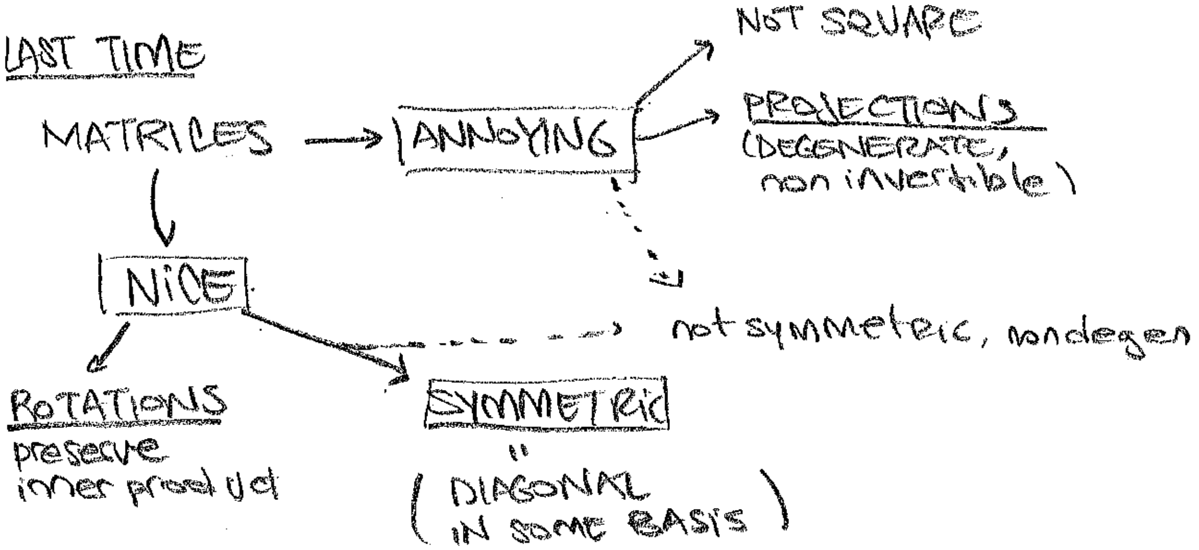
TO DISCUSS

- SHORT HW → finding eigen.
- DETERMINANTS, TRACES ; DIAG & MATRICES → inverses
- SYM, HERM, UNITARY

LAST TIME



SPECIAL BASIS VECTORS: EIGENVECTORS $\underline{\xi}_{(i)}$

DIAGONAL ELEMENTS: EIGENVALUES, λ_i

s.t. $A\,\xi_{(i)} = \lambda_i\,\xi_{(i)}$

this is IMPORTANT

eg. MATRIX INVERSES ARE HARD
SCALAR INVERSES ARE EASY

$$\text{if } A\,\xi_{(i)} = \lambda_i\,\xi_{(i)} \Rightarrow \underbrace{A^{-1}\,\xi_{(i)} = \frac{1}{\lambda_i}\,\xi_{(i)}}_{\text{EASY!}}$$

HOW TO USE: if $\underline{V} = V^1\,\xi_{(1)} + V^2\,\xi_{(2)} + \cdots$

then $A^{-1}\underline{V} = V^1(\lambda_1)^{-1}\,\xi_{(1)} + V^2(\lambda_2)^{-1}\,\xi_{(2)} + \cdots$

EASY! once you know the EIGENVECTORS & EIGENVALUES.

[do an example]

SYMMETRIC MATRICES

- $A^T = A$ ⟵ $$\underbrace{\underset{\underline{v}^T A \underline{w}}{\overset{\shortparallel}{\langle \underline{v}, A\underline{w}\rangle}} = \underset{\underline{v}^T A^T \underline{w}}{\overset{\shortparallel}{\langle A\underline{v}, \underline{w}\rangle}}}_{} = \langle \overset{\text{BRA}}{v|} A \overset{\text{KET}}{|w}\rangle$$

  in abstract vec space (eg ∞ dimensional), THIS WILL BE OUR DEFINITION OF SYMMETRIC

  (there's A VERSION of "SYMMETRIC" for ℂ VEC SPACES: HERMITIAN)

- THERE IS AN ORTHONORMAL BASIS WHERE A IS DIAGONAL:

