SHORT HW 5: **Commutators, Degenerate Eigenvalues**

COURSE: Physics 017, *Linear Algebra for Physics* (S22)
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DUE BY: **Thursday**, April 28

Note that this short assignment is due by class on Thursday. You have only *two days* to do it. This should be quick, I recommend doing it right after class on Tuesday.

The **commutator** of two matrices A and B is

$$[A, B] \equiv AB - BA \ . \tag{0.1}$$

When $[A, B] = 0$, we say that the two matrices commute.

1 Degenerate Eigenvectors

Suppose a matrix has degenerate eigenvalues, $\lambda_i = \lambda_j$ for $i \neq j$. This leads to an ambiguity for the choice of eigenvectors. To see, this consider the symmetric matrix A,

$$A = \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} = R_x \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} R_x^T \ . \tag{1.1}$$

Here R_x is the rotation by angle $\pi/4$ with respect to the x-axis,

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \ . \tag{1.2}$$

Show that the two eigenvectors with eigenvalue 2 are not unique. That is to say, there are an infinite number of ways of writing a pair of eigenvectors of A that (i) have eigenvalue $3/2$, (2) are orthonormal to each other, and (3) are orthonormal to the eigenvector with eigenvalue 1. HINT: THE PRODUCT OF TWO ROTATION IS A ROTATION.

2 Simultaneously Diagonalizable Matrices

Suppose A and B are both symmetric matrices.

2.1 Commutation Relation

Show that if A and B are diagonalized by the same rotation, then the two matrices commute.

$$[A, B] = 0 \tag{2.1}$$

HINT: diagonal matrices commute.

COMMENT: it is also true that if A and B commute, then they can be diagonalized by the same rotation. That is a little more tricky to prove, though.

2.2 Breaking the Degeneracy

Suppose A is defined as in (1.1), and B is define to be

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = R_x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} R_x^T \ . \tag{2.2}$$

Observe that B, like A, has a degeneracy: two eigenvectors have the same eigenvalue, $\lambda = 1$. Thus these two eigenvectors can be rotated into one another and the result are equally valid eigenvectors.

A and B can simultaneously be diagonalized. Let $\lambda_{A,i}$ be the i^{th} eigenvalue of A, and similarly with $\lambda_{B,i}$ for B. We can label the eigenvectors in ket notation:

$$|\lambda_{A,i}, \lambda_{B,i}\rangle \tag{2.3}$$

is the eigenvector of both A and B that satisfies

$$A|\lambda_{A,i}, \lambda_{B,i}\rangle = \lambda_{A,i}|\lambda_{A,i}, \lambda_{B,i}\rangle \qquad\qquad B|\lambda_{A,i}, \lambda_{B,i}\rangle = \lambda_{B,i}|\lambda_{A,i}, \lambda_{B,i}\rangle \ . \tag{2.4}$$

Write out the eigenvectors $|\lambda_{A,i}, \lambda_{B,i}\rangle$ that are the simultaneous eigenvectors of A and B. Explain why there is no more degeneracy in how these eigenvectors are defined, even though both A and B had degeneracies in how to define the eigenvectors.