P017 LEC 10 | TANEDO 322 | 26 APRIL '22

TODAY: HW discussion

→ SHORT HW: DEGENERATE EIGVAL

eg: $R = \begin{pmatrix} | & | & | \\ \xi_1 & \xi_2 & \xi_3 \\ | & | & | \end{pmatrix}$ $\qquad A = R \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} R^T$

$$R^T = \begin{pmatrix} - & \xi_1 & - \\ - & \xi_2 & - \\ - & \xi_3 & - \end{pmatrix}$$

thus: $A \xi_1 = R \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \underline{R^T \xi_1}$

$$\underline{R^T \xi_1} = R^T \begin{pmatrix} \boxed{\begin{matrix} | \\ \xi_1 \\ | \end{matrix}} & \begin{matrix} | \\ \xi_2 \\ | \end{matrix} & \begin{matrix} | \\ \xi_3 \\ | \end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} \xi_1 \cdot \xi_1 \\ \xi_1 \cdot \xi_2 \\ \xi_1 \cdot \xi_3 \end{pmatrix} \leftarrow \begin{pmatrix} - & \xi_1 & - \\ - & \xi_2 & - \\ - & \xi_3 & - \end{pmatrix} \begin{pmatrix} | \\ \xi_1 \\ | \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= R \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \lambda_1 R \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \lambda_1 \begin{pmatrix} | & | & | \\ \xi_1 & \xi_2 & \xi_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} \xi_{(1)}^1 \\ \xi_{(1)}^2 \\ \xi_{(1)}^3 \end{pmatrix}$$

$$= \lambda_1 \xi_1 \quad \checkmark$$

$\boxed{\text{SOL}}$ W/RT $A = R \left(\begin{array}{c|cc} \lambda_1 & & \\ \hline & \lambda_2 & \\ & & \lambda_2 \end{array} \right) R^T$

$B = R \left(\begin{array}{cc|c} \rho_1 & & \\ & \rho_1 & \\ \hline & & \rho_2 \end{array} \right) R^T$

SAME ROT, CAN PICK SAME EIGENVECTORS.

$\rightarrow$ NB IF ONLY A, then you can ROTATE to diff BASIS

$$\underbrace{R \left(\begin{array}{c|cc} 1 & & \\ \hline & c & s \\ & -s & c \end{array} \right)}_{R'} \underbrace{\left(\begin{array}{c|cc} \lambda_1 & & \\ \hline & \lambda_2 & \\ & & \lambda_2 \end{array} \right) \left(\begin{array}{c|cc} 1 & & \\ \hline & c & s \\ & -s & c \end{array} \right)^T}_{\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)} R$$

ξ_2' & ξ_3' differ from ξ_2, ξ_3

W/RT ξ_1, ξ_2, ξ_3 :

$|\lambda_1, \rho_1\rangle \quad |\lambda_2, \rho_1\rangle \quad |\lambda_2, \rho_1\rangle$

uniquely defined.

$A|\lambda_1, \rho_1\rangle = \lambda_1 |\lambda_1, \rho_1\rangle \leftrightarrow A\xi_1 = \lambda_1 \xi_1$

$B|\lambda_1, \rho_1\rangle = \rho_1 |\lambda_1, \rho_1\rangle \leftrightarrow B\xi_1 = \rho_1 \xi_1$

do example: SPRING THEORY, P7 of LEC 9 notes

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① VECTOR SPACES:

same as $\mathbb{R}$, but: • coeff can be $\mathbb{C}$
• metric contains conjugate
• $T \rightarrow \dagger$
• Rotation (ORTHOG) $\rightarrow$ UNITARY $U^\dagger U = \mathbb{1}$
• Symmetric $\rightarrow$ Hermitian

$\mathbb{R}$: $\langle \underline{w}, \underline{v} \rangle = \underline{w} \cdot \underline{v} = \underline{w}^T \underline{v}$

$\mathbb{C}$: $\langle \underline{w}, \underline{v} \rangle = \qquad = \underline{w}^\dagger \underline{v}$

$w^\dagger = w^{*T}$

$\mathbb{R}$: ROTATIONS: $\langle R\underline{w}, R\underline{v} \rangle = \langle \underline{w}, \underline{v} \rangle$

$\underline{w}^T \underbrace{R^T R}\, \underline{v} = \underline{w}^T \underline{v}$

$\boxed{R^T R = \mathbb{1}}$

$\mathbb{C}$: UNITARY TRANSF.