  $\underline{e}_{(i)} \to R\underline{e}_{(i)} \equiv \underline{\xi}_{(i)}$ for some ROTATION R

  $A \to RAR^T \equiv \hat{A}$ ⟵ hat is shorthand for "DIAGONAL"

  s.t. $\boxed{\hat{A}\,\underline{\xi}_{(i)} = \lambda_i\, \underline{\xi}_{(i)}}$ ⟵ niiice! essence of transformation

  EIGENVALUE (↑ λ_i) EIGENVECTOR (↑ $\underline{\xi}_{(i)}$)

  REAL ; nb: IDENTICAL EIGVAL → SUBSPACE w/ ROT. INVARIANCE

  $\hat{R}\hat{A}\xi = \hat{R}\lambda\xi = \lambda \underbrace{R\xi}_{\xi'}$

- WHAT ABOUT LESS NICE MATRICES?
  - → PROJECTIONS: SOME EIGENVALUES = 0,
  - → NON-SYMMETRIC MATRICES
    - → EIGENVALUES MAY BE ℂ
    - → EIGENVECTORS NOT AN ORTHONORMAL BASIS

DETERMINANTS

↳ see LEC 6 NOTES

SUMMARY: if COLUMNS of A ARE VECTORS:

$$A = \begin{pmatrix} | & | & \\ \underline{a}_1 & \underline{a}_2 & \cdots \\ | & | & \end{pmatrix}$$

then det A is the VOLUME of the PARALLELPIPED FORMED BY $\{\underline{a}_i\}$

↳ generalization of CROSS PRODUCT

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\begin{pmatrix} a \\ c \end{pmatrix} \times \begin{pmatrix} b \\ d \end{pmatrix} = ad - bc$$

convenient definition: LEVI-CIVITA TENSOR

↳ "totally antisymmeteric tensor in d-DIM"

CONVENTION: $\varepsilon^{123\cdots d} = +1$

($\varepsilon_{123\cdots d} = +1$ from meteric)

AND: $\varepsilon^{\cdots ij \cdots} = -\varepsilon^{\cdots ji \cdots}$

eg IN 2D: $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

then: $\det A = \varepsilon^{i_1 i_2 \cdots i_n} a^1{}_{i_1} a^2{}_{i_2} \cdots a^n{}_{i_n}$

$= \frac{1}{n!} \varepsilon^{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n} a^{j_1}{}_{i_1} \cdots a^{j_n}{}_{i_n}$

eg IN 2D: $\det A = \varepsilon^{ij} a^1{}_i a^2{}_j = a^1{}_1 a^2{}_2 - a^1{}_2 a^2{}_1$ ✓

$= \frac{1}{2!}(\varepsilon^{ij}\varepsilon_{k\ell} a^k{}_i a^j{}_\ell) = \frac{1}{2}(a^1{}_1 a^2{}_2 - a^1{}_2 a^2{}_1 - a^2{}_1 a^1{}_2 + a^1{}_1 a^2{}_2)$ ✓

⇒ LEVI-CIVITA TENSOR: in d-DIM, WE CAN USE LEVI-CIVITA TO FORM INVARIANTS BY CONTRACTING INDICES

↳ RELATED to MEASURING VOLUMES & the d-DIM VOLUME ELEMENT d^dV

eg the JACOBIAN is A DETERMINANT

GOOD HW PROB: POLAR COOR VOL EL.

eg the DETERMINANT IS INDEX-FREE → INVARIANT
IT IS AN INTRINSIC PROPERTY of the TRANSFORMATION, NO MATTER WHAT BASIS.

$$\det A = \det RAR^T$$

$\det A^{-1} = (\det A)^{-1}$ OBVIOUS FOR DIAGONAL MATRICES

<u>FACT</u>: $\det(AB) = (\det A)(\det B)$

<u>PROOF</u> (not for in-class)

$$(AB)^i{}_j = a^i{}_k b^k{}_j$$

$$\det AB = \varepsilon^{i_1 \cdots i_n}(a^1{}_{k_1} b^{k_1}{}_{i_1})(a^2{}_{k_2} b^{k_2}{}_{i_2})\cdots(a^n{}_{k_n} b^{k_n}{}_{i_n})$$

$$= (a^1{}_{k_1}\cdots a^n{}_{k_n})\,\underbrace{\varepsilon^{i_1\cdots i_n}(b^{k_1}{}_{i_1}\cdots b^{k_n}{}_{i_n})}$$

