# LEC 11: DISPERSION RELATIONS

17 OCTOBER

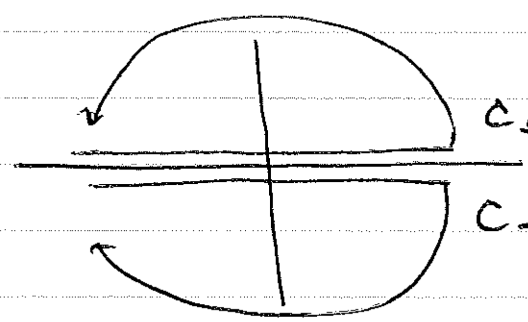
## AGENDA:

- EULER $\Gamma$ function $\longrightarrow$ DISCUSSION OR IF WE HAVE TIME
- CORRECTION – MY FOURIER SIGNS
- DISPERSION

## CORRECTION TO LAST TIME

FOURIER CONVENTION: — note: CHOICE OF SIGN IS CONVENTION

$$G(t) = \int \tilde{G}(k)\, e^{ikt}\, dk$$



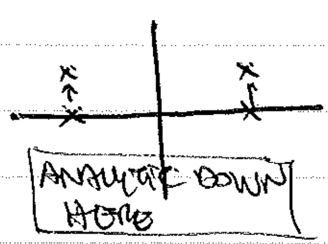
$\mathbb{K}$

$C_+ \leftarrow e^{ikt}$ CONV. WHEN $\boxed{t>0}$

$C_- \leftarrow$ —— " —— $\boxed{t<0}$

GOOD GREEN'S FUNCTION: $G^{(r)} \longleftrightarrow$ CAUSAL: $G^{(r)}(t<0) = 0$, $G^{(r)}(t>0) \neq 0$

SO WE PUSH THE POLES INTO UPPER HALF PLANE s.t. the $t>0$ CONTOUR PICKS UP RESIDUES WHILE $t<0$ GETS NOTHING.




$\rightsquigarrow$ this is the main takeaway from last week!

n.b. if we used $G(t) = \int \tilde{G}(k)\, e^{-ikt}\, dk$,
then we swap the $\cancel{G^{(a)} \longleftrightarrow G^{(r)}}$ $C_+ \longleftrightarrow C_-$ contours
(& the $G^{(a)} \longleftrightarrow G^{(r)}$ pole conventions)

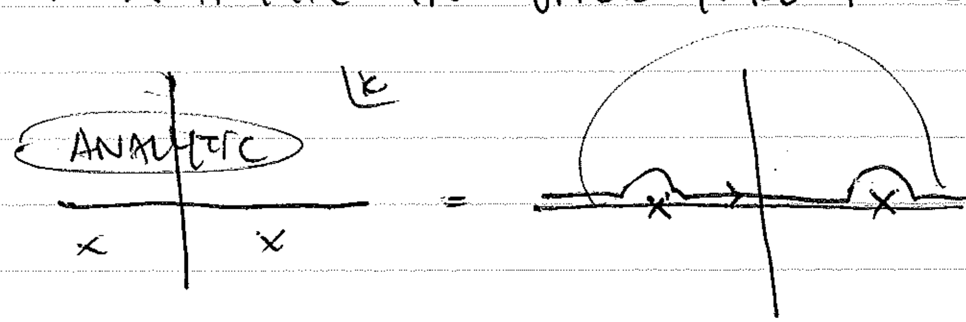
So: in what follows (& apparently what's the standard convention), WE DEFINE THE FOURIER TRANSFORM w/rt TIME AS

