a normal process, a sample function $X(t)$ follows a normal distribution with mean 0 and variance $= R_{XX}(0) = 4$.

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{\frac{-x^2}{8}}, \qquad -\infty < x < \infty$$

Now, $Y(t)$ is to be uniform in (6, 9)

$$\therefore \qquad f_Y(y) = \frac{1}{3}, \qquad 6 < y < 9 \qquad \text{(i)}$$

$\therefore$ The distribution function y is given by

$$F_Y(y) = \int_6^y f_Y(y)\,dy$$

$$= \frac{1}{3}(y-6) \qquad \text{(i)}$$

Again, $F_Y(y) = P[Y(t) \leq y] = P[g\{X(t)\} \leq y]$

$= P[X(t) \leq g^{-1}(y)] = P[X(t) \leq x]$ $\qquad$ [since $y = g(x)$]

$= F_X(x)$

But from Eq. (i)

$$F_Y[g(x)] = \frac{1}{3}[g(x) - 6]$$

$$\therefore \qquad \frac{1}{3}[g(x) - 6] = F_X(x)$$

$$\therefore \qquad g(x) = 6 + 3F_X(x) \Rightarrow g(x) = 6 + 3\int_{-\infty}^{x} \frac{1}{2\sqrt{2\pi}} e^{\frac{-x^2}{8}}\,dx$$

6.9 RANDOM TELEGRAPH PROCESS

It is defined as a discrete state, continuous parameter process $\{X(t)/-\infty < t < \infty\}$, with state space $\{-1, 1\}$. Assume that these two values are equally likely

i.e. $$P[X(t) = 1] = \frac{1}{2} = P[X(t) = -1], \quad -\infty < t < \infty$$

A typical sample function of the process is shown in Figure 6.6.

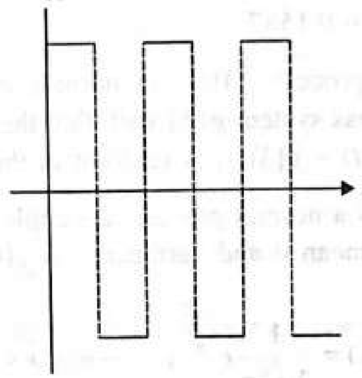

Figure 6.6 Function of random telegraph process.

6.9.1 Semirandom Telegraph Process

If $N(t)$ represents the number of occurrences of a specified event in $(0, t)$ and $X(t) = (-1)^{N(t)}$, then $\{X(t)\}$ is called a *semirandom telegraph signal process.*

6.9.2 Random Telegraph Signal Process

The random process $\{Y(t)\}$ defined by $Y(t) = \alpha X(t)$ where $X(t)\}$ is a semirandom telegraph signal process, α is a random variable which is independent of $X(t)$ and assumes the values +1 and −1 with equal probability is called a *random telegraph signal process.*

To Find the Mean of Random Telegraph Signal Process

The distribution of $N(t)$ is Poisson with mean λt and the process $\{N(t)\}$ is a Poisson process with probability law

$$P[N(t) = r] = \frac{e^{-\lambda t}(\lambda t)^r}{r!}, \quad r = 0, 1, 2, 3, \ldots$$

If $\{X(t)\}$ is the semirandom telegraph signal process, then $X(t)$ can take the values +1 and −1 only.

$$P[X(t) = 1] = P[N(t) \text{ is even}]$$

$$= P[N(t) = 0] + P[N(t) = 2] + P[N(t) = 4] + \cdots$$

$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right]$$

$$= e^{-\lambda t} \cosh \lambda t$$

$$P[X(t) = -1] = P[N(t) \text{ is odd}]$$

$$= P[N(t) = 1] + P[N(t) = 3] + P[N(t) = 5] + \cdots$$

$$= e^{-\lambda t}\left[\frac{\lambda t}{1!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots\right]$$

$$= e^{-\lambda t} \sinh \lambda t$$

$$E[X(t)] = \sum x(t)\, P[X(t)]$$

$\therefore$

$$= (1)\, e^{-\lambda t} \cosh \lambda t + (-1) e^{-\lambda t} \sinh \lambda t$$

$$= e^{-\lambda t}[\cosh \lambda t - \sinh \lambda t]$$

$$= e^{-\lambda t}\left[\left(\frac{e^{\lambda t} + e^{-\lambda t}}{2}\right) - \left(\frac{e^{\lambda t} - e^{-\lambda t}}{2}\right)\right]$$

$$= e^{-\lambda t}(e^{-\lambda t}) = e^{-2\lambda t}$$

To Find the ACF of Random Telegraph Signal

To find $E[X(t_1)X(t_2)]$, we consider the joint probability distribution of $\{X(t_1), X(t_2)\}$.

$$P[X(t_1) = 1, X(t_2) = 1] = P[X(t_1) = 1/X(t_2) = 1]\; P[X(t_2) = 1]$$

$= P$[an even number of occurrences of the event in $(t_1 - t_2)] \cdot P[X(t_2) = 1]$

$$= (e^{-\lambda \tau} \cosh \lambda \tau)(e^{-\lambda t_2} \cosh \lambda t_2)$$

Similarly,

$$P[X(t_1) = -1, X(t_2) = -1] = (e^{-\lambda \tau} \cosh \lambda \tau)(e^{-\lambda t_2} \sinh \lambda t_2)$$

and $$P[X(t_1) = -1, X(t_2) = 1] = (e^{-\lambda \tau} \sinh \lambda \tau)(e^{-\lambda t_2} \cosh \lambda t_2)$$

$$P[X(t_1) \cdot X(t_2) = 1] = P[X(t_1) = 1, X(t_2) = 1] + P[X(t_1) = -1, X(t_2) = -1]$$

$$= e^{-\lambda \tau} \cosh \lambda \tau\, e^{-\lambda t_2} \cosh \lambda t_2 + e^{-\lambda \tau} \cosh \lambda \tau\, e^{-\lambda t_2} \sinh \lambda t_2$$

$$= (e^{-\lambda \tau} \cosh \lambda \tau) e^{-\lambda t_2} (\cosh \lambda t_2 + \sinh \lambda t_2)$$

$$= e^{-\lambda \tau} \cosh \lambda \tau (e^{-\lambda t_2}\, e^{\lambda t_2})$$

$$= e^{-\lambda \tau} \cosh \lambda \tau$$

and $$P[X(t_1) \cdot X(t_2) = -1] = e^{-\lambda \tau} e^{-\lambda t_2} \sinh \lambda \tau [\cosh \lambda t_2 + \sinh \lambda t_2]$$

$$= e^{-\lambda \tau} \sinh \lambda \tau (e^{-\lambda t_2} e^{\lambda t_2})$$

$$= e^{-\lambda \tau} \sinh \lambda \tau$$

$\therefore$ The ACF of the process $\{X(t)\}$ is given by

$$R(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$= (1)e^{-\lambda\tau}\cosh\lambda\tau + (-1)e^{-\lambda\tau}\sinh\lambda\tau$$
$$= e^{-\lambda t}(\cosh\lambda\tau - \sinh\lambda\tau)$$
$$= e^{-\lambda\tau}e^{-\lambda\tau} = e^{-2\lambda\tau}$$

$\therefore$ $$R_{XX}(t_1, t_2) = e^{-2\lambda(t_1 - t_2)}$$

Note: The semirandom telegraph process $\{X(t)\}$ is an evolutionary process since $E[X(t)]$ is not a constant.

EXAMPLE 6.75 Prove that random telegraph process $\{Y(t)\}$ is a wide sense stationary process.

Solution Consider the random process $\{Y(t)\}$ where $Y(t) = \alpha X(t)$ and α is a random variable which assumes values -1 and $+1$ with equal probability $\frac{1}{2}$.

$\therefore$ $$P(\alpha = 1) = \frac{1}{2}$$

and $$P(\alpha = -1) = \frac{1}{2}, \quad E[X(t)] = e^{-2\lambda t}, \quad R_{XX}(t_1, t_2) = e^{-2\lambda(t_1 - t_2)}$$

$\therefore$ $$E(\alpha) = 1\left(\frac{1}{2}\right) - 1\left(\frac{1}{2}\right) = 0$$

and $$E(\alpha^2) = (1)^2 \times \left(\frac{1}{2}\right) + (-1)^2 \times \frac{1}{2} = 1$$

$\therefore$ $$E[Y(t)] = E(\alpha)\, E[X(t)]$$
$$= 0 \times e^{-2\lambda t} = 0$$

since α and $\{X(t)\}$ are independent

Also, $$E[Y(t_1)\, Y(t_2)] = E(\alpha^2)\, E[X(t_1)\, X(t_2)]$$
$$= (1)e^{-2\lambda(t_1 - t_2)}$$
$$= e^{-2\lambda(t_1 - t_2)}$$

Since $R_{YY}(t_1, t_2)$ is a function of $(t_1 - t_2)$ and $E[Y(t)] = 0$, the process $[Y(t)]$ is a wide sense stationary process.

EXAMPLE 6.76 Let $\{X(t): t \geq 0\}$ be a random process where $X(t)$ is the total number of points in the interval $(0, t) = k$ say and $X(t) = \begin{cases} 1, & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd} \end{cases}$.

Find the ACF of $X(t)$. Also if $P(A = 1) = P(A = -1) = \frac{1}{2}$ and A is independent of $X(t)$, find the ACF of $Y(t) = AX(t)$.

Solution To find $E[X(t_1)X(t_2)]$, we consider the joint probability distribution of $\{X(t_1), X(t_2)\}$.

$$P[X(t_1) = 1, X(t_2) = 1] = P[X(t_1) = 1/X(t_2) = 1]\ P[X(t_2) = 1]$$

$$= P[\text{an even number of occurrences of the event in } (t_1 - t_2)] \cdot P[X(t_2) = 1]$$

$$= (e^{-\lambda\tau} \cosh \lambda\tau)(e^{-\lambda t_2} \cosh \lambda t_2)$$

Similarly,

$$P[X(t_1) = -1, X(t_2) = -1] = (e^{-\lambda\tau} \cosh \lambda\tau)(e^{-\lambda t_2} \sinh \lambda t_2)$$

$$P[X(t_1) = -1, X(t_2) = 1] = (e^{-\lambda\tau} \sinh \lambda\tau)\ (e^{-\lambda t_2} \cosh \lambda t_2)$$

and

$$P[X(t_1) \cdot X(t_2) = 1]$$
$$= P[X(t_1) = 1, X(t_2) = 1] + P[X(t_1) = -1, X(t_2) -1]$$
$$= e^{-\lambda\tau} \cosh \lambda\tau\, e^{-\lambda t_2} \cosh \lambda t_2 + e^{-\lambda\tau} \cosh \lambda\tau\, e^{-\lambda \tau_2} \sinh \lambda t_2$$
$$= (e^{-\lambda\tau} \sinh \lambda\tau)\, e^{-\lambda t_2} (\cosh \lambda t_2 + \sinh \lambda t_2)$$
$$= e^{-\lambda\tau} \cosh \lambda\tau (e^{-\lambda t_2} e^{\lambda t_2})$$
$$= e^{-\lambda\tau} \cosh \lambda\tau$$

and

$$P[X(t_1) \cdot X(t_2) = -1]$$
$$= e^{-\lambda\tau} e^{-\lambda t_2} \sinh \lambda\tau (\cosh \lambda t_2 + \sinh \lambda t_2)$$
$$= e^{-\lambda\tau} \sinh \lambda\tau (e^{-\lambda t_2} e^{\lambda t_2})$$
$$= e^{-\lambda\tau} \sinh \lambda\tau$$

The ACF of the process $\{X(t)\}$ is given by

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$= (1)\ e^{-\lambda\tau} \cosh \lambda\tau + (-1)\ e^{-\lambda\tau} \sinh \lambda\tau$$
$$= e^{-\lambda\tau}(\cosh \lambda\tau - \sinh \lambda\tau)$$
$$= e^{-\lambda\tau}\, e^{-\lambda\tau} = e^{-2\lambda\tau}$$

$$\therefore \quad R_{XX}(t_1, t_2) = e^{-2\lambda(t_1 - t_2)}$$

To find the ACF of $Y(t) = A[X(t)]$:

$$E(A) = AP(A = 1) + AP(A = -1)$$
$$= 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$$
$$E(A^2) = A^2P(A = 1) + A^2P(A = -1)$$
$$= 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$$
$$R_{YY}(\tau) = E[Y(t)\, Y(t + \tau)]$$
$$= E[AX(t)\ AX(t + \tau)]$$
$$= E[A^2X(t)\ X(t + \tau)]$$
$$= E(A^2)R_{XX}(\tau)$$
$$\therefore \quad = R_{XX}(\tau)$$
$$R_{YY}(\tau) = R_{XX}(\tau) = e^{-2\lambda\tau}$$

EXERCISES

Basics of Random Process

1. State the four types of stochastic processes.
2. Define a stationary process and give an example.
3. Is the autocorrelation of a random process the same as the correlation coefficient of the process? Why?
   [**Ans.** No. Their definitions are entirely different]
4. Define a strict sense stationary process and provide an example.
   [**Ans.** Bernoulli process]
5. When is a complex random process $\{Z(t)\}$, where $Z(t) = X(t) + iY(t)$, said to be a SSS process?
6. What is the difference between a SSS process and a WSS process?
7. If $X(t) = Y \cos t + Z \sin t$ for all t where Y and Z are independent binary random variables, each of which assumes the values -1 and $+2$ with probabilities 2/3 and 1/3 respectively, prove that $\{X(t)\}$ is wide sense stationary. [**Ans.** $E\{X(t)\} = 0$; $R(t_1, t_2) = 2\cos(t_1 - t_2)$]
8. What is the difference between a random variable and a random process?
9. Define a random process and give an example of a random process.
10. Explain the terms 'state space' and 'parameter set' associated with a random process.
11. If $\{X(s, t)\}$ is a random process, what is the nature of $X(s, t)$ when
    (i) s is fixed and
    (ii) t is fixed?
12. What is the difference between a random sequence and random process?
13. What is a discrete random sequence? Give an example.
14. What is a continuous random sequence? Give an example.
15. What is a discrete random process? Give an example.
16. What is a continuous random process? Give an example.
17. How is a random process described mathematically?
18. Name three classes of random processes into which random processes are generally divided.
19. Name two important random processes with independent increments.
20. What do you mean by the mean and variance of a random process?
21. Define the autocorrelation of a random process $\{X(t)\}$ and explain its concept.
22. Define the autocovariance of a random process $\{X(t)\}$.
23. Define the correlation coefficient of a random process $\{X(t)\}$.

24. Is the autocorrelation of a random process the same as the correlation coefficient of the process? Why?
25. Define the cross-correlation of two random processes.
26. When are two random processes said to be orthogonal?
27. Define the cross-covariance of two random processes.
28. Define the cross-correlation coefficient of two random processes.
29. Define a strict sense stationary process and give an example.
30. Define a kth-order stationary process. When will it become a SSS process?
31. Prove that the first-order density function of a SSS process $\{X(t)\}$ is independent of t.
32. If $\{X(t)\}$ is a SSS process, prove that $E\{X(t)\}$ is a constant.
33. If $\{X(t)\}$ is a SSS process, prove that the joint PDF of $X(t_1)$ and $X(t_2)$ is a function of $(t_1 - t_2)$.
34. Prove that the autocorrelation of a SSS process $\{X(t)\}$ is a function of $(t_1 - t_2)$.
35. When are $\{X(t)\}$ and $\{Y(t)\}$ said to be jointly stationary in the strict sense?
36. When is a complex random process $\{Z(t)\}$, where $Z(t) = X(t) + iY(t)$, said to be a SSS process?
37. Define wide sense stationary process. Give an example.
38. What is the difference between a SSS process and a WSS process?
39. When is a random process said to be evolutionary? Give an example of an evolutionary process.
40. When are the processes $\{X(t)\}$ and $\{Y(t)\}$ said to be jointly stationary in the wide sense?
41. If $\{X(t)\}$ is a stationary process in any sense, prove that $\text{Var}\{X(t)\}$ is a constant.
42. Define a random process. Explain how you would classify a random process. Give an example to each type of random process. **[AU May/June '07]**
43. If $X(t) = R\cos(\omega t + \phi)$, where R and ϕ are independent random variables and ϕ is uniformly distributed in $(-\pi, \pi)$, prove that $R(t_1, t_2) = \frac{1}{2}E(R^2)\cos(t_1 - t_2)$.
44. Verify whether the random process $\{X(t)\} = A\cos(\omega t + \theta)$ is wide sense stationary when A and ω are constants and θ is uniformly distributed on the interval $0 < \theta < \frac{\pi}{2}$. **[AU May/June '07]** **[Ans. No]**
45. Show that the random process $X(t) = A\cos(\omega_0 t + \theta)$ is wide sense stationary if A and ω_0 are constants and θ is uniformly distributed random variable in $(0, 2\pi)$. **[AU November/December '05]**

46. Prove that the random process $\{X(t)\}$ is WSS if $X(t) = A\sin(\omega_0 t + \theta)$ where A and ω_0 are constants and θ is uniformly distributed random variable in $(0, 2\pi)$.

$$\left[\textbf{Ans.}\quad E[X(t)] = 0,\ R(t_1, t_2) = \frac{A^2}{2}\cos\omega(t_1 - t_2)\right]$$

47. A stochastic process is described by $X(t) = A\sin t + B\cos t$, where A and B are independent random variables with zero means and equal standard deviations. Show that the process is stationary of the second order.

[Ans. $E\{X(t)\} = 0;\ R(t_1, t_2) = \sigma^2\cos(t_1 - t_2)$]

48. If the random variables A_i are uncorrelated with zero mean and $E\{|A_i|^2\} = \sigma_i^2$, prove that the process $X(t) = \sum_{i=1}^{n} A_i e^{j\omega_i t}$ is wide sense stationary with zero mean. Show also that for $X(t)$, $R(\tau) = \sum_{i=1}^{n} \sigma_i^2 e^{j\omega x}$.

[Hint: For a complex-valued random process $\{X(t)\}$, the autocorrelation is defined as $R(t_1, t_2) = E\{X(t_1) \times X^*(t_2)\}$, where $X^*(t_2)$ is a complex conjugate of $X(t_2)$.]

49. If $U(t) = X\cos t + Y\sin t$ and $V(t) = Y\cos t + X\sin t$ where X and Y are independent random variables such that $E(X) = 0 = E(Y)$, $E(X^2) = E(Y^2) = 1$, show that $\{U(t)\}$ and $\{V(t)\}$ are individually stationary in the wide sense, but they are not jointly wide sense stationary.

[Ans. $R_{XY}(t_1, t_2) = \sin(t_1 + t_2)$]

50. If $X(t) = A\sin\omega(\omega t + \theta)$ where A and ω are constants and θ is a random variable, uniformly distributed over $(-\pi, \pi)$, find the autocorrelation of $\{Y(t)\}$ where $Y(t) = X^2(t)$.

$$\left[\textbf{Ans.}\quad R(t_1, t_2) = \frac{A^4}{8}\{2 + \cos 2\omega(t_1 - t_2)\}\right]$$

51. If X and Y are random variables such that $Y = aX + b$ where a and b are real constants, show that the correlation coefficient $r(X, Y)$ between them has magnitude one. **[AU May/June '06]**

52. Consider a random process $Z(t) = X_1\cos\omega_0 t - X_2\sin\omega_0 t$ where X_1 and X_2 are independent Gaussian random variables with zero mean and variance σ^2. Find $E[Z(t)]$ and $E[Z^2(t)]$. **[Ans.** $E(Z) = 0$, $E(Z^2) = \sigma^2$]

53. If $X(t) = \cos(\lambda t + \alpha)$ where α is uniformly distributed in $(-\pi, \pi)$, show that $\{X(t)\}$ is stationary in wide sense.

$$\left[\textbf{Ans.}\quad E[X(t)] = 0,\ R_{XX}(t, t+\tau) = \frac{\cos\lambda\tau}{2}\right]$$

54. Consider the process $W(t) = X(t)\cos\omega t + Y(t)\sin\omega t$, where $X(t)$ and $Y(t)$ are two real jointly stationary processes. What are the conditions for $W(t)$

to be WSS? In case $W(t)$ is wide sense stationary, what is its autocorrelation in terms of autocorrelations of $X(t)$ and $Y(t)$?

$$\left[\begin{aligned}\textbf{Ans. } & R_{XX}(\tau) = R_{YY}(\tau) \text{ and } R_{XY}(-\tau) = -R_{XY}(\tau); \\ & R_{\omega\omega}(\tau) = R_{XX}(\tau)\cos\omega\tau + R_{YX}(\tau)\sin\omega\tau \\ & \text{or } R_{XX}(\tau)\cos\omega\tau - R_{XY}(\tau)\sin\omega\tau\end{aligned}\right]$$

55. Show that, if the process $X(t) = a\cos\omega t + b\sin\omega t$ is SSS where a and b are independent random variables, then they are normal.

56. If $X(t) = 5\cos(10t + \theta)$ and $Y(t) = 20\sin(10t + \theta)$ where θ is a random variable uniformly distributed in $(0, 2\pi)$, prove that the processes $\{X(t)\}$ and $Y\{(t)\}$ are jointly wide sense stationary.

[**Ans.** $R_{XY}(t_1, t_2) = 50\sin 10(t_1 - t_2)$]

Markov Process and Markov Chain

57. Define a Markov process.
58. Define a Markov chain and give an example of a Markov chain.
59. Prove that the Poisson process is a Markov process.
60. When is a Markov chain called homogeneous?
61. When is a homogeneous Markov chain said to be regular?
62. Define transition probability matrix of a Markov chain.
63. What is a stochastic matrix? When is it said to be regular?
64. Prove that the TPM of a Markov chain is a stochastic matrix.
65. Define n-step transition probability in a Markov chain.
66. State Chapman–Kolmogorov theorem.
67. What do you mean by probability distribution of a Markov chain?
68. When is a Markov chain completely specified?
69. What is meant by steady-state distribution of a Markov chain?
70. Write down the relation satisfied by the steady-state distribution and the TPM of a regular Markov chain.
71. If the TPM of a Markov chain is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ find the steady-state distribution of the chain.

$\left[\textbf{Ans. } \pi_1 = \frac{1}{2},\ \pi_2 = \frac{1}{2}\right]$

72. When is a Markov chain said to be irreducible or ergodic?
73. Prove that the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ is the TPM of an irreducible Markov chain.

74. What do you mean by an absorbing Markov chain. Give an example.

75. If the initial state probability distribution of a Markov chain is $p^{(0)} = \left(\frac{5}{6}, \frac{1}{6}\right)$ and the TPM of the chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, find the probability distribution of the chain after two steps.

[**Ans.** $p^{(2)} = \left(\frac{11}{24}, \frac{13}{24}\right)$]

76. The TPM of a Markov chain with three states 0, 1, 2 is $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ and the initial state distribution of the chain is $P\{X_0 = i\} = 1/3$, $i = 0, 1, 2$. Find

   (i) $P\{X_2 = 2\}$

   (ii) $P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$ [**Ans.** (i) 1/6, (ii) 3/64]

77. A man is at an integral point on the x axis between the origin and the point 3. He takes a unit step to the right with probability 1/3 or to the left with probability 2/3, unless he is at the origin, where he takes a step to the right to reach the point 1 or is at the point 3, where he takes a step to the left to reach the point 2. What is the probability that

   (i) he is at the point 1 after 3 walks, and

   (ii) he is at the point 1 in the long run?

[**Ans.** (i) 22/27, (ii) 3/7]

78. Suppose that the probability of a dry day following a rainy day (state 1) is 1/3 and that probability of a rainy day following a dry day is 1/2. Given that May 1 is a dry day, find the probability that

   (i) May 3 is also a dry day, and

   (ii) May 5 is also a dry day. [**Ans.** (i) 5/12, (ii) 173/432]

79. A gambler has ₹ 3 at each play of the game, he loses ₹ 1 with probability 3/4, but wins ₹ 2 with probability 1/4. He stops playing if he has lost his initial amount of ₹ 3 or he has won at least ₹ 3. Write down the TPM of the associated Markov chain. Find the probability that there are at least 4 rounds to the game. [**Ans.** 27/64]

80. A fair coin is tossed until 3 heads occur in a row. Let X_n be the sequence of heads ending at nth trial. What is the probability that there is at least 8 tosses of the coin? [**Ans.** 81/128]

81. A housewife buys 3 kinds of cereals A, B and C. She never buys the same cereal in successive weeks. If she buys cereal A, the next week she buys

cereal B. However, if she buys B or C, the next week she is 3 times as likely to buy A as the other cereal. In the long run, how often she buys each of the three cereals?

$$\left[\textbf{Ans. } P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix} \text{ and } P = \left(\frac{15}{35}, \frac{16}{35}, \frac{4}{35}\right)\right]$$

82. A gambler's luck follows a pattern such that if he wins a game, the probability of winning the next game is 0.6 and if he loses a game, the probability of losing the next game is 0.7. There is an even chance that the gambler wins the first game. What is the probability that he wins
    (i) the second game,
    (ii) the third game, and
    (iii) in the long run? [**Ans.** (i) 9/20, (ii) 87/200, (iii) 3/7]

83. The three-state Markov chain is given by the TPM $P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

    Prove that the chain is irreducible and all the states are aperiodic and non-null persistent. Find also the steady-state distribution of the chain.

$$\left[\textbf{Ans. } \left(\frac{9}{27}, \frac{10}{27}, \frac{8}{27}\right)\right]$$

84. Assume that the weather in a certain locality can be modelled as a homogeneous Markov chain whose TPM is given as

Today weather	**Tomorrow weather**		
	Fair	Cloudy	Rainy
Fair	0.8	0.15	0.05
Cloudy	0.5	0.3	0.2
Rainy	0.6	0.3	0.1

If the initial state distribution is given by $P^{(0)} = (0.7, 0.2, 0.1)$, find $P^{(2)}$ and $\lim\limits_{n \to \infty} P^{(n)}$.

$$\left[\textbf{Ans. } (0.7245, 0.1920, 0.2727); \left(\frac{114}{157}, \frac{30}{157}, \frac{13}{157}\right)\right]$$

85. There are 2 white marbles in urn A and 4 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles

selected are interchanged. The state of the related Markov chain is the number of red balls in A after the interchange. What is the probability that there are 2 red balls in urn A

(i) after 3 steps, and
(ii) in the long run?

$\left[\textbf{Ans. } \frac{3}{8}, \frac{2}{5}\right]$

86. A man tosses a fair coin until 3 heads occur in a row. Let $X_n = k$ if at the nth trial, the last tail occurred at the $(n - k)$th trial i.e. X_n denotes the longest string of heads ending at the nth trial. Show that the process is Markovian. Find the transition matrix and classify the states.

Ans.
$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$
The chain is not irreducible.

State 3 is absorbing and other states are aperiodic.

87. Define a Markov chain. Explain how you would clarify the states and identify different classes of a Markov chain. Give an example to each class. **[AU December '04]**

88. The transition probability matrix of a Markov chain $\{X_n\}$, $n = 1, 2, \ldots$ having 3 states 1, 2, 3 is

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

and the initial distribution $P^{(0)} = (0.7, 0.2, 0.1)$. Find $P(X_2 = 3, X_1 = 3, X_0 = 2)$. **[AU November '03]**

[Ans. 0.012**]**

89. Assuming that a computer system is in any one of the three states: busy, idle and under repair respectively denoted by 0, 1, 2. Observing its state at 2 P.M., its transition probability matrix is given as

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{pmatrix}$$

Prove that the chain is irreducible and find the steady-state probabilities.

[AU December '07]

[Ans. 1/3, 1/3, 1/3]

Poisson Random Process

90. What is a point process? Give an example.
91. Define a Poisson process.
92. What are the postulates of a Poisson process?
93. State the probability law of a Poisson process.
94. When is a Poisson process said to be homogeneous?
95. If $\{X(t)\}$ is a homogeneous Poisson process, find $P\{X(t_1) = n_1, X(t_2) = n_2\}$, $t_2 > t_1$.
96. Find the autocorrelation $R_{XX}(t_1, t_2)$ of the Poisson process $\{X(t)\}$.
97. State and prove the additive property of a Poisson process.
98. Prove that the difference of two independent Poisson processes is not a Poisson process.
99. Prove that the interarrival time of a Poisson process follows an exponential distribution.
100. If the customers arrive at a bank according to a Poisson process with mean rate of 2 per minute, find the probability that during 1-minute interval, no customer arrives. **[Ans.** e^{-2}]
101. On the average, a submarine on patrol sights 6 enemy ships per hour. Assuming that the number of ships sighted in a given length of time is a Poisson variate, find the probability of sighting
    (i) 6 ships in the next half an hour,
    (ii) 4 ships in the next 2 hours,
    (iii) at least 1 ship in the next 15 minutes, and
    (iv) at least 2 ships in the next 20 minutes.

    [Ans. (i) 0.0504, (ii) 0.0054, (iii) 0.7769, (iv) 0.5941]
102. Assume that an office switchboard has 5 telephone lines and that starting at 8 a.m. on Monday, the time that a call arrives on each line is an exponential random variable with parameter λ. Also assume that the calls arrive independently on the lines. Show that the time of arrival of the first call is exponential with parameter 5λ.
103. A radioactive source emits particles at the rate of 6 per minute in accordance with Poisson process. Each particle emitted has a probability of 1/3 of being rewarded. Find the probability that at least 5 particles are recorded in a 5 minute period. **[Ans.** 0.9707]
104. Patients arrive randomly and independently at a doctor's consulting room from 8 a.m. at an average rate of one in 5 minutes. The waiting room can hold 12 persons. What is the probability that the room will be full when the doctor arrives at 9 a.m.? **[Ans.** 0.1144]

105. Passengers arrive at a terminal after 9 a.m.. The time of their arrival are Poisson with mean density $\lambda = 1$ per minute. The time interval from 9 a.m. to the departure of the next bus is a random variable T. Find the mean number of passengers in this bus

(i) if T has an exponential density with mean 30 minutes, and
(ii) if T is uniform between 0 and 60 minutes.

[Ans. (i) 30, (ii) 30]

106. Find the first-order characteristic function of a Poisson process.

[Ans. $e^{-\lambda t}(1 - e^{i\omega})$]

107. Messages arrive at a telegraph office in accordance with the laws of a Poisson process with a mean rate of 3 messages per hour.

(i) What is the probability that no message will hence arrived during the morning hours, that is, between 8 a.m and 12 noon?
(ii) What is the distribution of the time at which the first afternoon message arrives?

[Ans. (i) e^{-12}, (ii) $1 - e^{-3(t-12)}$, $t \geq 12$]

108. Assume that a circuit has an IC whose time to failure is an exponentially distributed random variable with expected lifetime of 3 months. If there are 10 spare ICs and time from failure to replacement is zero, what is the probability that the circuit can be kept operational for at least 1 year?

[Ans. 0.9972]

109. Suppose that customers arrive at a counter independently from 2 different sources. Arrivals occur in accordance with a Poisson process with mean rate of 6 per hour from the first source and 4 per hour from the second source. Find the mean interval between any 2 successives arrivals.

[Ans. 6 minutes]

110. Assume that a device fails when a cumulative effect of k shocks occur. If the shocks occur according to a Poisson process with parameter λ, find the density function for the life T of the device.

[Hint: Refer to Worked Example]

$$\left[\textbf{Ans.}\quad f_T(t) = \frac{\lambda^k \cdot t^{k-1} e^{-\lambda t}}{\underline{|k-1}}, t > 0\right]$$

111. Passengers arrive at a terminal for boarding the next bus. The times of their arrival are Poisson with an average arrival rate of 1 per minute. The times of departure of each bus are Poisson with an average departure rate of 2 per hour. Assume that the capacity of the bus is large. Find the average number of passengers in

(i) each bus, and
(ii) the first bus that leaves after 9 a.m.

