LAST TIME:

CHK: short HW?

VECTORS

$$\underline{V} = \begin{pmatrix} v^1 \\ \vdots \\ v^N \end{pmatrix}$$ ← UPPER INDEX

main character

sometimes we say v^i "IS A VECTOR"

↳ BUT THIS IS LIKE THE MAGRITTE PAINTING

Ceci n'cest pas un pipe

VECTOR SPACE: SET OF ALL POSSIBLE VECTORS (includes SUM + rescaling of EVERY vector)

$\underline{v} \in V$

MATRICES

$$A = A^i{}_j$$

← horiz & vertical position matters

AGAIN, not REALLY A MATRIX, A COMPONENT OF A MATRIX

RECALL: $f(\underline{V}) = A\underline{V}$ IS THE GENERAL FORM OF A LINEAR TRANSFORMATION

↳ $f(\underline{v}+\underline{w}) = f(\underline{v}) + f(\underline{w})$

$f(\alpha\underline{v}) = \alpha f(\underline{v})$

SUMMATION:

IF: TWO INDICES ARE THE SAME AND ONE IS UPPER, ONE IS LOWER

call this a dummy index

THEN: SUM OVER ALL ALLOWED VALUES OF THIS INDEX (CONTRACT)

i is uncontracted → FREE index

$$A^i{}_j V^j = A^i{}_1 V^1 + A^i{}_2 V^2 + \cdots + A^i{}_N V^N$$

why this is convenient

$$A\underline{V} = \begin{pmatrix} \vdots \\ A^i{}_1 V^1 + A^i{}_2 V^2 + \cdots \\ \vdots \end{pmatrix}$$

i^{th} COMPONENT IS $A^i{}_j V^j$

eg MATRIX MULTIPLICATION $\boxed{(AB)^i{}_j = A^i{}_k B^k{}_j}$ ← IMPORTANT

eg. $\text{Tr}\, A = A^i{}_i$

(eg. PROVE TR IS CYCLIC $\text{Tr}(ABC) = \text{Tr}(BCA)$)

eg $\mathbb{1} = \delta^i{}_j \equiv \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

"KRONECKER DELTA"

$$\delta^i{}_j A^j{}_k = \underbrace{\delta^i{}_1 A^1{}_k + \delta^i{}_2 A^2{}_k + \delta^i{}_3 A^3{}_k + \cdots}$$

every term = 0 except the one where $j = i$

$$= A^i{}_k$$

CAN THINK ABOUT MATRIX AS A MACHINE

$A^i{}_j$

CONTRACTS with any UPPER INDEX (eg vector index)

once j is contracted w/ a vector... the only leftover index is an upper one

→ left w/ a vector!

A: input: an upper index
output: an upper index

eg: takes in a vector $\underline{V}$
outputs a vector $(A\underline{V})$

→ the output is LINEAR in the input

(if you double the input, double output & so forth)

$\boxed{Q}$: WHAT IS INDEX STRUCTURE OF (A^{-1})?

$(A^{-1})^i{}_j \quad (A^{-1})_i{}^j \quad (A^{-1})_j{}^i \quad (A^{-1})_{ij}$ etc.

→ how is (A^{-1}) defined? (MATRIX MULT)

WHAT DO INDICES INDEX? <u>DIRECTIONS</u> [in space]

→ what should the LAWS OF PHYSICS look like w/rt DIRECTIONS?

<u>usually</u> no preferred direction (not even time)

maxwell's eqns never mention an explicit DIRECTION.

SCHRÖDINGER: $i\hbar \frac{d}{dt} \psi = \hat{H} \psi$

$= -\frac{\hbar^2}{2m} \nabla^2 \psi$

depends on direction symmetrically in all directions by summing

$\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2$

<u>CONTRACTING</u> $\boxed{\sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i}} \stackrel{?}{=} \left(\frac{\partial}{\partial x}\right)_i \left(\frac{\partial}{\partial x}\right)^i$

<u>nb</u>: not symmetric in time (eg $t \to -t$)

$\partial_x^2 + \partial_y^2 + \partial_z^2 = \nabla^2$ has "no free indices"

ROTATIONALLY SYMMETRIC
∇^2 MEANS THE SAME THING EVEN IF WE <u>ROTATE</u> OUR COORDINATE SYSTEM.

(compare to just ∂_y or ∂_y^2)

So if you expect the universe to be <u>ROTATIONALLY</u> symmetric, then you probably want to write the phy in EQUATIONS w/ no free indices...

