Path Integrals w/ FERMIONS

ANTICOMMUTING, so NEED NEW KIND OF ANTICOMMUTING # SYSTEM

Grassman #'s: let θ_i be a 'vector' of GRASSMAN #

$i = 1, \dots, n$

$$\boxed{\{\theta_i, \theta_j\} = 0} \qquad \theta_i \theta_j = -\theta_j \theta_i$$

$\Rightarrow \theta_i^2 = 0$ makes Taylor exp easy...

$$f(\underline{\theta}) = a + a^i \theta_i + a^{ij} \theta_i \theta_j + \cdots$$

$a^i \theta_i = \theta^i a_i$
WHERE a^i IS A c-NUMBER

CALCULUS W/ GRASSMAN # (GRASS. VAR)

for simplicity, $n = 2$ (anticipating Weyl spinors)

$$f(\theta) = a + b_1 \theta_1 + b_2 \theta_2 + c\,\theta_1 \theta_2$$

$$\frac{\partial}{\partial \theta_1} f = b_1 + c\theta_2$$

$$\frac{\partial}{\partial \theta_2} f = b_2 + c \underbrace{\frac{\partial}{\partial \theta_2}(\theta_1 \theta_2)}_{-\theta_1 \frac{\partial}{\partial \theta_2} \theta_2} = b_2 - c\,\theta_2$$

$$\rightarrow \left\{ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\} = 0$$

eg. $\theta_j \frac{\partial}{\partial\theta_j} f = b_1\theta_1 + c\theta_1\theta_2$

$$\frac{\partial}{\partial\theta_1}(\theta_1 f) = \frac{\partial}{\partial\theta_1}(a\theta_1 + b_2\theta_1\theta_2)$$
$$= a + b_2\theta_2$$

$$\{\theta_1, \frac{\partial}{\partial\theta_1}\} = 1 \quad \leftarrow \text{in gen: } \{\theta_i, \frac{\partial}{\partial\theta_j}\} = \delta_{ij}$$

<u>INTEGRATION</u>: $\int d\theta_i = 0$, $\int d\theta_i\, \theta_i = 1$ — ? motiv.

transl. inv.: eg $\int d\theta\, \theta = \int d\theta\, (\theta + \theta_0)$

$$\Rightarrow \int d\theta\,(a + b\theta) = b = \frac{\partial}{\partial\theta}(a + b\theta)$$

integration & differentiation are the same!

PICK CONVENTION: $d^n\theta = \underbrace{d\theta_n\, d\theta_{n-1} \cdots d\theta_1}$

order matters b/c $d\theta_i d\theta_j = -d\theta_j d\theta_i$

if $f(\underline{\theta}) = a\theta + a_i\theta_i + \frac{1}{2}a_{ij}\theta_i\theta_j + \cdots + \frac{1}{n!}a_{i_1\cdots i_n}\theta_{i_1}\cdots\theta_{i_n}$

nec. antisym.

<u>then</u>: $\int d^n\theta\, f(\theta) = \frac{1}{n!}\varepsilon_{i_1\cdots i_n} a_{i_1\cdots i_n}$ — similarities to differential n-forms is built in! (<u>not</u> a coincidence)

(or $\int d^n\theta\, \theta_{i_1}\cdots\theta_{i_n} = \varepsilon_{i_1\cdots i_n}$

CHANGE OF VARS:

$$\theta'_i = A_{ij}\theta_j$$

$$\Rightarrow \int d^n\theta \, f(A\underline{\theta}) = \int d^n\theta \quad a_{12\cdots n} A_{1i_1} \cdots A_{ni_n} \theta_{i_1} \cdots \theta_{i_n}$$

$$= a_{12\cdots n} \underbrace{A_{1i_1} \cdots A_{ni_n} \varepsilon_{i_1 \cdots i_n}}_{\det A}$$

$$\int d^n\theta \, f(\underline{A}\theta) = \det A \int d^n\theta \, f(\theta)$$

$$\shortparallel$$

$$\int d^n\theta \, f(\theta') = \det A \underbrace{\int d^n\theta' \, f(\theta')}_{\text{just relabel dummy var}}$$

$$\Rightarrow \boxed{d^n\theta' = (\det A)^{-1} d^n\theta}$$

this innocuous looking result is quite strange.

