Homework 6: **Form Follows Function**

Course: Physics 231, *Methods of Theoretical Physics* (2016)
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The title doesn't refer to modernist architecture[1], but rather differential forms which have followed our study of Green's functions. By the way, Stone and Goldbart is pretty good for differential geometry; it may be a good reference as we go through our lightning tour of this subject.

Update: 11/01. Please disregard Problem 1.1; this problem turned out to be much more subtle than I originally intended. Problem 1.4 has also been fixed for confusing typos and added clarity. Thanks to Adam G. and Cliff C. Problem 1.6 now has a new hint. Further: everything after problem 2.1 is now extra credit.

1 Green's Functions on Spacetime

In this problem, we solve for the Green's function of wave equation for electromagnetism:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right]\varphi(\mathbf{r},t) = \rho(\mathbf{r},t) \qquad\qquad \left[\frac{1}{c^2}\frac{\partial^2}{\partial t2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right]\mathbf{A}(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t)\ . \tag{1.1}$$

We make the speed of light, c, explicit and have chosen **Lorentz gauge** and are working in vacuum, $\varepsilon_0 = \mu_0 = 1$. You should recognize all the physical quantities here. The following steps probably be painfully familiar. The key difference is that now we're talking about propagation in both space and time.

1.1 A quick digression

[**Update: 11/01**). Please disregard this problem!]

Write (1.1) in a fancy-math-y way with respect to the one-form potential $A(x) = A_\mu(\mathbf{r},t)dx^\mu$ on spacetime.

Remarks: (Updated 11/01 You want to 'apply two derivatives' to A, but recall that $d^2 = 0$ so $d^2A \equiv 0$. You'll also want to promote ρ and $\mathbf{j}$ into a spacetime current j. Usually we talk about the current as a one-index object. This problem may convince you that it's 'naturally' a three-index object. Up to a possible sign, the answer is $(d\delta + \delta d)A = j$, where $\delta \sim *d*$ is the co-derivative. The d'Alembertian/4-Laplacian is $\Box = \pm d\delta \pm \delta d$ where the sign depends on the signs in the metric and whether the k-form on which it acts has odd or even k.[2]

[1] https://en.wikipedia.org/wiki/Form_follows_function

[2] See, e.g., http://www.people.vcu.edu/~rgowdy/phys591/pdf/difforms.pdf or section 3.8 of one of my favorite books, *Differential geometry, gauge theories, and gravity* by Göckeler & Schücker.

1.2 Setting up the problem

Use spacetime coordinates such that $x = (ct, \mathbf{r})$. Define the one-form $A_\mu(x) = (\varphi(x), \mathbf{A}(x))$, where x is a point in Minkowski space. Let $G(x, x')$ be the Green's function for each component of A_μ. Ignoring the curiosity in the previous sub-problem, let's assume that the current is also a one-form, $j_\mu(x')$.

Convince yourself that the potential is given by

$$A_\mu(x) = \int d^4x\, G(x, x')\, j_\mu(x') \ . \tag{1.2}$$

This isn't anything deep, but make sure you're comfortable with why the indices are where hey are. Why doesn't $G(x, x')$ have indices? Why shouldn't it be a 4-component object or a matrix?

1.3 Fourier Transform

The Green's function equation for each component is

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right] G(x, x') = \delta^{(4)}(x - x') \equiv \delta(x - x')\delta(y - y')\delta(z - z')\delta(t - t') \ , \tag{1.3}$$

where I hope there's no ambiguity between $\delta^{(4)}(x - x')$ and $\delta(x - x')$. The covariant Fourier transform in Minkowski space is[3]

$$\widetilde{G}(k, x') = \int d^4x\, e^{ik\cdot x} G(x, x') \qquad\qquad G(x, x') = \int đ^4k\, e^{-ik\cdot x}\widetilde{G}(k, x') \ , \tag{1.4}$$

where we've used the notation $đ = d/2\pi$. Further, $k = (E/c, \mathbf{k})$ and we recall that $k \cdot x \equiv k_\mu x^\mu = Et - k_x x - k_y y - k_z z$. Observe that momenta 'naturally' come with lower indices; we haven't used the metric. Step back for a moment: why is k_0 written as E?

