Homework 2a: **Self-Adjointness**

Course: Physics 231, *Methods of Theoretical Physics* (2021)
Instructor: Professor Flip Tanedo (flip.tanedo@ucr.edu)

Due by: Wed, October 13

1 The i in the momentum operator

In one-dimensional quantum mechanics the momentum operator is defined to be

$$\hat{p} = -i\frac{d}{dx} \ , \tag{1.1}$$

where we set $\hbar = 1$. Suppose the vector space of wavefunctions is composed of complex functions $\psi(x)$ over the real line that are square integrable[1]. The inner product on this space is

$$\langle \psi, \chi \rangle = \int_{-\infty}^{\infty} dx \ \psi^*(x)\, \chi(x) \ . \tag{1.2}$$

Recall that an operator $\mathcal{O}$ is self-adjoint (Hermitian) if $\mathcal{O}^\dagger = \mathcal{O}$. This is defined by:

$$\langle \mathcal{O}\psi, \chi \rangle = \langle \psi, \mathcal{O}\chi \rangle \ . \tag{1.3}$$

Show that $\hat{p}$ is self-adjoint. Comment on why it is important that $\hat{p}$ is defined to have a factor of i in it. Comment on whether or not the overall sign is meaningful.

Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

1 A two-dimensional function space

Let us ignore the subtleties of defining a function space (metric, domain, boundary conditions). Instead, let's construct a cute two-dimensional function space that gives us a shortcut to calculate a particular *indefinite* integral.

Consider a two dimensional vector space spanned by the functions

$$|f_1\rangle = f_1(x) = e^{ax}\cos bx \qquad\qquad |f_2\rangle = f_2(x) = e^{ax}\sin bx \ , \tag{1.1}$$

where a and b are constants. Forget orthonormality or boundary conditions for this problem. The derivative d/dx is a linear operator that acts on this space. Write down the derivative as a 2×2 matrix in the above basis, D.

Invert D in the usual way that you learned to invert 2×2 matrices during your childhood[2]. Call this matrix D^{-1}.

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[1] This basically means that $\psi(x) \to 0$ sufficiently fast for $x \to \pm\infty$. This means that you can integrate by parts without worrying about boundary terms.

[2] Stuck? Here's a life pro tip: http://bfy.tw/KG2Z

Now stop and think: the inverse of a derivative is an indefinite integral[3]. Thus acting with D^{-1} on the vector $|f_1\rangle$ should be understood as an integral of $f_1(x)$. Show that, indeed,

$$D^{-1}|f_1\rangle = \int dx\, e^{ax} \cos bx \ . \tag{1.2}$$

Feel free to use *Mathematica* to do the indefinite integral on the right-hand side. Pat yourself on the back if you can do it without a computer.

[3]Ignore the constant term.