LAST TIME: GHOSTS!

↳ cancel unphysical DOF in gauge loops w/ NON-ABELIAN GAUGE THY

⇝ come from $\det \frac{\delta G}{\delta \alpha}$ ← gauge fixing functional

"for free" upon GAUGE FIXING.

→ ONLY TALKS TO GAUGE FIELD SO APPEARS IN LOOPS LIKE

from $F^a_{\mu\nu} F^{a\mu\nu}$ vertices

from $\frac{\delta G}{\delta \alpha}$ vertices

WE ARGUED QUALITATIVELY:

even though $c, \bar{c}$ seem to have bona fide kinetic terms, they do not exist in the classical action & cannot be external states.

↳ anyway, they violate spin-statistics & are overall very unsavory characters...

WOULDN'T IT BE NICE TO HAVE A QUANTITATIVE WAY TO DEFINE THE PHYSICAL STATE SPACE?

BRS(T) ← Tyutin never published

↳ "extension" (or "PIECE OF" depending on how you look at it)

of GAUGE REDUNDANCY that defines a PROJECTION onto PHYSICAL STATES

FADDEEV - POPOV LAGRANGIAN

$$\mathcal{L} = \underbrace{-\frac{1}{4}(F^a_{\mu\nu})^2}_{\text{class. kinetic (not invertible)}} - \underbrace{\frac{1}{2\xi}(\partial A)^2}_{\text{from Gaussian weight over gauge fixing functionals, } \omega: \ G = \partial A - \omega} + \underbrace{\bar{\Psi}(i\not{D} - m)\Psi}_{\text{matter}} + \underbrace{\bar{c}^a(-\partial D)^{ac} c^c}_{\text{ghost "kin term" from functional determinant that come w/ } \delta(G[A])}$$

one q from last time: can we write this as a Lagrange multiplier?

↳ we will actually introduce a $\mathcal{L}$-mult field for other reasons:

AUXILIARY FIELD

DO NOT ERASE

(★) $$\mathcal{L} = -\frac{1}{4}(F)^2 + \bar{\Psi}(i\not{D} - m)\Psi + \underbrace{\frac{\xi}{2}(B^a)^2 + B^a \partial \cdot A^a}_{} + \bar{c}(-\partial D)c$$

BTW: you are allowed to do this at the QUANTUM level. IT'S A FIELD REDEF. THAT MAY INTRODUCE HIGHER ORDER TERMS.

↑ DID JOSE COVER THIS?

EOM: $\xi B^a = -\partial \cdot A^a$

impose EOM on action: $B = \frac{-1}{\xi}\partial A$

then these terms are:

$$\frac{1}{2}\xi\left(\frac{-1}{\xi}\partial A\right)^2 - \frac{1}{\xi}(\partial A)^2 = \underbrace{\frac{-1}{2\xi}(\partial A)}_{}$$

exactly the GAUSSIAN WEIGHT TERM ABOVE.

⇒ ★ is completely equivalent to THE FP LAGRANGIAN

(following P & S)

now define a fermionic transformation infinitesimal
transformation parameter, ε is an anticommuting scalar

don't need aux field

$$\delta A^a_\mu \equiv \varepsilon D^{ac}_\mu c^c$$

$$\delta \psi \equiv i g \varepsilon c^a T^a \psi \quad \leftarrow \text{REP INDICES SUPPRESSED}$$

$$\delta c^a \equiv -\tfrac{1}{2} g \varepsilon f^{abc} c^b c^c \leftarrow \sim \tfrac{1}{2} g\varepsilon [c,c]^a$$

DEF W/ AUX field for convenience

$$\delta \bar{c}^a \equiv \varepsilon B^a$$

$$\delta B^a \equiv 0$$

OBSERVE: (εc) IS BOSONIC

special for the GAUGE FIELD

RECALL: $$A^a_\mu \longrightarrow A^a_\mu + \tfrac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c = \boxed{A + \tfrac{1}{g} D\alpha}$$

$$\psi \longrightarrow \psi + i\alpha T \psi \qquad \text{(cf PS 15.46)}$$

$\Rightarrow$ these are just ordinary GAUGE transformations on the physical fields,

$$\boxed{\alpha(x) = g\varepsilon c^a(x)}$$

ghost is part of GAUGE transformation parameter.

