HOMEWORK 5: **Mid-Season Clip Show**[1]

COURSE: Physics 231, *Methods of Theoretical Physics* (2016)
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If this class were to have a final exam to show that you've learned the 'need to know' mathematical methods required of a physicist, this is the problem I would ask.

1 The Damped Spring

One of California's problems is a prolonged drought that led to a very dry spring. This homework has the opposite problem: what happens when a spring is damped? The goal of this problem is to take a familiar system and solve it from beginning to end using the Green's functions techniques that we've spent the last four weeks developing.

Suppose you have a weight on a spring with spring constant $k = \omega_0^2$ and damping coefficient γ. The spring constant tells us about the system's resonant frequency and the damping coefficient tells us how the system loses energy to the environment, for example through drag. Let $x(t)$ be the displacement of the spring from its stationary state and $F(t)$ be some driving force applied to the spring. For example, we can imagine $F(t)$ to be a sinusoidal driving force from someone trying to 'bounce' the weighted spring like a cheap yo-yo. Or we can impart a force $F(t)$ on the spring that starts at $t = 0$ and dies off exponentially in time as a model of a sudden impulse.

The system obeys the following second-order differential equation in time:

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t) \ . \tag{1.1}$$

We will solve this system to determine $x(t)$ in response to the driving force $F(t)$. There are no tricks to this problem: you've probably solved it in an undergraduate mechanics course, so you know (at least qualitatively) what the behavior is. Let's see how this all works in the context of Green's functions.

1.1 Dimensional Analysis

The damping coefficient can be written as a product of the natural frequency, ω, times a dimensionless ratio, ζ; that is $\gamma = \omega_0\zeta$. Notice, by dimensional analysis, that we've implicitly set mass to unity, $m = 1$. As a warm up, restore factors of m in (1.1).

1.2 Green's Function Equation

We want to solve (1.1) for some general driving force $F(t)$. As an intermediate step, we want to solve it for $F(t) = \delta(t - t')$. The solution is defined to be the Green's function, $G(t, t')$:

$$\ddot{G}(t,t') + 2\gamma\dot{G}(t,t') + \omega_0^2 G(t,t') = \delta(t - t') \ . \tag{1.2}$$

[1] https://en.wikipedia.org/wiki/Clip_show

Here the infinite set of distributions $\delta(t-t')$ for all possible t' form a "basis" for the driving force $F(t)$. The idea is that by sticking together a bunch of these $\delta(t-t')$, you can trivially construct any driving force function

$$F(t) = \int_{-\infty}^{\infty} F(t')\,\delta(t-t')\,dt' \ .$$

The solution to the physical problem, (1.1), is then given by the analogous construction of Green's functions

$$x(t) = \int_{-\infty}^{\infty} G(t,t')\,F(t')\,dt' \ . \tag{1.3}$$

Make a mental note that this is completely analogous to constructing the electrostatic potential at point $\Phi(\mathbf{r})$ by summing together the contributions from a charge distribution:

$$\Phi(\mathbf{r}) = \sum_i \frac{eq_i}{|\mathbf{r}-\mathbf{r}'_i|} \to \int d\mathbf{r}'\,\frac{e\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'_i|} \ . \tag{1.4}$$

Identify the Green's function in this electrostatic analogy. The Green's function $G(t,t')$ in (1.2) takes two arguments, where as the electrostatic Green's function, $G(\mathbf{r}-\mathbf{r}')$ is only a function of a single argument. Do you expect $G(t,t')$ to similarly depend on $(t-t')$? Should it be $(t-t')$, $|t-t'|$, or something different? Comment briefly on the physical significance. Hint: feel free to revise and finalize your answer only after you've worked through the problem, but I want you to think about this before moving forward.

1.3 Sanity check: dimensional analysis again

Does (1.2) make sense dimensionally? Compared to (1.1), the right-hand side does not have the dimensions of a force. What are the dimensions of $\delta(t-t')$? What are the dimensions of $G(t,t')$? Confirm that (1.3) corroborates your dimensional analysis.

