Short HW 10: **No cloning theorem**

Course: Physics 017, *Linear Algebra for Physics* (S22)
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Due by: **Thursday**, June 2

Note that this short assignment is due by class on Thursday. You have only *two days* to do it. This should be quick, I recommend doing it right after class on Tuesday.

This problem draws from the lovely text by Mertens and Moore, *The Nature of Computation* (chapter 15.3). We use the basis where $|\uparrow\rangle = |0\rangle$ and $|\downarrow\rangle = |1\rangle$.

1 No Cloning

It turns out that it is not possible to simply copy a qubit. This poses a set of challenges for doing classical computing on a quantum computer.

Classical computing is built on logic gates. One of the logic gates is the controlled NOT gate, or CNOT. This gate takes in two bits of data, $|a\rangle \otimes |b\rangle$ and spits out the following:

$$\text{CNOT}|a\rangle \otimes |b\rangle = \begin{cases} |a\rangle \otimes |b\rangle & \text{if } |a\rangle = |0\rangle \\ |a\rangle \otimes |0\rangle & \text{if } |a\rangle \otimes |b\rangle = |1\rangle \otimes |1\rangle \\ |a\rangle \otimes |1\rangle & \text{if } |a\rangle \otimes |b\rangle = |1\rangle \otimes |0\rangle \end{cases} \tag{1.1}$$

In other words, $\text{CNOT}|a\rangle \otimes |b\rangle$ will flip the value of $|b\rangle$ if $|a\rangle = |1\rangle$.

1.1 CNOT as a quantum cloning gate

Suppose you have a qubit $|\psi\rangle$ that is in some S_z eigenstate: it is either $|0\rangle$ or $|1\rangle$. If you happen to have a $|0\rangle$ qubit, then show that you can *clone* ψ using the CNOT gate:

$$\text{CNOT}|\psi\rangle \otimes |0\rangle = |\psi\rangle \otimes |\psi\rangle \ . \tag{1.2}$$

Hint: simply try the two cases for $|\psi\rangle$ and check what comes out.

1.2 CNOT cannot clone superpositions

Let us instead suppose that $|\psi\rangle$ is a superposition of S_z eigenstates, e.g.

$$|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle \ . \tag{1.3}$$

Calculate $|\psi\rangle \otimes |\psi\rangle$ and argue that there is no unitary *linear* operator that can possibly give this. We then conclude that $\text{CNOT}|\psi\rangle \otimes |0\rangle$ does not clone superpositions of S_z eigenstates.

Partial answer: the proposed cloned state is

$$|\psi\rangle \otimes |\psi\rangle = \psi_0^2|0\rangle \otimes |0\rangle + \psi_0\psi_1|0\rangle \otimes |1\rangle + \psi_1\psi_0|1\rangle \otimes |0\rangle + \psi_1\psi_1|1\rangle \otimes |1\rangle \ . \tag{1.4}$$

The right-hand side is written with respect to an eigenbasis of the tensor product. Compare this to the initial state, $|\psi\rangle \otimes |0\rangle$. Is there any linear operator that can take something linear in ψ_1 and ψ_2 and then turn it into something that is not linear in ψ_1 and ψ_2?