STORY SO FAR

function SPACE :

vectors are functions $f(x) \to |f\rangle$

... SUBJECT to BOUNDARY CONDITIONS

MATRICES ARE DIFFERENTIAL OPERATORS

we will just say "OPERATORS"

motivated: DYNAMICS LARGELY ENCODED IN LAPLACIAN

eg $\left(\frac{\partial}{\partial t}\right)^2$, ∇^2, ∂^2

second derivative is SYMMETRIC op

↳ diagonal in basis of eigenfunctions

BUT: ∇^2 takes many diff forms in diff COORDINATES

SPHERICAL $$\nabla^2 f = \frac{1}{r^2}\partial_r\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\partial_\theta\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f$$

CYLINDR $$\nabla^2 f = \frac{1}{s}\partial_s\left(s\,\partial_s f\right) + \frac{1}{s^2}\partial_\phi^2 f + \partial_z^2 f$$

↳ Different eigenfunctions in diff coords

↳ BEST TO USE COORDS WHERE THE SYM. OF THE SYSTEM ARE MANIFEST

LET'S START W/ 1-D.

OUR "HISTOGRAM BASIS" for function space is SILLY

for SEVERAL REASONS, it turns out!

eg. Stone & Goldbart

APPEL CH.9 → convergence issues

WHAT'S A BETTER BASIS?

maybe POLYNOMIALS: $|n\rangle = x^n$

BUT: not normalized? $|n\rangle = \mathcal{N}_n x^n$

↳ ok. $\langle n, n\rangle = \mathcal{N}^2 \underbrace{\int_a^b}_{\text{define space}} dx\, (x^n)^2 = \mathcal{N}^2\, 2n\, (x^{2n-1})\Big|_a^b$

$\mathcal{N}_n = \sqrt{\dfrac{1}{2n(b^{2n-1} - a^{2n-1})}}$

BUT: not ORTHOGONAL?

$$\langle n, m\rangle = \mathcal{N}_n \mathcal{N}_m \int_a^b dx\, x^{n+m} \neq \delta_{nm}$$

turns out you can make it orthonormal.

↳ GRAM SCHMIDT PROCEDURE

get: LEGENDRE POLYNOMIALS

shows up again in $\nabla^2 \big|_{\text{spherical}}$

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \end{aligned}\right\} \text{ def for } x \in [-1, 1]$$

CHECK : $\langle 2, 1\rangle = \int_{-1}^{1} dx\ P_2(x)\, P_1(x)\ dx$

$$= \frac{1}{2}\int_{-1}^{1} (3x^3 - x)\ dx$$

$$= \frac{1}{2}\left[\frac{3}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{1}$$

$$= 0 \quad \text{by even-ness of } [\cdots]$$

$$\langle 2, 2\rangle = \int_{-1}^{1} dx\ P_2(x)^2\ dx$$

$$= \frac{1}{4}\int_{-1}^{1} 9x^4 - 6x^2 + 1$$

$$= \frac{1}{4}\left[\frac{9}{5}x^5 - 2x^3 + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\underbrace{\left(\frac{9}{5} - \frac{10}{5} + \frac{5}{5}\right)}_{\frac{4}{5}}$$

$$= \boxed{\frac{2}{5}}$$

$\Leftarrow$ not normalized?!

for some reason we don't define the Legendre polyn as normalized functions.

you can now figure out the approp norm for each.

MORE COMMON BASIS : FOURIER → MOMENTUM SPACE

[eigenstates of momentum

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x)$$

$$a_0 = a_0 \cos(k_n x)$$

TRIG FUNCTIONS AS BASIS

WE ALREADY RECOGNIZE k_n AS A WAVE NUMBER. ANGULAR FREQUENCY ω

eg. ENERGY of LIGHT is $\hbar\omega$

Define a function space:

$\boxed{0 \le x \le L}$; $f(0) = f(L) = 0$

what is my basis?

↳ $a_0 = 0$ b/c cannot satisfy B.C.

similarly, $\cos(0) = 1 \Rightarrow a_n = 0$ (can treat each basis separately)

left w/ $C_n \sin(k_n x)$

↑ normalize, ↑ tbd

$\sin(0) = 0$ ✓

$\sin(k_n L) = 0 \Rightarrow k_n L = 0, \pi, 2\pi, \ldots$

$\boxed{k_n = \frac{n\pi}{L}}$ ← why not negative vals?

↑ hi n → hi freq.

$$\int_0^1 dx \, \sin^2(n\pi x) = \frac{1}{2}$$

$$\int_0^1 dx \, \sin(n\pi x)\sin(m\pi x) = 0 \qquad n \neq m$$

NORMALIZATION:

$y = x/L \qquad dx = L\,dy$

$$\int_0^L dx \, C_n^2 \sin^2\left(\frac{n\pi}{L}x\right) = C_n^2 \int_0^1 L\,dy \, \sin^2(n\pi y)$$

$$= C_n^2 \frac{L}{2}$$

$$\Rightarrow C_n = \sqrt{\frac{2}{L}}$$

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now insert 11 $\longrightarrow$ see sheet HW