LAST TIME:

$\boxed{v^i \quad \text{ceci n'est pas un vector}}$

(apologies to Magritte)

these indexed objects are COMPONENTS

the VECTOR-NESS is carried in an index-free object.

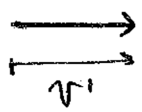
$$\underset{\text{VECTOR}}{\underline{V}} = \underset{\substack{\text{COMPONENT} \\ \text{OF VECTOR} \\ \text{(just \#s!)}}}{v^i}\ \underline{e}_{(i)}$$

BASIS VECTORS translate btwn components & "vectors"

eg: $\underline{e}_{(1)}$ could BE $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}1 & 0\end{pmatrix}$, $\frac{\partial}{\partial x}$, ...

then v^1 counts how many of these.

$$\begin{pmatrix}v^1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}v^1\\0\end{pmatrix} \text{ or } \begin{pmatrix}v^1 & 0\\0 & 0\end{pmatrix} \text{ or } \underbrace{v^1 \frac{\partial}{\partial x}}_{\substack{\text{DIRECTIONAL}\\ \text{DERIVATIVE}}}$$



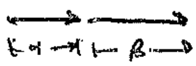
eg you often see

$\underline{v} \cdot \underline{\nabla}$ = derivative in the $\underline{v}$ DIRECTION (w/ MAGNITUDE)

all of these are LINEAR.

if $v^1 = \alpha + \beta$

$$\begin{pmatrix}v^1\\0\end{pmatrix} \Rightarrow$$



$$\text{or } \begin{pmatrix}\alpha & 0\end{pmatrix} + \begin{pmatrix}\beta & 0\end{pmatrix} \text{ or } \alpha \frac{\partial}{\partial x^1} + \beta \frac{\partial}{\partial x^1}$$

WE CAN BE MORE EXPLICIT w/ the LAST EXAMPLE

VECTOR SPACE of LINEAR DIFF OPERATORS

FOR SIMPLICITY : $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ etc. } BASIS

CONSIDER A DIFF. OPERATOR

$$D = d^1 \partial_x + d^2 \partial_y$$

IS THIS A VECTOR SPACE? YES.

~~COORDINATES~~ COMPONENTS ARE

$$D^i = \begin{pmatrix} d^1 \\ d^2 \end{pmatrix}$$

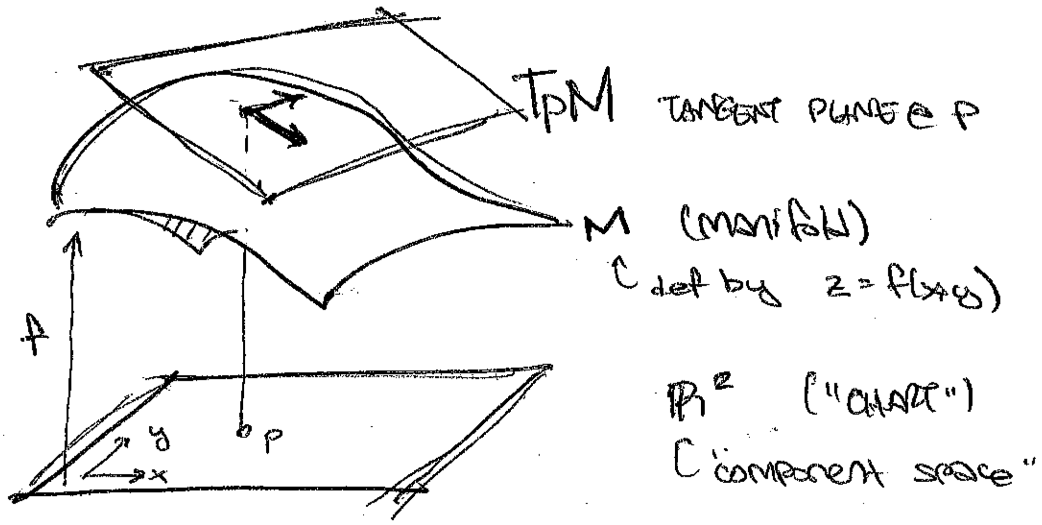
~~If $d^1\partial_x + d^2\partial_y$ is a diff op then~~

