HW46: LEGENDRE NORM

BIRD'S EYE VIEW of FOURIER & ALL THAT

↳ OUR BASIS: e^{ikx} ~> "PLANE WAVE"

note: $e^{\pm ikx}$ BOTH VALID

b/w, mat is: CAN think of as FUNC of x or k

GENERALIZE to HIGHER DIM:

$$e^{\pm i k_x x}\, e^{\pm i k_y y} \cdots \text{etc.}$$

(really just separation of vars)

... to SPACETIME

$$e^{\pm i\omega t}\, e^{\pm i k_x x} \cdots$$

convenient to choose opposite signs:

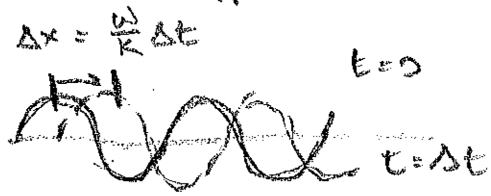
$$\underbrace{e^{-i\omega t + ikx}}_{\text{ENERGY \& MOMENTUM}} \sim \underbrace{\cos(-\omega t + kx)}_{\text{PLANE WAVE}} + i \sin \cdots$$

PLANE: WAVE FRONT IS A PLANE IN yz DIRS

WAVELENGTH $\lambda = \frac{k}{2\pi}$

PERIOD $\tau = \frac{\omega}{2\pi}$

velocity $v = \frac{\omega}{k}$



so we often call FOURIER BASIS ⟷ PLANE WAVES in fact

$$-i\omega t + ikx = -ip_\mu x^\mu = -ip^\mu \eta_{\mu\nu} x^\nu$$

$$p^\mu = (\omega, k) \qquad x^\mu = (t, x)$$

$$p_\mu = (\omega, -k)$$

so we write: $\boxed{e^{-ip\cdot x}}$ (or $e^{+ip\cdot x}$)

Stone & Goldbart App. B

FOURIER: from DISCRETE to CONTINUOUS

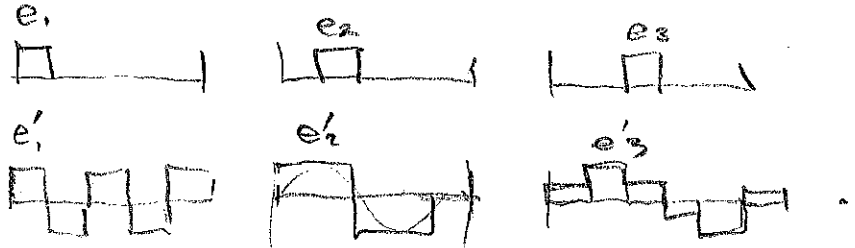
SERIES: c_m TRANSFORM: $\tilde{f}(k)$

PROTO-FOURIER: $\leftrightarrow$ Δx SPACING, $L = n\Delta x$

DISCRETE SAMPLING of FINITE DOMAIN

(back to HISTOGRAM space!)

FOURIER: TRADE BASIS of unit blocks to a basis of correlated blocks.



case: $\Delta n = n - n' = 0$

then: $\sum_{M=0}^{N-1} \langle n | k_m \rangle \langle k_m | n' \rangle = \sum_{M=0}^{N-1} \delta_{nn'} = N \, \delta_{nn'}$

$\times \, \delta_{nn'}$ to ensure $n = n'$

$\left(\text{so: } \sum |k_m\rangle\langle k_m| = N \, \mathbb{1} \right)$

we won't normalize... need to be able to take $N \to \infty$ lim.

GOING BETWEEN BASES:

$$f(x) = f(n\Delta x) = \sum_{n'=0}^{N-1} f(n'\Delta x) \underbrace{\delta_{n'n}}_{e_{n'}}$$

n: discrete index of pos space

$f(0) \times e_{n'}$

BUT WE CAN REPLACE $\delta_{n'n}$ w/ MOMENTUM COMPLETENESS...

$$f(n\Delta x) = \sum_{n'=0}^{N-1} f(n'\Delta x) \sum_{M=0}^{N-1} e^{ik_M(n'-n)\Delta x} \times \underbrace{\frac{1}{N}}_{\text{to normalize}}$$

collect n' dependence

$$= \sum_{m=0}^{N-1} \boxed{\sum_{n'=0}^{N-1} \frac{1}{N} f(n'\Delta x) e^{ik_m n' \Delta x}} \; e^{-ik_M n \Delta x}$$