[Ans. (i) 30, (ii) 60]

Bernoulli, Binomial, Sine Wave and Ergodic Random Processes

112. Define ensemble average and time average of a random process $\{X(t)\}$.
113. What is the difference between ensemble average and time average of a stochastic process $\{X(t)\}$?
114. When is a random process said to be ergodic? Give an example for an ergodic process.
115. Distinguish between stationarity and ergodicity.
116. What do you mean by mean ergodicity of a random process?
117. State mean ergodic theorem.
118. State the sufficient conditions for the mean ergodicity of a random process $\{X(t)\}$.
119. State two different sufficient conditions for $\{X(t)\}$ with constant mean to be mean ergodic.
120. Examine if the process $\{X(t)\}$, where $X(t) = X$, a random variable is mean ergodic.
121. Give an example of a WSS process which is not mean ergodic.
 [**Ans.** Random binary transmission process]
122. If $\bar{X}_T$ is the time-average of a stationary random process $\{X(t)\}$ over $(-T, T)$, express Var$(\bar{X}_T)$ in terms of the autocovariance function of $\{X(t)\}$ and, hence, state the sufficient condition for the mean ergodicity of $\{X(t)\}$.
123. When is a random process said to be correlation ergodic?
124. When is a random process said to be distribution ergodic?
125. If $\{X(t)\}$ is a random telegraph signal process with $E[X(t)] = 0$ and $R(\tau) = e^{-2\lambda|\tau|}$, find the mean variance of the time average of $\{X(t)\}$ over $(-T, T)$. Is it mean ergodic? [**Ans.** Yes]
126. If $X(t) = A$, where A is a random variable, prove that $\{X(t)\}$ is not mean ergodic.
127. If $S = \int_0^{10} X(t)\,dt$, show that $E(S^2) = \int_{-10}^{10} (10-|\tau|)\, R_{XX}(\tau)\,d\tau$, find also the mean and variance of S, if $E\{X(t)\} = 8$ and $R_{XX}(\tau) = 64 + 10e^{-2|\tau|}$.
 [**Ans.** 80, $5(e^{-20} + 19)$]
128. A stationary zero mean random process $\{X(t)\}$ has the autocorrelation function $R_{XX}(\tau) = 10e^{-0.1\tau^2}$ find the mean and variance of $\bar{X}_T = \frac{1}{5}\int_0^5 X(t)dt$.

$$\left[\textbf{Ans. } \sqrt{10},\ 4\int_4^5 e^{-x^2/10}dx + 4e^{-2.5} - 14\right]$$

129. If $\{X(t)\}$ is a WSS process with $E\{X(t)\} = 2$ and $R_{XX}(\tau) = 4 + e^{-|\tau|/10}$, find the mean and variance of $S = \int_0^1 X(t)dt$.

[**Ans.** 2, $20(10e^{-0.1} - 9)$]

130. If $\{X(t)\}$ is the random telegraph signal process with $E\{X(t)\} = 0$ and $R(\tau) = e^{-2\lambda|\tau|}$, find the mean and variance of the time average of $\{X(t)\}$ over $(-T, T)$. Is it mean ergodic?

$$\left[\textbf{Ans.}\ \ 0, \frac{1}{2\lambda T} - \frac{1}{8\lambda^2 T^2}(1 - e^{-4\lambda T}), \text{Yes}\right]$$

131. The random process $\{X(t)\}$ is stationary with $E\{X(t)\} = 1$ and $R(\tau) = 1 + e^{-2|\tau|}$, find the mean and variance of $S = \int_0^1 X(t)\,dt$.

$$\left[\textbf{Ans.}\ \ 1, \frac{1}{2}(1 + e^{-2})\right]$$

132. If the autocovariance function of a stationary process $\{X(t)\}$ is given by $C(\tau) = Ae^{-\alpha|\tau|}$, prove that $\{X(t)\}$ is mean ergodic. Also find $\text{Var}(\bar{X}_T)$, where $\bar{X}_T$ is the time average of $\{X(t)\}$ over $(-T, T)$

[**Hint:** $C(\tau) \to 0$ as $\tau \to \infty$]

$$\left[\textbf{Ans.}\ \ \frac{A}{\alpha T}\left[1 - \frac{(1 - e^{-2\alpha T})}{2\alpha T}\right]\right]$$

133. If the autocorrelation function of a WSS process $\{X(t)\}$ is $R(\tau)$, show that $P[|X(t + \tau) - X(t)| \geq a] \leq 2[R(0) - R(\tau)]/a^2$.

[**Hint:** Use Tchebycheff's inequality.]

134. Show that, if $\{X(t)\}$ is normal with $\mu_X = 0$ and $R_{XX}(\tau) = 0$ for $|\tau| > a$, then it is correlation-ergodic.

[**Hint:** $Z(t) = X(t + \lambda)\,X(t)$; $R_{ZZ}(\tau) = R^2(\lambda) + R^2(\tau) + R(\lambda + \tau)\,R(\lambda - \tau)$, since $R_{ZZ}(\tau) \to 0$ as $\tau \to \infty$, $C_{ZZ}(\tau) \to 0$ as $\tau \to \infty$.]

135. If the autocorrelation function of a stationary Gaussian process $\{X(t)\}$ is $R(\tau) = 10e^{-|\tau|}$, prove that $\{X(t)\}$ is ergodic both in mean and correlation.

136. In the fair coin experiment, we define the process $\{X(t)\}$ as follows: $X(t) = \sin \pi t$ if head shows and $= 2t$ if tail shows.

Find

(i) $E\{X(t)\}$ and

(ii) $F(X, t)$ for $t = 0.25$.

$$\left[\begin{array}{l}\textbf{Ans.} \quad \text{(i) } E\{X(t)\} = t + \dfrac{\sin \pi t}{2}, \\ \quad \text{(ii) } F(X,t) = 0 \text{ if } X < \dfrac{1}{2}; = \dfrac{1}{2} \text{ if } \dfrac{1}{2} < X < \dfrac{1}{\sqrt{2}} \\ \quad \text{and} = 1 \text{ if } X \geq \dfrac{1}{\sqrt{2}} \text{ when } t = 0.25\end{array}\right]$$

137. Suppose that $X(t)$ is a process with mean $\mu(t) = 3$ and autocorrelation $R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$. Determine the mean, variance and the covariance of the random variables $Z = X(5)$ and $W = X(8)$.

[**Ans.** $E(Z) = E(W) = 3$, $V(Z) = V(W) = 4$, $\text{Cov}(t_1, t_2) = 2.195$]

Normal or Gaussian Process

138. Define a Gaussian process.
139. When is a random process said to be a normal?
140. Give the nth-order density of Gaussian process or the nth-order normal density function.
141. State the properties of a Gaussian process.
142. If a Gaussian process is WSS, prove that it is also SSS.
143. If the member functions of a Gaussian process are uncorrelated, prove that they are independent.
144. Define square law detector process.
145. Define full-wave linear detector process.
146. Define half-wave linear detector process.
147. Define hard limiter process.
148. Define a band pass process or band pass signal.
149. If $Z(t) = X \cos \omega_0 t + Y \sin \omega_0 t$, where X and Y are independent Gaussian random variables with zero mean and unit variance, and ω_0 is a constant, show that $\{Z(t)\}$ is a Gaussian random process.

    [**Ans.** $Z(t)$ is Gaussian random variable for all t and $\{Z(t)\}$ is a Gaussian process]
150. If $\{X(t)\}$ is Gaussian process with $\mu_X = 0$ and $R_{XX}(\tau) = 0$ for $|\tau| > a$, prove that it is correlation ergodic.
151. Given a normal process $\{X(t)\}$ with $\mu_X = 0$ and $R_X(\tau) = 4e^{-2|\tau|}$, find the PDF of the random variable $Z = X(t + 1)$. Also find $P(Z < 1)$.

    $$\left[\textbf{Ans.} \quad \frac{1}{2\sqrt{2\pi}} e^{-z^2/8}, -\infty < z < \infty; 0.6915\right]$$

152. It is given that $R_X(\tau) = e^{-|\tau|}$ for a certain stationary Gaussian random process $\{X(t)\}$. Find the joint PDF of the random variables $X(t)$, $X(t+1)$, $X(t+2)$.

$$\left[\textbf{Ans.}\quad f(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}\left(1-\frac{1}{e^2}\right)} \exp\left\{-\frac{1}{2\left(1-\frac{1}{e^2}\right)} \times \left[x_1^2 - \frac{2}{e}x_1x_2 + \left(1+\frac{1}{e^2}\right)x_2^2 - \frac{2}{e}x_2x_3 + x_3^2\right]\right\}\right]$$

153. Prove that the random variables $X_1, X_2, \ldots, X_n$ are jointly normal, if the sum $a_1X_1 + a_2X_2 + \cdots + a_nX_n$ is a normal random variable for any set of constants $a_1, a_2, \ldots, a_n$.

[**Hint:** From the given condition:
$Z = \omega_1X_1 + \omega_2X_2 + \cdots + \omega_nX_n$ is a normal random variable
$E(Z) = 0$ and $E(Z^2) = \sigma_z^2$ if $E(X_i) = 0$
$\phi(Z) = e^{-\frac{\sigma_z^2}{2}}$.]

154. Let $Y_n = \sum_{k=1}^{n} X_k$, where the X_k's are a set of independent random variables each normally distributed with mean μ and variance σ^2. Show that $\{Y_n;\ n = 1, 2, \ldots\}$ is a normal (Gaussian) process.

155. Show that, if the random variables X, Y, Z are jointly normal and independent in pairs, then they are independent.

[**Hint:** Prove that $f_{XYZ}(x, y, z) = f_X(x)\, f_Y(y)\, f_Z(z)$]

156. If $Z(t) = X\cos\omega_0 t + Y\sin\omega_0 t$, where X and Y are independent Gaussian random variables with zero mean and unit variance and ω_0 is a constant.

(i) Show that $\{Z(t)\}$ is a Gaussian random process.
(ii) Find the joint PDF of $Z(t_1)$ and $Z(t_2)$.
(iii) Is the process WSS?
(iv) Is the process SSS?
(v) Find $E\{Z(t_2)/Z(t_1)\}$, $t_2 > t_1$.

[**Ans.** (ii) If $(Z_1, Z_2) = \dfrac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\dfrac{1}{2(1-r^2)} Z_1^2 - 2rZ_1Z_2 + Z_2^2\right)$
where $r = \cos\omega(t_1 - t_2)$
(iii) yes, (iv) yes), (v) $R_{Z2}(\tau)$, $2(t_1)$, $\tau = (t_1 - t_2)$]

7

Correlation and Spectral Densities

7.1 AUTOCORRELATION FUNCTION

If $\{X(t)\}$ is a stationary process either in the strict sense or in the wide sense, $E[X(t)X(t + \tau)]$ represents the autocorrelation function (ACF) of the random process $\{X(t)\}$ denoted by $R_{XX}(\tau)$, i.e.

$$R(\tau) = E[X(t)X(t + \tau)]$$

Time Average Approach

$$R_{XX}(t, t + \tau) = \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, X(t + \tau)dt$$

Note: If $\{X(t)\}$ is a complex random process, then

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)\, X^*(t_2)] \\ &= E[X^*(t_1)\, X(t_2)] \end{aligned}$$

7.1.1 Properties of Autocorrelation Function

1. The autocorrelation function is an even function of τ.

Proof By definition,

$$R(\tau) = E[X(t)X(t + \tau)]$$

Put $t + \tau = t_1 \Rightarrow t = t_1 - \tau$

$$R(\tau) = E[X(t_1 - \tau)\, X(t_1)]$$

$\therefore$

$$= R(-\tau)$$

$\therefore$ The autocorrelation function is an even function of τ.

2. The mean square value of the random process $\{X(t)\}$ is $R_{XX}(0)$, i.e.

$$R_{XX}(0) = E[X^2(t)]$$

Proof By definition,

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

Put $\tau = 0$

$$R_{XX}(0) = E[X(t)X(t)] = E[X^2(t)]$$

Hence proved.

Note: If $R_{XX}(\tau)$ is given, to find the mean square value $E[X^2(t)]$ replace τ by 0 in $R_{XX}(\tau)$.

3. $R_{XX}(\tau)$ is maximum at $\tau = 0$.
In other words, the autocorrelation function (ACF) evaluated at the origin $\tau = 0$ will be its maximum magnitude and it will be equal to or greater than evaluated at any other point τ
i.e. If $\{X(t)\}$ is a stationary process then

$$|R_{XX}(\tau)| \le R_{XX}(0)$$

Proof Since $\{X(t)\}$ is a stationary process, $E[X(t)]$ is a constant and its autocorrelation is a function of time difference

$$R_{XX}(0) = E[X^2(t)] = E[X^2(t-\tau)] \tag{i}$$

We know that

$$E\{X(t) \pm X(t+\tau)^2] \ge 0$$

$$\Rightarrow \quad E[X^2(t) + X^2(t+\tau) \pm 2X(t)\, X(t+\tau)] \ge 0$$

$$\Rightarrow \quad E[X^2(t)] + E[X^2(t+\tau)] \pm 2E[X(t)\, X(t+\tau)] \ge 0$$

Using Eq. (i)

$$R_{XX}^{(0)} + R_{XX}^{(0)} \pm 2R_{XX}^{(\tau)} \ge 0$$

$$2R_{XX}^{(0)} \pm 2R_{XX}^{(\tau)} \ge 0$$

$$R_{XX}^{(0)} \pm R_{XX}^{(\tau)} \ge 0$$

$\Rightarrow$ Hence proved.

$$|R_{XX}^{(\tau)}| \le R_{XX}^{(0)}$$

4. If the autocorrelation function $R(\tau)$ of a real stationary process $\{X(t)\}$ is continuous at $\tau = 0$, then it is continuous at every other point.

Proof Given $R(\tau)$ is continuous at $\tau = 0$

$\therefore$

$$\lim_{h \to 0}[R(\tau + h) - R(\tau)] = 0$$

$$\lim_{h \to 0} \{E[X(t)\,X(t+\tau+h)] - E[X(t)\,X(t+\tau)]\} = 0$$

$$\lim_{h \to 0} \{E[X(t)][X(t+\tau+h) - X(t+\tau)]\} = 0$$

$$E\{X(t) \lim_{h \to 0} [X(t+\tau+h) - X(t+\tau)]\} = 0$$

$$\Rightarrow \qquad \lim_{h \to 0} [X(t+\tau+h) - X(t+\tau)] = 0 \qquad \text{(i)}$$

Now,
$$R(\tau+h) - R(\tau) = E\{X(t)\,X[t-(\tau+h)]\} - E[X(t)X(t-\tau)]$$
$$= E\{X(t)\,[X(t-(\tau+h)) - X(t-\tau)]\}$$

Taking the limit as $h \to 0$ on both sides,

$$\lim_{h \to 0} [R(\tau+h) - R(\tau)] = E\{X(t) \lim_{h \to 0} [X(t-\tau-h) - X(t-\tau)]\} \text{ using Eq. (i)}$$
$$= E[X(t) \times 0] = E(0) = 0$$

$$\therefore \qquad \lim_{h \to 0} R(\tau+h) = R(\tau)$$

$R(\tau)$ is continuous at all points τ.

5. If $R(\tau)$ is the autocorrelation function of a stationary process $\{X(t)\}$ with no periodic component, then $\lim_{\tau \to \infty} R(\tau) = (\mu_X)^2$, provided the limit exists.

Proof By definition,

$$R(\tau) = E[X(t)X(t+\tau)]$$

Let $X(t)$ and $X(t+\tau)$ are two sample functions of the stationary process $\{X(t)\}$ which are observed at a very long intervals of time. Since $X(t)$ is a stationary process with no periodic component, $X(t)$ and $X(t+\tau)$ are independent sample functions.

$$R(\tau) = E[X(t)]\,E[X(t+\tau)]$$

Taking limit $\tau \to \infty$ on both sides,

$$\lim_{\tau \to \infty} R(\tau) = \lim_{\tau \to \infty} E[X(t)]\,E[X(t+\tau)]$$

Since for a stationary process, expectation is a constant.

$$E[X(t)] = E[X(t+\tau)] = \mu$$

$$\lim_{\tau \to \infty} R(\tau) = \lim_{\tau \to \infty} \mu_X \mu_X = \mu_X^2$$

Note: If the autocorrelation function $R(\tau)$ is given, then the mean of the random process is $\mu_X = \sqrt{\lim_{\tau \to \infty} R(\tau)}$.

6. The autocorrelation function (ACF) of a random process $\{X(t)\}$ is a finite energy function

$$R_{XX}(t_1 - t_2) = \int_{-\infty}^{\infty} R_{XX}(\tau) d\tau < \infty$$

7. The autocorrelation function (ACF) of a random process $\{X(t)\}$ cannot have an arbitrary shape.

8. If the random process $\{X(t)\}$ has a periodic component, then the autocorrelation function will also have a periodic component with same period.

EXAMPLE 7.1 Find the mean, variance and root mean square value of the process whose ACF is given as follows:

(i) $R_{XX}(\tau) = 2 + 4e^{-2|\tau|}$

(ii) $R_{XX}(\tau) = \dfrac{25\tau^2 + 36}{6.25\tau^2 + 4}$ **[AU December '03]**

(iii) $R_{XX}(\tau) = 25 + \dfrac{4}{1 + 6\tau^2}$. **[AU December '08; '09]**

Solution We know that, mean of the random process with no periodic component is given by

$$\mu = E[X(t)] = \sqrt{\lim_{\tau \to \infty} R_{XX}(\tau)} \quad \text{(i)}$$

Root mean square value (RMS) = $E[X^2(t)] = R_{XX}(0)$

(i) $$R_{XX}(\tau) = 2 + 4e^{-2|\tau|}$$

$$\lim_{\tau \to \infty} R_{XX}(\tau) = 2 + 4 \lim_{\tau \to \infty} e^{-2|\tau|}$$

$$\mu_X^2 = 2 \quad [\because e^{-\infty} = 0]$$

$$\mu_X = E[X(t)] = \sqrt{\lim_{\tau \to \infty} R_{XX}(\tau)} = \sqrt{2} \quad \text{[from Eq. (i)]}$$

The RMS value of $X(t)$ is $E[X^2(t)] = R_{XX}(0)$

$$\therefore \quad R_{XX}(0) = 2 + 4e^0 = 2 + 4 = 6 = E[X^2(t)]$$

$$E[X^2(t)] = 6 = \text{RMS value}$$

$$\text{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2$$

$$= 6 - (\sqrt{2})^2 = 6 - 2 = 4$$

(ii) $$R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4} = \frac{25 + \dfrac{36}{\tau^2}}{6.25 + \dfrac{4}{\tau^2}}$$

$$\therefore \lim_{\tau\to\infty} R_{XX}(\tau) = \frac{25}{6.25} = 4$$

$$\mu = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)} = \sqrt{4} = 2$$

$$R_{XX}(0) = E[X^2(t)] = \frac{36}{4} = 9$$

$$\text{Var}[X(t)] = 9 - 2^2 = 9 - 4 = 5$$

(iii) $$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

$$\mu^2 = \lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{\tau\to\infty}\left(25 + \frac{4}{1+6\tau^2}\right) = 25 \Rightarrow \mu = E[X(t)] = 5$$

and $$E[X^2(t)] = R_{XX}(0) = 25 + \frac{4}{1+6\times 0} = 29$$

$$\text{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2 = 29 - 25 = 4$$

EXAMPLE 7.2 If there are two random processes $\{X(t)\}$ and $\{Y(t)\}$ such that $Z(t) = X(t) + Y(t)$, then prove that

$$R_{(X+Y)(X+Y)}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

Solution
$$\begin{aligned} R_{(X+Y)(X+Y)}(\tau) &= R_{ZZ}(\tau) = E[Z(t+\tau)\,Z(t)] \\ &= E\{[X(t+\tau) + Y(t+\tau)]\,[X(t) + Y(t)]\} \\ &= E[X(t+\tau)X(t) + X(t+\tau)Y(t) + Y(t+\tau)X(t) \\ &\qquad + Y(t+\tau)Y(t)] \\ &= E[X(t+\tau)X(t)] + E[X(t+\tau)Y(t)] \\ &\qquad + E[Y(t+\tau)X(t)] + E[Y(t+\tau)Y(t)] \\ R_{ZZ}(\tau) &= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau) \end{aligned}$$

Hence proved.

EXAMPLE 7.3 If $\{X(t)\}$ is a complex WSS random process, then prove that $E[|X(t+\tau) - X(t)|^2] = 2\text{Re}[R(0) - R(\tau)]$.

Solution Given: $\{X(t)\}$ is a complex WSS random process.
$\therefore$ Let $X(t) = Y(t) + iZ(t)$ where $Z(t)$ and $Y(t)$ are real random processes.
By definition,

$$R_{XX}(0) = E[X^*(t)\,X(t)]$$

where $X^*(t)$ is a complex conjugate of $X(t)$

$$\begin{aligned} E[X^*(t)\,X(t)] &= E\{[Y(t) - iZ(t)]\,[Y(t) + iZ(t)]\} \\ &= E[Y^2(t) + Z^2(t)] \\ &= \text{Re}[R_{XX}(0)] \qquad \text{(i)} \end{aligned}$$

Since $\{X(t)\}$ is a WSS process, $E\{X(t)\}$ is a constant.

$$\therefore \quad E[X^*(t)\times X(t)] = E[X^*(t+\tau)\times X(t+\tau)] = \text{Re}[R_{XX}(0)] \qquad \text{(ii)}$$

$$E[X^*(t+\tau) \times X(t+\tau)] = E[Y^2(t+\tau) + Z^2(t+\tau)], \text{ from (i)}$$

$$\begin{aligned}
R_{XX}(\tau) &= E[X^*(t+\tau) \times X(t)] \\
&= E\{Y(t+\tau) - iZ(t+\tau)]\,[Y(t) + iZ(t)]\} \\
&= E[Y(t+\tau)Y(t) - iZ(t+\tau)Y(t) + iY(t+\tau)Z(t) \\
&\qquad + Z(t+\tau)Z(t)] \\
&= E[Y(t+\tau)Y(t) + Z(t+\tau)Z(t)] - iE[Z(t+\tau)Y(t) \\
&\qquad - Y(t+\tau)Z(t)]
\end{aligned}$$

$$\text{Re}[R_{XY}(\tau)] = E[Y(t+\tau)Y(t) + Z(t+\tau)Z(t)] \qquad \text{(iii)}$$

$$\begin{aligned}
E[|X(t+\tau) - X(t)|^2] &= E[|Y(t+\tau) + iZ(t+\tau) - Y(t) - iZ(t)|^2] \\
&= E\{[|Y(t+\tau) - Y(t)]^2 + i[Z(t+\tau) - Z(t)]|^2\} \\
&= E\{[Y(t+\tau) - Y(t)]^2 + [(Z(t+\tau) - Z(t)]^2\} \\
&\qquad [\because |x+iy|^2 = x^2 + y^2] \\
&= E\{[Y^2(t+\tau) + Y^2(t) - 2Y(t+\tau)Y(t)] \\
&\qquad + [Z^2(t+\tau) + Z^2(t) - 2Z(t+\tau)Z(t)]\} \\
&= E[Y^2(t+\tau)] + E[Y^2(t)] - 2E[Y(t+\tau)Y(t)] \\
&\qquad + E[Z^2(t+\tau)] + E[Z^2(t)] - 2E[Z(t+\tau)Z(t)] \\
&= E[Y^2(t+\tau) + Z^2(t+\tau)] + E[Y^2(t) + Z^2(t)] \\
&\qquad - 2E[Y(t+\tau)Y(t) + Z(t+\tau)Z(t)]
\end{aligned}$$

Using Eqs. (ii) and (iii), we get

$$\begin{aligned}
E[|X(t+\tau) - X(t)^2] &= \text{Re}[R(0)] + \text{Re}[R(0)] - 2\text{Re}[R(\tau)] \\
&= 2\text{Re}[R(0) - R(\tau)]
\end{aligned}$$

Hence proved.

7.2 CROSS-CORRELATION FUNCTION

The cross-correlation of the random processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

or

$$R_{YX}(\tau) = E[Y(t)X(t+\tau)]$$

7.2.1 Properties of Cross-correlation Function

1. Cross-correlation is not an even function in general but it is symmetric, i.e. $R_{XY}(-\tau) = R_{YX}(\tau)$.

Proof By definition,

$$R_{XY}(-\tau) = E[X(t)Y(t-\tau)]$$

Let $t - \tau = t_1$, then $t = t_1 + \tau$

$$\begin{aligned}
R_{XY}(-\tau) &= E[X(t_1+\tau)Y(t_1+\tau-\tau)] \\
&= E[X(t_1+\tau)Y(t_1)] = E[Y(t_1)X(t_1+\tau)] \\
&= R_{YX}(\tau)
\end{aligned}$$

Hence proved.

2. If $X(t)$ and $Y(t)$ are wide sense stationary random processes, then

(i) $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\, R_{YY}(0)}$

(ii) $|R_{XY}(\tau)| \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$. **[AU May '03]**

Proof To prove this result we know that,

(i) $R_{XX}(0) = E[X^2(t)] \ge 0$ and $R_{YY}(0) = E[Y^2(t)] \ge 0$

$$E\left[\left(\frac{X(t)}{\sqrt{R_{XX}(0)}} - \frac{Y(t+\tau)}{\sqrt{R_{YY}(0)}}\right)^2\right] = E\left[\frac{X^2(t)}{R_{XX}(0)} + \frac{Y^2(t)}{R_{YY}(0)} - \frac{2X(t)Y(t+\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}}\right] \ge 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)}\right] + E\left[\frac{Y^2(t)}{R_{YY}(0)}\right] - 2E\left[\frac{X(t)\,Y(t+\tau)}{\sqrt{R_{XX}(0)\,R_{YY}(0)}}\right] \ge 0$$

[$\because E[Y(t)]$ is a constant]

$$\frac{E[X^2(t)]}{R_{XX}(0)} + \frac{E[Y^2(t)]}{R_{YY}(0)} - \frac{2E[X(t)Y(t+\tau)]}{\sqrt{R_{XX}(0)R_{YY}(0)}} \ge 0$$

$$E[X^2(t)] = R_{XX}(0) \text{ gives}$$

We know that

$$1 + 1 - \frac{2E[X(t)\,Y(t+\tau)]}{\sqrt{R_{XX}(0)\,R_{YY}(0)}} \ge 0 \;\Rightarrow\; 2 - \frac{2E[X(t)Y(t+\tau)]}{\sqrt{R_{XX}(0)\,R_{YY}(0)}} \ge 0$$

$$\Rightarrow\; 1 \ge \frac{E[X(t)\,Y(t+\tau)]}{\sqrt{R_{XX}(0)\,R_{YY}(0)}}$$

$$\Rightarrow\; \sqrt{R_{XX}(0)\,R_{YY}(0)} \ge \{E[X(t)Y(t+\tau)]\}$$

$$E[X(t)\,Y(t+\tau)] \le \sqrt{R_{XX}(0)\,R_{YY}(0)}$$

$$R_{XY}(\tau) \le \sqrt{[R_{XX}(0)\,R_{YY}(0)]}$$

$$|R_{XY}(\tau)| \le [R_{XX}(0)R_{YY}(0)] \qquad (7.1)$$

(ii) We know that

$$E\{[X(t) \pm Y(t+\tau)]^2\} \ge 0$$

$$E[X^2(t) + Y^2(t+\tau) \pm 2X(t)Y(t+\tau)] \ge 0$$

$$E[X^2(t)] + E[Y^2(t+\tau)] \pm 2E[X(t)Y(t+\tau)] \ge 0$$

But $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$, $R_{XX}(0) = E[X^2(t)]$, $R_{YY}(0) = E[Y^2(t)]$

$\therefore\; R_{XX}(0) + R_{YY}(0) \pm 2R_{XY}(\tau) \ge 0$

$$\therefore\quad \frac{1}{2}[R_{XX}(0) + R_{YY}(0)] \ge |R_{XY}(\tau)|$$

$$\Rightarrow \qquad |R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)] \tag{7.2}$$

From Eqs. (7.1) and (7.2)

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) + R_{YY}(0)}$$

3. The cross-correlation function of two random processes $\{X(t)\}$ and $\{Y(t)\}$ does not have a maximum value at the origin.

4. If $\{X(t)\}$ and $\{Y(t)\}$ are two random processes, $R_{XX}(\tau)$ and $R_{YY}(\tau)$ are their respective autocorrelation functions, then $|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(\tau)\, R_{YY}(\tau)]$.

5. If the random processes $\{X(t)\}$ and $\{Y(t)\}$ are independent to each other, then their cross-correlation will be $R_{XY}(\tau) = \mu_X \mu_Y$ and $C_{XY}(\tau) = 0$.

6. If the two random processes $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal to each other, then $R_{XY}(\tau) = 0$.

7. If $\{X(t)\}$ and $\{Y(t)\}$ are two random processes with zero mean, then $\lim_{|\tau| \to \infty} R_{XY}(\tau) = 0$ and $\lim_{|\tau| \to \infty} R_{YX}(\tau) = 0$.

8. If $\{X(t)\}$ and $\{Y(t)\}$ are two independent random processes, then $R_{XY}(\tau) = R_{YX}(\tau)$.

EXAMPLE 7.4 If $\{X(t)\}$ and $\{Y(t)\}$ are two WSS random processes and $E[|X(0) - Y(0)|^2] = 0$ then prove that $R_{XX}(\tau) = R_{XY}(\tau) = R_{YY}(\tau)$.

Solution Let $\quad Z(t) = X(t + \tau)$

$\qquad\qquad W(t) = X(t) - Y(t)$

By Cauchy–Schwartz inequality,

$$E[(XY)^2] \leq E(X^2)E(Y^2)$$

$$E[|ZW|^2] \leq E[|Z|^2]E[|W|^2]$$

$$E\{|X(t + \tau)[X(t) - Y(t)]|^2\} \leq E[|X(t + \tau)|^2]\; E[|X(t) - Y(t)|^2]$$

$$E\{|X(t + \tau)\, X(t)|^2\} - E\{|X(t + \tau)\, Y(t)|^2\} \leq R_{XX}(0)\; E[|X(t) - Y(t)|^2]$$

$$R_{XX}^2(\tau) - R_{XY}^2(\tau) \leq R_{XX}(0)\; E[|X(t) - Y(t)|^2]$$

Now, let $t = 0$

$$\therefore\; R_{XX}^2(\tau) - R_{XY}^2(\tau) \leq R_{XX}(0) \times E[|X(0) - Y(0)|^2] \quad \Rightarrow \quad R_{XX}^2(\tau) - R_{XY}^2(\tau) = 0$$

$$R_{XX}^2(\tau) = R_{XY}^2(\tau) \quad \Rightarrow \quad R_{XX}(\tau) = R_{XY}(\tau)$$

Similarly, it can be proved that $R_{YY}(\tau) = R_{XY}(\tau)$.

EXAMPLE 7.5 Consider two random processes $X(t) = 3\cos(\omega t + \theta)$ and $Y(t) = 2\cos(\omega t + \phi)$ where $\phi = \theta - \frac{\pi}{2}$ and θ is uniformly distributed random variable over $(0, 2\pi)$. Verify that $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)\, R_{YY}(0)}$.