"EFF" GAUGE TRANSF PARAM FROM BRST transf.

BRST transf. PARAM.

nb: c^a also transforms like an adjoint (no ∂_μ term b/c c^a is not a gauge field)

GO BACK to $\mathcal{L}$: GAUGE INVT TERMS ARE OBVIOUSLY BRST INVARIANT (1st 2 terms)

- $\frac{\xi}{2}B^2$ term invariant (B invariant)
- $\underline{B\,\partial\cdot A + \bar{c}(-\partial D)c}$ transformations cancel

$$\underbrace{+B(\partial\varepsilon Dc) \quad +(\varepsilon B)(-\partial Dc)}_{=0} \quad + \bar{c}(-\partial D)(\overbrace{-\tfrac{1}{2}g\varepsilon f^{\cdots}c^{\cdot}c^{\cdot}}^{\delta c})$$

$$+\ \bar{c}(-\partial \underbrace{g f^{\cdots}\delta A^{\cdot}}_{\delta A \text{ in } D})c^{\cdot}$$

check that these cancel: the "$-\bar{c}\partial$" is common

$$\delta(Dc) = D\delta c + g f^{\cdots}\delta A^{\cdot} c^{\cdot}$$

$$= -\tfrac{1}{2}g\varepsilon\,\partial(f^{\cdots}c^{\cdot}c^{\cdot}) - \tfrac{1}{2}g^2\varepsilon f^{abc}A^b(f^{cde}c^d c^e) \leftarrow D\delta c$$

$$+\ g\varepsilon f^{\cdots}(\partial c)^{\cdot}c^{\cdot} + g\varepsilon f^{abc}(g f^{bde}A^d c^e)c^c$$

$\delta A = \varepsilon(Dc)$

the $O(g^2)$ terms cancel also

cancel

gives: $-\frac{1}{2}\varepsilon g^2 f^{abc}f^{cde}(A^b c^d c^e + A^d c^e c^b + A^e c^b c^d)$

USE: JACOBI IDENTITY:

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0$$

from $[T^a,[T^b,T^c]]$ + cyclic $= 0$

OK. so $\mathcal{L}$ is BRST INVARIANT — so what?

↳ seems to "generalize" GAUGE symmetry, accounting for extra fields.

(BTW, didn't need B field)

A USEFUL FEATURE (seems coincidental @ this point):

BRST transformation is nilpotent: $\boxed{\delta^2 = 0}$

or $\delta_c^2 = 0$ to make BRST clear

generic field transf: $\delta\phi = \varepsilon Q \phi$

some (anticom) operator (that goes like c)

nilpotent: $Q^2 \phi = 0$

↳ some of these are obvious to see, others are trickier

eg $Q^2 A = \delta(Dc) = 0$ from Jacobi argument in p4

$$Q^2 \psi = ig T^a (\delta c^a \psi + c^a \delta\psi)$$

$$= (-\tfrac{1}{2} g f^{abc} c^b c^c \psi + c^a ig c^b T^b \psi)$$

nb Q^2 vs. δ^2 b/c $\varepsilon^2 = 0$ trivially ... the nilpotency is deeper than that

turn this into $\frac{1}{2}[T^a, T^b] = -if^{abc}T^c$ to get cancellation

$$Q^2 c = \tfrac{1}{2} g^2 f^{abc} f^{bde} c^c c^d c^e \quad (\text{use JACOBI})$$

$$Q^2 \bar{c} = QB = 0$$

$$QB = 0$$

ε const.

so: new global sym: BRST

↳ nilpotent

↳ looks like GAUGE transf w/ $\alpha \sim \varepsilon C(x)$

1. so what?
2. where does it come from?

