NST 1A Mathematical Biology
2023 Paper
Model answers

A1

(a) The random variable Y has the probability density function f(y) given by:

$$f(y) = \begin{cases} k(y^2 + 2y + 2) & 1 \leq y \leq 3 \\ 0 & otherwise \end{cases}$$

Where k is a positive constant

(i) Find the value of k **[3 marks]**

Use the fact that total area under the pdf is equal to 1

$$\int_{-\infty}^{\infty} f(y)dy = 1$$

Only need to integrate between 1 and 3 as elsewhere area under pdf is zero

$$\int_1^3 k(y^2 + 2y + 2)dy = 1$$

$$k\left[\frac{y^3}{3} + y^2 + 2y\right]_1^3 = 1$$

$$[9 + 9 + 6] - \left[\frac{10}{3}\right] = \frac{1}{k}$$

$$k = \frac{3}{3 \times 24 - 10} = \frac{3}{62} = 0.0484$$

Note: full marks for either fractional or decimal form

(ii) Specify fully the cumulative distribution function of Y **[3 marks]**

$$F(a) = P(Y \leq a) = \int_{-\infty}^{a} f(y)dy$$

$$F(a) = \int_1^a k(y^2 + 2y + 2)dy$$

$$F(a) = k\left[\frac{a^3}{3} + a^2 + 2a\right] - k\left[\frac{10}{3}\right]$$

$$F(a) = \frac{3}{62}\left[\frac{a^3}{3} + a^2 + 2a\right] - \frac{3}{62}\left[\frac{10}{3}\right]$$

$$F(a) = \frac{3}{62}\left[\frac{a^3}{3} + a^2 + 2a\right] - \frac{10}{62}$$

(iii) Evaluate $P(Y \leq 2)$ **[2 marks]**

Can either integrate pdf between 1 and 2 or use expression for $F(a)$

$$F(2) = \frac{3}{62}\left[\frac{2^3}{3} + 2^2 + 2 \times 2\right] - \frac{10}{62}$$

$$= \frac{3}{62}\left[\frac{32}{3}\right] - \frac{10}{62} = \frac{22}{62} = \frac{11}{31} = 0.3548$$

Note: full marks for either fractional or decimal form

(b) A gardener sets up a challenge for visiting squirrels. When a squirrel stands on the starting platform, each of 5 doors in two parallel tubes are randomly set as 'open' with probability p_i and 'shut' with probability $q_i = 1 - p_i$ where $i = 1 \ldots 5$ is the door number as shown in the diagram:

(i) Write down an expression for the probability that, in any given trial, the squirrel will be able to reach the reward in the end chamber (assume that if the squirrel reaches a closed door in one tunnel it will try the other tunnel). **[3 marks]**

Let A = top path; B = lower path
Squirrel only needs one path to be clear to get to the reward i.e. either path A or path B

$$p(A \cup B) = p(A) + P(B) - p(A \cap B)$$

Top path fully open: $p(A) = p_1 p_2$;
Lower path fully open: $P(B) = p_3 p_4 p_5$;
Both paths fully open: $p(A \cap B) = p_1 p_2 p_3 p_4 p_5$

$$p(A \cup B) = p_1 p_2 + p_3 p_4 p_5 - p_1 p_2 p_3 p_4 p_5$$

(ii) The experiment is set up so that the probability of reaching the reward is 0.15. After each trial, the squirrel exits the contraption before returning for another go (resetting it each time).

What is the expected number of attempts required for the squirrel to get the reward? Explain your answer. **[3 marks]**
Geometric distribution: number of attempts need to get one success
Let X = number of attempts need to get one success

$$E[X] = \frac{1}{p} = \frac{1}{0.15} = \frac{20}{3} = 6.6\dot{6}$$

This is the expected number of attempts over lots of trials of the same experiment. On any given trial (where a trial is the number of attempts needed to get a success) the number of attempts will clearly be an integer value.

(iii) What is the probability it takes at least three attempts for the squirrel to get the reward? **[2 marks]**

Geometric distribution

$$p(X \geq 3) = 1 - \big(p(X = 1) + p(X = 2)\big) = 1 - (p + qp) = 1 - (0.15 + 0.85 \times 0.15)$$
$$= 0.7225$$

(c) The squirrel manages to collect and bury 50 nuts over an area of 100m^2. Another squirrel comes to the same patch of land hoping to steal some nuts. If the thief searches an area of 4m^2 what is the probability that it finds exactly 5 nuts? What assumptions did you make? **[4 marks]**

Poisson
In 4m^2 expect 2 nuts

$$\rho = \frac{50}{100} = 0.5; A = 4 \rightarrow \theta = \rho A = 0.5 \times 4 = 2$$

$$p(X = k) = \frac{e^{-\theta}\theta^k}{k!}$$

$$p(X = 5) = \frac{e^{-2}2^5}{5!} = \frac{4e^{-2}}{15} = 0.0361$$

Assumptions:
Original squirrel

- Buries the nuts **randomly** across the 100m^2
- There is no limit on how close the nuts can be buried

Thief squirrel

- keeps going until finds everything
- doesn't select area it searches based on watching the other squirrel
- none of the nuts have rotted away

A2

(a)
(i) We get the following matrix:

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

(ii) These vectors satisfy **Mx** = **y.**

Thus inserting **y** = (3, 10, 1, 7)^T from the data given, we get a linear system of equations for **x** which can be solved (by Gaussian elimination or otherwise) to get **x** = (6, 0, 12, 3)^T, so the inputs are

input 1: 6.0 µl/s
input 2: 0 µl/s
input 3: 12.0 µl/s
input 4: 3.0 µl/s

(1 mark for including the units)

(b) The characteristic equation:

$$det(A - \lambda I) = (5 - \lambda)(3 - \lambda)^2 = 0$$

This has roots at 5 and 3 (a double root). (These can be found using the rational root theorem after obtaining the unfactorized equation $-\lambda^3 + 11\lambda^2 + 39\lambda + 45$; e.g. 3 is a factor of 45.)

