Refs: Tong LECS on GAUGE THY, Shifman AQFT text

<u>LAST TIME</u> : <u>Anomaly</u>: symmetry of S, but not of Z.

e.g. <u>2D QED</u> $\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

BASIS:
$\gamma^0 = \sigma^2$
$\gamma^1 = i\sigma^1$
$\gamma^5 = \sigma^3$ ← chk sign

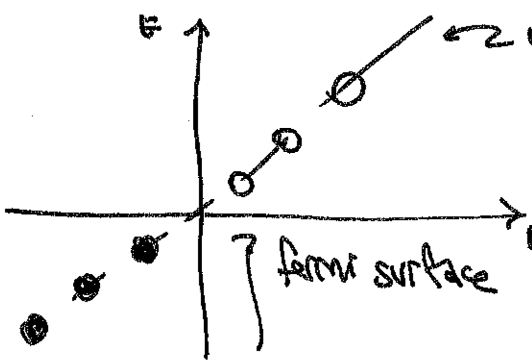
$$S = \int d^2x \; i\psi_+^\dagger \partial_- \psi_+ + i\psi_-^\dagger \partial_+ \psi_-$$

$$\partial_\pm = \partial_t \pm \partial_x$$

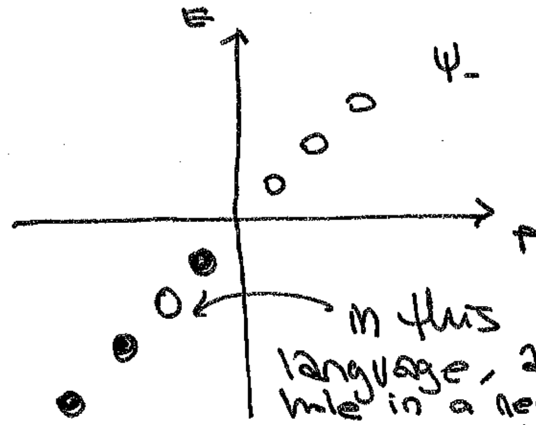
s.t. $\partial_- \psi_+ = 0 \Rightarrow \psi_+(t,x) = \underbrace{\psi_+(t+x)}_{\text{LEFT MOVING}} \equiv \psi_L$

$\partial_+ \psi_- = 0 \Rightarrow \psi_-(t,x) = \underbrace{\psi_-(t-x)}_{\text{RIGHT MOVING}} \equiv \psi_R$

Dispersion relation: $E = |p|$

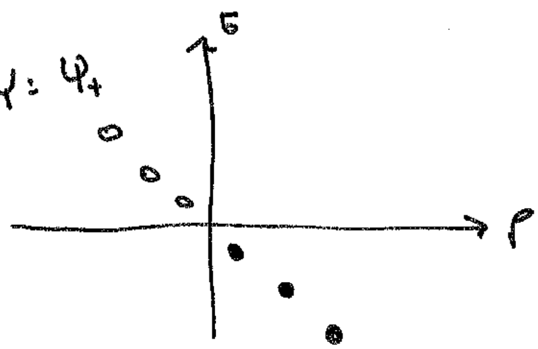


DIRAC SEA : filled negative energy states



in this language, a hole in a neg E state is an antiparticle: this is a $\psi_-^\dagger$.

