<u>LAST WEEK</u>: EIGENVECTORS for $\mathbb{R}$ VECTOR SPACES

<u>TODAY</u>: commutation relations
degenerate eigenvalues

in QM, the commutator of 2 matrices (OPERATORS) IS SIGNIFICANT. ↳ $[A,B] = AB - BA$

Where does this come from?

eg. Rotations do not commute

↳ the way in which they do not commute tells us about their mathematical relation to each other

do example w/ an asymmetric book

BUT: ∃ an ∞ # of ROTATIONS
→ BUT A FINITE # of rotation axes

the mathematical structure of "rotations" is encoded in the comm. relation wrt infinitesimal rotations.

$R(\theta) = e^{i\theta T}$ eg $T = \begin{pmatrix} & -i \\ i & \end{pmatrix}$ for 2D ROT.

$R_{x,y,z}(\theta) = e^{i\theta T_{x,y,z}}$ ← "GENERATOR"
→ HERMITIAN: $T^\dagger = T$

$$[R_x(\phi), R_z(\theta)] \underset{\theta,\phi \ll 1}{\approx} (1+i\theta T_x)(1+i\phi T_z) - (1+i\phi T_z)(1+i\theta T_x)$$

$$= -\theta\phi\,[T_x, T_z]$$

shows up in GROUP thy.

if $[T_x T_z] = 0$, then the rotations commute

FACT: if 2 SYMMETRIC MATRICES ARE DIAGONALIZED BY THE SAME ROTATIONS, THEN THEY COMMUTE

$$A = R\hat{A}R^T \qquad \Rightarrow \quad [A,B] = 0$$
$$B = R\hat{B}R^T$$

w/ $\hat{A}, \hat{B}$ DIAGONAL

why: $[A,B] = R\underbrace{[\hat{A},\hat{B}]}_{=0 \text{ for DIAGONAL MATRICES}}R^T$

turns out: converse is true as well (HOW WE USUALLY USE THIS)

$[A,B]=0 \Rightarrow$ can diagonalize BOTH A & B by doing one rotation

EIGENVECTORS SATISFY

$$A\,\xi_{(i,j)} = \lambda_i\,\xi_{(i,j)}$$
$$B\,\xi_{(i,j)} = \rho_j\,\xi_{(i,j)}$$

MORE CONVENIENT: LABEL A KET BY ITS EIGENVALUES W/RT A & B

$$\xi_{(i,j)} = |\lambda_i, \rho_j\rangle$$

s.t. $A|\lambda_i,\rho_j\rangle = \lambda_i|\lambda_i,\rho_j\rangle$

$B|\lambda_i,\rho_j\rangle = \rho_j|\lambda_i,\rho_j\rangle$

if this vector is a QUANTUM STATE, it is characterized by its eigenvalues w/rt COMMUTING OPERATORS.

DEGENERATE EIGENVALUES

$$A = R\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}R^T = \lambda RR^T = \lambda \mathbb{1}$$

ANY ORTHONORMAL BASIS IS A BASIS OF EIGENVECTORS; EACH W/ EIGENVALUE λ

eg $\underline{e}_{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad e_{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$

$$\underline{e}'_{(1)} = \begin{pmatrix}c\\-s\end{pmatrix} \quad \underline{e}'_{(2)} = \begin{pmatrix}c\\s\end{pmatrix}$$

RECALL: COLUMNS OF ROT MATRIX TO DIAGONALIZE A ARE THE EIGENVECTORS.

ANY ROTATION DIAGONALIZES A.

SAME IS TRUE FOR SUBSPACES:

$$A = R\left(\begin{array}{c|cc}\lambda_1 & & \\ \hline & \lambda_2 & \\ & & \lambda_2\end{array}\right)R^T$$

$$= R\tilde{R}\left(\begin{array}{c|cc}\lambda_1 & & \\ \hline & \lambda_2 & \\ & & \lambda_2\end{array}\right)\tilde{R}^T R^T$$

$$\tilde{R} = \left(\begin{array}{c|cc}1 & & \\ \hline & c & s \\ & -s & c\end{array}\right)$$

so if R diagonalizes A, so does $(R\tilde{R})$.
the columns of R & $(R\tilde{R})$ ARE EQUALLY VALID EIGENVECTORS.