$\langle U\underline{w}, U\underline{v} \rangle = \langle \underline{w}, \underline{v} \rangle$

$\underline{w}^\dagger \underbrace{U^\dagger U}\, \underline{v} = \underline{w}^\dagger \underline{v}$

$\boxed{U^\dagger U = \mathbb{1}}$ unitarity condition

$\mathbb{R}$: A IS SYMMETRIC: $\langle A\underline{w}, \underline{v}\rangle = \langle \underline{w}, A\underline{v}\rangle$

$$\underbrace{\underline{w}^T A^T \underline{v} = \underline{w}^T A \underline{v}}$$

$$A^T = A$$

$\mathbb{C}$: A IS HERMITIAN: $\langle A\underline{w}, \underline{v}\rangle = \langle \underline{w}, A\underline{v}\rangle$

def: ADJOINT OF A: $A^\dagger$ s.t.

$\langle A\underline{v}, \underline{w}\rangle = \langle \underline{v}, A^\dagger \underline{w}\rangle$

$$\underbrace{\underline{w}^\dagger A^\dagger \underline{v} = \underline{w}^\dagger A \underline{v}}$$

$$\boxed{A^\dagger = A}$$

HERMITIAN: SELF-ADJOINT

$\mathbb{R}$: DIAGONAL MATRIX: $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$ $\boxed{\lambda_i \in \mathbb{R}}$

OBVIOUSLY SYM

$\mathbb{R}$ EIGENVALS FOR SYMM. MATRICES

$\mathbb{C}$: DIAGONAL, HERMITIAN MATRIX: $\boxed{\lambda_i \in \mathbb{R}}$

not obvious must impose

$D^\dagger = \begin{pmatrix} \lambda_1^* & & \\ & \ddots & \\ & & \lambda_N^* \end{pmatrix}$

Why? WANT HERMITIAN MATRICES

$$(UDU^\dagger)^\dagger = UD^\dagger U^\dagger$$

NEED $D = D^\dagger$

$\Rightarrow$ EIGENVALUES OF HERMITIAN MATRICES ARE $\mathbb{R}$.

$\rightarrow$ physical observables are $\mathbb{R}$!

IN QM: OBSERVABLES ARE HERMITIAN OPS, the OBSERVED VALUE IS EIGENVAL.

RECALL: EIGENVECS ARE ORTHOGONAL
(normalized eigenbasis) are (ORTHONORMAL!

$$\underbrace{\begin{pmatrix} \text{—} \xi_1 \text{—} \\ \text{—} \xi_2 \text{—} \\ \vdots \\ \text{—} \xi_N \text{—} \end{pmatrix}}_{R^T} \underbrace{\begin{pmatrix} | & | & & | \\ \xi_1 & \xi_2 & \cdots & \xi_N \\ | & | & & | \end{pmatrix}}_{R} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} \xi_1 \cdot \xi_1 & \xi_1 \cdot \xi_2 & \cdots \\ \xi_2 \cdot \xi_1 & \xi_2 \cdot \xi_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

ea element is $\langle \xi_i, \xi_j \rangle$

$\Rightarrow$ the $\xi_{(i)}$ ARE ORTHONORMAL!

SIMILARLY: $U^\dagger U = \mathbb{1}$

$$\begin{pmatrix} \text{—} \xi_1^\dagger \text{—} \\ \text{—} \xi_2^\dagger \text{—} \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \xi_1^\dagger \xi_1 & \xi_1^\dagger \xi_1 \\ \xi_2^\dagger \xi_1 & \xi_2^\dagger \xi_2 & \cdots \end{pmatrix}$$

ea element is $\langle \xi_i, \xi_j \rangle$

$\Rightarrow$ so EIGENVECTORS of HERMITIAN MATRIX ARE ORTHONORMAL w/ $\mathbb{R}$ eigenvalues