LAST TIME: GRAM-SCHMIDT

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# MY WEIRD MENTAL MAP OF TRANSFORMATIONS

MATRICES

WEIRD / NOT NICE

$N \times M$ $\quad (V^M \to V^N)$

DIFF SPACES

$N \times N$ $\quad (V \to V)$

"DEGENERATE"

non-invertible PROJECTIONS

sometimes very important

eg $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

YOU LOSE INFORMATION

INVERTIBLE (non degenerate)

ROTATIONS

preserve metric

does "nothing"

NOT ROTATIONS

"does something"

DIAGONAL

SYMMETRIC

IMPORTANT IN QM

other

USUALLY DOESN'T COME UP... LOTS OF WEIRD CASES.

<u>ROTATIONS</u> : change of ref frame
orthonormal basis → orthon. basis

For simplicity, let's drop up/low notation
(notation should be in service to us, not vice versa!)

Metric is the UNIT MATRIX $\mathbb{1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$
in EUCLIDEAN SPACE

ROTATION: $\underline{v} \to R\underline{v}$

<u>Def</u>: "ROTATION PRESERVES METRIC"

$$\langle \underline{v}, \underline{w} \rangle = \langle R\underline{v}, R\underline{w} \rangle = (R\underline{v})^T R\underline{w}$$
$\underline{v}\cdot\underline{w}$
$$= \underline{v}^T \underbrace{R^T R}_{\equiv \mathbb{1}} \underline{w} \stackrel{?}{=} \underline{v}^T \underline{w}$$

<u>ORTHOGONAL</u> MATRIX: $R^T = R^{-1}$
"Rotation" ($\mathbb{C}$ version: unitary)

ACTION OF ROTATIONS:

$g_{ij} v^j$

$$\underline{v} \to R\underline{v} \quad \Rightarrow \quad \underline{v}^T \to (R\underline{v})^T = \underline{v}^T \boxed{R^T}$$
$\underline{v}^T R^{-1}$

Then objs w/ no indices are invt:

$$\underbrace{\underline{v}^T A \underline{w}}_{} \to (\underline{v}^T R^T) \boxed{A'}^{\,?} R\underline{w} = \underline{v}^T R^T A' R \underline{w}$$

eg $\langle \underline{v}, A\underline{w} \rangle$

invariant if $A' = RAR^T$
b/c:
$\underline{v}^T A \underline{w} \Rightarrow \underline{v}^T \underbrace{R^T R}_{\mathbb{1}} A \underbrace{R^T R}_{\mathbb{1}} \underline{w}$

vs $\langle A\underline{v}, \underline{w} \rangle = \underline{v}^T A^T \underline{w}$

So: $\langle \underline{v}, A\underline{w} \rangle = \langle A\underline{v}, \underline{w} \rangle$ for <u>SYMMETRIC</u> A btw: index intuition

UPPER INDEX: $v^i \to R^i{}_j v^j$ $\quad (\underline{v} \to R\underline{v})$

LOWER INDEX: $w_i \to w_j (R^{-1})^j{}_i$

MATRIX: $A^i{}_j \to \underbrace{(R^i{}_k)(R^{-1})^l{}_j A^k{}_l}$

& similarly for tensors.

$$R^i{}_k A^k{}_l (R^{-1})^l{}_j$$

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IOW $\boxed{A \to RAR^T}$

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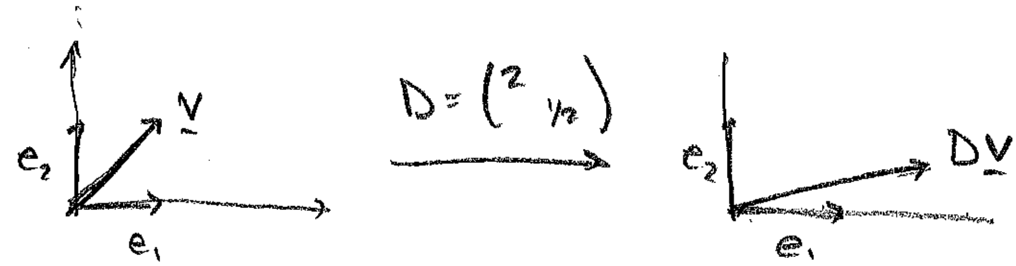
NICEST of NON-ROTATION MATRICES:

