REMINDER : 2D QED : Schwinger Model

$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\Leftarrow$ classically indep.

$\Downarrow$

two cons. currents

QUANTUMLY : one combination is ANOMALOUS

- eg $\mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}$ is not invariant $\rightarrow Z$ not invariant
- vac. shifts under LARGE GAUGE TRANSFORMS

DIRAC SEA IS "MOVING"

$\rightarrow$ seen in $\boxed{\text{UV and IR}}$ $\leftarrow$ intrinsic property of the theory!

$\hookrightarrow$ not like mass, that can be "small" in UV or properties that are "relics" of REGULATION

@ CUTOFF (regulator picture), LH modes disappear & RH modes come in

$\hookrightarrow$ REALLY LIKE A "HILBERT HOTEL"

@ FERMI SEA HOLE & PARTICLE "POP OUT"

Anomalous symmetries : not a symmetry.

famous eg: $U(1)_A$ PROBLEM:

APPROX GLOBAL SYM

$$U(3)_L \times U(3)_R \Rightarrow U(3)_V \times U(3)_A$$

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \quad \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

$SU(3)_V \rightarrow$ PRESERVED ("flavor")

$\times\, SU(3)_A \rightarrow$ SPONT. BROKEN

$\times\, U(1)_V \rightarrow$ PRESERVED (baryon #)

$\times\, U(1)_A \rightarrow$ SPONT BROKEN

VEV: $\langle q_L q_R \rangle$

SSB $\Rightarrow$ 1 Goldstone per broken sym

EXPECT 8 + 1 GOLDSTONES.

$SU(3)_A$ GENERATORS; $U(1)_A \rightarrow \eta'$?!

... only see 8. (π, K, η)

SOLUTION: $U(1)_A$ is not a symmetry, no reason to expect a low mass mode

$\alpha(x) = \frac{2\pi}{L} n x$

LAST TIME in SCHWINGER MODEL, BREAKS UP INTO DISTINCT HILBERT SPACE SECTORS RELATED BY LARGE GAUGE TRANSFORMATIONS

$$\underbrace{\Psi_n}_{|VAC\rangle_n} = \underbrace{\prod_{k=n}^{\infty} |1,k\rangle_L \prod_{k=n-1}^{-\infty} |1,k\rangle_R}_{\text{FILLED NEG E. STATES}} \otimes \Psi_0\left(A_1 - \frac{2\pi}{L} n\right)$$

$$E_k^{(L)} = -\left(k+\tfrac{1}{2}\right)\frac{2\pi}{L} + A_1$$

$$E_k^{(R)} = +\left(k+\tfrac{1}{2}\right)\frac{2\pi}{L} - A_1$$

$|0,k\rangle_{L/R}$ for + ENERGY

fermion vacuum

gauge vacuum

Vacuum state labelled by "FOURIER" angle:

$$\Psi_\theta = \sum_n e^{in\theta} \Psi_n$$

TOPOLOGY of U(1)

