Weekly HW 3: **Symmetry Breaking**

Course: Physics 165, *Introduction to Particle Physics* (2020)
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This is the main weekly homework set. Unless otherwise stated, give all responses in natural units where $c = \hbar = 1$ and energy is measured in electron volts (usually MeV or GeV).

Quantum Numbers of the Standard Model, so far

A summary of recent results.

Gauge quantum numbers

The three fundamental forces have three quantum numbers:

- Hypercharge is a conserved phase. We will sometimes write a subscript $Y = q_Y$ to remind us that a particle has hypercharge q_Y, e.g. $\psi_{Y=q_Y}$. The fancy mathematical name for this type of symmetry is U(1).
- The electroweak force has doublet indices, which we write as upper or lower $i, j, k, \cdots$. These indices are either 1 or 2. Upper and lower indices may *contract*, e.g. Q^i, then its antiparticle has a lower index, $(Q^\dagger)_i$. The fancy mathematical name for this type of symmetry is SU(2). It is the complex generalization of two-dimensional real rotations. Sometimes we will call it $\text{SU}(2)_\text{L}$ because only *left-chiral matter* carries electroweak charge.
- The color force has triplet indices, $a, b, c, \cdots$ that run from 1 to 3. These, too, may be upper or lower. If a particle has an upper index, e.g. Q^a, then its antiparticle has a lower index, $(Q^\dagger)_a$. The fancy mathematical name for this type of symmetry is SU(3). It is the complex generalization of three-dimensional real rotations. Sometimes we will call it $\text{SU}(3)_\text{c}$ because the values $a = 1, 2, 3$ are sometimes referred to as *colors*: red, green, and blue.

Pairs of upper and lower gauge indices may be *contracted* (summed over) as long as the indices are of the same type. Thus you could contract the upper color index of a quark doublet Q^a with the lower color index of an anti-quark doublet $(Q^\dagger)_b$ via

$$Q^a(Q^\dagger)_a \equiv \sum_{a=1}^{3} Q^a(Q^\dagger)_a,$$

however you *cannot* contract the color index of an anti-right-handed up quark $\bar{u}_a$ with the doublet index of a lepton, L^i. The a and the i indices are are totally different and cannot be contracted.

The symmetries also have *invariants* that let us raise/lower and otherwise contract indices. Each of the $\text{SU}(N)$ symmetries have the N-dimensional Levi–Civita tensor available. This means:

- For $\text{SU}(2)_\text{L}$, you may use ε_{ij} and ε^{ij} as invariant tensors. Because these have two lower or two upper indices and they happen to contract to $\varepsilon_{ij}\varepsilon^{jk} = \delta_i^k$, these can be understood as metrics on SU(2).

- For $\mathrm{SU}(3)_c$, you may use ε_{abc} and ε^{abc} as invariant tensors.

Note that the Levi–Civita tensor is totally antisymmetric. This means that it is only non-zero when each index takes on a different value.

Spin quantum numbers

Spin-1/2 particles come in two types:

- **Left-chiral** (left handed) particles have an index $\alpha, \beta, \cdots$ that runs from 1 to 2. These may be raised and lowered with $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$.
- **Right-chiral** (right handed) particles have an index $\dot{\alpha}, \dot{\beta}, \cdots$ that runs from 1 to 2. These indices may also be raised and lowered with $\varepsilon_{\dot{\alpha}\dot{\beta}}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$.

Tip: for the SU(2) and SU(3) indices, it will be important that we use the Levi–Civita tensor to trade upper and lower indices. For spin-1/2, this is less important for us and it's sufficient to remember that you can raise and lower the spin-1/2 index as needed.

The dotted and undotted indices are completely different: it is as if they come from different alphabets. In the same way that you wouldn't contract a SU(3) color index with an SU(2) electroweak index, you cannot contract the dotted and undotted indices with each other unless you have an invariant to tie them together. It turns out that nature (conservation of angular momentum) gives us such invariants:

$$(\sigma^\mu)^{\dot{\alpha}}{}_{\alpha} \qquad\qquad (\bar{\sigma}^\mu)^{\alpha}{}_{\dot{\alpha}} \ . \tag{1}$$

Comment: there are some notable similarities between spin-1/2 and SU(2). This is not a coincidence, but it is a very subtle story[1].

