Laboratory Exercises

- **Set 1 (21-01-2025): Logistic modelling of economic data**

  1. *The growth of IBM*: This is a brief set of guidelines for modelling the growth of *IBM*. The data have been provided in two data files. One is named `groibm.dat`, and it contains data in three columns. The first column gives the annual revenue (measured in millions of dollars) generated by *IBM*. The second column gives the annual human resource count (employees) of *IBM*. The third column gives the year count, starting from the year 1914 (this is the first year for which the revenue record exists). The last year up to which data are available is the 93rd year (2006). The second data file is named `prof_ibm.dat`. This data file has two columns. The first column is the year count, and the second column gives the net annual profit (in millions of dollars) made by *IBM*. Along with the data, a paper (file named `lm_refer.pdf`) has been provided. In this paper, Section I (*The Logistic Function*) gives the general theory of the logistic equation and Section III (*The Logistic Dynamics of a Company*) contains information about the modelling.

     A. Refer to Section I for understanding the logistic modelling and reproduce the four plots in Section III (Figs.3, 4, 5 & 6) of the paper. In so doing, improvisations to improve the modelling quality are encouraged. Examples of such improvisations are better scale setting of plots for the revenue/human resource versus time and statistical analyses of the model fitting. B. State three salient conclusions drawn from the modelling exercise.

  2. *GDP and trade of national economies*: This is a brief set of guidelines for modelling the GDP and trade dynamics of six countries with the highest GDP in the world. These countries are the USA, China, Japan, Germany, India and the UK. The data required for the modelling exercise have been provided in files with the generic name `Country_GDP_Trade.dat` for China, Japan, Germany, India and the UK. In each file, the data have been provided in three columns. The first column gives the time (in years), the second column gives the annual GDP and the third column gives the annual trade. For USA, there are two data files in a separate directory named `USA_GDP_Trade`. One file, named `US_GDP.dat`, gives time in the first column and the annual GDP in the second column. The other data file, named `US_Trade.dat`, gives time in the first column and the trade value in the second column. For all the six countries, the GDP and trade values are given in US dollars. The year count starts from 1960 for all the countries except Germany (for which the year count starts from 1970). Along with the data files, three papers have been provided. One of them (file named `kr0822arX.pdf`) and its journal version (file named `IJMPC_GDP_trade.pdf`) provide information about the modelling work and its implications. The third paper (file named `gdp_GEJ24.pdf`) gives a forecasting analysis of the GDP growth.

     A. Reproduce all the plots and tables in the papers. Improving improvisations are encouraged. B. Using parameter values relevant to India, predict the years in which India's GDP will be $\$4$ trillion and $\$5$ trillion. C. Assume that India's GDP grows exponentially (as $x = x_0 e^{at}$) and use your parameter values to find the GDP in 2047. D. State three salient conclusions drawn from the modelling exercise.

- **Set 2 (04-02-2025): Compartment modelling of linear systems**[1]

  1. The concentration $C(t)$ of pollutants in a lake follows the equation $\dot{C} = a - bC$, in which $a = FC_{\text{in}}/V$ and $b = F/V$. Here C_{in} in the constant pollutant concentration of inflow into the

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lake, F is the fixed volumetric flow rate and V is the fixed volume of the lake (as the lake also drains out). Take $F = 5 \times 10^8\,\mathrm{m^3/day}$, $V = 10^{12}\,\mathrm{m^3}$ and $C_{\mathrm{in}} = 3\,\mathrm{unit}$ and $C(0) = C_0 = 10\,\mathrm{unit}$.

A. Plot $C(t)$ versus t. B. Estimate the time taken (in days/years) for $C = 0.5C_0$. C. Test for the case of clean fresh water flow into the lake ($C_{\mathrm{in}} = 0$).

2. A single dose of a medicine is administered to a patient. The dynamics of the medicine follows the equation $\dot{x} = -k_1 x$, $x(0) = x_0$ in the GI tract, and $\dot{y} = k_1 x - k_2 y$, $y(0) = 0$ in the blood stream. Take $k_1 = 0.6931\,\mathrm{hr}^{-1}$, $k_2 = 0.0231\,\mathrm{hr}^{-1}$ and $x_0 = 1\,\mathrm{unit}$.

