LOOPS & DIVERGENT DIAGRAMS

let's put away the formal diagrammatica for a moment & do some "nuts & bolts" calculations.

WICK ROTATION

Feynman PROPAGATOR:

$$\frac{i}{k^2 - m^2 \underbrace{+ i\varepsilon}}$$

Banks p. 21

can think of this from convergence p.o.v.

$$Z \sim \int \mathcal{D}\phi \; e^{iS + i\int J\phi}$$

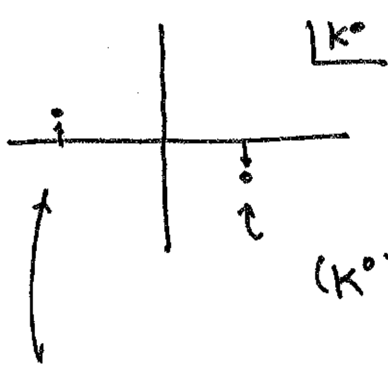
$$S = \int \tfrac{1}{2}(\partial\phi)^2 \underbrace{- V(\phi)}_{-\frac{1}{2}m^2\phi^2}$$

gives oscillating integral (steepest descent)

OR: MAKE REPLACEMENT $m^2 \to m^2 - i\varepsilon$

to get $e^{i\int (i\varepsilon)\phi^2} \sim \underbrace{e^{-\int \varepsilon\phi^2}}_{\text{exp damp.}}$

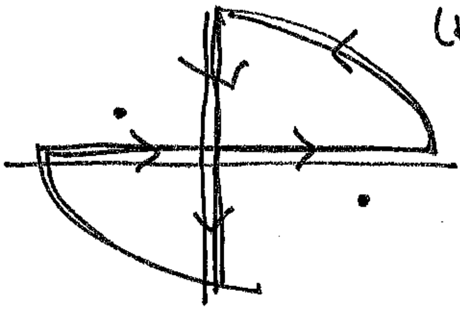
the $\varepsilon \to 0$ limit is a pole-pushing prescription



$$(k^0) = \underbrace{\sqrt{\vec{k}^2 + m^2}}_{E_k} - i\varepsilon$$

$k^0 = -E_k + i\varepsilon$

POLE STRUCTURE MEANS YOU CAN ROTATE INTEGRAL OVER dk^0



← integrand = 0

arcs = 0

$$\Rightarrow \int_{-\infty}^{\infty} dk^0 + \int_{i\infty}^{-i\infty} dk^0 = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} dk^0 = \int_{-\infty}^{\infty} d(ik^0)$$

EUCLIDEAN: $\boxed{k_E^0}$

then: $$\frac{i}{(k^0)^2 - \underline{k}^2 - m^2 + i\epsilon} \longrightarrow \frac{-i}{(k_E^0)^2 + \underline{k}^2 + m^2}$$

$$d^d k \longrightarrow i\, d^d k_E$$

UV DIVERGENCES come from loops ← EA. LOOP = UNCONSTR. MOMENTUM

IR DIVERGENCES ARE A SEPARATE, EQUALLY INTERESTING STORY!

tree level diagrams have enough vertices to impose no unconstr. momenta.

focus on scalar fields

$$\int \frac{d^{dL} l}{(2\pi)^{dL}} \frac{1}{k^{2I}}$$

← LOOP MOMENTUM

L # LOOPS

$k \sim l + \cdots$

[Superficial] Degree of Divergence : estimate based on POWER COUNTING

$D = dL - 2I$ ← integral is divergent if $D \geq 0$

Divergences are a consequence of the continuum limit
↳ $l \to \infty \iff$ lattice spacing $\to 0$

↑ ASSUMES no new microphysics becomes relevant.

modern view : theory comes w/ cutoff, Λ
↳ not nec. statement about SPACETIME, but usually about UV dynamics.

1^{st} step for DIVERGENT integrals: quantify them.
give some parameterization of the divergence st the integral is infinite when some parameter goes to 0 or ∞.

