Set-4: Constrained growth beyond the logistic model

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This report discusses two mathematical models: the Gompertz equation for tumor growth and the Allee effect for population dynamics. Rescaling the Gompertz equation results in a parameter-free differential equation, which is plotted for three initial values. The Allee effect is demonstrated by plotting $\dot{x}$ versus x and integrating using Euler's method with two initial values. The limiting values of x when $t \to \infty$ are observed, and the Allee effect is discussed.

I. EQUATIONS

Gompertz equation models for tumor growth

The Gompertz equation models tumour growth as,

$$\dot{x} = -axln(bx) \tag{1}$$

rescaling the factors as, $X = x/b^{-1}$ and $T = at$

$$\frac{d(\frac{x}{b^{-1}})}{d(at)} = -\frac{x}{b^{-1}} \ln\left(\frac{x}{b^{-1}}\right) \tag{2}$$

$$\dot{X} \equiv \frac{dX}{dT} = -X \ln(X) \tag{3}$$

The analytical solution is given as

$$X = e^{(\log(X_0)\cdot e^{-t})} \tag{4}$$

The Euler solution is given as,

$$X_{n+1} = X_n - X_n \cdot \log(X_n) \cdot dt \tag{5}$$

The Relative error calculation is as follows,

$$\text{error} = \frac{x_{\text{euler}} - x_{\text{analytical}}}{x_{\text{analytical}}} \tag{6}$$

Allee Effect model for population growth

$$\dot{x} = x[r - a(x-b)^2] \tag{7}$$

where $a, b, r > 0$,with values of $a = 1$, $r = 1$, and $b = 2$. For $\dot{x} = 0$, we get:

$$x = 0, \quad x = b - \frac{\sqrt{r}}{a}, \quad \text{and} \quad x = b + \frac{\sqrt{r}}{a}.$$

The Euler solution is given as:

$$x_{n+1} = x_n + \big(x_n \cdot \big(r - a\cdot(x_n - b)^2\big)\big)\cdot dt. \tag{8}$$

II. GRAPHS

A. Gompertz equation model for tumor growth

1. *Plot of $\dot{X}$ v/s X*

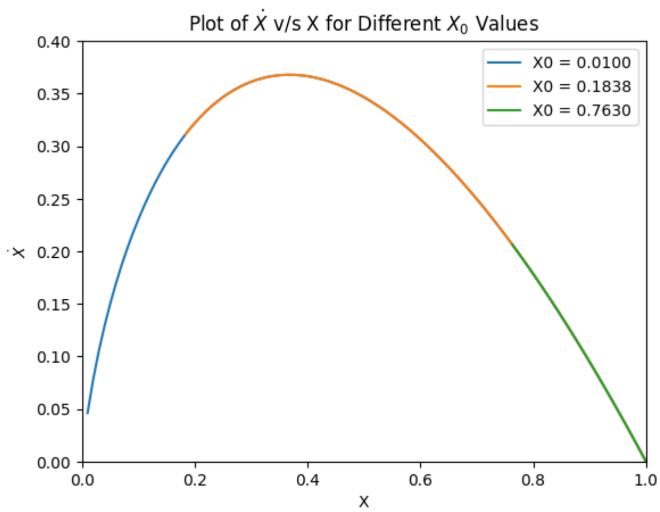

FIG. 1: The graph shows $\dot{X}$ v/s X for three different initial values, one of which is close to zero and another one is greater than e^{-1}. The third one have an intermediate value.

The three initial values are: $X = 0.01$ (very close to 0),$x = 0.183$ (intermediate value), $X = 0.763$ (greater than e^{-1}). Step size is $\Delta X = 0.01005$.

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2. *Euler and Analytical Solution for Gompertz Equation*

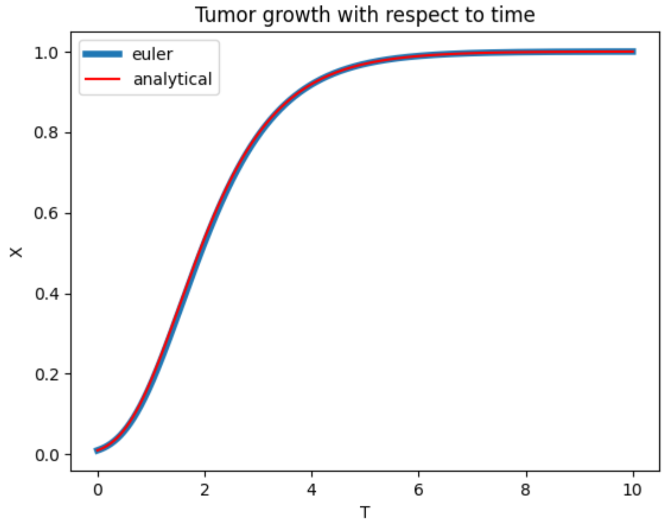

FIG. 2: The graph shows the tumor growth with respect to time both by Euler method and analytical solution.

