Renormalization

REVIEW: LOOP DIAGRAMS MAY DIVERGE

↳ need to REGULATE DIVERGENCE
not physical; just quantify
WE USED DIM REG.

→ Renormalize: define a meaningful theory: this requires introducing a mass scale at which parameters are defined

↳ that mass scale, μ, is now part of the definition of n-point functions @ loop level

!! ① that's weird, does μ have physical significance? is it measurable?

shouldn't matter

② I get logs of ratios like p^2/μ^2.
If we took such a theory & studied it at $p^2 \ll \mu^2$ or $p^2 \gg \mu^2$, then the loop-level terms may be larger than tree-level terms

... perturbation series breaks!

Ans: Renormalization Group / RUNNING COUPLINGS
(changing theories judiciously)

BUT NOW there's also divergence in $A(\Lambda, p^2)$

↳ near the pole:

$$\frac{i}{Ap^2 - m_0^2 - B} \longrightarrow \frac{iZ}{p^2 - m_R^2}$$

$$G_0^{(2)} = Z\, G_R^{(2)}$$

$$Z\left(\lambda_0(\Lambda), \frac{\Lambda}{m_R}\right) = \left[\frac{d}{d(p^2)} A(\Lambda, p^2)p^2\right]^{-1}_{p^2 = m_R^2}$$

def @ m_R

from TAYLOR EXP w/RT $(p^2 - m_R^2)$

dim analysis: Z, can only depend on Λ through Λ/m_R

RENORMALIZED FIELD, ϕ_R:

$$\phi_0(x, \Lambda) = \sqrt{\underbrace{Z(\lambda_0(\Lambda), \Lambda/m_R)}_{}}\ \phi_R(x)$$

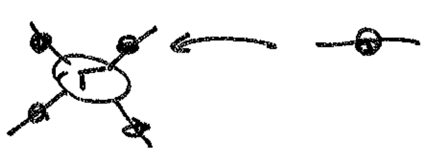
carries all of our sins.

wavefunc/field RENORM.

same Z as in LSZ REDUCTION.

↳ $Z^{1/2}$ disappears in LSZ formula if we use ϕ_R instead of ϕ_0

@ this point: 2pt func is finite

 is finite

Maggiore § 5.6 - 5.9 ← BEST, SUCCINCT DISCUSSION THAT I'VE SEEN

BARE : $\mathcal{L} = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$

everything dep on Λ .. in principle @ this stage (BARE)

PROPAGATOR : $\dfrac{i}{A(\Lambda, p^2)\, p^2 - m_0^2(\Lambda) - B(\Lambda)}$

 $= \dfrac{\lambda_0(\Lambda)}{32\pi^2}\left(\Lambda^2 + O(\log\Lambda) \ldots\right)$

$1 +$  $= 1 + \lambda_0^2(\Lambda)\left(c_1 \log \Lambda^2/p^2 + c_2\right)$

@ 1-loop : $\dfrac{i}{p^2 \underbrace{- m_0^2(\Lambda) - B(\Lambda)}_{-m_R^2}}$ ← REN MASS SET @ POLE

this is observable

but then: $m_0^2(\Lambda)$ dep on Λ

to cancel the div. in Λ in $B(\Lambda)$

@ 2-loop : m_R^2 still def by

$$A(\Lambda, p^2)\, p^2 - m_0^2(\Lambda) - B(\Lambda) = 0$$

$p^2 = m_R^2$

4-POINT:

if we do an expt: $i\mathcal{M}(\underbrace{\text{some kinematics}}_{\text{e.g. } (p_1+p_2)^2 = 4m_e^2}) \equiv -i\lambda_R$

physical (RENORM) coupling

$$= -i\lambda_0(\Lambda)\left[1 - \lambda_0(\Lambda)\left(\frac{\beta_0}{2}\log\frac{\Lambda^2}{M_e^2}\right) + \cdots\right)$$

some coef.

★ this defines $\lambda_0(\Lambda)$ w/rt OBSERVED λ_R

on the other hand, @ $(p_1+p_2)^2 = s \gg 4m_e^2$

$$i\mathcal{M}(s) = -i\lambda_0\left[1 - \lambda_0\left(\frac{\beta_0}{2}\log\frac{\Lambda^2}{s}\right) + \cdots\right]$$

$$\log\frac{\Lambda^2}{s} = \log\frac{\Lambda^2}{M_e^2} + \log\frac{M_e^2}{s}$$

$$= -i\lambda_R\left[1 - \lambda_R\frac{\beta_0}{2}\log\frac{M_e^2}{s}\right] + \mathcal{O}(\lambda_R^3)$$

should be λ_0, but $\lambda_0 = \lambda_R + \mathcal{O}(\lambda_R^2)$

but: $\log\frac{M_e^2}{s}$ can be large

... bigger than $\frac{2}{\lambda_R\beta_0}$!!

Nb: sign of β_0 determines how λ_R changes @ diff E

BACK TO OUR 2 CONCERNS

1. thy predictions shouldn't (CANNOT) depend on choice of REN. SCALE μ

   ... perturbativity of a given calc might!

2. maybe by changing μ we get better perturbativity?
   but how do we change μ & "stay in the same physics"?

   RIFE thy.

