finish: SPRING THY

where we were: $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ displacements

$$(*) \quad \ddot{\underline{x}} = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\underline{x}$$

① $\lambda_1 = -\frac{k}{m}$ $\qquad \lambda_2 = -3\frac{k}{m}$

how? $\det(A - \lambda \mathbb{1}) = 0$

② $\xi_{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \xi_{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

how? $(A - \lambda_i \mathbb{1})\xi_{(i)} = 0$

AND $\langle \xi_{(i)}, \xi_{(i)} \rangle = 1$

③ $\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \underbrace{\varphi_1(t)\,\xi_{(1)} + \varphi_2(t)\,\xi_{(2)}}$

$x_1(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ combine into a vector in the eigenbasis

$$= \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}_E \quad \text{EIGENBASIS}$$

then $(*)$ in EIGENBASIS is

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in std. basis

$$\ddot{\underline{x}} = A\,\underline{x} = \underbrace{\left(-\frac{k}{m}\right)}_{-\omega_1^2}\varphi_1(t)\,\xi_1 + \underbrace{\left(-3\frac{k}{m}\right)}_{-\omega_2^2}\varphi_2\,\xi_2$$

ANGULAR FREQ

$\frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$$\begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{pmatrix}_E = \begin{pmatrix} -\omega_1^2 & \\ & -\omega_2^2 \end{pmatrix}\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}_E$$

in EIGENBASIS

$$④ \Rightarrow \boxed{\ddot{\varphi}_i = -\omega_i^2\,\varphi_i}$$

$$|\underline{x}\rangle = \varphi_1(t)\left(\langle e_1|\xi_1\rangle |e_1\rangle + \langle e_2|\xi_1\rangle |e_2\rangle\right)$$
$$+\varphi_2(t)\left(\langle e_1|\xi_2\rangle |e_1\rangle + \langle e_2|\xi_2\rangle |e_2\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\varphi_1(t)+\varphi_2(t)\right)|e_1\rangle + \frac{1}{\sqrt{2}}\left(\varphi_1(t)-\varphi_2(t)\right)|e_2\rangle$$

$$\varphi_i = \underline{\varphi_i(0)}\cos(\omega_i t)$$

⑥ What are these? WANT W/RT STANDARD BASIS

$$@\ t=0: \quad x = \begin{pmatrix}\varphi_1(0)\\ \varphi_2(0)\end{pmatrix}_E = \begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}$$ ic in STD BASIS

$$\varphi_1(0) = \langle \xi_1 | \underline{x}(0)\rangle \quad \leftarrow \text{eg } (1\ \ 0)_E\begin{pmatrix}\varphi_1(0)\\ \varphi_2(0)\end{pmatrix}_E$$
$$= \frac{1}{\sqrt{2}}(1\ \ 1)\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left(x_1(0)+x_2(0)\right)$$

$$\varphi_2(0) = \frac{1}{\sqrt{2}}(1\ \ -1)\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left(x_1(0)-x_2(0)\right)$$

⑦
$$|x\rangle = \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}\left(x_1(0)+x_2(0)\right)\cos(\omega_1 t)$$
$$+\frac{1}{\sqrt{2}}\left(x_1(0)-x_2(0)\right)\cos(\omega_2 t)\Big)|e_1\rangle$$
$$+\frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}\left(x_1(0)+x_2(0)\right)\cos(\omega_1 t)$$
$$-\frac{1}{\sqrt{2}}\left(x_1(0)-x_2(0)\right)\cos(\omega_2 t)\Big)|e_2\rangle$$

$$= \frac{1}{2}\left((c_1+c_2)x_1(0)+(c_1-c_2)x_2(0)\right)\begin{pmatrix}1\\0\end{pmatrix}$$
$$+\frac{1}{2}\left((c_1-c_2)x_1(0)+(c_1+c_2)x_2(0)\right)\begin{pmatrix}0\\1\end{pmatrix}$$

IMPOSE I.C.: $\dot{\underline{x}}(0) = 0$

$\Rightarrow$ solution is $\ddot{\varphi}_i = \varphi_i(0) \underbrace{\cos(\omega_i t)}$

@ $t=0$, $\frac{d}{dt}\varphi_i \sim \sin 0 = 0$

$$\underline{x} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_E = \begin{pmatrix} \varphi_1(0) \cos(\omega_1 t) \\ \varphi_2(0) \cos(\omega_2 t) \end{pmatrix}_E$$