Solve for $\widetilde{G}(k, x')$. It should look very similar, perhaps with factors of c as required by dimensional analysis.

HINT: It may be useful to note that $\delta^{(4)}(x)$ has the following Fourier transform:

$$\delta^{(4)}(x - x') = \int đ^4k e^{-ik\cdot(x-x')} \ . \tag{1.5}$$

Observe that there is an $e^{ikx'}$ in $\widetilde{G}(k, x')$.

Write $G(x, x')$ as a 4D integral over $\widetilde{G}(k, x')$. ANSWER: You should get

$$G(x, x') = \int đ^4k\, \frac{c^2}{c^2\mathbf{k}^2 - E^2} e^{-ik\cdot(x-x')} \ . \tag{1.6}$$

Observe that $G(x, x') = G(x - x')$, as we noted in Homework 5 parts 1.2 and 1.6.

[3] I believe choice of signs in the exponentials or (2π)'s are at all consistent with what we eventually settled on in class. Compare to (1.5) of Homework 5.

1.4 Angular integrals in hyper-cylindrical coordinates

Update, 11/1: this sub-problem has been updated with corrections (typos) and a few explanatory sentences. Sorry for the confusion!

For convenience, write $y \equiv x - x' = (cu, \mathbf{s})$. (11/1) This means that u is a [shifted] time coordinate and $\mathbf{s}$ is a [shifted] space 3-vector. First note that $d^4x = d^4y$. In order to integrate (1.6), use 4D 'hyper-cylindrical coordinates' over k where time/energy is treated linearly and the space/momentum directions are treated in 3D spherical coordinates, $|\mathbf{k}|$, $\cos\theta$, and φ. Recall that θ is the azimuthal angle with respect to the k_z-axis. The volume element in these coordinates is

$$d^4k = dE\, d^3\mathbf{k} = |\mathbf{k}|^2\, dE\, d|\mathbf{k}|\, d\cos\theta\, d\varphi \ . \tag{1.7}$$

Since we are integrating over all values of k^μ, we are free to align our axes however we want. A particularly convenient choice is to align k_z to be aligned with $\mathbf{s}$. In this case, $\mathbf{k}\cdot\mathbf{s} = ks\sin\theta$, where it should be understood that k and s are magnitudes of the spatial 3-vectors[4]. Then it is convenient to note that

$$-ik\cdot y = iks\cos\theta - iEu \ . \tag{1.8}$$

Perform the angular integrals in (1.6). Show that you are left with

$$G(y) = \frac{c^2}{4\pi^3 s}\int_0^\infty \sin\, ks \left(\int_{-\infty}^{\infty} \frac{k}{c^2k^2 - E^2} e^{-iEu} dE\right) dk \ . \tag{1.9}$$

1.5 Practical Pole Pushing for Poor Physics People

Observe the familiar-looking dE integral in parenthesis in (1.9). As we usually do, we'd like to evaluate this using a contour integral. Identify the location of the two poles in the complex E plane.

You're now an expert on these integrals, so the following steps shoudl be routine. Refer back to Homework 5 if they are not.

Following the notation of Homework 5, you have a choice of contours: $\mathcal{C}_+$ and $\bar{\mathcal{C}}_-$. Comment on which contour one should choose as a depending on the sign of $u \equiv t - t'$.

Now we have to pick a prescription for how to navigate the poles. There are two physically-motivated choices: one can push the poles into the upper half plane and into the area enclosed by $\mathcal{C}_+$, or one can push the poles into the lower half plane and into the area enclosed by $\bar{\mathcal{C}}_-$. Which one corresponds to the causal solution? This is the retarded Green's function, $G^{(r)}(y)$. The a-causal solution is the advanced Green's function, $G^{(a)}(y)$. Perform the dE integral for both.

[4]That is, from now on I write $k = |\mathbf{k}|$. There should not be any ambiguity with the 4-vector, $k^2 = E^2/c^2 - \mathbf{k}^2$, which no longer shows up in our expressions.

