LINEAR ALGEBRA for PHYSICS

eg QM

→ TODAY REVIEW of BASICS + BIG PICTURE

linear, eg a function is linear when

$$f(\alpha \underline{v}) = \alpha f(\underline{v})$$
$$f(\underline{v}+\underline{w}) = f(\underline{v}) + f(\underline{w})$$

$\Big\}$ $f(\alpha \underline{v} + \beta \underline{w}) = \alpha f(\underline{v}) + \beta f(\underline{w})$

eg. consider functions of a real variable x

is $\sin(x)$ linear?

x^2 ?

e^x ?

end up w/ $\boxed{f(x) = ax}$

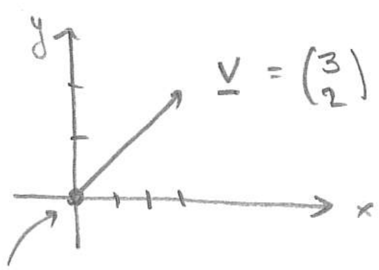
nb: CALCULUS IS OFTEN ABOUT "LINEAR APPROXIMATIONS"...

IN LINEAR ALGEBRA WE WILL GENERALIZE THIS TO ABSTRACT ARGUMENTS.

VECTORS

why? IN MATH + PHYS WE WILL NEED TO WORK W/ MORE GENERAL OBJECTS THAN ℝ numbers ... eg the QUANTUM STATE of a system

BASIC PICTURE:

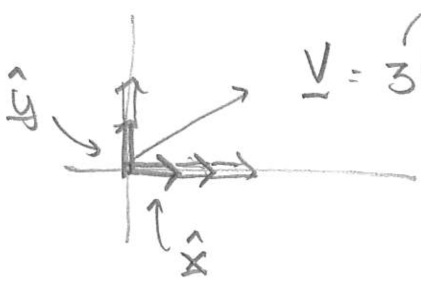


$\underline{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

\# ticks in x-DIR

y-DIR

"ticks" ⟷ unit vectors



$\underline{V} = 3\hat{\underline{x}} + 2\hat{\underline{y}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (components)

WE WILL WRITE $A\underline{v}$ ⟵ of $a x$

A ← MATRIX

$\underline{v}$ ← vector variable

eg $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

a ← COEFFICIENT ($\mathbb{R}$ #)

x ← VARIABLE ($\mathbb{R}$ #)

$\boxed{\text{RETURNS A \#}}$

in our case, a 2x2 matrix, eg

$$A = \begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}$$

We will explain the odd upper & lower indices

Just think of a letter w/ all of its indices as a variable

eg $\underline{V} = \begin{pmatrix} x \\ y \end{pmatrix}$ w/ $x = v^1$, $y = v^2$

$A = \begin{pmatrix} q & r \\ s & t \end{pmatrix}$ w/ $r = a^1{}_2$, etc.

why 2x2? A takes a 2-component vector & spits out a 2-component vector.
(cf: a takes a $\mathbb{R}$ # & spits out $\mathbb{R}$ #)

CHECK: is 2x2 matrix multiplication on 2 component vectors LINEAR?

$$f(\underline{v}) \equiv A\underline{v}$$

$$\left.\begin{aligned} f(\alpha\underline{v}) &\stackrel{?}{=} \alpha A\underline{v} \\ f(\underline{v}+\underline{w}) &\stackrel{?}{=} A\underline{v} + A\underline{w} \end{aligned}\right\}$$ check this! explicitly w/ #s if needed

↑ REQUIRES: $\alpha\underline{v}$ AND $\underline{v}+\underline{w}$ ARE VECTORS

INDEX NOTATION

mathematicians tease us about this.

notation is a simplification of an idea, not an idea itself! WE'RE INVENTING A LANGUAGE, not yet "doing math"

RULE: WRITE THE COMPONENTS of A VECTOR WITH UPPER INDICES.

eg a vector $\underline{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ has $\begin{cases} v^1 = -1 \\ v^2 = 3 \end{cases}$

(or $\vec{v}$) $= \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

note: most beginner texts use lower indices.

WE WILL NOT DO THIS IN OUR CLASS FOR A GOOD REASON (that you will see soon)

we are combining some "basic" stuff with more advanced stuff inspired by relativity.

it is easy enough to translate from your favorite textbook ... BUT DO BE CAREFUL!

in this class, vectors have UPPER INDICES BECAUSE A DIFFERENT OBJECT WILL HAVE LOWER INDICES.

[again: this "RULE" is a convention for WRITING THINGS – we're not "doing anything yet!]

<u>RULE</u>: THE COMPONENTS of A MATRIX HAVE THE FIRST INDEX (ROW #) UPPER & SECOND INDEX (COL #) LOWER

$$\text{eg } A = \begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}$$

<u>so far</u>, $a_1{}^2$, a^{22}, a_{21} DO NOT MAKE ANY SENSE!

