

P165: These are lecture notes from P231 (Fall 2017), provided as a hint for P165 (Win 2018) homework #4b.

GAUSSIAN INTEGRALS

↑ something different. we will relate to DIFF EQ.
↑ PROBABILITIES @ END.

REF: Zee, QFT in a Nutshell APPENDIX 1

$$G = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} \quad \leftarrow \text{how to solve?}$$

TRICK:

$$\begin{aligned} G^2 &= \int dx dy e^{-\frac{1}{2}(x^2+y^2)} \\ &\quad \underbrace{\quad}_{r dr d\theta} \quad \underbrace{\quad}_{r^2} \\ &= \int_0^\infty du e^{-u} (2\pi) \\ &= (-e^{-u} + 1) (2\pi) = 2\pi \Rightarrow \boxed{G = \sqrt{2\pi}} \end{aligned} \quad \begin{array}{l} \text{let } u = \frac{1}{2}r^2 \\ du = r dr \end{array}$$

SIMILARLY: SUPPOSE $x = \sqrt{a}y$

$$G = \int_{-\infty}^{\infty} \sqrt{a} dy e^{-\frac{1}{2}ay^2}$$

$$\rightarrow \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}ay^2}$$

$$= \sqrt{2\pi}$$

$$= \boxed{\sqrt{\frac{2\pi}{a}}}$$

↑
could guess $a^{-1/2}$
from dimensional
analysis.

MORE VARIANTS

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2 + Jx}$$

complete the square

$$= -\left(\frac{a}{2}\right)\left(x^2 - 2Jx/a\right) = -\left(\frac{a}{2}\right)\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a}$$

$$y = x - J/a$$

$$= \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}ay^2} \underbrace{e^{J^2/2a}}_{\text{const}}$$

$$= \boxed{\sqrt{\frac{2\pi}{a}} e^{J^2/2a}}$$

invertible

SYMMETRICIN N DIMENSIONS: $N \times N$ MATRIX A

$$\int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_N e^{-\frac{1}{2} \underbrace{\underline{x} \cdot \underline{A} \cdot \underline{x}}_{x_i A_{ij} x_j} + \underbrace{\underline{J} \cdot \underline{x}}_{J_i x_i}}$$

LET A BE DIAGONALIZED BY AN ORTHOGONAL TRANSFORMATION, R .

$$\underline{A} = \underline{R}^{-1} \cdot \underbrace{\hat{\underline{A}}}_{\text{DIAGONAL}} \cdot \underline{R}$$

$$\hat{\underline{A}} = \begin{pmatrix} \hat{a}_1 & & \\ & \hat{a}_2 & \\ & & \hat{a}_3 \dots \end{pmatrix}$$

THEN CHANGE VARIABLES: $\underline{y} = \underline{R} \cdot \underline{x}$

MEASURE UNCHANGED

$$\prod dx_i = \prod dy_i$$

$$= \int_{-\infty}^{\infty} dy_1 \dots dy_N e^{-\frac{1}{2} \underbrace{\underline{y} \cdot \hat{\underline{A}} \cdot \underline{y}}_{\hat{a}_{11} y_1^2 + \hat{a}_{22} y_2^2 + \dots} + \underbrace{\underline{J} \cdot (\underline{R}^{-1} \underline{y})}_{\underline{J}' \cdot \underline{y} = J'_1 y_1 + J'_2 y_2 + \dots}}$$

$$\hat{a}_{11} y_1^2 + \hat{a}_{22} y_2^2 + \dots$$

$$J'_1 y_1 + J'_2 y_2 + \dots$$

$$= \left(\int dy_1 e^{-\frac{1}{2} \hat{a}_1 y_1^2 + \underline{J}'_1 y_1} \right) \left(\int dy_2 \dots \right) \dots$$

$$= \prod_{i=1}^N \sqrt{\frac{2\pi}{\hat{a}_i}} e^{\frac{(\underline{J}'_i)^2}{2\hat{a}_i}}$$

$$\prod_i \hat{a}_i = \det \hat{A} \\ = \det A$$

$$\sum_i (\underline{J}'_i)^2 =$$

$$\underline{J}' \cdot \underline{J}' \\ = \underline{J}'^T \underline{R}^T \underline{R} \underline{J}' \\ = \underline{J}'^T \underline{J}$$