1.6 The Green's Function

You now have $G^{(r)}(y)$ and $G^{(a)}(y)$ as expressions with a single integral over dk. Go ahead and perform this integral. You did a similar integral in Homework 5, the most straightforward calculation involved breaking the sine into exponentials.

HINT: Use, once again, the fact that $\int e^{ikx}dk = 2\pi\delta(x)$.

Update 11/1, HINT: A useful intermediate step is to show that

$$\int_{-\infty}^{\infty} \frac{e^{-iEu}}{(E - ck - i\varepsilon)(E + ck - i\varepsilon)} dE = \frac{-i\pi}{ck}\left(e^{icku} - e^{-icku}\right) \ . \tag{1.10}$$

Another useful tip is to use

$$\int_0^{\infty} \left(e^{ikX} + e^{-ikX}\right) dk = \int_{-\infty}^{\infty} e^{ikX} dk = 2\pi\delta(X) \ . \tag{1.11}$$

INTERMEDIATE STEP: The penultimate step is (writing $s = |\mathbf{s}|$):

$$G^{(r)}(y) = \frac{c}{4\pi r}\begin{cases} \delta(s - cu) - \delta(s + cu) & \text{if } u = t - t' \geq 0 \\ 0 & \text{if } u = t - t' < 0 \end{cases} \tag{1.12}$$

$$G^{(a)}(y) = \frac{c}{4\pi r}\begin{cases} 0 & \text{if } u = t - t' \geq 0 \\ \delta(s - cu) - \delta(s + cu) & \text{if } u = t - t' < 0 \end{cases} \ . \tag{1.13}$$

Then observe that the radial coordinate $s \geq 0$ so that one of the δ functions in each expression will always be zero. The fancy way of saying this is that one of the δ functions has no support.

Taking $x' = 0$, show the final expression as:

$$G^{(r)}(x) = \frac{c}{4\pi r}\delta(r - ct) \ . \tag{1.14}$$

What's the corresponding advanced Green's function?

1.7 Understanding

What is the physical interpretation of $G^{(r)}(x - x')$ for, say, $x' = 0$?

ANSWER: The Green's function solves the electromagnetic wave equation for a 'blip' impulse at $t = 0$ and $\mathbf{r} = 0$.

Draw the support[5] of the retarded and advanced Green's function on a spacetime diagram[6].

Now make the diagram fancy by indicating not only the support of the Green's function on the spacetime diagram, but also giving some indication of its magnitude. Possibilities include: use colors or line thickness to indicate $|G^{(r)}(x)|$. (Include a legend.) This doesn't have to be done precisely, but you should end up with a figure which conveys the propagation of an electromagnetic wave through spacetime.

[5]Support: the region of a function's domain where the function is non-zero.

[6]The horizontal axis is the radial distance from the origin and the vertical axis is time.

1.8 Covariance

One thing you might be concerned about is the covariance of the δ-function with respect to Lorentz transformations. Specifically: the argument of $\delta(ct \pm r)$ is not Lorentz invariant. Recall, however, the rule for δ-functions:

$$\delta\left(g(x)\right) = \sum_{x_i} \frac{\delta(x - x_i)}{|g'(x_i)|} \qquad\qquad g(x_i) = 0 \ . \tag{1.15}$$

Writing $x = (ct, \mathbf{r})$, show that

$$\delta(x^2) = \frac{1}{2r}\left[\delta(r - ct) - \delta(r + ct)\right] \ . \tag{1.16}$$

Compare this to (1.13). Thus, for example,

$$G^{(r)}(x) = \frac{c}{2\pi}\delta(x^2)\Theta(t) \tag{1.17}$$

where $\Theta(t)$ is the Heaviside step function. What's the expression for $G^{(a)}(x)$? Are these expressions Lorentz invariant?

1.9 Another quick digression

It turns out that when calculating matrix elements (amplitudes) for quantum mechanical scattering a relativistic particle of mass m with four-momentum $p = (E, \mathbf{p})$, you often end up with expressions in momentum space where an integrand $f(p)$ needs to be integrated in a way where the on-shell conditions are fixed:

$$p^2 = E^2 - \mathbf{p}^2 = m^2 \qquad\qquad E > 0 \ . \tag{1.18}$$

In other words: you have an integral over the four directions in momentum space, but 'physicality' imposes a constraint. You should be dreaming of Lagrange multipliers[7].

