SHORT HW : FOURIER

the "USUAL" PROBLEM IN E&M:
↳ PHYSICAL CONFIGURATION → $\boxed{\text{BOUNDARY CONDITIONS}}$

(*) { range of x $\quad a \leq x \leq b$
& behavior of functions @ boundaries

PROBLEM: solution to some DIFF EQ (**)
(almost always LAPLACIAN)

↳ "find the ELECTROSTATIC POTENTIAL"

subject to B.C.

APPROACH: (*) DEFINES A FUNCTION SPACE
(**) DEFINES AN OPERATOR

⇒ find a BASIS of functions that are EIGENVECTORS $|n\rangle$ of the operator.
ie look up

- ANY LINEAR COMBINATION WILL SATISFY the DIFF EQ.
- FIND CONDITIONS ON
  1. any PARAMETERS LEFT OVER (eg wave #)
  2. coefficients (COMPONENTS)

eg. quantization

to satisfy B.C.

⇒ the QUANTIZATION in QM is from B.C.
(see LONG HW)

all #'s can now be $\mathbb{C}$

COMPLEX SPACES

"ROTATIONS" : $R^T R = \mathbb{1}$ $\qquad U^\dagger U = \mathbb{1}$ (UNITARY)

$$A^\dagger = A^{*T}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

"SYMMETRIC" : $A^T = A$ $\qquad A^\dagger = A$ (HERMITIAN)

eg $\begin{pmatrix} & i \\ -i & \end{pmatrix}^\dagger = \begin{pmatrix} & i \\ -i & \end{pmatrix}$

can be diagonalized : $A = U \hat{A} U^\dagger$

$$\hat{A} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

nb: $A^\dagger = (U^\dagger)^\dagger \hat{A}^\dagger U^\dagger$
$\qquad = U \hat{A}^\dagger U^\dagger$

HERMITIAN : $\hat{A}^\dagger = \hat{A}$

if A is HERMITIAN,
then EIGENVALUES ARE $\mathbb{R}$! $\lambda_i^* = \lambda_i$

→ still true: COLUMNS of U ARE EIGENVECTORS.

FORMAL DEF OF ADJOINT, $\dagger$

$$\langle Aw, v \rangle \equiv \langle w, A^\dagger v \rangle$$

for $\mathbb{R}$, $\dagger = T$

$\mathbb{R}$. eg. $\langle Aw, v \rangle = (w_1 \;\; w_2) A^T \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

$$= \langle w, A^T v \rangle$$

so HERMITIAN OPERATOR/MATRIX SATISFIES

$$\langle Aw, v \rangle = \langle w, Av \rangle \qquad A^\dagger = A$$

$$\overset{|||}{\langle w, A^\dagger v \rangle}$$

① function space

$$\langle f, g \rangle = \underbrace{\int dx}_{\substack{\text{int over} \\ \text{SPACE} \to \text{however} \\ \text{it is def in your} \\ \text{problem!} \\ \text{may be 2,3,4 DIM...}}} \; f^*(x) \, g(x)$$

why $*$? B/C WANT $\underbrace{\langle f, f \rangle}_{\int dx \, \underbrace{|f(x)|^2}_{\text{positive def.}}} = \|f\|^2 > 0$

(is the DERIVATIVE HERMITIAN ?)

ADJOINTS of DIFFERENTIAL OPERATORS

$$\langle f, \underset{\text{eg } \partial_x}{\mathcal{O}} g\rangle = \int dx\, f^*(\mathcal{O}g)$$

$$\stackrel{?}{=} \langle \mathcal{O} f, g\rangle = \int dx\, (\mathcal{O}f)^* g$$

$$\int_a^b dx\; f^* \,\partial_x g = -\int_a^b dx\, (\partial_x f)^* g + \underbrace{f^* g\Big]_a^b}_{\substack{=0 \\ \text{BY B/C}}}$$

ASSUME DIRICHLET

$\partial_x = \partial/\partial x$

$\partial_x^* = \partial/\partial x$

$$= (-)\langle \mathcal{O} f, g\rangle$$

∂_x is <u>not</u> hermitian.

2. BUT $-i\partial_x$ is. ← RELATED TO MOMENTUM

eg : $|k\rangle \sim e^{ikx}$

then $\hat{p}|k\rangle = -i(ik)|k\rangle = k|k\rangle$

$\hat{p} = -i\partial_x$ momentum operator

eigenvalue is the wave #

EIGENVALUES of HERMITIAN ops ARE $\mathbb{R}$

in QM: possible observable

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what is a bra $\langle f|$?

↳ linear function(al) that
acts on kets (functions)
& spits out #'s in a linear way

$$\langle f | \left(\alpha|g\rangle + \beta|h\rangle \right) = \alpha\underbrace{\langle f|g\rangle}_{\equiv\langle f, g\rangle} + \beta\underbrace{\langle f|h\rangle}_{\equiv\langle f, h\rangle}$$

nb inner product is not
technically a bra-ket...
but defines how to
"create" a ket.

ie, LOWERING AN INDEX