<u>reboot</u> : CLASSICAL FIELD THEORY

LET'S PULL BACK FROM OUR BACCHANAL OF INDICES.

big open questions:

what is $\langle H \rangle$, how does it give mass?

how is $\rightarrow\!\!\otimes\!\!\leftarrow$ ~~⊗-- "MASS"?

this is related to:

- → PARTICLE OR $\boxed{\text{WAVE?}}$ ← in a <u>FIELD</u>
- → WHY IS EVERY ELECTRON IDENTICAL?

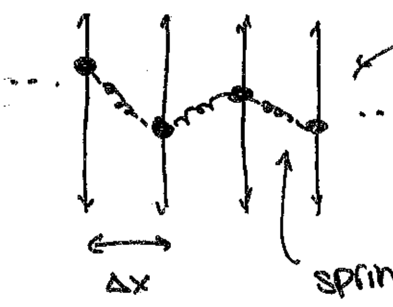
⇝ to be clear: our "QUANTUM" intuition in our Feynman diagram approach fails when dealing with <u>mass</u>

$$\leftarrow \quad m^2 = E^2 - \underline{p}^2 \qquad \text{(on shell)}$$

has to do with <u>propagation</u> ... "CLASSICAL" notion.

so let's reboot & understand <u>classical</u> fields.

... rather than starting w/ <u>fields</u> (abstract), lets imagine a system of harmonic oscillators.



masses, m, fixed to move on tracks. the position on the track is $q_i(t)$.

spring constant k

KINETIC ENERGY : $\sum_i \frac{1}{2} m \dot{q}_i^2$

POTENTIAL E : $\sum_i \frac{1}{2} K (q_{i+1} - q_i)^2$ ← HOOKE'S LAW this is a coupled HARMONIC OSCILLATOR @ EACH POSITION, i

IN QM: H.O. $\longrightarrow$ $a, a^\dagger$ ops FOR ENERGY LEVELS

IN QFT: H.O. $\longrightarrow$ $a, a^\dagger$ ops FOR PARTICLE NUMBER

$$L = \frac{1}{2}\sum_i \Delta x \left[\underbrace{\frac{m}{\Delta x}}_{\rho} \dot{q}_i^{\,2} - K\Delta x \underbrace{\frac{(q_{i+1}-q_i)^2}{\Delta x^2}}_{\text{LOOKS LIKE DERIVATIVE!}} \right]$$

continuum limit: $\sum_i \Delta x \longrightarrow \int dx$

$$\frac{q_{i+1}-q_i}{\Delta x} \to \frac{\partial q(x)}{\partial x}$$

$$\boxed{i \longrightarrow x}$$

$$L = \frac{1}{2}\int dx \left[\rho \frac{\partial q}{\partial t}^2 - (K\Delta x)\left(\frac{\partial q}{\partial x}\right)^2 \right]$$

$$= \frac{1}{2}\int dx\, \rho \left[\left(\frac{\partial q}{\partial t}\right)^2 - \left(\frac{K\Delta x}{\rho}\right)\left(\frac{\partial q}{\partial x}\right)^2 \right]$$

PICK UNITS s.t. THIS IS $c^2 = 1$

$$= \int dx \underbrace{\frac{1}{2}\left((\partial_t Q)^2 - (\partial_x Q)^2\right)}_{\text{LAGRANGIAN DENSITY}} \qquad Q \equiv \sqrt{\rho}\, Q$$

"CAME" FROM T − V MINUS SIGN...

$$S = \int \underbrace{dt\, dx}_{d^2x} \; \frac{1}{2}(\partial Q)^2 \longleftarrow \underbrace{(\partial_\mu Q)(\partial^\mu Q)}_{\text{LORENTZ INVARIANCE!}}$$

generalize: $[Q] = 1$ from $[S] = 0$

$$S = \int \underbrace{dt\, d^3x}_{d^4x} \; \frac{1}{2}(\partial Q)^2$$

New some poetry: DETAILS ARE IN P231 class

S encodes classical equation of motion

↳ of WHAT? ripples in the field of springs

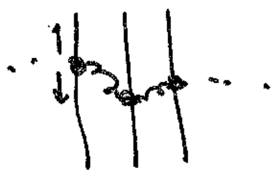
turns out to be: (EULER-LAGRANGE)

$$\partial^2 Q = \partial_t^2 Q - \underbrace{\partial_x^2}_{\nabla^2} Q = 0$$

this is the WAVE EQN.
what is the velocity of the wave?
(PROPERTY OF THE "MEDIUM")

$\rightarrow v = 1$ (speed of light)

conceptual checkpoint



I DO NOT CARE HOW FAST THE BEADS ARE MOVING ALONG THEIR TRACKS.