DIAGONAL MATRIX

$$D = \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}}_{\text{EIGENVALUES}} \longleftarrow D\begin{pmatrix} v^1 \\ v^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 v^1 \\ \lambda_2 v^2 \\ \vdots \end{pmatrix}$$

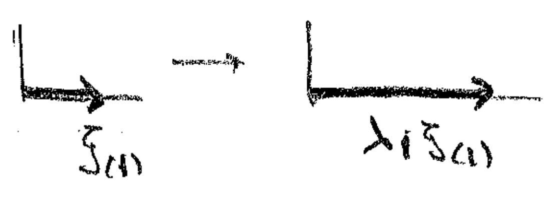
"proper" / "essence"

just rescaling of EA COMP BY EIGENVALS'




in fact: the canonical/std basis is esp. useful:

$$\underbrace{\xi_{(i)}}_{\text{EIGENVECTOR}} = e_{(i)} \quad \Rightarrow \quad D\xi_{(i)} = \lambda_i \xi_{(i)}$$



IN FACT, BECAUSE EIGENVECTORS of A DIAG MATRIX ARE JUST THE STANDARD BASIS,

↳ the EIGENVECTORS ARE A BASIS (obvious)

$$\underline{V} = V^1 \underline{\xi}_{(1)} + V^2 \xi_{(2)} + \cdots$$

$$D\underline{V} = V^1 D\underline{\xi}_{(1)} + V^2 D\xi_{(2)} + \cdots$$

$$= V^1 \lambda_1 \xi_{(1)} + V^2 \lambda_2 \xi_{(2)} + \cdots$$

$$= \begin{pmatrix} \lambda_1 V_1 \\ \vdots \end{pmatrix}$$ AS WE SAID BEFORE

WOULDN'T IT BE NICE IF THIS WERE TRUE MORE GENERALLY — SAY, for SOME CLASS OF TRANSFORMATIONS OF PHYSICAL SIGNIFICANCE — THAT THERE IS A SPECIAL BASIS THAT ONE COULD $\underbrace{\text{ROTATE}}_{\text{NICE}}$ TO FOR WHICH THE TRANSF IS A DIAGONAL MATRIX?

↳ well, we can construct THIS CLASS of TRANSFORMATIONS: JUST ROTATE THE DIAGONAL MATRIX.

$$D e_{(i)} \rightarrow \underbrace{(RDR^T)}_{D'} \underbrace{(R e_{(i)})}_{\xi'_{(i)}}$$

new MATRIX NOT DIAGONAL

NEW BASIS, no longer standard BASIS

then just REGROUP:

$$D\underline{e}_{(i)} \to D'\xi_{(i)} = RD\underbrace{(R^TR)}_{\mathbb{1}}\underline{e}_{(i)}$$

$$= RD\underline{e}_{(i)}$$

$$= R\,\lambda_i\,\underline{e}_{(i)}$$

$$= \lambda_i\,R\,\underline{e}_{(i)}$$

$$= \lambda_i\,\xi'_{(i)} \leftarrow \text{eigenvector of } D'$$

so: $\underset{\substack{\uparrow\\\text{not diag}}}{D'}\ \underset{\substack{\uparrow\\\text{not standard BASIS}}}{\xi'_{(i)}} = \underbrace{\lambda_i\,\xi'_{(i)}}_{\text{still RESCALING BY EIGENVALUES (SAME EIGENVALUES AS IN DIAG FORM)}}$

BUT WE KNOW IT AS A ROT of STD BASIS

these #'s are "INTRINSIC" to the transFORMATION

so: DIAG MAT CAN BE CONVERTED INTO NOT-DIAG MATRICES THAT INHERIT THE NICENESS of DIAG MATRIX, esp IN <u>eigenvector</u> BASIS

nb: b/c $\xi'_{(i)}$ IS JUST A ROTATION of $\underline{e}_{(i)}$, it IS A GOOD (ORTHONORMAL) BASIS

<u>BUT</u>: not all non-diag matrices can be <u>DIAGONALIZED</u> by a rotation.

<u>want</u>: what class of matrices may be <u>decomposed</u> as

$$M = RDR^T \quad ?$$

What properties does this imply?

$$M^T = (RDR^T)^T$$
$$= (R^T)^T D^T R^T$$
$$= RDR^T = M$$

B/C DIAGONAL

obvious in index not if we lower all matrices ... tricky otherwise

⇒ ie: DIAGONALIZABLE ⟶ $\boxed{\text{SYMMETRIC}}$

turns out: this is basically it. symmetric $\mathbb{R}$ matrix may be decomposed into $RDR^T$

then life is good.