$$f(t) = \int \tilde{f}(k)\, e^{-ikt}\, dt$$

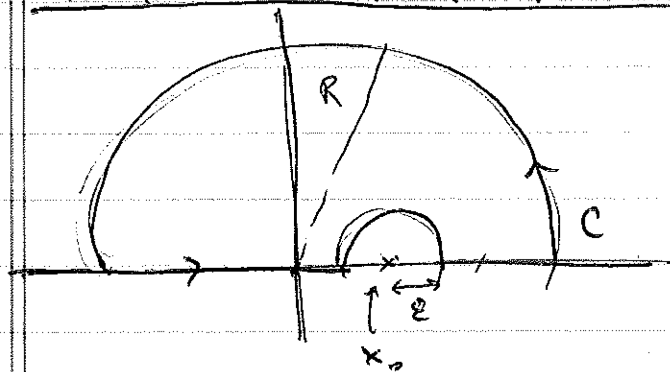
[ note: THIS IMPLIES A + SIGN FOR THE SPATIAL TRANSFORM SINCE: $f(x) = \int \tilde{f}(k)\, e^{-ik^\mu x_\mu}\, d^4x$ ]

<u>IN THIS CONVENTION</u>:

CAUSAL $\longleftrightarrow$ POLES SHIFTED <u>DOWN</u>
$\longleftrightarrow$ ANALYTIC IN UPPER HALF PLANE



APPROACH THIS FROM PRINCIPAL VALUE P.O.V.




nb. if f were analytic everywhere, it would be a boring function

SUPPOSE $f(z)$ ANALYTIC IN UPPER HALF. PLANE (like a good causal Green's function!)

& $f(z) \to$ on UPPER ARC; ie $f(Re^{i\theta}) \xrightarrow[R\to\infty]{} 0$

CAUCHY THM: $$\oint_C \frac{f(z)\,dz}{z-x_0} = 0$$

b/c $\frac{1}{z-x_0}$ IS ANALYTIC OVER THIS REGION.

$$\oint_C \frac{f(z)\,dz}{z-x_0} = \Big( \underbrace{\int_{-\infty}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{\infty}}_{\text{PRINCIPAL VALUE}} + \int_{\substack{\text{SMALL}\\ \text{ARC}}} + \underbrace{\int_{\substack{\text{BIG}\\ \text{ARC}}}}_{=0 \text{ BY ASSUMP}} \Big) \frac{f(z)\,dz}{z-x_0}$$

$z = \varepsilon e^{i\theta} + x_0$

$$\int_{\pi}^{0} \frac{f(\varepsilon e^{i\theta}+x_0)}{\cancel{\varepsilon e^{i\theta}}} \cancel{\varepsilon} i \cancel{e^{i\theta}}\, d\theta$$

$$= -i\pi f(x_0)$$

~~nb. if~~

$$\boxed{P\int_{-\infty}^{\infty} \frac{f(x')}{x'-x_0}\,dx' = i\pi f(x_0)}$$

DECOMPOSE INTO $\mathbb{R}$ & $\mathbb{I}m$ PARTS : $f = u + iv$

$$\mathcal{P}\int \frac{u(x) + iv(x)}{x - x_0}\,dx = i\pi u(x) - \pi v(x)$$

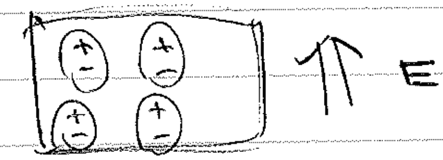
$$\Rightarrow \begin{cases} u(x) = \frac{1}{\pi}\mathcal{P}\int \frac{v(x)}{x - x_0}\,dx \\ v(x) = \frac{-1}{\pi}\mathcal{P}\int \frac{u(x)}{x - x_0}\,dx \end{cases}$$

↳ kind of like an integrated version of the Cauchy-Riemann eqs.

the $\mathbb{R}$ & $\mathbb{I}m$ parts are related!

↳ also gets back to the notion that analytic functions "want" to be single variable, but $\mathbb{C}$ plane is 2D.

