ANNOUNCE

> TA OFFICE HR on MON
> CAPY PARADE
> LONG HW DUE DATE

the BESTIARY

METRIC SPACE

v^i VECTOR

w_j ROW VEC

$A^i{}_j$ "MATRIX"

$\boxed{g_{ij} \text{ METRIC}}$

VECTOR SPACE

converts $v^i \to v_i \equiv g_{ij} v^j$

PROPERTY: $g_{ij} = g_{ji}$

INVERSE (g^{-1}) ... what indices?

$\to (g^{-1})^{ij}$

s.t. $(g^{-1})^{ij} g_{jk} = \delta^i{}_k$

VECTORS $\to$ VECTORS

convention: $g^{ij} \equiv (g^{-1})^{ij}$

usually: $g_{ij} = \delta_{ij}$ in EUCLIDEAN SPACE

RELATIVITY: $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ (Minkowski)

GR: even more DIFFERENT
$\to$ EXPLAINS WHAT HAPPENS IN A BH.

MULTILINEAR MAPS : $T^{ij\cdots}{}_{k\ell\cdots}$

takes in some # of vec & row vec,
spits out a # and is linear

$f(\underline{v}, \underline{w}, \cdots, \underset{\sim}{s}, \underset{\sim}{t}, \cdots)$ s.t. $f(\cdots, \alpha x, \cdots) = \alpha f(\cdots, x, \cdots)$

$f(\cdots, x+y, \cdots) = f(\cdots, x, \cdots) + f(\cdots, y, \cdots)$

<u>BASIS</u> almost like a set of coordinates

for us: <u>ALWAYS</u> ORTHONORMAL

$\perp$ and normalized to unit length

$\{ \underline{e}_{(1)}, \underline{e}_{(2)}, \dots \}$

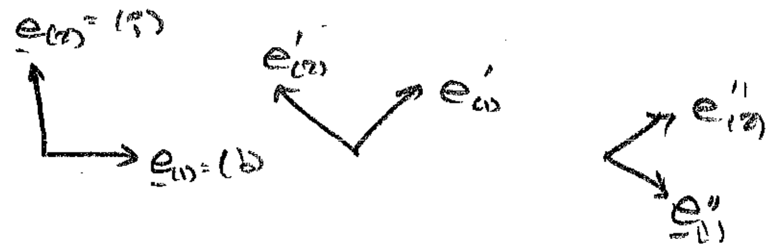
how many? = dimension of your VECTOR SPACE

METRIC gives a dot product ← PROJECTION of unit norm vector onto another

so $\underline{e}_{(i)} \cdot \underline{e}_{(j)} = \delta_{ij}$

eg in $\mathbb{R}^2$

nb notation: UNDERLINE is VECTOR-NESS, (i) is INDEX



VECTOR: LINEAR COMB. of BASIS VECTORS

$$\underline{v} = \underbrace{v^1 \underline{e}_{(1)} + v^2 \underline{e}_{(2)} + \dots}$$

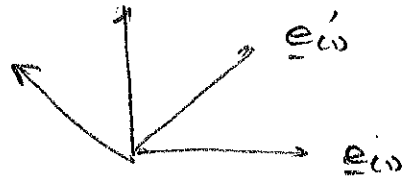
$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \end{pmatrix}$ ← what we usually call "the vector" are the COEFFICIENTS W/RT A BASIS

BUT WE MAY WRITE THE <u>SAME</u> VECTOR W/RT A DIFFERENT BASIS

$$\underline{v} = \underbrace{v'^1 \underline{e}'_{(1)} + v'^2 \underline{e}'_{(2)} + \dots}$$

$\begin{pmatrix} v'^1 \\ v'^2 \end{pmatrix}$ ← looks silly to write $\underline{v} = \binom{v^1}{v^2} = \binom{v'^1}{v'^2}$ when $v^i \neq v'^i$

note : $\underline{e}_{(i)} \cdot e'_{(j)} \neq \delta_{ij}$



CHANGE of BASIS IN GORY DETAIL $\mathbb{R}^2$ CASE

$$\bar{v} = v^i \underline{e}_{(i)}$$

change order for convenience

$$= \underline{e}_{(i)} v^i$$

$$= \left(\underline{e}_{(k)} \underbrace{(R^{-1})^k{}_i\right)\left(R^i{}_j}_{=\delta^k{}_j} v^j\right)$$

$$= \boxed{\underline{e}'_{(i)} \; v'^i}$$

$$\boxed{\underline{e}'_{(i)} = \underline{e}_{(k)} (R^{-1})^k{}_i} \quad \text{PASSIVE}$$