btw: now it matters that GAUGE GROUP is COMPACT vs $\mathbb{R}$

HOMOTOPY GROUP / fundamental group

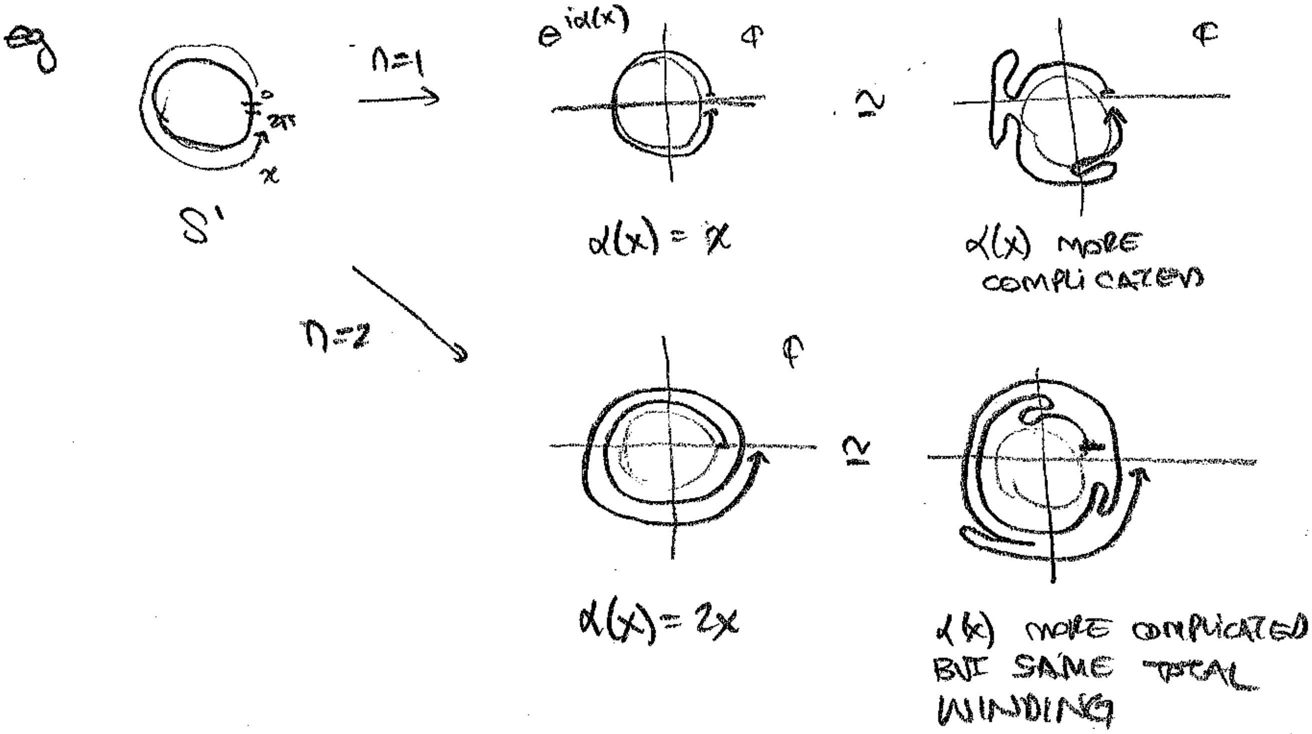
$\pi_i(X)$: different ways S^i can be mapped to X

for our purposes : $\boxed{\pi_1(U(1)) = \mathbb{Z}}$

how circle can be mapped onto $e^{i\alpha(x)}$ = WINDING NUMBER of MAPPING

GAUGE!

eg



CLUSTER DECOMPOSITION $\leftrightarrow$ causality

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$$

when $\mathcal{O}_1$ & $\mathcal{O}_2$ are causally independent.

some product of fields

CAN IMAGINE: $\mathcal{O}_{1,2}$ ARE CHIRAL CURRENTS s.t.:

$\mathcal{O}_2$ CREATES PARTICLE & HOLE, int over space
$\mathcal{O}_1 = \mathcal{O}_2^\dagger$, ANNIHILATES THEM, int over space

so $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \neq 0 \Rightarrow \langle \mathcal{O}_{1,2} \rangle = 0$

BUT $\mathcal{O}_i$ SHIFTS TO DIFFERENT n VACUUM.

in other words, Ψ_n vacua are no good.
BUT IF VAC IS A LIN COM OF ALL Ψ_n, then there is a piece

$$\langle \Psi_{n+1} | \mathcal{O}_2 | \Psi_n \rangle \neq 0.$$

$\Rightarrow$ forced to Θ vacua.

BEFORE MOVING ON TO 4D triangle graphs,
A HINT of how it works in 2D.

