Set-12: The Solow Theory of Economic Growth

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This assignment explores the mathematical principles underlying Robert Solow's theory of economic growth, emphasizing the application of nonlinear dynamics to economic modeling. The study focuses on analyzing the growth dynamics of the capital-to-labor ratio r(t) under the Solow growth model, incorporating Harrod's natural growth model for labor and a Cobb-Douglas production function. Using analytical and numerical techniques, key growth trends and their transitions are examined.

I. EQUATIONS :

The Solow growth model explains long-term economic growth through capital accumulation, labor growth, and productivity. This lab models the capital-to-labor ratio $r(t)$, focusing on interactions between capital and labor without technological progress.

- Output (Y) depends on capital (C) and labor (L) via a Cobb-Douglas function.
- Labor grows exponentially as per Harrod's natural growth model.
- A fraction (s) of output is reinvested to grow capital.

Production Function

The production output is given by the Cobb-Douglas production function:

$$Y(t) = F(C, L) = C^{\alpha} L^{1-\alpha}, \quad 0 < \alpha < 1$$

Labor Growth (Harrod's Natural Growth Model)

Labor grows exponentially:

$$\dot{L} = nL, \quad L(t) = L_0 e^{nt}, \quad 0 < n < 1$$

Capital Growth

The rate of capital accumulation is proportional to the savings from production:

$$\dot{C} = sY, \quad 0 < s < 1$$

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Capital-to-Labor Ratio Dynamics

By defining $r = \frac{C}{L}$, the dynamics of r are described by:

$$\dot{r} = sr^{\alpha} - nr$$

Integral Solution for r(t)

The analytical solution to the above differential equation is:

$$r(t) = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}} \left[1 + Ae^{-n(1-\alpha)t}\right]^{\frac{1}{1-\alpha}}$$

where A is an integration constant given by:

$$A = \left(\frac{n}{s}\right) r_0^{1-\alpha} - 1$$

Transition Time

The transition from early power-law growth to stabilization occurs at:

$$t_{\text{trans}} = \frac{1}{n(1-\alpha)} \ln\left(\frac{-A}{1-\alpha}\right)$$

II. RESULTS

The following parameter values were used throughout the assignment:

- $s = 0.15$ (Fraction of production for capital)
- $n = 0.03$ (Natural growth rate of labor)
- $\alpha = 0.5$ (Cobb-Douglas parameter)

A. Question 1

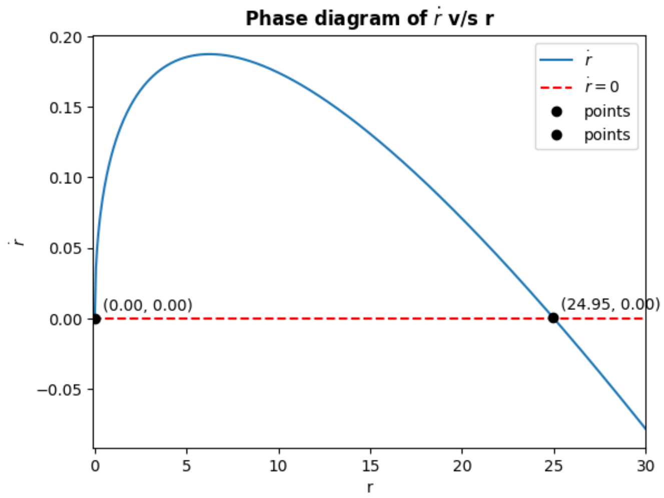

FIG. 1: Phase diagram of $\dot{r}$ versus r

We plot $\dot{r}$ versus r to analyze the phase diagram, identify fixed points, and verify intersections by graphing

$$y_1 = sr^{\alpha} \quad \text{and} \quad y_2 = nr.$$

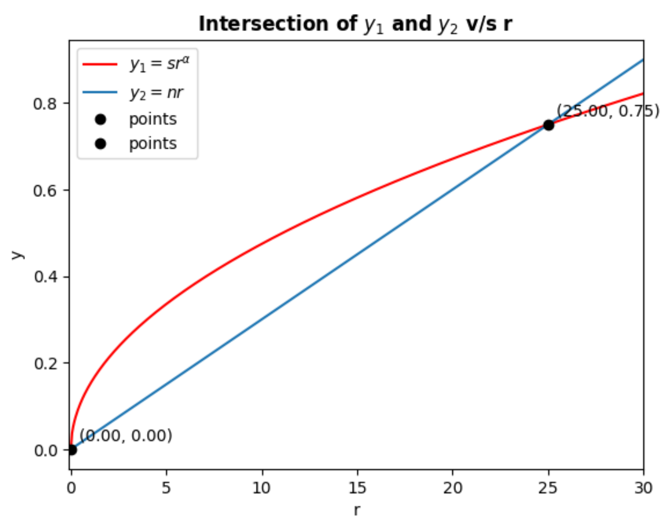

FIG. 2: Graphs of y_1 and y_2 vs r

The phase diagram reveals two fixed points, which correspond to the intersections of y_1 and y_2, aligning with theoretical predictions.