[AU May '03; '06, June '09]

Solution Since θ is uniformly distributed random variable over $(0, 2\pi)$, we have

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

By definition,

$$\begin{aligned} R_{XX}(\tau) &= E[X(t)X(t+\tau)] \\ &= E\{9\cos(\omega t+\theta)\cos[\omega(t+\tau)+\theta]\} \\ &= \frac{9}{2}E\{\cos(\omega t+\theta)\cos[\omega(t+\tau)+\theta]\} \\ &= \frac{9}{2}E(\cos\omega\tau) + \frac{9}{2}E[\cos(2\omega t+\omega\tau+2\theta)] \\ &= \frac{9}{2}E(\cos\omega\tau) + \frac{9}{4\pi}\int_0^{2\pi}\cos(2\omega t+\omega\tau+2\theta)d\theta \end{aligned} \quad \text{(i)}$$

$$\therefore \quad R_{XX}(\tau) = \frac{9}{2}\cos\omega\tau + 0$$

Given:

$$Y(t) = 2\cos(\omega t+\phi) = 2\cos\left(\omega t+\theta-\frac{\pi}{2}\right) = 2\sin(\omega t+\theta)$$

$$\begin{aligned} R_{YY}(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E\{4\sin(\omega t+\theta)\sin[\omega(t+\tau)+\theta]\} \\ &= 4E[\sin(\omega t+\theta)\sin[\omega(t+\tau)+\theta] \\ &= \frac{4}{2}E[\cos(\omega t+\theta-\omega t-\omega\tau-\theta)-\cos(2\omega t+\omega\tau+2\theta)] \\ &= 2E(\cos\omega\tau) - 2E[\cos(2\omega t+\omega\tau+2\theta)] \\ &= 2E(\cos\omega\tau) - 0 = 2\cos\omega\tau, \quad \text{using Eq. (i)} \end{aligned}$$

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\ &= E\{6\cos(\omega t+\theta)\sin[\omega(t+\tau)+\theta]\} \\ &= 3E(\sin\omega\tau) + 3E[\sin(2\omega t+\omega\tau+2\theta)] \\ &= 3\sin\omega\tau + 0 = 3\sin\omega\tau \end{aligned}$$

$$\therefore \quad |R_{XY}(\tau)| = 3|\sin\omega\tau| \le 3 \qquad (\because |\sin\omega\tau| \le 1)$$

But $R_{XX}(0)R_{YY}(0) = \frac{9}{2}\times 2 = 9 \Rightarrow \sqrt{R_{XX}(0)\,R_{YY}(0)} = 3$

Hence $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,R_{YY}(0)}$

EXAMPLE 7.6 Two jointly WSS random processes have sample functions $X(t) = A\cos(\omega t+\theta)$ and $Y(t) = B\cos(\omega t+\theta+\phi)$ where θ is uniformly distributed random variable in $(0, 2\pi)$ and A, B, ω and ϕ are constants.

Find the cross-correlation function for $X(t)$ and $Y(t)$. Also find the value of ϕ, when $X(t)$ and $Y(t)$ are orthogonal. **[AU December '09]**

Solution By definition,

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$
$$= E[AB\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta+\phi)]$$
$$= \frac{AB}{2}E[2\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta+\phi)]$$
$$= \frac{AB}{2}E[\cos(\omega\tau+\phi)+\cos(2\omega t+\omega\tau+2\theta+\phi)]$$
$$= \frac{AB}{2}E[\cos(\omega\tau+\phi)]+\frac{AB}{4\pi}\int_0^{2\pi}\cos(2\omega t+\omega\tau+2\theta+\phi)d\theta$$
$$= \frac{AB}{2}\cos(\omega\tau+\phi)+\frac{AB}{4\pi}(0)$$
$$\therefore \quad R_{XY}(\tau) = \frac{AB}{2}\cos(\omega\tau+\phi)$$

Given: $X(t)$ and $Y(t)$ are orthogonal

$$\therefore \quad R_{XY}(\tau) = 0$$
$$\therefore \quad \cos(\omega\tau+\phi) = 0 = \cos\frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, \ldots$$
$$\Rightarrow \quad \omega\tau+\phi = (2n-1)\frac{\pi}{2}$$
$$\Rightarrow \quad \phi = (2n-1)\frac{\pi}{2}-\omega\tau$$

EXAMPLE 7.7 If $X(t) = 5\sin(\omega t+\phi)$, $Y(t) = 2\cos(\omega t+\theta)$ and ϕ is a random variable uniformly distributed in $(0, 2\pi)$ where ω is a constant and $\theta+\phi = \frac{\pi}{2}$, find $R_{XX}(\tau)$, $R_{YY}(\tau)$, $R_{XY}(\tau)$ and $R_{YX}(\tau)$. Verify two properties of autocorrelation function and cross-correlation function. **[AU May '06]**

Solution Given: $X(t) = 5\sin(\omega t+\phi)$, $\quad f(\phi) = \begin{cases} \frac{1}{2\pi}, & 0<\phi<2\pi \\ 0, & \text{otherwise} \end{cases}$

$$R_{XX}(t, t+\tau) = E\{5\sin(\omega t+\phi)\,5\sin[\omega(t+\tau)+\phi)]\}$$
$$= \frac{25}{2}E(\cos\omega\tau)-\frac{25}{2}E[\cos(2\omega t+\omega\tau+2\phi)]$$

$$= \frac{25}{2}\cos\omega\tau - \frac{25}{4\pi}\int_0^{2\pi}\cos(2\omega t + \omega\tau + 2\phi)d\phi$$

$$R_{XX}(\tau) = \frac{25}{2}\cos\omega\tau - 0 = \frac{25}{2}\cos\omega\tau$$

$\therefore$

$$Y(t) = 2\cos(\omega t + \theta) = 2\cos\left(\omega t + \frac{\pi}{2} - \phi\right) = -2\sin(\omega t - \phi)$$

$$R_{YY}(t, t + \tau) = E[Y(t)\ Y(t + \tau)]$$

$$= E[2\sin(\omega t - \phi)2\sin(\omega t + \omega\tau - \phi)]$$

$$= 2E(\cos\omega\tau) + 2E[\cos(2\omega t + \omega\tau - 2\phi)]$$

$$R_{YY}(\tau) = 2E(\cos\omega\tau) = 2\cos\omega\tau$$

$$R_{XY}(t, t + \tau) = E[X(t)\ Y(t + \tau)]$$

$$= E[5\sin(\omega t + \phi)\ 2\cos(\omega t + \omega\tau + \theta)]$$

$$= 5E\left[2\sin(\omega t + \phi)\cos\left(\omega t + \omega t + \frac{\pi}{2} - \phi\right)\right]$$

$$= 5E[-2\sin(\omega t + \phi)\sin(\omega t + \omega\tau - \phi)]$$

$$= 5E[\cos(2\omega t + \omega\tau) - \cos(\omega\tau - 2\phi)]$$

$$= 5\cos(2\omega t + \omega\tau) - \frac{5}{2\pi}\int_0^{2\pi}\cos(\omega\tau - 2\phi)\,d\phi$$

$$= 5\cos(2\omega t + \omega\tau) + \frac{5}{4\pi}[\sin(\omega\tau - 2\phi)]_0^{2\pi}$$

$$= 5\cos(2\omega t + \omega\tau) + \frac{5}{4\pi}(\sin\omega\tau - \sin\omega\tau)$$

$\therefore$ $$R_{XY}(\tau) = 5\cos(2\omega t + \omega\tau)$$

Similarly, $$R_{YX}(t, t + \tau) = E[Y(t)\ X(t + \tau)]$$

$$= 5E[-2\sin(\omega t - \phi)\sin(\omega t + \omega\tau + \phi)]$$

$$= 5E[\cos(2\omega t + \omega\tau) - \cos(\omega\tau + 2\phi)]$$

$$= 5\cos(2\omega t + \omega t) - \frac{5}{2\pi}\int_0^{2\pi}\cos(\omega\tau + 2\phi)\,d\theta$$

$\therefore$ $$R_{YX}(\tau) = 5\cos(2\omega t + \omega\tau) + 0$$

$$= \text{a function of time}$$

Therefore, $X(t)$ and $Y(t)$ are not jointly stationary.

Verification of two properties:

Autocorrelation function:

(i) $R_{XX}(\tau) = \frac{25}{2}\cos\omega\tau$ and $R_{XX}(-\tau) = \frac{25}{2}\cos\omega(-\tau) = \frac{25}{2}\cos\omega\tau$

$R_{XX}(\tau)$ is an even function of τ.

Since $|\cos\omega\tau| \le 1$

(ii) $|R_{XX}(\tau)| = \left|\frac{25}{2}\cos\omega\tau\right| \le \frac{25}{2}$ and $R_{XX}(0) = \frac{25}{2}$

$\therefore \quad |R(\tau)| \le R(0)$

Similarly, $R_{YY}(\tau)$ is also an even function of τ.

Cross-correlation function:

$$|R_{XY}(\tau)| = 5|\cos(2\omega t + \omega\tau)| \le 5 \quad [\because |\cos(2\omega t + \omega\tau)| \le 1]$$

$$\sqrt{R_{XX}(0)\,R_{YY}(0)} = \sqrt{\frac{25}{2}\times 2} = 5$$

$$\therefore \quad |R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,R_{YY}(0)},$$

Also, $\frac{1}{2}[R_{XX}(0) + R_{YY}(0)] = \frac{1}{2}\left(\frac{25}{2}+2\right) = \frac{29}{4} = 7.25$

But $\quad |R_{XY}(\tau)| \le 5$

$$\therefore \quad |R_{XY}(\tau)| \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

7.3 POWER SPECTRAL DENSITY FUNCTION

The autocorrelation function $R(\tau)$ gives the information about how rapidly random signal $X(t)$ can be expected to change as a function of time. If the ACF decays rapidly (slowly), it indicates that the process can be expected to change rapidly (slowly). If the ACF has periodic components, then the corrsponding process will also have periodic components. Therefore, the ACF contains the information about the expected frequency content of the random process. Then, $S(\omega)$ which is called the *power spectral density (PSD) function* or *spectral density*, or *power spectrum of the stationary process* $\{X(t)\}$ gives the distribution of power of $\{X(t)\}$ as a function of frequency.

If $\{X(t)\}$ is a stationary process (either SSS or WSS) with ACF $R_{XX}(\tau)$, then the power spectral density of $\{X(t)\}$ is the Fourier transform of $R_{XX}(\tau)$, i.e.

$$S_{XX}(\omega) = S(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}\,d\tau$$

If the spectral density $S_{XX}(\omega)$ is given, then the ACF is

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\,d\omega$$

The power spectral density of the random process $\{X(t)\}$ is also defined by

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E(|X_T(\omega)|^2)}{2T}$$

where

$$X_T(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-i\omega t} dt = \int_{-T}^{T} X(t)\, e^{-i\omega t} dt$$

and

$$X_T(t) = \begin{cases} X(t), -T < t < T \\ 0, \quad \text{elsewhere} \end{cases}$$

The average power of the random process $\{X(t)\}$ is defined by

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

It is always given by the time average of its second moment

$$P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]dt$$

7.4 CROSS-SPECTRAL DENSITY FUNCTION†

The cross-spectral density of the random processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$$

and the cross-correlation is defined as

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega$$

7.5 WIENER–KHINCHINE THEOREM†

If $X_T(\omega)$ is the Fourier transform of the truncated random process defined as

$$X_T(t) = \begin{cases} X(t) \text{ for } |t| \le T \\ 0 \quad\;\; \text{for } |t| > T \end{cases}$$

where $\{X(t)\}$ is a real WSS process with power spectral density function $S(\omega)$, then

†AU May '06, December '07

$$S(\omega) = \lim_{T \to \infty} \left\{ \frac{1}{2T} E[|X_T(\omega)|^2] \right\}$$

Proof Given: $X_T(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-i\omega t} dt = \int_{-T}^{T} X(t) e^{-i\omega t} dt$

Since $\{X(t)\}$ is real

$$|X_T(\omega)|^2 = X_T(\omega)\, X_T(-\omega)$$

$$= \int_{-T}^{T} X(t_1) e^{-i\omega t_1} dt_1 \int_{-T}^{T} X(t_2) e^{i\omega t_2} dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} X(t_1)\, X(t_2) e^{-i\omega(t_1 - t_2)} dt_1 dt_2$$

$$\therefore E[|X_T(\omega)|^2] = \int_{-T}^{T}\int_{-T}^{T} E\{X(t_1) X(t_2)\} e^{-i\omega(t_1 - t_2)} dt_1 dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R(t_1 - t_2) e^{-i\omega(t_1 - t_2)}\, dt_1 dt_2 \text{ [since } \{X(t)\} \text{ is a WSS process]}$$

Taking $\phi(t_1 - t_2) = R(t_1 - t_2) e^{-i\omega(t_1 - t_2)}$, we get

$$E[|X_T(\omega)|^2] = \int_{-T}^{T}\int_{-T}^{T} \phi(t_1 - t_2)\, dt_1 dt_2, \text{ (say)} \tag{7.3}$$

The double integral [Eq. (7.3)] can be evaluated over the area of the square $ABCD$ bounded by $t_1 = -T$, T and $t_2 = -T$, T as shown in Figure 7.1.

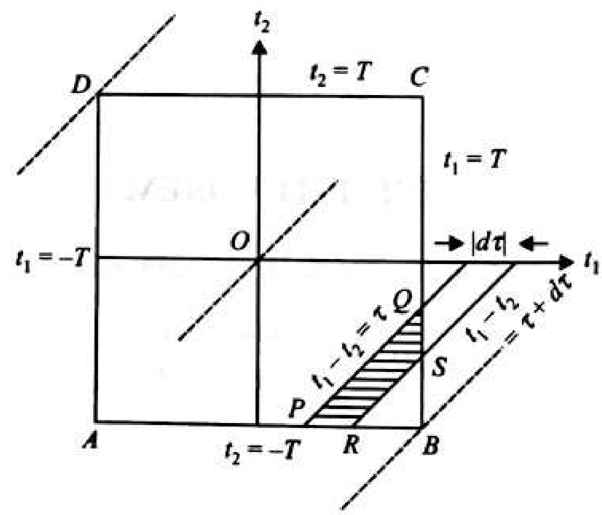

Figure 7.1 Area of the integral (7.3).

We divide the area of the square $ABCD$ into a number of strips like $PQRS$, where PQ is given by $t_1 - t_2 = \tau$ and RS is given by $t_1 - t_2 = \tau + d\tau$.

When $PQRS$ is at the initial position D, $t_1 - t_2 = -2T$, when it is at the final position B, $t_1 - t_2 = 2T$. Hence when τ varies from $-2T$ to $2T$, the area $ABCD$ is covered.

Now,
$$dt_1dt_2 = \text{elemental area of the } t_1t_2 \text{ plane}$$
$$= \text{area of } PQRS \tag{7.4}$$
$$(t_1)_P = \tau - T \text{ and } PB(=BQ) = T - (\tau - T)$$
$$= 2T - \tau, \text{ if } \tau > 0$$
$$= 2T + \tau, \text{ if } \tau < 0$$

When $\tau > 0$,
$$\text{Area of } PQRS = \Delta PBQ - \Delta RBS$$
$$= (2T - \tau)^2 - \frac{1}{2}(2T - \tau - d\tau)^2$$
$$= (2T - \tau)d\tau, \text{ omitting } (d\tau)^2 \tag{7.5}$$

From Eqs. (7.4) and (7.5),
$$dt_1dt_2 = (2T - |\tau|)d\tau \tag{7.6}$$

Using Eq. (7.6) in Eq. (7.3), we get
$$E[|X_T(\omega)|^2] = \int_{-2T}^{2T} \phi(\tau)(2T - |\tau|)d\tau$$
$$\therefore \quad \frac{1}{2T}E[|X_T(\omega)|^2] = \int_{-2T}^{2T} \phi(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau$$
$$\therefore \quad \lim_{T\to\infty}\left\{\frac{1}{2T}E[|X_T(\omega)|^2]\right\}$$
$$= \lim_{T\to\infty}\int_{-2T}^{2T}\phi(\tau)d\tau - \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}|\tau|\phi(\tau)d\tau$$
$$= \int_{-\infty}^{\infty}\phi(\tau)d\tau \qquad \left[\text{assuming that } \int_{-\infty}^{\infty}|\tau|\phi(\tau)d\tau \text{ is bounded}\right]$$
$$= \int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}\,d\tau \qquad \left[\text{provided } \int_{-\infty}^{\infty}|\tau|R(\tau)e^{-i\omega\tau}\,d\tau \text{ is bounded}\right]$$
$$= S(\omega), \text{ by definition of } S(\omega)$$

Note: This theorem provides an alternative method for finding PSD function, $S(\omega)$ of a WSS process.

Fourier Transforms of Some Important Functions

Table 7.1

	$X(\tau)$	$X(\omega) = F[X(\tau)]$
1.	$\alpha\delta(\tau)$	α
2.	$\frac{\alpha}{2\pi}(1)$	$\alpha\delta(\omega)$, $[F(\alpha) = 2\pi\alpha S(\omega)]$
3.	$u(\tau)$	$\pi\delta(\omega) + \frac{1}{i\omega}$
4.	$e^{-i\omega_0\tau}$	$2\pi\delta(\omega + \omega_0)$
5.	$e^{i\omega_0\tau}$	$2\pi\delta(\omega - \omega_0)$
6.	$\cos\omega_0\tau$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
7.	$\sin\omega_0\tau$	$-\pi i[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
8.	$e^{-\alpha\tau}u(\tau)$, $\alpha > 0$	$\frac{1}{\alpha + i\omega}$
9.	$\tau e^{-\alpha\tau}u(\tau)$, $\alpha > 0$	$\frac{1}{(\alpha + i\omega)^2}$
10.	$\tau^2 e^{-\alpha\tau}u(\tau)$, $\alpha > 0$	$\frac{2}{(\alpha + i\omega)^3}$
11.	$e^{-\alpha\lvert\tau\rvert}$, $\alpha > 0$	$\frac{2\alpha}{\alpha^2 + \omega^2}$
12.	$e^{-\alpha\tau^2}$	$\sqrt{\frac{\pi}{\alpha}}e^{\frac{-\omega^2}{4\alpha}}$

1. Find Fourier transform of $\cos p\tau$ and $\sin p\tau$.

Solution Take

$$F^{-1}[\delta(\omega + p) + \delta(\omega - p)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\delta(\omega + p) + \delta(\omega - p)]e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\delta(\omega + p)e^{i\omega\tau}d\omega + \int_{-\infty}^{\infty}\delta(\omega - p)e^{i\omega\tau}d\omega\right]$$

$$= \frac{1}{2\pi}\left[e^{i\tau(-p)} + e^{+i\tau p}\right]$$

$$= \frac{1}{\pi}\left(\frac{e^{i\tau p} + e^{-i\tau p}}{2}\right) = \frac{\cos p\tau}{\pi}$$

$$\pi F^{-1}[\delta(\omega + p) + \delta(\omega - p)] = \cos p\tau$$

$$F^{-1}\{\pi[\delta(\omega + p) + \delta(\omega - p)]\} = \cos p\tau$$

$$\therefore \quad F(\cos p\tau) = \pi[\delta(\omega + p) + \delta(\omega - p)]$$

: Take

$$F^{-1}[\delta(\omega+p)-\delta(\omega-p)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\delta(\omega+p)-\delta(\omega-p)]e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\delta(\omega+p)e^{i\omega\tau}d\omega - \int_{-\infty}^{\infty}\delta(\omega-p)e^{i\omega\tau}d\omega\right]$$

$$= \frac{1}{2\pi}\left[e^{i\tau(-p)} - e^{+i\tau p}\right] = \frac{-i}{\pi}\left(\frac{e^{i\tau p}-e^{-i\tau p}}{2i}\right)$$

$$= \frac{-i\sin p\tau}{\pi} = \frac{1}{\pi i}\sin p\tau$$

$$\pi i F^{-1}[\delta(\omega+p)-\delta(\omega-p)] = \sin p\tau$$

$$F^{-1}\{\pi i[\delta(\omega+p)-\delta(\omega-p)]\} = \sin p\tau$$

$$\therefore \quad F(\sin p\tau) = \pi i[\delta(\omega+p)-\delta(\omega-p)]$$

EXAMPLE 7.8 Find the autocorrelation function whose spectral density is given by

$$S(\omega) = \begin{cases}\pi, & |\omega| \le 1 \\ 0, & \text{otherwise}\end{cases}. \qquad \textbf{[AU April '03]}$$

Solution By definition,

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-1}^{1}\pi e^{i\omega\tau}d\omega$$

$$= \frac{\pi}{2\pi}\int_{-1}^{1}(\cos\omega\tau + i\sin\omega\tau)d\omega$$

$$= \frac{1}{2}\int_{-1}^{1}\cos\omega\tau\,d\omega + i\frac{1}{2}\int_{-1}^{1}\sin\omega\tau\,d\omega \qquad (\because \sin\omega\tau \text{ is an odd function})$$

$$= \frac{2}{2}\int_{0}^{1}\cos\omega\tau\,d\omega = \left[\frac{\sin\omega\tau}{\tau}\right]_0^1 = \frac{\sin\tau}{\tau}$$

EXAMPLE 7.9 Find the power density spectral of a stationary process $\{X(t)\}$ with $R_{XX}(\tau) = 6 + e^{-2|\tau|}$.

Solution

$$S_{XX}(\omega) = \int_{-\infty}^{\infty}R_{XX}(\tau)e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty}(6+e^{-2|\tau|})e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} 6e^{-i\omega\tau} d\tau + \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau = F(6) + F(e^{-2|\tau|})$$

$$= 2\pi(6)\delta(\omega) + \frac{2\times 2}{2^2+\omega^2} \Rightarrow 12\pi\delta(\omega) + \frac{4}{4+\omega^2}. \quad \text{[refer the Table 7.1]}$$

EXAMPLE 7.10 Find the power spectral denstiy of a stationary random process for which the ACF is $R_{XX}(\tau) = \sigma^2 e^{-\alpha|\tau|}$.

Solution By definition,

$$S_{XX}(\omega) = S(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$S(\omega) = \int_{-\infty}^{\infty} \sigma^2 e^{-\alpha|\tau|} e^{-i\omega\tau} d\tau = \sigma^2 \int_{-\infty}^{\infty} e^{-\alpha|\tau|} e^{-i\omega\tau} d\tau$$

$$= \sigma^2 F(e^{-\alpha|\tau|}) = \sigma^2 \frac{2\alpha}{\alpha^2+\omega^2} = \frac{2\sigma^2\alpha}{\alpha^2+\omega^2}$$

Another method:

$$S(\omega) = \sigma^2 \int_{-\infty}^{\infty} e^{-\alpha|\tau|}(\cos\omega\tau - i\sin\omega\tau) d\tau$$

$$= \sigma^2 \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos\omega\tau\, d\tau - \sigma^2 \int_{-\infty}^{\infty} i\sin\omega\tau e^{-\alpha|\tau|} d\tau$$

$$= \sigma^2 2\int_{0}^{\infty} e^{-\alpha\tau} \cos\omega\tau\, d\tau - i(0)$$

$$(\because \sin\omega\tau\, e^{-\alpha|\tau|} \text{ is an odd function})$$

$$= 2\sigma^2 \left[\frac{e^{-\alpha\tau}}{\alpha^2+\omega^2}(-\alpha\cos\omega\tau - \omega\sin\omega\tau)\right]_0^{\infty}$$

$$= 2\sigma^2 \left\{e^{-\infty} - \frac{e^0}{\alpha^2+\omega^2}[(-\alpha) - \omega(0)]\right\}$$

$$= \frac{2\sigma^2\alpha}{\alpha^2+\omega^2}$$

EXAMPLE 7.11 A wide sense stationary noise process $N(t)$ has an ACF $R_{NN}(\tau) = \rho e^{-3|\tau|}$ where τ is a constant. Find its power spectrum.

[AU December '07]

Solution Given: $R_{NN}(\tau) = \rho\, e^{-3|\tau|}$

$$S_{NN}(\tau) = \int_{-\infty}^{\infty} R_{NN}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} \rho e^{-3|\tau|} e^{-i\omega\tau} d\tau$$

$$= \rho \int_{-\infty}^{\infty} e^{-3|\tau|} e^{-i\omega\tau} d\tau = \rho F(e^{-3|\tau|})$$

$$= \rho \frac{2 \times 3}{3^2 + \omega^2} = \frac{6\rho}{9 + \omega^2}$$

EXAMPLE 7.12 Find the power spectral density of the random process $\{X(t)\}$ if $E[X(t)] = 1$ and $R_{XX}(\tau) = 1 + e^{-\alpha|\tau|}$. **[AU December '06; '09]**

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} (1 + e^{-\alpha|\tau|}) e^{-i\omega\tau} d\tau$$

$$= F(1 + e^{-\alpha|\tau|}) = F(1) + F(e^{-\alpha|\tau|})$$

$$= 2\pi\delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}$$

EXAMPLE 7.13 Find the spectral density of the random process $\{X(t)\}$ whose autocorrelaion is

$$R(\tau) = \begin{cases} -1, & -3 < \tau < 3 \\ 0, & \text{otherwise} \end{cases}.$$

Solution

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau = \int_{-3}^{3} (-1) e^{-i\omega\tau} d\tau$$

$$= -\int_{-3}^{3} (\cos \omega\tau - i \sin \omega\tau) d\tau = -2\int_{0}^{3} \cos \omega\tau \, d\tau + 0$$

$$= -2\left[\frac{\sin \omega\tau}{\omega}\right]_0^3 = \frac{-2 \sin 3\omega}{\omega}$$

3rd

EXAMPLE 7.14 If the autocorrelation function of a WSS process is $R(\tau) = \rho e^{-\rho|\tau|}$, show that its spectral density is

$$S(\omega) = \frac{2}{1 + \left(\frac{\omega}{\rho}\right)^2}.$$

[AU December '05]

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \rho e^{-\rho|\tau|} e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} \rho e^{-\rho|\tau|} (\cos \omega\tau - i \sin \omega\tau) d\tau$$

$$= \rho \int_{-\infty}^{\infty} e^{-\rho|\tau|} (\cos \omega\tau) d\tau - (i\rho) \int_{-\infty}^{\infty} e^{-\rho|\tau|} (\sin \omega\tau) d\tau$$

$$= \rho \int_{-\infty}^{\infty} e^{-\rho|\tau|} (\cos \omega\tau) d\tau + 0 = 2\rho \int_{0}^{\infty} e^{-\rho\tau} (\cos \omega\tau) d\tau$$

$$= 2\rho \left[\left(\frac{e^{-\rho\tau}}{\rho^2 + \omega^2} \right) (-\rho \cos \omega\tau + \omega \sin \omega\tau) \right]_0^{\infty} \qquad (\because \; e^{-\infty} = 0)$$

$$= 2\rho \left[\left(\frac{-1}{\rho^2 + \omega^2} \right) (-\rho) \right] = 2\rho \left(\frac{\rho}{\rho^2 + \omega^2} \right)$$

$$S(\omega) = \frac{2\rho^2}{\rho^2 \left[1 + \left(\frac{\omega}{\rho} \right)^2 \right]} = \frac{2}{1 + \left(\frac{\omega}{\rho} \right)^2}$$

Aliter

$$S(\omega) = F[R(\tau)] = F(\rho e^{-\rho|\tau|}) = \rho F(e^{-\rho|\tau|})$$

$$= \rho \left(\frac{2\rho}{\rho^2 + \omega^2} \right) = \frac{2\rho^2}{\rho^2 + \omega^2} = \frac{2\rho^2}{\rho^2 \left[1 + \left(\frac{\omega}{\rho} \right)^2 \right]}, \qquad \text{[refer the table 7.1]}$$

$$= \frac{2}{1 + \left(\frac{\omega}{\rho} \right)^2}$$

EXAMPLE 7.15 Find the autocorrelation function of the random process $\{X(t)\}$ for which the spectral density is given by

$$S(\omega) = \begin{cases} A, & |\omega| \le \beta \\ 0, & |\omega| > \beta \end{cases}.$$

Solution By definition,

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-\beta}^{\beta} A e^{i\omega\tau} d\omega = \frac{A}{2\pi} \int_{-\beta}^{\beta} (\cos \omega\tau + i \sin \omega\tau) d\omega$$

$$= \frac{2A}{2\pi}\int_0^{\beta} \cos\omega\tau\, d\omega + i\cdot 0 = \frac{A}{\pi}\left[\frac{\sin\omega\tau}{\tau}\right]_0^{\beta}$$

$$= \frac{A}{\pi}\left(\frac{\sin\beta\tau}{\tau}\right)$$

EXAMPLE 7.16 If $\{X(t)\}$ is a constant random process with $R(\tau) = m^2, \forall\tau$ where m is a constant, show that the spectral density of the process is $S(\omega) = 2\pi m^2\,\delta(\omega)$.

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\,d\tau = \int_{-\infty}^{\infty} m^2 e^{-i\omega\tau}\,d\tau$$

$$= m^2\int_{-\infty}^{\infty} e^{-i\omega\tau}\,d\tau = m^2 F(1) = m^2\, 2\pi\delta(\omega) = 2\pi m^2\,\delta(\omega),\quad \text{[refer Table 7.1]}$$

EXAMPLE 7.17 Find the spectral density function of a WSS random process $\{X(t)\}$ whose autocorrelation function is given as

$$R(\tau) = \begin{cases} \pi, & -\omega_0 < \tau < \omega_0 \\ 0, & \text{elsewhere} \end{cases}.$$

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\,d\tau$$

$$= \pi\int_{-\omega_0}^{\omega_0} (\cos\omega\tau - i\sin\omega\tau)\,d\tau$$

$$= \pi\int_{-\omega_0}^{\omega_0} \cos\omega\tau\,d\tau - i\pi\int_{-\omega_0}^{\omega_0} \sin\omega\tau\,d\tau = 2\pi\int_0^{\omega_0} \cos\omega\tau\,d\tau - 0$$

$$= 2\pi\left[\frac{\sin\omega\tau}{\omega}\right]_0^{\omega_0} = \frac{2\pi}{\omega}\sin(\omega\omega_0) = \frac{2\pi\sin(\omega\omega_0)}{\omega}$$

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EXAMPLE 7.18 Find the power spectral density of the random process, if its autocorrelation function is given by $R_{XX}(\tau) = e^{-\alpha|\tau|}\cos\beta\tau$. **[AU December '06]**

Solution Given: $R_{XX}(\tau) = e^{-\alpha|\tau|} \cos \beta\tau$

Then $S_{XX}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R_{XX}(\tau) d\tau$

$$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos \beta\tau (\cos \omega\tau - i \sin \omega\tau) d\tau$$

$$= 2\int_{0}^{\infty} e^{-\alpha\tau} \cos \omega\tau \cos \beta\tau d\tau$$

$$= \int_{0}^{\infty} e^{-\alpha\tau} \cos(\beta+\omega)\tau d\tau + \int_{0}^{\infty} e^{-\alpha\tau} \cos(\beta-\omega)\tau d\omega$$

$$= \left[\frac{e^{-\alpha\tau}}{\alpha^2+(\beta+\omega)^2}[-\alpha\cos(\beta+\omega)\tau + (\beta+\omega)\sin(\beta+\omega)\tau]\right]_0^{\infty}$$

$$+ \left[\frac{e^{-\alpha\tau}}{\alpha^2+(\beta-\omega)^2}[-\alpha\cos(\beta-\omega)\tau + (\beta-\omega)\sin(\beta-\omega)\tau]\right]_0^{\infty}$$

$$= \frac{\alpha}{(\beta+\omega)^2 + T\alpha^2} + \frac{\alpha}{(\beta-\omega)^2+\alpha^2} \qquad \text{[refer Table 7.1]}$$

Aliter

We know that

$$\cos \beta\tau = \frac{e^{i\beta\tau} + e^{-i\beta\tau}}{2} \qquad \therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R_{XX}(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau} e^{-\alpha|\tau|}\left(\frac{e^{i\beta\tau}+e^{-i\beta\tau}}{2}\right) d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-\alpha|\tau|} e^{-i(\omega-\beta)\tau} d\tau + \frac{1}{2}\int_{-\infty}^{\infty} e^{-\alpha|\tau|} e^{-i(\omega+\beta)\tau} d\tau$$

$$= \frac{1}{2}[F(e^{-\alpha|\tau|})]_{\omega \to \omega-\beta} + \frac{1}{2}[F(e^{-\alpha|\tau|})]_{\omega\to\omega+\beta}$$

$$= \frac{1}{2}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right]_{\omega\to\omega-\beta} + \frac{1}{2}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right]_{\omega\to\omega+\beta}$$

$$= \frac{\alpha}{\alpha^2+(\omega-\beta)^2} + \frac{\alpha}{\alpha^2+(\omega+\beta)^2}$$

EXAMPLE 7.19 The autocorrelation function of the random telegraph signal is given by $R(\tau) = a^2e^{-2\gamma|\tau|}$. Determine the spectral density function of the random telegraph signal.

