ENTANGLEMENT (BERNHARDT ch. 4)

↳ tensor products

ALICE: QUBIT IN BASIS $\{|a_0\rangle, |a_1\rangle\}$ | V=

BOB: ———"——— $\{|b_0\rangle, |b_1\rangle\}$

A: $|v\rangle = c_0|a_0\rangle + c_1|a_1\rangle$

B: $|w\rangle = d_0|b_0\rangle + d_1|b_1\rangle$

HOW TO DESCRIBE "A has $|v\rangle$ & B has $|w\rangle$"?

$$\underbrace{|v\rangle \otimes |w\rangle}_{\text{tensor}} = c_0 d_0 |a_0\rangle \otimes |b_0\rangle + c_0 d_1 |a_0\rangle \otimes |b_1\rangle + c_1 d_0 \underbrace{|a_1\rangle \otimes |b_0\rangle}_{\text{BASIS of tensor PRODUCT}} + c_1 d_1 |a_1\rangle \otimes |b_1\rangle$$

eg $A^i{}_j \underbrace{|i\rangle\langle j|}_{|i\rangle \otimes \langle j|}$

↳ nb: 1st ket is always A's
2nd ket is always B's

$$|v\rangle \otimes |w\rangle \neq |w\rangle \otimes |v\rangle$$

↳ can now drop explicit $\otimes$

coefficients are amplitudes: eg $\underbrace{c_0 d_0}\, |a_0\rangle|b_0\rangle$

$$\underbrace{P(a_0, b_0)}_{\text{PROB A READS 0, B READS 0}} = |c_0 d_0|^2$$

new notation:

$$|v\rangle|w\rangle = r|a_0\rangle|b_0\rangle + s|a_0\rangle|b_1\rangle + t|a_1\rangle|b_0\rangle + u|a_1\rangle|b_1\rangle$$

conservation of PROBABILITY → NORMALIZE $r^2 + s^2 + t^2 + u^2 = 1$

in above example $ru = st$ b/c both $= c_0 c_1 d_0 d_1$

↳ not nec. the case!

in our EXAMPLE, $P(a_1, b_0) = \underbrace{P(a_1)}_{|c_1|^2} \underbrace{P(b_0)}_{|d_0|^2}$; INDEP. MEAS.

BUT: this needn't be the case : the two qubits can be entangled

$\hookrightarrow r^2 + s^2 + t^2 + u^2 = 1$ from cons. prob

BUT $ru \neq st$

UNENTANGLED

eg: $\underbrace{\frac{1}{2\sqrt{2}}}_{r} |a_0\rangle|b_0\rangle + \underbrace{\frac{1}{2}\sqrt{\frac{3}{2}}}_{s} |a_0\rangle|b_1\rangle + \underbrace{\frac{1}{2\sqrt{2}}}_{t} |a_1\rangle|b_0\rangle + \underbrace{\frac{1}{2}\sqrt{\frac{3}{2}}}_{u} |a_1\rangle|b_1\rangle$

$ru = st \longrightarrow$ unentangled $\longrightarrow$ indep.

SUPPOSE A makes measurement - what implications?
for unentangled, we can factor:

$$|a_0\rangle\left(\frac{1}{2\sqrt{2}}|b_0\rangle + \frac{1}{2}\sqrt{\frac{3}{2}}|b_1\rangle\right) + |a_1\rangle\left(\frac{1}{2\sqrt{2}}|b_0\rangle + \frac{1}{2}\sqrt{\frac{3}{2}}|b_1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|a_0\rangle(\cdots) + \frac{1}{\sqrt{2}}|a_1\rangle(\cdots)$$ $\leftarrow$ factor out $1/\sqrt{2}$ s.t. $(\cdots)$ is unit vec.

$$= \underbrace{\left(\frac{1}{\sqrt{2}}|a_0\rangle + \frac{1}{\sqrt{2}}|a_1\rangle\right)}_{\text{A's qubit}} \underbrace{\left(\frac{1}{2}|b_0\rangle + \frac{\sqrt{3}}{2}|b_1\rangle\right)}_{\text{B's qubit}}$$

completely indep! if A measures 0 or 1, does not affect B's qubit $\rightarrow$ AMPS for B the same