   A. Plot $x(t)$ and $y(t)$ versus t (in hours) in the same graph. Note the peak value of $y(t)$ and its time. B. Test for $k_1 = k_2$ using both values above. Plot the results in two separate graphs.

3. A course of medicine is administered to a patient. The dynamics of the medicine follows the equation $\dot{x} = I - k_1 x$, $x(0) = 0$ in the GI tract, and $\dot{y} = k_1 x - k_2 y$, $y(0) = 0$ in the blood stream. Take $k_1 = 0.6931\,\mathrm{hr}^{-1}$, $k_2 = 0.0231\,\mathrm{hr}^{-1}$ and $I = 1\,\mathrm{unit}$.

   A. Plot $x(t)$ and $y(t)$ versus t (in hours) in two separate graphs. Note the limiting values of $x(t)$ and $y(t)$. B. Test for $k_1 = k_2$ using both values above. Plot the results in separate graphs.

- **Set 3 (04-02-2025): The logistic equation and its modifications**

1. Worldwide, the human population $x(t)$ grows according to the logistic equation $\dot{x} = ax - bx^2$. Take $a \simeq 0.03$ per annum and $b \simeq 3 \times 10^{-12}$ as per the global census data of 1961.
   (a) Estimate the limiting value of the world population. Compare this value with the present global population (to be found by a Google search).
   (b) Plot $x(t)$ using its analytical solution starting with an initial value of $x_0 \simeq 3 \times 10^9$.
   (c) Choosing an appropriately small time step Δt, numerically integrate by Euler's method. Plot the numerical solution along with the analytical solution to compare for accuracy. Separately plot the relative error between the two solutions.

2. Refer to the dynamical equation $\dot{x} = f(x)$ for the spread of agricultural innovations among farmers through both personal and impersonal communications. Perform the following tasks:

   A. Defining $X = x/N$, $T = cNt$, $A = C'/CN$ and $\dot{X} \equiv \mathrm{d}X/\mathrm{d}T$, recast the dynamical equation to the form $\dot{X} = (X + A)(1 - X)$. B. Rescale the integral solution $x(t)$ as $X(T)$, in which A will be the only parameter. C. Plot $\dot{X}$ versus X for $A = 0, 0.2, 0.5$ in the same graph. D. Plot the integral solution $X(T)$ for $A = 0, 0.2, 0.5$ in the same graph. Take $X(0) = 0.0001$.

3. The logistic equation is modified as $\dot{x} = rx\,[1 - (x/k)] - h$, in which h is the "harvesting" rate. Choose $r = 1$ and $k = 1000$.

   A. Plot $\dot{x}$ versus x in the same graph with $h = 0, 100, 500$. B. Given the initial value of $x(0) = 10$, integrate by Euler's method with a suitably chosen Δt to obtain $x(t)$ for all the aforementioned values of h. Plot x versus t for all three cases in the same graph. C. For $h = 0$ compare the analytical solution and the numerical solution in the same graph for quality testing. Plot the relative error between the two solutions in another graph.

- **Set 4 (18-02-2025): Constrained growth beyond the logistic model**[2]

1. The Gompertz equation models tumour growth as $\dot{x} = -ax\ln(bx)$ $(a, b > 0)$.

   A. Rescale $X = x/b^{-1}$ and $T = at$ to obtain a parameter-free differential equation $\dot{X} \equiv \mathrm{d}X/\mathrm{d}T = -X\ln X$. B. Plot $\dot{X}$ versus X for three different initial values of your choice. One of these should be very close to zero and another should be greater than e^{-1}. The third one should have an intermediate value. Show all three plots in the same graph. C. With the intermediate initial value, integrate by Euler's method with a suitably chosen Δt to obtain $X(T)$. Plot the

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analytical solution in the same graph for quality testing. D. Plot the relative error between the two solutions in another graph.