Gauge particles

For each of the fundamental gauge symmetries, we have a gauge boson. These are all spin-1 particles that carry a Lorentz index, $\mu, \nu, \cdots$. The U(1) hypercharge boson is special in that (1) it does not carry any of its own charge, and (2) it doesn't have any indices. Thus we have:

- Hypercharge: Y_μ
- Weak: $(W_\mu)^i{}_j$
- Color: $(g_\mu)^a{}_b$.

There is one hypercharge gauge boson, three electroweak gauge bosons, and eight color gauge bosons. In general, for an SU(N) gauge symmetry, there are $N^2 - 1$ gauge bosons corresponding to each rotation axis.

Gauge symmetry: these gauge bosons have a surprising transformation law under their symmetries. For simplicity, we focus only on U(1) boson and argue that something similar happens for the others. Under a *gauge transformation* (a 'rotation' in this abstract space), the gauge bosons transform as a shift: $Y_\mu \to Y_\mu + \partial_\mu \alpha(x)$ where $\alpha(x)$ is a *gauge function* that we are free to choose. It is related to

[1] Advanced: `www.maths.ed.ac.uk/~jmf/Teaching/Projects/Poincare/`

the transformation angle θ. This transformation by a shift implies that there is no way to write a mass term for the gauge bosons since $Y_\mu Y^\mu$ is not invariant.

Matter Particles

The matter particles of the Standard Model are 'cuddly'. The quark doublet is:

$$Q^{\alpha i a}_{Y=1/6} = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \ . \tag{2}$$

The right-handed anti-up and right-handed anti-down are:

$$(\bar{u}^\alpha_a)_{Y=-2/3} = u_R^\dagger \qquad\qquad (\bar{d}^\alpha_a)_{Y=1/3} = d_R^\dagger \ . \tag{3}$$

The lepton doublet is

$$L^{\alpha i}_{Y=-1/2} = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \ . \tag{4}$$

The right-handed positron is:

$$(\bar{e}^\alpha)_{Y=1} = e_R^\dagger \ . \tag{5}$$

Together these are the 'cuddly' ('QudLe') particles.

The Higgs vev

The Higgs is a scalar particle with indices that acquires a **vacuum expectation value** (vev), $\langle H \rangle$. This is a background value for the Higgs *field* that exists everywhere. The vev *spontaneously breaks* electroweak symmetry (SU(2)×U(1)).

$$H^i_{Y=1/2} \qquad\qquad \langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \qquad\qquad v = 246 \text{ GeV} \ . \tag{6}$$

Anywhere you can put the Higgs, you can put the Higgs vev as an **order parameter** for electroweak symmetry breaking. It's like a non-invariant tensor that we get to use as a free 'index contractor.'

1 Higgs vev: masses to the fermions

1.1 Yukawa couplings

Write out the three allowed Yukawa couplings in the Standard Model, make all indices explicit. To keep track of them, write a dimensionless coefficient in front of each one: y_u, y_d, and y_e. (Hopefully it's obvious which one goes with which interaction.) For example:

$$y_d \, Q^{\alpha i a} (H^\dagger)_i (d_R^\dagger)_{\alpha a} + \text{h.c.} \,, \tag{7}$$

where 'h.c.' stands for Hermitian conjugate. We've written $(d_R^\dagger)_{\alpha a}$ for $\bar{d}^\beta_a \varepsilon_{\beta\alpha}$.

1.2 Mass terms

Now insert the Higgs vev in place of the Higgs in the Yukawa couplings, $H \to \langle H \rangle$. Explicitly out the SU(2) doublets in terms of components, and explicitly write out the 'h.c.' terms so that you have a total of six terms. Perform the matrix multiplication to identify a set of 6 contractions that glue together pairs of spin-1/2 particles.

For example, the first term written above gives:

$$y_d \frac{v}{\sqrt{2}} d_L^{\alpha a} \left(d_R^\dagger\right)_{a\alpha} + y_d \frac{v}{\sqrt{2}} (d_L^\dagger)_{\dot{\alpha} a} d_R^{\dot{\alpha} a} \ . \tag{8}$$

The two particle that get glued together are *anti-particles* of one another. The coefficient of the two particles is the mass.

Observe that for each 'type of particle' (e.g. 'down quark') there are actually two pairs of matter particles. The left-handed down connects to the anti-right-handed down. The anti-left-handed down connects to the right-handed down. Also note that the d_L and $d_L^\dagger$ are both part of doublets, so they interact with the W bosons, whereas the d_R and $d_R^\dagger$ only talk to hypercharge.

1.3 Numbers

Look up the masses of the up quark, down quark, and electron. Write out the values of y_u, y_d, and y_e. Make sure the dimensions are correct.