HINT of things to come

INDICES tell US HOW OBJECTS TRANSFORM (eg WRT ROTATIONS & generalizations of ROTATIONS)

eg LORENTZ TRANSFORMATIONS

[to do: LOWER INDEX, METRIC, ROT AS $\langle , \rangle$ PRESERVING]

MORE OBJECTS

RELATIVITY — $|v\rangle$; QUANTUM

$\underline{v}$ → VECTOR / CONTRAVARIANT VECTOR / KET v^i

$\underset{\sim}{v}$ → $\boxed{\text{ROW VECTOR / COVARIANT VECTOR / BRA} \quad v_i}$

also: "1-FORM" — $\langle v|$ — LOWER INDEX

so far: totally independent ideas ← $\underset{\sim}{v} \in V^*$ SPACE OF ROW VECTORS

BUT: we know ROW VEC HAS LOWER INDEX

↳ & so can contract an upper index

eg: ROW VEC is a linear function: $V \to \#$ (eg $\mathbb{R}$)

a row vector $\underset{\sim}{w}$ can be fed a vector $\underline{V}$ and spit out a # by contraction

$$\underset{\sim}{w}\,\underline{V} = \underbrace{w_i V^i}_{\text{no ambiguity about vec/row vec}} = w_1 V^1 + w_2 V^2 + \dots$$

Q: is this a dot product? NO

eg: $\underset{\sim}{w}: A \to \underset{\sim}{w}A = w_i A^i{}_j = (wA)_j$

eg $\underset{\sim}{w} = (3\ 7)$ $\qquad \underset{\sim}{w}A = (3+14,\ 6+7)$

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad = (17, 13)$

nb: $\underset{\sim}{w}$ & $\underline{V}$ are both N-comp arrays, but "totally different"

BESTIARY SO FAR

vectors	v^i	can contract these
ROW VECTORS	w_j	eg $\underbrace{w_i A^i{}_j v^j}_{\text{no free indices, just a \#}} = \underset{\sim}{w} A \underline{v}$
MATRICES	$A^i{}_j$	

METRIC: dot product, inner product

last time: × product is weird! it is actually a VOLUME FORM in exterior ALGEBRA — can talk about Stokes' thm

g_{ij} OR $\eta_{\mu\nu}$ in SPECIAL RELATIVITY

REQ: symmetric $g_{ij} = g_{ji}$

$(\mu = (0, 1, 2, 3)$ for $t\ x\ y\ z$

A SPACE W/ A METRIC IS A "metric space"

ADDITIONAL DEFINITION

EUCLIDEAN SPACE: $\mathbb{R}^N$ w/ $g_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

nb not δ^i_j (index structure!)

MINKOWSKI SPACE

$$\eta_{\mu\nu} = \begin{cases} -1 & \text{if } \mu=\nu=0 \\ 1 & \text{if } \mu=\nu\neq 0 \\ 0 & \text{other} \end{cases}$$

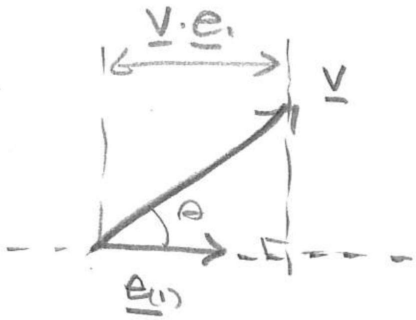
$$\eta_{\mu\nu} = \left(\begin{array}{c|ccc} -1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right)$$

$\boxed{g_{ij}: V \times V \to \#}$ via $\boxed{g_{ij} v^i w^j \equiv \underline{v} \cdot \underline{w} \equiv \langle \underline{v}, \underline{w} \rangle}$

DEFINES DOT PRODUCT

METRIC — for MEASURING LENGTH

eg given a REFERENCE vector that we assume has length "1", tells you how many "lengths" another vector has in that DIRECTION.



REFERENCE VECTOR

$\underline{v} \cdot \underline{e}_{(1)}$ = length of PROJECTION of $\underline{v}$ onto axis of $\underline{e}_{(1)}$ in units of $\underline{e}_{(1)}$

"BASIS" VECTOR

PARENTHESES: careful when counting indices

$\underline{e}_{(1)}, \underline{e}_{(2)}, \ldots$

choose?

in fact: let us define a set of BASIS vectors that define unit length in each DIRECTION

→ CHOOSE THESE TO BE ORTHOGONAL

$$\underline{e}_{(i)} \cdot \underline{e}_{(j)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

then: can write every vector $\underline{V}$ w/RT THIS BASIS:

$$\underline{V} = v^1 \underline{e}_{(1)} + v^2 \underline{e}_{(2)} + v^3 \underline{e}_{(3)} + \cdots$$

just the components [in this basis]

$$\boxed{v^i = \underline{v} \cdot \underline{e}_{(i)}}$$

nb: as opposed to defining $\underline{v}$ by its components, we define components w/rt basis.