ENTANGLED QUBITS

$*\quad \frac{1}{2}|a_0\rangle|b_0\rangle + \frac{1}{2}|a_0\rangle|b_1\rangle + \frac{1}{\sqrt{2}}|a_1\rangle|b_0\rangle + 0|a_1\rangle|b_1\rangle$

$\quad\; r \qquad\qquad s \qquad\qquad t \qquad\qquad u$

$ru \neq st \longrightarrow$ ENTANGLED.
nb: $r^2+s^2+t^2+u^2=1$

$\neq$!

$$= |a_0\rangle\left(\underbrace{\tfrac{1}{2}|b_0\rangle + \tfrac{1}{2}|b_1\rangle}_{\text{normalize}}\right) + |a_1\rangle\left(\underbrace{\tfrac{1}{\sqrt{2}}|b_0\rangle + 0|b_1\rangle}_{\text{normalize}}\right)$$

$$= \frac{1}{\sqrt{2}}|a_0\rangle\left(\frac{1}{\sqrt{2}}|b_0\rangle + \frac{1}{\sqrt{2}}|b_1\rangle\right) + \frac{1}{\sqrt{2}}|a_1\rangle\left(|b_0\rangle + 0|b_1\rangle\right)$$

MEASUREMENT: A measures 0, 1 w/ EQUAL PROB

if A meas 0: PROJECT onto $|a_0\rangle(\cdots)$

$\Rightarrow$ B now in $\frac{1}{\sqrt{2}}|b_0\rangle + \frac{1}{\sqrt{2}}|b_1\rangle$ state

Now entanglement is "done"

if A meas 1, PROJECT onto $|a_1\rangle(\cdots)$

$\Rightarrow$ B now in $|b_0\rangle \Rightarrow$ B must meas 0!

(& entanglement is done)

what if B measured first?

$\Rightarrow *\quad |\,\rangle|\,\rangle = \left(\frac{1}{2}|a_0\rangle + \frac{1}{\sqrt{2}}|a_1\rangle\right)|b_0\rangle + \left(\frac{1}{2}|a_0\rangle + 0|a_1\rangle\right)|b_1\rangle$

$$= \left(\frac{1}{\sqrt{3}}|a_0\rangle + \sqrt{\frac{2}{3}}|a_1\rangle\right)\frac{\sqrt{3}}{2}|b_0\rangle + \left(|a_0\rangle + 0|a_1\rangle\right)\frac{1}{2}|b_1\rangle$$

if B meas 0 $\longrightarrow$ A in this state
if B meas 1 $\longrightarrow$ A in $|a_0\rangle$

& entanglement is "done"

TENSOR PRODUCT BASIS

$$\underbrace{\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}}_{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}},\quad \underbrace{\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}}_{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}},\quad \underbrace{\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}}_{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}},\quad \underbrace{\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}}_{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}$$

s.t. $\begin{pmatrix}a_0\\a_1\end{pmatrix}\otimes\begin{pmatrix}b_0\\b_1\end{pmatrix} = \begin{pmatrix}a_0\begin{pmatrix}b_0\\b_1\end{pmatrix}\\ a_1\begin{pmatrix}b_0\\b_1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}a_0b_0\\a_0b_1\\a_1b_0\\a_1b_1\end{pmatrix}$

$\left.\begin{array}{l}\text{2 qubits} \to \text{4 states}\\ \text{3 qubits} \to \text{8 states}\\ \text{4 qubits} \to \text{16 states}\end{array}\right\}$ N qubits $\to \boxed{2^N \text{ states}}$

ENTANGLEMENT w/ CNOT GATE : controlled "not" :

x	y	x (CNOT out)	$x \oplus y$ (CNOT out)
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

exclusive or

control BIT

$$\text{CNOT} = \left(\begin{array}{cc|cc}1 & & & \\ & 1 & & \\ \hline & & 0 & 1\\ & & 1 & 0\end{array}\right)$$

eg: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\times\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right)$

$(\text{CNOT})\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

$ru = st = 0$ UNENTANGLED

$ru = 1 \neq st = 0 \Rightarrow$ ENTANGLED

A HINT of GROUP THEORY (ADDITION of ANGULAR MOMENTUM)

imagine 2 spins : $\underbrace{|s_1\rangle \otimes |s_2\rangle}$

how many states in system?