C_m FINITE FOURIER COEFF

evidently we used the opp. convention

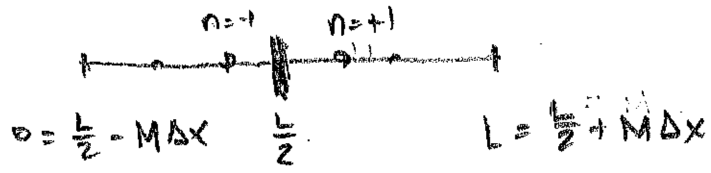
$$|n\rangle = \sum_m \frac{1}{N} |k_m\rangle\langle k_m|n\rangle$$

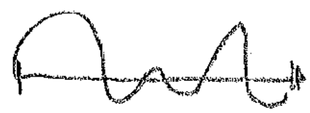
$\frac{1}{N}$ to normalize

So we have derived the FINITE FOURIER REP of A DISCRETIZED INTERVAL

$$f(na) = \sum_{m=0}^{N-1} c_m \, e^{-ik_m n \Delta x}$$

note : STRAIGHTFORWARD to do this for A SYMMETRIZED INTERVAL

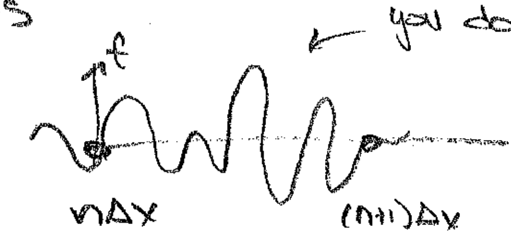


PASS TO CONTINUUM INTERVAL 

↳ Δx so small (rel to, eg. measurement) that we see f as continuous function

⇝ wiggles in f are $\gg \Delta x$
features

← you don't resolve this

(vs. 

SUM ↔ INTEGRAL: $\Delta x \sum_n f(n\Delta x) \to \int \Delta x \, dn \, f(n\Delta x)$

$$\to \boxed{\int dx \, f(x)}$$

KRONECKER δ : $\Delta x \sum_n f(n\Delta x) \boxed{\frac{\delta_{nn'}}{\Delta x}} = f(n'\Delta x)$

↓ ↓

$$\int dx \, f(x) \boxed{\delta(x-y)} = f(y)$$

DIRAC DELTA

$\delta(x-y)$ is not a function ∴ it is a distribution

only makes sense when integrated over.

→ many limiting functional forms.

↳ ∞ narrow, ∞ tall, integrates to one.

DISTRIBUTION: BRA / ROW VECTOR

$$\langle f, g \rangle = \int f^* g \, dx$$

$$\langle f | = \int f^* \_\_\_ \, dx$$

$$\langle n' | = \sum_n \delta_{nn'} (\_\_\_)_n$$

$$\downarrow$$

$$\langle y | = \int dx \, \delta(x-y) \, (\_\_\_)_x$$

BTW: DERIVATIVE of δ function?

$$\int_{-a}^{a} f(x) \, \delta'(x) \, dx = \underbrace{f\delta \Big|}_{\substack{\text{ZERO b/c} \\ \delta \text{ is only} \\ \text{nonzero @ 0}}} - \int f' \delta \, dx = f'(0)$$

nb: $\delta(x-y) \sim \frac{\delta_{nn'}}{\Delta x}$ ← physical significance IMPLICIT LATTICE SPACING

("UV")

DISCRETE SAMPLING: $x = n\,\Delta x$
(position space) $\quad$ n indexes position

(momentum space): $K_m = \underbrace{\frac{2\pi}{L}}_{\frac{2\pi}{N\Delta x} = \Delta K} m$ $\quad$ m indexes momentum

momentum basis: $e^{-iK_m(n\Delta x)} = |K_m\rangle$ (check)
NOT NORMALIZED

check: Histog basis is normalized
$\langle n | n' \rangle = \delta_{nn'}$
for NORMALIZED basis $\underbrace{\sum_i |e_i\rangle\langle e_i| = \mathbb{1}}_{\text{"completeness" of basis}}$