What one ends up doing is writing down integrals of the form

$$\int d^4p\, \delta(p^2 - m^2)\Theta(E) f(p) \ . \tag{1.19}$$

Writing $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$, show that

$$\int d^4p\, \delta(p^2 - m^2)\Theta(E) f(p) \ . = \int \frac{d^3\mathbf{p}}{2E(\mathbf{p})} f(E(\mathbf{p}), \mathbf{p}) \ . \tag{1.20}$$

This funny-looking differential element $d^3\mathbf{p}/2E(\mathbf{p})$ doesn't look Lorentz invariant, but we have derived that it is. Note that in the function, f, we fix E to be the required value by 'on-shell-ed-ness.'

[7]We won't use Lagrange multipliers, but a generalization of this method where the δ-function is promoted to a Lagrange multiplier shows up in gauge theory and is called the Fadeev-Popov procedure.

1.10 Radiation

Looking at the solutions for $G^{(r)}$ and $G^{(a)}$ and the form of the spacetime diagrams in sub-problem 1.7, you should now have some intuition for the meaning of the potential A from each of these cases. Suppose that the current $j_\mu(x)$ is localized in space and time. This means you have something electromagnetic that appears then disappears, and is never 'infinitely' large. The potential from this electromagnetic wibbly-wobbly[8] can be written in two ways

$$A_\mu(x) = A_\mu^{(\text{in})}(x) + \int G^{(r)}(x-x')\, j_\mu(x') d^4x' \tag{1.21}$$

$$A_\mu(x) = A_\mu^{(\text{out})}(x) + \int G^{(a)}(x-x')\, j_\mu(x') d^4x' \;. \tag{1.22}$$

Here $A^{(\text{in})}$ and $A^{(\text{out})}$ are solutions to the *free* wave equation. In $A^{(r)}$ we note that taking $t \to -\infty$ causes the integral to vanish—this leaves only $A^{(\text{in})}$ which we thus interpret as any incoming field coming in from the asymptotic past $t = -\infty$. Similarly, $A^{(\text{out})}$ is any outgoing field which exists in the asymptotic future.

We can interpret $A^{(\text{in})}$ as some electromagnetic field that existed before the wibbly-wobbly of the source $j_\mu(x)$, which the source later disturbs. This disturbed field then propagates into the future as $A^{(\text{out})}$.

The difference between $A^{(\text{in})}$ and $A^{(\text{out})}$ is the *radiated* field, $A^{(\text{rad})}$, from the source $j(x)$. This is

$$A_\mu^{(\text{rad})}(x) = A_\mu^{(\text{out})} - A_\mu^{(\text{in})}. \tag{1.23}$$

Write $A_\mu^{(\text{rad})}(x)$ in terms of the retarded and advanced Green's functions (trivial!) and reflect upon the fact that the advanced Green's function seems to be important for physics, after all.

2 Fun with Tensors

[**Update: 11/1**] Due to the exams and various mid-quarter hectic activities, everything except problem 2.1 is now **extra credit**. Go Lakers.

2.1 Symmetrization and Antisymmetrization

Please do this problem.

Consider a tensor T_{ij}. Decompose it into a symmetric and antisymmetric parts, $(T_{ij} \pm T_{ji})/2$. Show that these parts transform separately (they don't mix) under a 'rotation'. Specifically: show that the transformation of a symmetric tensor is symmetric and that the transformation of an anti-symmetric tensor is anti-symmetric. Convince yourself that this decomposition into symmetric and antisymmetric parts generalizes to multiple indices.

[8] https://www.youtube.com/watch?v=mDsN5lWLKU0

2.2 Electromagnetic Field of a Moving Charge

This problem and all subsequent problems are **extra credit**.

Consider the electric field of a point charge at rest. Write the electromagnetic field strength tensor $F_{\mu\nu}(x)$ for this field. Perform a Lorentz transformation to a frame where the point charge has velocity β in the positive x-direction. Determine $F_{\mu\nu}(x)$ in this reference frame. What are the electric and magnetic fields in this boosted frame?

