TODAY: QUESTIONS? eg on HW...

INDICES – the index game

THE LEPTONIC ELECTROWEAK SECTOR

collect short HW after class

- HW 2 CORRECTION

INDICES:

- given a vector space (V) w/ DIMENSION d, there are objects "native" to this vector space with d components.

$$\underline{v} = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^d \end{pmatrix} = \sum_i v^i \, \underline{e}_{(i)}$$

or $|v\rangle$

v^i: array of #'s → then the array of #'s carries all the info of $\underline{v}$.

$\underline{e}_{(i)}$: eg $\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$ ← BASIS. CARRIES "vector-ness"

WE WILL ASSUME THAT WE HAVE AGREED ON A BASIS.

- IT IS COMMON TO ALSO HAVE lower index objects

$$\underline{w}^T = \underbrace{(w_1 \; w_2 \cdots w_d)}_{w_i} = \sum_i w_i \, \underline{\tilde{e}}^{(i)}$$

or $\langle w|$

$\underline{\tilde{e}}^{(i)}$: eg $\underbrace{(0 \; 1 \; 0 \ldots)}_{\text{BASIS OF "ROW VECTORS"}}$

technical jargon: $\underline{v}$ (or "v^i" colloquially) is in vector space $\in V$

$\underline{w}^T$ is in dual vector space $\in V^*$

DUAL VECTOR SPACE: LINEAR MAP from VECTOR → #

given w_i, think: w_i : (vector space) → #

this is "obvious":

$$w_i(\underline{v}) \equiv w_i v^i = \underbrace{w_1 v^1 + w_2 v^2 + \cdots}_{\text{just a \#}}$$

same way that $\langle w|$ is a linear function that acts on $|v\rangle$ to give a # (amplitude)

<u>LINEAR</u>: $w_i(\underline{v} + \underline{x}) = w_i v^i + w_i x^i$, etc.

why make a big deal about this? somewhere "under the hood" we're doing a Taylor expansion & going to linear order.

also obvious: $\underline{v}$ is a linear map $V^* \to \#$.

- UPPER / LOWER INDICES MAY BE CONTRACTED

<u>nb</u>: for some vector spaces there is no need to distinguish upper & lower indices & so any indices may be contracted

$v^i w_i = w^i v_i$ = some # ← no indices

$$\underbrace{M^i{}_j v^j}_{} = (Mv)^i$$

3 indices
2 contracted

effectively one index

this is just matrix mult.

$$\begin{array}{cc} M^1{}_1 & M^1{}_2 \\ \downarrow & \downarrow \end{array}$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right) = \begin{pmatrix} av^1 + bv^2 \\ cv^1 + dv^2 \end{pmatrix}$$

nb index heights!

compare to $M^{i=1}{}_j v^j = M^1{}_1 v^1 + M^1{}_2 v^2$

- WE CARE ABOUT INVARIANCE

  <u>why?</u> symmetries are important … somehow related to conservation laws

  eg "length of a vector" is conserved under rotations

  how does this show up?

  eg. Newtonian gravity

  given ROT. SYM, want to ask what is potential $\underline{r}$ away from a point source?

  $\underline{r} = (x, y, z)$ … many components.
  not invariant under rot.

  $|\underline{r}|^2$ <u>is</u> invariant
  so potential can only depend on x in the combination

  $(x^2 + y^2 + z^2)$

  [eg can then make measurements along just one axis]

INDICES TELL YOU HOW AN OBJECT TRANSFORMS

$$\underbrace{\boxed{}^{i\,\cdots}{}_{\cdots}}_{\text{other indices}} \longrightarrow R^i{}_k (\cdots)^{\cdots}{}_{\cdots} \boxed{}^{k\,\cdots}{}_{\cdots}$$

other indices

other rotations

how the 1st index transforms

& so forth. eg.

$$T^i{}_{jk} \mapsto R^i{}_l \, (R^{-1})^m{}_j \, (R^{-1})^n{}_k \, T^l{}_{mn}$$

MANIPULATING INDICES

UPPER INDICES CONTRACT LOWER INDICES. THAT'S IT.