I CARE ABOUT THE VELOCITY OF A WAVE. ← what I will assoc. w/ PARTICLES.

nb: DISPLACEMENT in the $q \sim Q$ direction is not moving in physical space — it is a ripple in the FIELD.

(like temp $T(x,t)$ @ each point in spacetime)

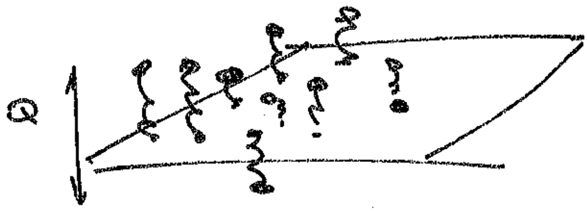
WAVE ANSATZ: $A e^{i(Et - p \cdot x)}$

$$\partial^2 Q = -(E^2 - p^2) = 0 \rightarrow \boxed{m = 0}$$

the ripples travel @ speed of light.

QUANTUM EXCITATIONS (quantum ripples) ARE PARTICLES w/ $m=0$ b/c they have speed $= c$.

LET'S BUILD A MYTHOLOGY AROUND THIS: SPACETIME IS A BOX SPRING.



LATTICE OF SPRINGS THAT ARE CONNECTED TO EACH OTHER.

@ spacetime point x^μ

EACH INDIVIDUAL BEAD, $Q(x)$ FEELS A H.O. POTENTIAL. QUANTUM MECHANICALLY,

$a^\dagger$ CREATES A PARTICLE (OR KILLS ANTIPARTICLE)

a ANNIHILATES —— " —— (OR CREATES —— " ——)

these are QUANTUM excitations of the box spring.

↳ these excitations $\boxed{\text{propagate}}$ through the mattress.

FREE FIELD: AS LONG AS S IS AS SIMPLE AS WE'VE WRITTEN, THE EQN IS LINEAR.

$$\partial^2 Q = 0$$

linear in Q

DEPENDED ON S BEING QUADRATIC IN Q.

this theory is solvable exactly.

Solution is something you've already done:

$$\langle q_i q_j \rangle \sim (A^{-1})_{ij} \quad \text{for } Z_0 = \int d^N q\, e^{\underbrace{q^T A q}_{\text{quad. part}} + Jq}$$

this looks like: $e^{iS_{\text{quadratic}}}$

imaginary...

$d^N q \leftrightarrow$ sum over spring configs.
(SUM OVER PATHS)

flow of concepts

$a^\dagger$ @ (x,t)

quantum field ⟷ excitations of this ⟷ PARTICLE

LAGRANGIAN DENSITY, func. of quantum field

$$S = \int d^4x \, \underbrace{\mathcal{L}[Q]}_{\text{function of the field}}$$

EOM from VARYING ACTION ⟷ PRINCIPLE OF LEAST ACTION. QUANTUM VERSION → PHASORS

comes from minimum of e^{iS}

we found (ARGUED)

$$\partial^2 Q = 0$$

DERIVATIVE IS ANALOG OF MATRIX IN DISCRETE SPACE.

btw: in full units, $e^{iS/\hbar}$ DOES TWIRL GET TIGHTER OR LOOSER AS $\hbar \to 0$?

$\langle q_i q_j \rangle$ correlation of an excitation @ spacetime point i w/ j

$\sim A^{-1}$ ← so we want inverse of ∂^2

FACT (Physics 231): the inverse of a differential operator is called a green's function.

for ∂^2, the GREEN'S FUNCTION is

$$G(x-y) = \int d^4p \, \frac{e^{i(x-y)\cdot p}}{p^2 + i\varepsilon}$$

← PROPAGATE FROM $x \to y$.

most important part: in momentum space

$$\boxed{G(p) = \frac{i}{p^2 + i\varepsilon}}$$

$\varepsilon \to 0$

DERIVATION IS BEYOND SCOPE OF CLASS

connection to Feynman diagrams :

$\overset{x}{\bullet} - - - - - \overset{y}{\bullet} \quad = G(x-y)$

$\xrightarrow{p} \quad = \frac{i}{p^2}$ } in momentum space (WHAT WE USE)

numerical factor in $\mathcal{M}$

OBSERVE :

$$S = \int \underbrace{d^4x}_{\text{SPACETIME}} \; \mathcal{L}[\text{fields}]$$

DIMENSIONLESS SCALAR ← w/rt all symmetries

just some # w/ mass dim +2

$$\mathcal{L} = \tfrac{1}{2}(\partial Q)^2 - \tfrac{1}{2}m^2 Q^2$$

PNEMONIC: $Q \to e^{ip\cdot x}$

WAVE EQ. W/ DAMPING

$$= \tfrac{1}{2} Q \underbrace{(-p^2 - m^2)}_{} Q$$

comes from $(\partial^2 + m^2) Q = 0$

Green's function / PROPAGATOR is

$$\boxed{G(p) = \frac{i}{p^2 - m^2}}$$

Einstein rel.