SIMILARLY: ψ_+



$$|\text{vac!}\rangle = \prod_{E>0} |0,p\rangle_L \prod_{E<0} |1,p\rangle_L$$

$$\prod_{E>0} |0,p\rangle_R \prod_{E<0} |1,p\rangle_R$$

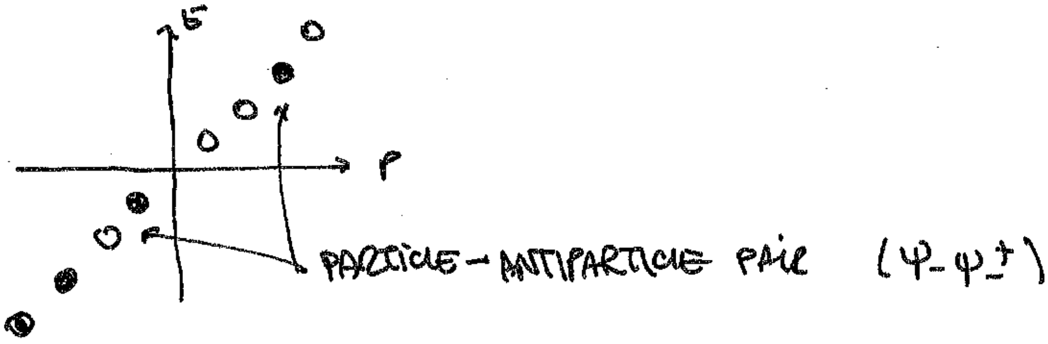
CHIRAL/AXIAL & VECTOR SYMMETRY

gauged : $\psi_\pm \to e^{i\theta}\psi_\pm$

$\psi_\pm \to e^{\pm i\theta}\psi_\pm$

$\Rightarrow$ CLASSICALLY, EXPECT # of $\psi_\pm$ to BE CONSERVED

eg



GAUGE VECTOR SYM : 2D QED $\quad \not\partial \to \not D$

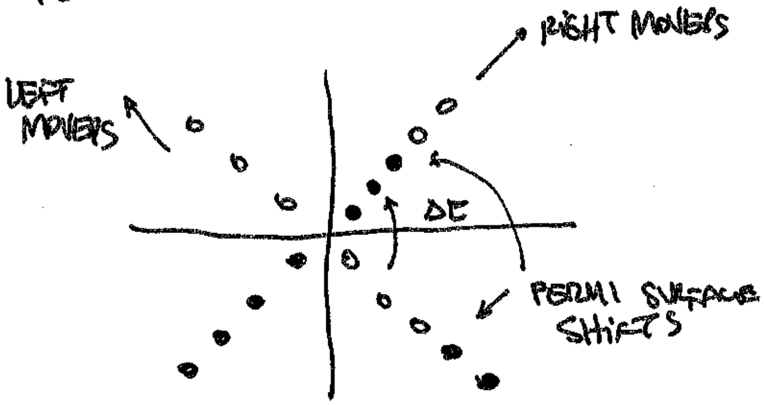
WE FOUND: $$\partial_\mu j^\mu_5 = \frac{e^2}{2\pi}\underbrace{\varepsilon^{\mu\nu}F_{\mu\nu}}_{2E}$$

first pass: what happens in the BG of an E field?

for $E>0$, pulls + charge to RIGHT.

$\Delta p = eEt \qquad$ (for const E)

FERMI SEA SHIFTS.



$\psi_\pm$ not cons!
(vector cons is)

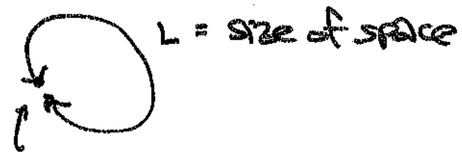
looks like LEFT MOVERS turned into RIGHT MOVERS (infrared interp)

... OR "HILBERT HOTEL" SHIFT w/ states moving @ ∞ (UV interp.)