For $\lambda_1 = 5$ we have $(\mathbf{A} - 5\mathbf{I})\mathbf{v}_1 = 0$, which reduces to

$x + z = 0$
$y - 2z = 0$

and hence

$$v_1 = \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

for any scalar α.

For $\lambda_2 = 3$ we have $(\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = 0$, which reduces to

$x + y - z = 0$

and hence e.g.

$$v_2 = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

for any scalars α and β.

B3

a) We don't have a-priori hypotheses about the frequencies, so we use an intrinsic (contingency table) chisq.
Chisq calculations:
Compute row sums:

```
cancer healthy
   1981    2219
```

Compute column sums:

```
 mutation ancestral
     1610      2590
```

Predicted values are obtained from the formula: (row_sum * col_sum)/total

```
     mutation ancestral
[1,] 759.3833  1221.617
[2,] 850.6167  1368.383
```

Now compute chisq as (E-O)^2/E

```
         mutation ancestral
cancer  0.7200320 0.4475874
healthy 0.6428046 0.3995812
```

Sum them up:

```
[1] 2.21
```

We compare our computed chisq value of 5.99 with 1df against the critical value in the table. As the observed value is SMALLER than the critical value, we conclude that we fail to reject the H_0 (there is no significant link between cancer and genotype)

b) There was no significant association between the cancer and the mutation (X_1=2.21, p>0.05).

c) Combine controls and patients (as we found no difference, but it could also be done just with the controls)

```
 mutation   ancestral
     1610        2590
Total number of 4200
```

Compute expected numbers given the proportion of 0.4

```
Expected values:
1680 2520
```

Now compute chisq as (E-O)^2/E

```
mutation ancestral
 2.916667  1.944444
```

And sum them to get the total chisq

```
4.861111
```

We compare our computed chisq value of 5.99 with 1df against the critical value in the table. As the observed value is SMALLER than the critical value, we conclude that we fail to reject the H_0 (There is no difference in the frequency of carriers between the two populations)

d) There is no difference in the frequency of carriers between the two populations (X_1=4.86, p>0.05).

e) Df = 44-42=2
Sum Sq = 176.2-64.5 = 111.7
MSq treat = 64.5 /2 = 32.25
MSq Res = 111.7 / 43 = 2.597
F = 32.25/2.597 = 12.41
The computed F is much bigger than the critical F for 2,42 df, so we reject the H0.

f) Zero no included in the first two comparisons (sign) but it is for the last (no sign). So, both treatments have a significant effect, but they don't differ from each other.

g) Treatment has a significant effect on the levels of PSA measured 6 months later ($F_{2,42}$ = 12.4, $p<0.05$): both chemo and radio therapy led to a decrease in PSA concentration (Tukey posthoc 95% CI chemo vs control: -3.4 to -0.5; radio vs control: -4.3 to -1.4), but there was no significant difference between them (Tukey post-hoc 95% CI: -2.4 to 0.5)

B4

a) The residual vs fitted plot should have a homogenous cloud, but it looks like there is a lack of positive values in the middle, and an excess on either side.
Scale location looks fine (expect even distribution); because of the modulus, it solves the non-linearity seen in the resid-fitted plot.
Strong deviation form 1:1 in the Q-Q plot
Nothing of note in leverage plot, Cook's distances are so small that we do not see the isoclines.

b) Df = N_obs-n_coeffs
Df_null = 67 -1 = 66
Df_model = 67-4=63

c) exp(1.888 + 0.841*0.4 +0+ 0.002*25)= 9.7
exp(1.888 + 0.841*0.4 +0.493+ 0.002*25)= 15.9

d) LRT_het = 5.089-0.422=4.667
LRT_bred = 44.079-0.422=43.657
LRT_n_days =0.576-0.422=0.154
Frame of reference is chisq with 1df for all predictors. First two computed LRT are LARGER than critical (significant), last one is not (NOT significant). We can drop n_days.

e) LRT_het = 5.362-0.576=4.786
LRT_bred = 45.748-0.576=45.172
Frame of reference is chisq with 1df for all predictors. Both computed LRT are LARGER than critical (significant). This is the minimal model, we can not drop anything.

f) Higher heterozygosity (X_1=4.78, $p<0.05$) was associated with an increase in breeding success, as was wild rearing (X_1=45.17, $p<0.05$). On the other hand, date of laying did not impact success (X_1=0.15, $p<0.05$).

g) The first 3 models are part of the set, but the second model should be discounted as it is more complex than the one above it. So, only models 4 and 2 should be considered.
A model with an AIC weight of 0.579 as 57.9% probability of being the best model within the available set.