↳ NILPOTENCY GIVES A WAY to PROJECT onto <u>PHYSICAL SPACE</u>

given a NILPOTENT op, Q

generator of SYM (really $\int d^3x\, j^0$)

DIVIDE STATES INTO:

$\mathcal{H}_1$: $Q|\psi\rangle \neq 0$

IMAGE OF Q

$\mathcal{H}_2$: $|\psi\rangle = Q|\phi\rangle$ for some $|\phi\rangle$

$\mathcal{H}_0$: $Q|\psi\rangle = 0$ <u>AND</u> $|\psi\rangle \neq Q|\phi\rangle \;\forall\, |\phi\rangle$

Kernel of Q

$\mathcal{H}_0$ is COHOMOLOGY OF Q

this should sound a lot like differential geometry w/ exterior derivative see VADIM'S BRST LEC @ UT

BRST TRANSFORM: $\left[\begin{matrix}\text{forward} \\ \text{comp of}\end{matrix}\; A \longrightarrow \text{GHOST}\right]$

$$QA = \partial c + \mathcal{O}(g)$$

take $g \to 0$ asymp lim

CHK: SEE SREDNICKI CH. 74

$$A^\mu = \sum_\lambda \int d^4k \left[\varepsilon_\lambda^{\mu *}(k)\, a_\lambda(k)\, e^{ikx} + \varepsilon_\lambda^\mu a_\lambda e^{-ikx} \right]$$

POLZ

k in z-dir

$$\sqrt{2}\, \varepsilon^\mu = \begin{matrix}(1,0,0,1) \\ (1,0,0,-1) \\ (0,1,-i,0) \\ (0,1,i,0)\end{matrix}$$

lightlike (longitudinal)

BACKWARD

LH POLZ

RH

$\varepsilon_{\lambda = >}$ FWD

IF YOU PLUG THIS IN,

$$Q a^\dagger_\lambda(k) = \sqrt{2} E_k\, \delta_{\lambda = >}\, c^\dagger(k)$$

Brward

(I believe other polz cancel)

so: Q: forward/longitudinal polz of $A \to$ ghost

Q: annihilates ghost (to $\mathcal{O}(g)$)

Q: takes anti-ghost to auxiliary field, B

$\hookrightarrow B^a \sim -\partial^\mu A^a_\mu$

polz st $k \cdot \varepsilon \neq 0$

$\to$ BACKWARD POLARIZED

$$\begin{aligned} Q A_{\lambda = \text{FWD}} &\sim c \\ Q\, c &\sim 0 \\ Q\, \bar{c} &\sim B \end{aligned}$$

FORWARD GAUGE
ANTIGHOST } $\in$ ~~[illegible]~~ $\mathcal{H}_1$ } + 3 - dof

GHOSTS
BACKWARD GAUGE } ~~[illegible]~~ $\mathcal{H}_2$

$\boxed{\text{Transverse} \quad \in \quad \mathcal{H}_0}$

SO WHAT:

REVIEW of p. 6

Kernel of Q: $Q|\psi\rangle = 0$

Image of Q: $|\psi\rangle = Q|\phi\rangle$

states in kernel of Q but NOT the IMAGE OF Q ARE CALLED THE COHOMOLOGY OF Q

really an identification $\ker Q / \operatorname{Im} Q$

$$|\psi\rangle \sim |\psi\rangle + Q|\phi\rangle$$

a state in cohomology is $|\psi\rangle$ w/ $Q|\psi\rangle = 0$
but $|\psi\rangle \neq Q|\phi\rangle$

$\mathcal{H}_0$ is PHYSICAL SPACE

MAY WORRY: IF S-MATRIX turns states in $\mathcal{H}_0$ INTO STATES OUTSIDE $\mathcal{H}_0$ ... then LOSE UNITARITY.

↳ "leaking" into unphysical states.