Let's squeeze the most out of this 2D theory
(Schwinger model: see Shifman AQFT text)

Convenient: compactify space on a circle

* important step w/ nuances!! (see Shifman caveat re: couplings)



IMPOSE BC @ $\pm L/2$
PERIODIC for BOSON
ANTIPER for FERMION

$$\boxed{\begin{aligned} A_\mu(t, x=-L/2) &= A_\mu(t, x=L/2) \\ \psi(t, x=-L/2) &= -\psi(t, x=L/2) \end{aligned}}$$

Mode expansion $(k \in \mathbb{Z})$

$$\psi = e^{i(k+\frac{1}{2})x\frac{2\pi}{L}}\,\psi_{(k)}(t)$$

$$A = e^{ikx\frac{2\pi}{L}}\,A_{(k)}(t) = \cos\left(kx\tfrac{2\pi}{L}\right)A_{(k)}(t)$$

Gauge invariance

$$\psi \to e^{i\alpha(x)}\psi \qquad A \to A + \partial_\mu\alpha(x)$$

OBS: most of $A_{\mu=1}$ can be ~~gauged away~~

$$\alpha(x) = -\frac{L}{2\pi k}A_k(t)\sin\left(kx\tfrac{2\pi}{L}\right)$$

$k \neq 0$ ... leaves ZERO MODE of A

almost fixes the GAUGE completely.
can still do LARGE GAUGE TRANSF.
(not continuously connected to identity)
↳ most intuitive analogy: PARITY in SPACETIME SYM

In our case: $\alpha(x) = \frac{2\pi}{L} n x \qquad n = \pm 1, \pm 2, \ldots$

why valid? $\partial_x \alpha = \frac{2\pi n}{L} = \text{const}$

s.t. $A(x = -L/2) = A(x = L/2)$
is preserved

$$\partial_t \alpha = 0$$

SIMILARLY, $\psi \to \exp(i\alpha)\psi = \psi$.

So now we have a set of ∞ gauge redundant ??
vacua that are identified:

$$A_1, \quad A_1 = \pm 2\pi/L, \quad A_1 = \pm 4\pi/L \quad \ldots$$

$\Rightarrow$ A_1 is only independent on $\boxed{A_1 \in (0, 2\pi/L]}$

REMARK: FLUCTUATIONS IN A_0 ARE
SMALL: $eL \ll 1$
GIVES COULOMB POT BTWN
CHARGES, not the PART
WE CARE ABOUT HERE

periodicity in space has
imposed a periodicity (TOPOLOGY)
in gauge space.

DIRAC SEA PICTURE: $\bar{\psi} i \not{D} \psi \to \gamma^0 i \not{D} \psi = 0$

"$e = 1$" (but careful)

EOM: $\gamma^0 [i\partial_t \gamma^0 + \gamma^1 (i\partial_x + A_1)]\psi = 0$

$iD = i\partial + A$ convention

$$= [i\partial_t + \gamma^5 (i\partial_x + A_1)]\psi = 0$$

$$\gamma^0\gamma^1 = \sigma^2(i\sigma^1) = \sigma^3 = \gamma^5$$

$$\Rightarrow \psi = e^{-iE_k t}\psi_k(x) \quad \text{w/ } E_k \psi_k = \gamma^5 (i\partial_x + A_1)\psi_k$$