GIVEN 2 DIFF. OPERATORS, D & $\tilde{D}$, YOU CAN ADD THEM

$$D = d^1\partial_x + d^2\partial_y$$
$$+\ \tilde{D} = \tilde{d}^1\partial_x + \tilde{d}^2\partial_y$$

$$\underbrace{D+\tilde{D}}_{\text{also a diff operator}} = (d^1+\tilde{d}^1)\partial_x + (d^2+\tilde{d}^2)\partial_y$$

IN fact, there's ANOTHER WAY TO INTERPRET THE VECTOR SPACE → tangent planes for a surface $z = f(x,y)$



the "VECTOR" ∂_x MAPS TO A TANGENT VECTOR IN 3D SPACE

$$\partial_x \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix}_p \qquad (\star)$$

nb not normalized w/rt Euclidean $\mathbb{R}^3$ metric

$$\partial_y \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix}_p \qquad (\star\star)$$

$(\star)$ & $(\star\star)$ ARE A BASIS for THE TANGENT PLANE of M @ the POINT $(p, f(p))$

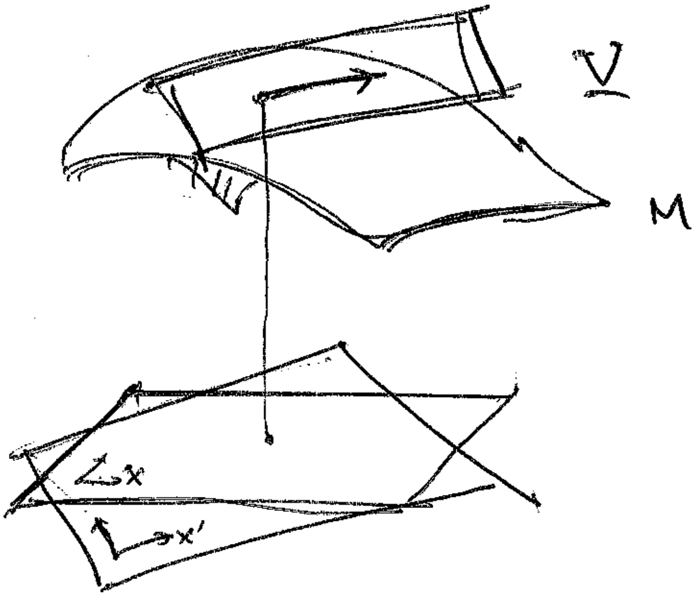
nb: NEAR A BLACK HOLE: TIME DILATION is MANIFESTED IN THE SPACETIME DIRECTION W/RT THE OBSERVER'S TIME DIRECTION.

(tangent space = local inertial frame)

A BIT OF GEOMETRY

VECTORS $\longleftrightarrow$ $V^\mu \frac{\partial}{\partial x^\mu}$

transformation W/RT CHANGE OF COORDINATES ?



What are components of $\underline{V}$ in $\underbrace{x' \text{ BASIS}}_{\frac{\partial}{\partial x'}}$?

$$V'^\mu \frac{\partial}{\partial x'^\mu} = V^\mu \frac{\partial}{\partial x^\mu}$$

$$= \underbrace{\left(V^\nu \frac{\partial x'^\mu}{\partial x^\nu}\right)}_{V'^\mu} \frac{\partial}{\partial x'^\mu}$$

$$\frac{\partial x'^1}{\partial x^1} = c \qquad \frac{\partial x'^1}{\partial x^2} = s \qquad \frac{\partial x'^2}{\partial x^1} = -s \qquad \frac{\partial x'^2}{\partial x^2} = c$$

