Short HW 7: Mass terms and equations of motion

Course: Physics 165, *Introduction to Particle Physics* (2022)
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This short homework motivates why invariants between two particles (and no derivatives) should be understood as mass terms. The discussion is necessarily qualitative.[1]

1 Equation of motion from matrix multiplication

Let $\mathbf{q} = (q_1, \cdots, q_N)^T$ be an N-dimensional vector in Euclidean space. Given a matrix A in this space, one can write an invariant under N-dimensional rotations,

$$L = \frac{1}{2}\mathbf{x}^T A\mathbf{x} = x_i A^i{}_j x^j = \sum_{i,j} \frac{1}{2} x_i A_{ij} x_j \ . \tag{1.1}$$

The factor of 1/2 is for convenience. In the last step we have used the fact that the Euclidean metric is δ_{ij} so that we may write all indices lower.[2]

Each variable x_i is an independent component. The *variation* of L with respect to x_i is

$$\frac{\delta}{\delta q_i} L = \frac{\partial}{\partial q_i} L \ . \tag{1.2}$$

Assuming that A is a symmetric matrix, $A_{ij} = A_{ji}$, show that the variation $\delta L/\delta x_i$ is

$$\frac{\delta L}{\delta q_i} = (A\mathbf{q})_i \tag{1.3}$$

Intermission: Discussion

The equation of motion for a particle (field) comes from a variational principle. We maintain that the action, $S = \int dt\, L$ is stationary with respect to variations of the dynamical variables. If L takes the form above and the x_i are the dynamical variables, one ends up with equations of motion of the form

$$\frac{\delta L}{\delta q_i} = A\mathbf{q} = 0. \tag{1.4}$$

Physical interpretation is that we should think of $\mathbf{q}$ as a field, ϕ. The individual components q_i correspond to values of the field at specific points in spacetime, $\phi(x)$. The matrix A is generally a differential operator. For a field like $\phi(x)$ with no indices, the differential operator takes a generic form

$$A = \partial^2 + m^2 \ . \tag{1.5}$$

[1] For details, see https://github.com/fliptanedo/Math-Methods-Notes/blob/master/P231_notes.pdf

[2] We deal with upper and lower indices when the distinction makes life easier. In this case, putting all indices lower makes life easier.

2 On-shell conditions for a scalar field

Let $\phi(x)$ be a scalar field. From the analogy to matrix multiplication, one form of the action is

$$S = \int dt\, d^3x\, \mathcal{L} = \int d^4x\, \frac{1}{2}\phi(x)\left(\partial^2 + m^2\right)\phi(x)\ . \tag{2.1}$$

The integral over d^3x is the analog of the sum over i, j in the matrix multiplication example.[3] Clearly we see that this 'invariant' corresponds to connecting two ϕ particles. We could write this as a vertex, but it turns out that the terms that are quadratic in fields can be solved exactly because the equation of motion, $(\partial^2 - m^2)\phi(x) = 0$ is linear.

Write $\phi(x)$ in a Fourier representation:

$$\phi(x) = \int đ^3p\, e^{-ip\cdot x}\widetilde{\phi}(p)\ . \tag{2.2}$$

In one quick line, show that the analog of $A\mathbf{x} = 0$ is

$$-(p^2 - m^2)\widetilde{\phi}(p) = 0\ . \tag{2.3}$$

Discussion: this tells us that the classical equation of motion imposes $p^2 = m^2$ for each Fourier mode of the field $\phi(x)$. This looks really familiar if we interpret m to be the mass of the particle: the classical equation of motion imposes that the particle is *on-shell*. In this way, we see that the $m^2\phi^2$ term controls the mass of the field ϕ.

Intermission: Discussion

So far we've taught ourselves to treat Lagrangian terms as rules for vertices. This is part of the idea that the Feynman rules are a Taylor expansion (perturbation expansion). It turns out that the terms in the Lagrangian that are quadratic in a field can be solved exactly. Thus we do not need Feynman rules with only two external lines.

Here's a sketch of what the exact solution looks like. It helps to insert a linear term, $J\phi$ into the Lagrangian. J is called the *source* of the field. It tells us that there's something causing excitations of this particle. The equation of motion ends up being (please forgive me for dropping all sorts of prefactors)

$$\left(\partial^2 + m^2\right)\phi(x) = J(x)\ . \tag{2.4}$$

We can then go to a Fourier basis for $\phi(x)$. The source is usually a δ-function, $J(x) = \delta^{(4)}(x)$, representing a specific moment in spacetime where a ϕ particle is created. Recalling the Fourier expansion of the δ-function, the equation of motion for a given Fourier mode is

$$(p^2 - m^2)\widetilde{\phi}(p) = i\ , \tag{2.5}$$

where I have probably made several sign mistakes. Poetically, the solution to this equation is:

$$\widetilde{\phi}(p) = \frac{i}{p^2 - m^2}\ , \tag{2.6}$$

which is exactly the propagator: the rule we used internal lines. We have skimmed over a *lot* here. For those interested in the nuts and bolts of how this actually works, this is called a Green's function. We go over Green's functions in gory detail in Physics 231.

[3] If you are observant, you may worry why there is only one integral rather than two. Excellent! The reason is that the 'matrix' $\partial^2 - m^2$ is diagonal. Feel free to ask about this in class.

3 Possible mass terms for a fermion

Fermion mass terms are funny. Recall that a left-chiral fermion ψ^α has one type of index, but its conjugate $(\psi^\dagger)_{\dot{\alpha}}$ has a different kind of index. Thus you can form invariants that look like $\psi^\alpha\psi^\beta\varepsilon_{\alpha\beta}$, but not $\psi^\dagger\psi$.

3.1 Charged fermions want to be massless

Argue that any fermion ψ with charge (e.g. indices, or even hypercharge) cannot have a mass term with itself. There is no invariant under all the symmetries—spin, SU(3), SU(2), U(1)—that connects two of the particle: either $\psi\psi$ or $\psi^\dagger\psi$.

3.2 Majorana mass terms

Argue, on the other hand, that as long as a fermion ψ has *no* charge, it may have a mass term. Write out the invariant. We call this a Majorana mass.