OBSERVE: $(\partial^2 + m^2)\, e^{ipx} \to \boxed{-p^2 + m^2 = 0} \Rightarrow \boxed{p^2 = m^2}$

so now you know mathematically what a MASS is from a Lagrangian density:

for a scalar field (no spin index)

$$\mathcal{L} = \frac{1}{2}(\partial Q)^2 - \frac{1}{2} m^2 Q^2$$

this is the MASS
it damps the wave eq.
s.t. fluctuations
in the quantum field
travel slower than c.

IN FACT: $\mathcal{L}$ contains everything about our theory.

- QUADRATIC PART $\longrightarrow$ we solved explicitly, obtain PROPAGATOR.
- BUT WE CAN THROW ANYTHING INTO $\mathcal{L}$!
  SUPPOSE WE INCLUDE A TERM $\boxed{\mu Q^3}$
  (μ: some #)

$$\mathcal{L} = \frac{1}{2}(\partial Q)^2 - \frac{1}{2} m^2 Q^2 - \frac{1}{3!}\mu Q^3$$

$$\text{EOM} \sim (\partial^2 - m^2)Q - \underbrace{\mu Q^2}_{\text{not linear!}} = 0$$

not linear!
cannot solve!

what we can do:

$$\langle Q_i \cdots Q_j \rangle = \int d^N Q \, (Q_i \cdots Q_j)\, e^{iS_{QUAD}} \underbrace{e^{iS_{TRI}}}$$

TAYLOR EXPAND
in nonlinear part

eg: (we won't do details)

$$\langle Q(x_1)Q(x_2)Q(x_3)\rangle = \int d^NQ \; Q_1Q_2Q_3 \; e^{iS_{quad}} \, e^{iS_{tri}}$$

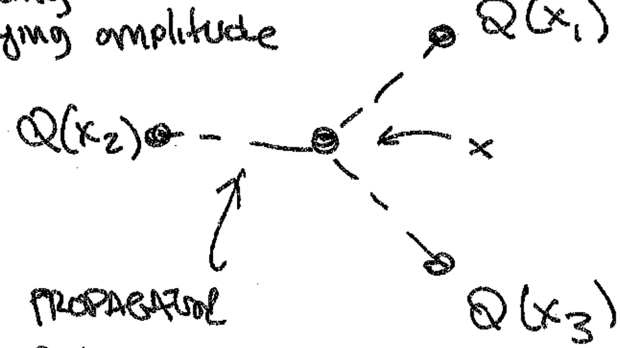
correlation of 3 points

amplitude

$$1 + \underline{\int d^4x \; \mu \, Q(x)Q(x)Q(x)}$$

$$\mu \int d^NQ \; Q_1Q_2Q_3 \; e^{iS_{quad}} \left(\int d^4x \, QQQ\right)$$

coeff. is something multiplying amplitude



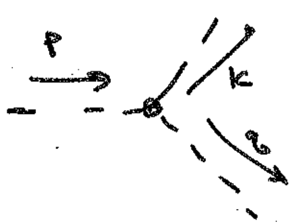
PROPAGATOR
$G(x_2 - x)$

INTEGRATE OVER d^4x

(integrate over all internal lines)

IN MOMENTUM SPACE:

CONVENTION: PEEL THESE FACTORS OFF FOR EXTERNAL LINES

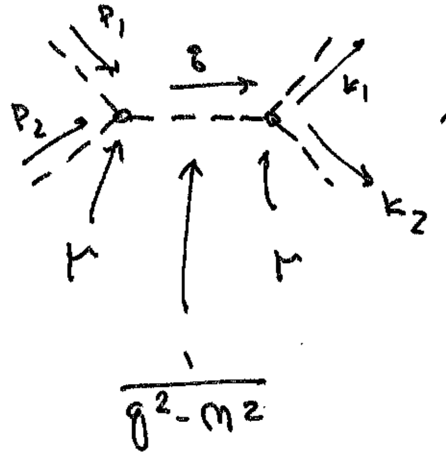


$$\sim \left[\frac{1}{p^2 - m^2} \; \frac{1}{k^2 - m^2} \; \frac{1}{q^2 - m^2} \right] \times \mu$$