... BUT SOME THEORIES GIVE US ADDITIONAL OBJECTS TO USE.

eg. one "useless" tensor is the identity matrix

$$\mathbb{1} = \delta^i{}_j \longrightarrow R^i{}_k (R^{-1})^l{}_j \, \delta^k{}_l$$

$$= R^i{}_k (R^{-1})^k{}_j = \delta^i{}_j$$

$\delta^i{}_j$: 1 iff $i=j$

(invariant)

eg. in EUCLIDEAN SPACE: Metric, δ_{ij}
w/ inverse, δ^{ij}

we abuse notation & call this δ as well because $\delta_{ij} = \delta^{ij} = 1$ iff $i=j$

so what?

$$V^i \delta_{ij} = (V\delta)_j$$

$(V\delta)_j$: lower index object

components:

$$(V\delta)_1 = V^1 \delta_{1(j=1)} + V^2 \cancel{\delta_{2(j=1)}}^{\,0}$$

$$= V^1$$

etc.

→ $(V\delta)_j$ HAS LOWER-INDEX COMPONENTS THAT ARE THE SAME AS V^i's UPPER INDEX COMPONENTS.

so LET'S JUST CALL IT V_j

the metric lets us <u>lower indices</u>

↳ but really just provides another <u>tensor</u> to contract.

Remarks

- it is not true that the metric will always have $V_j = V^j$

  ↳ eg SR: $p^\mu = (E, \underline{p})$, $p_\mu = (E, -\underline{p})$ } s.t. $p^\mu p_\mu = E^2 - \underline{p}^2$

- only one metric
- inverse metric pulls indices up. defined by

$$(\text{metric})_{ij} (\text{metric}^{-1})^{jk} \equiv \delta_i^k$$

↳ SR: $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

OTHER INVARIANTS

without going into the details of validity for the symmetries that we will explore in this class,

the d-dimensional LEVI-CIVITA tensor is "invariant"

ε_{ij} in 2D
ε_{ijk} in 3D
$\varepsilon_{\mu\nu\rho\sigma}$ in 4D } etc.

why? HAS TO DO W/ DEFINITION OF VOLUME.

eg VOLUME of PARALLELPIPED W/ EDGES $\underline{V}, \underline{W}, \underline{Z}$

$$\text{Vol} = \varepsilon_{ijk} V^i W^j Z^k = \underline{V} \cdot \underbrace{(\underline{W} \times \underline{Z})}$$

$\varepsilon_{ijk} W^j Z^k = (\underline{W} \times \underline{Z})_i$

convert 2 upper into one lower.

other generalizations of ROTATIONS will introduce other objects w/ indices.

idea: DIFFERENT KINDS OF INDICES WILL REPRESENT "ROTATIONS" IN DIFFERENT VECTOR SPACES.

these indices are related to the QUANTUM NUMBERS of particles.

eg. electron

$$| m_e, \underline{p}, Q=-1, \underbrace{S=\tfrac{1}{2}, S_z=+\tfrac{1}{2}}_{\text{spin}} \rangle$$

z is axis def by p

spinor: $\begin{pmatrix} c_+ \\ c_- \end{pmatrix}$ ← spin $+1/2$ ← spin $-1/2$

ψ_α ← $\alpha = 1, 2$ for $\pm 1/2$

HOW DO YOU ROTATE A SPINOR?

$$R(\theta \hat{z}) = e^{i\frac{\theta}{2}\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$(\sigma^z)^\alpha{}_\beta$ spinor indices (ROTATION IN SPIN SPACE)

ROTATION IN 3-SPACE INDICES

$$R(\theta \hat{\underline{r}}) = e^{i\frac{\theta}{2}\hat{r}_i \sigma^{i\,\alpha}{}_\beta} \psi^\beta$$