C5

(a) **[2 marks]**

Use the chain rule, setting $Y = \log(u)$ where $u = p + e^{-qt}$	**[1]**
Hence $$\frac{dY}{dt} = \frac{dY}{du}\frac{du}{dt} = \frac{1}{u}(-qe^{-qt}) = \frac{-qe^{-qt}}{p + e^{-qt}}$$	**[1]**

(b) **[1 marks]**

Decay/death/loss of infectivity of sclerotia	**[0.5]**
At relative/per capita rate γ (or stating decay is proportional to density of sclerotia)	**[0.5]**

(c) **[2 marks]**

Separating variables $$\int \frac{1}{X} dX = \int -\gamma dt$$	**[0.5]**
$$\log(X) = -\gamma t + C$$	**[0.5]**
Using $X(0) = X_0$ $$\log(X_0) = -\gamma \times 0 + C = C$$	**[0.5]**
Therefore $$X(t) = X_0 e^{-\gamma t}$$	**[0.5]**

(d) **[4 marks]**

The rate of change of the number of infected plants is proportional to the number of plants not yet infected $(N - I)$ (i.e., there is no secondary infection term)	**[1]**
The relative infection rate $\frac{\beta X}{\alpha + X}$ depends on the density of viable sclerotia in the soil	**[1]**
The density of sclerotia is fed into a nonlinear function to model the idea that sclerotia might not be the only possible limiting factor on the rate of infection/increases in infection rate not always proportional to density/any other sensible explanation of why infection rate might asymptote in the limit $X \to \infty$	**[1]**
The parameter β is the maximum rate at which susceptible plants can be infected in the limit $X \to \infty$	**[0.5]**
The parameter α is the density of sclerotia at which half this rate is attained.	**[0.5]**

(e) **[6 marks]**

The differential equation to be solved is

$$\frac{dI}{dt} = \frac{\beta X}{\alpha + X}(N - I),$$

Using the solution from part (c)

$$\frac{dI}{dt} = \frac{\beta X_0 e^{-\gamma t}}{\alpha + X_0 e^{-\gamma t}}(N - I)$$ **[0.5]**

Separating the variables and integrating

$$\int \frac{1}{N - I} dI = \beta \int \frac{X_0 e^{-\gamma t}}{\alpha + X_0 e^{-\gamma t}}\, dt$$ **[0.5]**

$$-\log(N - I) = -\frac{\beta}{\gamma} \int \frac{-\gamma e^{-\gamma t}}{(\alpha/X_0) + e^{-\gamma t}}\, dt$$ **[1]**

Rewriting the RHS allows us to use the results from part (a) to make further progress, taking $p = \alpha/X_0$ and $q = \gamma$

$$-\log(N - I) = -\frac{\beta}{\gamma} \log\left(\frac{\alpha}{X_0} + e^{-\gamma t}\right) + C$$ **[1.5]**

Since no plants are infected initially,

$$-\log(N) = -\frac{\beta}{\gamma} \log\left(\frac{\alpha}{X_0} + 1\right) + C$$ **[0.5]**

and so

$$\log(N) - \log(N - I) = \frac{\beta}{\gamma} \log\left(\frac{\alpha}{X_0} + 1\right) - \frac{\beta}{\gamma} \log\left(\frac{\alpha}{X_0} + e^{-\gamma t}\right)$$

Hence

$$\log\left(\frac{N}{N - I}\right) = \frac{\beta}{\gamma} \log\left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right) = \log\left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right)^{\frac{\beta}{\gamma}}$$ **[1]**

But then

$$\frac{N}{N - I} = \left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right)^{\frac{\beta}{\gamma}}$$

and so

$$I = N\left(1 - \left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right)^{-\frac{\beta}{\gamma}}\right)$$ **[1]**

(f) **[5 marks]**

Behaviour when $t = 0$

$$I = N\left(1 - \left(\frac{\alpha + X_0}{\alpha + X_0}\right)^{-\frac{\beta}{\gamma}}\right) = N(1-1) = 0$$

as required.

Checking the given expression in the differential equation **[1]**

Left hand side

$$\frac{dI}{dt} = \frac{d}{dt}\left(N\left(1 - \left(\frac{\alpha + X_0}{\alpha + X_0}\right)^{-\frac{\beta}{\gamma}}\right)\right) = -\frac{d}{dt}\left(N\left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right)^{-\frac{\beta}{\gamma}}\right)$$

Using the chain rule, take

$$u = \frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}} = \frac{\alpha + X_0}{v}$$ **[0.5]**

and so (where have used chain rule again to differentiate u …)

$$\frac{du}{dt} = (\alpha + X_0)\left(-\frac{1}{v^2}\right)\frac{dv}{dt} = \frac{\gamma X_0 e^{-\gamma t}(\alpha + X_0)}{(\alpha + X_0 e^{-\gamma t})^2} = \frac{\gamma X_0 e^{-\gamma t}}{\alpha + X_0 e^{-\gamma t}}u$$

Therefore, since

$$\frac{dI}{dt} = -\frac{d}{dt}\left(Nu^{-\frac{\beta}{\gamma}}\right) = -\frac{d}{du}\left(Nu^{-\frac{\beta}{\gamma}}\right)\frac{du}{dt}$$ **[1]**

it must be that

$$\frac{dI}{dt} = -\left(-\frac{\beta}{\gamma}Nu^{-\frac{\beta}{\gamma}-1}\right)\left(\frac{\gamma X_0 e^{-\gamma t}}{\alpha + X_0 e^{-\gamma t}}u\right) = \frac{\beta N X_0 e^{-\gamma t}}{\alpha + X_0 e^{-\gamma t}}u^{-\frac{\beta}{\gamma}}$$

