Short HW 7: **Fourier Space**

Course: Physics 017, *Linear Algebra for Physics* (S22)
Instructor: Prof. Flip Tanedo (flip.tanedo@ucr.edu)
Due by: **Thursday**, May 12

Note that this short assignment is due by class on Thursday. You have only *two days* to do it. This should be quick, I recommend doing it right after class on Tuesday.

Consider the space of functions $f(x)$ on interval from $x = 0$ to $x = L$ with Dirichlet boundary conditions,

$$f(0) = 0 \qquad\qquad f(L) = 0 \; . \tag{0.1}$$

The **Fourier series** is a way of writing these functions in terms of a basis of sines or cosines. The choice of sines or cosines (or both) depends on the boundary conditions. For the above boundary conditions, the following basis of functions/kets/vectors is appropriate:

$$|n\rangle = C_n \sin(k_n x) \qquad\qquad n = 1, 2, 3, \cdots \; . \tag{0.2}$$

We sometimes call these basis vectors *modes* of the Fourier series. We are writing $|n\rangle$ instead of $\mathbf{e}_{(n)}$ or $|e_n\rangle$ for convenience; it's just a basis function. The inner product on this space between two functions $f(x)$ and $g(x)$ is

$$\langle f, g\rangle = \int_0^1 dx\, f(x) g(x) \; . \tag{0.3}$$

1 The basis vectors

The following integrals may be handy:

$$\int_0^1 dx\ \sin^2(\pi x) = \frac{1}{2} \qquad\qquad \int_0^1 dx\ \sin(n\pi x)\sin(m\pi x) = 0 \; , \tag{1.1}$$

where in the second relation we assume $m \neq n$. You can think about how to prove these relations. One clever way is to use $\cos^2\theta + \sin^2\theta = 1$ and make and argument based on periodicity.

1.1 Finding the frequencies

The k_n is related to the angular frequency or momentum of each mode. In order to satisfy the boundary condition $f(L) = 0$, there are restrictions on the possible values of k_n: the frequencies must take on *discrete* values such that $|n\rangle$ has a *node* (it is zero) at $x = L$. In other words, *the modes of a Fourier series have quantized frequencies.*

There are an infinite number of ways to pick k_n to satisfy $f(L) = 0$. The index n enumerates the infinite allowed values of k_n. Derive the expression for the possible values of k_n. You may assume that $k_n > 0$.[1]

Answer: The answer is $k_n = n\pi/L$ for $n = 1, 2, 3, \cdots$. You should explain *how* one arrives at this answer.

[1] You should ask: why not negative values of k_n? This is a great question. Think what do negative frequency basis functions look like? Are those linearly independent from the positive frequency basis functions?

1.2 Normalizing the Fourier basis

Use the normalization condition $\langle n, n \rangle = 1$ to determine the basis prefactors C_n.

ANSWER: The answer is $C_n = \sqrt{2/L}$. You should explain *how* one would arrive at this answer.

1.3 What do the basis vectors look like?

Sketch a graph of the first three basis vectors. Make sure your sketch only goes over the appropriate domain of x.

2 Fourier Series

As is often the case, the key step to representing a function $f(x)$ in its Fourier series is to multiply by one:

$$\mathbb{1} = \sum_{n=1}^{\infty} |n\rangle\langle n| \ , \tag{2.1}$$

where we recall that the bra/row vector $\langle n|$ acts on a function $f(x) = |f\rangle$ as

$$\langle n|f\rangle = \langle n, f\rangle = \int_0^L dx\, C_n \sin(k_n x) f(x) \ . \tag{2.2}$$

The Fourier series representation of a function $f(x)$ in our function space is:

$$|f\rangle = \sum_{n=1}^{\infty} \langle n|f\rangle |n\rangle \ , \tag{2.3}$$

where $\langle n|f\rangle$ are just numbers that are called the Fourier coefficients. To write it out more explicitly, this simply says:

$$f(x) = \sum_{n=1}^{\infty} A_n \times C_n \sin(k_n x) \qquad\qquad A_n = \langle n|f\rangle \ , \tag{2.4}$$

where the A_n are the Fourier coefficients.

Find the first three Fourier coefficients of the function $f(x) = x(x - L)$.[2] You do not have to perform the integral, just write out what the A_n are. The first one, for example, is

$$A_1 = \int_0^L x(x-1) C_1 \sin(k_1 x) \ , \tag{2.5}$$

where you already know what the C_1 and k_1 are. You can always perform these integrals—for example, in *Mathematica* if need be—but usually you can leave them implicit until you actually need a number. After all, these coefficients are just numbers with no functional dependence on x.

[2]We have written $f(x) = x(x - 1)$ rather than $x^2 - x$ as a reminder that this is a function that satisfies the Dirichlet boundary conditions of our vector space.

3 So what?

The Fourier series is convenient because it is a basis of *eigenfunctions of the one-dimensional Laplacian*, $(d/dx)^2$. As we discussed in class, the Laplacian shows up all the times in physics because it is the rotationally symmetric [second] derivative that connects nearby points in space. What is the eigenvalues λ_n of the n^{th} Fourier basis vector, $|n\rangle$, with respect to $(d/dx)^2$?

COMMENT: The eigenvalues of a Hermitian operator like the Laplacian typically carries physical significance. For example, these may be the allowed harmonics on a guitar string, or the Kaluza–Klein modes of a particle in an extra dimension.