REM: Stone & Goldb. are pretty good for this ↓ cu. n ish.
(I'LL PROB DRAW REM. BOOK FOR HW)

LEC 15: S. RELATIVITY, FORMS

OCT 26

LET'S GET SOME GROUNDING W/ TENSORS & METRICS
W/ A SIMPLE, FAMILIAR EXAMPLE

2D SPECIAL RELATIVITY (1+1 dim)
easy to generalize to 1+3 dim
START W/ ASSERTION: imagine (1+1) dim Minkowski

this is a flat 2D space with the metric

$$ds^2 = dt^2 - dx^2$$

← relative sign!
(overall sign is conventio

INVARIANT DISTANCE

$$= \eta_{\mu\nu}\, \underbrace{ds^\mu\, ds^\nu}_{\text{BASIS DUAL VECTORS / 1-FORMS}}$$

$\eta_{\mu\nu} = \text{diag}(1, -1)$

$\eta^{\mu\nu} = \text{diag}(1, -1)$

INVARIANTS ON THIS SPACE:
DOES NOT TRANSFORM UNDER "ROTATIONS"
what are the "rotations" of 2D minkowski?

~~TRANSLATIONS [illegible]~~

things w/ no indices
BECAUSE "INDICES TRANSFORM"

eg consider vectors

$$q^\mu = \begin{pmatrix} t \\ x \end{pmatrix} \quad \& \quad p^\mu = \begin{pmatrix} s \\ y \end{pmatrix}$$

then an invariant is:

$$q^\mu \underbrace{p^\nu \eta_{\mu\nu}}_{p_\mu} = \underbrace{\eta_{\mu\nu} q^\mu}_{q_\nu} p^\nu = q^\mu p_\mu = q_\mu p^\mu = \text{etc}$$

LOWERS INDEX

$$q^\mu p_\mu = q \cdot p = st - xy$$

even better, the length of q:

$$\|q\|^2 = q^2 = q^\mu q_\mu = q^\mu q^\nu \eta_{\mu\nu} = \boxed{t^2 - x^2}$$

((analog in $\mathbb{R}^2$: $t^2 + x^2$))

↳ this is invariant under: $t \to t' = \cos\theta\, t + \sin\theta\, x$
$x \to x' = -\sin\theta\, t + \cos\theta\, x$

from $\cos^2\theta + \sin^2\theta = 1$.

$$\cos x / \sin x = \frac{1}{2(i)}\left(e^{ix} \pm e^{-ix}\right)$$

NOW RECALL <u>*HYPERBOLIC*</u> TRIG FUNCTIONS

$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x})$$
$$\sinh x = \tfrac{1}{2}(e^{x} - e^{-x})$$

satisfy: $\boxed{\cosh^2 x - \sinh^2 x = 1}$

SO GUESS: $\begin{pmatrix} t \\ x \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$

is an <u>invariance</u>.

REMARK: NO MINUS SIGN ON LOWER-LEFT! in representation theory parlance: this "rotation" is <u>non-compact</u>.

NB: IN $\mathbb{R}^2$: $\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ MINUS SIGN MEANT: AS ONE THING GETS BIGGER, SOMETHING ELSE IS GETTING SMALLER. IN $\mathbb{R}^{1,1}$, CAN JUST KEEP MAKING THINGS 'BIGGER'

CHECK: $q \to q' = \begin{pmatrix} ct + sx \\ st + cx \end{pmatrix}$ where now $c = \cosh r$ etc.

$$q^2 = (ct + sx)^2 - (st + cx)^2 = \begin{matrix} c^2t^2 + \cancel{2cs\,tx} + s^2x^2 \\ - s^2t^2 - \cancel{2cs\,tx} - c^2x^2 \end{matrix}$$
$$= t^2 - x^2 \quad \checkmark$$

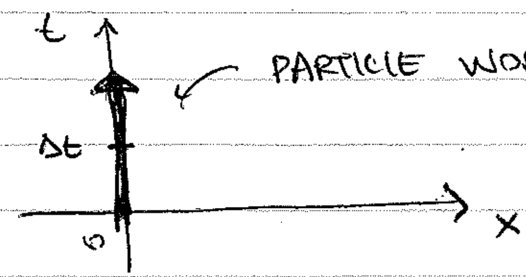
x IS PARTICLE POSITION.

CONNECT THIS TO PHYSICS

START W/ A FRAME WHERE PARTICLE IS AT REST
COORDINATES!

$\Rightarrow$ velocity $= 0$ $\leftarrow$ $\Delta x/\Delta t = 0$

rest frame



NOW TRANSFORM: ("ROTATE")

$$\begin{pmatrix} t \\ x \end{pmatrix} \longrightarrow \begin{pmatrix} t\cosh r + x\sinh r \\ t\sinh r + x\cosh r \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$$

still $(0,0)$ for $t = x = 0$.