↳ interpretation later.

simplest choice : strict cutoff. all moments $\leq \Lambda$
↑ turns out to be clunky
you lose translation invariance
... & also not gauge invariant.

want a better REGULARIZATION scheme
↑ quantify divergence

Perhaps the most practical: $\boxed{\text{DIM-REG}}$

for sufficiently small d, SUPERF. DEG. OF DIV. $\to 0$ $\longrightarrow$ continuously change the dimension of spacetime

divergence in $d \to 4$ limit

mathematical aside: the Γ function

$$\Gamma(\alpha) \equiv \int_0^\infty dx\, x^{\alpha-1} e^{-x} \quad \longleftarrow \quad \Gamma(n) = (n-1)! \text{ for INTEGERS}$$

finite for $\alpha > 0$
divergent for $\alpha \leq 0$

pf: $\Gamma = \frac{1}{\alpha}\int dx \left(\frac{d}{dx}x^\alpha\right) e^{-x}$

+ int by parts

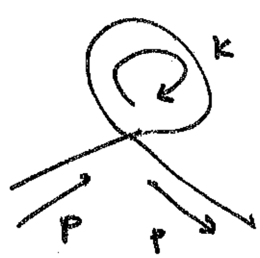
so: $\Gamma(1) = 1$, $\boxed{\Gamma(\alpha \to 0) \sim \frac{1}{\alpha}}$

we will use this as our parameterization.

where does Γ show up?

consider: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2 - \frac{1}{4!}\lambda\phi^4$

calculate



$$= \frac{-i\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\, \frac{i}{k^2 - m^2 + i\epsilon}$$

general DIMENSION

symmetry factor

$$= \frac{\lambda}{2}\int \frac{i\,d^d k_E}{(2\pi)^d}\ \underbrace{\frac{-1}{k_E^2+m^2}}_{\text{no angular dep in } d\text{-DIM EUCLIDEAN SPACE}}$$

hyperspherical coords: $d^d K_E = S_d\, k^{d-1}\, dk$

angular int in d dimensions

$k = |K_E|$

$$S_1 = 2$$
$$S_2 = 2\pi$$
$$S_3 = 4\pi$$

$$\boxed{S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

pf: HO (4.44) FROM GAUSSIAN INT.

$$\left(\frac{\pi}{\lambda}\right)^{d/2} = \int d^d K_E\, e^{-\lambda K_E^2}$$
$$= S_d \int_0^\infty dk_E\, k_E^{d-1}\, e^{-\lambda k_E^2}$$
$$= \frac{1}{2} S_d \frac{\Gamma(d/2)}{\lambda^{d/2}}$$

$$= -\frac{i\lambda}{2} S_d \int \frac{dk}{(2\pi)^d} \frac{k^{d-1}}{k^2+m^2}$$

$$\frac{2}{\Gamma(d/2)} \frac{\pi^{d/2}}{(2\pi)^d} = \frac{1}{(4\pi)^{d/2}} \frac{2}{\Gamma(d/2)}$$

$$= \frac{-i\lambda}{\Gamma(d/2)} \frac{1}{(4\pi)^{d/2}} \int dk\, \frac{k^{d-1}}{k^2+m^2}$$

<u>TRICK</u> $\frac{1}{k^2+m^2} = \int_0^\infty d\alpha \; e^{-\alpha(k^2+m^2)}$ ← I'm going to stop writing k_E

 $= \frac{-i\lambda}{\Gamma(d/2)} \frac{1}{(4\pi)^{d/2}} \int_0^\infty d\alpha \int_0^\infty dk \; k^{d-1} e^{-\alpha(k^2+m^2)}$

write: $u = k^2$

$$= \int d\alpha \int \frac{du}{2} \underbrace{k(u)^{d-2}}_{u^{d/2-1}} e^{-\alpha(u+m^2)}$$

$$= \frac{1}{2}\int d\alpha \underbrace{\int^\infty du \; u^{\frac{d}{2}-1} e^{-\alpha u}}$$