(5) BUT NOW WE WANT THIS BACK IN THE STANDARD BASIS!

$$\underline{x} = \underbrace{\varphi_1(0) \cos(\omega_1 t)}_{\varphi_1(t)} \underbrace{\xi_1}_{|\xi_1\rangle} + \underbrace{\varphi_2(0) \cos(\omega_2 t)}_{\varphi_2(t)} \underbrace{\xi_2}_{|\xi_2\rangle}$$

CONVERT BACK BY MULTIPLYING BY 1.

$$\mathbb{1} = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$$

we are using BRA-KET NOTATION b/c it makes the most sense here

$$A = A^i{}_j \; |e_i\rangle\langle e_j|$$

$$\mathbb{1} = \delta^i{}_j \; |e_i\rangle\langle e_j|$$

$$\mathbb{1}\,|x\rangle = \varphi_1(t) \left(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| \right) |\xi_1\rangle + \varphi_2(t) \left(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| \right) |\xi_2\rangle$$

$$\langle e_1 | \xi_1 \rangle = (1 \; 0) \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tfrac{1}{\sqrt{2}}$$

$$\langle e_2 | \xi_1 \rangle = \tfrac{1}{\sqrt{2}}$$

$$\langle e_1 | \xi_2 \rangle = (0 \; 1) \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \tfrac{1}{\sqrt{2}}$$

$$\langle e_2 | \xi_2 \rangle = -\tfrac{1}{\sqrt{2}}$$

(8) CAN BEAUTIFY THIS

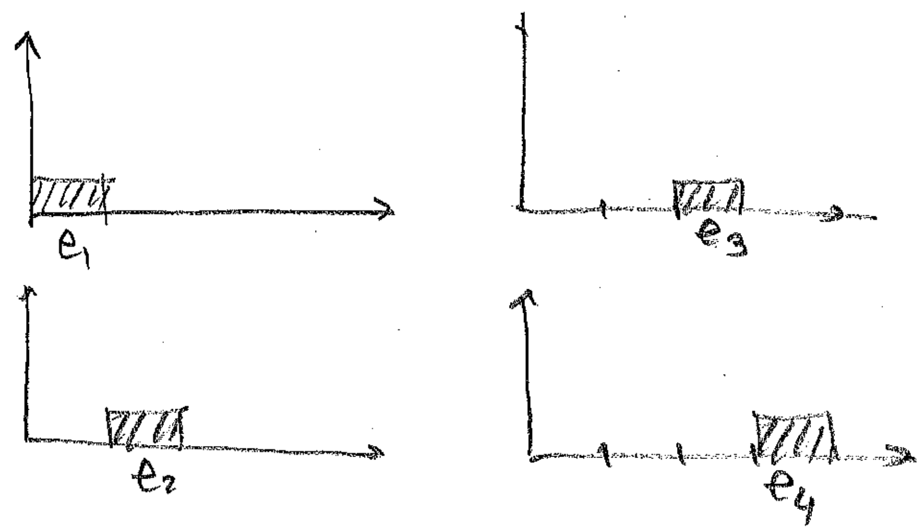
init conditions

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = B(t) \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$B(t) = \frac{1}{2} \begin{pmatrix} \cos(\omega_1 t) + \cos(\omega_2 t) & \cos(\omega_1 t) - \cos(\omega_2 t) \\ \cos(\omega_1 t) - \cos(\omega_2 t) & \cos(\omega_1 t) + \cos(\omega_2 t) \end{pmatrix}$$

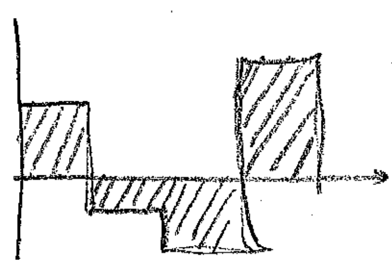
SOMETHING A BIT DIFFERENT (but actually very similar)
linear algebra to study "function space"

eg <u>HISTOGRAM SPACE.</u>



ANY 4-BIN HISTOGRAM IS A LIN COMB of THESE 4 BASIS VECTORS:

$$|f\rangle = \sum v^i |e_i\rangle$$

eg $v^i = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 3 \end{pmatrix} \longleftrightarrow$ 

"OBVIOUSLY" this is a crude approx for A FUNCTION.
CAN GENERALIZE TO LARGE # OF "BINS"

$$|f\rangle = \sum_{i=1}^{N} f^i |e_i\rangle$$

WE DON'T HAVE "MATRICES" ACTING ON FUNCTIONS.
↳ we have derivatives ← (why?)
PHYSICS Q, NOT MATH Q.