1.4 Fourier Transform

Let us use the Fourier transform convention[2] for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\widetilde{f}(k)\,e^{-ikt} \qquad\qquad \widetilde{f}(k) = \int_{-\infty}^{\infty} dt\,f(t)\,e^{ikt} \ . \tag{1.5}$$

Now Fourier transform both sides of (1.2). The left-hand side is

$$\int_{-\infty}^{\infty} \left(\frac{d^2G(t,t')}{dt^2} + 2\gamma\frac{dG(t,t')}{dt} + \omega_0^2 \right) e^{ikt}dt \ . \tag{1.6}$$

[2] (1.5) is a Fourier Transform convention common in high energy physics. Keeping the 2π under the dk turns out to be convenient for determining the validity of certain Taylor expansions. Convince yourself that this is a valid choice. In fact, you can choose any signs for the exponentials as long as the Fourier transform and its inverse have opposite signs. There has to be an overall factor of $1/2\pi$ split however you want between the dk and the dt. To check this, use $\delta(t-t') = (2\pi)^{-1}\int dk\,\exp[ik(t-t')]$ and that δ is symmetric for $(t-t') \leftrightarrow (t'-t)$.

What is the right-hand side?

Show that by integration by parts the left-hand side of this expression becomes

$$\left(-k^2 - 2i\gamma k + \omega_0^2\right) \widetilde{G}(k,t) \ . \tag{1.7}$$

In doing this integration by parts, you had to make some implicit assumptions about the values of $x(t)$ at $t = \pm\infty$. Explicitly state what these assumptions are and explain why they are justified. Your justification should be physically motivated: interpret (1.2) as a the differential equation for the system with some infinitesimal impulse applied.

1.5 Why did it turn complex?

(1.7) is funny looking because the differential operator is not real. The imaginary part of this operator is associated with γ—what is its physical significance. Does it make sense that this piece should be imaginary?

An analogy: the Hamiltonian for a single Higgs boson[3] at rest takes the form $H = M - i\Gamma$. As a quantum mechanic, this should make you unhappy—why is H not Hermitian[4]? What does the Γ physically signify, and how do we interpret the time evolution by e^{-iHt}? (Hint: we did *not* discover the Higgs boson by ever observing a Higgs boson smack into a detector element, like we would for an electron. What did we observe?) Comment on how this analogy relates to the imaginary piece in (1.7).

1.6 Solution in Fourier Space

Now show that the solution to (1.2) is

$$\widetilde{G}(k,t) = \frac{e^{ikt'}}{-k^2 - 2i\gamma k + \omega_0^2} \ . \tag{1.8}$$

Hint: This step takes one line if you wrote out the right-hand side correctly in step 1.4. What we really want is Fourier transform back to $x(t)$. Show that this is given by

$$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikt'}}{-k^2 - 2i\gamma k + \omega_0^2} e^{-ikt}\, dk \ . \tag{1.9}$$

Combine the exponentials to show that indeed, $G(t,t')$ can be written as a function of a single combination of t and t', namely $G(t-t')$ as anticipated in the discussion below (1.4).

[3] For the purposes of this example, you can replace the Higgs boson by a muon or a radioactive americum-237 nucleus that you can find in a smoke detector. However, (this is a hint) you cannot replace it with a proton. Or, according to this, a diamond: `https://youtu.be/QFSAWiTJsjc`.

