LAST TIME

$\boxed{\text{SYMMETRY}}$ ← ABSTRACT, GENERAL, but you can think of indices from RELATIVITY as a concrete example

→ OBJECTS THAT TRANSFORM UNDER THE SYMMETRY HAVE INDICES. INDEX TELLS YOU HOW IT TRANSFORMS

$V^i \to R^i{}_j V^j$ (col. vector)

$W_i \to W_j (R^{-1})^j{}_i$ (row vector)

$= (R^{-1})^j{}_i W_j$

nb: $\boxed{R^{-1} = R^T}$ for ROTATIONS

general object has upper & lower indices

$$A^{ij}{}_k \to R^i{}_\ell R^j{}_m (R^{-1})^n{}_k A^{\ell m}{}_n$$

OBJECT W/ NO INDICES DOES NOT TRANSFORM → INVARIANT

eg $\underbrace{W_i A^i{}_j V^j}$ = $\underline{W}^T \cdot A \underline{V}$ is invariant

no uncontracted indices.

NB: A SYMMETRY CAN HAVE DIFFERENT KINDS OF INDICES!
→ WE'LL HAVE TO USE DIFFERENT KINDS OF LETTERS

eg: vectors & spinors transform under rotations → BUT not by the SAME MATRIX!

$V^\mu \to \Lambda^\mu{}_\nu V^\nu$ — 4x4, $\mathbb{R}$

$\psi^\alpha \to (\tilde{\Lambda})^\alpha{}_\beta \psi^\beta$ — 2x2 $\mathbb{C}$

these different kinds of indices are called <u>REPRESENTATIONS</u> of THE SYMMETRY GROUP

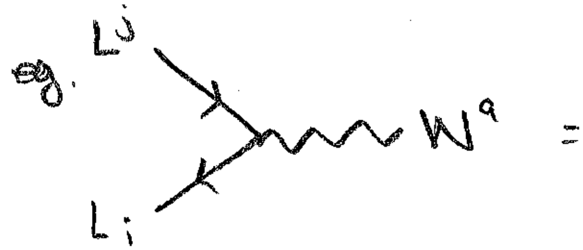
↳ what we're doing is
"representation thy of <u>Lie Groups</u>"

continuous symmetry, like rotations.

UNLIKE DISCRETE SYM, LIKE POLYHEDRAL GROUPS.

<u>PUNCHLINE</u>: symmetry (mathematical entity) gives us a bunch of different kinds of indices & rules for how they transform.

eg.

SU(2) 

$= ig(T^a)^i_{\ j}$

"vector" index

"ADJOINT" index

$(T^a)^i_{\ j}$ is called a <u>GENERATOR</u> of the symmetry.

a index is SPECIAL: runs from $a = \underbrace{1,2,3}$

DIMENSION OF SYMMETRY

SU(2) has 3 kinds of ROTATIONS (like 3-space)

GENERATORS: one of the "SPECIAL TENSORS" that the SYMMETRY GIVES US.

general form: $(T^a)^i{}_j$

is DIMENSION of SYMM. { indexes the generator: the "direction" of ROTATION

one upper, one lower

# of values IS DIMENSION OF REPRESENTATION

how it is used: $U(\underline{\theta})^i{}_j = \left(e^{i\theta^a T^a}\right)^i{}_j$

or R

$= \mathbb{1} + i\theta^a T^{a\,i}{}_j$

$- \frac{1}{2}\theta^a\theta^b (T^a)^i{}_k (T^b)^k{}_j$

$+ i \cdots$

eg. 2D ROTATIONS $SO(2) = U(1)$ ← fancy name

$(T^1)^i{}_j = \begin{pmatrix} & -i \\ i & \end{pmatrix}$

one generator; $i,j \in \{1,2\}$; nb: HERMITIAN MATRIX

$\underbrace{e^{i\theta\begin{pmatrix} & -i \\ i & \end{pmatrix}}} = 1 + \theta\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} + \frac{1}{2}\theta^2\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} + \frac{1}{3!}\theta^3\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} + \cdots$