Given <u>symmetric matrix</u> $S$, find rotation $\underline{R}$

↪ $\xi_{(i)} = R\underline{e}_{(i)}$

BASIS WHERE MAT IS DIAGONAL

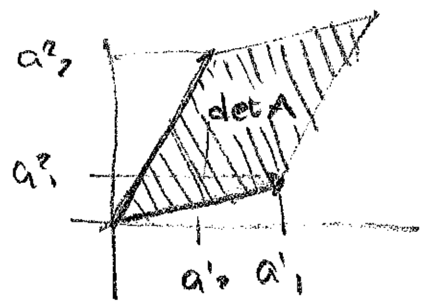
$$S = RDR^T \rightarrow R^T S R = \text{diag}(\lambda_1, \lambda_2, \ldots)$$

## DETERMINANTS & TRACES ← PROPERTIES of MATRICES.

Det $A$ = volume of parallelpiped w/ edges given by columns of $A$ ... up to sign (oriented)

$$A = \begin{pmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \\ | & | & & | \end{pmatrix}$$

eg $\underline{2\times 2}$: $A = \begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}$




Row, x product is more of this than a vec.

in 2D: $a^1{}_1 a^2{}_2 - a^1{}_2 a^2{}_1$ ← every upper index appears exactly once; every lower index appears exactly once

$$= \sum_i \varepsilon^{ij} a^1{}_i a^2{}_j \qquad \text{(SUM)}$$

$\varepsilon^{12} = 1, \quad \varepsilon^{ij} = -\varepsilon^{ji}$

Levi-Civita tensor

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$= \frac{1}{2!} \varepsilon^{ij} \varepsilon_{kl} a^k{}_i a^l{}_j$$

IN GENERAL DIM:

$$\det A = \varepsilon^{i_1 \cdots i_n} a^1{}_{i_1} a^2{}_{i_2} \cdots a^n{}_{i_n}$$

$$= \frac{1}{n!} \varepsilon^{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n} a^{j_1}{}_{i_1} a^{j_2}{}_{i_2} \cdots a^{j_n}{}_{i_n}$$

accounts for $n!$ terms added REDUNDANTLY

FACT: $\det(AB) = \det A \, \det B$
cf. AXLER §10.B
↳ we can do this later.

$$\det D = \prod_k \lambda_k$$

$$\det D' = \underbrace{\det R \, \det R^{-1}}_{= 1 \times 1} \det D = \boxed{\prod_k \lambda_k}$$

intrinsic PROP. of MATRIX

TRACE: $\operatorname{tr} A = A^i{}_i = A^1{}_1 + A^2{}_2 + \cdots$

UNDER ROT: 
$$\operatorname{tr}(RAR^T) = R^i{}_j A^j{}_k (R^T)^k{}_i$$

$$= \underbrace{(R^T)^k{}_i R^i{}_j}_{\delta^k{}_j} A^j{}_k$$

$$= A^j{}_j = \boxed{\operatorname{tr} A}$$

ALSO INTRINSIC PROP of MATRIX

## CHARACTERISTIC EQN

"characteristic = eigenvalue"

How to find $\lambda_i$?

first: $D - \lambda \mathbb{1} = \begin{pmatrix} \lambda_1 - \lambda & & \\ & \lambda_2 - \lambda & \\ & & \ddots \end{pmatrix}$

s.t. $\det(D - \lambda \mathbb{1}) = \prod_i (\lambda_i - \lambda)$

if we set $\det(D - \lambda \mathbb{1}) = 0$,
then solutions are $\lambda = \lambda_i$ for ea $\lambda_i$

↳ now an algebraic polynomial eqn.

then transform to other basis:

$$(D - \lambda \mathbb{1}) \longrightarrow R(D - \lambda \mathbb{1})R^T$$

$$= RDR^T - \lambda RR^T$$

$$= D' - \lambda \mathbb{1}$$

so this still works:

$\det(D' - \lambda \mathbb{1}) = 0 \longrightarrow$ polynomial eqn whose solutions are $\lambda = \lambda_i$

[ eg hw: use det & tr to invert 2x2
& note matrix inverses are easy
in eigenbasis ]