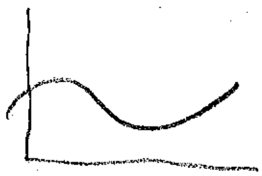
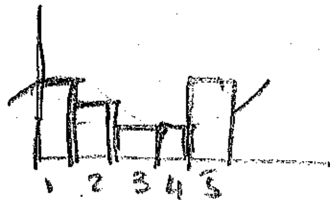
DISCRETIZED function space

$$f(x) \longrightarrow f^i \quad \text{for some fixed } \underline{\text{samples}} \quad i = 1, \dots, N$$





$$= \begin{pmatrix} 2 \\ 1.5 \\ .8 \\ .8 \\ 2 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

DERIVATIVE

$$(Df)^i_+ = \frac{f^{i+1} - f^i}{\Delta x} \qquad \text{BUT } (Df)^N = \frac{f^{N+1} - f^N}{\Delta x} \; ??$$

$$(Df)^i_- = \frac{f^i - f^{i-1}}{\Delta x} \qquad \text{BUT } (Df)^1 = \frac{f^1 - f^0}{\Delta x} \; ??$$

BOUNDARY CONDITIONS: PART OF DEFINING the MATRIX

btw, we can start calling these (LINEAR) OPERATORS

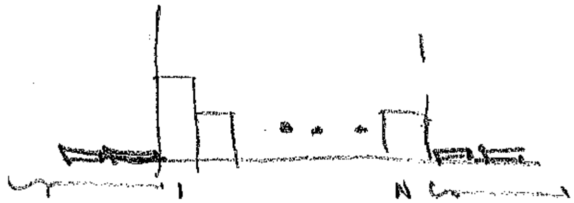
MIDDLE of DERIVATIVE

$$D_+ \text{ (FWD)} = \frac{1}{\Delta x}\begin{pmatrix} \ddots & & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 & \ddots \end{pmatrix}$$

$$D_- \text{ (BKWD)} = \frac{1}{\Delta x}\begin{pmatrix} \ddots & & & & \\ & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 & \\ & & & & & \ddots \end{pmatrix}$$

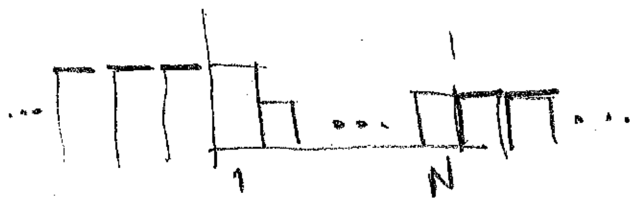
- not symmetric → not nec $\mathbb{R}$ eigenvalues

DIRICHLET BC: fix ends to zero



$f^{i<1} = 0$ $\qquad$ $f^{i>N} = 0$

NEUMANN BC: fix ends to have zero change
(ZERO DERIVATIVE)



$f^{i<1} = f^1$ $\qquad$ $f^{i>N} = f^N$

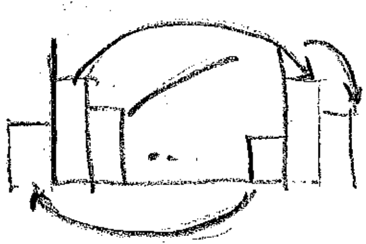
REMARK: why not $f^{i<1} = 5$?

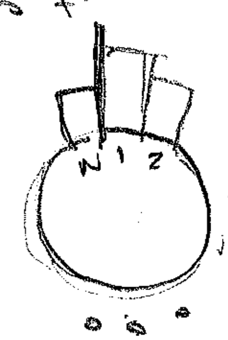
↳ not linear.

If $\underline{f}$ & g are vectors/discr functions w/ this BC, then $(f+g)$ has $\underline{(f+g)^{i<1} = 10}$

not in func. space.