$\hat{r}_i = \delta_{ij}\hat{r}^j$

PAULI MATRICES: INVARIANTS. ROSETTA STONE FOR CONVERTING "VECTOR" ⟷ SPINOR INDICES

later: this is why we say 720° rotation req to bring a spinor back

IN FACT: in QED: e $\quad \dot{\alpha} \longrightarrow \alpha$

electron has spin

hint of things to come

DOT IS WHOLE DIFFERENT THING CANNOT CONTRACT w/ α

$(e^{+})_{\dot{\alpha}}$ $\qquad$ e^{α}

RH SPIN $\qquad$ LH SPIN

this is most directly related to the vector potential A_μ

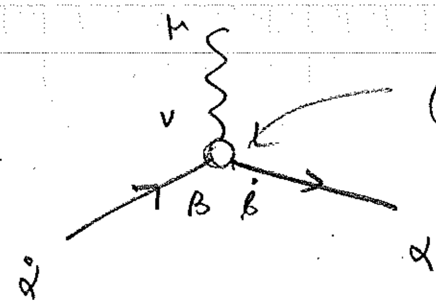
γ: $\mu \sim\sim\sim \nu$

4D generalization of ROTATIONS: PROVIDES AN INVARIANT

$$(\sigma^{\mu})^{\dot{\alpha}}{}_{\alpha}$$

in this class: TREAT THESE OBJECTS AS PART OF DEF. OF SYMMETRY.

SO: WE CAN CONNECT THESE



$(\sigma^{\nu})^{\dot{\beta}}{}_{\beta}$

because we could write

$$\underbrace{A_\nu (e^{+})_{\dot{\beta}}\, e^{\beta}\, \sigma^{\nu \dot{\beta}}{}_{\beta}}_{\text{all indices contract}}$$

FEYNMAN RULES :