2.3 Gauge transformations

In problem 1 we familiarized ourselves a bit with the electromagnetic potential, A. Recall that the electromagnetic fields are derived from A as $F = dA$. Show that geometrically there is a redundancy[9] in the definition of A spanned by the space of functions (0-forms).

2.4 Half of Maxwell's Equations

Write $F = dA$ as field equations relating the electric and magnetic fields to the scalar and vector potentials. Show that $F = dA$ reproduces two of the four Maxwell's equations.

2.5 Hodge duality is electromagnetic duality

Recall the Hodge star operator acting on a k-form living on an n-dimensional space:

$$\star dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{1}{(n-k)!}\sqrt{\det g}\, g^{i_1 j_1} \cdots g^{i_k j_k} \varepsilon_{j_1 \cdots j_n} dx^{j_{k+1}} \wedge \cdots dx^{j_n}. \tag{2.1}$$

Defining $\star F = \frac{1}{2}\widetilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu$, show that $\widetilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu}$. (There may be a factor of 2 difference.) Show, further, that the components of $\widetilde{F}^{\alpha\beta}$ are related to those of $F_{\mu\nu}$ by a simple transformation on the electric on the magnetic fields; what is this transformation?

3 More fun with tensors

3.1 Induced metric, volume in polar coordinates

Recall that if an n-dimensional manifold is a surface in a larger Euclidean space $\mathbb{R}^{m>n}$, then the Euclidean metric induces a metric on the surface. Let x^i be the usual Cartesian coordinates on $\mathbb{R}^m$ and y^a parameterize the surface. Clearly $i \in 1, \cdots m$ and $a \in 1 \cdots n$. The surface is parameterized by a mapping $x^i(y)$. The induced metric is

$$g_{ab} = \sum_{i=1}^{m} \frac{\partial x^i}{\partial y^a} \frac{\partial x^i}{\partial y^b} \ . \tag{3.1}$$

[9] Called a gauge symmetry.

The volume form of the surface is $\sqrt{\det g_{ab}} dy^1 \wedge \cdots \wedge dy^n$.

Derive the induced metric of the unit two-sphere embedded in $\mathbb{R}^3$ parameterized by the spherical angles θ and φ. Determine the differential area ('2-volume') element, confirm that it is $d(\cos\theta)d\varphi$.

3.2 Arc length vs. integrating a 1-form

Suppose live in a manifold $\mathcal{M}$. In lecture we talked about integrating a 1-form df over a curve, $\mathcal{C}$. Stokes' theorem told us that

$$\int_{\mathcal{C}} df = \int_{\partial C} f = f(p_1) - f(p_0) \;, \tag{3.2}$$

where p_0 and p_1 are the beginning and endpoints of C. This gives the usual notion of a line integral.

We can write this with respect to a parameterization of $\mathcal{C}$. Let $\mathbf{x} : [0,1] \to \mathcal{C} \in \mathcal{M}$ parameterize the path. Then we may write

$$\int_{\mathcal{C}} df = \int_0^1 dt \left(\frac{df(\mathbf{x}(t))}{dt} \right) \;. \tag{3.3}$$

You can think of t as some kind of internal clock of a traveller whose progress along $\mathcal{C}$ is given by $\mathbf{x}(t)$. Observe that if $\mathcal{C}$ has some 'switchback' in $\mathcal{M}$ where the path folds back on itself, those two legs cancel.

Compare this to the case where $df = ds$ is simply the infinitesimal distance traveled. Recall that $ds^2 = g_{ij} dx^i dx^j$. Then consider the slightly different integral:

$$\int_0^1 dt \left| \frac{ds}{dt} \right| = \int_0^1 \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \;. \tag{3.4}$$

Observe that in this case the 'switchbacks' do not cancel. Indeed, this expression measures the arc length of a path. It is not the integral of a differential form that can be solved using Stokes' theorem.

Consider the case where g_{ij} is the Minkowski space metric. This introduces relative signs in in the terms summed under the square root sign of the integrand, (3.4), so that terms may cancel. What does it mean when the arc length in Minkowski space is zero? Comment on how this is relevant to the statement that 'photons will never live to see their first birthday, yet they live forever.'