LENGTH of A VECTOR $|\underline{v}| = v$ is

$$v^2 = \underline{v}\cdot\underline{v} = v^i \underline{e}_{(i)} \cdot v^j \underline{e}_{(j)}$$

#'s   vectors (BASIS VEC)

$$= v^i v^j \underbrace{(\underline{e}_{(i)}\cdot\underline{e}_{(j)})}_{\text{"}\delta_{ij}\text{"}}$$

← NOT REALLY A TENSOR

$$= (v^1)^2 + (v^2)^2 + \cdots$$

OR, IN BRAKET NOTATION: $\underline{e}_{(i)} = |i\rangle$

$$\underline{e}_{(i)}\cdot\underline{e}_{(j)} = \langle i, j\rangle = \delta_{ij}$$

$$|v\rangle = v^i |i\rangle \quad \text{(SUM)}$$

<u>UNIT VECTOR</u>: $\hat{\underline{v}} = \left(\frac{1}{v}\right)\underline{v}$ s.t. $\hat{\underline{v}}\cdot\hat{\underline{v}} = 1$

<u>ANGLE btwn 2 VECTORS</u>:

$$\hat{\underline{v}}\cdot\hat{\underline{w}} \equiv \cos\theta \iff \underline{v}\cdot\underline{w} = \underset{|\underline{v}|}{v}\,\underset{|\underline{w}|}{w}\cos\theta$$

ROW VECTOR from COLUMN VECTOR

$\langle \underline{v}, \_\_ \rangle$ is a machine/func
plug in vector, get # $\langle \underline{v}, \underline{w} \rangle$

AND it is LINEAR

$\underline{v} \cdot (\underline{w} + \underline{z}) = \underline{v} \cdot \underline{w} + \underline{v} \cdot \underline{z}$ etc.

... this is exactly a
ROW VEC / 1-FORM / COVARIANT ...

metric $\leftrightarrow$ DOT PROD
$\hookrightarrow$ CONVERTS VECTOR $\rightarrow$ ROW VEC

BASIS for ROW VECTORS

$\underset{\sim}{e}^{(i)}$ s.t. $\underset{\sim}{w} = w_i \underset{\sim}{e}^{(i)}$ $\leftarrow$ or $\langle i|$

and $\underbrace{\underset{\sim}{e}^{(i)} [\underline{e}_{(j)}]}_{\text{linear function that EATS vectors}} = \delta^i_j$

$$\underset{\sim}{w} \underline{v} = w_i \underset{\sim}{e}^{(i)} v^j \underline{e}_{(j)}$$

$$= \underbrace{w_i v^j}_{\#'s} \underbrace{\underset{\sim}{e}^{(i)} \underline{e}_{(j)}}_{\delta^i_j} = w_i v^i$$

METRIC AS LOWERING: $g_{ij} \underbrace{\underset{\sim}{e}^{(i)} \underset{\sim}{e}^{(j)}}_{\text{BASIS for}}$

$$\underset{\sim}{w} = g \underline{w} = g_{ij} w^k \underset{\sim}{e}^{(i)} \underbrace{\boxed{\underset{\sim}{e}^{(j)} \underline{e}_{(k)}}}_{\delta^j_k}$$

$w^i \underline{e}_{(i)}$

$$= \boxed{g_{ij} w^j \underset{\sim}{e}^{(i)}} \equiv w_i \underset{\sim}{e}^{(i)}$$

aphorism: any time there is an index, there is probably a continuous symmetry that you care about

can transform by a lot or A little or any in btwn

indices tell you how the objects transform w/RT the symm

eg ROTATION $R^i{}_j$ eg $R^i{}_j = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

c = cos θ, s = sin θ

vector: $V^i \mapsto (RV)^i = R^i{}_j V^j$

PROPERTIES $(R^T R)^i{}_j = \mathbb{1} \Leftrightarrow (R^T) = R^{-1}$

$(R^T)^i{}_j = R^j{}_i$

ROW VECTOR?

we know $\underline{V} \cdot \underline{W}$ is ROT. INVARIANT

$= \underset{\sim}{W} \underline{V} \to \underset{\sim}{W} \boxed{?} (R \underline{V})$

if $\boxed{?} = R^T$, then invariant

claim $\underset{\sim}{W} \to \underset{\sim}{W} R^T$

$\omega_i \to (\underset{\sim}{W} R^T)_i = \omega_j (R^T)^j{}_i$

eg. $\underset{\sim}{W} A \underline{V} \to \underset{\sim}{W} (R^T) \boxed{?} A \boxed{?} R \underline{V}$

TRANSF of METRIC

$A \to R A R^{-1}$

$A^i{}_j \to R^i{}_k A^k{}_l (R^T)^l{}_j$