2. The Allee effect models high growth rate of a population when the initial population size has an intermediate value. The model equation is $\dot{x} = x\left[r - a(x-b)^2\right]$ $(a, b, r > 0)$. The model is effective only when $r < ab^2$.

   A. Plot $\dot{x}$ versus x by hand, and note the values of x when $\dot{x} = 0$ (the fixed points). For suitably chosen values of a, b and r (eg. $a = r = 1$ and $b = 2$), verify your plotting result on a computer and show the plot. B. Integrate by Euler's method with a suitably chosen Δt to obtain $x(t)$. If the initial value is $x(0) = x_0$, then take two different values of x_0. They are $x_0 < (b - \sqrt{r/a})$ and $x_0 > (b - \sqrt{r/a})$. However, both initial values should be very close to $b - \sqrt{r/a}$. Plot the numerical solutions for both initial values in the same graph. C. In both solutions note the limiting values of x when $t \longrightarrow \infty$ and comment on the Allee effect.

- **Set 5 (25-02-2025): Modelling data with power laws (Pareto's law and Zipf's law)**

1. <u>*Pareto distribution of wealth in India*</u>: This is a brief set of guidelines for modelling the *Pareto* distribution of wealth in India. In 2021 India had 330000 people with at least \$1 million, 21000 people with at least \$10 million, 1074 people with at least \$100 million and 120 people with at least \$1 billion. These four data points are given in the file named `pareto_in.dat`. Using the data, reproduce the graph in the file named `graph_Pareto_India.pdf`. Given that x is the amount of wealth and $N(x)$ is the frequency distribution of wealth holders, apply the function $N(x) = A + Bx^{-\alpha}$ (with $A = 60$ and $\alpha = 5/4$) to model the power law that the data follow.

2. <u>*Zipf's law in the dependency network of Debian*</u>: This is a brief set of guidelines for modelling dependency distributions in *Debian*. The data have been provided in six data files, covering three successive releases of *Debian* — *Etch* (Debian 4.0), *Lenny* (Debian 5.0) and *Squeeze* (Debian 6.0). For each release there are two files, one for the distribution of incoming dependency links and the other for the distribution of outgoing dependency links. The names of the files are self-explanatory. In every data file the first column contains the number of links and the second column contains the frequency distribution of the links. Along with the data files, a paper (file named `nnr_compsys.pdf`) provides all relevant information about the modelling approach.

   A. Reproduce the first six plots in the paper. For this purpose refer to Equations (2) & (5) in the paper and the caption of every relevant plot for the numerical values of the parameters. B. State three salient conclusions drawn from the modelling exercise.

- **Set 6 (25-02-2025): Modelling traffic flow data with a bimodal function**

1. <u>*Bimodality in traffic flows*</u>: This is a brief set of guidelines for modelling bi-directional traffic flows with a bimodal function. There is one data file, named `traffic_time.dat`. The data contained in this file have been taken from the repository of the *Alabama Department of Transportation (ALDOT)*, specifically pertaining to the city of *Jackson* in *Alabama State, U.S.A.* The data show the volume of vehicular traffic as a function of time (measured in hours). There are three columns of data. The first column gives the average volume of traffic flowing towards the east. The second column gives the hour of the day, with the 0 hour being set at midday, 12 pm. So all forenoon hours are negative. The third column gives the average volume of the traffic flowing towards the west. Along with the data file, a paper (file named `am_akr_jphys.pdf`) provides all relevant information about the modelling approach.

   A. Reproduce the first two plots in the paper. For this purpose refer to Equation (1) in the paper and the captions of the two plots for the numerical values of the parameters. B. Find the mean and the standard deviation of the *relative* variation of the model fitting for each plot.