by comparison, for ordinary commuting variables:

$$\underbrace{d^n(Ax)}_{d^n y} = \boxed{\det A} d^n x$$

positive power! (Jacobian)

GAUSSIAN INTEGRAL - *fermionic*

$$\int d^n\theta \, e^{\frac{1}{2} A_{ij} \theta_i \theta_j}$$

ANTISYM : $A_{ij} = -A_{ji}$

take $n = 2m$ (even)

HOW TO EVALUATE? GAUSSIAN TRICK? no!

$$e^{\frac{1}{2} A_{ij}\theta_i\theta_j} = 1 + \tfrac{1}{2}A_{ij}\theta_i\theta_j + \tfrac{1}{2}\left(\tfrac{1}{2}\underline{\underline{A}}\,\underline{\theta}\,\underline{\theta}\right)^2 + \cdots$$

only the term w/ $2m$ θ's SURVIVES.
this is the m^{th} term in exp.

$$= \left(\begin{matrix}\text{vanishes under}\\ d^n\theta\end{matrix}\right) + \frac{1}{m!}\frac{1}{2^m} A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{n-1} i_n} \times \theta_{i_1} \cdots \theta_{i_n}$$

$$\int d^n\theta \, e^{\frac{1}{2} A_{ij}\theta_i\theta_j} = \frac{1}{m!}\frac{1}{2^m} A_{i_1 i_2} \cdots A_{i_{n-1} i_n} \varepsilon_{i_1 \cdots i_n}$$

$$\equiv \mathrm{Pf}(A)$$ PFAFFIAN

Pfaffian w/rt det: $\underline{\theta}' = \underline{\underline{B}}\,\underline{\theta}$

$$\underbrace{\int d^n\theta'\, e^{\frac{1}{2}A_{ij}\theta'_i\theta'_j}}_{\substack{= \mathrm{Pf}(A) \\ \text{(JUST RELABEL DUMMY} \\ \text{VARIABLES)}}} = \underbrace{\frac{1}{\det B}}_{\substack{\text{SURPRISING} \\ \text{DET. RELATION} \\ \text{from R3}}} \int d^n\theta\, e^{\frac{1}{2}A_{ij}(B\theta)_i(B\theta)_j}$$

$A_{ij}B_{ik}B_{jl}$

$$= \frac{\mathrm{Pf}(B^T A B)}{\det B}$$

"standard result"
$\exists$ B s.t.:

$$\underline{B}^T\underline{A}\,\underline{B} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0 & 1 \\ & & -1 & 0 \ \ddots \end{array}\right) \equiv J$$

symplectic form

$(Sp(2m))$

$$\Rightarrow \boxed{\det{}^2 B \det A = 1}$$

fact: $\mathrm{Pf}(J) = 1$ $\quad \frac{1}{m!}\frac{1}{2^m}J_{..}\cdots J_{..}\,\varepsilon_{\dots} = \frac{m!\,2^m}{m!\,2^m}$

(combinatoric exercise)

<u>then</u>: $\mathrm{Pf}(A) = \frac{\mathrm{Pf}(B^T A B)}{\det B} = \frac{\mathrm{Pf}(J)}{1/\sqrt{\det A}} = \boxed{\pm\sqrt{\det A}}$

$\mathbb{C}$ version : θ_i + $\bar\theta_i$ indep, conjugate

$$\overline{\theta_1 \cdots \theta_n} : \bar\theta_n \bar\theta_{n-1} \cdots \bar\theta_1$$

$$\{\theta_i, \bar\theta_j\} = 0$$

$$\int \underbrace{d^n\theta\, d^n\bar\theta}_{d\theta_1 d\bar\theta_1 \cdots d\theta_n d\bar\theta_n} \; e^{\bar\theta_i C_{ij} \theta_j} \qquad C_{n\times n}$$