## <u>RECALL</u>: EM WAVES IN <u>DIELECTRICS</u>




$$\underline{\nabla} \cdot \underline{E} = 4\pi(\rho_{FREE} + \rho_{BOUND})$$

↑ WANT TO TREAT THIS AS "ENVIRONMENT"

$$= -\underline{\nabla} \cdot \underline{P} \quad \text{(POLARIZATION)}$$

$$\Rightarrow \underline{\nabla} \cdot (\underbrace{\underline{E} + 4\pi \underline{P}}_{\underline{D}}) = 4\pi \rho_{FREE}$$

(DIELECTRIC DISPLACEMENT)

for not-too-big $\underline{E}$, many materials obey

$$\underline{P} = \chi \underline{E}$$

↑ ELECTRIC SUSCEPTIBILITY

$$\Rightarrow \underline{D} = \underbrace{(1 + 4\pi\chi)}_{\varepsilon} \underline{E}$$

← DIELECTRIC CONST.

FOR AN EM WAVE PASSING THROUGH THE MEDIUM THE VALUE OF $\chi$ (OR $\varepsilon$) IS <u>FREQUENCY DEPENDENT</u>.

↳ PHYSICALLY: $\exists$ time scale for the molecules to flip

$$\boxed{P(\omega) = \chi(\omega) E(\omega)}$$ ← UP TO FACTORS OF $2\pi$

IN FACT, $\chi$ IS A GREEN'S FUNCTION:

$$\underline{P}(t) = \int \chi(t-t')\, \underline{E}(t')\, dt'$$

CHK

$$\int P(\omega) e^{i\omega t}$$

$$= \int dt' \int d\omega\, e^{i\omega(t-t')}\, \chi(\omega) \int d\omega' e^{i\omega' t'} E(\omega')$$

$$= \underbrace{\int dt' \, d\omega\, d\omega'\, e^{i(\omega'-\omega)t'}}_{2\pi\, \delta(\omega'-\omega)} e^{i\omega t}\, \chi(\omega)\, E(\omega')$$

$$= \int d\omega\, e^{i\omega t}\, \chi(\omega) E(\omega) \quad //$$

WHAT DO WE KNOW ABOUT $\chi$ (AS A GREEN'S F'N)?

- $\chi(t<0) = 0 \longleftrightarrow$ ANALYTIC IN UHP

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}(\chi(\omega)) &= \frac{1}{\pi} P\int \frac{\mathrm{Im}[\chi(\omega')]}{\omega' - \omega}\, d\omega' \\ \mathrm{Im}(\chi(\omega)) &= \frac{-1}{\pi} P\int \frac{\mathrm{Re}[\chi(\omega')]}{\omega' - \omega}\, d\omega' \end{aligned}}$$

## SO WHAT

HECHT & ZAJAC p. 63

EM WAVE $\sim e^{iKx - i\omega t}$

$K = \omega / v$ ← velocity (units of c)

BUT: $n = \frac{1}{v} = \sqrt{\varepsilon \mu} \to \sqrt{\varepsilon}$ ($\mu = 1$) "PURE" DIELEC.

NOW $\varepsilon = 1 + 4\pi\chi$ HAS Re & Im PART

FREQ. DEP: DISPERSION
DIFF. WAVELENGTHS
SEPARATE.

$e^{i(iK)x} \to e^{-Kx}$

DISSIPATION
ENERGY LOST TO
THE MEDIUM.

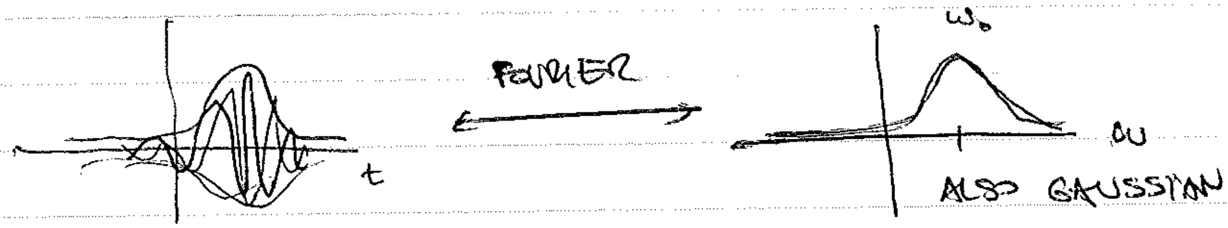
So KRAMERS-KRÖNIG:
CAUSALITY/ANALYTICITY ⇒ RELATES DISPERSIVE & DISSIPATIVE PARTS OF PROPAGATION

~~[illegible]~~?