PERIODIC: connect f^N to f^1
f^0 to f^N





D_+, DIRICH $= \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}$ $\quad f^{N+1} = 0$
$\uparrow$
FWD

$(D_+f)^N = \frac{f^{N+1} - f^N}{\Delta x}$

D_-, DIRICH $= \frac{1}{\Delta x}\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{pmatrix}$ $\quad f^0 = 0$

$(D_-f)^1 = \frac{f^1 - f^0}{\Delta x}$

D_+, NEU $= \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & 0 \end{pmatrix}$ $\quad (D_+f)^N = \frac{f^{N+1} - f^N}{\Delta x} = 0$

D_-, NEU $= \frac{1}{\Delta x}\begin{pmatrix} 0 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{pmatrix}$ $\quad (D_-f)^1 = \frac{f^1 - f^0}{\Delta x} = 0$

D_+, PER $= \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ 1 & & & -1 \end{pmatrix}$ $\quad (D_+f)^N = \frac{f^{N+1} - f^N}{\Delta x}$ $\quad f^1$

D_-, PER $= \frac{1}{\Delta x}\begin{pmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{pmatrix}$ $\quad (D_-f)^1 = \frac{f^1 - f^0}{\Delta x}$ $\quad f^N$

notion of FORWARD / BACKWARD derivative

↳ DERIVATIVE HAS A DIRECTION (ORIENTATION)

eg $\underline{\nabla}$ points in a direction

most of the time, the laws of physics do not have a preferred direction, so you do not see 1st DERIVATIVES IN SPACE

eg HEAT EQ: $\frac{\partial}{\partial t} u = K \nabla^2 u$

DIMENSIONS: $\left(\frac{L^2}{T}\right)$

BETTER: WAVE EQ: $\left(\frac{\partial}{\partial t}\right)^2 u = K \nabla^2 u$

DIM: $\left(\frac{L^2}{T^2}\right) = (vel)^2$

$$\underbrace{(\partial_t^2 - c^2 \partial_x^2)}_{} u$$

INVARIANT WRT SPECIAL RELATIVITY

$$\partial_\mu = (\partial_t, c\partial_x)$$

$$\partial^\mu = \eta^{\mu\nu} \partial_\nu = (\partial_t, -c\partial_x)$$

$$\partial^2 = \Box = \partial_t^2 - c^2 \partial_x^2$$

second derivative is symmetric

$$(D^2 f)^i = \frac{f^{i+1} - 2f^i + f^{i-1}}{\Delta x^2} = \frac{1}{\Delta x}\left(\underbrace{\frac{f^{i+1} - f^i}{\Delta x}}_{} - \frac{f^i - f^{i-1}}{\Delta x} \right)$$

$$\frac{1}{\Delta x}\left[(D_+ f)^i - (D_- f)^i \right]$$

SAME DEAL w/ BC:

DEFINE

$$(D^2 f)^1 = \frac{f^2 - 2f^1 - f^0}{\Delta x^2}$$

HIGHER DERIVATIVES

$$(D^3_+ f)^i = \frac{1}{\Delta x}\left((D^2 f)^{i+1} - (D^2 f)^i\right)$$

$$= \frac{1}{\Delta x}\left[\frac{f^{i+2} - 2f^{i+1} - f^i}{\Delta x^2} - \frac{f^{i+1} - 2f^i - f^{i-1}}{\Delta x^2}\right]$$

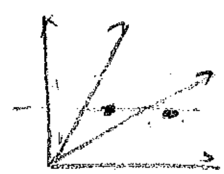
$$= \frac{f^{i+2} - 3f^{i+1} - 3f^i + f^{i-1}}{\Delta x^3}$$

$$= \begin{array}{c c}
 & \quad\quad i \\
i & \begin{pmatrix} \ddots & -3 & -3 & & & \\ & 1 & -3 & -3 & 1 & \\ & & 1 & -3 & -3 & 1 \\ & & & & \ddots & \end{pmatrix}
\end{array}$$

HIGHER DERIVATIVES POPULATE ELEMENTS FURTHER AWAY FROM DIAGONAL

$\hookrightarrow$ BECOMES MORE NONLOCAL $\leftarrow$ ① TAYLOR
② NONLOCALITY

physics reason for LOCALITY



SIMULTANEITY

① this nonlocality is exactly a taylor exp.

$$f(x) = \underbrace{f(x_0)}_{@\,x_0} + \underbrace{f'(x_0)(x-x_0)}_{\text{neighbor of } x_0} + \underbrace{\tfrac{1}{2}f''(x_0)(x-x_0)^2}_{\text{later neighbors}} + \cdots$$

APPROX very good for $x - x_0$ small

MORE TERMS needed (POSSIBLY ∞) to account for LARGER $|x-x_0|$

(DISCRETE)
for $\mathbb{R}$ function space,

<u>metric</u>: $g_{ij} = \delta_{ij}$

$$f_i = \delta_{ij} f^j = f^i$$

$$(f_1, f_2, \dots) = (f^1, f^2, \dots)$$

<u>INNER PRODUCT</u>: $\langle f, g\rangle = g_{ij} f^i g^j \underbrace{(\Delta x)}_{\text{usually set to 1}}$

$$= \sum_i f^i g^i \, \Delta x \approx \int_{x_1}^{x_N} dx\, f(x)\, g(x)$$

continuous space limit!