$$\psi_k \sim e^{i(k+\frac{1}{2})\frac{2\pi x}{L}}$$

antiperiodic

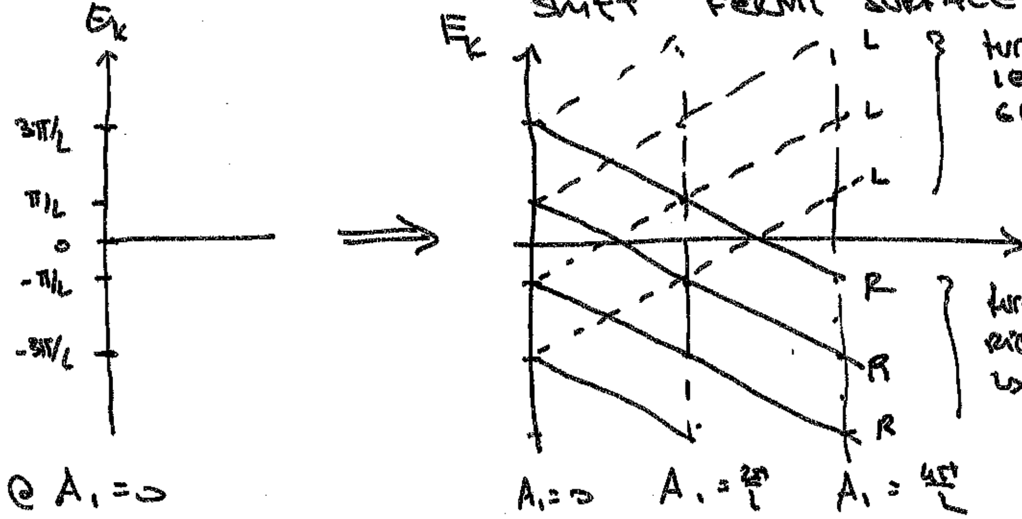
so given: $\psi_k = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ & $\gamma^5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$\gamma^5 (i\partial_x + A_1)\, e^{i(k+\frac{1}{2})\frac{2\pi x}{L}} = \gamma^5 \left[-(k+\tfrac{1}{2})\tfrac{2\pi}{L} + A \right] e^{-\!\!-}$$

$$\Rightarrow \boxed{\begin{aligned} E_k^{(L)} &= -(k+\tfrac{1}{2})\tfrac{2\pi}{L} + A_1 \\ E_k^{(R)} &= +(k+\tfrac{1}{2})\tfrac{2\pi}{L} - A_1 \end{aligned}}$$

neg E states filled
L movers have k < 0

as I turn on A_1, shift fermi surface ✓



turn up A, left movers gain E

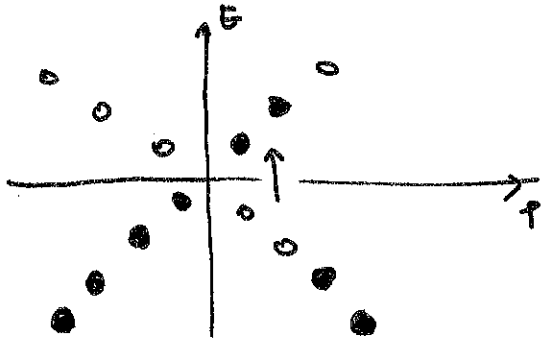
turn on A, right movers lose E

physical regime

degen w/ $A_1 = 0$, but we've shifted entire fermi sea

compare this to hole picture:

@ $A_1 \approx 0$

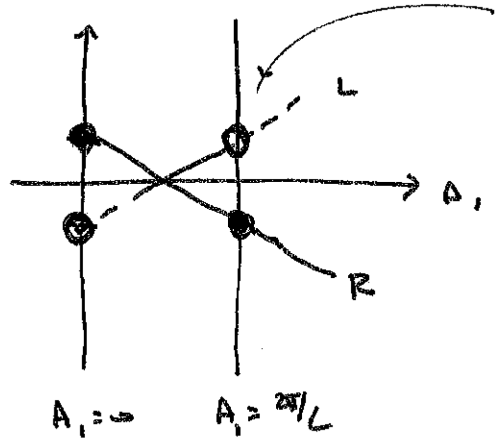


$|VAC\rangle =$

$$\prod_{k=0,1,\ldots} |1,k\rangle_L \prod_{k=-1,-2,\ldots} |0,k\rangle_L$$

$$\underbrace{\prod_{k=-1,\ldots} |1,k\rangle_R}_{\text{filled states}} \prod_{k=0,1,2\ldots} |0,k\rangle_R$$

AS WE INCREASE A_1 to $2\pi/L$, $|VAC?\rangle$ now describes a particle-antiparticle pair:



the 'new' vacuum.

relative to this, the "old vacuum" is a particle-antiparticle pair.