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$4\gamma^2+\omega^2$

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty} a^2e^{-2\gamma|\tau|}e^{-i\omega\tau}d\tau$$

$$= a^2F(e^{-2\gamma|\tau|})$$

$$= a^2\frac{2(2\gamma)}{4\gamma^2+\omega^2} = \frac{4a^2\gamma}{4\gamma^2+\omega^2}$$

Aliter

By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} a^2e^{-2\gamma|\tau|}e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty} a^2e^{-2\gamma|\tau|}(\cos\omega\tau - i\sin\omega\tau)d\tau$$

$$= a^2\int_{-\infty}^{\infty} e^{-2\gamma|\tau|}(\cos\omega\tau)d\tau - ia^2\int_{-\infty}^{\infty} e^{-2\gamma|\tau|}(\sin\omega\tau)d\tau$$

$$= 2a^2\int_{0}^{\infty} e^{-2\gamma\tau}(\cos\omega\tau)d\tau + 0$$

$$= 2a^2\left[\left(\frac{e^{-2\gamma\tau}}{(2\gamma)^2+\omega^2}\right)(-2\gamma\cos\omega\tau + \omega\sin\omega\tau)\right]_0^{\infty}$$

$$= 2a^2\left[\left(\frac{-1}{4\gamma^2+\omega^2}\right)(-2\gamma)\right] \qquad (\because\ e^{-\infty} = 0)$$

$$= 2a^2\left(\frac{2\gamma}{4\gamma^2+\omega^2}\right) = \left(\frac{4a^2\gamma}{4\gamma^2+\omega^2}\right)$$

Revise sml

EXAMPLE 7.20 Find the power spectral density of the random process if its autocorrelation function is $R(\tau) = e^{-\alpha\tau^2}\cos\omega_0\tau$.

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau^2} \cos\omega_0\tau \, e^{-i\omega\tau} d\tau$$

Using $\cos\omega_0\tau = \dfrac{e^{i\omega_0\tau} + e^{-i\omega_0\tau}}{2}$, we get

$$S(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-i(\omega-\omega_0)\tau} d\tau + \frac{1}{2}\int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-i(\omega+\omega_0)\tau} d\tau$$

$$= \frac{1}{2}\left[F(e^{-\alpha\tau^2})\right]_{\omega\to\omega-\omega_0} + \frac{1}{2}\left[F(e^{-\alpha\tau^2})\right]_{\omega\to\omega+\omega_0}$$

$$= \frac{1}{2}\left(\sqrt{\frac{\pi}{\alpha}} e^{\frac{-\omega^2}{4\alpha}}\right)_{\omega\to\omega-\omega_0} + \frac{1}{2}\left(\sqrt{\frac{\pi}{\alpha}} e^{\frac{-\omega^2}{4\alpha}}\right)_{\omega\to\omega+\omega_0}$$

$$= \frac{1}{2}\left[\sqrt{\frac{\pi}{\alpha}} e^{\frac{-(\omega-\omega_0)^2}{4\alpha}}\right] + \frac{1}{2}\left[\sqrt{\frac{\pi}{\alpha}} e^{\frac{-(\omega+\omega_0)^2}{4\alpha}}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\left[e^{\frac{-(\omega-\omega_0)^2}{4\alpha}} + e^{\frac{-(\omega+\omega_0)^2}{4\alpha}}\right]$$

EXAMPLE 7.21 Find the spectral density of a WSS random process $\{X(t)\}$ whose ACF is $e^{\frac{-\alpha^2\tau^2}{2}}$.

Solution By definition,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{\frac{-\alpha^2\tau^2}{2}} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{\frac{-1}{2}(\alpha^2\tau^2 + 2i\omega\tau)} d\tau \qquad \begin{bmatrix} a^2 = \alpha^2\tau^2 & ab = i\omega\tau \\ 2ab = 2i\omega\tau & b = \dfrac{i\omega}{\alpha} \end{bmatrix}$$

$$= \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[\left(\alpha\tau + \frac{i\omega}{\alpha}\right)^2 - \left(\frac{i\omega}{\alpha}\right)^2\right]} d\tau = \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[\left(\alpha\tau + \frac{i\omega}{\alpha}\right)^2 + \left(\frac{\omega^2}{\alpha^2}\right)\right]} d\tau$$

$$= \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\alpha\tau + \frac{i\omega}{\alpha}\right)^2} e^{\left(\frac{-\omega^2}{2\alpha^2}\right)} d\tau = e^{\left(\frac{-\omega^2}{2\alpha^2}\right)} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\alpha\tau + \frac{i\omega}{\alpha}\right)^2} d\tau$$

Put $\alpha\tau + \dfrac{i\omega}{\alpha} = x \Rightarrow \alpha d\tau + 0 = dx$

$$\therefore \qquad d\tau = \frac{dx}{\alpha}$$

$$= e^{\frac{-\omega^2}{2\alpha^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x)^2} \frac{dx}{\alpha} = \frac{1}{\alpha} e^{\frac{-\omega^2}{2\alpha^2}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$$

$$= \frac{2}{\alpha} e^{\frac{-\omega^2}{2\alpha^2}} \int_{0}^{\infty} e^{\frac{-x^2}{2}} dx = \frac{2}{\alpha} e^{\frac{-\omega^2}{2\alpha^2}} \sqrt{\frac{\pi}{2}}$$

$$S_{XX}(\omega) = \frac{\sqrt{2\pi}\, e^{\frac{-\omega^2}{2\alpha^2}}}{\alpha}$$

Aliter

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{\frac{-\alpha^2\tau^2}{2}} e^{-i\omega\tau} d\tau$$

$$= F\left(e^{-\frac{\alpha^2}{2}\tau^2}\right)$$

$$= \sqrt{\frac{\pi}{\frac{\alpha^2}{2}}}\, e^{\frac{-\omega^2}{4\left(\frac{\alpha^2}{2}\right)}}, \qquad \text{[using Table 7.1]}$$

$$= \frac{\sqrt{2\pi}}{\alpha} e^{\frac{-\omega^2}{2\alpha^2}}$$

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EXAMPLE 7.32 A random process $\{X(t)\}$ has the autocorrelation function $R(\tau) = e^{-\alpha|\tau|}(1 + \alpha|\tau|)$. Find the power spectrum of the random process $\{X(t)\}$.

[AU May '08]

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-\alpha|\tau|}(1 + \alpha|\tau|)\, e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} (1+\alpha|\tau|)e^{-\alpha|\tau|}(\cos\omega\tau - i\sin\omega\tau)\,d\tau$$

$$= 2\int_0^{\infty} (1+\alpha\tau)e^{-\alpha\tau}(\cos\omega\tau)\,d\tau + 0$$

$$= 2\int_0^{\infty} (1+\alpha\tau)e^{-\alpha\tau}\,(\text{RP of } e^{i\omega\tau})\,d\tau$$

$$= \text{RP of } 2\int_0^{\infty} (1+\alpha\tau)e^{-\alpha\tau}e^{i\omega\tau}\,d\tau$$

$$= \text{RP of } 2\int_0^{\infty} (1+\alpha\tau)e^{-\tau(\alpha-i\omega)}\,d\tau$$

$$= \text{RP of } 2\left[(1+\alpha\tau)\left(\frac{e^{-\tau(\alpha-i\omega)}}{-(\alpha-i\omega)}\right) - \alpha\left(\frac{e^{-\tau(\alpha-i\omega)}}{(\alpha-i\omega)^2}\right)\right]_0^{\infty}$$

$$= \text{RP of } 2\left[0+\left(\frac{1}{\alpha-i\omega}\right)+0+\alpha\left(\frac{1}{(\alpha-i\omega)^2}\right)\right]$$

$$= \text{RP of } 2\left[\left(\frac{1}{\alpha-i\omega}\right)+\left(\frac{\alpha}{(\alpha-i\omega)^2}\right)\right]$$

$$= \text{RP of } 2\left[\left(\frac{\alpha+i\omega}{\alpha^2+\omega^2}\right)+\left(\frac{\alpha(\alpha+i\omega)^2}{(\alpha^2+\omega^2)^2}\right)\right]$$

$$= \text{RP of } 2\left[\left(\frac{\alpha+i\omega}{\alpha^2+\omega^2}\right)+\left(\frac{\alpha(\alpha^2+i2\alpha\omega-\omega^2)}{(\alpha^2+\omega^2)^2}\right)\right]$$

$$= 2\left\{\left(\frac{\alpha}{\alpha^2+\omega^2}\right)+\left[\frac{\alpha(\alpha^2-\omega^2)}{(\alpha^2+\omega^2)^2}\right]\right\} = 2\left[\frac{\alpha(\alpha^2+\omega^2)+\alpha(\alpha^2-\omega^2)}{(\alpha^2+\omega^2)^2}\right]$$

$$= 2\left(\frac{2\alpha^3}{(\alpha^2+\omega^2)^2}\right) = \left(\frac{4\alpha^3}{(\alpha^2+\omega^2)^2}\right)$$

EXAMPLE 7.23 Find the power spectral density of a random binary transmission process where autocorrelation function is

$$R(\tau) = \begin{cases} 1-\dfrac{|\tau|}{T}, & |\tau| \le T \\ 0, & \text{otherwise} \end{cases}.$$

[AU April '04, May '08]

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau} d\tau = \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) e^{-i\omega\tau} d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) (\cos\omega\tau - i\sin\omega\tau) d\tau$$

$$= 2\int_{0}^{T} \left(1 - \frac{|\tau|}{T}\right) \cos\omega\tau\, d\tau + 0 = 2\int_{0}^{T} \left(1 - \frac{\tau}{T}\right) \cos\omega\tau\, d\tau$$

$$= 2\left[\left(1 - \frac{\tau}{T}\right)\frac{\sin\omega\tau}{\omega} - \left(\frac{-1}{T}\right)\frac{(-\cos\omega\tau)}{\omega^2}\right]_0^T$$

$$= 2\left(0 - 0 - \frac{\cos\omega T}{T\omega^2} + \frac{1}{T\omega^2}\right) = 2\left(\frac{1 - \cos\omega T}{T\omega^2}\right)$$

$$= 4\left[\frac{\sin^2\left(\frac{\omega T}{2}\right)}{T\omega^2}\right]$$

EXAMPLE 7.24 Given the power spectral density $S_{XX}(\omega) = \frac{1}{\omega^2 + 4}$, find the average power of the process. **[AU December '09]**

Solution Given: $S_{XX}(\omega) = \dfrac{1}{\omega^2 + 4}$

$$\therefore \qquad \text{ACF} = R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$= F^{-1}[S_{XX}(\omega)] = F^{-1}\left(\frac{1}{\omega^2 + 4}\right)$$

$$R_{XX}(\tau) = \frac{1}{4}e^{-2|\tau|}, \text{ using } F(e^{-2|\tau|}) = \frac{2\alpha}{\alpha^2 + \omega^2}$$

$\therefore$ The average power of the process

$$R_{XX}(0) = \frac{1}{4}e^{-2\times 0} = \frac{1}{4}$$

EXAMPLE 7.25 Find the power spectrum of a WSS random process $\{X(t)\}$ whose

$$R(\tau) = \begin{cases} 1 + \tau^2, & |\tau| < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\,d\tau = \int_{-1}^{1}(1+\tau^2)\,e^{-i\omega\tau}\,d\tau$$

$$= \int_{-1}^{1}(1+\tau^2)\,(\cos\omega\tau - i\sin\omega\tau)\,d\tau$$

$$= \int_{-1}^{1}(1+\tau^2)\cos\omega\tau\,d\tau - i\int_{-1}^{1}(1+\tau^2)\sin\omega\tau\,d\tau$$

$$= 2\int_{0}^{1}(1+\tau^2)\cos\omega\tau\,d\tau - i\times 0$$

$$= 2\left[(1+\tau^2)\left(\frac{\sin\omega\tau}{\omega}\right) - 2\tau\left(\frac{-\cos\omega\tau}{\omega^2}\right) + 2\left(\frac{-\sin\omega\tau}{\omega^3}\right)\right]_0^1$$

$$= 2\left(\frac{2\sin\omega}{\omega} + \frac{2\cos\omega}{\omega^2} - \frac{2\sin\omega}{\omega^3}\right)$$

$$\therefore \quad S(\omega) = \frac{4\sin\omega}{\omega} + \frac{4\cos\omega}{\omega^2} - \frac{4\sin\omega}{\omega^3}$$

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EXAMPLE 7.26 The power spectral density of a zero mean process $\{X(t)\}$ is given by

$$S(\omega) = \begin{cases} 1, & |\omega| \le \omega_0 \\ 0, & \text{elsewhere} \end{cases}$$

Find $R(\tau)$ and also prove that $X(t)$ and $X\left(t + \frac{\pi}{\omega_0}\right)$ are uncorrelated.

[AU May '07; '09, December '08]

Solution Given $E[X(t)] = 0$. By definition,

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}(\cos\omega\tau + i\sin\omega\tau)\,d\omega = \frac{2}{2\pi}\int_{0}^{\omega_0}\cos\omega\tau\,d\omega + 0$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega\tau}{\tau}\right]_0^{\omega_0} = \frac{1}{\pi}\left(\frac{\sin\omega_0\tau}{\tau} - 0\right)$$

$$R(\tau) = \frac{\sin\omega_0\tau}{\pi\tau}$$

To show that $X(t)$ and $X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated, we have to show that

$$C\left[X(t)X\left(t+\frac{\pi}{\omega_0}\right)\right]=0$$

$$C\left[X(t)X\left(t+\frac{\pi}{\omega_0}\right)\right]=E\left[X(t)\,X\left(t+\frac{\pi}{\omega_0}\right)\right]-E[X(t)]E\left[X\left(t+\frac{\pi}{\omega_0}\right)\right]$$

$$=R_{XX}\left(\frac{\pi}{\omega_0}\right)-0=R_{XX}\left(\frac{\pi}{\omega_0}\right)\qquad [\because\ E[X(t)]=0]$$

But $$R_{XX}(\tau)=\frac{\sin\omega_0\tau}{\pi\tau}$$

$\therefore$ $$R_{XX}\left(\frac{\pi}{\omega_0}\right)=\frac{\sin\omega_0\left(\frac{\pi}{\omega_0}\right)}{\pi\left(\frac{\pi}{\omega_0}\right)}=\frac{\sin\pi}{\left(\frac{\pi^2}{\omega_0}\right)}=0$$

$\therefore$ $$C\left[X(t)X\left(t+\frac{\pi}{\omega_0}\right)\right]=0$$

$\therefore$ $X(t)$ and $X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated.

EXAMPLE 7.27 The power spectral density of a WSS process is given by

$$S(\omega)=\begin{cases}\frac{b}{a}(a-|\omega|), & |\omega|\le a\\ 0, & |\omega|>a\end{cases}$$

Find the autocorrelation function of the process.

[AU May '08; '09, June '09]

Solution By definition,

$$R_{XX}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)e^{i\omega\tau}\,d\omega$$

$$=\frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|)e^{i\omega\tau}\,d\omega$$

$$=\frac{1}{2\pi}\frac{b}{a}\int_{-a}^{a}(a-|\omega|)(\cos\omega\tau+i\sin\omega\tau)\,d\omega$$

$$= \frac{b}{2\pi a}\int_{-a}^{a}(a-|\omega|)\cos\omega\tau\,d\omega + \frac{ib}{2\pi a}\int_{-a}^{a}(a-|\omega|)\sin\omega\tau\,d\omega$$

$$= \frac{2b}{2\pi a}\int_{0}^{a}(a-\omega)\cos\omega\tau\,d\omega + 0$$

(since the second integrand is an odd function)

$$= \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\omega\tau}{\tau} - \frac{-\cos\omega\tau}{\tau^2}\right]_0^a$$

$$= \frac{b}{\pi a}\left(\frac{\cos a\tau}{\tau^2} - \frac{1}{\tau^2}\right)$$

$$\therefore \quad R_{XX}(\tau) = \frac{b}{\pi a}\left(\frac{\cos a\tau - 1}{\tau^2}\right)$$

EXAMPLE 7.28 Find the autocorrelation of a WSS random process whose power spectral density is given by

$$S(\omega) = \begin{cases} p + \dfrac{iq}{p}\omega, \text{ where } -p<\omega<p \\ 0, \quad \text{otherwise} \end{cases}.$$

[AU December '06]

Solution By definition,

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,d\omega$$

$$R(\tau) = \frac{1}{2\pi}\int_{-p}^{p}\left(p + \frac{iq}{p}\omega\right)e^{i\omega\tau}\,d\omega$$

$$= \frac{p}{2\pi}\int_{-p}^{p}(\cos\omega\tau + i\sin\omega\tau)\,d\omega + \frac{iq}{2\pi p}\int_{-p}^{p}\omega(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

Since $\sin\omega\tau$ and $\omega\cos\omega\tau$ are odd functions, we have

$$R(\tau) = \frac{2p}{2\pi}\int_0^p \cos\omega\tau\,d\omega + 0 + \frac{2iq}{2\pi p}\left(\omega\times 0 + i\int_0^p \omega\sin\omega\tau\,d\omega\right)$$

$$= \frac{p}{\pi}\left[\frac{\sin\omega\tau}{\tau}\right]_0^p - \frac{q}{\pi p}\int_0^p \omega\sin\omega\tau\,d\omega$$

$$= \frac{p}{\pi}\frac{\sin p\tau}{\tau} - \frac{q}{\pi p}\left[\omega\left(\frac{-\cos\omega\tau}{\tau}\right) - \left(\frac{-\sin\omega\tau}{\tau^2}\right)\right]_0^p$$

$$= \frac{p}{\pi}\left(\frac{\sin p\tau}{\tau}\right) + \frac{q}{\pi}\left(\frac{\cos p\tau}{\tau}\right) - \frac{q}{\pi p}\left(\frac{\sin p\tau}{\tau^2}\right)$$

$$\therefore \quad R(\tau) = \frac{1}{\pi\tau}(p\sin p\tau + q\cos p\tau) - \frac{q}{\pi p}\left(\frac{\sin p\tau}{\tau^2}\right)$$

EXAMPLE 7.29 Find the autocorrelation function of the random process $\{X(t)\}$ for which the spectral density is given by

$$S(\omega) = \begin{cases} \omega^2, |\omega| \le 1 \\ 0, \quad |\omega| > 1 \end{cases}.$$

Solution By definition,

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \omega^2 e^{i\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-1}^{1}\omega^2(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$R_{XX}(\tau) = \frac{2}{2\pi}\int_0^1 \omega^2\cos\omega\tau\,d\omega + 0$$

$$= \frac{1}{\pi}\left[\frac{\omega^2(\sin\omega\tau)}{\tau} - \frac{2\omega(-\cos\omega\tau)}{\tau^2} + \frac{2(-\sin\omega\tau)}{\tau^3}\right]_0^1$$

$$\therefore \quad R(\tau) = \frac{1}{\pi}\left(\frac{\sin\tau}{\tau} + \frac{2\cos\tau}{\tau^2} - \frac{2\sin\tau}{\tau^3}\right) = \frac{1}{\pi\tau^3}(\tau^2\sin\tau + 2\tau\cos\tau - 2\sin\tau)$$

EXAMPLE 7.30 Find the autocorrelation of the process $\{X(t)\}$ for which the spectral density is given by

$$S(\omega) = \begin{cases} 1+\omega^2, |\omega| \le 1 \\ 0, \quad\quad |\omega| > 1 \end{cases}.$$

Solution By definition,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}(1+\omega^2)(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}(1+\omega^2)\cos\omega\tau\, d\omega + \frac{i}{2\pi}\int_{-1}^{1}(1+\omega^2)\sin\omega\tau\, d\omega$$

Since $\cos\omega\tau$ is an even function of ω and $\sin\omega\tau$ is an odd function of ω

$$R_{XX}(\tau) = \frac{2}{2\pi}\int_{0}^{1}(1+\omega^2)\cos\omega\tau\, d\omega + 0$$

$$= \frac{1}{\pi}\left[(1+\omega^2)\left(\frac{\sin\omega\tau}{\tau}\right) - (2\omega)\left(\frac{-\cos\omega\tau}{\tau^2}\right) + 2\left(\frac{-\sin\omega\tau}{\tau^3}\right)\right]_0^1$$

$$= \frac{1}{\pi}\left[\left(\frac{2\sin\tau}{\tau}\right) + \left(\frac{2\cos\tau}{\tau^2}\right) - \left(\frac{2\sin\tau}{\tau^3}\right)\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin\tau}{\tau}\right) + \left(\frac{\cos\tau}{\tau^2}\right) - \left(\frac{\sin\tau}{\tau^3}\right)\right]$$

$$= \frac{2}{\pi\tau^3}(\tau^2\sin\tau + \tau\cos\tau - \sin\tau)$$

EXAMPLE 7.31 Find the power spectral density of a WSS process with autocorrelation function $R(\tau) = e^{-\alpha\tau^2}$, $\alpha > 0$.

[AU December '06; '07; '08, November '08]

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau^2}e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha}\tau + \frac{i\omega}{2\sqrt{\alpha}}\right)^2} e^{\frac{-\omega^2}{4\alpha}}\, d\tau \qquad \left[\begin{array}{ll} a^2 = \alpha\tau^2 & 2ab = i\omega\tau \\ a = \sqrt{\alpha}\,\tau & 2\sqrt{\alpha}\;\; \tau b = i\omega\tau \\ & b = \dfrac{i\omega}{2\sqrt{\alpha}} \end{array}\right]$$

Put $\left(\sqrt{\alpha}\tau + \dfrac{i\omega}{2\sqrt{\alpha}}\right) = x \;\Rightarrow\; \sqrt{\alpha}d\tau = dx \;\Rightarrow\; d\tau = \dfrac{dx}{\sqrt{\alpha}}$

and when $\tau = \infty$, $x = \infty$, and $\tau = -\infty$, $x = -\infty$

$$\therefore \qquad S(\omega) = \int_{-\infty}^{\infty} e^{-(x)^2} e^{\frac{-\omega^2}{4\alpha}}\frac{dx}{\sqrt{\alpha}}$$

$$= \frac{e^{\frac{-\omega^2}{4\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-(x)^2} dx = \frac{e^{\frac{-\omega^2}{4\alpha}}}{\sqrt{\alpha}} \sqrt{\pi}$$

$$= \sqrt{\frac{\pi}{\alpha}} e^{\frac{-\omega^2}{4\alpha}}$$

Aliter

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-i\omega\tau} d\tau$$

$$= F(e^{-\alpha\tau^2})$$

$$= \sqrt{\frac{\pi}{\alpha}} e^{\frac{-\omega^2}{4\alpha}} \qquad \text{[refer Table 7.1)}$$

EXAMPLE 7.32 A random process $\{X(t)\}$ is given by $X(t) = A\cos pt + B\sin pt$ where A and B are independent random variables such that $E(A) = E(B) = 0$, $E(A^2) = E(B^2) = \sigma^2$. Find the power spectral density of the process.

Solution To find the power spectrum, first we have to find the autocorrelation function.

By definition,

$$R(\tau) = E[X(t)X(t+\tau)]$$

$$= E\{(A\cos pt + B\sin pt)[A\cos p(t+\tau) + B\sin p(t+\tau)]\}$$

$$= E[A^2 \cos pt \cos p(t+\tau) + AB\cos pt \sin p(t+\tau) + AB\sin pt \cos p(t+\tau) + B^2 \sin pt \sin p(t+\tau)]$$

$$= \cos pt \cos p(t+\tau)\, E(A^2) + \cos pt \sin p(t+\tau)\, E(AB) + \sin pt \cos p(t+\tau)\, E(AB) + \sin pt \sin p(t+\tau)\, E(B^2)$$

Since A and B are independent $E(AB) = E(A)E(B) = 0$,

$$R(\tau) = [\cos pt \cos p(t+\tau) + \sin pt \sin p(t+\tau)]\sigma^2$$

$$= \cos[p(t+\tau) - pt]\sigma^2 = \sigma^2 \cos p\tau$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} (\sigma^2 \cos p\tau) e^{-i\omega\tau} d\tau$$

$$= \sigma^2 \int_{-\infty}^{\infty} \cos p\tau\, e^{-i\omega\tau} d\tau = \sigma^2 F(\cos p\tau)$$

$$= \sigma^2 F(\cos p\tau)$$

$$\therefore\ S(\omega) = \sigma^2 \pi[\delta(\omega + p) + \delta(\omega - p)]$$

EXAMPLE 7.33 Find the power density spectrum of the random process $X(t) = A\cos(\omega_0 t + \theta)$, where A and ω_0 are constants and θ is uniformly distributed in $(0, 2\pi)$. **[AU December '05]**

Solution Since θ is uniformly distributed in $(0, 2\pi)$

$$f(\theta) = \begin{cases} \dfrac{1}{2\pi}, 0 < \theta < 2\pi \\ 0, \quad \text{otherwise} \end{cases}$$

and Given: $\qquad X(t) = A\cos(\omega_0 t + \theta)$

By definition,

$$\begin{aligned} R_{XX}(\tau) &= E[X(t)\,X(t+\tau)] \\ &= E[A\cos(\omega_0 t + \theta)\,A\cos(\omega_0 t + \omega_0\tau + \theta)] \\ &= \frac{A^2}{2}E[\cos(2\omega_0 t + \omega_0\tau + 2\theta) + \cos(\omega_0 t + \theta - \omega_0\tau - \omega_0 t - \theta)] \\ &= \frac{A^2}{2}\{E[\cos(2\omega_0 t + \omega_0\tau + 2\theta)] + E[\cos(-\omega_0\tau)]\} \end{aligned}$$

$$[\because \cos(-\omega_0\tau) = \cos\omega_0\tau]$$

$$\begin{aligned} &= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau + 2\theta)\,d\theta + \frac{A^2}{2}E(\cos\omega_0\tau) \\ &= \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t + \omega_0\tau + 2\theta)}{2}\right]_0^{2\pi} + \frac{A^2}{2}(\cos\omega_0\tau) \\ &= \frac{A^2}{8\pi}[\sin(2\omega_0 t + \omega_0\tau + 4\pi) - \sin(2\omega_0 t + \omega_0\tau)] + \frac{A^2}{2}(\cos\omega_0\tau) \\ &= 0 + \frac{A^2}{2}\cos\omega_0\tau \end{aligned}$$

$$\therefore \quad R_{XX}(\tau) = \frac{A^2}{2}(\cos\omega_0\tau)$$

$$\begin{aligned} \therefore \quad S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}\,d\tau \\ &= \frac{A^2}{2}\int_{-\infty}^{\infty}\cos(\omega_0\tau)e^{-i\omega\tau}\,d\tau = \frac{A^2}{2}F[\cos(\omega_0\tau)] \\ &= \frac{A^2}{2}\{\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]\} \end{aligned}$$

EXAMPLE 7.34 Show that PSD of a real random process $\{X(t)\}$ is real and verify that $S_{XX}(-\omega) = S_{XX}(\omega)$. **[AU May '05]**

Solution To show $S_{XX}(\omega)$ is real for a real random process.
By definition,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau) (\cos \omega\tau - i \sin \omega\tau)\, d\tau$$

We know that $R_{XX}(\tau)$ is always an even function by property

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) (\cos \omega\tau) d\tau - i \int_{-\infty}^{\infty} R_{XX}(\tau) (\sin \omega\tau) d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) (\cos \omega\tau) d\tau - i \times 0$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) (\cos \omega\tau) d\tau \Rightarrow \text{(Real function of } e^{-i\omega\tau})$$

$\therefore$ $S_{XX}(\omega)$ is also a real function.
Again, by definition,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau} d\tau$$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i(-\omega)\tau} d\tau$$

Replace $\tau = -\tau$, $d\tau = -d\tau$

When $\tau = -\infty$, $-\tau = -\infty \Rightarrow \tau = \infty$
$\tau = \infty$, $-\tau = \infty \Rightarrow = \tau = -\infty$

$$S_{XX}(-\omega) = \int_{\infty}^{-\infty} R_{XX}(-\tau)\, e^{i\omega(-\tau)} (-d\tau)$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau} d\tau$$

$$= S_{XX}(\omega)$$

$\therefore$ PSD is also an even function.

EXAMPLE 7.35 Consider a random process $\{X(t)\}$ that assumes the value $\pm A$ with equal probability. Let Z be the random variable representing the number of 0 crossings. If α is the average number of crossings per unit time, then probability of having exactly A crossings in time τ is given by

$$P(Z=k)=\frac{e^{-\alpha\tau}(\alpha\tau)^k}{k!}, \qquad k=0,1,2,3,\cdots.$$

Find ACF and PSD of $X(t)$.

Solution $R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$
$= E[X(t)$ and $X(t+\tau)$ have same sign or $X(t)$ and $X(t+\tau)$ have different sign]
$= A^2P[X(t)$ and $X(t+\tau)$ have same sign]
$+(-A^2)P[X(t)$ and $X(t+\tau)$ have different sign]

$$= A^2 \sum_{k=0,2,4\ldots} \frac{e^{-\alpha\tau}(\alpha\tau)^k}{k!} + (-A^2) \sum_{k=1,3,5\ldots} \frac{e^{-\alpha\tau}(\alpha\tau)^k}{k!}$$

$$= A^2 \sum_{k=0,1,2,\ldots} \frac{e^{-\alpha\tau}(\alpha\tau)^k}{k!}(-1)^k$$

$$= A^2e^{-\alpha\tau}\left[1-\frac{(\alpha\tau)}{1!}+\frac{(\alpha\tau)^2}{2!}-\frac{(\alpha\tau)^3}{3!}+\cdots\right]$$

$$= A^2e^{-\alpha\tau}e^{-\alpha\tau} = A^2e^{-2\alpha|\tau|}$$

Even if we consider the case $\tau < 0$, we get $R_{XX}(\tau) = A^2e^{-2\alpha|\tau|}$.

$$\therefore \qquad S_{XX}(\omega) = F[R_{XX}(\tau)] = F(A^2e^{-2\alpha|\tau|}) = \frac{A^2 4\alpha}{4\alpha^2+\omega^2}$$

EXAMPLE 7.36 Find the autocorrelation function of the WSS process $\{X(t)\}$ where spectral density function is given by

$$S(\omega) = \left[\frac{1}{(1+\omega^2)^2}\right].$$

Find its average power. **[AU Model]**

Solution Given:

$$S(\omega) = \left[\frac{1}{(1+\omega^2)^2}\right] = \left[\frac{1}{(1+i\omega)\,(1-i\omega)}\right]^2$$

$$= \frac{1}{4}\left[\frac{1}{(1-i\omega)}+\frac{1}{(1+i\omega)}\right]^2, \qquad \text{using partial fraction}$$

$$= \frac{1}{4}\left[\frac{1}{(1-i\omega)^2}+\frac{1}{(1+i\omega)^2}+\frac{2}{(1-i\omega)\,(1+i\omega)}\right]$$

$$\therefore \qquad R(\tau) = F^{-1}[S(\omega)]$$

$$= \frac{1}{4}F^{-1}\left[\frac{1}{(1-i\omega)^2} + \frac{1}{(1+i\omega)^2} + \frac{2}{(1-i\omega)(1+i\omega)}\right]$$

$$= \frac{1}{4}F^{-1}\left[\frac{1}{(1-i\omega)^2} + \frac{1}{(1+i\omega)^2} + \frac{2}{(1+\omega^2)}\right]$$

$$= \frac{1}{4}\left[(\tau e^{\tau} + \tau e^{-\tau})\, u(\tau) + e^{-|\tau|}\right]$$

$$= \frac{1}{4}(\tau e^{-|\tau|} + e^{-|\tau|}) = \frac{1}{4}\left[(1+\tau)e^{-|\tau|}\right]$$

$\therefore$ The average power $R(0) = \frac{1}{4}$.

EXAMPLE 7.37 If $Y(t) = X(t + a) - X(t - a)$, prove that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau - 2a) - R_{XX}(\tau + 2a)$. Hence show that the power spectral density of $Y(t)$ is $S_{YY}(\omega) = 4\sin^2 a\omega\, S_{XX}(\omega)$, where $X(t)$ is a WSS random process.