So another way of thinking of BRA / ROW VECTOR IS AN OBJECT BORN TO "CONTRACT" w/ KET IN A LINEAR WAY

$\langle f|$ is a "functional": takes function, spits out # in a LINEAR way

f_i or $f_i \approx$

$$\langle f| = \int_{x_1}^{x_N} dx\, f(x) \,\_\_\_$$

insert argument (KET) here

nb: obviously linear

$$\langle f| \big(\alpha|g\rangle + \beta|h\rangle\big) = \alpha\langle f|g\rangle + \beta\langle f|h\rangle$$

$$\int dx\, f(x)\big(\alpha g(x) + \beta h(x)\big) = \alpha \int dx\, f(x) g(x) + \beta\int dx\, f(x) h(x)$$

OPERATOR/MATRIX

EIGENSTUFF: $\mathcal{O}|f_i\rangle = \lambda_i |f_i\rangle$

for SYMMETRIC $\mathcal{O}$, then λ_i is REAL, $|f_i\rangle$ <u>ORTHOGONAL</u>

ORTHOGONAL? $\langle f_i | f_j\rangle = \delta_{ij} = \int_{x_1}^{x_N} dx\, f_i(x)\, f_j(x)$

example: $\mathcal{O} = \partial_x^2$ SPACE : $x \in [0,1]$

W/ BOUNDARY CONDITION : $f(0) = f(1) = 0$

EIGENFUNCTION: $f_n = \underset{\text{NORM}}{\mathcal{N}} \cos \underbrace{(n\pi x)}_{\text{for B.C.}}$

$\partial_x \cos(n\pi x) = -n\pi \sin(n\pi x)$
∂_x^2 ——"—— $= \underbrace{-(n\pi)^2} \cos(n\pi x)$

$$\boxed{\lambda_n = -n^2\pi^2} \in \mathbb{R}$$
EIGENVALS

$\langle f_n, f_n \rangle = 1$ $u = n\pi x$ $dx = \frac{du}{n\pi}$

$= \int_0^1 dx \cos^2(n\pi x) \mathcal{N}^2$

$= \int_0^{n\pi} \frac{du}{n\pi} \mathcal{N}^2 \cos^2 u$

$= n \int_0^{\pi} \frac{du}{n\pi} \mathcal{N}^2 \cos^2 u$

$= \frac{\mathcal{N}^2}{\pi} \underbrace{\int_0^{\pi} du \cos^2 u}$

CHECK: $\int_0^{\pi} du \cos^2 u = \int_0^{\pi} du \sin^2 u$
$= \frac{1}{2} \int_0^{\pi} du \underbrace{(c^2 + s^2)}_{1} = \frac{\pi}{2}$

$= \frac{\mathcal{N}^2}{2} \Rightarrow \mathcal{N} = \sqrt{2}$

BTW: BETTER : $x \mapsto x/L$ then $\mathcal{N} = \sqrt{2/L}$

s.t. dimless in sine.

and eg : $\int_0^1 dx \cos(3\pi x) \cos(2\pi x) = 0$ w/o work

$\mathbb{C}$ VECTOR SPACE

RECALL

$$\boxed{\text{MATRICES}} \xrightarrow{\text{nice}} \underbrace{(N\times N)}_{\text{transf of space to itself}}$$

MATRICES $\xrightarrow{\text{subs}}$ $N\times M$

$(N\times N) \xrightarrow{\text{subs}}$ non invertible

$(N\times N) \to$ ROTATIONS PRESERVE $\langle\cdot,\cdot\rangle$, $R^TR = \mathbb{1}$

$(N\times N) \to$ SYMMETRIC: $A^T = A$
DIAGONAL IN SOME BASIS,
$\mathbb{R}$-eigenvals,
orthon. eigenvec

complex: $\mathbb{C}$ elements, $\mathbb{C}$ coefficients

$\boxed{\text{MATRICES}} \to N\times N$

MATRICES $\to N\times M$

$N\times N \to$ non-inv.