$d\bar\theta_i$ to the RIGHT by convention

nb you can move PAIRS of GRASSMAN #'S w/o PICKING UP SIGNS

$$= \int d^n\theta\, d^n\bar\theta \; \frac{1}{n!}\left(\bar\theta_i C_{ij} \theta_j\right)^n = \frac{1}{n!} \varepsilon_{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n} C_{i_1 j_1} \cdots C_{i_n j_n}$$

$$= \boxed{\det C}$$

vs. $+\sqrt{\det A}$ for $\mathbb{R}$ case

FERMIONIC H.O.

creation/ann ops: $\hat b$, $\hat b^+$, $\hat b^2 = (\hat b^+)^2 = 0$

$\hat b|0\rangle = 0$ $\quad \hat b^+|0\rangle = |1\rangle \quad \{\hat b, \hat b^+\} = \mathbb{1}$

states:

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad \hat b^+ = \begin{pmatrix} & \\ 1 & 0\end{pmatrix}$ $\qquad \hat b^+|1\rangle = 0$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $\qquad \hat b = \begin{pmatrix}0 & 1\\ & \end{pmatrix}$ $\qquad \hat b|1\rangle = |0\rangle$

$$\hat H = \omega \hat b^+ \hat b = \begin{pmatrix}0 & \\ & \omega\end{pmatrix}$$

Path integral :

states

$$|\theta\rangle = |0\rangle + \theta|1\rangle$$
$$\langle\bar\theta| = \langle 0| + \bar\theta\langle 1|$$

$\theta^2 = \bar\theta^2 = 0$

$$\boxed{\hat b|\theta\rangle = \theta|\theta\rangle} \Leftarrow$$

c.f. $\hat a|z\rangle = z|z\rangle$ for bosons

$\underbrace{\theta}_{0}|0\rangle + \theta|0\rangle$ $\qquad$ $\theta|0\rangle + \underbrace{\theta^2}_{0}|1\rangle$

TIME EVOLUTION

$$\langle\bar\theta|e^{-i\hat H t}|\theta\rangle = \langle 0|0\rangle + \bar\theta\theta\langle 1|e^{-i\omega t}|1\rangle$$
$$= 1 + \bar\theta\theta e^{-i\omega t}$$
$$= \boxed{e^{\bar\theta\theta e^{-i\omega t}}}$$

★

completeness : $\int d\bar\theta\, d\theta\, e^{\theta\bar\theta}\, |\theta\rangle\langle\bar\theta|$

$$= \int d\bar\theta\, d\theta\, e^{\theta\bar\theta}\left(|0\rangle\langle 0| + \theta\bar\theta|1\rangle\langle 1|\right)$$
$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

no lin. term w.r.t. $d\bar\theta d\theta$

$$= \mathbb{1}$$

$$= \int d\bar\theta' d\theta'\, e^{\theta'\bar\theta'}\langle\bar\theta|e^{-i\hat H(t-t')}|\theta'\rangle\langle\bar\theta'|e^{-i\hat H t'}|\theta\rangle$$

introduce int. state @ t'

INSERTING LOTS OF $\mathbb{1}$'s:

$$\langle\bar\theta|e^{-i\hat H T}|\theta_0\rangle = \int \prod_{r=1}^{N} d\bar\theta_r d\theta_r \, e^{\theta_r\bar\theta_r} \langle\bar\theta_r| e^{-i\hat H \delta t}|\theta_{r-1}\rangle$$

$$= \int \prod_r d\bar\theta_r d\theta_r \, e^{iS}$$

$$iS = \sum_{r=1}^{N} \big(\underbrace{\theta_r\bar\theta_r}_{e^{\theta_r\bar\theta_r}} + \bar\theta_r\theta_{r-1} e^{-i\omega\delta t} \big) .$$

from (*) on p.7

$$\approx \sum_r \big(\theta_r\bar\theta_r + \bar\theta_r\theta_{r-1}(1 - i\omega\delta t)\big)$$

$$S = \delta t \sum_r \Big(i\bar\theta_r \frac{\theta_r - \theta_{r-1}}{\delta t} - \omega\bar\theta_r\theta_{r-1} \Big)$$