Initial value: $X_0 = 0.01$ and $\Delta t = 0.01$.

3. *Relative error between Analytical and Euler solution*

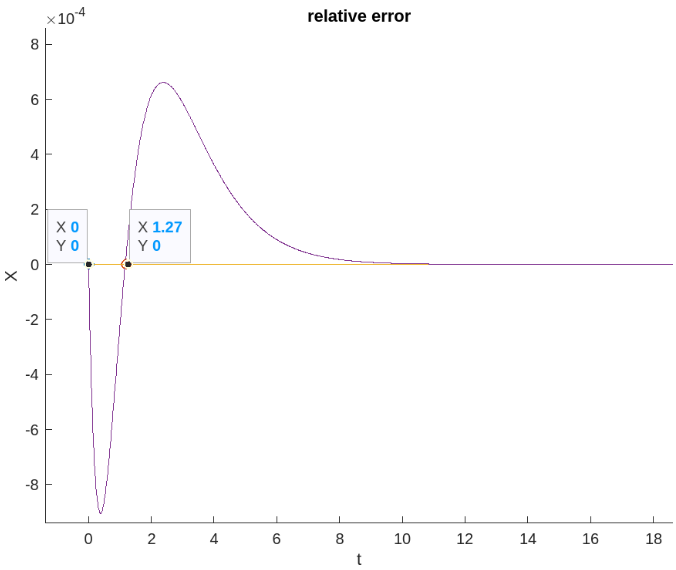

FIG. 3: This graph shows relative error between analytical and numerical solution.

The graph has zero value at $t = 0$ and $t = 1.27$. Step size, $\Delta t = 0.01$ unit

B. Allee Effect model for population growth

1. *Plot of $\dot{X}$ v/s X*

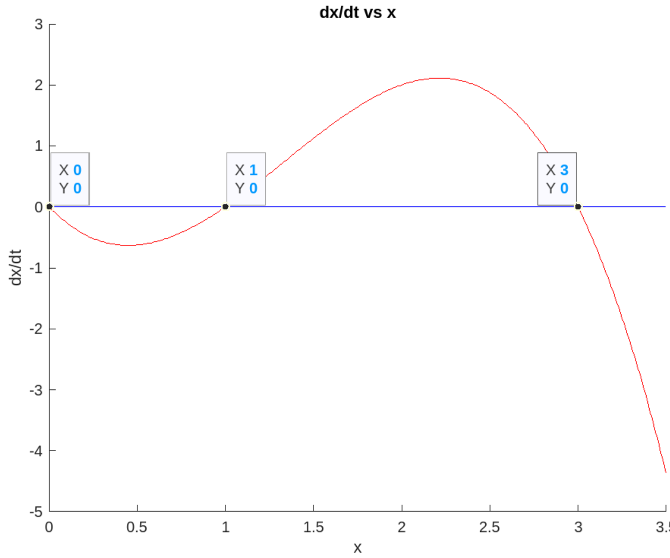

FIG. 4: The graph shows $\dot{X}$ v/s X for an Allee effect.

Here, $a = 1$,$b = 2$,$r = 1$. Graph value becomes zero at $t = 0$,$t = 1$ and $t = 3$ units. Step size, $\Delta t = 0.01$ unit.

2. *Plot the numerical solutions for both initial values*

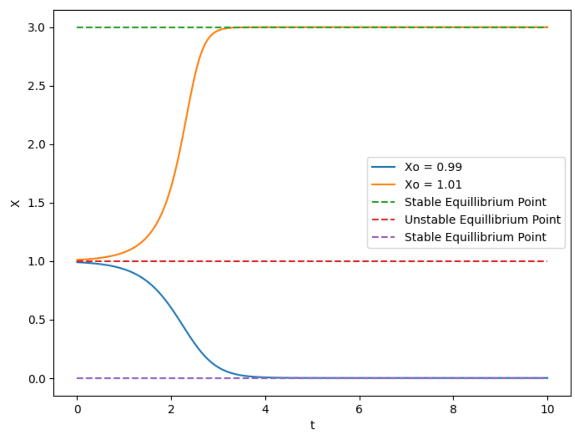

FIG. 5: The two initial values taken are 0.99 and 1.01. $\Delta x = 0.01$ unit. Limiting values of x when x_0 is 0.99 and 1.01 are 0 and 3 respectively