Replacing u with the relevant function of time

$$\frac{dI}{dt} = \frac{\beta N X_0 e^{-\gamma t}}{\alpha + X_0 e^{-\gamma t}}\left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right)^{-\frac{\beta}{\gamma}}$$ **[1]**

Right hand side

Substituting the expression for $I(t)$ into the right-hand side of the model

$$\frac{\beta X}{\alpha + X}(N - I) = \frac{\beta X_0 e^{-\gamma t}}{\alpha + X_0 e^{-\gamma t}}\left(N - N\left(1 - \left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right)^{-\frac{\beta}{\gamma}}\right)\right)$$

So lots of terms cancel in the big bracket

$$\frac{\beta X}{\alpha + X}(N - I) = \frac{\beta X_0 e^{-\gamma t} N}{\alpha + X_0 e^{-\gamma t}}\left(\frac{\alpha + X_0}{\alpha + X_0 e^{-\gamma t}}\right)^{-\frac{\beta}{\gamma}}$$

Conclusion **[1]**

The two sides match, and so we have a solution. **[0.5]**

C6

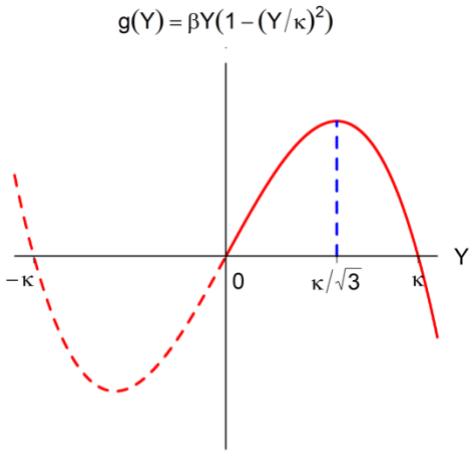
(a) **[5 marks]**

Substituting $Y(t) = \bar{Y} + h(t)$ into the differential equation $$\frac{d}{dt}(\bar{Y} + h) = F(\bar{Y} + h)$$ Since $\bar{Y}$ is constant $$\frac{d}{dt}(\bar{Y} + h) = \frac{dh}{dt}$$	**[1]**
Since h is small $$F(\bar{Y} + h) \approx F(\bar{Y}) + h\frac{dF}{dY}(\bar{Y})$$	**[1]**
Since $\bar{Y}$ is an equilibrium $$F(\bar{Y} + h) \approx h\frac{dF}{dY}(\bar{Y}) = hD$$ (where introduced $D = \frac{dF}{dY}(\bar{Y})$ to emphasise that this quantity is just a constant)	**[1]**
Equating the two sides, the perturbation follows the differential equation $$\frac{dh}{dt} \approx hD$$ and so $$h(t) = h_0 e^{Dt}$$	**[1]**
Hence the perturbation either grows (initially at least) or decays off, depending on the sign of $D = \frac{dF}{dY}(\bar{Y})$	**[1]**

(b) **[2 marks]**

Clear description of absolute and/or relative growth rates, with correct naming	**[0.5]**
Notion that the model is density-dependent	**[0.5]**
β is the relative rate of growth when Y is very small	**[0.5]**
κ is the population size at which growth ceases (accept "carrying capacity" with no further comment)	**[0.5]**

(c) **[2 marks]**

Equilibria at $\bar{Y} = 0, \pm\kappa$ (fine if negative equilibrium not considered; no problem if roots not noted explicitly as "equilibria" and can be implicit from sketch) $G(Y)$ is cubic, with negative Y^3 term, and so $G(Y) \to -\infty$ as $Y \to +\infty$ (again, no problem if not noted explicitly and can be implicit from sketch)  	**[0.5]** **[0.5]** **[1]**

(d) **[3 marks]**

Differentiating the differential equation with respect to time $$\frac{d^2Y}{dt^2} = \frac{d}{dt}\left(\beta Y - \beta\frac{Y^3}{\kappa^2}\right) = \beta\left(1 - 3\left(\frac{Y}{\kappa}\right)^2\right)\frac{dY}{dt}$$ A necessary condition for population to be growing most rapidly is $\frac{d^2Y}{dt^2} = 0$ Setting aside any equilibria with $\frac{dY}{dt} = 0$, that occurs when $$1 - 3\left(\frac{Y}{\kappa}\right)^2 = 0$$ i.e., $$Y = \pm\frac{\kappa}{\sqrt{3}}.$$ It follows $Y = \kappa/\sqrt{3}$ is the population size with maximum growth (need to justify this; easiest is appealing to graph sketched in previous part).	**[0.5]** **[0.5]** **[0.5]** **[0.5]** **[1]**

(e) **[3 marks]**

Equilibria $$0 = Y\left(\beta\left(1 - \left(\frac{Y}{\kappa}\right)^2\right) - \alpha\right)$$ i.e. $\bar{Y} = 0$ or $$1 - \left(\frac{Y}{\kappa}\right)^2 = \frac{\alpha}{\beta}$$	**[0.5]**

and so

$$\bar{Y} = \pm\kappa\sqrt{1-\frac{\alpha}{\beta}}$$

(where this root can be taken since $\alpha < \beta$).