WILL BE
EXPLICIT
W/ T
FOR THIS TIME

note:
 $\underline{J}'^T \rightarrow \underline{J}$

FURTHER:

$$\sum_i (\underline{J}'_i) \frac{1}{\hat{a}_i} = \underline{J}'^T \underline{R}^T (\hat{A})^{-1} \underline{R} \underline{J}$$

$$\underline{A}^{-1} = \underline{R}^{-1} \cdot \hat{\underline{A}}^{-1} \cdot \underline{R}$$

$$\sum_i (\underline{J}'_i) \frac{1}{\hat{a}_i} = \underline{J} \cdot \underline{A}^{-1} \underline{J}$$

$$= \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2} \underline{J} \cdot \underline{A}^{-1} \underline{J}}$$

cute result. so what?

Gaussians show up all the time as distributions

WE OFTEN TAKE MOMENTS OR CORRELATION FUNCTIONS
OF DISTRIBUTIONS,

eg. $\int_{-\infty}^{\infty} dx \cdot x \cdot e^{-\frac{1}{2}ax^2} = 0$ by symmetry
 NEGATIVE PART
 CANCELS POSITIVE

eg. $\int_{-\infty}^{\infty} dx \cdot (x^2) \cdot e^{-\frac{1}{2}ax^2} = ?$

↑
 USE AN EXPECTATION OF x^2

OBSERVE: $\frac{d}{da} \int_{-\infty}^{\infty} dx \cdot e^{-\frac{1}{2}ax^2} = -2 \frac{d}{da} \left[\int_{-\infty}^{\infty} dx \cdot e^{-\frac{1}{2}ax^2} \right]$
 $= -2 \frac{d}{da} \sqrt{\frac{2\pi}{a}}$
 $= \sqrt{2\pi} \left[a^{-3/2} \right]$

↑
 COULD GUESS
 FROM DIM. ANALYSIS

DEFINE EXPECTATION VALUE OF x^2 WRT DIST

$\langle x^2 \rangle = \frac{1}{Z} \left(-2 \frac{d}{da} \right) Z$
 \downarrow
 $= \left[\frac{1}{a} \right]$
 \uparrow
 $Z = \int dx \cdot e^{-\frac{1}{2}ax^2}$

SIMILARLY

$\langle f(x^2) \rangle = \frac{1}{Z} f \left(-2 \frac{d}{da} \right) Z$
 (as Taylor expansion)

when we take the INTEGRAL w/ A SOURCE:

$Z = \int dx \cdot e^{-\frac{1}{2}ax^2 + Jx}$

$\langle x \rangle = \frac{1}{Z} \int dx \cdot x \cdot e^{-\frac{1}{2}ax^2 + Jx} = \frac{1}{Z} \frac{d}{dJ} Z = \left(\frac{J}{a} \right)$

J IS A SOURCE FOR x

LET ME NOW CHANGE VARIABLES FROM x TO q
CONSIDER OUR MOST COMPLICATED GAUSSIAN

$$Z = \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T A q + J^T q} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2} J^T A^{-1} J}$$

↑
some
kind of
distribution

↑
bunch of objects
like sequence of coupled springs
that have heights distributed
according to Z

Z encodes physics

THEN CAN ASK: $\langle q_i q_j \rangle = \frac{1}{Z} \int dq_1 \dots dq_N (q_i q_j) e^{-\frac{1}{2} q^T A q + J^T q}$

What is the correlation btwn spring i & j ?
IF SPRING i IS SQUISHED, IS SPRING j
UPON TO BE

SQUISHED ALSO
PULLED INSTEAD
COMPLETELY INDEP

$$\begin{aligned} \langle q_i q_j \rangle &> 0 \\ \langle q_i q_j \rangle &< 0 \\ \langle q_i q_j \rangle &= 0 \end{aligned}$$

$$\langle q_i q_j \rangle = \frac{1}{Z} \frac{d}{dJ_i} \frac{d}{dJ_j} Z$$

$$= \frac{1}{Z} \sqrt{\frac{(2\pi)^N}{\det A}} \frac{d}{dJ_i} \frac{d}{dJ_j} \exp\left[\frac{1}{2} J_a A^{-1}_{ab} J_b\right]$$

$$\dots + \frac{1}{2} J_i A^{-1}_{ij} J_j + \frac{1}{2} J_j A^{-1}_{ji} J_i + \dots$$

(no sum over repeated)

$$\text{nb: } A^{-1}_{ij} = A^{-1}_{ji}$$

BY SIM. OF A

$$= \left(\frac{1}{Z} \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2} J^T A^{-1} J} \right) \boxed{A^{-1}_{ij}}$$

$= 1$

So (A^{-1}_{ij}) TELLS YOU HOW INFORMATION AT q_i PROPAGATES TO q_j .