[AU May '06, December '06; '09]

Solution Given that $X(t)$ is a WSS random process. We know that, by definition,

$$\begin{aligned}
R_{YY}(\tau) &= E[Y(t)\,Y(t + \tau)] \\
&= E\{[X(t + a) - X(t - a)]\,[X(t + \tau + a) - X(t + \tau - a)]\} \\
&= E[X(t + a)\,X(t + \tau + a) - X(t + a)\,X(t + \tau - a) \\
&\qquad - X(t - a)\,X(t + \tau + a) + X(t - a)\,X(t + \tau - a)] \\
&= E[X(t + a)\,X(t + \tau + a)] - E[X(t + a)\,X(t + \tau - a)] \\
&\qquad - E[X(t - a)\,X(t + \tau + a)] + E[X(t - a)\,X(t + \tau - a)] \\
&= R_{XX}(\tau) - R_{XX}(\tau - 2a) - R_{XX}(\tau + 2a) + R_{XX}(\tau) \\
&= 2R_{XX}(\tau) - R_{XX}(\tau - 2a) - R_{XX}(\tau + 2a)
\end{aligned}$$

By definition of power spectral density function,

$$\begin{aligned}
S_{YY}(\omega) &= \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-i\omega\tau}\,d\tau \\
&= \int_{-\infty}^{\infty} [2R_{XX}(\tau) - R_{XX}(\tau - 2a) - R_{XX}(\tau + 2a)]e^{-i\omega\tau}\,d\tau \\
&= 2\int_{-\infty}^{\infty} [R_{XX}(\tau)]e^{-i\omega\tau}\,d\tau - \int_{-\infty}^{\infty} [R_{XX}(\tau - 2a)]e^{-i\omega\tau}\,d\tau \\
&\qquad - \int_{-\infty}^{\infty} [R_{XX}(\tau + 2a)]e^{-i\omega\tau}\,d\tau
\end{aligned}$$

$$= 2S_{XX}(\omega) - \int_{-\infty}^{\infty} [R_{XX}(\tau - 2a)]e^{-i\omega\tau} d\tau - \int_{-\infty}^{\infty} [R_{XX}(\tau + 2a)]e^{-i\omega\tau} d\tau \quad \text{(i)}$$

In the third integral term,

Put $\tau + 2a = \tau_2 \Rightarrow \tau = \tau_2 - 2a \Rightarrow d\tau = d\tau_2$

$$\therefore \quad \int_{-\infty}^{\infty} [R_{XX}(\tau + 2a)]e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau_2)e^{-i\omega(\tau_2 - 2a)} d\tau_2$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau_2)e^{-i\omega\tau_2}e^{i\omega 2a} d\tau_2 = e^{i2\omega a} \int_{-\infty}^{\infty} R_{XX}(\tau_2)e^{-i\omega\tau_2} d\tau_2$$

$$= e^{i\omega 2a} \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau} d\tau \qquad \text{(by changing the variable, } \tau_2 \to \tau)$$

$$= e^{i\omega 2a} S_{XX}(\omega)$$

In the second integral term,

Put $\tau - 2a = \tau_1 \Rightarrow \tau = \tau_1 + 2a \Rightarrow d\tau = d\tau_1$

$$\therefore \quad \int_{-\infty}^{\infty} [R_{XX}(\tau - 2a)]e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau_1)e^{-i\omega(\tau_1 + 2a)} d\tau_1$$

$$= e^{-i2\omega a} \int_{-\infty}^{\infty} R_{XX}(\tau_1)e^{-i\omega\tau_1} d\tau_1 = e^{-i\omega 2a} \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau} d\tau$$

$$= e^{-i2\omega a}[S_{XX}(\omega)] \qquad \text{(by changing the variables, } \tau_1 \to \tau)$$

Therefore, Eq. (i) becomes

$$\begin{aligned}
S_{YY}(\omega) &= 2S_{XX}(\omega) - e^{-i2\omega a}S_{XX}(\omega) - e^{+i2\omega a}S_{XX}(\omega) \\
&= 2S_{XX}(\omega) - S_{XX}(\omega)\,(e^{i2\omega a} + e^{-i2\omega a}) \\
&= 2S_{XX}(\omega) - S_{XX}(\omega)\,(2\cos 2\omega a) \\
&= 2S_{XX}(\omega)(1 - \cos 2\omega a) = 2S_{XX}(\omega)\,2\sin^2(\omega a) \\
&= 4S_{XX}(\omega)\sin^2(a\omega)
\end{aligned}$$

EXAMPLE 7.38 If $\{X(t)\}$ is a stationary random process with PSD $S_{XX}(\omega)$ and $\{Y(t)\}$ is another independent random process $Y(t) = A\cos(\omega_0 t + \theta)$ where θ is a random variable uniformly distributed over $(-\pi, \pi)$, find the PSD of $\{Z(t)\}$ where $Z(t) = X(t) \cdot Y(t)$.

Solution Given: $Y(t) = A\cos(\omega_0 t + \theta)$

$$f(\theta) = \frac{1}{2\pi}, \quad -\pi < \theta < \pi$$
$$= 0, \quad \text{otherwise}$$

Therefore,

$$R_{YY}(t, t+\tau) = \frac{A^2}{2} E[2\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)]$$

$$= \frac{A^2}{2}\{E(\cos\omega_0\tau) + E[\cos(2\omega_0 t + \omega_0\tau + 2\theta)]\}$$

$$= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{4\pi}\int_{-\pi}^{\pi}\cos[2\omega_0 t + \omega_0\tau + 2\theta)]d\theta$$

$$= \frac{A^2}{2}\cos\omega_0\tau + 0$$

$$R_{YY}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$$

$$R_{ZZ}(t, t+\tau) = E[X(t)Y(t)\,X(t+\tau)Y(t+\tau)]$$

Since the processes $\{X(t)\}$ and $\{Y(t)\}$ are independent

$$R_{ZZ}(t, t+\tau) = E[X(t)X(t+\tau)]\;E[Y(t)Y(t+\tau)]$$

$$\therefore \qquad R_{ZZ}(\tau) = R_{XX}(\tau)\,R_{YY}(\tau) = \frac{A^2}{2}R_{XX}(\tau)\cos\omega_0\tau$$

PSD of $Z(t) = S(\omega) = \dfrac{A^2}{2}\displaystyle\int_{-\infty}^{\infty} R_{XX}(\tau)\cos\omega_0\tau\, e^{-i\omega\tau}d\tau$

$$= \frac{A^2}{4}\int_{-\infty}^{\infty} R_{XX}(\tau)(e^{i\omega_0\tau} + e^{-i\omega_0\tau})e^{-i\omega\tau}d\tau$$

$$= \frac{A^2}{4}\left\{\int_{-\infty}^{\infty} R_{XX}(\tau)[e^{-(\omega-\omega_0)i\tau}d\tau] + \int_{-\infty}^{\infty} R_{XX}(\tau)[e^{-(\omega+\omega_0)i\tau}d\tau]\right\}$$

Using $S_{XX}(\omega) = \displaystyle\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}\,d\tau$, we get

$$S_{ZZ}(\omega) = \frac{A^2}{4}[S_{XX}(\omega-\omega_0) + S_{XX}(\omega+\omega_0)]$$

7.6 CAUCHY'S RESIDUE THEOREM

If $f(Z)$ is analytic within and on a simple closed curve C except at a finite number of poles $Z_1, Z_2, \ldots, Z_n$, then $\displaystyle\int_C f(Z)dz = 2\pi i\,\Sigma R$, where ΣR is the sum of the residues at its poles which lies inside C.

Case 1

If $z = a$ is a simple pole of $f(Z)$, then residue at $z = a$ is

$$[\text{Res} f(Z)]_{z=a} = \lim_{z \to a} (z-a) f(Z)$$

Case 2

If $z = a$ is a pole of order 2, then

$$[\text{Res} f(Z)]_{z=a} = \lim_{z \to a} \frac{d}{dz}[(z-a)^2 f(Z)]$$

Case 3

If $z = a$ is a pole of order m, then

$$[\text{Res} f(Z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(Z)]$$

If C is the contour consisting of the real axis $-R$ to R and upper half of the circle Γ, then

$$\int_C f(Z)dz = \int_{-R}^{R} f(Z)dz + \int_\Gamma f(Z)dz$$

As $R \to \infty$, $\int_\Gamma f(Z)dz \to 0$, by Cauchy's Lemma

$$\int_{-\infty}^{\infty} f(Z)dz = \int_C f(Z)dz$$

Note: Poles with negative imaginary part lies outside C.

7.7 ACF USING CAUCHY'S RESIDUE THEOREM

EXAMPLE 7.39 Find the autocorrelation function of a WSS process $\{X(t)\}$ where

$$S(\omega) = \left[\frac{1}{(\omega^2+4)^2}\right].$$

Also find its average power.

Solution By definition,

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(\omega^2+4)^2}\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_C \frac{e^{i\omega\tau}}{(\omega^2+4)^2}\, d\omega$$

$$= \frac{1}{2\pi}\int_C \frac{e^{i\omega\tau}}{(\omega^2+4)^2}\, d\omega$$

$$= \frac{1}{2\pi} 2\pi i\, \Sigma R = i\Sigma R$$

$\therefore \quad R(\tau) = i$ [sum of the residues of $f(\omega)$ at its poles which lies inside C] (i)

To find the poles of $f(\omega) = \dfrac{e^{i\omega\tau}}{(\omega^2+4)^2}$

$$(\omega^2+4)^2 = 0 \;\Rightarrow\; \omega^2 = -4 \qquad [\because\ \omega^2 + 4 = (\omega+2i)(\omega-2i)]$$

$$\Rightarrow \qquad \omega = \pm 2i$$

$\therefore \quad \omega = \pm 2i$ are the poles of order 2, and $\omega = 2i$ is the only pole which lies inside the circle C.

$$[(\text{Res}\, f(\omega)]_{\omega=2i} = \lim_{\omega\to 2i}\frac{d}{d\omega}\left[(\omega-2i)^2\frac{e^{i\omega\tau}}{(\omega-2i)^2(\omega+2i)^2}\right]$$

$$= \lim_{\omega\to 2i}\frac{d}{d\omega}\left[\frac{e^{i\omega\tau}}{(\omega+2i)^2}\right]$$

$$= \lim_{\omega\to 2i}\left\{\frac{(\omega+2i)^2 e^{i\omega\tau}(i\tau) - e^{i\omega\tau}[2(\omega+2i)]}{(\omega+2i)^4}\right\}$$

$$= \lim_{\omega\to 2i}\left[\frac{(\omega+2i)(i\tau)e^{i\omega\tau} - 2e^{i\omega\tau}}{(\omega+2i)^3}\right]$$

$$= \left[\frac{-4\tau e^{-2\tau} - 2e^{-2\tau}}{(4i)^3}\right]$$

$$= \frac{(2\tau+1)e^{-2\tau}}{32i}$$

Substituting in Eq. (i), we get

$$R(\tau) = i\left[\frac{(1+2\tau)e^{-2\tau}}{32i}\right]$$

$$R_{XX}(\tau) = \frac{(1+2\tau)e^{-2\tau}}{32}$$

$$\therefore \qquad \text{Average power} = R(0) = \frac{1}{32}$$

EXAMPLE 7.40 The autocorrelation of a Poisson incrementation process is given by

$$R(\tau) = \begin{cases} \lambda^2, & |\tau| > \varepsilon \\ \lambda^2 + \dfrac{\lambda}{\varepsilon}\left(1 - \dfrac{|\tau|}{\varepsilon}\right), & |\tau| \le \varepsilon \end{cases}$$

Prove that its spectral density function is given by

$$S(\omega) = \lambda^2 2\pi\delta(\omega) + \frac{4\lambda \sin^2 \dfrac{\omega\varepsilon}{2}}{\varepsilon^2\omega^2}$$

[AU May '07, June '09]

Solution By definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{-\varepsilon} R(\tau)e^{-i\omega\tau}\,d\tau + \int_{-\varepsilon}^{\varepsilon} R(\tau)e^{-i\omega\tau}\,d\tau + \int_{\varepsilon}^{\infty} R(\tau)e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{-\varepsilon} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{-\varepsilon}^{\varepsilon}\left[\lambda^2 + \frac{\lambda}{\varepsilon}\left(1 - \frac{|\tau|}{\varepsilon}\right)\right]e^{-i\omega\tau}\,d\tau + \int_{\varepsilon}^{\infty} \lambda^2 e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{-\varepsilon} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{-\varepsilon}^{\varepsilon} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{\varepsilon}^{\infty} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{-\varepsilon}^{\varepsilon} \frac{\lambda}{\varepsilon}\left(1 - \frac{|\tau|}{\varepsilon}\right)e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{-\varepsilon}^{\varepsilon} \frac{\lambda}{\varepsilon}\left(1 - \frac{|\tau|}{\varepsilon}\right)e^{-i\omega\tau}\,d\tau$$

$$= \lambda^2 \int_{-\infty}^{\infty} e^{-i\omega\tau}\,d\tau + \frac{\lambda}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(1 - \frac{|\tau|}{\varepsilon}\right)(\cos\omega\tau - i\sin\omega\tau)\,d\tau$$

$$= \lambda^2 F(1) + \frac{\lambda}{\varepsilon}2\int_{0}^{\varepsilon}\left(1 - \frac{\tau}{\varepsilon}\right)(\cos\omega\tau)\,d\tau - i \times 0$$

$$= \lambda^2 2\pi\delta(\omega) + \frac{2\lambda}{\varepsilon}\left[\left(1-\frac{\tau}{\varepsilon}\right)\frac{\sin\omega\tau}{\omega} - \left(\frac{-1}{\varepsilon}\right)\left(\frac{-\cos\omega\tau}{\omega^2}\right)\right]_0^{\varepsilon}$$

$$= 2\pi\lambda^2\delta(\omega) + \frac{2\lambda}{\varepsilon}\left[(0-0) - \left(\frac{1}{\omega^2\varepsilon}\right)(\cos\omega\varepsilon - 1)\right]$$

$$= 2\pi\lambda^2\delta(\omega) + \frac{2\lambda}{\varepsilon^2\omega^2}(1-\cos\omega\varepsilon)$$

$$= 2\pi\lambda^2\delta(\omega) + \frac{2\lambda}{\varepsilon^2\omega^2}\left[1-\cos\left(2\frac{\omega\varepsilon}{2}\right)\right]$$

$$S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda}{\varepsilon^2\omega^2}\sin^2\left(\frac{\omega\varepsilon}{2}\right)$$

Hence proved.

EXAMPLE 7.41 Given the power spectral density of a continuous process as

$$S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4},$$

find the mean square value of the process.

Solution Mean square value of the process = $R_{XX}(0)$

where $$R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)]$$

Given: $$S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4} = \frac{\omega^2+9}{(\omega^2+4)(\omega^2+1)}$$

By partial fraction method,

$$\frac{\omega^2+9}{(\omega^2+4)(\omega^2+1)} = \frac{x+9}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1} \quad [\text{where } x=\omega^2]$$

$$\therefore \quad x+9 = A(x+1) + B(x+4) \qquad \text{(i)}$$

Put $x = -4$ in Eq. (i), $\quad -3A = 5 \Rightarrow A = -\frac{5}{3}$

Put $x = -1$ in Eq. (i), $\quad 3B = 8 \Rightarrow B = \frac{8}{3}$

$$\therefore \quad \frac{\omega^2+9}{(\omega^2+4)(\omega^2+1)} = \frac{8}{3}\frac{1}{\omega^2+1} - \frac{5}{3}\frac{1}{\omega^2+4}$$

$$\therefore \quad F^{-1}\left[\frac{\omega^2+9}{(\omega^2+4)(\omega^2+1)}\right] = \frac{8}{3}F^{-1}\left(\frac{1}{\omega^2+1}\right) - \frac{5}{3}F^{-1}\left(\frac{1}{\omega^2+4}\right)$$

Using $$F^{-1}\left(\frac{1}{\omega^2+a^2}\right) = \frac{e^{-a|\tau|}}{2a},$$

$$F^{-1}[S_{XX}(\omega)] = \frac{8}{3}\frac{e^{-|\tau|}}{2} - \frac{5}{3}\frac{e^{-2|\tau|}}{4}$$

$$\therefore \qquad R_{XX}(\tau) = \frac{4}{3}e^{-|\tau|} - \frac{5}{12}e^{-2|\tau|}$$

Hence mean square value of the process $= R_{XX}(0) = \frac{4}{3} - \frac{5}{12} = \frac{11}{12}$

7.8 CROSS-SPECTRAL DENSITY

If $X(t)$ and $Y(t)$ are jointly WSS, then the cross-correlation functions are $R_{XY}(\tau)$ and $R_{YX}(\tau)$ for the two-dimensional random process $\{X(t), Y(t)\}$. The Fourier transform of these cross-correlation functions define the cross-spectral density functions $S_{XY}(\omega)$ and $S_{YX}(\omega)$ as

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-i\omega\tau}\,d\tau$$

and

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-i\omega\tau}\,d\tau$$

7.8.1 Alternate Definition

If $X(t)$ and $Y(t)$ are random processes, then

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[\bar{X}_T(\omega)Y_T(\omega)]}{2T}$$

and

$$S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[\bar{Y}_T(\omega)\,X_T(\omega)]}{2T}$$

7.8.2 Relationship between Cross-correlation and Cross-Spectral Densities

If the random processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS, then the cross-correlations and cross-spectral densities form Fourier transform pairs. That is, if

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-i\omega\tau}\,d\tau, \text{ then}$$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{i\omega\tau}\,d\omega$$

Also if $$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-i\omega\tau}\,d\tau, \text{ then}$$

$$R_{YX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)e^{i\omega\tau}\,d\omega$$

Since $$R_{XY}(\tau) = R_{YX}(-\tau), \text{ we obtain}$$

$$S_{XY}(\omega) = S_{YX}(-\omega) = \bar{S}_{YX}(\omega)$$

Note: The cross-powers P_{XY} and P_{YX} of the random processes $X(t)$ and $Y(t)$ are defined by

$$P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\,d\omega$$

and $$P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)\,d\omega$$

Also if $X(t)$ and $Y(t)$ are two random processes, then the time average cross-correlation function

$$\langle R_{XY}(t, t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\,dt$$

The cross-power spectrum $S_{XY}(\omega)$ is defined by

$$S_{XY}(\omega) = \int_{-\infty}^{\infty}\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\,dt\right\}e^{-i\omega\tau}\,d\tau$$

$\therefore$ $S_{XY}(\omega)$ is FT of $\displaystyle\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\,dt$

By inverse FT, we then have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{i\omega\tau}\,d\omega$$

For jointly wide-sense stationary processes, the time average ACF is just $R_{XY}(\tau)$,

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{i\omega\tau}\,d\omega$$

7.9 PROPERTIES OF CROSS-POWER DENSITY SPECTRUMS

1. $S_{XY}(\omega) = S_{YX}(-\omega) = \bar{S}_{YX}(\omega)$.
2. $\mathrm{Re}[S_{XY}(\omega)]$ and $\mathrm{Re}[S_{YX}(\omega)]$ are even functions of ω.
3. $\mathrm{Im}[S_{XY}(\omega)]$ and $\mathrm{Im}[S_{YX}(\omega)]$ are odd functions of ω.
4. $S_{XY}(\omega) = 0$ and $S_{YX}(\omega) = 0$ if $X(t)$ and $Y(t)$ are orthogonal.
5. If $X(t)$ and $Y(t)$ are uncorrelated and have constant means $\bar{X}$ and $\bar{Y}$, $S_{XY}(\omega) = S_{XY}(\omega) = 2\pi\overline{X}\overline{Y}\ \delta(\omega)$.
6. $\langle R_{XY}(t, t+\tau)\rangle \leftrightarrow S_{XY}(\omega)$ and $\langle R_{YX}(t, t+\tau)\rangle \leftrightarrow S_{YX}(\omega)$, i.e. the cross-power density spectrum and the time average of the cross-correlation function are a Fourier transform pair.

EXAMPLE 7.42 A random process is given by $Z(t) = AX(t) + BY(t)$ where A and B are real constants and $X(t)$ and $Y(t)$ are jointly WSS process. Find

(i) the power spectrum $S_{ZZ}(\omega)$ of $Z(t)$.
(ii) $S_{ZZ}(\omega)$ if $X(t)$ and $Y(t)$ are uncorrelated.
(iii) $S_{XZ}(\omega)$ and $S_{YZ}(\omega)$.

Solution

(i) To find the ACF of $Z(t)$:
By definition,

$$R_{ZZ}(t, t+\tau) = E[Z(t)Z(t+\tau)]$$

$$\begin{aligned}\therefore\quad R_{ZZ}(t, t+\tau) &= E\{[AX(t) + BY(t)]\,[AX(t+\tau) + BY(t+\tau)]\}\\ &= A^2E[X(t)X(t+\tau)] + AB\,E[X(t)Y(t+\tau)]\\ &\qquad + ABE[Y(t)X(t+\tau)] + B^2E[Y(t)Y(t+\tau)]\\ &= A^2R_{XX}(t, t+\tau) + AB\,R_{XY}(t, t+\tau)\\ &\qquad + AB\,R_{YX}(t, t+\tau) + B^2R_{YY}(t, t+\tau)\end{aligned}$$

$$\therefore\quad R_{ZZ}(\tau) = A^2R_{XX}(\tau) + AB\{R_{XY}(\tau) + R_{YX}(\tau)\} + B^2R_{YY}(\tau)$$

Taking Fourier transform on both sides of the above expression,

$$S_{ZZ}(\omega) = A^2S_{XX}(\omega) + AB[S_{XY}(\omega) + S_{YX}(\omega)] + B^2S_{YY}(\omega)$$

(ii) If $X(t)$ and $Y(t)$ are uncorrelated, then

$$S_{XY}(\omega) = S_{YX}(\omega)$$

$$\therefore\quad S_{ZZ}(\omega) = A^2S_{XX}(\omega) + B^2S_{YY}(\omega) + 2ABS_{XY}(\omega)$$

(iii)
$$\begin{aligned}R_{XZ}(t, t+\tau) &= E[X(t)Z(t+\tau)]\\ &= E\{X(t)[AX(t+\tau) + BY(t+\tau)]\}\\ &= AE[X(t)X(t+\tau)] + BE[X(t)Y(t+\tau)]\end{aligned}$$

$$\Rightarrow\quad R_{XZ}(\tau) = AR_{XX}(\tau) + BR_{XY}(\tau)$$

Taking Fourier transform on both sides of the above expression,

$$S_{XZ}(\omega) = AS_{XX}(\omega) + BS_{XY}(\omega)$$

In a similar manner, we can easily prove

$$S_{YZ}(\omega) = BS_{YY}(\omega) + AS_{YX}(\omega)$$

EXAMPLE 7.43 The cross-power spectrum of real random process $\{X(t)\}$ and $\{Y(t)\}$ is given by

$$S_{XY}(\omega) = \begin{cases} a + jb\omega, & |\omega| < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the cross-correlation function. **[AU December '06]**

Solution Here $i = j$

By definition,

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-1}^{1}(a + jb\omega)e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}(a + jb\omega)(\cos\omega\tau + j\sin\omega\tau)d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{1}(a + jb\omega)\cos\omega\tau\, d\omega + j\int_{-1}^{1}(a + jb\omega)\sin\omega\tau\, d\omega\right]$$

$$= \frac{1}{2\pi}\left(\int_{-1}^{1}a\cos\omega\tau\, d\omega + \frac{jb}{2\pi}\int_{-1}^{1}\omega\cos\omega\tau\, d\omega + j\int_{-1}^{1}a\sin\omega\tau\, d\omega - b\int_{-1}^{1}\omega\sin\omega\tau\, d\omega\right)$$

$$= \frac{2}{2\pi}\int_{0}^{1}a\cos\omega\tau\, d\omega + 0 + 0 - \frac{2b}{2\pi}\int_{0}^{1}\omega\sin\omega\tau\, d\omega$$

$$= \frac{1}{\pi}\left\{a\left(\frac{\sin\omega\tau}{\tau}\right) - b\left[\omega\left(\frac{-\cos\omega\tau}{\tau}\right) - 1\left(\frac{-\sin\omega\tau}{\tau^2}\right)\right]\right\}_0^1$$

$$= \frac{1}{\pi}\left(\frac{a\sin\tau}{\tau} + \frac{b\cos\tau}{\tau} - \frac{b\sin\tau}{\tau^2}\right)$$

$$= \frac{1}{\pi\tau^2}[(a\tau - b)\sin\tau + b\tau\cos\tau]$$

EXAMPLE 7.44 The cross-correlation function of two processes $\{X(t)\}$ and $\{Y(t)\}$ is given by

$$R_{XY}(t, t+\tau) = \left(\frac{AB}{2}\right)[\sin \omega_0 \tau + \cos \omega_0 (2t + \tau)]$$

where A and B are constants. Find the cross-power spectrum $S_{XY}(\omega)$ and find $\langle R_{XY}(t, t+\tau)\rangle$. **[AU May '04]**

Solution Time average function of $X(t)$ and $Y(t)$ is given by $\langle R_{XY}(t, t+\tau)\rangle$

$$\langle R_{XY}(t, t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t+\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{AB}{2}\right)[\sin \omega_0 \tau + \cos \omega_0 (2t+\tau)]\, dt$$

$$= \lim_{T\to\infty} \frac{AB}{4T} \left\{ \int_{-T}^{T} (\sin \omega_0 \tau)\, dt + \int_{-T}^{T} [\cos \omega_0 (2t+\tau)]\, dt \right\}$$

$$= \lim_{T\to\infty} \frac{AB}{4T} \left\{ \sin \omega_0 \tau [t]_{-T}^{T} + \left[\frac{\sin \omega_0 (2t+\tau)}{2\omega_0}\right]_{-T}^{T} \right\}$$

$$= \lim_{T\to\infty} \frac{AB}{4T} \left\{ \sin \omega_0 \tau (2T) + \frac{1}{2\omega_0}[\sin \omega_0 (2T+\tau) - \sin \omega_0 (-2T+\tau)] \right\}$$

$$= \lim_{T\to\infty} \frac{AB}{2} \sin \omega_0 \tau + \operatorname*{lt}_{T\to\infty} \frac{AB}{4T} \left\{ \frac{1}{2\omega_0}[\sin \omega_0 (2T+\tau) - \sin \omega_0 (-2T+\tau)] \right\}$$

$$= \frac{AB}{2} \sin \omega_0 \tau + 0$$

$$\therefore\ R_{XY}(t, t+\tau) = \frac{AB}{2} \sin \omega_0 \tau$$

Again, cross-power spectrum,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} \langle R_{XY}(t, t+\tau)\rangle\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(\frac{AB}{2} \sin \omega_0 \tau\right) e^{-i\omega\tau}\, d\tau = \frac{AB}{2} F(\sin \omega_0 \tau)$$

$$= \frac{AB}{2}\{-i\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\}$$

$$= -\frac{iAB\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

EXAMPLE 7.45 Let $\{X(t)\}$ and $\{Y(t)\}$ be both zero mean and WSS random processes. Consider the random process defined by $Z(t) = X(t) + Y(t)$.

(i) Find the ACF and power spectrum of $Z(t)$ if $X(t)$ and $Y(t)$ are jointly WSS random process.

(ii) Find the power spectrum of $Z(t)$ if $X(t)$ and $Y(t)$ are orthogonal.

(iii) Show that if $X(t)$ and $Y(t)$ are orthogonal, then the mean square of $Z(t)$ = the sum of the mean square of $X(t)$ and $Y(t)$.

Solution Given both $\{X(t)\}$ and $\{Y(t)\}$ are zero mean and WSS random processes.

$\therefore$ $E[X(t)] = E[Y(t)] = 0$ and their ACF are functions of time difference.

(i) ACF:

$$\begin{aligned}R_{ZZ}(\tau) &= E[Z(t)Z(t+\tau)]\\ &= E\{[X(t)+Y(t)]\,[X(t+\tau)+Y(t+\tau)]\}\\ &= E[X(t)X(t+\tau)+Y(t)X(t+\tau)+X(t)Y(t+\tau)+Y(t)Y(t+\tau)]\\ &= E[X(t)X(t+\tau)]+E[Y(t)X(t+\tau)]+E[X(t)Y(t+\tau)]\\ &\qquad +E[Y(t)Y(t+\tau)]\\ &= R_{XX}(\tau)+R_{YX}(\tau)+R_{XY}(\tau)+R_{YY}(\tau)\end{aligned}$$

Power spectrum:

$$S_{ZZ}(\omega) = \int_{-\infty}^{\infty} R_{ZZ}(\tau)e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} [R_{XX}(\tau)+R_{YX}(\tau)+R_{XY}(\tau)+R_{YY}(\tau)]e^{-i\omega\tau}\,d\tau$$

$$= S_{XX}(\omega)+S_{XY}(\omega)+S_{YX}(\omega)+S_{YY}(\omega)$$

(ii) If $X(t)$ and $Y(t)$ are orthogonal, then

$$R_{YX}(\tau) = 0 = R_{XY}(\tau)$$

Then the power spectrum,

$$S_{XY}(\omega) = 0 = S_{YX}(\omega)$$

$$\therefore \quad S_{ZZ}(\omega) = S_{XX}(\omega) + S_{YY}(\omega)$$

(iii) If $X(t)$ and $Y(t)$ are orthogonal, then

$$R_{YX}(\tau) = 0 = R_{XY}(\tau)$$
$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

$\therefore$

Mean square of $Z(t) = E[Z^2(t)] = R_{ZZ}(0)$

$\therefore$ $\quad R_{ZZ}(0) = R_{XX}(0) + R_{YY}(0)$

i.e. $\quad E[Z^2(t)] = E[X^2(t)] + E[Y^2(t)]$

= sum of the mean square of $X(t)$ and $Y(t)$

EXERCISES

Autocorrelation and Cross-correlation Functions

1. What is meant by ACF?
2. State any two properties of ACF.
3. Define ACF of a stationary process.
4. What is the ACF of a stochastic process measure?
5. Define cross-correlation function.
6. State any two properties of cross-correlation function.
7. If $R(\tau)$ is the ACF of a complex process, prove that $R^*(\tau) = R(-\tau)$.
8. If $R(\tau)$ is the ACF of a stationary process $\{X(t)\}$, prove that $\lim_{\tau \to \infty} [R(\tau)] = \mu_X^2$. Is it true for any stationary process?

   [Ans. No, it is true only when the stationary process does not contain periodic components]
9. Find the variance of the stationary process $\{X(t)\}$ whose ACF is given by $R(\tau) = 2 + 4e^{-2|\tau|}$. **[Ans.** $E[X(t)] = \sqrt{2}, E\{X^2(t)\} = 6, \text{Var}\{X(t)\} = 4$]
10. When the jointly stationary processes $\{X(t)\}$ and $\{Y(t)\}$ are independent, prove that $R_{XY}(\tau) = \mu_X\mu_Y$.

    [Ans. $R_{XY}(\tau) = E[X(t)Y(t-\tau)] = E[X(t)]E[Y(t-\tau)] = \mu_X\mu_Y$]
11. What is the difference between ensemble average and time average of a stochastic process $\{X(t)\}$?

    [Ans. The ensemble average is given by $E\{X(t)\} = \sum x_i p_i$ or $\int_x xf(x)dx$, the time average is $\bar{X}_T = \dfrac{1}{2T}\int_{-T}^{T} X(t)\,dt$. To compute $E\{X(t)\}$, we should know the probability distribution or density function of $X(t)$. To compute $\bar{X}_T$, it is enough we know a single sample function of the process]

12. Prove that the ACF $R(\tau)$ of a real process is an even function of τ.
13. If $R(\tau)$ is the ACF of a stationary process, prove that $|R(\tau)| \le R(0)$.
14. Find the mean of the stationary process $\{X(t)\}$ whose ACF is given by

$$R(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}.$$ **[Ans.** 2]

15. Find the variance of the stationary process $\{X(t)\}$ whose ACF is given by

$$R(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}.$$ **[Ans.** 9]

16. Find the cross-correlation function of two stationary processes that are orthogonal.
17. If $\{X(t)\}$ is a WSS process with autocorrelation function $R_{XX}(\tau)$ and if $Y(t) = X(t + a) - X(t - a)$, show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a)$.
18. If $\{X(t)\}$ and $\{Y(t)\}$ are independent WSS processes with zero means, find the autocorrelation function of $\{Z(t)\}$ when
    (i) $Z(t) = a + bX(t) + cY(t)$
    (ii) $Z(t) = aX(t)\, Y(t)$.
    [Ans. (i) $a^2 + b^2R_{XX}(\tau) + c^2R_{YY}(\tau)$, (ii) $a^2R_{XX}(\tau)\, R_{YY}(\tau)$]
19. If $\{X(t)\}$ is a WSS process with $E\{X(t)\} = 2$ and $R_{XX}(\tau) = 4 + e^{\frac{-|\tau|}{10}}$, find the mean and variance of $S = \int_0^1 X(t)dt$. **[Ans.** 2, $20(10e^{-0.1} - 9)$]
20. If the autocorrelation of a process $\{X(t)\}$ is $R(t_1, t_2)$ and if $Y(t) = X(t + a) - X(a)$ where a is a constant, express $R_{YY}(t_1, t_2)$ in terms of $R(t_1, t_2)$.
21. If $\{X(t)\}$ is a WSS process with autocorrelation function $R_{XX}(\tau)$ and if $Y(t) = X(t + a) - X(a)$, show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a)$.
22. Statistically independent zero mean random processes $\{X(t)\}$ and $\{Y(t)\}$ have ACF $R_{XX}(\tau) = e^{-|\tau|}$ and $R_{YY}(\tau) = \cos 2\pi\tau$ respectively. Find the ACF of the sum $Z(t) = X(t) + Y(t)$.