if $|K_m\rangle$ is not normalized (but orthog)
then $\sum |K_m\rangle\langle K_m| = \# \mathbb{1}$

$$\langle n | \left(\sum_{m=0}^{N-1} |K_m\rangle\langle K_m| \right) | n' \rangle = \sum_{m=0}^{N-1} e^{-iK_m(n\Delta x)} e^{+iK_m(n'\Delta x)}$$

$$= \sum_{m=0}^{N-1} e^{iK_m(n'-n)\Delta x} = \sum \underbrace{e^{\frac{i(2\pi(n-n'))}{N}}}_{a = e^{\frac{2\pi i \Delta n}{N}}}{}^{m}$$

$M = N-1$

$$\boxed{\begin{aligned} &\text{geom series: } \sum_{m=0}^{M} a^m = \Sigma \\ &\text{then: } \Sigma - a\Sigma = a^0 - a^{M+1} \\ &\Rightarrow \Sigma = \frac{1 - a^{M+1}}{1 - a} \end{aligned}}$$

$$= \frac{1 - e^{2\pi i \Delta n}}{1 - e^{2\pi i \Delta n / N}} \begin{cases} 0 & \text{if } \Delta n \neq 0 \\ ? & \text{if } \Delta n = 0 \end{cases}$$

<u>continuum limit continued</u>

$$f(na) = \sum_{m=-M}^{m=M} c_m \, \underbrace{e^{-\frac{2\pi i m}{N\Delta x} \cdot n\Delta x}}$$

$$\downarrow$$

$$f(x) = \sum_{m=-M}^{m=M} c_m \, e^{-\frac{2\pi i m}{L} x} \longleftarrow \quad x = n\Delta x$$

$M \to \infty$ as $\Delta x \to 0$ (∞ # samplings)

$$c_m = \sum_{n'=0}^{N-1} \underbrace{\frac{1}{N} \cdot \frac{\Delta x}{\Delta x}} \underbrace{f(n'\Delta x)}_{x'} \underbrace{e^{i\frac{2\pi i m}{N\Delta x} n'\Delta x}}$$

$$\frac{\Delta x}{N\Delta x} = \frac{\Delta x}{L} \qquad e^{+\frac{2\pi i m}{L} x'}$$

$$= \int_0^L \frac{dx}{L} f(x') \, e^{2\pi i m x'/L}$$

giving us the expression for the FOURIER SERIES coefficients

$\Rightarrow$ ASSUMPTIONS: f is sufficiently <u>nice</u> for convergence of series

(ultimately this is why histogram basis was used)

see eg. Stone & Goldbart, Appel

FOURIER TRANSFORM

shift to a symmetric interval: $\frac{L}{2} \le x \le \frac{L}{2}$

$$f(x) = \underbrace{\sum_{m=-\infty}^{\infty}}_{\substack{\infty \text{ \# of} \\ \text{discrete} \\ \text{modes}}} c_m \, e^{-\frac{2\pi i m x}{L}}$$

$$c_m = \frac{1}{L}\int_{-L/2}^{L/2} f(x')\, e^{\frac{2\pi i m x'}{L}}\, dx$$

$$\text{AS } L \to \infty \qquad \underbrace{\Delta k = \frac{2\pi}{L} \to 0}_{\substack{\text{mode spacing} \to 0 \\ \text{; } \Delta k \to dk}}$$

$$k_m = \frac{2\pi}{L} m \to dk = \frac{2\pi}{L}\Delta m$$

$$\sum_m \to \int \Delta m = \int \frac{L\, dk}{2\pi}$$ ← there's that silly 2π factor!

BUT $L \to \infty$ ...

$$\boxed{f(x) = \frac{L}{2\pi}\int dk\; c(k)\, e^{-ikx}}$$

$$c(k) = \frac{1}{L}\int_{-L/2}^{L/2} f(x')\, e^{ikx'}\, dx'$$

AH! def $c(k) = \frac{1}{L}\tilde{f}(k)$

cancel L's

$đ = d/2\pi$

$$\boxed{\begin{aligned} f(x) &= \int đk\; \tilde{f}(k)\, e^{-ikx} \\ \tilde{f}(k) &= \int dx\; f(x)\, e^{ikx} \end{aligned}}$$

next: solving ODE w/ this

eg SHO