BUT NOW CONSIDER SOME $t = \Delta t$
$x = \Delta x = 0$

then: $\begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \Delta t\cosh r \\ \Delta t\sinh r \end{pmatrix}$

& the velocity in this frame is

$$\boxed{\frac{\Delta x'}{\Delta t'} = \tanh r}$$

$\leftarrow$ r is the RAPIDITY

VELOCITY $= \beta$ ($= v/c$ for the DIMENSIONFUL)

BUT NOW I WANT TO CONNECT THIS TO MORE FAMILIAR THINGS:

$$\Lambda = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix}$$

$$\cosh^2 \eta - \sinh^2 \eta = 1$$

$$\Rightarrow \quad 1 - \underbrace{\tanh^2 \eta}_{\beta^2} = \frac{1}{\cosh^2 \eta}$$

top left element

$$\Rightarrow \Lambda = \begin{pmatrix} \gamma & \cdots \\ \cdots & \gamma \end{pmatrix} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

RECALL $\beta < 1 \Rightarrow$ gives <u>time dilation</u>

$$\Delta t' = \Delta t \, \gamma$$

also: $\sinh \eta = \tanh \eta \cosh \eta$
$= \beta \quad \gamma$

$$\Rightarrow \boxed{\Lambda = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}}$$

2D LORENTZ TRANSFORM

FORMS

CONSIDER DUAL VECTORS. WE'VE SEEN 2 EXAMPLES OF TENSORS w/ LOWER INDICES:

METRIC $\eta_{\mu\nu}$ $\leftarrow$ $ds^2 = \underbrace{\eta_{\mu\nu}\, dx^\mu dx^\nu}_{\text{SYMMETRIC}}$

DIFFERENTIAL OF A FUNCTION: $d_p f = \frac{\partial f}{\partial x^i} dx^i$

NOW WE WILL FOCUS ON A SPECIFIC CLASS OF DUAL VECTORS: THOSE w/ ANTISYMMETRIC INDICES.

FACT (pf is HW):

$$T_{ij\dots}^{\dots} = \underbrace{\tfrac{1}{2}\left(T_{ij\dots}^{\dots} + T_{ji\dots}^{\dots}\right)}_{\text{SYMMETRIC PART}} + \underbrace{\tfrac{1}{2}\left(T_{ij\dots}^{\dots} - T_{ji}\right)}_{\text{ANTISYMMETRIC PART}}$$

$$\equiv S_{ij}^{\dots} \qquad\qquad \equiv A_{ij}^{\dots}$$

further: under a "rotation",

$S \longrightarrow S'$ another symmetric matrix

$A \longrightarrow A'$ another antisym. matrix

SO WE CAN TALK ABOUT ~~[illegible]~~ ANTISYM. MATRICES "BY THEMSELVES"

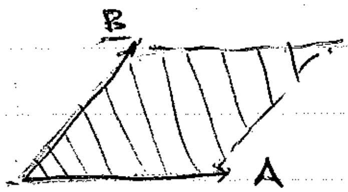
"Why?" will be clear soon.

GIVEN 2 VEC IN

IMAGINE THE CROSS PRODUCT : (in $\mathbb{R}^2$)

$\underline{A} \times \underline{B} = AB \sin\theta$ ←

$= -\underline{B} \times \underline{A}$



$\underline{A} \times \underline{B} = (\pm)\,\underline{\underline{\text{AREA}}}$

det.

$$\underline{A} \times \underline{B} = \begin{vmatrix} a^1 & b^1 \\ a^2 & b^2 \end{vmatrix}$$

$= a^1 b^2 - a^2 b^1$ ← antisymmetric

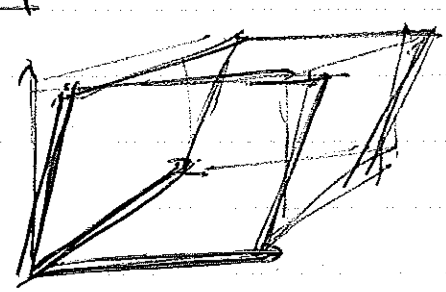
$= \varepsilon_{ij}\, a^i b^j$

IN FACT: THIS GENERALIZES

given 3 3-vectors $A, B, C \in \mathbb{R}^3$

VECTOR TRIPLE PRODUCT

$\underline{A} \cdot (\underline{B} \times \underline{C}) =$



volume of parallelepiped

what a weird combination of 3-vector operations. . .

TURNS OUT: (on $\mathbb{R}^3$)

$$\underline{A} \cdot (\underline{B} \times \underline{C}) = \begin{vmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \leftarrow \text{det}$$

$$\equiv \underbrace{\varepsilon_{ijk}}_{\text{3D LEVI-CIVITA TENSOR}} a^i b^j c^k$$

LEVI-CIVITA: one person

& so forth: the volume of an n-dim parallelpiped is ← w/ sides $\underline{A}_{(i)}$

$$\boxed{\varepsilon_{i_1 \dots i_n} a^{i_1}_{(1)} \dots a^{i_n}_{(n)}}$$

ANTISYMMETRIC FORMS ⟷ VOLUMES

⇝ integration?

