① VECTOR SPACES REVIEW

focus on $N \times N$ matrices

"① ROTATIONS" : UNITARY MATRIX : $U^\dagger U = \mathbb{1}$

HERMITIAN : $A^\dagger = A$ ← "nice" transformations

→ DIAGONALIZABLE $A = U D U^\dagger$

→ EIGENVALUES ARE REAL

now to FOURIER TRANSFORM → ∞ interval ⟷ continuum
→ ① functions

FOURIER SERIES

$$f(x) = a_0 + \sum_{n=1} a_n \cos(k_n x) + \sum_{n=1} b_n \sin(k_n x)$$

UP to BOUNDARY COND, WHICH KILLS SOME TERMS, FIXES k_n

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

SO WE COULD WRITE eg $\frac{n\pi}{L}$ for $-L < x < L$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}$$

LOWER LIMIT

for REAL functions, c_n & c_{-n} are related

↳ eg sine series: $c_{-n} = -c_n$ and c_n are pure imaginary

BUT MORE GENERALLY, c_n COULD ALL BE INDEPENDENT ⇝ ① function

I may have ∓ wrong

eg. for $-L < x < L$

$$c_n = \int_{-L}^{L} f(x) \underbrace{\sqrt{\tfrac{1}{2L}}\, e^{-i\frac{n\pi}{L}x}}_{\text{BASIS FUNCTION } |n\rangle} dx$$

$\langle n|f\rangle$ ← $\langle n|$

check: $\langle n|m\rangle = \int_{-L}^{L} dx \left(\frac{1}{2L}\right) e^{-i\frac{(n-m)\pi}{L}x}$

$$= \frac{1}{(2L)} \frac{1}{\frac{-i(n-m)\pi}{L}} \underbrace{\left[e^{-i\frac{(n-m)\pi}{L}x} \right]_{-L}^{L}}$$

$$\underbrace{e^{-i(n-m)\pi}} - e^{+i(n-m)\pi}$$

BUT $\begin{cases} e^{\pm i\pi} = e^{\pm i3\pi} = e^{\pm i5\pi} = -1 \\ e^{\pm 2i\pi} = e^{\pm 4i\pi} = 1 \end{cases}$

So no matter what n & $m \in \mathbb{Z}$ are, this vanishes ✓

... unless $n=m$:

$$\langle n|n\rangle = \int_{-L}^{L} dx \left(\frac{1}{2L}\right) e^{0x} = 1 \quad ✓$$

eg Felder? 8.9.5 p. 469

$$f(x) = x^2+3 \qquad L=\pi \quad -\pi < x < \pi$$

$$f(x) = \sum_n \boxed{c_n \sqrt{\tfrac{1}{2\pi}}\, e^{inx}} = \sum_n \bar{c}_n e^{inx}$$

$|n\rangle$ convenient to def $\bar{c}_n$

$$\bar{c}_n = c_n \sqrt{\tfrac{1}{2\pi}}$$

$$c_n = \langle n|f\rangle = \int_{-\pi}^{\pi} dx \sqrt{\tfrac{1}{2\pi}}\, e^{-inx} f(x)$$

$$\bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$\Rightarrow \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x^2+3)\, e^{-inx}\, dx$$

$$\underbrace{\left(3\int_{-\pi}^{\pi} e^{-inx}\, dx + \right)}_{=0 \text{ by } *} \quad \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx = uv\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} v\, du$$

$u = x^2 \qquad du = 2x\,dx$

$dv = e^{-inx}\,dx \qquad v = \frac{i}{n}e^{-inx}$

$$= \frac{i\pi^2}{n}\underbrace{\left(e^{-in\pi} - e^{in\pi}\right)}_{=0 \ \forall n} - \frac{2i}{n}\int_{-\pi}^{\pi} x\, e^{-inx}$$

$u = x \qquad du = dx$

$dv = \text{SAME} \qquad v = \text{SAME}$

$$= -\frac{2i}{n}\cdot\left[\frac{ix}{n}e^{-inx}\right]_{-\pi}^{\pi} + \left(\frac{2i}{n}\right)^2 \underbrace{\int_{-\pi}^{\pi} e^{-inx}\, dx}_{\frac{i}{n}\left(e^{in\pi} - e^{-in\pi}\right) = 0} \quad *$$

$$= \frac{2}{n^2}\left(\pi e^{-in\pi} + \pi e^{in\pi}\right)$$

$$= \frac{2\pi}{n^2}\left(e^{-in\pi} + e^{in\pi}\right)$$

$$= \frac{4\pi}{n^2}(-)^n \quad \Rightarrow \boxed{\bar{c}_n = \frac{2(-)^n}{n^2}}$$