$\hookrightarrow$ DEGENERATE EIGENVALUES $\rightarrow$ REDUNDANT OPTIONS FOR EIGENVECTORS

BREAKING the DEGENERACY if YOU HAVE A COMMUTING MATRIX to SIMULTANEOUSLY DIAGONALIZE.

EASY CASE:

$$A = R\left(\begin{array}{c|cc} \lambda_1 & & \\ \hline & \lambda_2 & \\ & & \lambda_2 \end{array}\right)R^T$$

$$C = R\begin{pmatrix} \rho_1 & & \\ & \rho_2 & \\ & & \rho_3 \end{pmatrix}R^T$$

then: once you diagonalize C, no more ambiguity.

$$R = \underbrace{\begin{pmatrix} | & | & | \\ \xi_1 & \xi_2 & \xi_3 \\ | & | & | \end{pmatrix}}_{\text{eigenvectors of } C \text{ AND } A}$$

$$\xi_{i} = |\lambda_i ; \rho_i\rangle$$

What happened to REDUNDANCY IN A? R vs. $(R\tilde{R})$?

$\hookrightarrow$ I CAN ROTATE $\underbrace{R\left(\tilde{R}\left(\begin{array}{c|cc} \lambda_1 & & \\ \hline & \lambda_2 & \\ & & \lambda_2 \end{array}\right)\tilde{R}^T\right)R^T}_{A \to A}$

BUT: this does not work on C

$$\underbrace{\tilde{R}\left(\begin{array}{c|cc} \rho_1 & & \\ \hline & \rho_2 & \\ & & \rho_3 \end{array}\right)\tilde{R}}_{\text{will be some off diagonal mess.}} \neq \left(\begin{array}{c|cc} \rho_1 & & \\ \hline & \rho_2 & \\ & & \rho_3 \end{array}\right)$$

in fact, two degenerate sym. matrices that commute w/ different degeneracies can totally lift (remove) the degeneracy

eg $$A = R\begin{pmatrix} \lambda_1 & \\ & \lambda_2 & \\ & & \lambda_2 \end{pmatrix}R^T$$

$$B = R\begin{pmatrix} \rho_1 & & \\ & \rho_1 & \\ & & \rho_2 \end{pmatrix}R^T$$

$$A \text{ HAS } R\underbrace{\tilde{R}\hat{A}\tilde{R}^T}_{=\hat{A}}R$$

$$B \text{ HAS } R\underbrace{R'\hat{B}R'^T}_{=\hat{B}}R \quad \text{when } R' = \begin{pmatrix} c & s & \\ -s & c & \\ & & 1 \end{pmatrix}$$

BUT: A is not degen w/rt R'; B —— " —— w/rt $\tilde{R}$ } must pick $\tilde{R} = R' = \mathbb{1}$ to diagonalize both

$\Rightarrow$ no ambiguity left in the "CORRECT" EIGENBASIS.

IN QM: HYDROGEN ATOM: 3 commuting matrices whose eigenvalues carry physical significance

$\boxed{H, \; L^2, \; L_z}$ $\longrightarrow$ EIGENBASIS: $\underbrace{|E_n, \ell, m\rangle}$ in QM these take DISCRETE VALUES

H: ENERGY (hamiltonian)

L^2: total ANGULAR MOMENTUM, squared

L_z: ANGULAR MOMENTUM IN Z-DIR

eg $$L^2 |E_n, \ell, m\rangle = \ell(\ell+1)\hbar\,|\ldots\rangle$$

$$L_z |\longrightarrow\rangle = m\hbar\,|\ldots\rangle$$

$$H\,|\longrightarrow\rangle = E_n |\longrightarrow\rangle$$

eg. in GROUND STATE, $n=1$

CAN HAVE $\underline{l=1}$ state

total angular momentum $=1$
DEGENERACY: 2 PARTICLES w/ SPIN ½
can be configured
in many ways to give $l=1$