[4] Those who are interested in this can see section 13.1 of `http://particle.physics.ucdavis.edu/hefti/members/lib/exe/fetch.php?media=students:flavor-grossman-extras-new.pdf`

1.7 Identifying Contours

We would now like to evaluate (1.9) as a contour integral. This means we have to complete the integration along the real k line into a loop in the complex k plane. There are two contours to consider that also include the real line: the contour that goes counter-clockwise over the upper-half plane ($\mathcal{C}^+$) and the contour which goes clockwise over the lower-half plane ($\bar{\mathcal{C}}^-$). The contours that you use depends on whether $t > t'$ or $t < t'$. For each case, identify what factors in the integrand determine which contour you should use. Form your answer as a sentence of the following form:

> *If $t < t'$ ($t > t'$), then you should use the [upper/lower]-half plane the integral over the arc vanishes as the arc radius becomes large.*

1.8 Poles

Where are the poles of the integrand on the complex k plane? If you haven't already, draw the contours $\mathcal{C}^+$ and $\bar{\mathcal{C}}^-$ as well as the poles of the integrand. Recalling that t' is the time of the δ-function 'blip' that caused the motion, causality requires that $G(t, t')$ is zero for $t < t'$. Check that this is true.

1.9 Why are the poles complex!?

Observe that the integrand is perfectly well behaved on the real axis and that the integration contour does not 'hit' any poles. We never had to talk about principal values or doing ϵ-sized deformations to the denominator of the integrand. This is very different from the simple harmonic oscillator that we did in class and in the homework. Compare the integrand to that of the undamped harmonic oscillator—what term in (1.2) caused the poles to become 'automatically' complex?

One byproduct of this is that there is no choice to be made about an advanced or retarded Green's function. There's only one Green's function.

You should be a little worried about this. It looks like changing the sign of γ would have pushed the poles into the upper half of the complex plane ($\text{Im } k > 0$) so that $G(t - t')$ would not be causal. But you don't have to worry about this. Comment on why the sign of the γ term in (1.1) is fixed and what would go wrong if it were flipped. In other words: what's unphysical about (1.1) if $\gamma \to -\gamma$?

Take a moment to reflect upon this in the context of part 1.5. We don't have to push the pole off the real axis because it's already complex. Any real spring has some (perhaps small) damping. In this sense, maybe all our manipulations invoking principal values for the ideal spring are all unnecessary. This turns out not to be true: there exist "harmonic oscillators" in physics whose poles must be on the real axis[5].

[5] One example is the photon. The photon does not decay—think about why this means the pole in the Green's function for a photon propagating from one spacetime point to another is on the real axis.

1.10 Residue Theorem

Apply Cauchy's residue theorem to calculate $G(t-t')$ for the cases $t > t'$ and $t < t'$. Be sure you include the minus sign for the orientation of $\bar{\mathcal{C}}^-$. If the poels are located at k_1 and k_2, show that this gives

$$G(t-t'>0) = -2\pi i \left[\frac{e^{-ik_1(t-t')}}{k_1-k_2} + \frac{e^{-ik_2(t-t')}}{k_2-k_1} \right] \tag{1.10}$$

$$G(t-t'<0) = 0\ . \tag{1.11}$$

Plug in the values of k_1 and k_2 that you found above to show that the $t-t'>0$ case integrates to

$$G(t-t') = \frac{e^{-\gamma(t-t')} \sin\left[\sqrt{\omega_0^2-\gamma^2}\,(t-t')\right]}{\sqrt{\omega_0^2-\gamma^2}}\ . \tag{1.12}$$

1.11 Check against a known limit

Since you've done so much hard work on this, check that the $\gamma \to 0$ limit reproduces the undamped case that we did together in class and that you did on previous homeworks.

1.12 Now what?

If you were to print a t-shirt for your progress thus far, it would say "I took four weeks of Physics 231 and I all I got was this stupid Green's function." Now that we have $G(t-t')$, how do we actually solve the problem that we care about?