CHECK FOR ROTATIONS

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} cx^1 + sx^2 \\ -sx^1 + cx^2 \end{pmatrix}$$

$$\Rightarrow V'^1 = V^1 \frac{\partial x'^1}{\partial x^1} + V^2 \frac{\partial x'^1}{\partial x^2} = V^1 c + V^2 s$$

BETTER (sometimes) NOTATION: BRA-KET

$| \rangle$ REPLACES $\underline{\quad}$ eg $|v\rangle = \underline{v}$

$\langle |$ REPLACES $\sim$ eg $\langle w| = \underset{\sim}{w}$

<u>BASIS</u>: $|i\rangle = \underline{e}_{(i)}$
$\langle j| = \underset{\sim}{e}^{(j)}$
} yeah, you lose the HEIGHTS.

$$\underline{v} = |v\rangle = v^i \, |i\rangle$$
$$\underset{\sim}{w} = \langle w| = w_j \, \langle j|$$
$$A = A = A^i{}_j \, |i\rangle\langle j|$$
$$g = g? = g_{ij} \, \langle i|\langle j|$$

← but we RARELY USE THIS IN QM ... BECAUSE WE HAVE AN INNER PRODUCT ... MORE LATER.

⌈GOOD HW: GR METRIC⌋

physics.stackx/359071

$\boxed{\text{TRANSFORMATIONS IN GENERAL}}$

↳ physics should not depend on your COORDINATE system

no indices, no vectors..

eg <u>LAWS</u> OF PHYSICS SHOULD BE W/RT INVARIANTS

eg OBSERVATIONS (that usually dep on obj) SHOULD BE INVARIANTS WHERE WE USE THE OBSERVER AS AN "INDEXED" OBJECT.

(cf SCHUTZ sec 2.2)

eg. SPECIAL RELATIVITY

throw a rock of mass m.
it has 4 momentum

$$p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \qquad \text{where } \underbrace{E^2 - \vec{p}^2}_{p^\mu p_\mu} = m^2$$

FRAME-DEPENDENT

using $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

what is the ENERGY of the ROCK?
OBVIOUSLY E.

now I'm watching you throw the rock while i'm on my scooter.

I see that the ROCK HAS

$$p'^\mu = \begin{pmatrix} E' \\ \vec{p}' \end{pmatrix} \qquad (E')^2 - (\vec{p}')^2 = m^2$$

SAME

but $E' \neq E$

HOW DO YOU CALCULATE WHAT ENERGY E' I MEASURE?

I AM MOVING w/ VELOCITY $v = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right)$

PRIME = MY FRAME

τ IS CALLED PROPER TIME (time I MEASURE)

note: $v' = (1, \vec{0})$ in my frame

$v^2 = 1$ (in ANY FRAME)

why this?

$v = (1+\varepsilon, 0, 0, \sqrt{2\varepsilon + \varepsilon^2})$

sign: moving in same dir AS the BALL.

OBSERVE: $v \cdot p$ is a SCALAR QUANTITY

"no indices" → invariant

$v \cdot p = v^\mu p^\nu g_{\mu\nu}$

physically: "what observer w/ 4-vel v measures the energy of object w/ 4-mom. p to have"

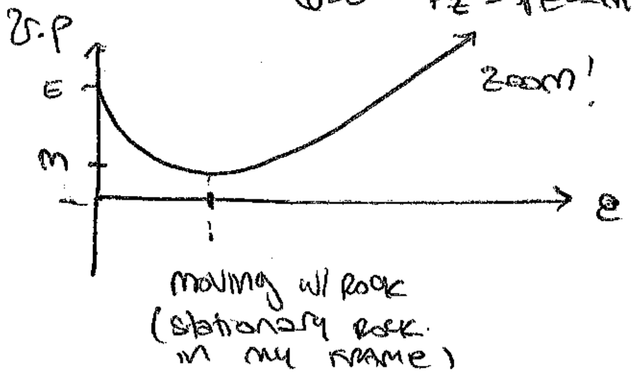
in MY frame $\quad v' \cdot p' = \boxed{E'}$

in YOUR frame $\quad v \cdot p = \boxed{(1+z)E - \sqrt{2z+z^2}\, p_z}$

EQUAL

ASSUME ALL MOTION IN Z-DIR

by the way: you can plot this (use $p_z = \sqrt{E^2 - m^2}$)



REMARK: $v_\mu p^\mu$ is invariant. WHY?