The two equilibria to consider are $\bar{Y} = 0$ and $\bar{Y} = \kappa\sqrt{1-\frac{\alpha}{\beta}}$. **[0.5]**

Taking $F(Y) = G(Y) - H(Y)$, we can reuse working from part (c) to see

$$\frac{dF}{dY} = \beta\left(1 - 3\left(\frac{Y}{\kappa}\right)^2\right) - \alpha$$ **[0.5]**

At $\bar{Y} = 0$, $D = \frac{dF}{dY}(\bar{Y}) = \beta - \alpha$

Since $\alpha < \beta$, we know $D > 0$ and so this equilibrium is unstable **[0.5]**

At $\bar{Y} = \kappa\sqrt{1-\frac{\alpha}{\beta}}$

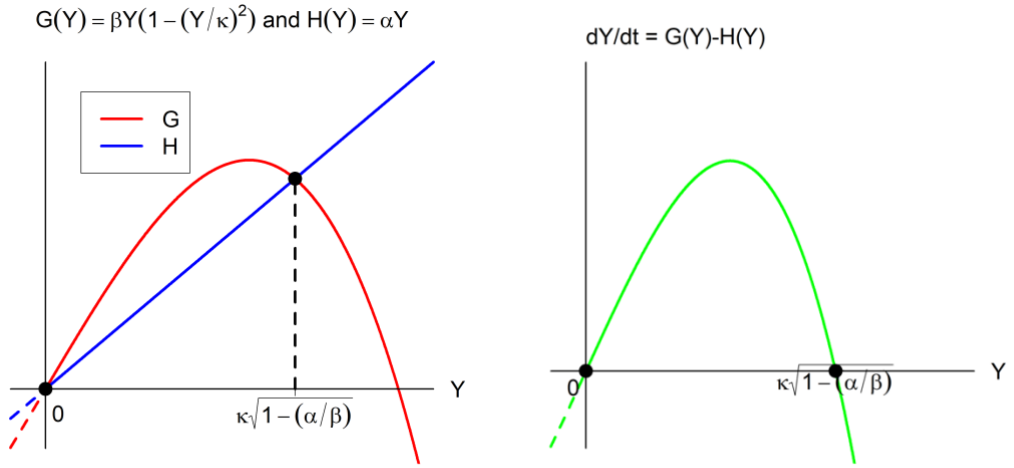
$$D = \frac{dF}{dY}(\bar{Y}) = \beta\left(1 - 3\left(1-\frac{\alpha}{\beta}\right)\right) - \alpha = -2\beta + 2\alpha = 2(\alpha - \beta)$$ **[0.5]**

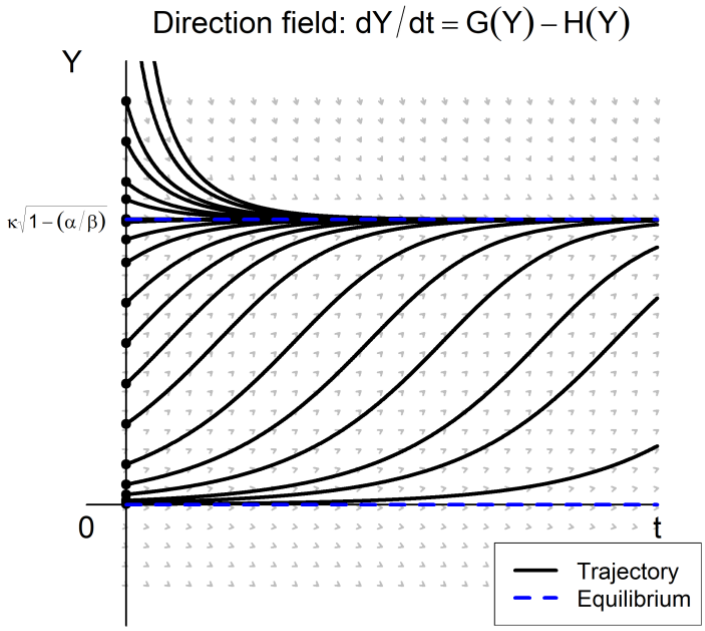
Since $\alpha < \beta$, we know $D < 0$ and so this equilibrium is stable **[0.5]**

(f) **[3 marks]**

Because $\alpha < \beta$ we know the two graphs must cross for positive Y (this needs to be noted, with some sort of explanation, either based on the non-zero equilibrium of the model being at a positive value (or in fact, existing at all)!, or the gradients of $G(Y)$ and $H(Y)$. **[0.5]**

And this allows us to infer the sign of $\frac{dY}{dt} = G(Y) - H(Y)$ as a function of Y



	**[1]**
Either clearly showing the inflexion point or showing behaviour for initial values of Y above the non-zero equilibrium is required for full credit here.	**[1]** **[0.5]**

(g) **[2 marks]**

When $\alpha < \beta$, the relative predation rate at low population size is smaller than the relative growth rate, and so the population should increase if Y is small However, as the population size increases, the relative growth rate decreases to zero, whereas the relative predation rate is unchanged, and so it makes sense for the population to settle to an equilibrium.	**[1]** **[1]**

D7

a) $\begin{cases} x' = xy^2 - 3xy = 0 \\ y' = y - x^2 + x - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} xy(y-3) = 0 \quad (1) \\ y - x^2 + x - 1 = 0 \quad (2) \end{cases}$

$(1) \Rightarrow x = 0$ (1a) or $y = 0$ (1b) or $y = 3$ (1c)