YOU CAN ALSO CALCULATE

$$\begin{pmatrix} i & j & k & l \\ k & j & i & l \end{pmatrix}$$

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl}$$

$$+ A^{-1}_{ik} A^{-1}_{jl}$$

$$+ A^{-1}_{il} A^{-1}_{jk}$$

$$(\equiv)$$

$$(\times)$$

$$(1 \ 1 \ 1)$$

PROS
FEYNMAN
DIAGRAMS

4-point correlation

breaks into pairs
of 2-point correl.

YOU MAY RECOGNIZE Z AS A PARTITION FUNCTION IN STATISTICAL MECHANICS.

$$Z = \sum_i e^{-\beta E_i}$$

↳ PROMOTED TO AN INTEGRAL

EXACT SAME STRUCTURE CARRIES OVER TO QUANTUM MECHANICS

↳ quantum randomness
~ thermal randomness

$$\text{CLAIM: } Z = \int dq_1 dq_2 \dots e^{iS(q)}$$

↑
 $q_i = q(t_i)$

$$S = \int_0^T dt \sum m \dot{q}^2 - V(q)$$

HOW TO DO THESE INTEGRALS:

$$I = \int dq \cdot e^{-\frac{1}{\hbar} f(q)}$$

\hbar SMALL PARAM

$$f(q) = f(a) + \frac{1}{2} f''(a) (q-a)^2 + \dots$$

\uparrow
MINIMUM

$$= \int dq e^{-\frac{1}{\hbar} f(a)} e^{-\frac{1}{\hbar} \frac{1}{2} f''(a) (q-a)^2} e^{\dots}$$

$$= e^{-\frac{1}{\hbar} f(a)} \sqrt{\frac{2\pi\hbar}{f''(a)}} e^{-O(\hbar^{1/2})}$$

\downarrow As $\hbar \rightarrow 0$

$$= e^{-\frac{1}{\hbar} f(a)} \sqrt{\frac{(2\pi\hbar)^N}{\det f''(a)}} e^{-O(\hbar^{1/2})}$$

can do quadratic part.

WAIT: WHAT ABOUT i ?! $Z = \int Dq e^{iS}$

\hookrightarrow PHASOR: STILL DOMINATED BY STEEPEST DESCENT.



RELATION OF
IR & IM
IS USED OFTEN
 \downarrow
Analytic
continuation

EXTREMUM OF $S \rightarrow$ EULER-LAGRANGE EQ.

\hookrightarrow WHERE ALL OF OUR GREEN'S FUNCTIONS
COME FROM.

CHAINS OF SPRINGS (like a bed mattress)

$$L = \frac{1}{2} \sum_i m \dot{q}_i^2 - \sum_{ij} \tilde{K}_{ij} q_i q_j$$

↑
 $[q_i(t) - q_j(t + \epsilon)]^2$
 [usually only nearest neighbors to good approx (coupled H.O.)]

$$\frac{1}{2} K_{ij} (q_i - q_j)^2$$

↓

$$\frac{1}{2} K (q(t, x) - q(t, x + \epsilon))^2$$

$$L = \int dx \frac{1}{2} \dot{q}(t, x)^2 - \frac{1}{2} q'(t, x)^2$$

UP TO
NORMALIZE

$$S = \int dx dt \frac{1}{2} \{ (\partial_t q)^2 - (\partial_x q)^2 \}$$

$$\underbrace{\quad}_{(\partial q)^2}$$

WHERE THE
WAVE EQ
COMES FROM
IT'S ALL SLO.

VARIATIONAL PRINCIPLE GIVES US KG EQ: $\boxed{\partial^2 q = 0}$

WHEN THERE IS A SOURCE

$$Z = e^{iS + Jq} \leftarrow \text{up to factors of } i$$

↓

$$\boxed{\partial^2 q = J}$$

we've sketched a functional approach to this problem.