Homework 1a: Dimensional Analysis

Course: Physics 231, *Methods of Theoretical Physics* (2021)
Instructor: Professor Flip Tanedo (flip.tanedo@ucr.edu)
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1 Identifying Dimensions

As a warm up, write out the dimensions of the following quantities in the form $[Q] = L^\alpha, M^\beta T^\gamma$, that is: write out the length, mass, and time dimensions.

(a) Electric charge, e. (In lecture we wrote out the answer; derive it.)

(b) Action ($S = \int dt L$, where L is the Lagrangian)

(c) Magnetic field, B

(d) Energy.

2 Lagrangian for a Scale-Invariant Theory

Work in *natural units* where the speed of light is $c = 1$ and the reduced Planck's constant is $\hbar = 1$. In these units, dimensional analysis is simply keeping track of a single unit, say energy. The Lagrangian for a free particle in one dimension is

$$L_{\text{free}} = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 ,$$

where $q(t)$ is the position of the particle.[1]

The theory described by L_{free} obeys two notable symmetries: (1) time translation invariance and (2) scale invariance. Under time translation invariance $t \to t + \tau$ for some constant τ. Time translation invariance is clear because $S = \int dt\, L_{\text{free}}$ only depends on dt and not t explicitly.[2] Scale invariance is the shift $t \to \alpha t$; this means that $dt \to \alpha\, dt$. Under this transformation, we *assume* that $q(t) \to \alpha^{1/2} q(t)$ so that S unchanged.

The most general theory of a single particle that obeys these two symmetries is $L_{\text{free}} - V[q]$. There is only one unique term in the potential, $V[q]$, that is scale invariant. Given that $V[q]$ should have no explicit time dependence (i.e. only depends on t through $q(t)$) and should not carry any time derivatives, derive the potential for the theory up to an overall dimensionless constant. Answer: $V[q] \sim 1/q^2$.

Remark: This is conformal quantum mechanics, see Extra Credit.

[1] You may be concerned that this does not look familiar from introductory [quantum/classical] mechanics. Don't worry: we have simply absorbed the constant mass into the definition of q; if you want, you can imagine that $q(t) = \sqrt{m}x(t)$ so that $L_{\text{free}} = \frac{1}{2}m\dot{x}^2$, as you would expect.

[2] Are you worried that $q(t)$ depends on t? Good. Observe that even though $q(t)$ transforms under a time translation, the action $S = \int dt\, L_{\text{free}}$ does not transform because one integrates over the same [transformed] region. If you're confused, try doing a definite integral $\int_a^b dt\, f(t)$ and then change the integration variable to $s = t + \tau$. You get the same result. You can wax poetic and reflect on how this connects to a notion of relativity: this theory offers no preferred coordinate system.

Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

1 Allometry

These two problems come from *Mathematical Methods in Classical Mechanics* by the eminent mathematician V.I. Arnold.

(a) A desert animal has to cover great distance between sources of water. How does the maximal time the animal can run depend on the size L of the animal?

(b) How does the height of an animal's jump depend on its size? Use the fact that the force applied by muscles is proportional to the strength of bones, which is itself is proportional to their cross section.

2 Conformal Quantum Mechanics

In Problem 2 we showed that the unique time-translation invariant and scale-invariant theory of a single particle is described by a Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{g}{2q^2} \ .$$

Here g is the overall dimensionless constant for the potential term.

1. Argue that a sensible theory has $g > 0$.

2. Show that this theory is actual invariant under a more general transformation:

$$t \to \frac{at+b}{ct+d} \qquad\qquad q \to \frac{q}{ct+d} \qquad\qquad ad - bc = 1 \ .$$

   Let us call these *conformal* transformations. Note that in order for the theory to be invariant, it is sufficient that the *action* is invariant, not that the *Lagrangian* is invariant. That is: the Lagrangian can change by a total time derivative, $L \to L + dG/dt$ for some $G[q]$.

3. The parameters a, b, c, d of the transformation may be represented as a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

   Show that successive conformal transformations correspond to the usual matrix multiplication. That is, if we do a conformal transformation by a, b, c, d and then a second conformal transformation by a', b', c', d', then this is equivalent to a conformal transformation given by

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

   The mathematical name for the transformations given by a 2×2 matrix of real numbers with unit determinant is SL(2,$\mathbb{R}$).

For more on conformal quantum mechanics, see "Conformal Invariance in Quantum Mechanics" by de Alfaro et al. *Il Nuovo Cimento A* **34**, 569 (1976).