

SHORT HW 10: A hint of loop diagrams

COURSE: Physics 165, *Introduction to Particle Physics* (2022)

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Please have all assignments turned in by the end of this week. This short assignment is especially short, but it is a peek into what a full quantum field theory calculation looks like. The full calculation is in Chapter 10 of Peskin & Schroeder's *Introduction to Quantum Field Theory*.

1 Defining the four-point interaction

Suppose you have a theory with a particle φ with a four-point self-interaction whose coupling strength is λ . That is, your lagrangian has a term that looks like $\mathcal{L} \sim \lambda \varphi^4$. If you calculate the scattering of 2 particles into 2 particles, your amplitude to leading order is just λ , up to some numerical factors.

At next-to-leading order, we introduce loop diagrams. Here are the diagrams up to ‘one loop’ level (from Peskin & Schroder):

The image shows the Feynman diagrams for a four-point interaction. On the left is a tree-level diagram, represented by a simple 'X' shape where two incoming lines from the left meet at a central vertex, and two outgoing lines go to the right. This is followed by a plus sign and a large left parenthesis. Inside the parenthesis are three one-loop diagrams, each followed by a plus sign. The first loop diagram is a bubble (a circle) with two external lines on the left and two on the right. The second loop diagram is a tadpole (a circle with a vertical line through its center) with two external lines on the left and two on the right. The third loop diagram is a sunset (two circles sharing a vertex) with two external lines on the left and two on the right. The entire set of diagrams is enclosed in a large right parenthesis.

The first term is the usual leading-order ‘tree-level’ diagram. The three terms in parenthesis are obviously the loop diagrams that go like λ^2 . Feel free to take a peek at Peskin & Schroder chapter 10 to see what’s going on in its full technical glory.

1.1 Distinct diagrams

Assume that each particle has a distinct momentum, say p_1 and p_2 for the incoming particles, and q_1 and q_2 for the outgoing particles. Explain why the three loop diagrams are all *distinct* diagrams that tell different spacetime histories.

1.2 Renormalization conditions

Let us now revise our notation. Let’s call the coupling in the Lagrangian λ_0 to distinguish it from the physical coupling, λ . Here’s how this works:

1. If we are calculating only to leading order, then $\lambda_0 = \lambda$.
2. If we are calculating to loop-level, then the sum of the three diagrams above should give λ (again, up to numerical factors).

Because the loop diagrams go like a higher power of λ and we’re doing a Taylor expansion in λ , we expect them to be small corrections, so that $\lambda_0 \approx \lambda$. However, we now know that this is not true: the loop diagrams seem to be infinite because of the integral over the internal loop momentum. This means that λ_0 differs from λ by a formally infinite amount.

In order to deal with this, we split λ_0 into the physical part and ‘everything else’: $\lambda_0 = \lambda + \delta_\lambda$. This last term is called a ‘counter term,’ but it is just our way of peeling off the non-physical piece of λ_0 .

The sum of the four diagrams above is

$$\lambda_{\text{phys}} = \lambda - \lambda^2 [V(s) + V(t) + V(u)] + \delta_\lambda . \quad (1.1)$$

Here V is the loop integral function that encodes the formally divergent integral. The arguments are the so-called Mandelstam variables that encode the Lorentz invariants that encode all of the kinematic data¹:

$$s = (p_1 + p_2)^2 \quad t = (q_1 - p_1)^2 \quad u = (q_1 - p_2)^2 . \quad (1.2)$$

[Flip: Corrected 6/1: q_1 was previously written as k_1 . Thanks Andrew C.] In order to define our theory, we have to make a measurement of λ . We now know that the properties of the measurement are part of the definition of the theory. Suppose we perform our experiment by scattering two *very* slow φ particles in the system’s rest frame. This means we can ignore their kinetic energies. We then measure the physical coupling² to be λ_{meas} .

Show that this defines δ_0 at the one-loop level:

$$\lambda_{\text{meas}} = \lambda - \lambda^2 [V(4m^2) + V(0) + V(0)] + \delta_\lambda , \quad (1.3)$$

where m is the mass of the φ . Note: since we are doing perturbation theory, you can set

This is just an exercise in kinematics and plugging in the appropriate values of s , t , and u as defined above. However, please appreciate what is happening here: the definition of the Lagrangian parameter λ_0 depends the properties of how we measured it. We now have λ_0 (equivalently δ_λ) defined unambiguously, which means we can now calculate (and *predict*) the loop-level rate for $2\varphi \rightarrow 2\varphi$ at, say, very high energies. Note that this rate will be different and will inherit the momentum dependence of the λ_0 -defining measurement at $s = 4m^2$.

1.3 An explicit example

We actually get to choose *how* we parameterize the apparent divergence in the loop diagrams. One convenient method is called dimensional regularization. In this scheme, the function $V(p^2)$ is

$$V(p^2) = \lim_{\varepsilon \rightarrow 0} \frac{-1}{32\pi^2} \int_0^1 dx \left[\frac{2}{\varepsilon} - \log(m^2 - x(1-x)p^2) \right] . \quad (1.4)$$

Show that once λ_0 is defined for any values of s , t , and u , the divergent $2/\varepsilon$ pieces cancel for any other values of s , t , and u . **HINT:** the divergent piece does not depend on the kinematics at all. Thus there is simply a term in δ_λ that is proportional to the total divergence with an overall minus sign.

¹There are four 4-momenta, they are constrained by total momentum conservation. Further, we know these should show up as Lorentz invariant objects. The particle mass and the Mandelstam variable encode all of these Lorentz invariants.

²Formally, we measure the rate or cross section, then infer the physical coupling from the appropriate cross section formula.