- FINDING EIGENVECTORS
  - > EIGENVALUES & NORMALIZE
  - > FROM ROTATION
- SIMULTANEOUSLY DIAGONALIZABLE MATRICES

- POPULATION DYNAMICS
- EXP.

LAST TIME symmetric matrix : $A = RDR^T$

D: DIAGONAL

$$\det A = \det D = \prod_i^N \lambda_i$$

$$\operatorname{tr} A = \operatorname{tr} D = \sum_i^N \lambda_i$$

det : PAIN TO CALCULATE IN GENERAL, EASY FOR DIAGONAL (& SYMMETRIC) MATRICES

HOW TO FIND EIGENVALUES & EIGENVECTORS

1. EIGENVALUES from CHARACTERISTIC EQN

   CHARACTERISTIC: synonym to "EIGENVALUE"

$$\det(A - \lambda \mathbb{1}) = \det\left(R\,(D - \lambda\mathbb{1})\,R^{-1}\right)$$

$$= \det\underbrace{(D - \lambda\mathbb{1})}_{\begin{pmatrix} (\lambda_1 - \lambda) & & \\ & (\lambda_2 - \lambda) & \\ & & \ddots \end{pmatrix}}$$

$$= \underbrace{\prod_i^N (\lambda_i - \lambda)}$$

N^{TH} ORDER POLYNOMIAL IN λ WHOSE ROOTS ARE THE N EIGENVALS

$\Rightarrow$ solutions to $\boxed{\det(A - \lambda\mathbb{1}) = 0}$ are the λ_i

eg. $A = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

$$A - \lambda \mathbb{1} = \begin{pmatrix} 5-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix}$$

$$\det(A - \lambda \mathbb{1}) = \underbrace{(5-\lambda)(1-\lambda) - 4}_{} = 0$$

$$\lambda^2 - 6\lambda + 1 = 0$$

$$\lambda_\pm = \frac{6 \pm \sqrt{36+4}}{2}$$

2. WHAT ABOUT EIGENVECTORS?

HOW TO THINK ABOUT THESE:

$$D \, \underline{e}_{(i)} = \lambda_i \, \underline{e}_{(i)}$$

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \qquad \text{eg } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_{(1)}$$

ROTATE

$$\underbrace{RDR^T}_{\substack{A \\ \text{SYMMETRIC}}} \underbrace{R\underline{e}_{(i)}}_{\equiv \underline{\xi}_{(i)}} = \lambda_i \underbrace{R\underline{e}_{(i)}}_{\equiv \underline{\xi}_{(i)}}$$

$$\begin{pmatrix} R^1{}_1 & R^1{}_2 & R^1{}_3 \\ R^2{}_1 & R^2{}_2 & R^2{}_3 \\ R^3{}_1 & R^3{}_2 & R^3{}_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R^1{}_1 \\ R^2{}_1 \\ R^3{}_1 \end{pmatrix}$$ 1st COLUMN of R

$$— \text{ " } — \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R^2{}_1 \\ R^2{}_2 \\ R^2{}_3 \end{pmatrix}$$ 2ND COLUMN of R

so: $\boxed{A\,\xi_{(i)} = \lambda_i\,\xi_{(i)}}$

the i^{th} COLUMN of R
is the i^{th} EIGENVECTOR.

$$R = \begin{pmatrix} | & | & \\ \xi_{(1)} & \xi_{(2)} & \cdots \\ | & | & \end{pmatrix}$$

SUPPOSE WE HAVE the EIGENVALUES, λ_i
How to find the EIGENVECTORS?

$$(\underbrace{A - \lambda_i \mathbb{1}}_{})\,\xi_{(i)} = 0$$

N×N MATRIX,
no UNKNOWNS

N component
vector / ALL UNKNOWN

⇒ N EQUATIONS for N UNKNOWNS.

eg. $A = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$ $\quad \lambda_\pm = 3 \pm \sqrt{8}$

$$(\underbrace{A - \lambda_+ \mathbb{1}}_{})\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-\sqrt{8} & -2 \\ -2 & -2-\sqrt{8} \end{pmatrix}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-\sqrt{8})\,\xi^1 - 2\xi^2 = 0 \longrightarrow \xi^2 = \tfrac{1}{2}(2-\sqrt{8})\,\xi^1$$

$$-2\,\xi^1 - (2+\sqrt{8})\,\xi^2 = 0 \longrightarrow \underbrace{\xi^1 = -\tfrac{1}{2}(2+\sqrt{8})\,\xi^2}_{}$$

$$\xi^2 = -\tfrac{1}{4}(4-8)\,\xi^2 \Rightarrow \xi^2 = \xi^2$$

no new info from
2ND eqn!