Power Spectral Density Function

23. Define the power spectral density (PSD) function of a stationary process.
24. Express each of ACF and PSD of a stationary process in terms of the other.
25. Write down the Wiener–Khinchine relations or theorem.

26. Define the cross power spectral density of the random processes $\{X(t)\}$ and $\{Y(t)\}$.
27. State any two properties of the PSD function of a stationary process.
28. What is average power of a WSS process $\{X(t)\}$ and express it in terms of the PSD function of the process. [**Ans.** $R(0) = E[X^2(t)]$]
29. Find the mean square value (or the average power) of the process $\{X(t)\}$, if its ACF is given by $R(\tau) = e^{-\tau^2/2}$. [**Ans.** 1]
30. Prove that the PSD function of a real stationary process is an even function.
31. Prove that the PSD function of a real or complex stationary process is a real function of ω.
32. Prove that the PSD function of a real WSS process is twice the Fourier cosine transform of its ACF.
33. Prove that the ACF of a real WSS process is half the Fourier inverse cosine transform of its PSD function.
34. Find the PSD function of a stationary process whose ACF is $e^{-|\tau|}$.

$$\left[\textbf{Ans.}\quad \frac{2}{1+\omega^2}\right]$$

35. For the random process $\{X(t)\}$ where $X(t) = a\cos(bt + Y)$ where Y is uniformly distributed over $(-\pi, \pi)$, find spectral density of the process $\{X(t)\}$.

$$\left[\textbf{Ans.}\quad \frac{\pi a^2}{2}[\delta(\omega - b) + \delta(\omega + b)]\right]$$

36. Find the power spectral density of the random process $\{X(t)\}$ where $X(t) = a\sin(bt + Y)$ and Y is uniformly distributed over $(0, 2\pi)$.

$$\left[\textbf{Ans.}\quad \frac{\pi a^2}{2}[\delta(\omega - b) + \delta(\omega + b)]\right]$$

37. Find the ACF of the random process $\{X(t)\}$ for which the PSD is given by

$$S(\omega) = 2\pi\delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}. \qquad [\textbf{Ans.}\quad 1 + e^{-\alpha|\tau|}]$$

38. Determine the PSD of the random process whose ACF is $R_{XX}(\tau) = Ae^{-\tau^2/2a^2}$ where A and a are constants. $\left[\textbf{Ans.}\quad A\sqrt{2\pi}e^{-a^2\omega^2/2}\right]$
39. An ergodic random process is known to have an ACF

$$R(\tau) = \begin{cases} 1 - |\tau|, & |\tau| < 1 \\ 0, & |\tau| > 1 \end{cases}$$

Find the PSD of the process. $\left[\textbf{Ans.}\quad \left(\frac{\sin \omega/2}{\omega/2}\right)^2\right]$

40. Find the power spectral density of the random process $\{X(t)\}$ if $E\{X(t)\} = 1$ and $R_{XX}(\tau) = 1 + e^{-\alpha|\tau|}$.

$$\left[\textbf{Ans.}\quad 2\pi\delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}\right]$$

41. If $\{X(t)\}$ is a stationary random process with PSD given by

$$S(\omega) = \begin{cases} S_0, & -W < \omega < W \\ 0, & \text{otherwise} \end{cases}$$

find the ACF and the mean square value of the process $\{X(t)\}$.

$$\left[\textbf{Ans.}\quad \frac{S_0 \sin W\tau}{\pi\tau}; \frac{WS_0}{\pi}\right]$$

42. For the process $\{X(t)\}$, where $X(t) = \sum_{i=1}^{n} (A_i \cos p_i t + B_i \sin p_i t)$, where A_i and B_i are uncorrelated random variables with mean zero and variance σ_i^2, show that the autocorrelation function is given by

$$R(\tau) = \sum_{i=1}^{n} \sigma_i^2 \cos p_i \tau.$$

Prove also that the power spectrum for this process is given by

$$S(\omega) = \pi \sum_{i=1}^{n} \sigma_i^2 [\delta(\omega - p_i) + \delta(\omega + p_i)]$$

43. Find the average power of the random process $\{X(t)\}$, if its power spectral density is given by

$$S(\omega) = \frac{10\omega^2 + 35}{(\omega^2 + 4)(\omega^2 + 9)}$$

(i) using $S(\omega)$ directly, and
(ii) using the autocorrelation function $R(\tau)$.

$$\left[\textbf{Hint:}\ \text{Average power} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega \text{ or } R(0)\right]$$

$$\left[\textbf{Ans.}\quad R(\tau) = \frac{11}{6}e^{-3|\tau|} - \frac{1}{4}e^{-2|\tau|},\ \text{average power} = \frac{19}{12}\right]$$

44. If $\{Y(t)\}$ is the moving time average of $\{X(t)\}$ over $\{t - T, t + T\}$, express $S_{YY}(\omega)$ in terms of $S_{XX}(\omega)$. Hence find the autocorrelation function of $\{Y(t)\}$ in terms of that of $\{X(t)\}$.

$$\left[\textbf{Ans.}\quad S_{YY}(\omega) + \frac{\sin^2 \omega T}{\omega^2 T^2} S_{XX}(\omega)\right.$$

$$\left. R_{YY}(\tau) = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\alpha|}{2T}\right) R_{XX}(\tau - \alpha)\, d\alpha \right]$$

45. A wide sense stationary noise process $W(t)$ has an ACF $R_{NN}(\tau) = \rho e^{-3|\tau|}$, $-\infty < \tau < \infty$ with ρ as a constant. Find the power density spectrum.

$$\left[\textbf{Ans.}\quad \frac{6\rho}{\omega^2 + 9}\right]$$

8

Linear Systems with Random Inputs

8.1 LINEAR SYSTEM

If $x(t)$ represents a sample function of a random process $\{X(t)\}$, the system produces an output or response $y(t)$ and the ensemble of the output functions forms a random porcess $\{y(t)\}$. This random process $\{y(t)\}$ can be considered as the output of the system or transformation 'f' with $\{X(t)\}$ as the input, the system is completely specified by the operator f.

Hence $X(t)$ means $X(s, t)$, $s \in S$ (sample space). If the system operates on both t and s, it is called stochastic and if the system operates only on 't' treating 's' as a parameter then the system is called a deterministic system.

In this chapter we are going to deal with deterministic systems only.

The functional relationship between the input $X(t)$ and the output $Y(t)$ is called a system, i.e.

$$Y(t) = f[X(t)], -\infty < t < \alpha$$

The system is said to be linear if

$$f[\alpha_1 X_1(t) + \alpha_2 X_2(t)] = \alpha_1 f[X_1(t)] + \alpha_2 f[X_2(t)]$$

8.2 LINEAR TIME INVARIANT SYSTEM

A linear system is said to be time invariant if the input $X(t)$ is time shifted by an amount, then the corresponding output $Y(t)$ should also be time shifted by the same amount, i.e. if $Y(t) = f[X(t)]$, then $Y(t + h) = f[X(t + h)]$ then the system 'f' is called a *time invariant system*. The system $Y(t) = f[X(t)]$ is said to be a *memoryless system*, if $Y(t)_{t\to t_1}$ depends only on $X(t)_{t\to t_1}$ not on the past or future values of $X(t)$, i.e. $Y(t_1) = f[X(t_1)]$.

The system is said to *be casual* if $Y(t)$ at t_1 depends only on the past values of $X(t)$ up to $t = t_1$, i.e. $Y(t_1) = f[X(t)]_{t\le t_1}$.

EXAMPLE 8.1 Examine whether the following systems are linear:

(i) $Y(t) = kX(t)$
(ii) $Y(t) = X^3(t)$.

Solution

(i) Given $Y(t) = kX(t) \;\Rightarrow\; Y(t) = f[X(t)] = kX(t)$

$\therefore$ $Y_1(t) = kX_1(t),\; Y_2(t) = kX_2(t)$

Let
$$\begin{aligned} Y_3(t) &= f[\alpha_1 X_1(t) + \alpha_2 X_2(t)] \\ &= k[\alpha_1 X_1(t) + \alpha_2 X_2(t)] \\ &= k\alpha_1 X_1(t) + k\alpha_2 X_2(t) \end{aligned}$$

$\therefore$ $Y_3(t) = \alpha_1 Y_1(t) + \alpha_2 Y_2(t)$

Hence the system is linear.

(ii) Given $Y(t) = X^3(t) \;\Rightarrow\; Y(t) = f[X(t)] = X^3(t)$

$\therefore$ $Y_1(t) = X_1^3(t),\; Y_2(t) = X_2^3(t)$

Let
$$\begin{aligned} Y_3(t) &= f[\alpha_1 X_1(t) + \alpha_2 X_2(t)] \\ &= [\alpha_1 X_1(t) + \alpha_2 X_2(t)]^3 \\ &= [\alpha_1 X_1(t)]^3 + [\alpha_2 X_2(t)]^3 + 3[\alpha_1 X_1(t)]^2\, \alpha_2 X_2(t) \\ &\qquad + 3[\alpha_2 X_2(t)]^2\, \alpha_1 X_1(t) \\ &\ne \alpha_1 Y_1(t) + \alpha_2 Y_2(t) \end{aligned}$$

Hence it is not linear.

8.3 SYSTEM IN THE FORM OF CONVOLUTION

If $h(t)$ is the system weighting function (unit impulse function), then the output $Y(t)$ is given in terms of the convolution of $h(t)$ and input $X(t)$ as

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)\,du$$

or

$$Y(t) = \int_{-\infty}^{\infty} h(t-u)X(u)\,du$$

If $h(t)$ is the unit impulse response of the system, then the Fourier transform of $h(t)$ is the system transfer function $H(\omega)$,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,dt$$

and
$$H^*(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{i\omega t}\,dt$$

Properties of the linear system

1. If a system is such that its input $X(t)$ and its output $Y(t)$ are related by a convolution integral $Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)\,du$, then the system is a linear time invariant system.

Proof To prove that the system is linear:

Let $X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$

Given
$$Y(t) = \int_{-\infty}^{\infty} h(u)\,X(t-u)\,du$$

$$= \int_{-\infty}^{\infty} X(u)h(t-u)\,du, \quad \text{by convolution definition}$$

$$= \int_{-\infty}^{\infty} [\alpha_1 X_1(u) + \alpha_2 X_2(u)]h(t-u)\,du$$

$$= \alpha_1 \int_{-\infty}^{\infty} X_1(u)h(t-u)\,du + \alpha_2 \int_{-\infty}^{\infty} X_2(u)h(t-u)\,du$$

$$= \alpha_1 Y_1(t) + \alpha_2 Y_2(t)$$

∴ The system is linear.

To prove that the system is time invariant:

Given:
$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)\,du$$

Then,
$$Y(t+h) = \int_{-\infty}^{\infty} h(u)X(t+h-u)\,du$$

∴ The system is time invariant.

Note: If $h(t)$ is absolutely integrable, then

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

In other words, for each bounded input, we get bounded output and, hence, the system is said to be stable.

If $h(t) = 0,\ t < 0$, then the system is said to be casual.

2. If the input to a time invariant stable linear system is a WSS process, then the output will also be a WSS process. **[AU December '06]**

Proof Given input $X(t)$ is a WSS random process i.e., $E[X(t)] = \mu$ is a constant and $R_{XX}(\tau)$ is a function of time difference

By definition,

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(u)\, E[X(t-u)]\,du$$

$$= \mu \int_{-\infty}^{\infty} h(u)\,du$$

= a constant, since $E[X(t)]$ is a constant

$\therefore$ $E[Y(t)]$ is a constant.

By definition,

$$R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u_1) X(t-u_1)\,du_1 \int_{-\infty}^{\infty} h(u_2) X(t+\tau-u_2)\,du_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)X(t-u_1)X(t+\tau-u_2)\,du_1 du_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-u_1)X(t+\tau-u_2)]\,h(u_1)h(u_2)\,du_1\,du_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+u_1-u_2)\,h(u_1)h(u_2)\,du_1 du_2$$

Since $R_{XX}(\tau + u_1 - u_2)$ is a function of time difference, $R_{YY}(t, t+\tau)$ will also be a function of time difference.

$\therefore$ $Y(t)$ is also a WSS random process.

3. If $X(t)$ is a WSS process and if $Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)\,du$, then prove

(i) $R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)$

(ii) $R_{YX}(\tau) = R_{XX}(\tau) * h(\tau)$ **[AU May '03]**

(iii) $R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$

Proof Given $X(t)$ is a WSS process, therefore $E[X(t)] = \mu$ is a constant and $R_{XX}(\tau)$ is a function of time difference

(i)
$$
\begin{aligned}
R_{XY}(\tau) &= E[X(t + \tau)Y(t)] \\
&= E\left[X(t-\tau)\int_{-\infty}^{\infty} h(u)X(t-u)\,du\right] \\
&= E\left[\int_{-\infty}^{\infty} h(u)X(t-\tau)\,X(t-u)\,du\right] \\
&= \int_{-\infty}^{\infty} h(u)E[X(t-\tau)X(t-u)]\,du \\
&= \int_{-\infty}^{\infty} h(u)R_{XX}(\tau-u)\,du
\end{aligned}
\tag{8.1}
$$

$\therefore \quad R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$

(ii)
$$
\begin{aligned}
R_{YX}(t) &= E[Y(\tau)\ X(t + \tau)] \\
&= E\left[\int_{-\infty}^{\infty} h(u)\ X(t-u)\ du\ X(t+\tau)\right] \\
&= E\left[\int_{-\infty}^{\infty} h(u)\ X(t-u)\ X(t+\tau)\ du\right] \\
&= \int_{-\infty}^{\infty} h(u)\ E[X(t-u)\ X(t+\tau)]du \\
&= \int_{-\infty}^{\infty} h(u)\ R_{XX}(\tau+u)\ du \\
&= \int_{-\infty}^{\infty} h(-v)\ R_{XX}(\tau-v)dv \quad [\text{taking } u = -v]
\end{aligned}
\tag{8.2}
$$

$\therefore \quad R_{YX} = R_{XX}(\tau) * h(-\tau)$

(iii) $$R_{YY}(\tau) = E[Y(t)\ Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, du\, Y(t+\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, y(t+\tau)\, du\right]$$

$$= \int_{-\infty}^{\infty} h(u)\, E[X(t-u)\, Y(t+\tau)]\, du$$

$$= \int_{-\infty}^{\infty} h(u)\, R_{XY}(\tau+u)\, dv$$

$$= \int_{-\infty}^{\infty} h(-v)\, R_{XY}(\tau-v)\, dv, \text{ (taking } u = -v)$$

$$\therefore \quad R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) \tag{8.3}$$

Using (8.1) in (8.3), we have

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

Hence proved

4. If $X(t)$ is a WSS process and if $Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u) du$, then $R_{YY}(\tau) =$

$R_{XX}(\tau) = R_{XX}(\tau) * k(\tau)$ where $k(t) = h(t) * h(-t) = \int_{-\infty}^{\infty} h(u)\, h(t+u)\, du$

Proof By definition

$$R_{YY}(t) = E[Y(t)\ Y(t+\tau)] = E[Y(t+\tau)\ Y(\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u_1)\, X(t+\tau-u_1)\, du_1 \int_{-\infty}^{\infty} h(u_2)\, X(t-u_2) du_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, X(t+\tau-u_1)\, X(t-u_2)\, du_1 du_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1) h(u_2)\, E[X(t+\tau-u_1) X(t-u_2)] du_1 du_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(u_1 - u_2 - \tau)du_1du_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\,h(u_2)R_{XX}(\tau + u_2 - u_1)du_1du_2 \quad [\because \ R_{XX}(\tau) = R_{XX}(-\tau)]$$

$$= \int_{-\infty}^{\infty} h(u_2)\left[\int_{-\infty}^{\infty} h(u_1)\,R_{XX}(\tau + u_2 - u_1)du_1\right]du_2$$

Replace $u_1 - u_2 = v$ [in the inner integral, u_2 is treated as a constant]

$$du_1 = dv, \quad u_1 = v + u_2$$

when $u_1 = -\infty$, $v = -\infty$ and $u_1 = \infty$, $v = \infty$

$$= \int_{-\infty}^{\infty} h(u_2)\left[\int_{-\infty}^{\infty} h(u_2 + v)\,R_{XX}(\tau - v)dv\right]du_2$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(u_2)\,h(u_2 + v)\,du_2\right]R_{XX}(\tau - v)\,dv$$

$$= \int_{-\infty}^{\infty} k(v)\,R_{XX}(\tau - v)\,dv$$

$$= R_{XX}(\tau) * k(\tau)$$

Hence proved.

5. If the output of the input $X(t)$ is defined as $Y(t) = \frac{1}{T}\int_{t-T}^{t} X(s)ds$, prove that $X(t)$ and $Y(t)$ are related by means of convolution integral. Find the unit impulse response of the system.

Proof Given: $Y(t) = \frac{1}{T}\int_{t-T}^{t} X(s)ds$

Put $s = t - u$, $ds = -du$ [treating t as a constant/parameter].
When $s = t - T$, $u = T$ and $s = t$, $u = 0$.

$$Y(t) = \frac{1}{T}\int_{T}^{0} X(t-u)\,(-du)$$

$$= \frac{1}{T}\int_{0}^{T} X(t-u)\,du$$

$$= \int_{0}^{T} \frac{1}{T} X(t-u)\,du$$

Let the unit impulse response of the system

$$h(t) = \begin{cases} \frac{1}{T}, 0 \le t \le T \\ 0, \text{ otherwise} \end{cases}$$

$\therefore$ $X(t)$ and $Y(t)$ are related by means of convolution integral as

$$Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\, du$$

6. If the input $X(t)$ and the output $Y(t)$ are connected by a differential equation $T\frac{dY(t)}{dt} + Y(t) = X(t)$, prove that they can be related by means of convolution integral. Assume that $X(t)$ and $Y(t)$ are zero for $T \le 0$.

Proof Given: $$T\frac{dY(t)}{dt} + Y(t) = X(t)$$

$$\frac{dY(t)}{dt} + \frac{1}{T}Y(t) = \frac{X(t)}{T}$$

which is a first-order linear differential equation in $Y(t)$

$\therefore$ $$\text{Integrating factor} = e^{\int p\,dt} = e^{\int \frac{1}{T}dt} = e^{\frac{t}{T}}$$

The solution is

$$Y(t)e^{\frac{t}{T}} = \int \frac{X(t)}{T} e^{\frac{t}{T}} dt = \int \frac{X(u)}{T} e^{\frac{u}{T}} du \qquad [\because \text{ replacing } t \text{ by } u]$$

$$Y(t) = \int X(u) \frac{e^{\frac{u}{T}}}{T} \cdot e^{\frac{-t}{T}} du$$

$$Y(t) = \int \frac{1}{T} X(u) \cdot e^{\frac{-1}{T}(t-u)} du$$

$$= \int X(u) \frac{e^{\frac{-1}{T}(t-u)}}{T} du$$

Given $X(t)$ and $Y(t)$ are zero when $T \le 0$

$\therefore$ $$Y(t) = \int_0^{\infty} X(u) \frac{e^{-\frac{1}{T}(t-u)}}{T} du$$

$$= \int_0^{\infty} X(u) \frac{1}{T} e^{-\frac{1}{T}(t-u)} du$$

Comparing it with $Y(t) = \int_{-\infty}^{\infty} X(u)\, h(t-u)\, du$, we have

$$h(t) = \begin{cases} \frac{1}{T} e^{-\frac{1}{T}t}, & 0 \le t \le \infty \\ 0, & \text{otherwise} \end{cases}$$

8.4 RELATION BETWEEN INPUT AND OUTPUT OF A LINEAR TIME INVARIANT SYSTEM

If $X(t)$ is the input and $Y(t)$ is the corresponding output of the system, then

$$Y(t) = \int_{-\infty}^{\infty} X(u)\, h(t-u)\, du = \int_{-\infty}^{\infty} X(t-u)\, h(u)\, du = X(t) * h(t)$$

i.e. output is the convolution of input $X(t)$ and impulse function $h(t)$.

The system is said to be stable, if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

The mean and autocorrelation of a output system:

Mean

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(u)\, h(t-u)\, du\right] = E\left[\int_{-\infty}^{\infty} X(t-u)\, h(u)\, du\right]$$

$$= \int_{-\infty}^{\infty} E[X(t-u)]\, h(u)\, du$$

$$= E[X(t)] * h(t)$$

$$\therefore \quad \mu_Y(t) = \mu_X(t) * h(t)$$

Autocorrelation

$$Y(t) = \int_{-\infty}^{\infty} X(t-u)\, h(u)\, du = \int_{-\infty}^{\infty} X(t-\alpha)\, h(\alpha)\, d\alpha = \int_{-\infty}^{\infty} X(t-\beta)\, h(\beta)\, d\beta$$

$$R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} X(t-\alpha)\, h(\alpha) d\alpha \int_{-\infty}^{\infty} X(t+\tau-\beta)\, h(\beta)\, d\beta\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\alpha)\, X(t+\tau-\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[X(t-\alpha)\, X(t+\tau-\beta)\right] h(\alpha) h(\beta) d\alpha d\beta$$

$$\therefore \quad R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [R_{XX}(\tau+\alpha-\beta)] h(\alpha) h(\beta) d\alpha d\beta$$

8.5 RELATION CONNECTING THE INPUT $X(t)$, OUTPUT $Y(t)$ AND ITS CROSS-CORRELATION

8.5.1 PSD of a System

If $X(t)$ is the input to a linear time invariant system and if $Y(t)$ is the corresponding output of the system, then the PSD of the output $Y(t)$ is

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau} d\tau = F[R_{YY}(\tau)]$$

and also

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

EXAMPLE 8.2 If $Y(t)$ is the output process, when an input process $X(t)$ is applied to the linear time invariant system with impulse response. The autocorrelation of the output system $Y(t)$ is

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

where $H(\omega)$ is the system transfer function. **[AU May '03; '04]**

Solution Given $Y(t)$ is the output process, when an input process $X(t)$ is applied to the linear time invariant system with impulse response. The relation between $X(t)$ and $Y(t)$ of a linear time invariant system is

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [R_{XX}(\tau+\alpha-\beta)] h(\alpha)\, h(\beta)\, d\alpha d\beta$$

By definition,

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[R_{XX}(\tau+\alpha-\beta)]\,h(\alpha)\,h(\beta)\,e^{-i\omega\tau}d\alpha d\beta d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}[R_{XX}(\tau+\alpha-\beta)]e^{-i\omega\tau}d\tau\right\}h(\alpha)\,h(\beta)\,d\alpha\,d\beta$$

Substitute $u = \tau + \alpha - \beta \Rightarrow \tau = u - \alpha + \beta \Rightarrow d\tau = du$

$$\therefore \quad S_{YY}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[R_{XX}(u)\,e^{-i\omega(u-\alpha+\beta)}du]\,h(\alpha)\,h(\beta)d\alpha d\beta$$

$$= \int_{-\infty}^{\infty}h(\alpha)\int_{-\infty}^{\infty}h(\beta)\int_{-\infty}^{\infty}[R_{XX}(u)]e^{-i\omega u}\,e^{i\omega\alpha}\,e^{-i\omega\beta}du\,d\alpha\,d\beta$$

$$= \int_{-\infty}^{\infty}h(\alpha)\,e^{i\omega\alpha}d\alpha\int_{-\infty}^{\infty}h(\beta)\,e^{-i\omega\beta}d\beta\int_{-\infty}^{\infty}R_{XX}(u)]\,e^{-i\omega u}\,du$$

$$= H(\omega)\,H^*(\omega)\,S_{XX}(\omega)$$
$$= |H(\omega)|^2\,S_{XX}(\omega)$$

Alternate method:

We know that

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

Taking Fourier transform on both sides and using convolution theorem on Fourier transform as

$$F[f(t) * g(t)] = F[f(t)]\,F[(g(t)],\text{ we get}$$
$$S_{YY}(\omega) = S_{XX}(\omega)\,H(\omega)\,H^*(\omega) = S_{XX}(\omega)\,|H(\omega)|^2$$

EXAMPLE 8.3 Show that $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ where $S_{XX}(\omega)$ and $S_{YY}(\omega)$ are the power spectral density functions of the input $X(t)$ and the output $Y(t)$ respectively and $H(\omega)$ is the system transfer function.

[AU May/June '09]

Solution Let $X(t)$ and $Y(t)$ be the input and output of the system respectively. Let the autocorrelation function of $X(t)$ be $R_{XX}(\tau)$ and $Y(t)$ be $R_{YY}(\tau)$, and the cross-correlation of $X(t)$ and $Y(t)$ be $R_{XY}(\tau)$. Let $h(t)$ be the unit impulse response of the system whose Fourier transform is the system transfer function $H(\omega)$. We know that by the convolution theorem on Fourier transform,

$$F[f(t) * g(t)] = F(\omega)\, G(\omega) \tag{i}$$

Using the relation between the input and output of the system,

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$$

We have

$$R_{XY}(\tau) = E[X(t-\tau)Y(t)]$$

$$= E\left[X(t-\tau)\int_{-\infty}^{\infty} h(u) X(t-u)\, du\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(u) X(t-\tau)\, X(t-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} h(u)\, E[X(t-\tau)\, X(t-u)]\, du$$

$$= \int_{-\infty}^{\infty} h(u)\, R_{XX}(\tau-u)\, du$$

$$\therefore \quad R_{XY}(\tau) = R_{XX}(\tau) * h(\tau) \tag{ii}$$

$$R_{YX}(t) = E[Y(\tau)\, X(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, du\, X(t+\tau)\right]$$

$$= \int_{-\infty}^{\infty} h(u)\, E[X(t-u)\, X(t+\tau)] du$$

$$= \int_{-\infty}^{\infty} h(u)\, R_{XX}(\tau+u) du$$

$$= \int_{-\infty}^{\infty} h(-v)\, R_{XX}(\tau-v)\, dv \quad \text{(taking } u = -v)$$

$$\therefore \quad R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau) \tag{iii}$$

$$R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)du\, Y(t+\tau)\right]$$

$$= \int_{-\infty}^{\infty} h(u)\, E[X(t-u)\, Y(t+\tau)]du$$

$$= \int_{-\infty}^{\infty} h(u)\, R_{XY}(\tau+u)\, du$$

$$= \int_{-\infty}^{\infty} h(-v)\, R_{XY}(\tau-v)dv \qquad \text{(taking } u = v\text{)}$$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) \qquad \text{(iv)}$$

Using (ii) in (iv), we get

$$R_{YY}(\tau) = R_{XY} * h(\tau) * h(-\tau)$$

Taking Fourier transform on both sides,

$$S_{YY}(\tau) = S_{XX}(\tau)\, H^*(\omega)\, H(\omega), \text{ using Eq. (i)}$$
$$= S_{XX}(\tau)\, |H(\omega)|^2$$

Hence proved.

Note: Here $\qquad H(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega\tau} d\tau$

and $\qquad H^*(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{i\omega\tau} d\tau$

Put $\tau = -t$, $d\tau = -dt$, when $\tau = -\infty$, $t = \infty$ and $\tau = \infty$, $t = -\infty$

$$H^*(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{i\omega\tau} d\tau = \int_{-\infty}^{\infty} h(-t)\, e^{-i\omega t}(-dt) = \int_{\infty}^{-\infty} h(-t)\, e^{-i\omega t} dt$$
$$= F[h(-t)]$$

Hence proved.

EXAMPLE 8.4 Suppose that $X(t)$ is the input to a linear time invariant system with impulse response $h_1(t)$ and that $Y(t)$ is the input to another linear time invariant system with impulse response $h_2(t)$. It is assumed that $X(t)$ and $Y(t)$ are jointly WSS. Let $V(t)$ and $Z(t)$ denote the random processes at the respective systems output. Find the cross-correlation function of $V(t)$ and $Z(t)$.

Solution $R_{VZ}(t_1, t_2) = E[V(t_1) Z(t_2)]$

where $V(t_1)$ and $Z(t_2)$ are the outputs of the inputs $X(t_1)$ and $Y(t_2)$ with impulse responses $h_1(t)$ and $h_2(t)$ respectively. Then we have,

$$V(t_1) = \int_{-\infty}^{\infty} X(t_1 - \alpha)\, h_1(\alpha)\, d\alpha$$

and
$$Z(t_2) = \int_{-\infty}^{\infty} Y(t_2 - \beta)\, h_2(\beta)\, d\beta$$

$\therefore$
$$R_{VZ}(t_1, t_2) = E[X(t_1)\, Z(t_2)]$$

$$= E\left[\int_{-\infty}^{\infty} X(t_1 - \alpha)\, h_1(\alpha)\, d\alpha \int_{-\infty}^{\infty} Y(t_2 - \beta)\, h_2(\beta)\, d\beta\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1 - \alpha)\, Y(t_2 - \beta)\, h_1(\alpha)\, h_2(\beta)\, d\alpha d\beta\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t_1 - \alpha)\, Y(t_2 - \beta)]\, h_1(\alpha)\, h_2(\beta)\, d\alpha d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XY}(t_2 - \beta - t_1 + \alpha)\, h_1(\alpha)\, h_2(\beta)\, d\alpha d\beta$$

Given that X and Y are jointly WSS random processes and so their cross-correlation is a function of time difference.