- **Set 7 (04-03-2025): Modelling strategic conflict between nations**

  1. *Richardson's mathematical model of conflict between nations*: Strategic conflict between two nations is modelled by the coupled equations $\dot{x} = ky + g - \alpha x$ and $\dot{y} = lx + h - \beta y$. Here x and y are the war-waging potential of the two nations, with $k, l, h, g, \alpha, \beta (> 0)$ being fixed parameters. Model the following strategic scenarios with moderate parameter values of your choice (say, of the order of unity). However, the parameter values should be different. State these values clearly.

     A. Mutual disarmament without grievance, $g = h = 0$. Under the condition of $\alpha\beta > kl$, integrate $x(t)$ and $y(t)$ by Euler's method. Choose the value of Δt in the Euler method with care. Take moderate and comparable initial values for $x(0)$ and $y(0)$. Plot $x(t)$ and $y(t)$ along the vertical axis and t along the horizontal axis in the same graph. Check the vertical axis on a logarithmic scale. In another graph plot y versus x.
     B. Mutual disarmament with grievance, $g, h \neq 0$. With initial values of $x(0) = y(0) = 0$, integrate $x(t)$ and $y(t)$ by Euler's method. Plot $x(t)$ and $y(t)$ along the vertical axis and t along the horizontal axis in the same graph. In another graph plot y versus x.
     C. Unilateral disarmament, $y(0) = 0$ and $x(0) \neq 0$. Integrate $y(t)$ by Euler's method, for a moderate initial value of $x(0)$. Plot $y(t)$ versus t and show that y grows again (rearmament).
     D. Arms race, $\alpha = \beta = g = h = 0$. With comparable initial values for $x(0)$ and $y(0)$, integrate by Euler's method. Plot $x(t)$ and $y(t)$ along the vertical axis and t along the horizontal axis in the same graph. Check the vertical axis on a logarithmic scale. In another graph plot y versus x.

- **Set 8 (18-03-2025): Battles and war games: Iwo Jima (land) and Trafalgar (naval)**[3]

  1. *Battle of Iwo Jima*: In 1945 a fierce battle was fought on the Japanese island of Iwo Jima between the Imperial Japanese Army and the US Army. At a time t, the number of Japanese troops is $J(t)$ and the number of American troops is $A(t)$. Applying the Lanchester model of conventional-conventional combat, the coupled equations for the engagement are $\dot{J} = -aA$ and $\dot{A} = -jJ$. Here $a = 0.0106$ and $j = 0.0544$ are the combat effectiveness parameters of the American and the Japanese troops, respectively. The initial number of Japanese soldiers is $J(0) = J_0 = 18274$ and American soldiers is $A(0) = A_0 = 66454$.

     A. Going by Lanchester's square law, predict the outcome of the battle.
     B. Taking a time step $\Delta t = 1\,\text{day}$, integrate the foregoing coupled equations by Euler's method. Stop the integration when the number of troops of one army falls below 1. Why should you stop the integration at this number? Count the number of days for this outcome. What are the numbers of active troops and casualties in the victorious army? What is the significance of these numbers?
     C. Plot J and A along the vertical axis, and t along the horizontal axis in the same graph. Separately plot with a logarithmic vertical axis and state the approximate slopes of J and A in this linear-log plot. Compare and state the closeness of these values to $\sqrt{aj}$.
     D. Plot A along the vertical axis and J along the horizontal axis in a graph with origin at $(0, 0)$.

  2. *Battle of Trafalgar*: In 1805 the battle of Trafalgar was fought between Napoleon's French Navy and the British Royal Navy, which was under the command of Lord Nelson. The British emerged victorious in this momentous naval battle. If at a time t, the number of French ships is $F(t)$ and the number of British ships is $B(t)$, then applying the Lanchester model of conventional-conventional combat, the coupled equations for the naval battle are $\dot{F} = -bB$ and $\dot{B} = -fF$. The combat effectiveness parameters are $f = b = 0.05$. The initial number of ships in the French fleet is $F(0) = F_0 = 33$ and in the British fleet is $B(0) = B_0 = 27$. Noting the numerical superiority of the French fleet, Lord Nelson divided his overall battle plan into three stages.