DERIVATIVE, DISCRETE form:

$$\frac{d}{dx} f(x_i) \equiv \langle e_i | D f_i \rangle = \frac{f^{i+1} - f^i}{\Delta x}$$ ← PICK SOME Δx eg $\Delta x = 1$

$$D|f\rangle = \frac{1}{\Delta x}\begin{pmatrix} \vdots \\ f^{i+1} - f^i \\ \vdots \end{pmatrix} \leftarrow i^{\text{th}} \text{ COMPONENT.}$$

$$|f\rangle = \begin{pmatrix} f^1 \\ f^2 \\ \vdots \end{pmatrix}$$

(btw, this is very familiar for PROGRAMMERS)

SO THE DERIVATIVE <u>MATRIX</u> IS

$$D = \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} & & i & i+1 & & \end{array} \\ \begin{pmatrix} \ddots & & & & \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & \\ & & & & \ddots \end{pmatrix} \end{array} \qquad \begin{pmatrix} \vdots \\ f^i \\ f^{i+1} \\ \vdots \end{pmatrix}$$

Q: what happens @ N? what is N^{th} comp of $D|f\rangle$?

$$\langle e_N | D | f \rangle \stackrel{?}{=} \frac{f^{N+1} - f^N}{\Delta x} \quad ??$$

hmm... maybe we could have defined not the FORWARD derivative, but instead the BACKWARD derivative...

$$\langle e_i | D | f \rangle = \frac{f^i - f^{i-1}}{\Delta x} \qquad D = \begin{array}{c} \\ i-1 \\ i \\ i+1 \end{array}\begin{array}{c} \begin{array}{cccc} i-1 & i & i+1 & \end{array} \\ \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{pmatrix} \end{array}$$

But then what is $\langle e_0 | D | f \rangle$? f^0 ?!

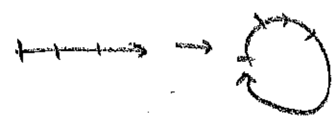
$\boxed{\text{LESSON}}$ BOUNDARY CONDITIONS ARE PART OF THE DEFINITION OF FUNCTION SPACE

DIRICHLET BC: $f^0 = f^{N+1} = 0$

NEUMANN BC: $f^{N+1} - f^N = 0 \Rightarrow f^{N+1} = f^N$

$f^1 - f^0 = 0 \Rightarrow f^0 = f^1$

PERIODIC BC: $f^{N+1} = f^1$
$f^0 = f^N$



eg $$\underset{\text{FORWARD PERIODIC}}{D} = \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \\ 1 & & & -1 \end{pmatrix} ; \quad D_{\substack{\text{BWD}\\\text{PER}}} = \begin{pmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & & \ddots & \\ & & -1 & 1 \end{pmatrix}$$

PERIODIC

eg $$D_{\substack{\text{FWD}\\\text{DIRICHLET}}} = \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

$f^1 - f^0 = f^1 - f^1 = 0$

eg $$D_{\substack{\text{BKWD}\\\text{NEUMAN}}} = \begin{pmatrix} 0 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & & \ddots \end{pmatrix}$$

<u>REMARKS</u>: IMPOSSIBLE TO DEF <u>"SYMMETRIC"</u> 1^{st} DERIVATIVE

BUT we like sym matrices...

All nonzero elem clustered along diagonal & adjacent $\longrightarrow$ LOCALITY

SECOND DERIVATIVE: can be defined symmetrically

$$\langle i | D_2 | f \rangle = \underbrace{\frac{1}{\Delta x^2}}_{\text{can set to } 1} \underbrace{\left(f^{i+1} - 2f^i + f^{i-1} \right)}_{\underbrace{(f^{i+1} - f^i) - (f^i - f^{i-1})}_{D_{FWD} D_{BKWD}}}$$

UP to BOUNDARY, this is sym.

$$D_2 \sim \begin{pmatrix} \ddots & & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & & \ddots \end{pmatrix}$$

APPLIC: start to solve linear eqns

$$D_2 |f\rangle = |g\rangle$$

$$\rightsquigarrow |f\rangle = \underbrace{(D_2)^{-1}}_{\text{GREEN's function}} |g\rangle$$