Taking $t_2 - t_1 = \tau$, we get

$$R_{VZ}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XY}(\tau - \beta + \alpha)\, h_1(\alpha)\, h_2(\beta)\, d\alpha d\beta = R_{VZ}(\tau)$$

EXAMPLE 8.5 A linear system is described by the impulse response $h(t) = \left(\frac{1}{RC} e^{\frac{-t}{RC}}\right) u(t)$. Assume an input process whose ACF is $A\delta(\tau)$. Find the mean and ACF of the output process. **[AU December '09]**

Solution Let
$$\beta = \frac{1}{RC}$$

$\therefore$
$$h(t) = \left(\frac{1}{RC} e^{\frac{-t}{RC}}\right) u(t) = \beta e^{-\beta t} u(t)$$

We know that

$$S_{YY}(\omega) = |H(\omega)|^2\, S_{XX}(\omega)$$

$\therefore |H(\omega)|^2 = H(\omega)\, H^*(\omega)$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} \beta e^{-\beta t}\, u(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} 0\, dt + \int_{0}^{\infty} \beta e^{-\beta t}(1) e^{-i\omega t}\, dt, \text{ using the definition of } u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$= \beta \int_{0}^{\infty} e^{-(\beta + i\omega)t}\, dt = \beta \left[\frac{e^{-(\beta + i\omega)t}}{-(\beta + i\omega)} \right]_0^{\infty}$$

$$= \beta \left[e^{-\infty} + \frac{e^0}{(\beta + i\omega)} \right] = \frac{\beta}{(\beta + i\omega)}$$

$$\therefore \quad H(\omega) = \frac{\beta}{(\beta + i\omega)}$$

and
$$H^*(\omega) = \frac{\beta}{(\beta - i\omega)}$$

$$|H(\omega)|^2 = \frac{\beta}{(\beta + i\omega)} \frac{\beta}{(\beta - i\omega)} = \frac{\beta^2}{(\beta^2 + \omega^2)}$$

$$\therefore \quad S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$S_{YY}(\omega) = \frac{\beta^2}{(\beta^2 + \omega^2)} S_{XX}(\omega) \qquad \text{(i)}$$

We know that

$$S_{XX}(\omega) = F[R_{XX}(\tau)]$$
$$= F[A\,\delta(\tau)]$$
$$= AF[\delta(\tau)] = A$$

Substituting in Eq. (i), we get

$$S_{YY}(\omega) = \frac{\beta^2}{\beta^2 + \omega^2} A$$

$$\therefore \quad R_{YY}(\tau) = F^{-1}\left(\frac{A\beta^2}{\beta^2 + \omega^2} \right)$$

$$= A\beta^2 F^{-1}\left(\frac{1}{\omega^2 + \beta^2} \right)$$

$$= A\beta^2 \frac{1}{2\beta} e^{-\beta|\tau|}$$

$$\therefore \quad R_{YY}(\tau) = \frac{A\beta}{2} e^{-\beta|\tau|}$$

EXAMPLE 8.6 Given $E[X(t)] = 0$, $H(\omega) = \dfrac{R}{R + iL\omega}$ and $R_{YY}(\tau) = e^{-\alpha|\tau|}$. Find $S_{YY}(\omega)$ and $R_{YY}(\tau)$. **[AU December '08]**

Solution We know that

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \tag{i}$$

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{R}{R + iL\omega}\frac{R}{R - iL\omega} = \frac{R^2}{R^2 + (L\omega)^2}$$

To find $S_{YY}(\omega)$:

$$S_{XX}(\omega) = F[R_{XX}(\tau)] = F[e^{-\alpha|\tau|}] = \frac{2\alpha}{(\alpha^2 + \omega^2)}$$

Using it Eq. (i), we get

$$\therefore \quad S_{YY}(\omega) = \frac{R^2}{R^2 + (L\omega)^2}\frac{2\alpha}{(\alpha^2 + \omega^2)} = \frac{2\alpha R^2}{[R^2 + (L\omega)^2](\alpha^2 + \omega^2)}$$

To find $R_{YY}(\tau)$:

$$R_{YY}(\tau) = F^{-1}[S_{YY}(\omega)] = F^{-1}\left\{\frac{2\alpha R^2}{[R^2 + (L\omega)^2](\alpha^2 + \omega^2)}\right\} \tag{i}$$

Let $$\frac{1}{[R^2 + (L\omega)^2](\alpha^2 + \omega^2)} = \frac{A}{[R^2 + (L\omega)^2]} + \frac{B}{(\alpha^2 + \omega^2)}$$

$$1 = A(\alpha^2 + \omega^2) + B[R^2 + (L\omega)^2] \tag{ii}$$

Put $\omega^2 = -\alpha^2$ in (ii)

$$1 = A(0) + B[R^2 + L^2(-\alpha^2)]$$

$$\therefore \quad B = \frac{1}{R^2 - L^2\alpha^2} = \frac{1}{L^2}\left(\frac{1}{\dfrac{R^2}{L^2} - \alpha^2}\right)$$

Put $\omega^2 = \dfrac{-R^2}{L^2}$ in (ii)

$$1 = A\left(\alpha^2 - \frac{R^2}{L^2}\right) + B(0)$$

$$\therefore \quad A = \frac{1}{\alpha^2 - \dfrac{R^2}{L^2}}$$

$$\therefore \quad R_{YY}(\tau) = 2\alpha R^2 F^{-1}\left\{\frac{\dfrac{1}{\alpha^2 - \dfrac{R^2}{L^2}}}{[R^2 + (L\omega)^2]} + \frac{\dfrac{1}{L^2\left(\dfrac{R^2}{L^2} - \alpha^2\right)}}{(\alpha^2 + \omega^2)}\right\}$$

$$= \frac{2\alpha R^2}{\alpha^2 - \dfrac{R^2}{L^2}} F^{-1}\left\{\frac{1}{[R^2 + (L\omega)^2]} - \frac{1}{L^2(\alpha^2 + \omega^2)}\right\}$$

$$= \frac{2\alpha R^2}{L^2\left[\alpha^2 - \left(\dfrac{R}{L}\right)^2\right]} F^{-1}\left\{\frac{1}{\left[\left(\dfrac{R}{L}\right)^2 + \omega^2\right]} - \frac{1}{(\alpha^2 + \omega^2)}\right\}$$

$$= \frac{2\alpha R^2}{L^2\left[\alpha^2 - \left(\dfrac{R}{L}\right)^2\right]} \left\{F^{-1}\left[\frac{1}{\left(\dfrac{R}{L}\right)^2 + \omega^2}\right] - F^{-1}\left[\frac{1}{\alpha^2 + \omega^2}\right]\right\}$$

$$= \frac{2\alpha\left(\dfrac{R}{L}\right)^2}{\alpha^2 - \left(\dfrac{R}{L}\right)^2}\left[\frac{1}{2\left(\dfrac{R}{L}\right)} e^{-\frac{R}{L}|\tau|} - \frac{1}{2\alpha} e^{-\alpha|\tau|}\right]$$

$$\therefore \quad R_{YY}(\tau) = \frac{1}{\alpha^2 - \left(\dfrac{R}{L}\right)^2}\left[\frac{\alpha R}{L} e^{-\frac{R}{L}|\tau|} - \left(\frac{R}{L}\right)^2 e^{-\alpha|\tau|}\right]$$

EXAMPLE 8.7 Given $R_{XX}(\tau) = Ae^{-\alpha|\tau|}$ and $h(t) = e^{-\beta t}u(\tau)$ where

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the spectral density of the output $Y(t)$ and hence, find $R_{YY}(\tau)$.

[AU December '08]

Solution We know that

$$S_{XX}(\omega) = F[R_{XX}(\tau)]$$

$$S_{XX}(\omega) = F(Ae^{-\alpha|\tau|}) = A\left(\frac{2\alpha}{\alpha^2 + \omega^2}\right)$$

$$H(\omega) = F[h(t)] = F[e^{-\beta t}u(t)] = \frac{1}{\beta + i\omega}$$

$$|H(\omega)|^2 = H(\omega)\, H^*(\omega) = \left(\frac{1}{\beta + i\omega}\right)\left(\frac{1}{\beta - i\omega}\right) = \frac{1}{\beta^2 + \omega^2}$$

Therefore, spectral density of the output $Y(t)$ is

$$S_{YY}(\omega) = \frac{1}{\beta^2 + \omega^2}\left[A\left(\frac{2\alpha}{\alpha^2 + \omega^2}\right)\right] = A2\alpha\left[\frac{1}{(\beta^2 + \omega^2)(\alpha^2 + \omega^2)}\right] \quad \text{(i)}$$

By resolving into partial fractions,

$$\frac{1}{(\beta^2 + \omega^2)(\alpha^2 + \omega^2)} = \frac{A}{(\alpha^2 + \omega^2)} + \frac{B}{(\beta^2 + \omega^2)}$$

$$1 = A(\beta^2 + \omega^2) + B(\alpha^2 + \omega^2) \quad \text{(ii)}$$

Put $\omega^2 = -\beta^2$ in (ii)	Put $\omega^2 = -\alpha^2$
$1 = A(0) + B(\alpha^2 - \beta^2)$	$1 = A(\beta^2 - \alpha^2) + B(0)$
$B = \dfrac{1}{\alpha^2 - \beta^2}$	$\therefore \quad A = \dfrac{1}{\beta^2 - \alpha^2}$

$$\therefore \quad \frac{1}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)} = \frac{\frac{1}{\alpha^2 - \beta^2}}{(\beta^2 + \omega^2)} + \frac{\frac{1}{\beta^2 - \alpha^2}}{(\alpha^2 + \omega^2)}$$

Substituting in Eq. (i)

$$\therefore \quad S_{YY}(\omega) = A2\alpha\left[\frac{\frac{1}{\alpha^2 - \beta^2}}{(\beta^2 + \omega^2)} + \frac{\frac{1}{\beta^2 - \alpha^2}}{(\alpha^2 + \omega^2)}\right]$$

$$= \frac{A2\alpha}{\alpha^2 - \beta^2}\left[\frac{1}{(\beta^2 + \omega^2)} - \frac{1}{(\alpha^2 + \omega^2)}\right]$$

Taking inverse Fourier transform on both sides, we get

$$F^{-1}[S_{YY}(\omega)] = F^{-1}\left\{\frac{A2\alpha}{\alpha^2 - \beta^2}\left[\frac{1}{(\beta^2 + \omega^2)} - \frac{1}{(\alpha^2 + \omega^2)}\right]\right\}$$

$$R_{YY}(\tau) = \frac{A2\alpha}{\alpha^2 - \beta^2}\left\{F^{-1}\left[\frac{1}{(\beta^2 + \omega^2)}\right] - F^{-1}\left[\frac{1}{(\alpha^2 + \omega^2)}\right]\right\}$$

$$= \frac{2\alpha A}{\alpha^2 - \beta^2}\left\{\frac{1}{2\beta}e^{-\beta|\tau|} - \frac{1}{2\alpha}e^{-\alpha|\tau|}\right\}$$

$$\therefore \quad R_{YY}(\tau) = \frac{\alpha A}{\beta(\alpha^2 - \beta^2)}e^{-\beta|\tau|} - \frac{A}{(\alpha^2 - \beta^2)}e^{-\alpha|\tau|}$$

EXAMPLE 8.8 The relation between input $X(t)$ and output $Y(t)$ of the diode is expressed as $Y(t) = X^2(t)$. Let $X(t)$ be a zero mean stationary Gaussian random process with ACF $R_{XX}(\tau) = e^{-\alpha|\tau|}$, $\alpha > 0$. Find the output autocorrelation $R_{YY}(\tau)$ and the output PSD $S_{YY}(\omega)$.

Solution Given $Y(t) = X^2(t)$ where $X(t)$ is the zero mean stationary Gaussian random process.

$$R_{YY}(t_1, t_2) = E[Y(t_1)Y(t_2)] = E[X^2(t_1)X^2(t_2)]$$

$$\therefore \quad R_{YY}(t_1, t_2) = E[X^2(t_1)]\, E[X^2(t_2)] + 2\{E[X(t_1)\, X(t_2)]\}^2$$

$$= R_{XX}(0)\, R_{XX}(0) + 2[R_{XX}(t_2 - t_1)]^2$$

$$= R_{XX}(0)\, R_{XX}(0) + 2[R_{XX}(\tau)]^2$$

$$\therefore \quad R_{YY}(t_1, t_2) = 1 + 2(e^{-\alpha|\tau|})^2, \text{ given } R_{XX}(\tau) = e^{-\alpha|\tau|} \text{ and so } R_{XX}(0) = 1$$

$$R_{YY}(\tau) = R_{YY}(t_1, t_2) = 1 + 2e^{-2\alpha|\tau|}$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau}\, d\tau = F[R_{YY}(\tau)]$$

$$= F[1 + 2e^{-2\alpha|\tau|}] = F(1) + 2F(e^{-2\alpha|\tau|})$$

$$= 2\pi\delta(\omega) + \frac{2 \times 4\alpha}{(2\alpha)^2 + \omega^2}$$

$$\therefore \quad S_{YY}(\omega) = 2\pi\delta(\omega) + \frac{8\alpha}{4\alpha^2 + \omega^2}$$

EXAMPLE 8.9 The input to a RC filter is a white noise process with ACF $R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$. If $H(\omega) = \frac{1}{1 + i\omega RC}$, find the autocorrelation and the mean square value of the output $Y(t)$.

Solution We know that

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

Given:

$$R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$$

and

$$H(\omega) = \frac{1}{1 + i\omega RC}$$

$$\therefore \quad S_{XX}(\omega) = F[R_{XX}(\tau)] = F\left[\frac{N_0}{2}\delta(\tau)\right] = \frac{N_0}{2} F[\delta(\tau)]$$

$$= \frac{N_0}{2}(1)$$

$$H(\omega)\, H^*(\omega) = |H(\omega)|^2 = \frac{1}{1+\omega^2 R^2 C^2}$$

$$\therefore \quad S_{YY}(\omega) = \left(\frac{1}{1+\omega^2 R^2 C^2}\right)\left(\frac{N_0}{2}\right)$$

$$= \frac{N_0}{2} \frac{1}{R^2C^2\left(\frac{1}{R^2C^2}+\omega^2\right)}$$

$$= \frac{N_0}{2R^2C^2} \frac{1}{\left(\frac{1}{R^2C^2}+\omega^2\right)}$$

$$R_{YY}(\omega) = F^{-1}[S_{YY}(\omega)] = F^{-1}\left[\frac{N_0}{2R^2C^2}\frac{1}{\left(\omega^2+\frac{1}{R^2C^2}\right)}\right]$$

$$= \frac{N_0}{2R^2C^2} F^{-1}\left[\frac{1}{\left(\omega^2+\frac{1}{R^2C^2}\right)}\right] = \frac{N_0}{2R^2C^2}\left[\frac{e^{\left(\frac{-1}{RC}\right)|\tau|}}{\frac{2}{RC}}\right]$$

$$R_{YY}(\tau) = \frac{N_0}{4RC} e^{\left(\frac{-1}{RC}\right)|\tau|}$$

$\therefore$ Mean square value of $Y(t)$ is, $R_{YY}(0) = E[Y^2(t)] = \dfrac{N_0}{4RC}$.

EXAMPLE 8.10 A circuit has an impulse response given by

$$h(t) = \begin{cases} \frac{1}{T}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Evaluate $S_{YY}(\omega)$ in terms of $S_{XX}(\omega)$. **[AU December '09]**

Solution Given: $h(t) = \begin{cases} \frac{1}{T}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$

$$\therefore \quad H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{i\omega t}\, dt = \frac{1}{T}\int_0^T {}^{-i\omega t}\, dt$$

$$= \frac{1}{T}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_0^T = -\frac{1}{T}\left(\frac{1-e^{-i\omega T}}{i\omega}\right)$$

$$= \frac{i}{T\omega}(1 - \cos\omega T + i\sin\omega T)$$

$$H(\omega) = \frac{1}{T\omega}[-\sin\omega T + i(1 - \cos\omega T)]$$

$$|H(\omega)|^2 = H(\omega)\,H^*(\omega) = \frac{1}{\omega^2 T^2}[\sin^2\omega T + (1 - \cos\omega T)^2]$$

$$= \frac{1}{\omega^2 T^2}[\sin^2\omega T + 1 - 2\cos\omega T + \cos^2\omega T]$$

$$= \frac{1}{\omega^2 T^2}(2 - 2\cos\omega T)$$

$$= \frac{2}{\omega^2 T^2}(1 - \cos\omega T)$$

$$= \frac{2}{\omega^2 T^2}\sin^2\frac{\omega T}{2}$$

$$\therefore \qquad S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \left(\frac{2}{\omega^2 T^2}\sin^2\frac{\omega T}{2}\right) S_{XX}(\omega)$$

EXAMPLE 8.11 A random process $X(t)$ having ACF $R_{XX}(\tau) = Ce^{-\alpha|\tau|}$ where C and α are real positive constants, applied to the input of the system with impulse response

$$h(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t > 0 \\ 0, & \text{for } t < 0 \end{cases}$$

where $\lambda > 0$. Find the ACF of the output response $Y(t)$ and cross-correlation function $R_{XY}(\tau)$.

Solution

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \tag{i}$$

$$S_{XX}(\omega) = F[R_{XX}(\tau)] = F(Ce^{-\alpha|\tau|})$$

$$\therefore \qquad S_{XX}(\omega) = C\frac{2\alpha}{\alpha^2 + \omega^2} = \frac{2C\alpha}{\alpha^2 + \omega^2}$$

$$H(\omega) = F[h(t)] = F[\lambda e^{-\lambda t}u(t)] = \lambda F[e^{-\lambda t}u(t)]$$

$$H(\omega) = \lambda\left(\frac{1}{\lambda + i\omega}\right)$$

$$|H(\omega)|^2 = \left(\frac{\lambda}{\lambda + i\omega}\right)\left(\frac{\lambda}{\lambda - i\omega}\right) = \left(\frac{\lambda^2}{\lambda^2 + \omega^2}\right)$$

Substituting in Eq. (i)

$$\therefore \qquad S_{YY}(\omega) = \left(\frac{\lambda^2}{\lambda^2 + \omega^2}\right)\left(\frac{2C\alpha}{\alpha^2 + \omega^2}\right) = 2C\alpha\lambda^2\left[\frac{1}{(\lambda^2 + \omega^2)(\alpha^2 + \omega^2)}\right] \tag{i}$$

Resolving into partial fraction

$$\frac{1}{(\lambda^2+\omega^2)(\alpha^2+\omega^2)} = \frac{A}{(\lambda^2+\omega^2)} + \frac{B}{(\alpha^2+\omega^2)}$$

$$1 = A(\alpha^2 + \omega^2) + B(\lambda^2 + \omega^2) \qquad \text{(ii)}$$

Substituting in Eq. (i)

$$\begin{array}{l|l} \omega^2 = -\alpha^2 & \omega^2 = -\lambda^2 \\ 1 = A(0) + B(\lambda^2 - \alpha^2) & 1 = A(\alpha^2 - \lambda^2) + B(0) \\ B = \dfrac{1}{\lambda^2 - \alpha^2} & A = \dfrac{1}{\alpha^2 - \lambda^2} \end{array}$$

$$\therefore \quad S_{YY}(\omega) = 2C\alpha\lambda^2\left[\frac{\frac{1}{\alpha^2-\lambda^2}}{(\lambda^2+\omega^2)} + \frac{\frac{1}{\lambda^2-\alpha^2}}{(\alpha^2+\omega^2)}\right]$$

$$= \frac{2C\alpha\lambda^2}{\alpha^2-\lambda^2}\left[\frac{1}{(\lambda^2+\omega^2)} - \frac{1}{(\alpha^2+\omega^2)}\right]$$

$$\therefore \quad F^{-1}[S_{YY}(\omega)] = R_{YY}(\tau)$$

$$= \frac{2C\alpha\lambda^2}{\alpha^2-\lambda^2}\left\{F^{-1}\left[\frac{1}{(\lambda^2+\omega^2)}\right] - F^{-1}\left[\frac{1}{(\alpha^2+\omega^2)}\right]\right\}$$

i.e.

$$R_{YY}(\tau) = \frac{2C\alpha\lambda^2}{\alpha^2-\lambda^2}\left(\frac{1}{2\lambda}e^{-\lambda|\tau|} - \frac{1}{2\alpha}e^{-\alpha|\tau|}\right)$$

$$R_{XY}(\tau) = \frac{C\alpha\lambda^2}{\alpha^2-\lambda^2}\left(\frac{e^{-\lambda|\tau|}}{\lambda} - \frac{e^{-\alpha|\tau|}}{\alpha}\right)$$

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau) = Ce^{-\alpha|\tau|} * h(\tau)$$

where

$$h(\tau) = \begin{cases} \lambda e^{-\lambda\tau}, & \text{for } \tau > 0 \\ 0, & \text{for } \tau < 0 \end{cases}$$

$$\therefore \quad R_{XY}(\tau) = \int_0^\infty Ce^{-\alpha|u|}\lambda e^{-\lambda(\tau-u)}\,du$$

$$= \int_0^\infty Ce^{-\alpha u}\lambda e^{-\lambda(\tau-u)}\,du = C\lambda e^{-\lambda\tau}\int_0^\infty e^{-\alpha u}e^{\lambda u}\,du$$

$$= C\lambda e^{-\lambda\tau}\int_0^\infty e^{-(\alpha-\lambda)u}\,du = C\lambda e^{-\lambda\tau}\left[\frac{e^{-(\alpha-\lambda)u}}{-(\alpha-\lambda)}\right]_0^\infty$$

$$\therefore \quad R_{XY}(\tau) = C\lambda e^{-\lambda\tau}\left[e^{-\infty} + \frac{e^0}{(\alpha-\lambda)}\right] = \frac{C\lambda e^{-\lambda\tau}}{(\alpha-\lambda)}$$

EXAMPLE 8.12 Given that $Y(t) = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} X(\alpha)d\alpha$ where $\{Y(t)\}$ is a WSS process, prove that $S_{YY}(\omega) = \frac{\sin^2 \varepsilon\omega}{\varepsilon^2\omega^2} S_{XX}(\omega)$. Find the output autocorrelation function.

Solution To find the output autocorrelation function of $Y(t)$:

Given $Y(t) = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} X(\alpha)d\alpha$

Let $\alpha = t - u$, then $d\alpha = -du$

When $\alpha = t - \varepsilon$, $u = \varepsilon$

and $\alpha = t + \varepsilon$, $u = -\varepsilon$

$\therefore$
$$Y(t) = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} X(t-u)du$$

Let us define

$$h(t) = \begin{cases} \frac{1}{2\varepsilon} & \text{for } |t| \le \varepsilon \\ 0 & \text{for } |t| > \varepsilon \end{cases}$$

Then $Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$

$\therefore$ PSD of the output $Y(t)$ is

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$H(\omega) = F[h(t)] = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} e^{-i\omega t} dt$$

$$= \frac{1}{2\varepsilon}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-\varepsilon}^{\varepsilon}$$

$$= \frac{1}{\varepsilon\omega}\left(\frac{e^{i\omega\varepsilon} - e^{-i\omega\varepsilon}}{2i}\right) = \frac{\sin \varepsilon\omega}{\varepsilon\omega}$$

$\therefore$
$$S_{YY}(\omega) = \frac{\sin^2 \varepsilon\omega}{\varepsilon^2\omega^2} S_{XX}(\omega) \qquad \text{(i)}$$

Taking inverse Fourier transform on both sides of the expression (i),

$$R_{YY}(\tau) = F^{-1}\left(\frac{\sin^2 \varepsilon\omega}{\varepsilon^2\omega^2}\right) * R_{XX}(\tau) \qquad \text{(ii)}$$

where * denotes the convolution.

Consider the function,

$$R(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{2\varepsilon} & \text{if } |\tau| \le 2\varepsilon \\ 0 & \text{if } |\tau| > 2\varepsilon \end{cases}$$

Then

$$F[R_{XX}(\tau)] = \int_{-2\varepsilon}^{2\varepsilon} \left(1 - \frac{|\tau|}{2\varepsilon}\right) e^{-i\omega\tau}\, d\tau$$

$$= \int_{-2\varepsilon}^{2\varepsilon} \left(1 - \frac{|\tau|}{2\varepsilon}\right) \cos\omega\tau\, d\tau + i \int_{-2\varepsilon}^{2\varepsilon} \left(1 - \frac{|\tau|}{2\varepsilon}\right) \sin\omega\tau\, d\tau$$

$$= 2\int_{0}^{2\varepsilon} \left(1 - \frac{\tau}{2\varepsilon}\right) \cos\omega\tau\, d\tau + 0$$

$$= 2\left[\left(1 - \frac{\tau}{2\varepsilon}\right)\left(\frac{\sin\omega\tau}{\omega}\right) + \frac{1}{2\varepsilon}\left(-\frac{\cos\omega\tau}{\omega^2}\right)\right]_0^{2\varepsilon}$$

$$= \frac{1}{\varepsilon\omega^2}(1 - \cos 2\varepsilon\omega)$$

$$= \frac{2\sin^2\varepsilon\omega}{\varepsilon\omega^2}$$

$$F[R_{XX}(\tau)] = 2\varepsilon\left(\frac{\sin^2\varepsilon\omega}{\varepsilon^2\omega^2}\right)$$

$$\therefore \quad F^{-1}\left(\frac{\sin^2\varepsilon\omega}{\varepsilon^2\omega^2}\right) = \frac{1}{2\varepsilon} R_{XX}(\tau)$$

$$= \begin{cases} \dfrac{1}{2\varepsilon}\left(1 - \dfrac{|\tau|}{2\varepsilon}\right) & \text{if } |\tau| \le 2\varepsilon \\ 0 & \text{if } |\tau| > 2\varepsilon \end{cases}$$

From Eq. (ii), we have

$$R_{YY}(\tau) = F^{-1}\left(\frac{\sin^2\varepsilon\omega}{\varepsilon^2\omega^2}\right) * R_{XX}(\tau)$$

$$= \frac{1}{2\varepsilon}\int_{-2\varepsilon}^{2\varepsilon}\left(1 - \frac{|u|}{2\varepsilon}\right) R_{XX}(\tau - u)\, du$$

EXAMPLE 8.13 An LTI system has an impulse response $h(t) = e^{-\beta t}u(t)$. Find the output autocorrelation function $R_{YY}(\tau)$ corresponding to an input $X(t)$.

Solution We know that

$$H(\omega) = \text{Fourier transform of } h(t)$$

$$= \int_0^{\infty} e^{-\beta t}\, e^{-i\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(\beta + i\omega)t}\, dt$$

$$= -\left[\frac{e^{-(\beta + i\omega)\,t}}{\beta + i\omega}\right]_0^{\infty} = \frac{1}{\beta + i\omega}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2 + \omega^2}$$

The output power density spectrum is given by

$$S_{YY}(\omega) = |H(\omega)|^2\, S_{XX}(\omega)$$

$$= \left(\frac{1}{\beta^2 + \omega^2}\right) S_{XX}(\omega)$$

$$S_{YY}(\omega) = \frac{1}{2\beta}\left(\frac{2\beta}{\beta^2 + \omega^2}\right) S_{XX}(\omega)$$

Taking inverse Fourier transform on both sides of the above expression,

$$R_{YY}(\tau) = \frac{1}{2\beta} F^{-1}\left(\frac{2\beta}{\beta^2 + \omega^2}\right) * R_{XX}(\tau), \text{ where } * \text{ denotes convolution}$$

$$= \frac{1}{2\beta} e^{-\beta|\tau|} * R_{XX}(\tau)$$

$\therefore$

$$R_{YY}(\tau) = \frac{1}{2\beta} \int_0^{\infty} e^{-\beta|u|}\, R_{XX}(\tau - u)\, du$$

EXAMPLE 8.14 Assume a random process $\{X(t)\}$ is given as input to a system with system transfer function $H(\omega) = 1, -W_0 < \omega < W_0$. If the ACF of the input process is $\frac{N_0}{2}\delta(\tau)$, find the ACF of the output process.

Solution Given $R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$ is the ACF of the input process. To find the ACF of the output process:

Taking Fourier transform,

$$S_{XX}(\omega) = F\left[\frac{N_0}{2}\delta(\tau)\right] = \frac{N_0}{2} \qquad \text{since } F[\delta(\tau)] = 1$$

Given: $H(\omega) = 1, \quad -W_0 < \omega < W_0 \quad \therefore \quad |H(\omega)|^2 = 1, \quad -W_0 < \omega < W_0$

We know that the PSD of the output process $Y(t)$ is

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{N_0}{2}, \quad -W_0 < \omega < W_0$$

$\therefore$ The output ACF is

$$R_{YY}(\tau) = F^{-1}[S_{YY}(\omega)]$$

$\Rightarrow$

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\left(\frac{N_0}{2}\right)\int_{-W_0}^{W_0} e^{i\omega\tau} d\omega$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-W_0}^{W_0}$$

$$= \frac{N_0}{2\pi\tau}\left(\frac{e^{iW_0\tau} - e^{-iW_0\tau}}{2i}\right)$$

$$= \frac{N_0 \sin W_0\tau}{2\pi\tau}, \quad -\infty < \tau < \infty$$

8.6 REPRESENTATION OF NOISE IN COMMUNICATION SYSTEM

In communication systems, the message to be transmitted to a long distant places is first converted into an electrical waveform called input signal, before being sent into the transmitter. The transmitter processes and modifies the input signal for better transmission. The transmitted signal is then sent through the channel which is just a medium such as wire, coaxial cable or optical fibre. The channel output or the received signal is then reprocessed by the receiver which sends out the output signal. The output signal is converted to its original form, namely the message.

When the message is communicated in this manner, the signal is not only distorted by the channel but also by the external source noise. The situation can be explained using Figure 8.1.

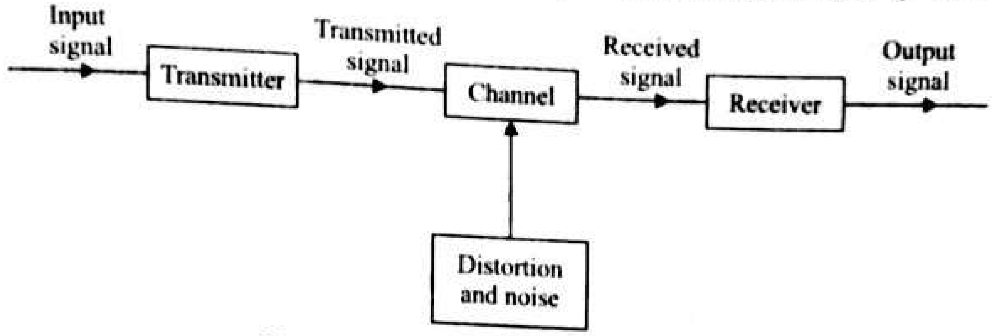

Figure 8.1 Communication of message

The term noise is used to designate unwanted waves that tend to disturb the transmission and processing of signals in communication system over which we have incomplete control. In practice, we find that there are many potential sources of noise in a communication system. The sources of noise may be external to the system (e.g. atmospheric noise, galactric noise, man-made noise) or internal to the system.

The internal sources of noise include an important type of noise that arises due to spontaneous fluctuations of current or voltage in electrical circuits. This type of noise, in one way or another, is present in every communication system and represents a basic limitation on the transmission or detection of signals. The two most common examples of spontaneous fluctuations in electrical circuits are shot noise and thermal noise.

8.6.1 Shot Noise

Shot noises arise in electronic devices because of the discrete nature of current flow in the device. An important characteristic of shot noise is that it is Gaussian-distributed with zero mean.

Poisson Points and Shot Noise

Given a set of Poisson points t_1 and a fixed point t_0, we form the random variable $z = t_1 - t_0$ where t_1 is the first random point to the right of t_0.

The random variable z has an exponential distribution $f_Z(z) = \lambda e^{-\lambda z}, z > 0$.

We note that the distance $x = x_n - x_{n-1} = t_n - t_{n-1}$ between two consecutive points t_{n-1} and t_n has an exponential distribution $f_X(x) = \lambda e^{-\lambda x}$ (Figure 8.2).

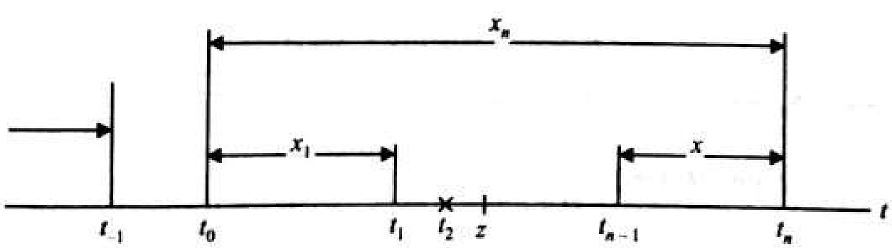

Figure 8.2 Input-poisson points

Definition of Shot Noise

Given a set of Poisson points t_i with average density λ and a real function $h(t)$, we form the sum

$$s(t) = \sum_i h(t - t_i) \tag{8.3}$$

This sum is an SSS process, known as shot noise.

From the definition, it follows that $s(t)$ can be represented as the output of a linear system, shown in Figure 8.3 with impulse response $h(t)$ and input the Poisson impulses

$$z(t) = \sum_i \delta(t - t_i) \tag{8.4}$$

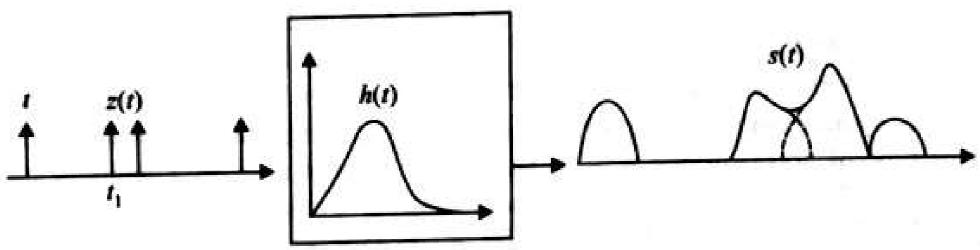

Figure 8.3 Output of a linear system.