Callan-Symanzik :

$$\underbrace{\Gamma_0(p; g_0(\Lambda), \Lambda)}_{\mu \text{ indep.}} = Z(g_0(\Lambda), \tfrac{\Lambda}{\mu})^{n/2}\, \Gamma_R(p, g_R, \mu)$$

so let's do something silly: $\underbrace{\mu \frac{d}{d\mu}}_{\frac{d}{d\ln\mu}} \Gamma_0 = 0 \quad \Rightarrow \underbrace{\frac{d}{d\ln\mu}\left[Z^{n/2}\Gamma_R\right] = 0}$

this tells me how to change μ & "stay in same physics" "dimensional transmutation"

@ high energies : m negligible

$$\Gamma_R(p_i; g_R, \mu) = \underset{\substack{\uparrow \\ [\Gamma]}}{\mu^{d_r}} \underbrace{f\left(g_R, \frac{p_i}{\mu}\right)}_{\text{some dimless function}}$$

$$\Gamma_R\left(p_i; g_R, \frac{\mu}{u}\right) = \left(\frac{\mu}{u}\right)^{d_r} f\left(g_R, \frac{u}{\mu} p_i\right)$$

$$= \frac{1}{u^{d_r}} \underbrace{\mu^{d_r} f\left(g_R, \frac{(u p_i)}{\mu}\right)}_{\Gamma(u p; g_R, \mu)}$$

$$\Rightarrow \Gamma(u p; g_R, \mu) = u^{d_r} \Gamma_R\left(p; g_R, \frac{\mu}{u}\right)$$

when I rescale the external momenta

$$= u^{d_r} Z_{eff}^{-n/2} \Gamma_R(p; g_{eff}, \mu)$$

solution to $\frac{1}{2} \frac{d \ln Z}{d \ln u} = -\gamma(g_{eff})$

$$\Rightarrow Z = \exp\left[-2\int_0^{\ln u} \gamma(g_{eff}(u')) \, d\ln u'\right]$$

$$= u^{d_r} \underbrace{e^{n\int_0^{\ln u} \gamma \, d\ln u'}} \Gamma(p; g_{eff}, \mu)$$

NDA is wrong!
γ contributes an ANOMALOUS scaling term.

$\rightarrow u^{d_r + n\gamma} \, \Gamma$ for γ const.

$$\mu\frac{d}{d\mu}\left[Z^{n/2}\Gamma_R\right] = \left(\mu\frac{d}{d\mu}Z^{n/2}\right)\Gamma_R + Z^{n/2}\,\mu\frac{d}{d\mu}\Gamma_R(g_R,\mu)$$

implicit dep on μ — explicit μ dep.

$$= \underbrace{\left(\frac{n}{2}\,\frac{\mu}{Z}\,\frac{dZ}{d\mu}\,Z^{n/2}\right)\Gamma_R}_{n\cdot\underbrace{\frac{1}{2}\frac{d\ln Z}{d\ln\mu}}_{\equiv\gamma(g_R)} Z^{n/2}\Gamma_R} + Z^{n/2}\left(\mu\frac{\partial}{\partial\mu} + \underbrace{\mu\frac{dg_R}{d\mu}}_{\equiv\beta(g_R)}\frac{\partial}{\partial g_R}\right)\Gamma_R$$

$$0 = \underbrace{Z^{n/2}}_{\neq 0}\underbrace{\left(\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} + n\gamma(g_R)\right)\Gamma_R}_{=0}$$

$\mu\frac{d\Gamma_0}{d\mu} = 0$

CALLAN-SYMANZIK

Solution (Method of characteristics)

define scale-dependent / <u>running</u> functions:

$g_{eff}(u)$ s.t. $u\frac{dg_{eff}}{du} = \beta(g_{eff}(u))$

u: DIMLESS $\qquad g_{eff}(1) = g_R$

$Z_{eff}(u)$ s.t. $\frac{1}{2}\frac{d\ln Z_{eff}}{d\ln u} = -\gamma(g_{eff}(u))$

$Z_{eff}(1) = 1$

<u>then</u>: $\Gamma_R\left(p; g_R, \frac{\mu}{u}\right) = Z_{eff}^{-n/2}(u)\,\Gamma_R(p; g_{eff}(u), \mu)$

rescaling of μ

dial in u to get

Γ @ $\mu_{eff} = \mu/u$

use the RUNNING COUPLING

$u = \mu/\mu_{eff}$

In practice

$$= i\lambda_0^2 \underbrace{\frac{3}{16\pi^2}}_{\beta(\lambda) = \beta_0 \lambda_0^2} \log\Lambda$$

$$\lambda_R = \lambda_0 + \lambda_0^2 \frac{3}{16\pi^2} \log\Lambda$$

RG EQ: $$E \frac{d}{dE} \lambda_{eff} = \beta_0 \lambda_{eff}^2$$

w/ $\lambda_{eff}(E=\mu) = \lambda_*$

INTEGRATE:

$$\int \frac{d\lambda_{eff}}{\lambda_{eff}^2} = \int \beta_0 \, d\ln E$$

$$= -\left(\frac{1}{\lambda_{eff}(E)} - \frac{1}{\lambda_*} \right) = \beta_0 \ln E/\mu$$

$$\frac{1}{\lambda_{eff}(E)} = \frac{1}{\lambda_*} - \beta_0 \ln E/\mu$$

$$\boxed{\lambda_{eff}(E) = \frac{\lambda_*}{1 - \lambda_* \beta_0 \ln E/\mu}}$$

RESUMMATION IN $\lambda_* \beta_0 \ln E/\mu$
("LEADING LOG" RESUMMATION)

→ saves perturbativity against large logs!

nb: this is good when $\beta_0 < 0$
then for $\ln E/\mu \gg 1$, $\lambda_{eff}(E) \ll 1$

BUT: if $\beta_0 > 0$ (as in $\lambda\phi^4$),
denominator can go to zero:

$$1 = \lambda_* \beta_0 \ln E/\mu$$

$$\Rightarrow E_* = \mu \, e^{1/\beta_0 \lambda_*}$$

LANDAU POLE

theory becomes strongly
coupled (non pert) near here.

Kind of neat: can predict new behavior
in the UV ... QED has a Landau
pole, but it is completed by EW thy
well below that scale !!