$2 \times 2 = 4.$

can choose EIGENSTATES of S_{z1} & S_{z2}

$S_z \otimes \mathbb{1}$ $\quad$ $\mathbb{1} \times S_z$ $\quad$ w/ $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

in fact: can define total spin in z-DIR.

$\underbrace{S_z} = S_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z$

WHAT ARE the EIGENVALUES?

$S_z |s_1\rangle \otimes |s_2\rangle = \{1, 0, -1\} |s_1\rangle \otimes |s_2\rangle$

multiplicity 2!

EACH SPINOR $|s_i\rangle$ IS A $\mathbb{C}$ 2-COMP VECTOR.

ROTATIONS ACT AS $e^{iS_j\theta^j} |s\rangle$

$S_x = \frac{\hbar}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ $\quad S_y = \frac{\hbar}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$ $\quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

Def: RAISING/LOWERING OPS $\quad S_\pm = S_x \pm i S_y$

$= \begin{cases} \hbar \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} & S_+ \\ \hbar \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} & S_- \end{cases}$

$S_+ |\uparrow\rangle = 0$ $\qquad$ $\boxed{S_+ |\downarrow\rangle = \hbar |\uparrow\rangle}$

$S_- |\downarrow\rangle = 0$ $\qquad$ $\boxed{S_- |\uparrow\rangle = \hbar |\downarrow\rangle}$

OPERATOR THAT INCREASES/DECR. SPIN

gives 0 if no allowed state

EIGENSTATE of S_z w/ "HIGHEST WEIGHT"

$\Rightarrow$ given a state $|\uparrow\rangle$ s.t. $S_+|\uparrow\rangle = 0$

$\hookrightarrow$ can fill out the other state: $S_-|\uparrow\rangle = |\downarrow\rangle$

this works for tensor product:

$$\mathcal{S}_+ = \underset{\substack{\uparrow \\ \text{ACTS on } 1^{st}}}{S_+} \otimes \mathbb{1} + \mathbb{1} \otimes \underset{\substack{\uparrow \\ \text{ACTS on } 2^{ND}}}{S_+}$$

$$\mathcal{S}_- = S_- \otimes \mathbb{1} + \mathbb{1} \otimes S_-$$

state w/ "MOST" SPIN in z-DIR:

$\underbrace{|\uparrow\rangle|\uparrow\rangle}_{|\uparrow\uparrow\rangle \text{ for simplicity}}$ w/ $\mathcal{S}_z|\uparrow\uparrow\rangle = \frac{1}{2}|\uparrow\uparrow\rangle + \frac{1}{2}|\uparrow\uparrow\rangle = \underset{\substack{\uparrow \\ \text{SPIN in z DIR} = 1}}{|\uparrow\uparrow\rangle}$

$|\uparrow\uparrow\rangle$ for simplicity, also set $\boxed{\hbar = 1}$

then find state w/ one lower value of $\mathcal{S}_z$

$\mathcal{S}_-|\uparrow\uparrow\rangle = \underbrace{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle}_{\text{what is spin in z DIR?}}$ $\leftarrow$ normalize w/ $\frac{1}{\sqrt{2}}$

$$\mathcal{S}_z\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) = \Big(\underbrace{-\tfrac{1}{2}|\downarrow\uparrow\rangle + \tfrac{1}{2}|\uparrow\downarrow\rangle + \tfrac{1}{2}|\downarrow\uparrow\rangle - \tfrac{1}{2}|\uparrow\downarrow\rangle}_{S_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z}\Big)$$

$$= 0\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

lower again: $\mathcal{S}_-\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) = \left(0|\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle + 0|\uparrow\downarrow\rangle\right)$

$= |\downarrow\downarrow\rangle$ (normalizing)

$\mathcal{S}_z|\downarrow\downarrow\rangle = -1$ & clearly $S_-|\downarrow\downarrow\rangle = 0$

so we found 3 states: $\begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{cases}$

what's missing?

