Long HW 6: Mass

Course: Physics 165, *Introduction to Particle Physics* (2018)
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Due by: **Tuesday**, February 20

This is the main weekly homework set. Unless otherwise stated, give all responses in natural units where $c = \hbar = 1$ and energy is measured in electron volts (usually MeV or GeV).

[**Flip**: 2/14: corrected eq (9). Thanks Adam G.]

Recall that a Lagrangian Density ("Lagrangian" from now on) is a function $\mathcal{L}(x)$ that is integrated over spacetime to give the action, S:

$$S = \int d^4x \mathcal{L}(x) \ . \tag{1}$$

In fact, to be precise, $\mathcal{L}$ is a *functional* of the **fields** in your theory, $\mathcal{L}[\varphi(x), \cdots]$. The quadratic part of the Lagrangian is special because it can be solved exactly to derive the **propagator** of a particle.

1 Why the mass term is a mass

In class we wrote down the Lagrangian for an infinite array of beads connected to their nearest neighbors by springs:

$$L = \frac{1}{2}\sum_i \Delta x \left[\frac{m_{\text{bead}}}{\Delta x}\dot{q}_i^2 - k\Delta x\left(\frac{q_{i+1}-q_i}{\Delta x}\right)^2\right] \ , \tag{2}$$

we then 'chose units' so that this became

$$L = \frac{1}{2}\sum_i \Delta x \left[\dot{q}_i^2 - \left(\frac{q_{i+1}-q_i}{\Delta x}\right)^2\right] \quad \rightarrow \quad \int dx\, \frac{1}{2}(\partial q)^2 \ . \tag{3}$$

We identify $\mathcal{L}[q] = (\partial q)^2/2$. In the case of a (3+1)-dimensional spacetime, we this generalizes to

$$S = \int dt\, d^3x\, \mathcal{L} = \int d^4x\, \frac{1}{2}(\partial q)^2 \ . \tag{4}$$

We propose adding another term to the Lagrangian,

$$\Delta\mathcal{L} = -\frac{1}{2}Aq^2 \ . \tag{5}$$

1.1 Dimensional Analysis

In this class, 'dimension' means "mass dimension." For example, an energy E has dimension 1, while a length ℓ has dimension -1. The notation for this is

$$[E] = 1 \qquad\qquad [\ell] = -1 \ . \tag{6}$$

This just means that you can write a quantity x as a number times $\text{GeV}^{[x]}$.

(a) Given that $[S] = 0$, what is the dimension of $\mathcal{L}$?

(b) What is the dimension of the field q?

(c) What is the dimension of the coefficient A?

1.2 What is mass?

Write A as some mass scale m to the appropriate power. We want to identify m with the mass of a quantum excitation. The equation of motion for

$$\mathcal{L} = \frac{1}{2}(\partial q)^2 - \frac{1}{2}Aq^2 \ , \tag{7}$$

is simply

$$(-\partial^2 - A)q = 0 \ . \tag{8}$$

To understand what this means, make the ansatz that q is a plane wave of definite momentum, $q(x) \propto \exp(ip^\mu x_\mu)$. Show that the equation of motion then implies that m is identified with the 'mass' of the plane wave. HINT: use the Einstein relation for an on-shell particle.

2 Mixed Mass

Consider the following quadratic Lagrangian written in terms of fields $\varphi_1(x)$ and $\varphi_2(x)$. [**Flip**: 2/14: corrected $\varphi_1 \to \varphi_2$ in the first term.]

$$\mathcal{L}[\varphi_1, \varphi_2] = \frac{1}{2}(\partial_\mu \varphi_1)(\partial^\mu \varphi_1) + \frac{1}{2}(\partial_\mu \varphi_2)(\partial^\mu \varphi_2) - \frac{1}{2}m^2 \varphi_1 \varphi_2 \ . \tag{9}$$

Observe that the mass term connects a φ_1 and a φ_2 field. This means that these field mix with one another. We are free to redefine fields. Show that the following redefinition diagonalizes the quadratic Lagrangian above; that is: each term contains only one type of field.

$$\varphi_1(x) = \frac{1}{\sqrt{2}}(\varphi_A(x) + \varphi_B(x)) \tag{10}$$

$$\varphi_2(x) = \frac{1}{\sqrt{2}}(\varphi_A(x) - \varphi_B(x)) \tag{11}$$

What is the mass-squared of the φ_A field? What is strange about the mass-squared of the φ_B field?