(1a) + (2): $y - 1 = 0$, $y = 1$ $\Rightarrow$ $(0,1)$ is an eq. point

(1b) + (2): $-x^2 + x - 1 = 0$

$x^2 - 1x + 1 = 0$

$x = \frac{1 \pm \sqrt{1^2 - 4(1)}}{2} = \frac{1 \pm \sqrt{-3}}{2}$ X

no sols

(1c) + (2): $3 - x^2 + x - 1 = 0$

$-x^2 + x + 2 = 0$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$\Rightarrow x = -1$ or $x = 2$

$\Downarrow$ $\Downarrow$

$(-1,3)$ $(2,3)$ are eq. points

Classification

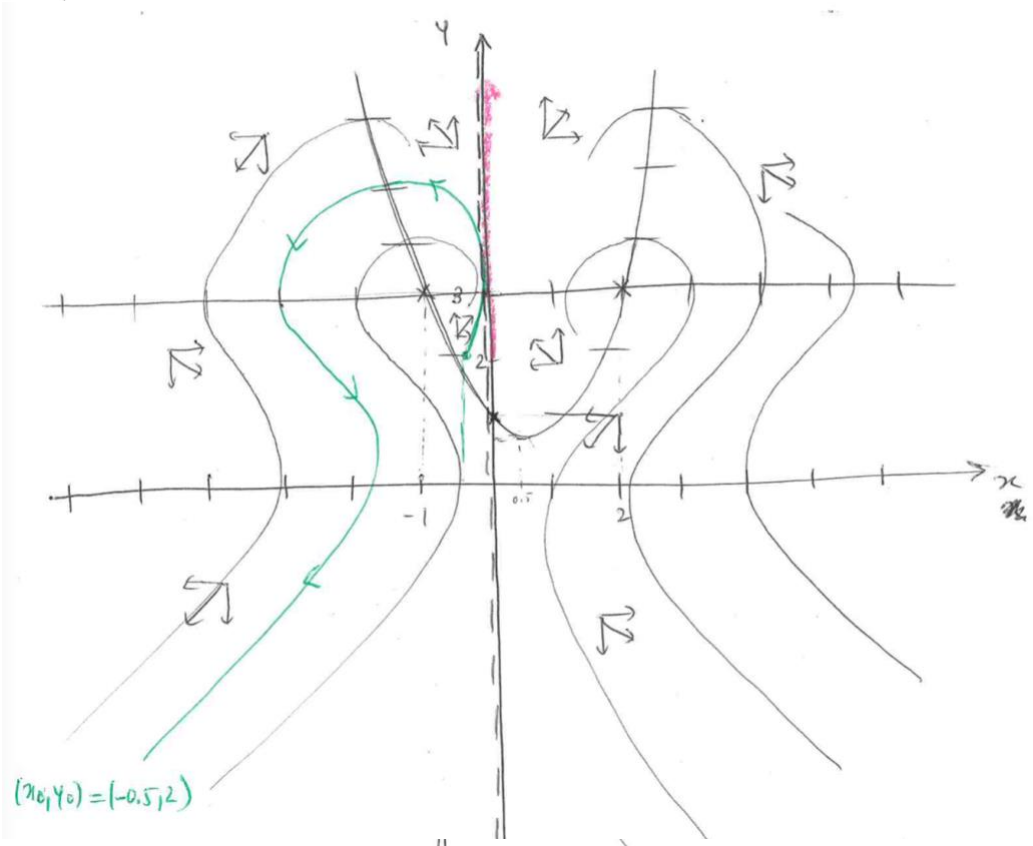
$$D = \begin{pmatrix} y^2 - 3y & 2xy - 3x \\ -2x + 1 & 1 \end{pmatrix}$$

	$(-1,3)$	$(2,3)$	$(0,1)$
$D_{11} = y^2 - 3y$	$3^2 - 3(3) = 0$	$3^2 - 3(3) = 0$	$1 - 3(1) = -2$
$D_{12} = 2xy - 3x$	$2(-1)(3) - 3(-1) = -6 + 3 = -3$	$2(2)(3) - 3(2) = 12 - 6 = 6$	$2(0)(1) - 3(0) = 0$
$D_{21} = -2x + 1$	$-2(-1) + 1 = 2 + 1 = 3$	$-2(2) + 1 = -3$	1
$D_{22} = 1$	1	1	1
T	$0 + 1 = 1$	$0 + 1 = 1$	$-2 + 1 = -1$
Δ	$0 - (-3)(3) = 9$	$0 - 6(-3) = 18$	$-2 - 0 = -2 < 0$
$T^2 - 4\Delta$	$1^2 - 4(9) = -35 < 0$ unstable focus	$1^2 - 4(18)$ unstable focus	Saddle point

b) x-clines $\quad x = 0$
$y = 0$
$y = 3$

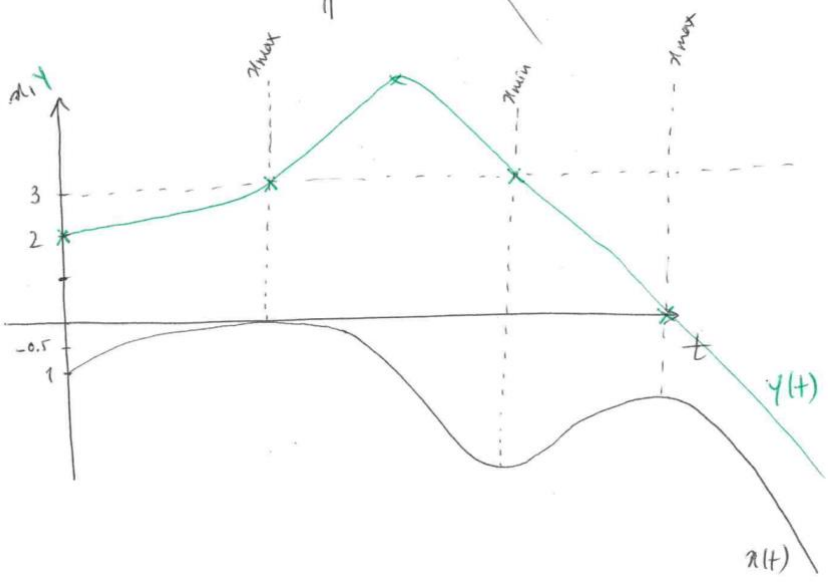
y-cline $\quad y - x^2 + x - 1 = 0$
$y = x^2 - x + 1$