is this weird? 2 eqns, 2 unknowns
... BUT THE EQNS ARE REDUNDANT!

why? if $A\xi_{(i)} = \lambda_i \xi_{(i)}$

then $A\xi'_{(i)} = \lambda_i \xi'_{(i)}$ for $\boxed{\xi'_{(i)} = a\,\xi_{(i)}}$

free to RESCALE EIGENVECTORS

BUT, there is A "CORRECT" NORMALIZATION

$$\langle \xi_{(i)}, \xi_{(i)} \rangle = 1$$

ORTHONORMAL BASIS

then the EIGENVECTORS COME TOGETHER AS A ROTATION MATRIX

$$R = \begin{pmatrix} | & | & | & \\ \xi_{(1)} & \xi_{(2)} & \xi_{(3)} & \cdots \\ | & | & | & \end{pmatrix}$$

so we can write

$$\xi = N\begin{pmatrix} 1 \\ \frac{1}{2}(2-\sqrt{8}) \end{pmatrix}$$

$$\xi^2 = \underbrace{N^2\left(1^2 + \tfrac{1}{4}(2-\sqrt{8})^2\right) = 1}_{\text{GIVES } N \text{ to NORMALIZE}}$$

then REPEAT for 2ND EIGENVALUE

eg. 3x3, just the first few steps

$$A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 6 & 7 \\ 3 & 7 & 5 \end{pmatrix} \rightarrow \det(A - \lambda \mathbb{1}) = 0$$

$$\uparrow$$

$$\begin{pmatrix} (2-\lambda) & -1 & 3 \\ -1 & (6-\lambda) & 7 \\ 3 & 7 & (5-\lambda) \end{pmatrix}$$

$$\det(A-\lambda\mathbb{1}) = (2-\lambda)\begin{vmatrix} 6-\lambda & 7 \\ 7 & 5-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 7 \\ 3 & (5-\lambda) \end{vmatrix} + 3\begin{vmatrix} -1 & (6-\lambda) \\ 3 & 7 \end{vmatrix}$$

$$= (2-\lambda)\big((6-\lambda)(5-\lambda)-49\big) + \big(-(5-\lambda)-21\big) + 3\big(-7-3(6-\lambda)\big)$$

= SOME 3RD O POLYNOMIAL

3 ROOTS ARE $\boxed{\lambda_i}$, EIGENVALUES

EIGENVECTORS:

$$\left(A - \lambda_i \mathbb{1}\right)\underbrace{\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}}_{\xi_{(i)}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (2-\lambda_i)\xi^1 - \xi^2 + 3\xi^3 \\ -\xi^1 + (6-\lambda)\xi^2 + 7\xi^3 \\ 3\xi^1 + 7\xi^2 + (5-\lambda)\xi^3 \end{pmatrix}$$

3 eqns, 1 will be redundant

solve for ξ^1, ξ^2, $\xi^3 = 1$

$\boxed{\text{then NORMALIZE} \quad \xi_{(i)}^2 = 1}$

the REAL WAY: USE A COMPUTER, or look UP THE EIGENVAL/VECs (eg for function spaces)

IF YOU KNOW THE NORMALIZED EIGENVECTORS, THEN YOU KNOW THE ROTATION BTWN STANDARD BASIS & EIGENBASIS.

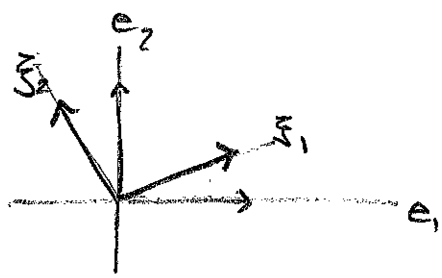
$$R = \begin{pmatrix} | & | & \\ \xi_1 & \xi_2 & \cdots \\ | & | & \end{pmatrix}$$

SO GIVEN $\underline{v} = e_{(i)} v^i = \underbrace{e_{(j)} (R^T)^j{}_i}_{\xi_{(i)}} \underbrace{R^i{}_k v^k}_{\tilde{v}^i}$