"value" of vertex
"coupling constant"

HOW QFT CALCULATIONS WORK:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{3!}\mu\phi^3$$ ← self-interaction



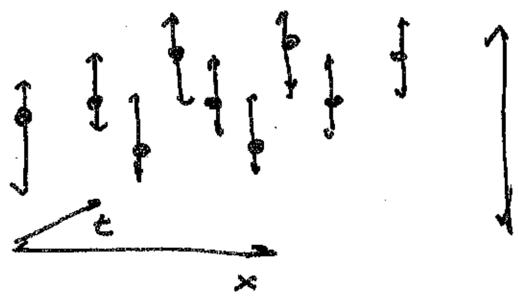
$$\sim \mu^2 \frac{1}{q^2 - m^2}$$

from momentum conservation

$q = p_1 + p_2$

<u>Question</u>: what if $(p_1+p_2)^2 = m^2$?!

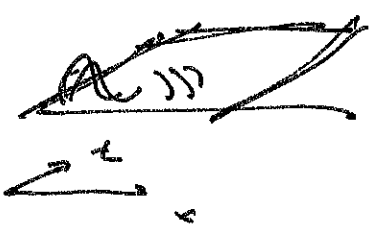
Cartoon of the quantum field:



<u>field value</u>

beads on a rail
"mass" of bead ≠ mass of particle.

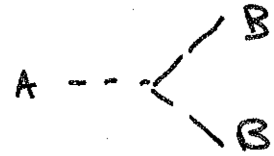
neighboring beads are connected by spring
→ Hooke's law potential



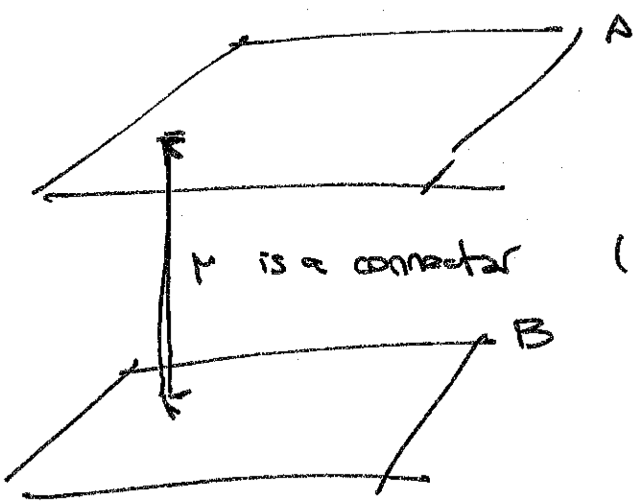
excitations of a spring $q(t_1, x_1)$ can <u>propagate</u> to other spacetime points $q(t_2, x_2)$

INTERACTING QUANTUM FIELDS

$$\mathcal{L} = \tfrac{1}{2}(\partial A)^2 - \tfrac{1}{2}m_A^2 A^2 \quad \} \text{ free A field}$$
$$+ \tfrac{1}{2}(\partial B)^2 - \tfrac{1}{2}m_B^2 B^2 \quad \} \text{ free B field}$$
$$+ \tfrac{1}{2}\mu A B^2 \longleftarrow \boxed{\text{interaction}}$$



$\sim \mu$



(attach springs @ EA SPACETIME POINT)

Wiggles in A field → wiggles in B field
how? a quantum ($a_A^\dagger$) of A wiggle
can connect to 2 quanta ($a_B^\dagger a_B^\dagger$) of
B wiggles

THEORY WRITING "model building"

① Define symmetries

A. SPACETIME (assumed)
- TRANSLATION → every particle has well defined momentum, conserved @ vertex
- LORENTZ → spin-0, spin-$\frac{1}{2}$, spin-1, ...

B. GAUGE → gives spin-1 bosons that talk to anything w/ indices/charge of that GAUGE symmetry.

C. GLOBAL → symmetry w/ no assoc. boson
eg FLAVOR

② Define fields (particles)

→ carry indices of their symmetry.
→ may be $\mathbb{C}$ or $\mathbb{R}$ according to symm.

③ WRITE ALL ALLOWED INTERACTIONS

↳ vertices

↔ terms in $\mathcal{L}$ that are invariant.

$\mathcal{L}$ is written w/rt FIELDS ← "classical" info about MASS, ...

PARTICLES are quantum excitations of the respective field.

- QUADRATIC TERM: SOLVABLE ⟷ PROPAGATOR
- HIGHER TERMS: TAYLOR EXP., LOCAL VERTICES.