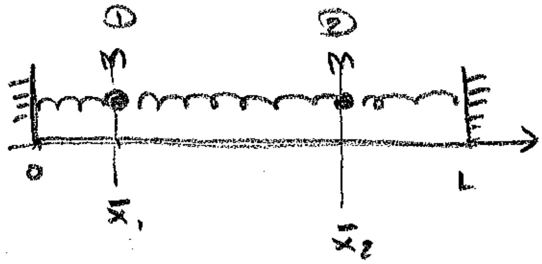
the L_z EIGENVALUES BREAK the DEGENERACY

$\uparrow\uparrow$	$l=1$	$m=1$
$\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)$	$l=1$	$m=0$
$\downarrow\downarrow$	$l=1$	$m=-1$

eg PROTON, e SPIN
UP OR DOWN

GIVES a fixed basis to describe the HYDROGEN states w/ DEGENERATE l.

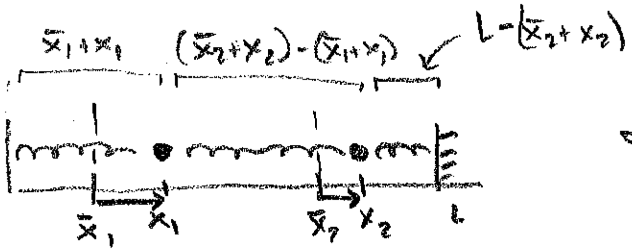
"SPRING THEORY" SHANKAR ex 1.6



EQUILIBRIUM
HOOKE'S LAW: $m\ddot{x} = -kx$

on ① $0 = -k\bar{x}_1 + k(\bar{x}_2 - \bar{x}_1)$

② $0 = -k(\bar{x}_2 - \bar{x}_1) + k(L - \bar{x}_2)$



out of EQUILIBRIUM

x_1 & x_2 MEAS. RELATIVE TO EQUILIBRIUM → DISPLACEMENTS

$$m\ddot{x}_1 = -k(\bar{x}_1 + x_1) + k\left((\bar{x}_2 + x_2) - (\bar{x}_1 + x_1)\right)$$

$$= -k\bar{x}_1 + k(\bar{x}_2 - \bar{x}_1) \quad \leftarrow = 0$$

$$-kx_1 + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k\left((\bar{x}_2 + x_2) - (\bar{x}_1 + x_1)\right) + k\left(L - (\bar{x}_2 + x_2)\right)$$

$$= -k(\bar{x}_2 - \bar{x}_1) + k(L - \bar{x}_2) \quad \longleftarrow = 0$$

$$-k(x_2 - x_1) - kx_2$$

COUPLED 2ND O. DIFF EQ.

$$\ddot{x}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2$$

$$= \ \underline{\ddot{x}} = \frac{k}{m}\underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}_{\text{SYMMETRIC MATRIX}}\underline{x}$$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

BASIS: $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ "displace 1st MASS BY 1 UNIT"

$|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ " —— 2ND —— 1 UNIT"

EIGEN STUFF

$$\ddot{\underline{x}} = \underbrace{\frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}_{A} \underline{x}$$

$\det(A - \lambda \mathbb{1}) = 0 \rightarrow$ solve for λ_1, λ_2

$(A - \lambda_i \mathbb{1})\, \xi = 0 \rightarrow$ solve for $\xi_{(1)}, \xi_{(2)}$.

write: $\lambda = \frac{k}{m} \bar{\lambda}$

$$\det\left[\frac{k}{m}(A - \bar{\lambda}\mathbb{1})\right] = \left(\frac{k}{m}\right)^2 \det\begin{pmatrix} (-2-\bar{\lambda}) & 1 \\ 1 & (-2-\bar{\lambda}) \end{pmatrix}$$

$$= \left(\frac{k}{m}\right)^2 \underbrace{\left(\bar{\lambda}^2 + 4\lambda + 3\right)}_{(\bar{\lambda}+3)(\bar{\lambda}+1)}$$

$$\Rightarrow \boxed{\lambda_1 = -\frac{k}{m}} \quad \boxed{\lambda_2 = -3\frac{k}{m}}$$

EIGENVAL EQ: $\ddot{\xi}_{(1)} = -\frac{k}{m}\xi_{(1)} \equiv -\omega_1^2 \xi_{(1)}$

$\ddot{\xi}_{(2)} = -3\frac{k}{m}\xi_{(2)} \equiv -\omega_2^2 \xi_{(2)}$

DEF NORMAL FREQ.