$\theta\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$= \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}$ ✓ as

ORTHOGONAL: $U^T U = \mathbb{1}$

SPECIAL: $\det U = 1$

GENERATORS of 3D ROTATIONS $SO(3) \simeq SU(2)$

$$L_x = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix} \qquad \tfrac{1}{2}\sigma_x = \tfrac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$L_y = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix} \qquad \tfrac{1}{2}\sigma_y = \tfrac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$$L_z = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \qquad \tfrac{1}{2}\sigma_z = \tfrac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$(T^a)^i{}_{\underbrace{j}_{\mathbb{R},\ 3\text{ space}}} \qquad\qquad (T^a)^i{}_{\underbrace{j}_{\mathbb{C},\ 2\text{ space}}}$$

BUT, AS YOU KNOW, SAME <u>COMMUTATION RELATIONS</u>

exactly what matters when studying symmetries

eg. ROTATIONS DO NOT COMMUTE in dim > 2.

So: that's GROUP THEORY :)

HOW WE WILL USE IT: SU(2) we will write out explicitly BECAUSE (1) ITS EASY (2) HIGGS

8 generators of SU(3) $\leftarrow$ SU(3) we will just write $(T^a)^i{}_j$ in the abstract... just make sure indices contract.

GAUGE THY "review" ← ψ IS AN ELECTRON FIELD

$U(1)$: $\psi \to e^{i\theta q}\psi$ (FIELD JUST MEANS FUNCTION ON SPACETIME)

q is CHARGE; diff charged particles REPHASE BY DIFF AMTS

$\psi^\dagger\psi$ is invariant, so is $\psi^\dagger\partial_\mu\psi$

GAUGED = LOCAL $U(1)$: $\psi \to e^{i\theta(x)q}\psi$

$\psi^\dagger\psi$ INVARIANT

$\psi^\dagger\partial_\mu\psi$ IS NOT → PICKS UP $\underline{i(\partial_\mu\theta(x))}\,q\,\psi^\dagger\psi$

cost of local transform

BUT IF WE "PROMOTE" $\partial_\mu \to D_\mu = \underbrace{\partial_\mu + iqA_\mu}_{\text{COVARIANT DERIV.}}$

WHERE A_μ IS A GAUGE FIELD that ALSO TRANSFORMS

$A_\mu \to A_\mu - \partial_\mu\theta(x)$

⇒ then $\psi^\dagger D_\mu\psi$ IS INVARIANT B/C EXTRA TERMS CANCEL!

$\boxed{A_\mu \text{ is the PHOTON}}$ ← "fields are particles"

the transformation $A_\mu \to A_\mu - \partial_\mu\underbrace{\theta(x)}_{\forall\,\theta(x)}$

is A GAUGE TRANSFORMATION: PHYSICS (MAXWELL'S EQN) IS TOTALLY UNCHANGED. ⇒ LOCAL SYM / GAUGE SYM IS A REDUNDANCY OF THE THEORY

⇒ mathematical ORIGIN of FORCES.

SU(2) :

SUM OVER a matrices, even if upper

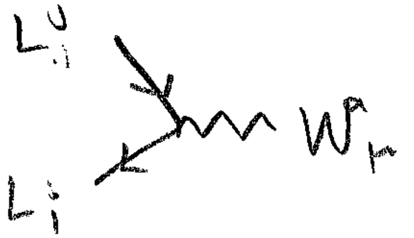
$$(L^\dagger)_i \left(\mathbb{1}^i{}_j \, \partial_\mu + ig (T^a)^i{}_j \, W^a_\mu \right) L^j$$

DOUBLET

antiparticle (LOWER INDEX)

"kinetic term"

W^a_μ : 3 TYPES OF VECTOR



$$= ig (T^a)^i{}_j$$