$$v'^i = R^i{}_j v^j \quad \text{ACTIVE}$$

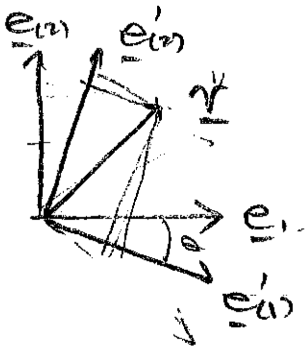
eg in $\mathbb{R}^2$ CARTESIAN

Dot w/ $\underline{e}_{(j)}$

$$\underline{e}'_{(i)} \cdot \underline{e}_{(j)} = \underbrace{\underline{e}_{(k)} \cdot \underline{e}_{(j)}}_{\delta_{kj}} (R^{-1})^k{}_i$$

$$= (R^{-1})^j{}_i$$ ← components of R^{-1}

eg:



WHAT ARE COMPONENTS of $\underline{v}$ in PRIMED BASIS?

← $\underline{e}'_{(1)} = c_\theta \underline{e}_{(1)} - s_\theta \underline{e}_{(2)}$

$\underline{e}'_{(2)} = s_\theta \underline{e}_{(1)} + c_\theta \underline{e}_{(2)}$

How to do this; 1st method

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v'^1 \underline{e}'_{(1)} + v'^2 \underline{e}'_{(2)}$$

$$= v'^1 \begin{pmatrix} c_\theta \\ -s_\theta \end{pmatrix} + v'^2 \begin{pmatrix} s_\theta \\ c_\theta \end{pmatrix}$$

$$\Rightarrow \begin{cases} v^1 = c_\theta v'^1 + s_\theta v'^2 \\ v^2 = -s_\theta v'^1 + c_\theta v'^2 \end{cases}$$

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} v'^1 \\ v'^2 \end{pmatrix}$$

then invert

<u>ALTERNATIVE</u>

$$v'^i = R^i{}_j v^j$$

$$(R^{-1})^j{}_i = \underline{e}'_{(i)} \cdot e_{(j)}$$

$$R^{-1} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix}$$

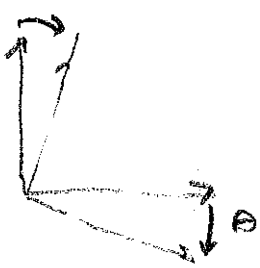
$$(R^{-1})^1{}_2 = \underline{e}'_{(2)} \cdot \underline{e}_{(1)}$$

$$s_\theta \underline{e}_{(1)} + c_\theta \underline{e}_{(2)}$$

$$\text{then} \rightarrow \quad R = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}$$

$$\boxed{v' = \begin{pmatrix} c_\theta v^1 - s_\theta v^2 \\ s_\theta v^1 + c_\theta v^1 \end{pmatrix}}$$

EASIER: ACTIVE TRANSF



"same" as

ACTIVE TRANSF.

IN $\mathbb{R}^2$: transformations that go from one ORTHONORMAL BASIS TO ANOTHER ARE ROTATIONS.

↑ isometries (from a different perspective)

in SPECIAL RELATIVITY: LORENTZ TRANSFORMATIONS

WE SEE that the INDICES ARE COMPONENTS WRT A GIVEN BASIS.

↳ tell us how the components transform WRT CHANGE OF BASIS

(also how they transform in general)

$\underline{e}_{(i)}$ ← index to connect to components ← in QM: $|i\rangle$

↑ CARRIES "VECTOR NESS" eg $|v\rangle = v^1|1\rangle + \cdots$

BASIS for ROW VECTORS

$\underset{\sim}{e}^{(j)}$ ← UPPER INDEX or $\langle j|$

RECALL: ROW VECTORS (EAT) VECTORS ...

↑ $\underline{v} = v^i \underline{e}_{(i)}$

so think of ROW BASIS AS

$\underset{\sim}{e}^{(i)}$

$\leftarrow e_{(j)} \Rightarrow \underset{\sim}{e}^{(i)}\left(e_{(j)}\right) = \delta^i{}_j$ $\Big\}$ $\langle i|j\rangle = \delta^i_j$

nb: DUALITY: $\underline{e}_{(i)}\left(\underset{\sim}{e}^{(j)}\right) = \delta^j_i$

WHAT ABOUT MATRICES?

$$A^i{}_j \rightsquigarrow A = A^i{}_j \; e_{(i)} \otimes \underset{\sim}{e}^{(j)}$$

$$= A^i{}_j \, |i\rangle\langle j|$$

$$\left(A^i{}_j |i\rangle\langle j| \right) \left(v^k |k\rangle \right) = A^i{}_j v^k \, |i\rangle \underbrace{\langle j|k\rangle}_{\delta^j_k}$$

$$= \underbrace{A^i{}_j v^j}_{(Av)^i} |i\rangle$$