2 The right-handed neutrino

Inspired by the previous problem and the fact that neutrinos have mass, we may want to introduce a right-handed neutrino, $\bar{N} = \nu_R^\dagger$. This would have a Yukawa coupling analogous to the up-type:

$$y_N \, \varepsilon_{ij} H^j L^{\alpha i} (\nu_R^\dagger)_\alpha + \text{h.c.} \ . \tag{9}$$

That's a hint for the previous problem if you had a hard time getting indices to contract and charges to cancel.

What are the quantum numbers of the right-handed anti-neutrino, $\bar{N} = \nu_R^\dagger$? Specifically, what is the hypercharge? (ANSWER: zero. Confirm this.)

2.1 A Majorana mass for the right-handed neutrino

A matter particle with no charges or other gauge indices may pick up a mass by itself. These are called *Majorana* masses and imply that the particle is its own anti-particle. Show that $\bar{N}\bar{N}$ is an invariant by checking that all of the gauge charges/indices cancel/contract and that the spin indices contract.

COMMENT: the fact that you can form a Majorana mass from $\bar{N}\bar{N}$ implies that this mass is 'unprotected'. Unlike the other matter particles that only pick up mass because of the Higgs, the right-handed neutrino can pick up large masses—indeed, this is what we expect should happen.

2.2 A Majorana mass for the left-handed neutrino, part 1

It turns out that one can form a Majorana mass for ν_L *without* using $\bar{N}$. This cannot come from a Yukawa coupling (why?). We know that such a mass term has two powers of the left-handed neutrino; since ν_L lives inside of L, this means that it comes with two powers of L. We know that it cannot have any other powers of additional particles, but that there are leftover indices that need to be contracted: the only tool that we have is the Higgs vev, $\langle H \rangle$. Show that you can create a mass term using L and $\langle H \rangle$: explicitly write out all indices and check charge conservation when $\langle H \rangle$ is replaced by H. Do the matrix multiplication in SU(2) space to show that you only pick up a mass for the ν_L, and not the e_L.

FUN FACT: this operator is called the Weinberg operator.

3 The Higgs Potential

At each point in spacetime, the Higgs *field* feels a potential

$$V[H] = \pm \mu^2 H^\dagger H + \frac{\lambda^2}{2}(H^\dagger H)^2 \ . \tag{10}$$

3.1 Pictures Help

Sketch the potential $V[v/\sqrt{2}]$ for the two signs above.

3.2 The Higgs vev

Write out $H = (0 \ v/\sqrt{2})^T$. Minimize the potential with respect to v to determine v as a function of μ and λ. What is the value of v when the μ^2 term has a positive sign? What is the value of v when the μ^2 term has a negative sign?

3.3 Extra credit: tachyonic mass

Note that the $H^\dagger H$ term is precisely what we would call a mass. It looks like electroweak symmetry breaking has to do with the Higgs appearing to have a negative squared mass term. Argue why a particle with a negative squared mass is *tachyonic*.

Note that the physical Higgs particle does *not* have a tachyonic mass. The apparent tachyonic mass comes from expanding the Higgs field about a position that is not the minimum of the potential.

4 Electrodynamics from the electroweak force

The Higgs vev $\langle H \rangle$ is the order parameter of electroweak symmetry breaking.

4.1 Transforming the vev

Write out the Higgs vev explicitly as a 2-vector. How does it transform under a hypercharge rotation by angle θ_Y? How does it transform under a weak transformation with respect to the third axis by angle θ^3? Recall that the weak transformation on a doublet is

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \exp\left(\frac{1}{2}\theta^A \sigma^A\right) \begin{pmatrix} a \\ b \end{pmatrix} \ . \tag{11}$$

Argue that the vev is clearly *not* invariant under rotations about the first two axes.

4.2 A leftover U(1)

The Higgs vev is invariant under a combination of U(1) hypercharge and SU(2) rotations about the third axis. What combination is this? (Explain why.)

ANSWER: The Higgs vev is invariant under a combined rotation where $\theta_Y = \theta^3$.

This leftover U(1) symmetry is electromagnetism.

4.3 Electric charges of matter

Determine the *electric charge* of each matter particle. This requires treating the components of the weak doublets separately: u_L and d_L, ν_L and e_L. You know the correct answers, but show how this is related to the 'leftover U(1)' above.

5 Protons

A proton is a particle with a spin-1/2 index and net electric charge +1. Show how three quarks can be combined to give an object with these quantum numbers and no net color charge. Use the SU(3) invariant tensor.