Let's go back to the complete problem. We would like to find the solution to the forced, damped spring with some force function $F(t)$, that is, (1.1). The solutions of this system are $x(t)$, given by

$$x(t) = Ax_1(t) + Bx_2(t) + \int_{t_1}^{t_2} \frac{e^{-\gamma(t-t')} \sin\left[\sqrt{\omega_0^2-\gamma^2}(t-t')\right] F(t')}{\sqrt{\omega_0^2-\gamma^2}} dt'\ , \tag{1.13}$$

where $x_{1,2}(t)$ are solutions to the *homogeneous* differential equation, that is (1.1) with $F(t) = 0$. If this isn't obvious, then check that this is the general solution by actually plugging (1.13) into (1.1). Observe, specifically, that the $x_{1,2}(t)$ drop out by their definition as solutions to the homogeneous equation.

Explain why we may take the upper limit of integration to be $t_2 = t$. The lower limit, t_1 can be taken to be the time at which the driving force is applied.

1.13 Initial Conditions

The coefficients A and B are determined by the initial conditions of the problem. OPTIONAL[6]: solve for $x_{1,2}(t)$; I suggest doing a Fourier transform and using *Mathematica* to check.

[6]Optional means: you should do this if it's not immediately clear to you that you can do this.

Without knowing the precise form of $x_{1,2}(t)$, we know that for a specific initial condition (say, at t_1), the coefficients A and B vanish. What are the initial conditions for which this is true? Specify $x(t_1)$ and $\dot{x}(t_1)$. You should answer this by thinking about it physically, and confirm your intuition by looking at (1.13) to see that it makes sense. In a complete sentence (using words, not equations), explain the state of the system at $t = t_1$.

1.14 Exponentially falling blip

For simplicity, set $t_1 = 0$. Suppose that the time-dependent force is

$$F(t) = F_0 e^{-\alpha t} \ . \tag{1.14}$$

Assume the initial conditions above for which $A = B = 0$. Solve (1.13) by performing the integral. No need to do anything fancy like a contour integral. You may find it useful to write the sine as a sum of exponentials. Show that the solution is

$$x(t) = \frac{F_0}{\sqrt{\omega_0^2 - \gamma^2}} \frac{\sin\left[\sqrt{\omega_0^2 - \gamma^2}t - \delta\right]}{\sqrt{\omega_0^2 + \alpha^2 - 2\alpha\gamma}} e^{-\gamma t} + \frac{F_0}{\omega_0^2 + \alpha^2 - 2\alpha\gamma} e^{-\alpha t} \ . \tag{1.15}$$

Here we've defiend

$$\tan\delta = \frac{\sqrt{\omega_0^2 - \gamma^2}}{\alpha - \gamma} \ . \tag{1.16}$$

If you don't believe this solution, check it by plugging into (1.1). Write out the limiting form when the damping goes to zero, $\gamma \to 0$. Show that in the limit of negligible damping and for 'late times', that

$$x(t) = \frac{F_0}{\omega_0} \frac{\sin(\omega_0 t - \delta)}{\sqrt{\omega_0^2 + \alpha^2}} \ . \tag{1.17}$$

What does 'late times' mean in this context? Identify the quantity with dimension T (time) that you can use to define 'late.' What does this mean physically? Explain what's happening as one of the terms in (1.15) vanishes.

1.15 Energy of the system

If the system initially has zero energy—confirm that this is consistent with the boundary conditions that set $A = B = 0$—show that the energy of the system at late times is

$$E = \frac{F_0^2}{2(\omega_0^2 + \alpha^2)} \ . \tag{1.18}$$

HINT: recall that the energy for the system is $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2$.

Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

2 Kramers–Kronig in Two Lines

Read Ben Yu-Kuang Hu's half-page paper "Kramers-Kronig in two lines" in the *American Journal of Physics*[7] As promised in the title, the Kramers–Kronig relations are derived in a slick, compact way. The brevity comes from assuming a clear understanding of the material covered thus far in our course. In your own words, flesh out their derivation in a way that would be pedagogically clear to you before you took this class.

[7] *Am. J. Phys.* **57**, 821 (1989); `http://dx.doi.org/10.1119/1.15901`