RODRIGUES FORMULA

$$P_n(x) = \underbrace{\frac{1}{2^n n!}}_{\equiv A_n} \left(\frac{d}{dx}\right)^n \left[(x^2-1)^n \right]$$

WANT TO SHOW:

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1} \delta_{nm}$$

USE: INTEGRATION BY PARTS

eg
$$\int_{-1}^{1} \left(\frac{d}{dx}\right)^n \left[(x^2-1)^n \right] \left(\frac{d}{dx}\right)^m \left[(x^2-1)^m \right]$$

$$= \left(\frac{d}{dx}\right)^{n-1} \left[(x^2-1)^n \right] \left(\frac{d}{dx}\right)^{m} \left[(x^2-1)^m \right] \Big|_{-1}^{1}$$

$$- \int_{-1}^{1} \left(\frac{d}{dx}\right)^{n-1} \left[(x^2-1)^n \right] \left(\frac{d}{dx}\right)^{m+1} \left[(x^2-1)^m \right]$$

- ARGUE THAT $\left(\frac{d}{dx}\right)^{n-1} \left[(x^2-1)^n \right] = 0$ AT $x = \pm 1$
  SO THAT THE SURFACE (BOUNDARY) TERM $= 0$

- THEN ARGUE THAT IF $n > m$, END UP WITH MORE DERIVATIVES THAN POWERS of x
  ($n < m$ CASE is ANALOGOUS)

WHAT ABOUT THE CASE $n=m$?

$$\int_{-1}^{1} P_n(m)^2 = A_n^2 \int_{-1}^{1} \left(\left(\frac{d}{dx}\right)^n \left[(x^2-1)^n \right] \right)^2 dx$$

$\vdots$

$$= (-)^n A_n^2 \int_{-1}^{1} dx \, (x^2-1)^n \underbrace{\left(\frac{d}{dx}\right)^{2n} \left[(x^2-1)^n \right]}$$

$$= \left(\frac{d}{dx}\right)^{2n} x^{2n}$$

$$= (2n)!$$

$$= (2n)! \, A_n^2 \int_{-1}^{1} dx \, \underbrace{(-)^n (x^2-1)^n}_{(1-x^2)^n}$$

not obvious step!

FROM Matthews & Walker, Math. Methods of Physics

u substitution: $x \equiv 2u - 1$ $\quad$ $x=1 \rightarrow u=1$, $x=-1 \rightarrow u=0$

$dx = 2du$

$$\int_{-1}^{+1} dx \, P_n(x)^2 = (2n)! \, A_n^2 \int_0^1 2du \left(1 - (2u-1)^2 \right)$$

$$= (2n)! \, A_n^2 \int_0^1 2du \cdot 4^n u^n (1-u)^n$$

$$= \frac{(2n)!}{\cancel{2^{2n}} (n!)^2} \cdot 2 \cdot \cancel{4^n} \int_0^1 u^n (1-u)^n \, du$$

$$\int_{-1}^{+1} dx\, P_n(x)^2 = 2\,\frac{(2n)!}{(n!)^2} \underbrace{\int_0^1 du\; u^n (1-u)^n}$$

this is a special integral

"BETA FUNCTION"

$$B(n+1, n+1) = \frac{(n!)^2}{(2n+1)!}$$

for POSITIVE INTEGERS, n

full definition:

$$B(r,s) = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)}$$

$\Gamma(n) = (n-1)!$ for POSITIVE INTEGER, n

GAMMA function

$$= 2\,\frac{(2n)!}{\cancel{(n!)^2}}\;\frac{\cancel{(n!)^2}}{(2n+1)!}$$

$$= \frac{2}{2n+1}$$