BUT : careful w/ $n=0$ (const term)

$$\bar{c}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2+3)\,e^{-i0x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{3}x^3+3x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{2}{3}\pi^3+6\pi\right]$$

$$= \boxed{\frac{1}{3}\pi^2+3}$$

LESSON : CAREFUL w/ $n=0$

$$\text{So}: \boxed{f(x) = x^2+3 = \left(\frac{\pi^2}{3}+3\right)+\sum_{n\neq 0}\frac{2(-)^n}{n^2}e^{inx}}$$

obs: $|k\rangle$ is an eigenstate of $\hat{p} = -i\hbar \frac{d}{dx}$

$$\hat{p}|k\rangle = \underbrace{k}_{\text{"momentum"}}|k\rangle$$

$$\boxed{\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{+ik\cdot x} \\ \tilde{f}(k) &= \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} \end{aligned}}$$

FOURIER & INVERSE FOURIER

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eg HEISENBERG UNCERTAINTY PRINCIPLE

$$\psi(x) = \underset{\sim}{\mathcal{N}}\, e^{-(x/a)^2} \quad \leftarrow \quad \psi(x) = \langle x|\psi\rangle$$

POSITION BASIS

$$\mathcal{N}^2 = \sqrt{\frac{2}{\pi}}\,\frac{1}{a}$$

LET A be an operator

if A hermitian, then this is clear

mean value / expectation: $\langle A\rangle = \langle\psi|A|\psi\rangle$

$$\langle x\rangle = \langle\psi|x|\psi\rangle = \int_{-\infty}^{\infty} x\,|\psi(x)|^2\,dx = 0$$

BY SYM

variance: $(\Delta A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle$

$$\langle\Delta x\rangle^2 = \langle\psi|(x - \langle x\rangle)^2|\psi\rangle$$

fact of GAUSSIAN

$$= \int dx\, x^2\, |\psi(x)|^2\, dx = \boxed{\frac{a^2}{4}}$$

<u>take $L \to \infty$ Limit</u> · fourier series $\to$ fourier TRANSFORM

$$f(x) = \sum_{n=-\infty}^{\infty} \underbrace{c_n \sqrt{\tfrac{1}{2L}}}_{\bar{c}_n} \underbrace{e^{+i\frac{n\pi x}{L}}}_{|n\rangle} \qquad c_n = \langle n|f\rangle = \int_{-L}^{L} dx \sqrt{\tfrac{1}{2L}}\, e^{\frac{in\pi x}{L}} f(x)$$

$-L \le x \le L$

$$\bar{c}_n = \frac{1}{2L}\int_{-L}^{L} dx\, e^{\frac{in\pi x}{L}} f(x)$$

SUPERPOSITION of WAVES
AS $L \to \infty$, SUM of ARB SHORT WAVELENGTHS

$\}$ btw: UV physics can be subtle

short LENGTH SCALES!

$$k_n = \frac{n\pi}{L} \qquad \Delta k = \pi/L$$

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L}\int_{-L}^{L} dx'\, e^{-ik_n x'} f(x')\right) e^{+ik_n x}$$

DUMMY VARIABLE

$$= \sum_n \underbrace{\left(\frac{\pi}{L}\frac{1}{2\pi}\right)}_{=\Delta k \frac{1}{2\pi}} \underbrace{\left(\int_{-L}^{L} dx'\, e^{-ik_n x'} f(x')\right)}_{\equiv \tilde{f}(k_n)} e^{+ik_n x}$$

ORIGIN of DREADED 2π convention.

no longer k_n

$$= \underbrace{\int dk}_{\sum_n \Delta k} \; \frac{1}{2\pi} \underbrace{\tilde{f}(k_n)}_{(\sqrt{2L}\langle f|k\rangle)} \underbrace{e^{+ikx}}_{|k\rangle}$$

$k_n \to k$

normaliz gets weird.
formally, the ∞ plane waves are not normalized

FOURIER TRANSFORM TO MOMENTUM SPACE

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \int dx \; \underbrace{\langle p|x\rangle}_{\frac{1}{2\pi\hbar} e^{-ipx/\hbar}} \underbrace{\langle x|\psi\rangle}_{\psi(x)}$$

⌈check norms⌋

$$= \tilde{N} \; e^{-(ap)^2/(2\hbar)^2}$$

$$\tilde{N}^2 = \sqrt{\frac{2}{\pi}} \; \frac{a}{\hbar}$$

$$(\Delta p)^2 = \langle\psi|(p - \cancel{\langle p\rangle})^2|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dp \; p^2 |\tilde{\psi}(p)|^2 = \frac{\hbar^2}{a^2}$$

$$\Rightarrow (\Delta x)^2 (\Delta p)^2 = \frac{a^2}{4} \frac{\hbar^2}{a^2} = \frac{\hbar^2}{4}$$

$$\Rightarrow \boxed{(\Delta x)(\Delta p) = \frac{\hbar}{2}}$$ for GAUSSIAN.

DIRAC

$$f(x) = \int dk \; \tilde{f}(k)\, e^{ikx} = \int dk \left(\int dy\, f(y)\, e^{-iky}\right) e^{ikx}$$

$dk/2\pi$

$$= \int dy \underbrace{\int dk \; e^{ik(x-y)}}_{\equiv\, \delta(x-y)} f(y)$$