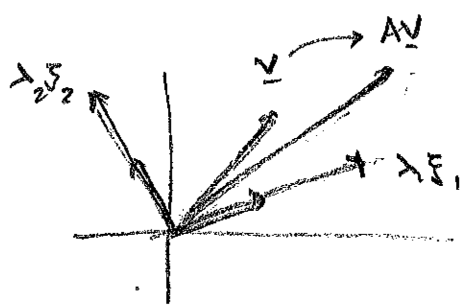
components of $\underline{v}$ in eigenbasis

$$A\underline{v} = A\left(\tilde{v}^i \xi_{(i)}\right)$$
$$= \tilde{v}^i A \xi_{(i)}$$
$$= \sum_i \tilde{v}^i \lambda_i \xi_{(i)} \quad \leftarrow \text{easy}$$

PICTURE



ACTION OF A:
RESCALE ALONG ξ_1 DIR BY λ_1
RESCALE ALONG ξ_2 DIR BY λ_2



SOME TRICKS

1. 2x2 USE det & tr
2. Exponentiation

(1) $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

$$\det A = \lambda_1 \lambda_2 = ad - b^2$$
$$\mathrm{tr}\, A = \lambda_1 + \lambda_2 = a + d$$

$$\lambda_1 (a+d-\lambda_1) = ad - b^2$$

quadratic eq for λ_1

then $\lambda_2 = a+d-\lambda_1$

(2) if YOU KNOW EIGENVALUES & EIGENVEC,

$$A = RDR^T$$
$$A^2 = RDR^TRDR^T = RD^2R^T$$
$$A^N = R\,D^N R^T$$

$$D^N = \begin{pmatrix} \lambda_1^N & & \\ & \lambda_2^N & \\ & & \ddots \end{pmatrix}$$

if $B = A^2 = RD^2R^T$

then $\sqrt{B} = A = RDR^T$

SQRT of MATRIX

eg
$$e^A = \mathbb{1} + A + \tfrac{1}{2}A^2 + \cdots$$
$$= R\left(\mathbb{1} + D + \tfrac{1}{2}D^2 + \cdots\right)R^T$$
$$= R\,\underbrace{e^D}_{\begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots \end{pmatrix}}\,R^T$$

POPULATION DYNAMICS EXAMPLE

$$\frac{df}{dt} = af \Rightarrow \frac{df}{f} = a\,dt$$

$$\log f = at + c$$

$$\boxed{f = Ce^{at}}$$

$$\boxed{\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= by + cx\end{aligned}} \Rightarrow \frac{d}{dt}\underbrace{\underline{v}}_{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\underline{v}$$

<u>solution</u>: $\underline{v}(t) = \underbrace{e^{\begin{pmatrix} a & b \\ b & c \end{pmatrix} t}}_{\mathbb{1} + \begin{pmatrix} a & b \\ b & c \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 t + \dots} \underline{v}_0$

chk $\left.\frac{d\underline{v}(t)}{dt}\right|_{t=0} = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \underline{v}_0$ ✓

now we know:

$$e^{\begin{pmatrix} a & b \\ b & c \end{pmatrix} t} = R\, e^{\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} t} R^T$$

$$= R \begin{pmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{pmatrix} R^T$$

some lin comb of x & y scales like $e^{\lambda_1 t}$
other — $e^{\lambda_2 t}$

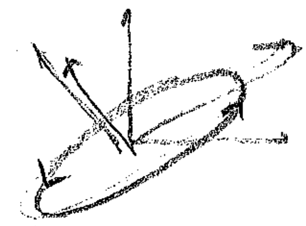
↳ eg $\lambda_1 > \lambda_2 \Rightarrow \lambda_1$ combo dominates over long times

more interesting behaviour
when we have a <u>not-symmetric</u>
matrix → EIGENVALUES MAY BE <u>COMPLEX</u>

& EIGENVEC NOT ORTHOG

exp growth / decay → rotations

eg



eg. Lotka-Volterra eqn s

RABBITS BREED

$$\frac{dr}{dt} = Ar - Brw$$

WOLVES EAT RABBITS, REDUCE RABBIT POP.

$$\frac{dw}{dt} = -Cw + \underline{Drw}$$

WOLVES THAT EAT WELL BREED

WOLVES COMPETE

ah. not linear :)

nevermind.