$\hookrightarrow$ by GRAM-SCHMIDT (OR JUST THINKING ABOUT IT)

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

$\uparrow$ orthogonal to others.

e.g. $\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

$$= \langle\uparrow\downarrow|\uparrow\downarrow\rangle - \langle\downarrow\uparrow|\downarrow\uparrow\rangle = 0$$

$$= 0 \left\{ + \langle\uparrow\downarrow|\downarrow\uparrow\rangle - \langle\downarrow\uparrow|\uparrow\downarrow\rangle \right.$$

$$\hookrightarrow \left(\langle\uparrow|\otimes\langle\downarrow|\right)\left(|\downarrow\rangle\otimes|\uparrow\rangle\right)$$

$$= \langle\uparrow|\downarrow\rangle\langle\downarrow|\uparrow\rangle = 0$$

clearly $S_z \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) = 0$ (no spin)

& can check $S_\pm (\text{——} \,\text{"}\, \text{——}) = 0$ no other states

What distinguishes the TRIPLET & SINGLET?

$\underline{2\otimes 2} = \underline{3} \oplus \underline{1}$

from your HW:

$S^2 = \underbrace{S_x^2 + S_y^2 + S_z^2}_{(\text{total spin})^2}$ satisfies $\underbrace{[S^2, S_z] = 0}_{\text{commuting obs!}}$

from HW: $S^2 = S_\pm S_\mp + S_z^2 \mp \underbrace{\hbar}_{\hbar=1} S_z$

$S^2|\uparrow\uparrow\rangle = (\underset{\text{choose lower}}{0} + 1^2 + 1)\ |\uparrow\uparrow\rangle = 2|\uparrow\uparrow\rangle$

$S^2|\uparrow\downarrow + \downarrow\uparrow\rangle = \boxed{} \quad 0 + 0$

$S_+S_-|\uparrow\downarrow\rangle = S_+(S_-\otimes 1 + 1\otimes S_-)|\uparrow\downarrow\rangle$

$= S_+|\downarrow\downarrow\rangle$

$= |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$

+ same for 2nd term $\Rightarrow 2|\uparrow\downarrow + \downarrow\uparrow\rangle$

$S^2|\downarrow\downarrow\rangle = (\underset{\text{choose upper}}{0} + 1^2 - (-1))\ |\downarrow\downarrow\rangle = 2|\downarrow\downarrow\rangle$

$\Rightarrow$ so all three have same S^2 eigenval!

similarly: $\boxed{S^2|\uparrow\downarrow - \downarrow\uparrow\rangle = 0}$

turns out: in general $S^2|\ldots\rangle = \underset{\text{total ang momentum quantum \#}}{l}(l+1) \quad l = 0, 1, \ldots$

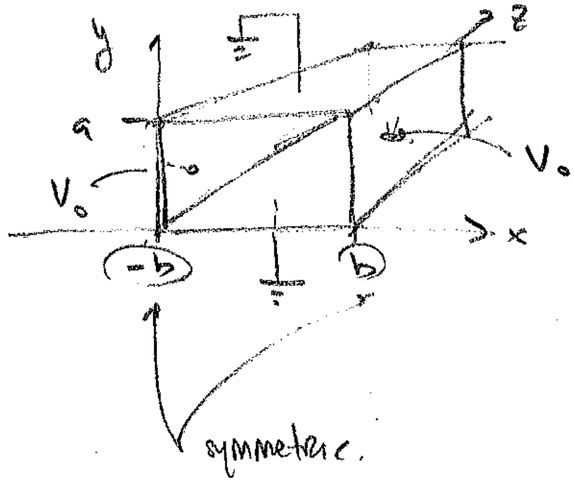
$S_z|\ldots\rangle = m \quad m = -l, \ldots, l$

for 2 spin system: tensor product $|S_1\rangle \otimes |S_2\rangle$

separates into "spin-1" & "spin-0"

$\Rightarrow$ rotations won't mix the two!

MORE FOURIER EXAMPLES (from GRIFFITHS E&M ch 3)



$(\partial_x^2 + \partial_y^2) V(x,y) = 0$

find V inside

(no infinite in z)

SEP of VARS: $V = X(x)\, Y(y)$ ANSATZ

$$\nabla^2 V = X'' Y + X Y'' = 0$$

$$\boxed{\frac{X''}{X} = -\frac{Y''}{Y} = C}$$

some const. that links $X(x)$ & $Y(y)$

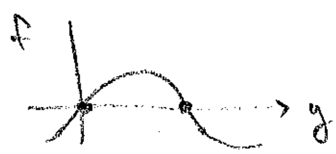
what is the sign of C?