$v^\nu g_{\mu\nu}$

under LORENTZ TRANSFORMS ("rotations of relativity")

$$v_\mu p^\mu \to \underbrace{v_\nu (\Lambda^{-1})^\nu{}_\mu}_{v'} \underbrace{\Lambda^\mu{}_\rho p^\rho}_{p'}$$

$(\Lambda^{-1})^\nu{}_\mu \Lambda^\mu{}_\rho = \mathbb{1}$

& can check: $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$ $\quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$

Λ^{-1} has $\beta \to -\beta$ then $\Lambda\Lambda^{-1} = \mathbb{1}$

TRANSFORMATIONS: BIG PICTURE

MATRICES

- WEIRD (eg $N \times M$)
- non invertible → "PROJECTIONS" (lose information)
- invertible → RESHUFFLES INFORMATION
  - "ROTATIONS" PRESERVES INNER PRODUCTS
    $\langle \underline{v}, \underline{w} \rangle = g_{ij} v^i w^j$
  - other (DOES SOMETHING)

nb: QUADRATIC FORM

given $\underline{x} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \end{pmatrix}$

a quadratic form is

$f(\underline{x}) = a_{ij} x^i x^j$ w/ a symmetric

$\underline{x}^T A \underline{x}$

when A is positive definite this could be a metric

this "TRANSPOSE" is part of "LOWERING INDEX" along w/ METRIC (cf CARROLL P. 12)
→ CARRIES INDEX LOWERING, WHILE A HAS the "#'s"

What kind of TRANSF WOULD PRESERVE THE INTERVAL?

$(\underline{x}' \to R\underline{x}) \to \underline{x}^T \underbrace{R^T \overset{\text{METRIC}}{\mathbb{1}} R}_{= \mathbb{1}} x \to \boxed{R^T = R^{-1}} \Rightarrow$ Rotation

"Reshuffles info" TRANSFORMATIONS → Rotations: PRESERVE METRIC.

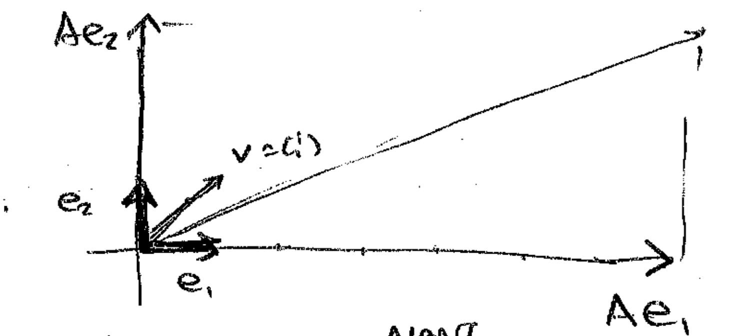
[good hw: DERIVE form of LORENTZ TRANSF]

DOES STUFF: ... "not Rotations"

→ What does this mean?

eg $\begin{pmatrix} 7 & \\ & 3 \end{pmatrix}$

clearly not a rotation



$Av = \begin{pmatrix} 7v^1 \\ 3v^2 \end{pmatrix}$

(EASY to FIGURE out.)

(rescaling ALONG eg basis

SUB CAT → $\boxed{\text{NICE}}$ 'self adjoint'
(OBSERVABLE?

→ $\boxed{\text{NOT NICE}}$ → RANDOM ASS MATRIX

for now: NICE MATRICES ARE SYMMETRIC

why so nice?

EIGENVALUES

NICE MATRIX = $R^T \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} R$ ← R IS A ROT. MAT. CONTAINING EIGENVECS