WANT TO ARGUE THAT THIS ISN'T POSSIBLE:

$$\sum_c \langle a, \text{phy} | S^\dagger | c, \text{phy} \rangle \langle c, \text{phy} | S | b, \text{phy} \rangle = \langle a, \text{phy} | \mathbb{1} | b, \text{phy} \rangle$$

all "phy" fields in $\mathcal{H}_0$

Q annihilates $|a, \text{phy}\rangle \in \mathcal{H}_0$

& commutes w/ HAMILTONIAN

$\Rightarrow$ time evolution produces state
in KERNEL of Q

$$Q \underbrace{S|A, \text{PHY}\rangle}_{\text{LIN COMB OF STATES IN } \mathcal{H}_0 \oplus \mathcal{H}_2} = 0$$

DON'T WANT

BUT: state in $\mathcal{H}_2$ HAS
ZERO INNER PRODUCT
W/ EA OTHER & W/ STATE IN $\mathcal{H}_0$

for $\psi_2 \in \mathcal{H}_2$ $\exists$ $|\phi\rangle$ s.t. $|\psi\rangle = Q|\phi\rangle$

$$\langle \psi_2 | \psi_2 \rangle = \langle \phi | \underbrace{Q|\psi_2\rangle}_{=0}$$

$$\langle \psi_2 | \underbrace{\psi_0}_{\in \mathcal{H}_0 :\ Q|\psi_0\rangle = 0} \rangle = \langle \phi | Q | \psi_0 \rangle = 0$$

so only states in $\mathcal{H}_0$

$$\Rightarrow \langle A, \text{PHYS} | S^\dagger S | B, \text{PHYS} \rangle = \sum_C \underbrace{\langle A, \text{PHYS} | S^\dagger | C, \text{PHYS}\rangle \langle C, \text{PHYS}}_{\text{s.t. GHOSTS CANCEL UNPHYS DOF}} | S | B, \text{PHYS} \rangle$$

Sketch of GEOMETRIC PICTURE (2 mins)

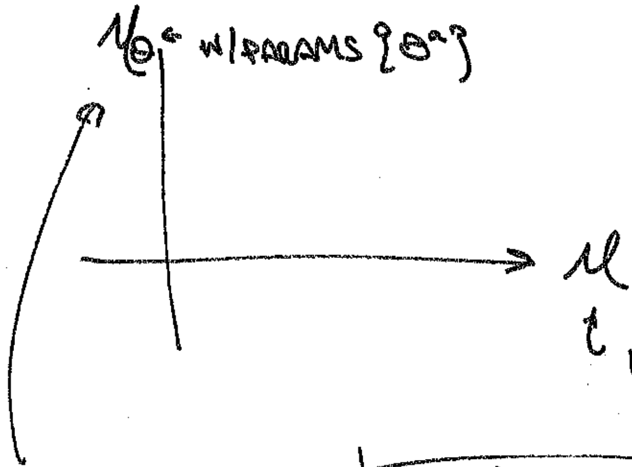
↳ see, e.g. BERTLMANN ANOMALIES IN QFT

extend spacetime to full fiber bundle

locally a product space $\mathcal{M} \times \mathcal{M}_\theta$

SPACETIME

GAUGE PARAMS ($\leftrightarrow G$)



HAS EXTERIOR DERIVATIVE

d : n-FORM → (n+1)-FORM

$dx^a \frac{\partial}{\partial x^a}$

EXTERIOR DERIVATIVE $\boxed{\delta = d\theta^a \frac{\partial}{\partial \theta^a}}$

now just "DO DIFFERENTIAL GEOMETRY"

FADDEEV-POPOV GHOST ~ MAURER-CARTAN 1-FORM

$$c = g^{-1}\delta g = c(x,\theta)\, d\theta$$

← 1-FORM

⇓

ANTICOMMUTES

$g(\theta, x)$ is "GAUGE"

RECALL: 1-FORMS "CONNECT" AS WEDGE PRODUCTS

$dx \wedge dy$

EXTERIOR DERIVATIVES WORK AS EXPECTED:

$\Delta = d + \delta$ ← "explains" $d\delta_0 + \delta d = 0$ used for ANOMALY EQNS.

$\Delta^2 = 0$

BRS : comes automatically from Maurer-Cartan structure eqns (geometry of extended space ... analogous to using vielbeins in GR)

eg. 1-form ω
2-form $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega[X,Y]$

for a lie group w 1-forms c, then

$dc^a(X_b, X_c) = -f^a{}_{bc}$

↑ ~~[illegible]~~

this starts to motivate, eg

$\delta C \sim f^{abc} c^b c^c$