ANTISYMMETRY IN {i} ⇒
—"— IN {k_n}

$$(\cdots b^{k_a}{}_{i_a}\cdots b^{k_b}{}_{i_b}\cdots) - \underbrace{(\cdots b^{k_a}{}_{i_b}\cdots b^{k_b}{}_{i_a}\cdots)} + \cdots$$

if $k_a = k_b$, these cancel → no symm. piece in {k}

SO: WE SUM OVER PERMUTATIONS OF K W/ SIGN
⇒ this is just contracting w/ $\varepsilon_{k_1\cdots k_n}$

$$= \underbrace{\sum_{\{k\}} \text{sgn}(\{k\})(a^1{}_{k_1}\cdots a^n{}_{k_n})}_{=\det A} \cdot \underbrace{\varepsilon^{i_1\cdots i_n}(b^{k_1}{}_{i_1}\cdots b^{k_n}{}_{i_n})}_{=\det B \text{ for EA } \{k\}}$$

$$= (\det A)(\det B) \checkmark$$

for DIAGONAL MATRIX, $\hat{A} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots)$

↳ $\det \hat{A} = \lambda_1 \lambda_2 \cdots \lambda_n = \prod_n \lambda_n$

invariant under rotations: $\det A = \det R\hat{A}R^T$

$$= (\det R)(\det \hat{A})(\det R^T)$$

$$= \det \hat{\lambda}$$

b/c $R^T = R^{-1} \rightarrow \det R^T = (\det R)^{-1}$

↳ so $\det A$ = PRODUCT OF EIGENVALUES IN ANY BASIS.

<u>TRACE IS SIMILAR.</u>

$$\mathrm{Tr}\, A = {A^i}_i = {A^1}_1 + {A^2}_2 + \cdots$$

$$\mathrm{Tr}\, \hat{A} = \lambda_1 + \lambda_2 + \cdots$$

claim: $\underline{\mathrm{Tr}\, AB} = \mathrm{Tr}\, BA$

$${A^i}_j {B^j}_i = {B^j}_i {A^i}_j$$

claim: $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(CAB)$ (cyclic)

$${A^i}_j {B^j}_k {C^k}_i = {C^k}_i {A^i}_j {B^j}_k \quad \checkmark$$

$\Rightarrow \mathrm{Tr}\, A = \mathrm{Tr}(R\hat{A}R^T) = \mathrm{Tr}\,(R^T R \hat{A}) = \mathrm{Tr}\, \hat{A} = \Sigma \lambda_i$

↳ trace is <u>invariant</u>.

EIGENVALUE / CHARACTERISTIC EQ

$\det A = \det \hat{A} = \prod \lambda_i$

$\leftarrow$ want to find λ_i

TRICK: unknown λ

then $\det(\underbrace{\hat{A} - \lambda \mathbb{1}}) = \underbrace{\prod (\lambda_i - \lambda)}$

$$\begin{pmatrix} (\lambda_1 - \lambda) & & \\ & (\lambda_2 - \lambda) & \\ & & \ddots \end{pmatrix} = 0 \text{ when } \lambda = \lambda_i$$

this gives an n^{th} O- polynomial whose n roots are $\{\lambda_i\}$

↳ FINDING EIGENVALUES REDUCES TO SOLVING ALGEBRAIC EQN

SINCE det IS INVARIANT:

$$\det(\hat{A} - \lambda \mathbb{1}) = \det(R(\hat{A} - \lambda\mathbb{1})R^T)$$
$$= \det(A - \lambda\mathbb{1}) \qquad = \prod (\lambda_i - \lambda)$$

eg $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ w/ eigenvals λ_1, λ_2

then: $(a-\lambda)(c-\lambda) - b^2 = 0$

has 2 solutions, $\lambda = \lambda_1 \,\&\, \lambda_2$

we also know: $a + c = \lambda_1 + \lambda_2$

chk: $\lambda^2 - (a+c)\lambda - b^2 \rightarrow \lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$$\lambda_{\pm} = \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 + 4b^2}}{2}$$

$\Rightarrow \lambda_+ + \lambda_- = a+c$ ✓