SIGNIFICANCE: $f'' = k^2 f \rightarrow f \sim e^{kx} + e^{-kx}$

$f'' = -k^2 f \rightarrow f \sim \underbrace{e^{ikx} + e^{-ikx}}_{\text{or sin/cos}}$

WHICH of X vs Y should be PERIODIC?

↳ clearly the one w/ GROUNDED BOUNDARIES (DIRICHLET) should be PERIODIC: we need func to be ZERO AT TWO PLACES.



in fact: DIRICHLET → sines

so: $\boxed{C = k^2}$

$$V(x,y) = \underbrace{(A e^{kx} + B e^{-kx})} \sin ky$$

nb. symmetry of problem wrt $\pm x$
→ x-dep must be symmetric
⇒ $A = B$ + use $e^{kx} + e^{-kx} = 2\cosh kx$

$$= C \cosh(kx) \sin(ky)$$

but what is k? in fact, ∞ # of k's that satisfy BC!

↳ so $V(x,y) = \sum_n C_n \cosh(k_n x) \underbrace{\sin(k_n y)}$

BC @ $y = 0, a$
⇒ $k_n = \frac{n\pi}{a}$

general form: $V(x,y) = \sum_n \underline{C_n} \cosh\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{a} y\right)$

↑
what coefficients?
we have the x BC.
(other one – we implicitly used one BC @ x)

$$V(b,y) = V_0 = \sum_n \underbrace{C_n \cosh\left(\frac{n\pi}{a} b\right)}_{D_n} \sin\left(\frac{n\pi}{a} y\right)$$

project out coefficients D_n using $\underbrace{\int \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{n\pi}{a} y\right) dy}_{= \frac{a}{2}}$

(or you can use normalized functions)

$$\boxed{\frac{a}{2} D_m = \int_0^a V(b,y) \sin\left(\frac{m\pi y}{a}\right) dy}$$

SPHERICAL SYM

$$\nabla^2 V = r^{-2}\partial_r\left(r^2\partial_r V\right) + \frac{1}{r^2 s_\theta}\partial_\theta\left(s_\theta \partial_\theta V\right) + \underbrace{\frac{1}{r^2 s_\theta^2}\partial_\phi^2 V}_{} = 0$$

AZIMUTHAL SYM: ASSUME no ϕ-DEP. (FOR SIMPLICITY)

Sep. of VARS: $V = R(r)\Theta(\theta)$

end up w/

$$\underbrace{\frac{1}{R}\partial_r\left(r^2 \frac{dR}{dr}\right)}_{} = \underbrace{-\frac{1}{\Theta s_\theta}\frac{d}{d\theta}\left(s_\theta \frac{d\Theta}{d\theta}\right)}_{} = \underbrace{\text{const}}_{= \ell(\ell+1)}$$

solution:

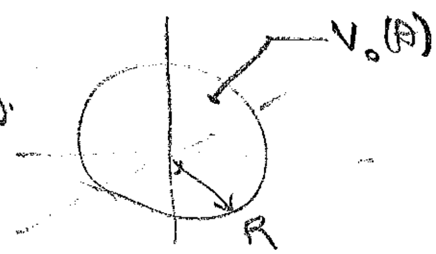
$R(r) = A r^{\ell} + \frac{B}{r^{\ell+1}}$

$\Theta(\theta) = P_\ell(\cos\theta)$

LEGENDRE!

$$V(r,\theta) = \left(A r^{\ell} + \frac{B}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

e.g.



find $V(r)$ inside

$\uparrow$

$B = 0$ (finiteness)

$$V(r,\theta) = A\, r^{\ell} P_\ell(\cos\theta) \quad w/ \quad V_0(\theta) = \underbrace{A R^{\ell}}_{C_\ell} P_\ell(\cos\theta)$$

$$\int_0^1 d(\cos\theta)\, V_0(\theta)\, P_{\ell'}(\cos\theta) = \frac{2}{2\ell'+1} C_\ell$$ $\leftarrow$ USING LEGENDRE ORTHOG. RELATION.