     A. In the first stage he kept 14 ships in reserve and intensely engaged a small French force of 3 ships with his 13 ships. Taking a time step $\Delta t = 1\,\text{unit}$, integrate the Lanchester equations

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by Euler's method with the initial values of the first stage of Lord Nelson's battle plan. Stop the integration when the number of French ships is $1 < F < 2$. Also note the value of B at this stage. Plot B and F along the vertical axis and t along the horizontal axis in the same graph.
B. In the second stage of Lord Nelson's battle plan, the surviving British ships of the first stage join forces with the 14 ships kept in reserve. A fractional number implies a badly damaged ship. Similarly the surviving French ships join a larger force of 17 French ships. Clearly state the initial conditions of the second stage. With these initial conditions, integrate the Lanchester equations once again by Euler's method with the same time step. Stop the integration when the number of French ships is $1 < F < 2$. Also note the value of B at this stage. Plot B and F along the vertical axis and t along the horizontal axis in the same graph.
C. In the third (and final) stage, the surviving French ships of the second stage join the last group of 13 French ships, all of which are engaged with the full force of the remaining British ships. State the initial conditions and using them integrate the Lanchester equations once again by Euler's method with the same time step. Stop the integration when the number of French ships is $F < 1$. Also note the value of B at this stage. Plot B and F along the vertical axis and t along the horizontal axis in the same graph.
D. In a war game show that the British would have lost the battle of Trafalgar if Lord Nelson had engaged the full French fleet of 33 ships with his 27 ships all at the same time. Use $f = b = 0.1$. Plot $B(t)$ and $F(t)$ against t in the same graph.

- **Set 9 (01-04-2025): Competitive exclusion and predator-prey dynamics**

1. *Competitive exclusion*: In Nature it happens often that two similar species compete for resources and living space within the same ecological territory. The outcome of this competition is usually the extinction of one species. This is known as the Principle of Competitive Exclusion. Let the population densities (number per unit area) of an X-species be $x(t)$ and of a Y-species be $y(t)$. The growth of x and y is modelled by the coupled equations $\dot{x} = Ax - Bx^2 - \alpha xy$ and $\dot{y} = Cy - Dy^2 - \beta xy$ $(A, B, C, D, \alpha, \beta > 0)$. Take $A = 0.21827$ and $C = 0.06069$ in all cases. The other parameters (B, D, α, β) vary from case to case.

   A. Starting with $x(0) = 0.5, y(0) = 1.5$ and $B = D = 0, \alpha = 0.05289, \beta = 0.00459$, integrate $x(t)$ and $y(t)$ by Euler's method (choose $\Delta t \leq 0.0001$). Plot x and y along the vertical axis and t along the horizontal axis. In a separate graph plot x along the horizontal axis and y along the vertical axis. B. Now take $x(0) = y(0) = 0.5$ and $B = 0.017, D = 0.010$. Using the previous values of α and β, carry out the same exercise as above. Note the maximum values of both x and y in time t. C. For no competition take $\alpha = \beta = 0$. With the second set of initial conditions and the values of B and D, carry out the same exercise as earlier.

2. *Predator-prey dynamics*: In the interaction between a prey species X and a predator species Y, let the population densities (number per unit area) of the X-species (prey) be $x(t)$ and of the Y-species (predator) be $y(t)$. The growth of x and y is modelled by the coupled equations $\dot{x} = Ax - Bxy - \epsilon x$ and $\dot{y} = -Cy + Dxy - \epsilon y$. Take $A = 1.0, B = 0.01, C = 0.5, D = 0.005$. Without human interference in Nature (like fishing, poaching, deforestation, etc.) $\epsilon = 0$.