3 Negative mass-squared

A field that has a negative value for m^2 is problematic. That would imply that the mass is imaginary. Not good. We need to make sense of this. Assume you have the following Lagrangian:

$$\mathcal{L}[\varphi] = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - V[\varphi] \qquad\qquad V[\varphi] = -\frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4}\varphi^4 \ . \tag{12}$$

Here $V[\varphi]$ is called the **potential** of the field φ. It gives the potential energy of a [classical] field configuration.

(a) Plot the potential $V[\varphi]$ as a function of φ.

(b) For what value(s) of φ is $V[\varphi]$ minimized? Call these values $\pm\varphi_0$. These are just some spacetime-independent constants.

(c) The problem with the field $\varphi(x)$ is that excitations of the field are not being expanded about the minimum energy (potential) configuration. Instead, define a shifted field $\phi(x)$ by $\varphi(x) = \varphi_0 + \phi(x)$. Expand $V[\varphi_0 + \phi(x)]$, show that the $\phi(x)$ field has a positive mass-squared term. What is the mass of the ϕ particle?

The lesson here is that in the absence of excitations, the field wants to take values $\varphi(x) = \varphi_0$. This is the constant value that minimized the potential. Quantum excitations of the field are 'wiggles' on top of this background value.

The background value, φ_0 is called the **vacuum expectation value** of $\varphi(x)$ and is often written $\langle\varphi\rangle = \varphi_0$. An analog of this is precisely whats happening with the Higgs field.

4 Reading

Read the article "Particle Physics and the Standard Model" by Raby, Slansky, and West in *Particle Physics: A Los Alamos Primer*[1]. This should be a nice, enjoyable article that contextualizes our discussions so far. Answer the following question: Let g be the coupling of the weak bosons (W^A) and g' be the coupling of the hypercharge boson (B). If I doubled g, what happens to the size of the photon coupling, e? What happens to the mass of the W boson? You do not have to read in thorough detail, but we will come back to this article soon.

Extra Credit

1 Dimensional analysis: propagator to long-range force

The propagator of an intermediate, massless particle with momentum p is

$$\Delta(p) = \frac{1}{p^2 - m^2} \ . \tag{13}$$

(We're dropping factors of $\pm i$ that are not our concern for this course.) Draw the Feynman diagrams for an electron with momentum p_1 and a positron with momentum p_2 scattering into an electron with momentum k_1 and a positron with momentum k_2,

$$e^-(p_1) + e^+(p_2) \to e^-(k_1) + e^+(k_2) \ . \tag{14}$$

Write down the propagator for the internal line for each of these diagrams.

(a) Can either of the denominators ever be zero?

(b) One of these diagrams represents the *long range* electric force of the electron and positron acting on one another. Based on the interpretation of the Feynman diagram as a spacetime trajectory, which diagram represents the long range force?

[1] https://archive.org/details/ParticlePhysicsAndTheStandardModel

(c) Let $q = p_1 - k_1$ be the exchanged 4-momentum between the electron and positron. The long range potential $V(r)$ between the two particles is given by the spatial Fourier transform of the amplitude $V(\mathbf{r}) \sim \int d^3\mathbf{q} \exp(\mathbf{q} \cdot \mathbf{r}) \mathcal{M}$. Assuming that $\mathcal{M}$ is simply the propagator of the diagram encoding the long-range force, what is the r dependence of the potential? Don't calculate, just use dimensional analysis.

2 Reading

Read the article "Lecture Notes: from simple field theories to the standard model" by Slansky in *Particle Physics: A Los Alamos Primer*[2]. Based on the discussion in lecture 8, does the electron have a bigger Yukawa coupling or a smaller Yukawa coupling than the muon? (The Yukawa is what Slansky calls G_Y.) You do not have to read in thorough detail, but we will come back to this article soon.

[2] `library.lanl.gov/cgi-bin/getfile?11-03.pdf`