DIFFERENTIAL K-FORMS

GENERALIZATION OF $d_p f$

sometimes $1/n!$

$$\boxed{\underbrace{\omega_{\mu_1 \dots \mu_n}}_{n \text{ lower indices}}(x)} \quad \underbrace{dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}}_{\text{basis of antisymmetric tensors}}$$

in gen, func. of position

$$dx^{\mu_1} \wedge dx^{\mu_2} = dx^{\mu_1} \otimes dx^{\mu_2} - dx^{\mu_2} \otimes dx^{\mu_1}$$

etc.

a special one: the <u>VOLUME FORM</u> $d\Omega^{(n)}$ in n-DIM
(NOT ANGULAR DIFFERENTIAL, SORRY FOR NOTATION)

$$\int_V d\Omega^{(n)} = \int_V dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} = \mathrm{Vol}(V)$$

<u>WEDGE PRODUCT</u>:

$$\omega = \omega_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$
$$\rho = \rho_{\nu_1 \dots \nu_p} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p}$$

$$\omega \wedge \rho = \omega \rho \, \underline{dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p}}$$

<u>note</u>: $dx^{\mu} \wedge dx^{\mu} = 0$
$\Rightarrow dx^{\mu_1} \dots dx^{\mu_n}$ for $n > \dim(M) = 0$

EXTERIOR DERIVATIVE

↳ DIFFERENTIAL OPERATOR

eg $df(x) \to \underbrace{\frac{\partial f}{\partial x^\mu}}_{\substack{\uparrow \\ \text{COMPONENT}}} \underbrace{dx^\mu}_{\text{BASIS 1-FORM}}$

eg if $u = u(x,y)$, $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

eg if $v = v(x,y)$ ——"——

$$du \wedge dv = (u_x\, dx + u_y\, dy) \wedge (v_x\, dx + v_y\, dy)$$
$$= u_x v_x \cancel{dx \wedge dx} + u_x v_y\, dx \wedge dy$$
$$+ u_y v_x\, dy \wedge dx + u_y v_y \cancel{dy \wedge dy}$$

$$= \underbrace{(u_x v_y - u_y v_x)}\, dx \wedge dy$$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \text{Jacobian}(u,v;x,y)$$

IN GENERAL: for k-form $\omega = \underbrace{\omega_{\mu_1 \dots \mu_k}}_{\omega_{\mu_1 \dots}(x)} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$ ← function of M

$\frac{1}{k!}$ ¥

$$\boxed{d\omega = \underbrace{(\partial_\nu \omega_{\mu_1 \dots \mu_k})}\, dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}}$$

(k+1) tensor

(or (0, k+1)-tensor to be precise)

d: $\underline{k\text{-forms "over } \mathcal{M}\text{"}} \longrightarrow (k+1)\text{-forms over } \mathcal{M}$

this space is related to
the "cotangent bundle"

<u>OBSERVE</u>: $d^2 = 0$ (d is <u>NILPOTENT</u>)

$$d\omega = \partial_\nu \omega_{\mu\ldots}\, dx^\nu \wedge dx^\mu \wedge \ldots \frac{1}{(\ldots)!}$$

$$d^2\omega = \underbrace{\partial_\rho \partial_\nu}_{\substack{\text{SYMMETRIC IN} \\ \rho \leftrightarrow \nu}} \omega_{\mu\ldots}\, \underbrace{dx^\rho \wedge dx^\nu}_{\text{ANTISYMMETRIC IN } \rho \leftrightarrow \nu} \wedge dx^\mu \ldots \frac{1}{(\ldots)!}$$

$$S_{\rho\nu} = \tfrac{1}{2}(S_{\rho\nu} + S_{\nu\rho}) \qquad A^{\rho\nu} = \tfrac{1}{2}(A^{\rho\nu} - A^{\nu\rho})$$

s.t. eg $S_{\rho\nu} \cdot \frac{1}{2}(A^{\rho\nu} - A^{\nu\rho})$

$$= \tfrac{1}{2} S_{\rho\nu} A^{\rho\nu} - \tfrac{1}{2} S_{\rho\nu} A^{\nu\rho}$$
$$- \tfrac{1}{2} S_{\nu\rho} A^{\nu\rho}$$
$$= 0 .$$

F is <u>EXACT</u>. (comes from potential)

A VERY NICE EXAMPLE: E & M $\qquad A = (\varphi, \vec{A})$

$$F = dA \quad \text{on } \mathbb{R}^{1,3} \text{ (generalizes)}$$

$$\frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = d(A_\nu\, dx^\nu)$$
$$= \underbrace{\partial_\mu A_\nu}\ \underbrace{dx^\mu \wedge dx^\nu}_{\text{ANTISYM.}} \ \cancel{}$$

effectively antisymmetrized
i.e. $(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \otimes dx^\nu$

$$F = \underbrace{\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ + & 0 & -B_z & B_y \\ + & + & 0 & B_x \\ + & + & + & 0 \end{pmatrix}}_{\text{antisymmetric}}$$

transformations of this give E & B in different frames
→ shows that E & B fields transform into each other under boosts.

<u>OBSERVE</u>: $dF = ddA = 0$
"F IS A CLOSED FORM"

for our purposes: CLOSED = EXACT.