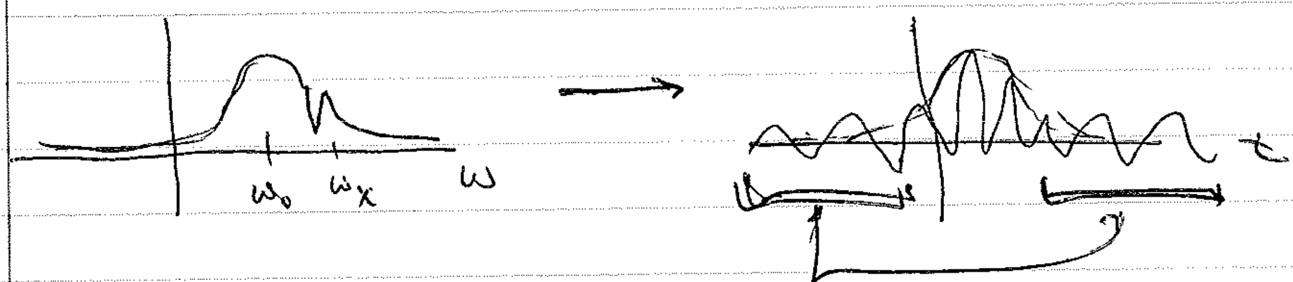
APPLICATION: MEASURING INDEX OF REFRACTION BY MEASURING ABSORPTION.

BUT <u>WHY</u> DO WE EXPECT THIS?

IMAGINE GAUSSIAN WAVEPACKET




NOW SEND THROUGH IDEALIZED ABSORBING MATERIAL THAT EFFECTIVELY PULLS OUT ONE FREQUENCY, $\omega_x$




NO LONGER A NICE WAVE PACKET!
GET THESE A-CAUSAL WAVES!

KRAMERS-KRÖNIG IS TELLING US THAT DISSIPATION (ABSORPTION) DOESN'T HAPPEN BY ITSELF → MUST COME W/ DISPERSION s.t. YOU DO <u>NOT</u> GET THIS NON-CAUSAL JUNK!

TO FURTHER UNDERSTAND, IT'S USEFUL TO SEPARATE $\chi$ INTO EVEN & ODD PIECES W/RT TIME REVERSAL:

$$\left.\begin{aligned} \chi_E &= \tfrac{1}{2}\left(\chi(t) + \chi(-t)\right) \\ \chi_O &= \tfrac{1}{2}\left(\chi(t) - \chi(-t)\right) \end{aligned}\right\} \text{ in time domain}$$

for $t>0$, these parts cancel in a causal function!