$e^{-\alpha m^2}$

$v = \alpha u$

$$\frac{dv}{\alpha} \alpha^{-\frac{d}{2}+1} \underbrace{v^{\frac{d}{2}-1} e^{-v}}_{\text{integrand of } \Gamma}$$

$$= \frac{1}{2}\int^\infty \underbrace{d\alpha \; e^{-\alpha m^2} \alpha^{-\frac{d}{2}}} \quad \Gamma(d/2)$$

$w = m^2\alpha$

$$\to \frac{dw}{m^2}(m^2)^{d/2} e^{-w} w^{-d/2} = (m^2)^{d/2-1} \underline{dw \, e^{-w} w^{(1-d/2)-1}}$$

$$= \frac{\Gamma(d/2)}{2} (m^2)^{d/2-1} \Gamma(1-d/2)$$

 $\boxed{= \frac{-i\lambda}{2} \frac{1}{(4\pi)^{d/2}} (m^2)^{d/2-1} \Gamma(1-d/2)}$

indep of p... in general,

 may have p-dependence

Some properties : Γ has poles @ $0, -1, -2, \cdots$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1} \left(1+\frac{z}{n}\right) e^{-z/n}$$

$\gamma \approx 0.6$
EULER-M CONST.

$$\boxed{\Gamma(\underset{\substack{\uparrow \\ n=0,1,2,\ldots}}{-n+\varepsilon}) = \frac{(-)^n}{n!}\Big(\underset{\substack{\uparrow \\ \text{pole}}}{\frac{1}{\varepsilon}} \underbrace{-\gamma+1+\tfrac{1}{2}\cdots+\tfrac{1}{n}}_{O(1)} + \underset{\substack{\uparrow \\ \text{goes to } 0}}{O(\varepsilon)}\Big)}$$

fact II: $A^{\varepsilon} = e^{\ln a^{\varepsilon}} = e^{\varepsilon \ln a}$

$$= 1 + \varepsilon \ln a + O(\varepsilon^2)$$

SO WE HAVE :

$$= \frac{-i\lambda}{2} \frac{m^2}{(4\pi)^2} \underbrace{\left(\frac{m^2}{4\pi}\right)^{\frac{d}{2}-2} \Gamma\left(1-\frac{d}{2}\right)}$$

let $d = 4-\varepsilon$

$\frac{d}{2}-2 = -\frac{\varepsilon}{2}$

$1-\frac{d}{2} = \frac{\varepsilon}{2}-1$

$$= \frac{-i\lambda}{2} \frac{m^2}{(4\pi)^2} \left(1 - \frac{\varepsilon}{2} \ln\left(\frac{m^2}{4\pi}\right) + \cdots\right)(-)\left(\frac{1}{\varepsilon} - \gamma + 1 + \cdots\right)$$

DIMENSIONS!

$$= \frac{+i\lambda}{2} \frac{m^2}{(4\pi)^2} \left[\frac{1}{\varepsilon} - \frac{1}{2} \ln \frac{m^2}{4\pi} - \gamma + 1 + O(\varepsilon)\right]$$

want to remove this
... and maybe other terms ...
there is a scale in here

So what now?

GO BACK TO DEFINITION of theory.

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4$$

subscript: bare parameters

in contrast to: what I measure.

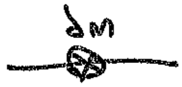
@ this order in pert thy (IGNORING FIELD STRENGTH)

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\underbrace{m^2 + \delta m}_{m_0^2})\phi^2 - \frac{1}{4!}(\underbrace{\lambda + \delta_\lambda}_{\lambda_0})\phi^4$$

written w.r.t. what I measure & a COUNTER TERM.

counter term is what must be there st @ loop level you get the observed couplings.

can draw Feynman rules for THEM:



$= -i\delta_m$

$= -i\delta_\lambda$

then: "— +" (kinda)

 $=$  $+$ 

$$= \frac{i\lambda}{2}\frac{m^2}{16\pi^2}\left[\frac{1}{\epsilon} + \cdots\right] - i\delta_m$$

we get to pick this

HOW TO CHOOSE!