("IR PICTURE" → everything happening @ $E \approx 0$)

<u>UV PICTURE</u> : REGULATE

for this system: impose DISCRETE SPACE-TIME
(Schwinger splitting)

$$j^\mu_{(5)} = \bar\psi(t,x)\gamma^\mu(\gamma^5)\psi(t,x) \to \bar\psi(t,x+\varepsilon)\gamma^\mu(\gamma^5)\psi(t,x)$$

shift a small amount
(position cutoff)

nb: not gauge inv.
so attach a <u>WILSON LINE</u>

$$j^\mu_{(5)\,reg} = \underbrace{\bar\psi(t,x+\varepsilon)\gamma^\mu(\gamma^5)\psi(t,x)}\, e^{i\int_x^{x+\varepsilon} dx\, A_1}$$

$$\to (-+)\, e^{-i\alpha(x+\varepsilon)} e^{i\alpha(x)} \qquad \leftarrow \text{compensates}$$

$$\begin{array}{l} Q = Q_L + Q_R \\ Q_5 = Q_R - Q_L \end{array} \to \left| \begin{array}{l} Q_L = \sum_k^{\text{FILLED}} e^{+i\varepsilon E_k^{(L)}} = \sum_k e^{-i\varepsilon[(k+\frac{1}{2})\frac{2\pi}{L} - A_1]} \\ Q_R = \sum_{k'}^{\text{FILLED}} e^{-i\varepsilon E_{k'}^{(R)}} = \sum_{k'} e^{-i\varepsilon[(k'+\frac{1}{2})\frac{2\pi}{L} - A_1]} \end{array} \right.$$

$$\boxed{\begin{array}{l}\text{from } \bar\psi(x+\varepsilon) \text{ vs } \psi(x) \\ \} \; e^{i\int dx A_1}\end{array}}$$

So near $A_1 = 0$ $\quad (|A_1| < \pi/L)$

$$Q_L = \sum_{k=0}^{\infty} e^{i\varepsilon E_k^{(L)}} \qquad Q_R = \sum_{k=-1}^{-\infty} e^{-i\varepsilon E_k^{(R)}}$$

$k \geq 0 \to$ neg E $\qquad\qquad$ $k < 0 \to$ neg E

PERFORM SUM: $(Q_L)_{VAC} = -(Q_R)_{VAC}$

$$(Q_L)_{VAC} = \frac{e^{i\varepsilon A_1}}{2i \sin(\varepsilon\pi/L)}$$

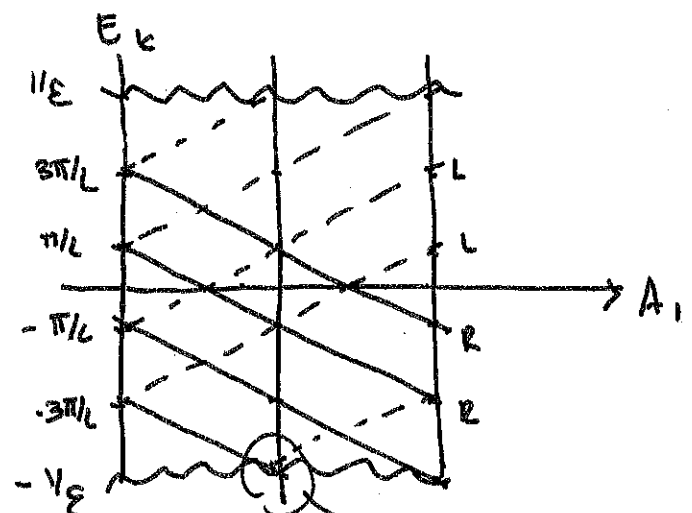
$$= \frac{L}{2\pi i \varepsilon} + \frac{1}{2\pi} A_1 + O(\varepsilon)$$

∞ const $\left(\sum_k 1\right)$

cuts off $|p + A_1| \gtrsim 1/\varepsilon$

LINEAR A_1 DEP.

@ $A_1 = 2\pi/L$, shift Q_5 by 2



one RH state disappears beyond cutoff, one LH state appears.