B. Question 2

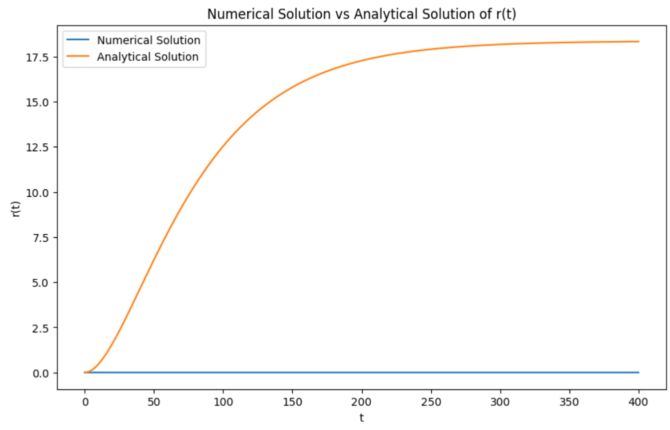

FIG. 3: Numerical and Analytical Solutions with $r_0 = 0$ and $\Delta t = 0.0333$.The value of $\mathrm{t_{trans}} = 46.2098\,\mathrm{s}$ (normal scale)

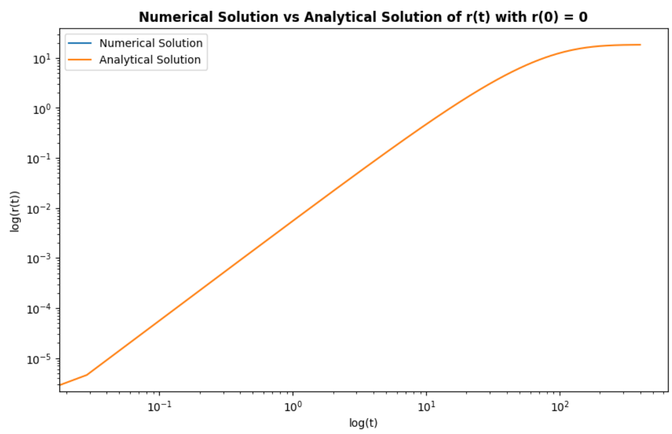

FIG. 4: Comparison of Numerical and Analytical Solutions for r(t) with $r_0 = 0$ and $\Delta t = 0.0333$. Transition time t_{trans} $= 46.2098$ s (log-log scale)

C. Question 3

In this step, we repeat the procedure with the initial condition $r(0) = r_0 = 0.001$, and explore the behavior for other small values of r_0 much smaller than the transition value

$$r_{\mathrm{trans}} = \left(\frac{\alpha s}{n}\right)^{\frac{1}{1-\alpha}}.$$

We also note that the admissible range for r_0 is

$$0 < r_0 < \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}},$$

as this ensures a valid solution within the model constraint.

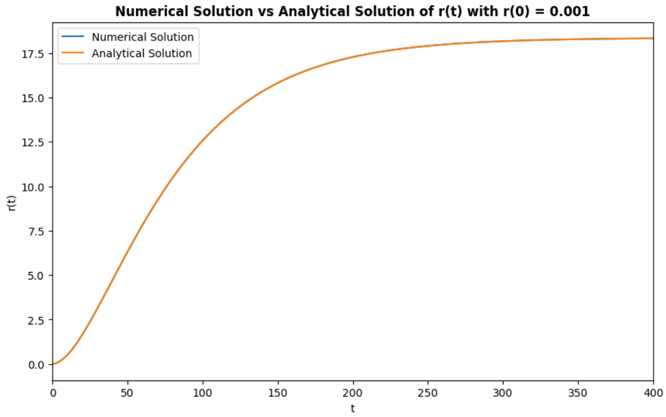

FIG. 5: Comparison of Numerical and Analytical Solutions for r(t) with $r_0 = 0.001$ and $\Delta t = 0.0333$. Transition time $t_{\mathrm{trans}} = 45.787 \quad \mathrm{s} \quad$ (normal scale)

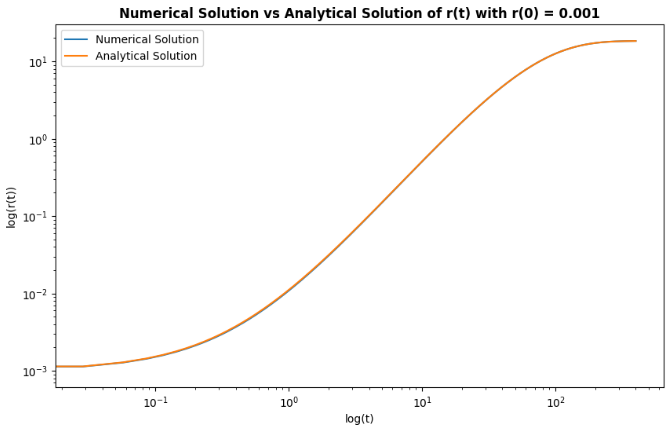

FIG. 6: Comparison of Numerical and Analytical Solutions for r(t) with $r_0 = 0.001$ and $\Delta t = 0.0333$. Transition time $t_{\mathrm{trans}} = 45.787 \quad \mathrm{s} \quad$ (log-log scale)

III. SALIENT FEATURES:

1. For $r(0) = 0$, the numerical solution remains constant at 0, unable to capture the initial growth dynamics.

2. For $r(0) > 0$, the numerical solution exhibits growth, closely matching the analytical solution.

3. Both numerical and analytical solutions converge to the same long-term behavior, demonstrating the stability of the system over time.