@ $(2,1)$: $\quad x' = 2(1)^2 - 3(2)(1) = 2 - 6 = -4 < 0$
$y' = 1 - 2^2 + 2 - 1 = -4 + 2 = -2 < 0$

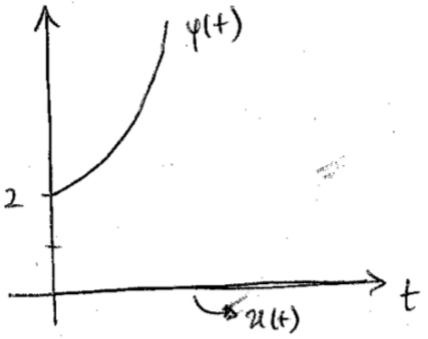


$(x_0, y_0) = (-0.5, 2)$

c)



d) for $(x_0, y_0) = (0, 2)$



Analytically:

$x = 0 \Rightarrow x' = 0 \Rightarrow x = \text{constant}$

$$y' = y - 1$$

$$\frac{dy}{dt} = y - 1$$

$$\frac{1}{y-1}\, dy = dt$$

$$\ln(y-1) = t + k$$

$$e^{\ln(y-1)} = e^{t+k}$$

$$y - 1 = e^t \cdot C$$

@ $t = 0$: $y = 2$

$$2 - 1 = 1 \cdot C \Rightarrow C = 1$$

$$\Rightarrow y(t) - 1 = e^t \cdot 1$$

$$y(t) = e^t + 1$$

$x(t)$ and $y(t)$ solutions match the graph above.

D8

a [2]. In the absence of rabies, the model is reduced to $S' = \alpha - \delta\, S$ [1], which is a monomolecular growth model with stationary point $S = \alpha/\delta$ [1].

b [2]. By analogy with the SIR model taught in Block D, the basic reproductive number can be written as $\mathcal{R}_0 = \frac{\beta N}{\mu}$, where βN is the number of new cases per unit of time in a fully susceptible population *N* and $1/\mu$ is the average duration of infection [1].

From question (a), $N = \alpha/\delta$ [1], hence the full expression as proposed.

c [6].

Three nullclines:

- S nullclines: $\frac{dS}{dt} = 0$ yields $I = \frac{\alpha - \delta S}{\beta S} = \frac{\alpha}{\beta S} - \frac{\delta}{\beta}$ for $S > 0$, which can also be written as $S = \frac{\alpha}{\beta I + \delta}$ [1]
- I nullclines; $\frac{dI}{dt} = 0$ yields two solutions: $I = 0$ [1] or $S = \frac{\mu}{\beta}$ [1].

One or two stationary points:

- $S = \frac{\alpha}{\delta}, I = 0$ [1]
- $S^* = \frac{\mu}{\beta}, I^* = \frac{\alpha}{\mu} - \frac{\delta}{\beta}$ [1] which is only relevant if $I^* > 0$, requiring $\beta\alpha > \delta\mu$, i.e. $\mathcal{R}_0 > 1$ [1].

d [5].

Here, $S^* = \frac{\mu}{\beta}, I^* = \frac{\alpha}{\mu} - \frac{\delta}{\beta} = \frac{\delta}{\beta}(\mathcal{R}_0 - 1)$, with $\mathcal{R}_0 = 2$, hence , $I^* = \frac{\delta}{\beta}$ [1]

Partial derivatives:

$$D_{11} = -\beta I^* - \delta$$
$$D_{12} = -\beta S^*$$
$$D_{21} = \beta I^*$$
$$D_{22} = \beta S^* - \mu$$

Hence the Jacobian matrix is:

$$\begin{pmatrix} -2\delta & -\mu \\ \delta & 0 \end{pmatrix}$$ [2]

$$T = -2\delta < 0$$
$$\Delta = \delta\mu > 0$$
$$T^2 - 4\Delta = 4\delta(\delta - \mu) < 0$$ [2]