I WANT  @ 1-LOOP TO BE $\dfrac{i}{p^2 - m^2}$ ← physical

WE WROTE theory w/r+ m^2 & δ

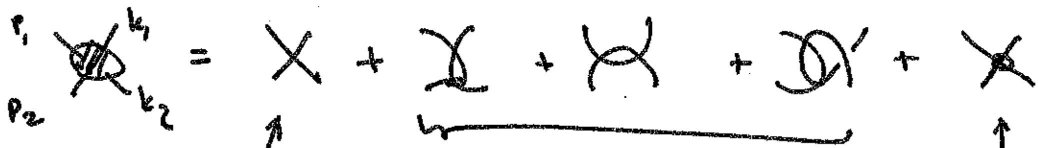
↳ so want the counter-term & loop correction pieces to cancel.

for this case, it's trivial: complete cancellation

$$\delta_m = \frac{\lambda}{2}\frac{m^2}{16\pi^2}\left[\frac{1}{\epsilon} + \cdots\right]$$

(RENORMALIZATION CONDITION) ⇝ but we're not doing renormaliz. yet.

PS P. 326 ANALOGOUS: λ AT LOOP LEVEL:



$-i\lambda$ $\qquad$ $(-i\lambda)\left(iV(s) + iV(t) + iV(u)\right)$ $\qquad$ $-i\delta_\lambda$

$$\left.\begin{aligned} s &= (p_1+p_2)^2 \\ t &= (p_1-k_1)^2 \\ u &= (p_1-k_2)^2 \end{aligned}\right\} \text{ Mandelstam vars for } 2\to 2$$

there's one last trick to evaluate these loops (FEYNMAN PARAMETERIZATION) momentum dep!

$$V(p^2) \to \frac{1}{32\pi^2}\int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\left[m^2 - x(1-x)p^2\right]\right)$$

HAVE TO CHOOSE REN. CONDITION

@ some scale / kinematic config.

LET US CHOOSE (can make any num of choices)

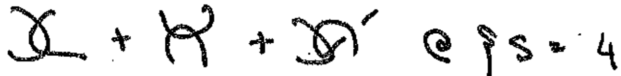
define λ to be the scattering of 2 ϕ particles in the zero 3-momentum limit

$$\Rightarrow \quad s = (p_1+p_2)^2 = 4M^2$$

$$t = (p_1-k_1)^2 = 0 \qquad \leftarrow \quad p_i = k_i = (M, \underline{0})$$

$$u = (p_1-k_2)^2 = 0$$

then:

δ_λ cancels  @ $\begin{cases} s = 4m \\ t = u = 0 \end{cases}$

$$\delta_\lambda = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\varepsilon} - \cdots - \log\left[m^2 - x(1-x)\underline{\underline{4m^2}}\right] - 2\log m^2\right)$$

IN GENERAL (away from $s=4M$, $t=u=0$), the AMPLITUDE is

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2} + \log\frac{m^2 - x(1-x)t}{m^2} + \log\frac{m^2 - x(1-x)u}{m^2}\right]$$

thus amplitude dep on kinematics (not surprising) in a way that dep on REN. COND $\leftarrow$ mildly surprising

Philosophy:

NEED to REGULATE theory: make sense in UV

dim reg does this ... could use anything.

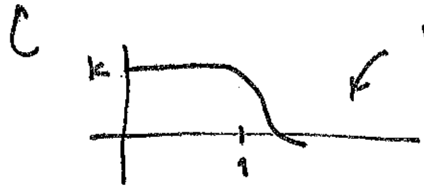
$\Rightarrow$ introduces a scale

↳ Q: where is the scale in DIM REG?

↓

can imagine HARD CUTOFF AS AN EXAMPLE

$$\mathcal{L} \to \frac{1}{2} K(\partial^2/\Lambda^2)\,(\partial\phi)^2$$

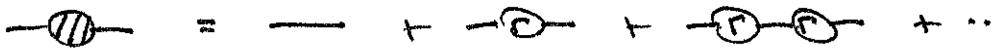


kills hi momentum modes.

the details of the REGULATOR SHOULDN'T MATTER (some are naturally easier to work with)

↳ BUT SCALE is PART OF THE DEFINITION OF THE THEORY.