This representation agrees with the generation of shot noise in physical problems. The process $s(t)$ is the output of dynamic system activated by a sequence of impulses (e.g. particle emissions) occurring at the random times t_i.

Hence the mean of the process $s(t)$ is given by

$$E[s(t)] = \lambda \int_{-\infty}^{\infty} h(t)\,dt = \lambda H(0)$$

Furthermore, since $S_{ZZ}(\omega) = 2\pi\lambda^2\delta(\omega) + \lambda$, it follows that the power spectral density of shot noise $s(t)$ is given by $S_{SS}(\omega) = 2\pi\lambda^2H^2(0)\,\delta(\omega) + \lambda|H(\omega)|^2$ because $|H(\omega)|^2\,\delta(\omega) = H^2(0)\,\delta(\omega)$.

The ACF is the inverse Fourier transform of the PSD $S_{SS}(\omega)$. Taking inverse Fourier transform in $S_{SS}(\omega)$, we get the ACF

$$R_{SS}(\tau) = \lambda^2H^2(0) + \lambda h(\tau)$$

Suppose the impulse response $h(t) = e^{-\alpha t}u(t)$, then

$$H(\omega) = F[h(t)] = \frac{1}{\alpha + i\omega} \qquad |H(\omega)|^2 = \frac{1}{\alpha^2 + \omega^2} \;\Rightarrow\; H^2(0) = \frac{1}{\alpha^2}$$

Then the power spectral density of the shot noise is given by

$$S_{SS}(\omega) = \frac{2\pi\lambda^2}{\alpha^2}\delta(\omega) + \frac{\lambda}{\alpha^2 + \omega^2}$$

8.6.2 Thermal Noise

Thermal noise is the name given to the electrical noise arising from the random motion of free electrons in a conducting media such as a resistor.

The mean square value of the thermal noise voltage appearing across the terminals of a resistor, measured in a bandwidth of Δf Hz is given by

$$E(V_{TN}^2) = 4kTR\,\Delta f \text{ volts}^2$$

where k is Boltzmann's constant equal to 1.38×10^{-23} J K^{-1}, T is the absolute temperature in kelvin and R is the resistance in ohms.

A noisy resistor is modelled by a noiseless resistor R in series with a voltage sources $n_e(t)$ or in parallel with a current source $n_i(t) = \dfrac{n_e(t)}{R}$ as in Figure 8.4.

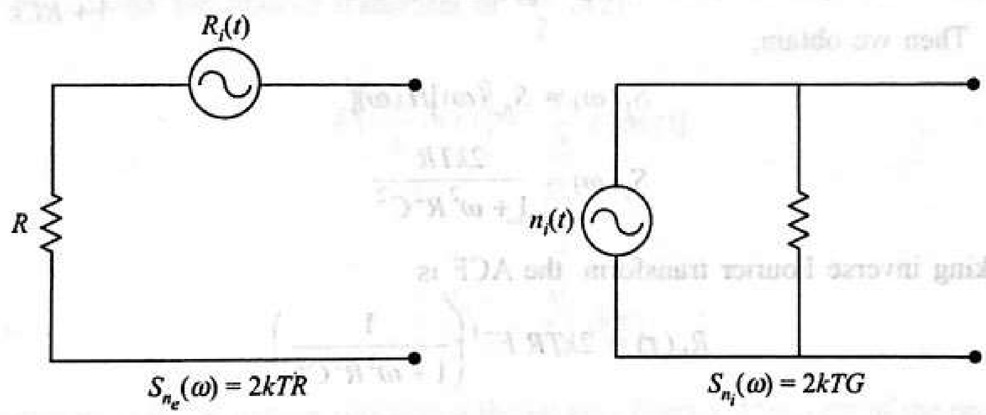

Figure 8.4 A noisy resistor.

It is assumed that $n_e(t)$ is a normal process with zero mean and flat spectrum $S_{n_e}(\omega) = 2kTR$.

The power spectral density of thermal noise is given by

$$S_{n_i}(\omega) = \frac{S_{n_e}(\omega)}{R^2} = \frac{2kTR}{R^2} = 2kTG$$

where $G = \dfrac{1}{R}$ is the conductance.

Thermal noise generated in resistors and semiconductors is assumed to be a zero mean stationary Gaussian random process $\{N(t)\}$ with a power spectral density that is flat over a wide range of frequencies, i.e. the graph of PSD $S_{NN}(\omega)$ is a straight line parallel to the ω axis.

For example:

Consider the circuit shown in Figure 8.5, consists of a resistor R and a capacitor C.

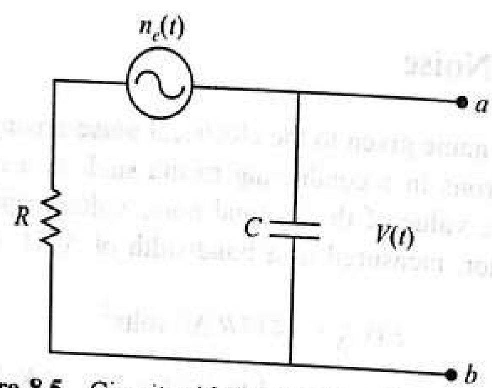

Figure 8.5 Circuit with input $n_e(t)$ and output $V(t)$.

We shall determine the spectrum of the voltage $V(t)$ across the capacitor due to thermal noise. The voltage $V(t)$ can be considered as the output of a system with input the noise voltage $n_e(t)$ and system function $H(s) = \dfrac{1}{1+RCs}$.

Then we obtain,

$$S_V(\omega) = S_{n_e}(\omega)\,|H(\omega)|^2$$

$$\Rightarrow \qquad S_V(\omega) = \frac{2kTR}{1+\omega^2R^2C^2}$$

Taking inverse Fourier transform, the ACF is

$$R_V(\tau) = 2kTR\, F^{-1}\left(\frac{1}{1+\omega^2R^2C^2}\right)$$

$$= \frac{2kT}{RC^2}\, F^{-1}\left(\frac{1}{\omega^2 + \frac{1}{R^2C^2}}\right)$$

$$= \frac{2kT}{RC^2}\, \frac{1}{2\frac{1}{RC}}\, e^{-\frac{1}{RC}|\tau|}$$

$$= \frac{kT}{C}\, e^{\frac{-|\tau|}{RC}}$$

$$\therefore \qquad R_V(\tau) = \frac{kT}{C}\, e^{\frac{-|\tau|}{RC}}$$

8.6.3 White Noise

White noise is also called *white Gaussian noise*. A noise process whose power spectral density is independent of the operating frequency is called white noise.

So in white noise, the power spectral density contains all frequencies in equal amount. The adjective 'white' is used in the sense that white light contains equal amounts of all frequencies within the visible band of electromagnetic radiation.

A sample function $X(t)$ of a WSS noise random process $\{X(t)\}$ is called white noise if the power spectral density of $\{X(t)\}$ is a constant at all frequencies.

In general, a process $\{X(t)\}$ is called white noise if its values $X(t_i)$ and $X(t_j)$ are uncorrelated, i.e. $C_{XX}(t_i, t_j) = 0$.

We denote the power spectral density of white noise $W(t)$ as

$$S_W(f) = \frac{N_0}{2}$$

or

$$S_W(\omega) = \frac{N_0}{2}$$

We consider the Fourier transform of $\frac{N_0}{2}\delta(\tau)$

$$F\left[\frac{N_0}{2}\delta(\tau)\right] = \frac{N_0}{2}F[\delta(\tau)]$$

$$= \frac{N_0}{2}$$

$$\Rightarrow \qquad F^{-1}\left(\frac{N_0}{2}\right) = \frac{N_0}{2}\delta(\tau)$$

Since the autocorrelation function is the inverse Fourier transform of the power spectral density, it follows that the autocorrelation function of white noise is

$$R_{WW}(\tau) = \frac{N_0}{2}\delta(\tau) \qquad \left[\delta(\tau) = \begin{cases} 1, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}\right]$$

We note that $R_{WW}(\tau)$ is zero for $\tau \neq 0$.

Note:

(i) Any two different samples of white noise are uncorrelated.

(ii) If the white noise $W(t)$ is also Gaussian, then the two samples are satistically independent.

(iii) The average power of the white noise is given by

$$R_{WW}(0) = \int_{-\infty}^{\infty} \frac{N_0}{2} d\omega \to \infty$$

Therefore, the white noise has infinite average power.

(iv) If the bandwidth of a noise process at the input of a system is appreciably larger than that of the system itself, then we may model the noise process as white noise.

Band-limited White Noise

Noise having a non-zero and constant power spectral density over a finite frequency band and zero elsewhere is called band-limited white noise.

∴ The PSD of the band-limited white noise is given by

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega| \le \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

Some Special White Noise Processes

1. Ideal low-pass filtered white noise: Suppose that a white Gaussian noise $W(t)$ of zero mean and power spectral density $\frac{N_0}{2}$ is applied to an ideal low-pass filter of bandwidth B and a passband amplitude response one.

The power spectral density of the noise $n(t)$ appearing at the filter output is given by

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & -B < f < B \\ 0, & |f| > B \end{cases}$$

It is shown in Figure 8.6.

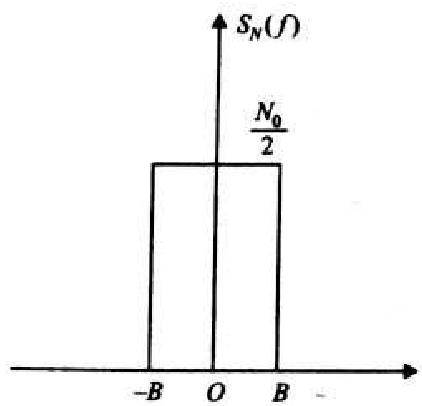

Figure 8.6 PSD of the noise $n(t)$.

The inverse Fourier transform of PSD $S_N(\omega)$ gives the ACF and is given by

$$R_N(\tau) = \frac{1}{2\pi}\int_{-B}^{B}\frac{N_0}{2}e^{i\omega\tau}\,d\omega$$

$$= \frac{N_0}{4\pi}\int_{-B}^{B}e^{i\omega\tau}\,d\omega = \frac{N_0}{4\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-B}^{B}$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{i\beta\tau}-e^{-i\beta\tau}}{i\tau}\right] = \frac{N_0}{2}\left[\frac{\sin\beta\tau}{\tau}\right]$$

$$\therefore \qquad R_N(t) = \frac{N_0 B}{2}\left[\frac{\sin B\tau}{B\tau}\right]$$

We see that as $\tau \to 0$, $R_N(\tau)$ has its maximum value $\dfrac{N_0 B}{2}$ and it passes through zero at $\tau = \pm\dfrac{n\pi}{B}$, $n = 1, 2, 3, \ldots$

Since the input noise $W(t)$ is Gaussian, it follows that the band-limited noise $n(t)$ at the filter output is also Gaussian. Suppose now that $n(t)$ is sampled at the rate of $2B$ times per second. From the graph of $R_N(\tau)$, we see that the resulting noise samples are uncorrelated and being Gaussian, they are statistically independent.

EXAMPLE 8.15 If $\{N(t)\}$ is a band-limited white noise such that

$$S_{NN}(\omega) = \begin{cases}\dfrac{N_0}{2}, & \text{for } |\omega| < \omega_B\\ 0, & \text{elsewhere}\end{cases}$$

find the autocorrelation function.

Solution By definition, the PSD is given by

$$S_{NN}(\omega) = \begin{cases}\dfrac{N_0}{2}, & |\omega| < \omega_\beta\\ 0, & \text{elsewhere}\end{cases}$$

The ACF is given by the inverse Fourier transform

$$\therefore \qquad R_{NN}(\tau) = \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B}\frac{N_0}{2}e^{i\omega\tau}\,d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\omega_B}^{\omega_B}e^{i\omega\tau}\,d\omega$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_B}^{\omega_B}$$

$$= \frac{N_0}{4\pi}\left(\frac{e^{i\omega_B\tau} - e^{-i\omega_B\tau}}{i\tau}\right)$$

$$= \frac{N_0}{2\pi\tau}\left(\frac{e^{i\omega_B\tau} - e^{-i\omega_B\tau}}{2i}\right)$$

$$= \frac{N_0}{2\pi}\left(\frac{\sin\omega_B\tau}{\tau}\right)$$

2. RC low-pass filtered white noise: Consider a white Gaussian noise $\omega(t)$ of zero mean and power spectral density $\frac{N_0}{2}$ applied to a low-pass RC filter, as shown in Figure 8.7.

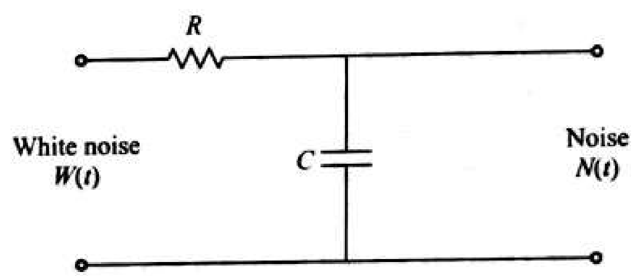

Figure 8.7 Low-pass RC filter.

The transfer function of the filter is

$$H(f) = \frac{1}{1 + i2\pi f\, RC} = \frac{1}{1 + i\omega RC}$$

where $\omega = 2\pi f$.

The power spectral density of the noise $N(t)$ appearing at the low-pass RC filter output is given by

$$S_{NN}(\omega) = S_{WW}(\omega)\, |H(\omega)|^2$$

$$= \frac{N_0}{2}\cdot\frac{1}{1 + (\omega RC)^2}$$

$$= \frac{N_0}{2}\cdot\frac{\beta^2}{\beta^2 + \omega^2}$$

where $\beta = \frac{1}{RC}$

$$= \frac{\beta N_0}{2} \cdot \frac{\beta}{\beta^2 + \omega^2}$$

Taking inverse Fourier transform, we get the ACF

$$R_{NN}(\tau) = F^{-1}\left(\frac{\beta N_0}{4} \frac{2\beta}{\beta^2 + \omega^2}\right) = \frac{\beta N_0}{4} e^{-\beta|\tau|} = \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}}$$

$$\left[\text{since } F(e^{-\alpha|\tau|}) = \frac{2\alpha}{\alpha^2 + \omega^2}\right]$$

Thus, if the noise appearing at the filter output is sampled, the resulting samples are uncorrelated and they are statistically independent.

3. Autocorrelation of a sine wave with white noise: Consider a random process $X(t)$ consisting of a sinusoidal wave component $A \cos(2\pi f_c t + \theta)$ and a white noise process $W(t)$ of zero mean and power spectral density $\frac{N_0}{2}$, defined by $X(t) = A \cos(2\pi f_c t + \theta) + W(t)$, where A and f_c are constants and θ is a random variable uniformly distributed in $(0, 2\pi)$.

The two components of $X(t)$ are clearly independent. The ACF of $X(t)$ is the sum of the individual autocorrelation functions of the sinusoidal wave component $A \cos(2\pi f_c t + \theta)$ and the white noise component.

The ACF of sinusoidal component is given by $\frac{A^2}{2}\cos(2\pi f_c \tau)$ and the ACF of the white noise component is equal to $\frac{N_0}{2}\delta(\tau)$.

Hence the ACF of $\{X(t)\}$ is given by

$$R_{XX}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{N_0}{2}\delta(\tau)$$

EXAMPLE 8.16 If $\{N(t)\}$ is a band-limited white noise centred at a carrier frequency ω_0 such that

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

find the autocorrelation of $\{N(t)\}$.

Solution Given: $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$

To find the autocorrelation of $\{N(t)\}$:

$$R_{NN}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\, e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\,\frac{N_0}{2}\int_{-\omega_B+\omega_0}^{\omega_B+\omega_0} e^{i\omega\tau}\, d\omega$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_B+\omega_0}^{\omega_B+\omega_0}$$

$$= \frac{N_0}{2\pi\tau}\left[\frac{e^{i\tau(\omega_B+\omega_0)} - e^{i\tau(-\omega_B+\omega_0)}}{2i}\right]$$

$$= \frac{N_0}{2\pi\tau}\, e^{i\omega_0\tau}\left(\frac{e^{i\omega_B\tau} - e^{-i\omega_B\tau}}{2i}\right)$$

$$= \frac{N_0}{2\pi\tau}\, e^{i\omega_0\tau}\cdot \sin\omega_B\tau$$

$$= \frac{N_0\omega_B}{2\pi}\left(\frac{\sin\omega_B\tau}{\omega_B\tau}\right) e^{i\omega_0\tau}$$

$$= \frac{N_0\omega_B}{2\pi}\left(\frac{\sin\omega_B\tau}{\omega_B\tau}\right)(\cos\omega_0\tau + i\sin\omega_0\tau)$$

Since $R_{NN}(\tau)$ is a real function,

$$R_{NN}(\tau) = \frac{N_0\omega_B}{2\pi}\left(\frac{\sin\omega_B\tau}{\omega_B\tau}\right)\cos\omega_0\tau$$

EXAMPLE 8.17 If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$ where A is a constant, θ is a random variable with a uniform distribution in $(-\pi, \pi)$ and $\{N(t)\}$ is band-limited Gaussian white noise with power spectral density

$$S_{NN}(\omega) = \begin{cases} \dfrac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

find the power spectral density of $\{Y(t)\}$. Assume that $\{N(t)\}$ and θ are independent.

Solution Given: $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$

Therefore,

$$Y(t_1)Y(t_2) = [A\cos(\omega_0 t_1 + \theta) + N(t_1)][A\cos(\omega_0 t_2 + \theta) + N(t_2)]$$
$$= A^2\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta) + N(t_1)N(t_2)$$
$$+ A\cos(\omega_0 t_1 + \theta)\, N(t_2) + A\cos(\omega_0 t_2 + \theta)\, N(t_1)$$

Since $N(t)$ and θ are independent,

$$R_{YY}(t_1, t_2) = A^2 E[\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta)] + R_{NN}(t_1, t_2)$$
$$+ AE[\cos(\omega_0 t_1 + \theta)]\, E[N(t_2)] + AE[\cos(\omega_0 t_2 + \theta)]\, E[N(t_1)]$$

By hypothesis,

$$E[N(t_1)] = 0, \quad E[N(t_2)] = 0$$

and

$$E\{\cos[\omega_0(t_1 + t_2) + 2\theta]\} = 0$$

We have

$$R_{YY}(\tau) = \frac{A^2}{2}\cos \omega_0 \tau + R_{NN}(\tau)$$

Taking Fourier transform,

$$\therefore \quad S_{YY}(\tau) = \frac{A^2}{2}\int_{-\infty}^{\infty} \cos \omega_0 \tau \, e^{-i\omega\tau} d\tau + S_{NN}(\omega)$$

$$= \frac{A^2}{2} F(\cos \omega_0 \tau) + S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + S_{NN}(\omega)$$

where

$$S_{NN}(\omega) = \begin{cases} \dfrac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

EXAMPLE 8.18 The impulse response of a low-pass filter is $\alpha e^{-\alpha t}U(t)$ where $\alpha = \frac{1}{RC}$. If a zero mean, white Gaussian process $\{N(t)\}$ is input into this filter, find the ACF of the output.

Solution The impulse response $h(t)$ is given by

$$h(t) = \begin{cases} \alpha e^{-\alpha t}, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$H(\omega) = \text{Fourier transform of } h(t) = F[h(t)]$$

$$= \int_0^{\infty} \alpha e^{-\alpha t} e^{-i\omega t} dt$$

$$= \alpha \int_0^{\infty} e^{-(\alpha + i\omega)t} dt$$

$$= \alpha \left[\frac{e^{-(\alpha + i\omega)t}}{-(\alpha + i\omega)}\right]_0^{\infty}$$

$$H(\omega) = \frac{\alpha}{\alpha + i\omega}$$

$$\therefore \qquad |H(\omega)|^2 = \frac{\alpha^2}{\alpha^2 + \omega^2}$$

The input is the zero mean Gaussian noise $\{N(t)\}$. So the PSD of $\{N(t)\}$ $S_{NN}(\omega) = \frac{N_0}{2}$.

The spectral density of the output $Y(t)$ is

$$S_{YY}(\omega) = |H(\omega)|^2 S_{NN}(\omega)$$
$$= \frac{\alpha^2}{\alpha^2 + \omega^2} \frac{N_0}{2}$$

The ACF function of the output $Y(t)$ is

$$R_{YY}(\tau) = F^{-1}[S_{YY}(\omega)]$$
$$= \frac{N_0 \alpha}{4} F^{-1}\left(\frac{2\alpha}{\alpha^2 + \omega^2}\right)$$
$$= \frac{N_0 \alpha}{4} e^{-\alpha|\tau|}$$

EXAMPLE 8.19 A white Gaussian noise $W(t)$ of zero mean and power spectral density $\frac{N_0}{2}$ is applied to the high-pass RL filter as shown in Figure 8.8. Determine the ACF of the output $N(t)$.

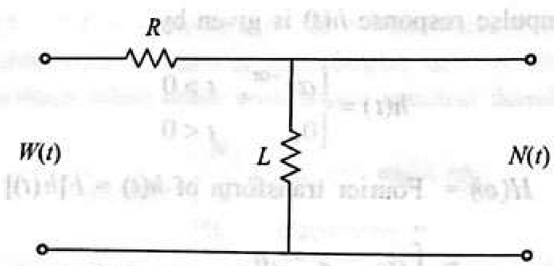

Figure 8.8 High-pass RL filter.

Solution The transfer function of the filter is

$$H(\omega) = \frac{1}{1 + i\frac{\omega L}{R}} = \frac{R}{R + i\omega L}$$

$$|H(\omega)|^2 = \frac{R^2}{R^2 + (\omega L)^2}$$

The PSD of the input Gaussian white noise $W(t)$ is

$$S_{WW}(\omega) = \frac{N_0}{2}$$

The PSD of the output noise $N(t)$ is given by

$$S_{NN}(\omega) = |H(\omega)|^2 S_{WW}(\omega)$$

$$= \frac{N_0}{2} \cdot \frac{R^2}{R^2 + (\omega L)^2}$$

$$= \frac{N_0}{2} \cdot \frac{\beta^2}{\omega^2 + \beta^2}$$

where $\beta = \frac{R}{L}$

$$\therefore \quad S_{NN}(\omega) = \frac{N_0 \beta}{4} \left(\frac{2\beta}{\beta^2 + \omega^2} \right)$$

Taking inverse Fourier transform,

$$R_{NN}(\tau) = \frac{N_0 \beta}{4} e^{-|\tau|\beta}$$

$$= \frac{N_0 R}{4L} e^{-\frac{|\tau| R}{L}}$$

EXERCISES

1. What do you mean by a system? When a system is called
   (i) a deterministic system, and
   (ii) a stochastic system?
2. Describe a linear system.
3. Give an example for a linear system.
4. Describe a linear system with a random input.
5. When the system is said to be linear?
6. Write a note on linear system.
7. When a system is called a time invariant system?
8. When a system is called a memoryless system?
9. Define a system. When is it called a causal system?
10. Define system weighting function.

11. If a system is defined as $Y(t) = \dfrac{1}{T}\int\limits_0^T X(t-u)du$, find its weighting function.

$$\left[\textbf{Ans. } h(t) = \begin{cases} \dfrac{1}{T}e^{-t/T}, & \text{for } 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}\right]$$

12. What is unit impulse response of a system? Why is it called so?

13. If the input $X(t)$ of the system $Y(t) = \int\limits_{-\infty}^{\infty} h(u)\, X(t-u)du$ is the unit impulse function, prove that $Y(t) = h(t)$.

14. If a system is defined as $Y(t) = \dfrac{1}{T}\int\limits_0^{\infty} X(t-u)e^{-u/T}du$, find its unit impulse response.

$$\left[\textbf{Ans. } \begin{cases} \dfrac{1}{T}e^{-t/T}, & \text{for } t \ge 0 \\ 0, & \text{elsewhere} \end{cases}\right]$$

15. Prove that the system $Y(t) = \int\limits_{-\infty}^{\infty} h(u)X(t-u)du$ is a linear time invariant system.

16. When is the system $Y(t) = \int\limits_{-\infty}^{\infty} h(u)X(t-u)du$ said to be stable?

17. If $\{X(t)\}$ and $\{Y(t)\}$ in the system $Y(t) = \int\limits_{-\infty}^{\infty} h(u)X(t-u)du$ are WSS processes, how are their ACFs related?

18. If the input and output of the system $Y(t) = \int\limits_{-\infty}^{\infty} h(u)X(t-u)du$ are WSS processes, how are their PSDFs related?

19. Define the power transfer function (or system function) of the system $Y(t) = \int\limits_{-\infty}^{\infty} h(u)X(t-u)du$.

20. If the system function of a convolution type of linear system is given by $h(t) = \begin{cases} \dfrac{1}{2c}, & \text{for } |\tau| \le c \\ 0, & \text{for } |\tau| > c \end{cases}$. Find the relation between PSDFs of the input and output processes.

21. What is a white noise?

22. Give two examples of a white noise.
23. What is the average power of the white noise?

$$\left[\textbf{Ans. } E(N^2(t)) = R_{NN}(0) = \int_{-\infty}^{\infty} \frac{N_0}{2}\, d\omega \to \infty\right]$$

24. The impulse response of a low-pass filter is $\alpha e^{-\alpha t}u(t)$ where $\alpha = \frac{1}{RC}$. If a zero mean white Gaussian process $\{N(t)\}$ is an input into this filter, find the autocorrelation function and mean square value of the output process.

$$\left[\textbf{Ans. } \frac{N_0\alpha}{4} e^{-\alpha|\tau|}, \frac{N_0\alpha}{4}\right]$$

25. For a narrow band process $X(t) = X_c(t)\cos\omega_0 t + X_s(t)\sin\omega_0 t$, where $\{X_c(t)\}$ and $\{X_s(t)\}$ are stationary, uncorrelated, lowpass processes with

$$S_{X_cX_c}(\omega) = S_{X_sX_s}(\omega) = \begin{cases} g(\omega), & |\omega| < \omega_B \\ 0, & |\omega| > \omega_B \end{cases}$$

show that $S_{XX}(\omega) = \frac{1}{2}\{g(\omega - \omega_0) + g(\omega + \omega_0)\}$.

26. (i) If $\{N(t)\}$ is a band-limited white noise such that

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

find the autocorrelation of $\{N(t)\}$.

(ii) If $\{N(t)\}$ is a band-limited white noise centred at a carrier frequency ω_0 such that

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

find the autocorrelation of $\{N(t)\}$.

$$\left[\textbf{Ans. } \text{(i) } \frac{N_0\omega_B}{2\pi}\left(\frac{\sin\omega_B\tau}{\omega_B\tau}\right), \text{(ii) } \frac{N_0\omega_B}{2\pi}\left(\frac{\sin\omega_B\tau}{\omega_B\tau}\right)\cos\omega_0\tau\right]$$

27. If the input to a linear time-invariant system is a zero mean, white Gaussian process $\{N(t)\}$ and $\{(Y(t)\}$ is the output, prove that

(i) $E[Y(t)] = 0$,

(ii) $R_{YY}(\tau) = \frac{N_0}{2}\delta(\tau) * h(\tau) * h(-\tau)$, and

(iii) $S_{YY}(\omega) = \frac{N_0}{2}|H(\omega)|^2$.

28. The impulse response of a low pass filter is $\alpha e^{-\alpha t}U(t)$, where $\alpha = \frac{1}{RC}$. If RC a zero mean, white Gaussian process $\{N(t)\}$ is input into this filter, find the mean square value and autocorrelation function of the output.

$$\left[\textbf{Ans. } \alpha\frac{N_0}{4},\ \alpha\frac{N_0}{4}e^{-\alpha\tau},\ \tau \geq 0\right]$$

29. A circuit has an impulse response given by

$$h(t) = \begin{cases} \frac{1}{T}, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Evaluate $S_{YY}(\omega)$ of the output process $\{Y(t)\}$ corresponding to an input process $\{X(t)\}$.

$$\left[\textbf{Ans. } S_{YY}(\omega) = \frac{\sin^2\left(\frac{\omega T}{2}\right)}{(\omega T/2)} S_{XX}(\omega)\right]$$

30. A random process $X(t)$ is applied to a network with impulse response $h(t) = te^{-bt}u(t)$, where $b > 0$ is a constant. The cross-correlation function of $X(t)$ with the output $Y(t)$ is known to have the same ACF $R_{XY}(\tau) = te^{-bt}u(t)$. Find the ACF of the output $Y(t)$.

$$\left[\begin{array}{l} \textbf{Hint: } S_{YY}(\omega) = H^*(\omega)S_{XY}(\omega) \ \Rightarrow\ S_{YY}(\omega) = \dfrac{1}{(b^2+\omega^2)^2} \\ \text{Taking inverse Fourier transform,} \\ R_{YY}(\tau) = \dfrac{1}{b^4}\left[u(\tau)\,\tau e^{-b\tau} + u(\tau)\tau e^{b\tau} + \dfrac{1}{b}e^{-b|\tau|}\right] \end{array}\right]$$

31. Let $X(t)$ is the input voltage to a circuit, $Y(t)$ is the output voltage, and $\{X(t)\}$ is a stationary random process with $\mu_X = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find the mean, autocorrelation and power spectral density of the output $Y(t)$ if the system function is given by $H(\omega) = \frac{1}{\omega + 2i}$.

$$\left[\textbf{Ans. } E[Y(t)] = 0,\ S_{YY}(\omega) = \frac{4}{(\omega^2+4)^2},\ R_{YY}(\tau) = \frac{1}{8}\left(1 + 2|\tau|e^{-2|\tau|}\right)\right]$$

32. Find the input autocorrelation, output autocorrelation of the RC low-pass filter, when the filter is subjected to a white noise of spectral density $\frac{N_0}{2}$.

$$\left[\textbf{Ans. } R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau),\ R_{YY}(\tau) = \frac{N_0}{4RC}e^{-|\tau|/RC}\right]$$

33. Let $X(t)$ is the input voltage to a circuit, $Y(t)$ is the output voltage, and $\{X(t)\}$ is a stationary random process with $\mu_X = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find μ_Y, $S_{YY}(\omega)$ and $R_{YY}(\tau)$ if the system function is given by $H(\omega) = \dfrac{1}{\omega + 2i}$.

$$\left[\textbf{Ans. } 0, \frac{4}{(\omega^2+4)^2}, \frac{1}{8}(1+2|\tau|)e^{-2|\tau|}\right]$$

34. For a linear system with random input $X(t)$, the impulse response $h(t)$ and output $Y(t)$, obtain the cross-correlation function $R_{XY}(\tau)$ and the output autocorrelaton function $R_{YY}(\tau)$. **[AU May '08]**

35. State and prove the fundamental theorem on the power spectrum of output of a linear system. **[AU May '05; '08]**

36. Describe linear systems with random inputs and give examples. **[AU May '08]**

37. Let $X(t)$ is the input voltage to a circuit (system), $Y(t)$ is the output voltage, and $\{X(t)\}$ is a stationary random process with $\mu_X = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find μ_Y, $S_{YY}(\omega)$ and $R_{YY}(\tau)$ if the power function is given by $H(\omega) = \dfrac{R}{R + iL\omega}$. **[AU December '08]**

38. If $\{X(t)\}$ is a WSS process with autocorrelation function $R_{XX}(\tau)$ and if $Y(t) = X(t + a) - X(t - a)$, prove that $S_{YY}(\omega) = 4 \sin^2(a\omega)\, S_{XX}(\omega)$. **[AU November '09]**

39. Describe a linear system. Show that the power spectrum $S_{YY}(\omega)$ of the output of a linear system with system function $H(j\omega)$ is given by $S_{YY}(\omega) = S_{XX}(\omega)\, |H(|\omega)|^2$ where $S_{XX}(\omega)$ is the power spectrum of the input. **[AU May '04]**

40. Write detailed notes on:
   (i) correlation integrals, and
   (ii) linear system with random inputs.

41. For a linear system with random input $X(t)$, impulse response $h(t)$, establish the Wiener–Khinchin relation. **[AU May/June '06]**

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