   A. Starting with $x(0) = 200, y(0) = 80$ and $\epsilon = 0$, integrate $x(t)$ and $y(t)$ by Euler's method (choose $\Delta t \leq 0.0001$). Plot x and y along the vertical axis and t along the horizontal axis. In a separate graph plot x along the horizontal axis and y along the vertical axis. Note the maximum value of y. B. When human interference occurs, choose $\epsilon = 0.1$. Repeat the same exercise as above, and compare the maximum values of y in both cases. C. Taking $x(0) = 200$ but $y(0) = 0$ and $\epsilon = 0$ (no predator and no human interference), integrate $x(t)$ and plot its logarithm against t. Comment on your result. D. Taking $y(0) = 80$ but $x(0) = 0$ and $\epsilon = 0$ (no prey and no human interference), integrate $y(t)$ and plot its logarithm against t. Comment.

- **Set 10 (01-04-2025): Modelling epidemics and endemic breakouts of infectious diseases**

  1. <u>*Epidemics*</u>: Consider a population size of N, through which an infection spreads. The population is divided into three classes — the infected class $x(t)$, the susceptible class $y(t)$, and the recovered class $z(t)$, so that $x(t) + y(t) + z(t) = N$ (constant). The coupled dynamics of these variables is given by $\dot{x} = Axy - Bx$, $\dot{y} = -Axy$, and $\dot{z} = Bx$. in which A is the infection rate and B is the removal rate ($A, B > 0$). At $t = 0$, $x(0) = x_0$ and $z(0) = 0$. Hence, $y(0) = y_0 = N - x_0$. The value of x_0 is small ($x_0 \ll N$). Apply this model to the following problem.

     The total number of students in a boarding school is 763. Initially a single student introduces an infectious disease in this population. Take $A = 2.18 \times 10^{-3}\,\mathrm{day}^{-1}$ and $B = 0.44\,\mathrm{day}^{-1}$.

     A. Integrate $x(t)$, $y(t)$ and $z(t)$ by Euler's method up to 25 days. Choose Δt carefully (maybe an hour). B. Plot x and y along the vertical axis (with and without a logarithmic scale) and t (in days) along the horizontal axis. Record the time when x reaches its maximum value. C. Plot z along the vertical axis (with and without a logarithmic scale) and t (in days) along the horizontal axis in a separate graph. D. Plot x along the *vertical* axis and y along the *horizontal* axis in a separate graph. Compare the plot with the threshold value $\rho = B/A$ and the reproduction number $R = Ay_0/B$. An epidemic breaks out when $R > 1$ or $y_0 > \rho$.

  2. <u>*Endemic diseases*</u>: Endemic diseases persist in a population and break out from time to time. In this case $N \equiv N(t)$, i.e. the total population size changes. If the per capita death rate is a and the per capita birth rate is b ($a, b > 0$), then the relevant coupled system of equations is given by $\dot{x} = Axy - Bx - ax$, $\dot{y} = bN - Axy - ay$, $\dot{z} = Bx - az$ and $\dot{N} = (b - a)N$.

     Consider the case of $a = b = 0.02\,\mathrm{year}^{-1}$ so that $\dot{N} = 0$, i.e. N is fixed. Take $A = 10^{-6}\,\mathrm{year}^{-1}$, $B = 0.333\,\mathrm{year}^{-1}$, $N = 10^6$, $x_0 = 10^5$ and $y_0 = 9 \times 10^5$.

     A. Integrate $x(t)$, $y(t)$ and $z(t)$ by Euler's method up to 150 years. Choose Δt carefully (maybe a day). B. Plot x along the vertical axis and t (in years) along the horizontal axis. Record the times when x reaches its peaks. These are the times when endemic breakouts occur. C. Plot y and z along the vertical axis and t (in years) along the horizontal axis in two separate graphs.