FIELD STR. REN. (Peskin 7.1) (in gen)



1PI loops
(only 1PI; non 1PI accounted for by other terms in the sum)

in general,  depends on p^2
$\Sigma(p^2)$ — scalar

$$= \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}\frac{\Sigma(p)}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}\left(\frac{\Sigma(p)}{p^2 - m_0^2}\right)^2 + \cdots$$

$$= \frac{i}{p^2 - m_0^2 - \Sigma(p^2)}$$

$$\equiv \frac{iZ}{p^2 - m^2}$$

field str. RENORM.

POLE MASS
(btw: this is why it's hard to define quark masses)

$$m: \quad p^2 - m_0^2 - \Sigma(m^2) = 0$$

NEAR THE POLE:

$$\frac{i}{p^2 - m_0^2 - \Sigma(p^2)} = \frac{i}{p^2 - m^2}\,\frac{1}{1 - \Sigma'(m^2)}$$

$-\Sigma(M^2) - (p^2 - M^2)\Sigma'(M^2) + \cdots$

Z^{-1}

 $= \dfrac{iZ}{p^2 - M^2}$

Sketch: LSZ REDUCTION

as if: $\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi = \frac{1}{2}\phi\left(-\partial^2 - m_0^2\right)\phi$

$$\rightarrow \phi\left(\frac{-\partial^2 - m^2}{2Z}\right)\phi$$

can canonically normalize fields by rescaling $\phi \rightarrow \sqrt{Z}\,\phi$: $\phi\left(\frac{-\partial^2 - m^2}{2}\right)\phi$

↳ BUT now correlation functions pick up powers of $\sqrt{Z}$

LSZ: how to relate CORRELATION functions to S-matrix elements:

final RESULT:

PS (7.42)

$$\prod_{i=\text{IN}} \int d^4x_i\, e^{ip_i\cdot x_i} \prod_{j=\text{OUT}} \int d^4y\, e^{-ik_j y_j}$$

$$\times \langle \phi(x_1)\cdots\phi(x_n)\,\phi(y_1)\cdots\phi(y_m)\rangle$$

TIME ORDERED etc.

$$\xrightarrow[\substack{p_i^0\to E_{p_i}\\ k_i^0 \to E_{k_i}}]{} \underbrace{\left(\prod_{i=\text{IN}} \frac{\sqrt{Z}\, i}{p_i^2 - M^2 + i\varepsilon}\right)\left(\prod_{j=\text{OUT}} \frac{\sqrt{Z}\, i}{k_i^2 - M^2 + i\varepsilon}\right)} \underbrace{\langle p_1 \cdots | S | k_1 \cdots\rangle}_{\text{S-MATRIX}}$$

take correlation func.
FIND POLE where ext particles are on shell. PULL OUT FIELD STR. REN.

what's left is an AMPLITUDE.

Regularization of ϕ^4 including Z:

↳ not relevant @ 1-loop. BUT SHOW HOW IT WORKS

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4$$

CANON. NORM OF ϕ FROM LSZ: $\phi \to \sqrt{Z}\,\phi_R$

$$= \frac{1}{2}Z(\partial\phi_R)^2 - \frac{1}{2}m_0^2 Z\phi_R^2 - \frac{\lambda_0}{4!}Z^2\phi_R^4$$

$$\equiv \frac{1}{2}(\partial\phi_R)^2 - \frac{1}{2}m^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4$$

$$+ \frac{1}{2}\delta_Z(\partial\phi_R)^2 - \frac{1}{2}\delta_M\phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4$$

$\delta_Z = Z - 1$ $\qquad m_0^2 Z - m^2$ $\qquad \delta_\lambda = \lambda_0 Z^2 - \lambda$

Z is something you calculate ORDER-BY-ORDER in PERT THY ~~⊕~~

so have to be consistent when writing counterterms to do everything to same ORDER.