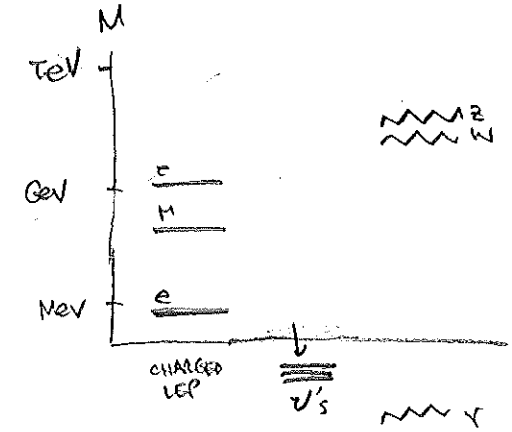
→ e, μ, τ } CHARGED LEPTONS

→ ν_e, ν_μ, ν_τ } neutrinos

FLAVORS

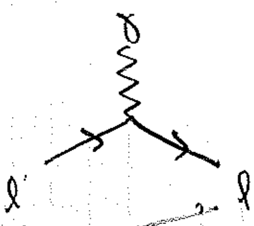
〰 γ, Z

〰 $W^\pm$ CHARGED



CAN IMAGINE w/ ARROW; convention: DON'T DRAW IT

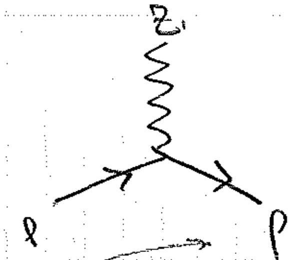
INTERACTIONS



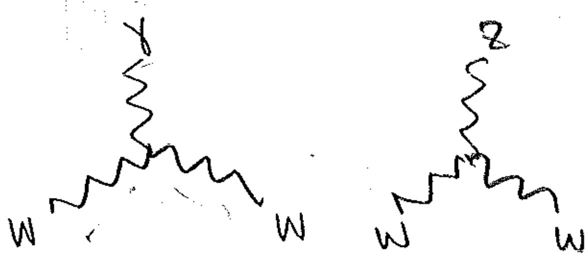
$l = e, \mu, \tau$

same charged lepton

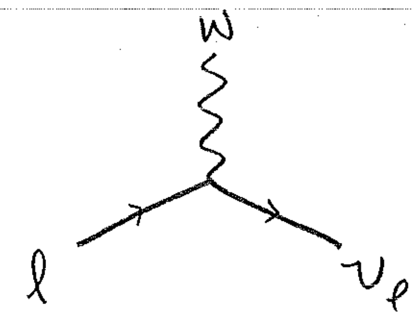
this is what electric charge means



same lepton, any $l = e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$



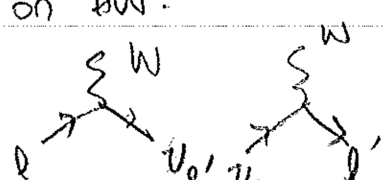
"W boson has charge"



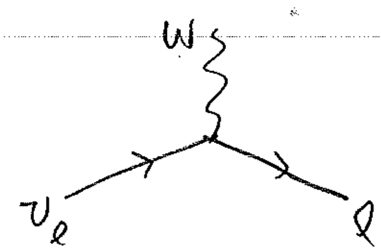
CHARGED LEPTON

ASSOCIATED NEUTRINO

on HW:



$l \neq l$



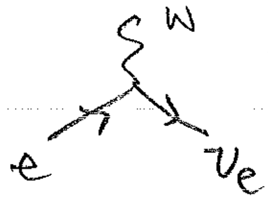
} DIFFERENT RULE! (BUT REQUIRED)

DISCUSS

- ELECTRIC CHARGE IS CONSERVED
- W COUPLINGS ARE WEIRD
  ↳ CHANGES the IDENTITY OF THE LEPTON!
  connects charged lepton w/ neutral!

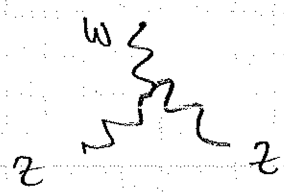
what is W charge?

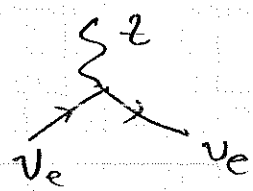
it almost looks like flavor charge



conserves electron #
if ν_e is defined to have electron #

for that matter, what is Z charge?





"heavy version of γ ... but apparently more!"

→ this will be something we need to untangle

$m_W \approx m_Z$ ... may have something to do w/ each other?!

RELATED TO: HIGGS MECHANISM
ELECTROWEAK SYMMETRY BREAKING

NEUTRINO OSCILLATION

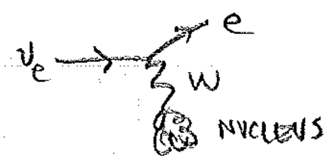
usually treated in QM
(THOUGH many subtle points)

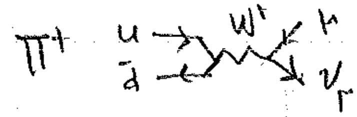
why: neutrinos are produced & observed
in a flavor basis

ν_e, ν_μ, ν_τ

experimentally more subtle
(but see, eg ICE CUBE expt)

detect: produce





BUT: neutrinos propagate in ENERGY eigenstates

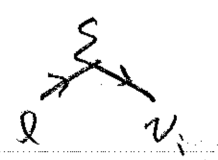
lesson from QM

why would ν's have different Energies?

$\hookrightarrow$ DIFFERENT MASSES

$$|\nu_i\rangle = U_i^{\ \alpha}\,|\nu_\alpha\rangle$$

MASS STATES flavor

OR:  in flavor basis

$\updownarrow$ do not match.

$\longrightarrow \nu_\alpha$ in MASS basis

subtle: E conservation