- **Set 11 (08-04-2025): Modelling stock price variations as a Bachelier-Wiener process**[4]

  1. This is a brief set of guidelines for modelling price variations of the stocks of companies listed in *NIFTY (National Stock Exchange, India)* as a Bachelier-Wiener Process. A paper (file named `IJMPC_stocks.pdf`) provides relevant information about the modelling work. The data for the modelling cover the period January, 1997 to April, 2019 (22 years), as registered in the *NIFTY* website. The data have been provided in five data files, pertaining to the six figures in the paper. The contents of the data files and their relation to the six figures are as follows:
     (a) In the file named `price.dat` the second column gives time and the third column gives the daily average price of the stock index. Refer to Fig. 1.
     (b) In the file named `fluct.dat` the second column gives time and the third column gives the daily percentage fluctuation of stock values. Refer to Fig. 2.
     (c) In the file named `gauss.dat` the second column gives the daily percentage fluctuation of stock values and the third column gives the frequency of the fluctuations. Refer to Fig. 3.
     (d) In the file named `wiene.dat` the first column gives time and the third column gives the monthly average of the *logarithm* of the stock values. Refer to Fig. 4.
     (e) In the file named `wiene.dat` the first column gives time and the fourth column gives the Wiener variance of the monthly average of the *logarithm* of the stock values. Refer to Fig. 5.
     (f) In the file named `trade.dat` the second column gives time and the third column gives the daily trade volume (number of daily transactions) of the stock index. Refer to Fig. 6.

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A. Clearly understand all the figures from their captions, the relevant equations and parameter values in Table I. B. Reproduce all the six plots in the paper. State the equations and parameter values for the plotting. C. State three salient conclusions drawn from the modelling exercise.

- **Set 12 (08/15-04-2025): The Solow model of economic growth (Robert Solow)**

1. Production output $Y(t)$ depends on capital $C(t)$ and labour $L(t)$. In the absence of technological innovations, labour grows according to Harrod's natural growth model $\dot{L} = nL$ $(0 < n < 1)$, whose integral solution is $L = L_0 e^{nt}$. Further, the growth rate of capital is $\dot{C} = sY$, in which s is a fraction of the production $(0 < s < 1)$. Defining $r = C/L$ and using the Cobb-Douglas function for $Y = F(C, L) = C^{\alpha} L^{1-\alpha}$ $(0 < \alpha < 1)$ give a first-order autonomous differential equation $\dot{r} = f(r) = sr^{\alpha} - nr$. Its integral solution is $r(t) = (s/n)^{1/(1-\alpha)} \left[1 + Ae^{-n(1-\alpha)t}\right]^{1/(1-\alpha)}$, in which A is the integration constant. For $r(0) = r_0$ at $t = 0$, one gets $A = (n/s)r_0^{1-\alpha} - 1$. Note that growth of $r(t)$, i.e. $\dot{r} > 0$, is ensured only when $A < 0$. The growth follows two types of dynamics. In the early stages $(t \longrightarrow 0)$ the growth follows a power law and on long time scales $(t \longrightarrow \infty)$ the growth approaches a terminal value. The transition from one trend to the other occurs when $t = t_{\text{trans}} = n^{-1}(1-\alpha)^{-1} \ln[-A/(1-\alpha)]$.

   A. Choose values of s and n within the ranges $0.1 < s < 0.2$ and $0.02 < n < 0.05$, respectively. State the values. Also take $\alpha = 0.5$. Using these values plot $\dot{r}$ versus r and show the two fixed points of r. Now define $y_1 = sr^{\alpha}$ and $y_2 = nr$. In a separate graph plot y_1, y_2 versus r and show the two intersection points of y_1 and y_2. Compare both the graphs and note their consistency with each other. B. Starting with the initial condition $r(0) = r_0 = 0$, numerically integrate $\dot{r} = f(r)$ by Euler's method for the same parameter values of s, n and α. Choose $\Delta t \ll n^{-1}$ (maybe $\Delta t = 0.001 n^{-1}$) and state its value. Plot the numerical values of $r(t)$ versus t. In the same graph also plot the analytical solution of $r(t)$ for $r_0 = 0$. Separately plot the numerical and analytical functions in a log-log graph. Note the value of $t = t_{\text{trans}}$ in both graphs and the behaviour of the functions before and after $t = t_{\text{trans}}$. C. Repeat the entire procedure of the previous step with the initial condition $r(0) = r_0 = 0.001$. Comment on the difference you may note in the respective results of the numerical integration when $r_0 = 0$ and $r_0 \neq 0$.