IDENTIFY $\theta_r \to \psi(t_r)$

$\bar\theta_r \to \bar\psi(t_r)$

$$= \int_0^T dt \quad i\bar\psi(t)\dot\psi(t) - \omega\bar\psi(t)\psi(t)$$

$$\langle\bar\theta|e^{-i\hat H t}|\theta_0\rangle = \int \mathcal{D}\bar\psi\mathcal{D}\psi \, e^{iS[\psi,\bar\psi]} \qquad \text{subj to B.C.}$$

VARIATIONAL PRINCIPLE: $i\dot\psi(t) = \omega\psi(t)$

$-i\dot{\bar\psi}(t) = \omega\bar\psi(t)$

CLASSICAL SOL: $\psi_c(t) = \theta_0 e^{-i\omega t}$

$\bar\psi_c(t) = \bar\theta \, e^{i\omega(t-T)}$

field theory:

$$\int \mathcal{D}\psi \mathcal{D}\bar\psi\, e^{iS} \qquad S = \int d^dx\ \bar\psi\,(\underbrace{\gamma\cdot\partial - m}_{D})\,\psi$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\downarrow$$

$$= \boxed{\det D}$$

free theory: norm s.t. $\det D = 1$

cf. BOSONIC THEORY!

free FERMIONS

$$\langle \psi_\alpha(x)\,\bar\psi_\beta(y)\rangle = \underbrace{\frac{1}{\det D}}_{\text{norm.}} \int \mathcal{D}\psi\mathcal{D}\bar\psi\ \psi_\alpha(x)\,\bar\psi_\beta(y)\ e^{iS}$$

claim: $D_x \langle \psi_\alpha(x)\bar\psi_\beta(y)\rangle = i\,\mathbb{1}_{\alpha\beta}\,\delta^{(d)}(x-y)$

$\gamma^\mu \frac{\partial}{\partial x^\mu} - m$

(as expected: this is GREEN'S FUNC.)

pf. $D_x\langle\psi(x)\bar\psi(y)\rangle = \frac{1}{\det D}\int \mathcal{D}\psi\mathcal{D}\bar\psi \left(\frac{\delta}{-i\delta\bar\psi(x)}e^{iS}\right)\bar\psi(y)$

$$D_x\psi(x) = \frac{\delta}{\delta\bar\psi(x)} S$$

$$= \frac{1}{\det D}\int \mathcal{D}\psi\mathcal{D}\bar\psi\ i e^{iS}\frac{\delta}{\delta\bar\psi(x)}\bar\psi(y)$$

$$= \frac{\det D}{\det D}\ i\,\mathbb{1}\,\delta^{(d)}(x-y)$$

✓

the RESULT is familiar :

$$\langle \psi_\alpha(x)\, \psi_\beta(y) \rangle = i\, S_F(x-y)_{\alpha\beta}$$

$$\tilde{S}_F(p) = \int d^4x\, e^{-ip\cdot x}\, S_F(x)$$

SATISFIES

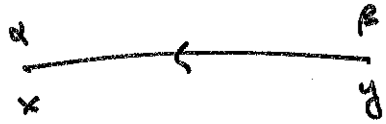
$$(\gamma\cdot p - m)\, \tilde{S}_F(p) = 1$$

mult both sides by $-\gamma\cdot p - m$

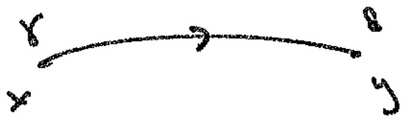
$$(p^2 - m^2)\, \tilde{S}_F = -(\not{p} + m)$$

$$-\tilde{S}_F = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}$$ ← FEYNM. PROP.

RULES :



$\langle \psi_\alpha(x)\, \bar{\psi}_\beta(y) \rangle = i S_F(x-y)_{\alpha\beta}$



$\langle \bar{\psi}_\gamma(x)\, \psi_\delta(y) \rangle$