WE will define them soon.

by the way: GENERALIZATION IS CLEAR

$$A = \begin{array}{c} \overset{j}{\longleftrightarrow} \\ \left(\cdots \, a^i{}_j \cdots \right) \end{array} \updownarrow i$$

element @ i^{th} ROW
j^{th} COLUMN

eg: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ what is $\boxed{a^1{}_3 \, ?}$

<u>RULE</u>: EINSTEIN SUMMATION CONVENTION (<u>CONTRACTION</u>)

IN ANY EXPRESSION WITH A <u>REPEATED UPPER & LOWER INDEX</u> THE REPEATED INDEX IS SUMMED OVER.

$$\text{eg } a^i{}_j v^j \equiv \sum_j a^i{}_j v^j = a^i{}_1 v^1 + a^i{}_2 v^2 + \underbrace{\cdots}_{\text{UP TO \# OF ELEMENTS ("DIMENSION")}}$$

seems random, right? BUT LET US RECALL MATRIX MULTIPLICATION

$$\begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} a^1{}_1 v^1 + a^1{}_2 v^2 \\ a^2{}_1 v^1 + a^2{}_2 v^2 \end{pmatrix} = \underbrace{\begin{pmatrix} a^1{}_j v^j \\ a^2{}_j v^j \end{pmatrix}}_{\text{SUMMATION CONVENTION!}}$$

note: NOTION OF "FREE INDEX"
↳ not contracted

INDICES & THE SUMMATION CONVENTION END UP BEING REALLY HELPFUL AS A TOOL TO GENERALIZE "LINEAR FUNCTIONS"

eg: $$B^i{}_j C^j{}_k = B^i{}_1 C^1{}_k + B^i{}_2 C^2{}_k + B^i{}_3 C^3{}_k + \cdots$$

(COMPARE to MATRIX

WE CAN EVEN IMAGINE OTHER OBJECTS (neither vectors nor matrices) WHOSE INDICES ARE MORE EXOTIC

THINK of THIS LIKE A $N \times N \times N \times N$ ARRAY w/ SPECIAL RULES

$$R^i{}_{jk\ell}$$

← UPPER
← LOWER

1st 2ND 3RD 4th

object like this shows up in GEN. RELATIVITY & encodes curvature of SPACETIME

WITHOUT SAYING ANYTHING ABOUT WHAT THIS OBJECT IS OR WHERE IT CAME FROM, WE KNOW HOW TO CONTRACT IT WITH OTHER OBJECTS

eg $$R^i{}_{jk\ell} V^j W^k T^\ell = \sum_j \sum_k \sum_\ell R^i{}_{jk\ell} V^j W^k T^\ell$$

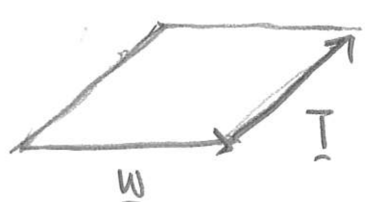
RESULT IS AN OBJECT THAT HAS ONE FREE UPPER INDEX

$\sum_{j,k,\ell}$

REPEATED INDICES ARE CALLED "DUMMY INDICES"

→ eg can call it $U^i = (RVWT)^i$

(IN GR: turns out this is is how much a vector $\underline{V}$ changes when moved in a parallelogram given by $\underline{W}$ & $\underline{T}$



examples

- $A^i{}_j B_{jk}$? not allowed

both lower indices
REPEATED INDEX DOESN'T MAKE SENSE

- $S_{ij}{}^k T_\ell{}^j$? allowed!

SUM OVER UPPER & LOWER INDEX

$$= \sum_j S_{ij}{}^k T_\ell{}^j = S_{i1}{}^k T_\ell{}^1 + S_{i2}{}^k T_\ell{}^2 + \cdots$$

$A^i{}_i$? allowed!

$$C = \sum_i A^i{}_i = \underbrace{A^1{}_1 + A^2{}_2 + \cdots}_{}$$

this is the TRACE OF MATRIX A

- CAN WE TAKE THE TRACE of AN OBJECT WITH COMPONENTS B_{ij} ?
  ↳ no! two lower indices → B_{ii} doesn't make sense!

  why? (ANSWER SOON)

nb: we will soon introduce new mathematical objects to permit this

will be related to the dot product

eg: $\underline{V} \cdot \underline{W} = \underbrace{V^1 W^1 + V^2 W^2 + \cdots}_{}$

looks like $V^i W^i$ ... doesn't make sense!

BIG PICTURE:

Contraction rules & index convention generalize matrix multiplication.

$\Rightarrow$ MULTI-LINEAR functions ("multilinear maps")

aka TENSORS

WHAT DO WE MEAN BY THIS?