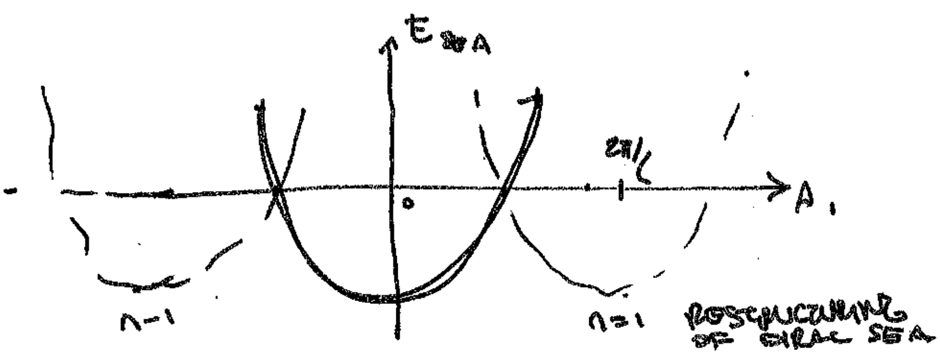
the Θ vacuum :

$$E_{LEFT} = \sum_{k=0}^{\infty} E_k^{(L)} e^{i\varepsilon E_k^{(L)}} \qquad E_{RIGHT} = \sum_{k=-1}^{-\infty} E_k^{(R)} e^{-i\varepsilon E_k^{(R)}}$$

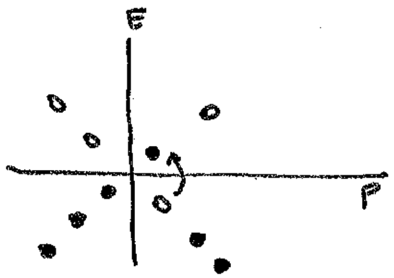
$$\frac{\partial}{\partial i\varepsilon} Q_L \qquad\qquad \frac{\partial}{\partial(-i\varepsilon)} Q_R$$

$$E_{SEA} = E_L + E_R = \frac{\partial}{\partial(i\varepsilon)} Q_L + \frac{\partial}{\partial(-i\varepsilon)} Q_R = 2\frac{\partial}{\partial i\varepsilon} Q_L$$

$$\sim \frac{L}{2\pi}\left(A_1^2 - \frac{\pi^2}{L^2}\right) + \text{const.}$$



BUT WHEN A_1 GOES FROM "near 0" to "near $2\pi/L \approx 0$" then vacuum shifts:



now we have a particle-antiparticle pair in what is supposed to be the "same" vacuum.

WHAT WE CALLED THE VACUUM near $A_1 = 0$ is not the VACUUM near $A_1 = 2\pi/L$.

HILBERT SPACE SPLITS into DISTINCT <u>SECTORS</u> :

$$\Psi_n = \prod_{k=n}^{\infty} |1,k\rangle_L \prod_{k=n-1}^{-\infty} |1,k\rangle_R \, \psi_0\left(A_1 - \frac{2\pi}{L} n\right)$$

start @ n

nb Ψ_n & $\Psi_{n'}$ are orthonormal

CAN WE CONSTRUCT A VACUUM that is invariant under LARGE GAUGE transformations? Yes.

<u>Θ - vacua</u>

new parameter describing vacuum

$$\Psi_\theta = \sum_n e^{in\theta} \Psi_n$$

under large gauge transform, Ψ_θ "intact"

$A_1 \to A_1 + 2\pi/L$

$\Psi_\theta \to \underbrace{e^{i\theta}}\Psi_\theta \qquad (\Psi_n \to \Psi_{n+1})$

? or minus?

but this overall phase of wavefunction is unobservable.

APPEARANCE IN $\mathcal{L}$:

$$\Delta\mathcal{L}_\theta = \underbrace{\frac{\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}} \leftarrow \text{total derivative}$$

topological density

(which topology?