freq. space

$$\chi(\omega) = \int e^{i\omega t}\, \chi(t)\, dt$$

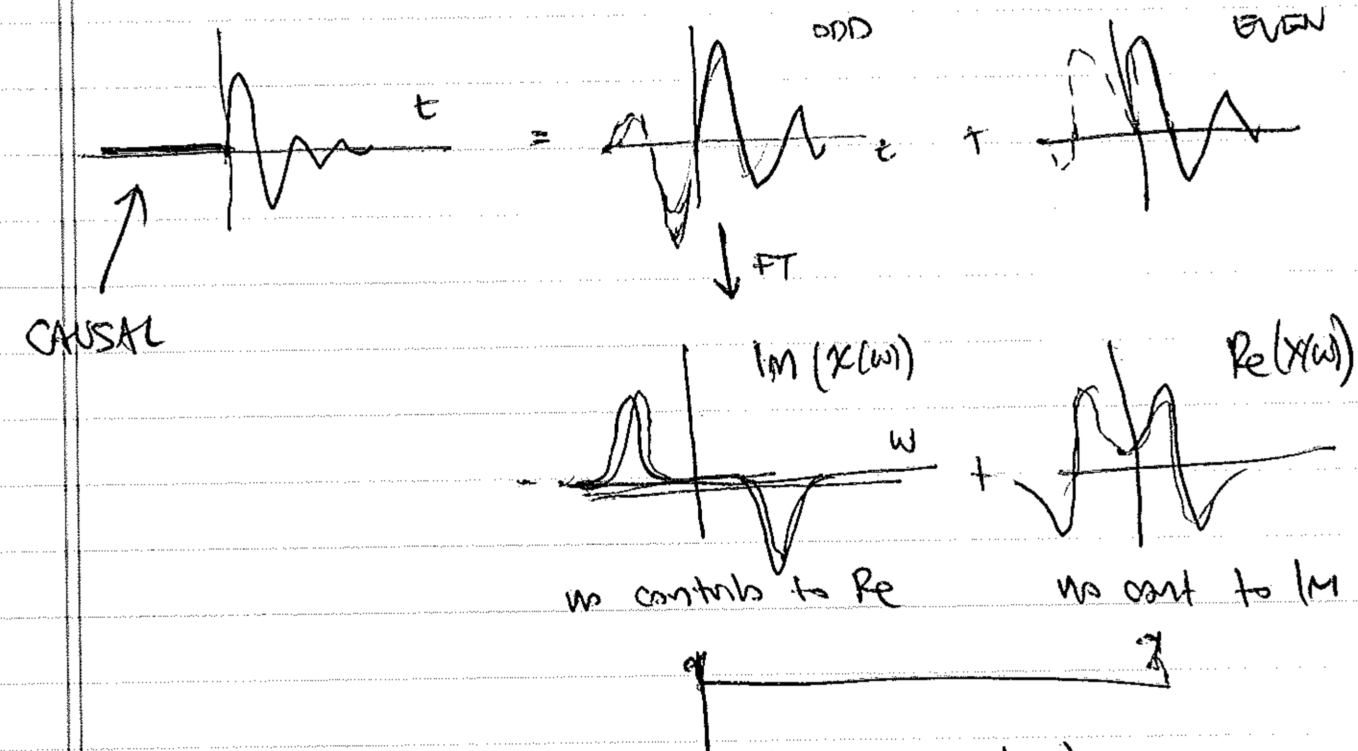
$$\underbrace{\cos \omega t}_{\text{EVEN IN } t} + i \underbrace{\sin \omega t}_{\text{ODD IN } t}$$

BUT $\int (\text{EVEN})(\text{ODD}) = 0$

$$\Rightarrow \begin{aligned} \mathrm{Re}\,\chi(\omega) &= \int e^{i\omega t}\, \chi_E(t)\, dt \\ \mathrm{Im}\,\chi(\omega) &= \int e^{i\omega t}\, \chi_O(t)\, dt \end{aligned}$$

THIS IS WHAT KRAMERS-KRONIG IS DOING

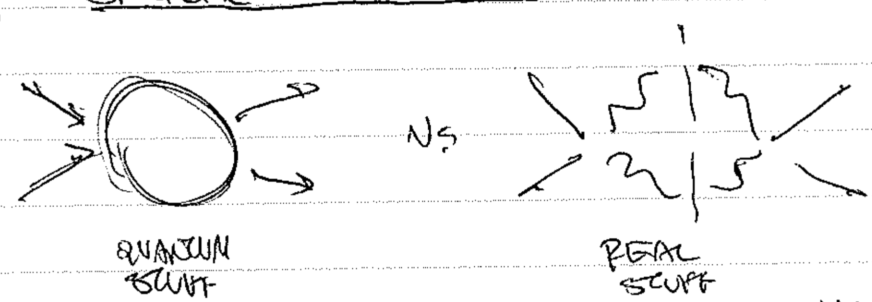
Sketch from WIKIPEDIA




Re & Im linked to one another.

TIES INTO LOTS OF OTHER THINGS

eg <u>OPTICAL THEOREM</u>




→ CAN TAKE YOU OUT OF YOUR HILBERT SPACE

→ ANALOGOUS TO DISSIPATION

$\text{Im}\,\mathcal{M} \sim \sigma_{\text{FWD}}$