RECALL $R^i{}_{jkl}$ "=" R ← a tensor (you may want to WRITE it as $\underline{R}$ to remind you that it is not a #)

this "EQUALITY" is an identification of the GENERIC COMPONENT w/ the WHOLE OBJECT; like saying

$$A^i{}_j = A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix}$$

if WE "FEED" R 3 VECTORS, IT SPITS OUT ANOTHER VECTOR W/ COMPONENTS

$$R^i{}_{jkl} \underbrace{V^j W^k T^l}_{\text{3 vectors that we feed}} \leftarrow (RVWT)^i \leftarrow i^{th} \text{ component of output vector}$$

analog for matrix: feed A a vector, get a vector

$$A^i{}_j V^j = (A\underline{V})^i \leftarrow i^{th} \text{ component of output vector}$$

MULTILINEARITY: $R^i{}_{jkl}(V+U)^j W^k T^l = R^i{}_{jkl} V^j W^k T^l + R^i{}_{jkl} U^j W^k T^l$

α a #

$$R^i{}_{jkl}(\alpha V)^j W^k T^l = \alpha R^i{}_{jkl} V^j W^k T^l$$

& SIMILARLY for EACH other "ARGUMENT": $\underline{V}, \underline{W}, \underline{T}$

In fact, imagine a 2 lower index tensor, g_{ij}

↳ can interpret this to be a function that takes an upper indexed object ↙ vector & turns it into a lower indexed object

$$g_{ij} V^j = (gV)_i$$

... OR GENERALIZE TO "LOWER" AN INDEX of an object w/ MANY INDICES

$$g_{ij} R^j{}_{k\ell m} = (gR)_{ik\ell m}$$

... OR TAKE 2 VECTORS & SPIT OUT PURE # ↙ no indices

$$\underbrace{g_{ij} V^i W^j} = (\underline{V} g \underline{W}) = g_{11} V^1 W^1 + g_{12} V^1 W^2 + g_{13} V^1 W^3 + g_{21} V^2 W^1 + g_{22} V^2 W^2 + \cdots$$

n/b ORDER DOESN'T MATTER:

$$= W^j g_{ij} V^i = V^i W^j g_{ij} = \text{etc}$$

but be careful $(\underline{V} g \underline{W})$ has no ambiguities!

<u>QUESTION</u>: if g encodes the <u>DOT PRODUCT</u>, what are its components?

[ASSUMING N DIM. Real, flat space]

<u>linearity</u>: $g_{ij} (\alpha \underline{V} + \beta \underline{U})^i (\delta \underline{W} + \gamma \underline{T})^j =$

$$\alpha\delta g_{ij} V^i W^j + \alpha\gamma g_{ij} V^i T^j$$
$$+ \beta\delta g_{ij} V^i W^j + \beta\gamma g_{ij} U^i T^j$$

So things are hopefully starting to fall in place

few other ideas in "JR HIGH" LIN ALG:

IDENTITY MATRIX: $\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix} = \delta^i{}_j$

KRONECKER-δ

$$\delta^i{}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\mathbb{1}\underline{v} = \underline{v} \longleftrightarrow \delta^i{}_j v^j = \delta^i{}_1 v^1 + \delta^i{}_2 v^2 + \delta^i{}_3 v^3 + \dots$$
$$= v^i$$

we can omit dummy index when sum is obvious

MATRIX MULTIPLICATION: $AB = A^i{}_j B^j{}_k = \underbrace{(AB)^i{}_k}_{\text{has index structure of a matrix}}$

gives notion of INVERSE matrix

Q: What should the index structure of an inverse matrix be?

if $A = A^i{}_j$

$A^{-1} = \begin{cases} (A^{-1})^i{}_j \\ (A^{-1})^j{}_i \\ (A^{-1})_i{}^j \\ \vdots \end{cases}$

inverse matrix is just a matrix s.t.

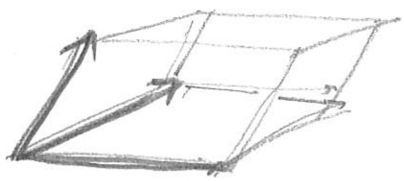
$(A^{-1})A = A(A^{-1}) = \mathbb{1}$

$(A^{-1})^i{}_j A^j{}_k = A^i{}_j (A^{-1})^j{}_k = \delta^i{}_k$

↑ you can solve this for the components of the inverse matrix. (good excercise for 2x2) the other thing we can do is take $\underline{\text{DETERMINANTS}}$

↳ we'll save this for a BIT LATER

BUT DETERMINANTS ARE HYPERVOLUMES OF PARALLELPIPEDS



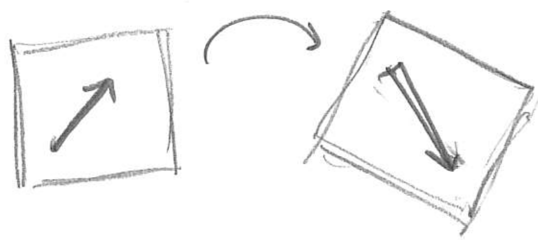
one more big picture idea:

what's so great about $\underline{\text{indices}}$?

↳ they tell us how an object $\underline{\text{transforms}}$

↑ w/rt $\underline{\text{SYMMETRIES}}$ of SPACE

eg



these are the same vector, we just ROTATED the page.

in general, $\underline{\text{any}}$ matrix transforms a vector

eg $A = \begin{pmatrix} 3 & 1/2 \end{pmatrix}$

but special transformations preserve something about the vector

↳ $\underline{\text{ROTATIONS}}$ (& their generalization)

eg $R^i{}_j = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $\quad \underline{V} \to R\underline{V}$

Q: how does a matrix transform UNDER ROTATIONS?