$N\times N \to \boxed{\text{UNITARY}}$ $U^\dagger U = \mathbb{1}$

$N\times N \to \boxed{\text{HERMITIAN}}$ $A^\dagger = A$
$\hookrightarrow$ DIAG in some basis
$\mathbb{R}$ eigenvals
orthon. eigenvec.

$\mathbb{R}$: $\langle \underline{w}, \underline{v}\rangle = \underline{w}^T \underline{v}$

$\mathbb{C}$: $\langle \underline{w}, \underline{v}\rangle = \underline{w}^\dagger \underline{v}$

$\quad \hookrightarrow \underline{w}^\dagger = \underline{w}^{*T}$

ROTATIONS: ($\mathbb{R}$) $\underbrace{\langle R\underline{w}, R\underline{v}\rangle}_{} = \langle \underline{w}, \underline{v}\rangle$

$$\underline{w}^T \underbrace{R^T R}_{} \underline{v} = \underline{w}^T\underline{v}$$

$$\boxed{R^TR = \mathbb{1}}$$

UNITARY TRANSF. ($\mathbb{C}$) $\langle U\underline{w}, U\underline{v}\rangle = \langle \underline{w}, \underline{v}\rangle$

$$\underline{w}^\dagger \underbrace{U^\dagger U}_{} \underline{v} = \underline{w}^\dagger \underline{v}$$

$$U^\dagger U = \mathbb{1}$$

$\mathbb{R}$: A IS SYMMETRIC: $\langle A\underline{w}, \underline{v}\rangle = \langle \underline{w}, A\underline{v}\rangle$

$$\underline{w}^T A^T \underline{v} = \underline{w}^T A \underline{v}$$

defines transpose

$\boxed{A^T = A}$ "self ADJOINT"

$\mathbb{C}$: A IS HERMITIAN: $\langle A\underline{w}, \underline{v}\rangle = \langle \underline{w}, A\underline{v}\rangle$

$$\underline{w}^T A^\dagger \underline{v} = \underline{w}^T A \underline{v}$$

def of ADJOINT, $\dagger$
$\langle A\underline{v}, \underline{w}\rangle \equiv \langle \underline{v}, A^\dagger \underline{w}\rangle$

$\boxed{A^\dagger = A}$

$\mathbb{R}$: DIAGONAL MATRIX: $D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}$ $\lambda_i \in \mathbb{R}$
(IS SYMMETRIC)

$\mathbb{C}$: DIAG HERMITIAN MATRIX: $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\lambda_i \in \mathbb{R}$

BC. $\lambda_i^* = \lambda_i$
BY HERMITICITY.

EIGENVALUES of HERMITIAN MATRICES ARE $\mathbb{R}$.

IN QM: OBSERVABLES ARE EIGENVALUES of HERMITIAN MATRICES

EIGENVEC ORTHOGONALITY

$$\underbrace{\begin{pmatrix} - \xi_1 - \\ - \xi_2 - \\ \vdots \end{pmatrix}\begin{pmatrix} | & | & \\ \xi_1 & \xi_2 & \cdots \\ | & | & \end{pmatrix}}_{R^T R} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$R^T R = \left(\begin{array}{c|c} \underbrace{\xi_1 \cdot \xi_1}_{=1} & \underbrace{\xi_1 \cdot \xi_2}_{=0} \\ & \xi_2 \cdot \xi_2 \quad \ddots \end{array}\right)$$

SO ORTHONORMAL!

<u>SIMILARLY</u> $U^\dagger U = \mathbb{1}$

$$\begin{pmatrix} - \xi_1^* - \\ - \xi_2^* - \\ \vdots \end{pmatrix}\begin{pmatrix} | & | & \\ \xi_1 & \xi_2 & \cdots \\ | & | & \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} \underbrace{\xi_1^\dagger \xi_1}_{=1} & \underbrace{\xi_1^\dagger \xi_2}_{=0} \\ & \xi_2^\dagger \xi_2 \quad \ddots \end{array}\right)}_{} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle \cdots \\ & \langle \xi_2, \xi_2 \rangle \quad \ddots \end{array}\right)$$