EIGENVECTORS :

$$A - \lambda_1 \mathbb{1} = \frac{k}{m}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}}_{-a+b\,=\,0} = 0$$

$$\xi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \langle \xi_1, \xi_1 \rangle = 1$$

$$A - \lambda_2 \mathbb{1} = \frac{k}{m}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}}_{a+b\,=\,0} = 0$$

$$\xi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \langle \xi_2, \xi_2 \rangle = 1$$

NOW WRITE $\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \chi_1(t)\,\xi_{(1)} + \chi_2(t)\,\xi_{(2)}$

$x_1(t)\begin{pmatrix}1\\0\end{pmatrix} + x_2(t)\begin{pmatrix}0\\1\end{pmatrix} = x_1 \underline{e}_1 + x_2 \underline{e}_2$

then: $\ddot{\underline{x}}(t) = \ddot{\chi}_1(t)\,\xi_{(1)} + \ddot{\chi}_2(t)\,\xi_{(2)}$

$$= \begin{pmatrix} \ddot{\chi}_1 \\ \ddot{\chi}_2 \end{pmatrix}_E$$

EIGENBASIS

$$= \begin{pmatrix} -\omega_1^2 & \\ & -\omega_2^2 \end{pmatrix}_E \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_E$$

$$\ddot{\chi}_i = -\omega_i^2 \chi_i$$

NOW ASSUME (INIT COND) $\dot{\chi}_i(0) = 0$

$$\chi_i(t) = \underbrace{\chi_i(0)}_{\text{init displ in EIG BASIS}} \cos \omega_i t$$

$\underline{x}(t) = \chi_1(t)\,\xi_1 + \chi_2(t)\,\xi_2$

$= \chi_1(0) \cos \omega_1 t\, |\xi_1\rangle + \chi_2(0) \cos \omega_2 t\, |\xi_2\rangle$

convert back to standard BASIS

multiply by $\mathbb{1} = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$

$= \chi_1(0)\cos\omega_1 t \left(|e_1\rangle \overbrace{\langle e_1|\xi_1\rangle}^{1/\sqrt{2}} + |e_2\rangle \overbrace{\langle e_2|\xi_1\rangle}^{1/\sqrt{2}} \right)$

$+ \chi_2(0)\cos\omega_2 t \left(|e_1\rangle \underbrace{\langle e_1|\xi_2\rangle}_{1/\sqrt{2}} + |e_2\rangle \underbrace{\langle e_2|\xi_2\rangle}_{-1/\sqrt{2}} \right)$

$\underline{\chi_1(0) \,\&\, \chi_2(0)}$: $|x(0)\rangle = \chi_1(0)|\xi_1\rangle + \chi_2(0)|\xi_2\rangle$

$$= \chi_1(0)\big(\underbrace{\langle e_1|\xi_1\rangle}_{1/\sqrt{2}}|e_1\rangle + \underbrace{\langle e_2|\xi_1\rangle}_{1/\sqrt{2}}|e_2\rangle\big)$$

$$+ \chi_2(0)\big(\underbrace{\langle e_1|\xi_2\rangle}_{1/\sqrt{2}}|e_1\rangle + \underbrace{\langle e_2|\xi_2\rangle}_{-1/\sqrt{2}}|e_2\rangle\big)$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left(\chi_1(0) + \chi_2(0)\right)}_{= x_1(0)}|e_1\rangle$$

$$+ \underbrace{\frac{1}{\sqrt{2}}\left(\chi_1(0) - \chi_2(0)\right)}_{= x_2(0)}|e_2\rangle$$

can invert to give $\chi_1(0)$ & $\chi_2(0)$
in terms of $x_1(0)$, $x_2(0)$