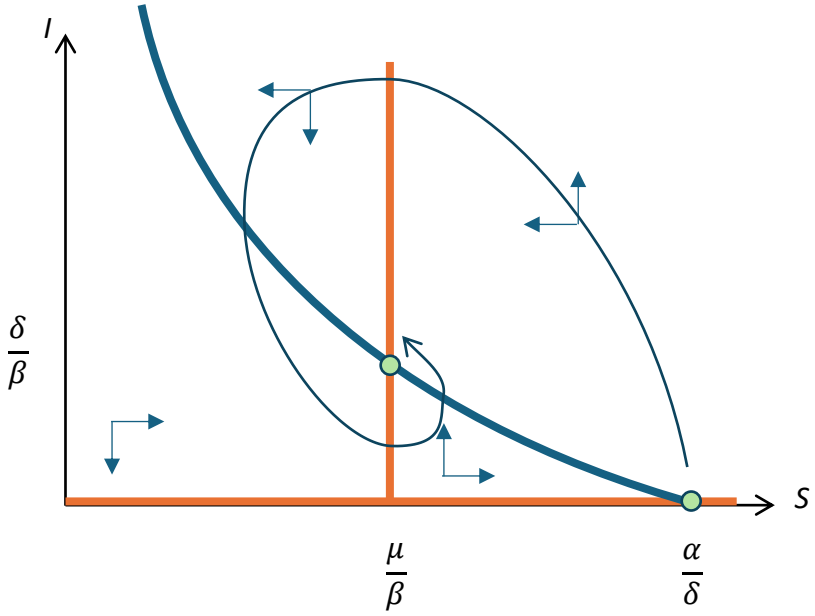
e [5].

Direction field: when S and I are close to 0, $\frac{dS}{dt} \approx \alpha > 0$ and $\frac{dI}{dt} \approx -\mu I < 0$. [1]

Correct sketch of nullclines showing intercepts and direction field [2]

Two stationary points highlighted [1]

Correct trajectory spiralling towards the endemic stationary point and showing horizontal/vertical crossings of the nullclines [1].



E9

(a) $k_1[EH] = k_2[E^-][H^+]$

Therefore $K_a = [E^-][H^+]/[EH] = k_1/k_2$

(b) $[E]_{TOT} = [EH] + [E^-] + [ES]$

(c) $d[ES]/dt = k_3[E^-][S^+] - [ES](k_4 + k_5)$

$d[P]/dt = k_5[ES]$

(d) Applying the steady state approximation to [ES] gives:

$d[ES]/dt = k_3[E^-][S^+] - [ES](k_4 + k_5) = 0$

Therefore:

$K_M = (k_4 + k_5)/k_3 = [E^-][S^+]/[ES]$

And:

$[ES] = [E^-][S^+]/K_M$

(e) $[E]_{TOT} = [EH] + [E^-] + [ES]$

$= [E^-][H^+]/K_a + [E^-] + [E^-][S^+]/K_M$

$= [E^-]\{ 1 + [S^+]/K_M + [H^+]/K_a \}$

Therefore:

$[E^-] = [E]_{TOT}/\{ 1 + [S^+]/K_M + [H^+]/K_a \}$

(f) $d[P]/dt = k_5[ES] = k_5[E^-][S^+]/K_M$

$= \{ k_5[S^+]/K_M \} \times [E]_{TOT} / \{ 1 + [S^+]/K_M + [H^+]/K_a \}$

$= \{ k_5[E]_{TOT}[S^+] \} / \{ K_M + [S^+] + K_M[H^+]/K_a \}$

$= \{ k_5[E]_{TOT}[S^+] \} / \{ K_M(1 + [H^+]/K_a) + [S^+] \}$

$= \{ V_{MAX}^{app}[S^+] \} / \{ K_M^{app} + [S^+] \}$

Where $V_{MAX}^{app} = k_5[E]_{TOT}$

and $K_M^{app} = K_M(1 + [H^+]/K_a)$

Under initial rate conditions we can assume that $[S^+] \approx [S^+]_0$, therefore:

$v_0 = \{ V_{MAX}^{app}[S^+]_0 \} / \{ K_M^{app} + [S^+]_0 \}$

(g) V_{MAX}^{app} doesn't depend on $[H^+]$, but K_M^{app} does, so this behaviour has the same characteristics as ***competitive*** inhibition.

(h) If $[H^+]$ is very low, then:

$V_{MAX}^{app} = k_5[E]_{TOT}$ and $K_M^{app} \approx K_M(1 + 0/K_a) = K_M$

On the other hand, if $[H^+]$ is very high, then:

$V_{MAX}^{app} = k_5[E]_{TOT}$ and $K_M^{app} \approx K_M(0 + [H^+]/K_a) = (K_M/K_a).[H^+]$

E10

a) The single parent's optimal care c_{single} must satisfy first-order optimisation condition

$$W'(c_{\text{single}}) = \frac{b}{(b+c_{\text{single}})^2} - k = 0$$

$$\rightarrow c_{\text{single}} = \sqrt{\frac{b}{k}} - b$$

b) $W(c_{\text{single}}) = \frac{c_{\text{single}}}{b+c_{\text{single}}} - kc_{\text{single}} = 1 + k\left(b - 2\sqrt{\frac{b}{k}}\right)$

c) A parent's best-response care $c_{\text{opt}}(c_{\text{mate}})$ must satisfy first-order optimisation condition

$\frac{\partial W(c_{\text{self}}, c_{\text{mate}})}{\partial c_{\text{self}}} = \frac{b}{(b+c_{\text{self}}+c_{mate})^2} - k = 0$ for $c_{\text{self}} = c_{\text{opt}}(c_{\text{mate}})$

$$\rightarrow c_{\text{opt}}(c_{mate}) = \sqrt{\frac{b}{k}} - b - c_{\text{mate}}$$

d) $c_{\text{opt}}(c^*) = c^*$

$$\rightarrow \sqrt{\frac{b}{k}} - b - c^* = c^*$$

$\rightarrow c^* = \frac{1}{2}\left(\sqrt{\frac{b}{k}} - b\right)$, which is equal to c_{single} / 2