REGULATORS + ANOMALIES
Shifman AQFT 33.7

want to see $\partial_\mu j_5^\mu = -\frac{1}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}$ ← in 4D: $\sim \varepsilon^{\cdots}F_{..}F_{..}$

<u>PAULI-VILLARS REGULATOR</u>:

$$\lim_{\Lambda\to\infty} \frac{i}{p^2-M^2} - \underbrace{\frac{i}{p^2-\Lambda^2}}_{}$$ ← fictitious REGULATOR field

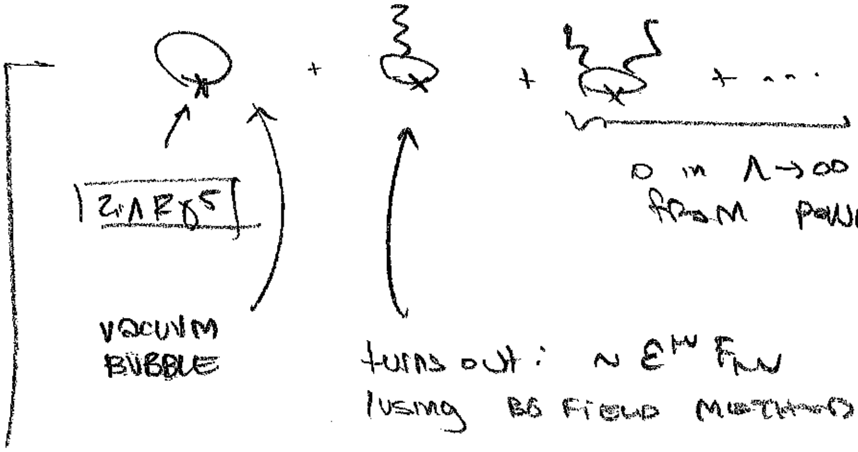
cancels loops @ $\sim\Lambda$ & higher momenta

$$j^\mu_5 = \bar\psi\gamma^\mu\gamma^5\psi + \bar R\gamma^\mu\gamma^5 R$$

$$\partial_\mu j^\mu_5 = 0 + 2i\Lambda \bar R\gamma^5 R$$

MASSLESS $\not p\,\psi(p)=0$

now contract these & take $\Lambda\to\infty$ limit
allow conversion into light fields



$\boxed{2i\Lambda\bar R\gamma^5}$

VACUUM BUBBLE

turns out: $\sim\varepsilon^{\mu\nu}F_{\mu\nu}$
(using BG FIELD METHODS)

0 in $\Lambda\to\infty$ limit from POWER COUNT

$$= 2i\Lambda_0\bar R\gamma^5 R \sim -\Lambda_0\,\mathrm{Tr}\left[\frac{\gamma^5}{i\not D-\Lambda}\right]$$

nb: in 3+1 D triangle diagram

TONG

4D triangles

$$\partial_\mu j_5^\mu = \frac{e^2}{16\pi^2} \underbrace{\varepsilon^{\cdots\cdot} F_{\cdot\cdot}}_{\tilde{F}^{\cdot\cdot}} F_{\cdot\cdot}$$

start w/ free theory : massless fermions

↳ 2 sym: $j^\mu = \bar\psi \gamma^\mu \psi$

$j_5^\mu = \bar\psi \gamma^\mu \gamma^5 \psi$

REGULATOR: one of these must be sacrificed

CALCULATE $\Gamma^{\mu\nu\rho}(x_1,x_2,x_3) = \langle 0 | j^\mu(x_1)\, j^\nu(x_2) \boxed{j_A^\rho(x_3)} | 0 \rangle$

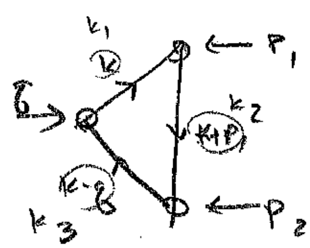
only eq for Minkowski & TIME ORDERED (IMPLICIT)

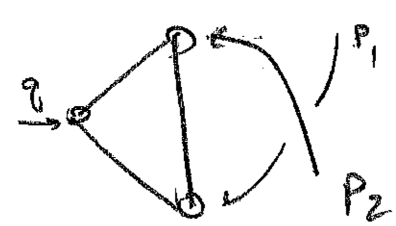
COULD CONSTR W/ PHOTONS ON RHS OF $\partial \cdot j_5$

CLASSICAL CURRENT CONS: $\partial_{x_1\mu} \Gamma^\mu = 0 \rightarrow (p_1)_\mu \Gamma^\mu(p_1,p_2,q) = 0$
& so forth

CALCULATE IN MOMENTUM SPACE

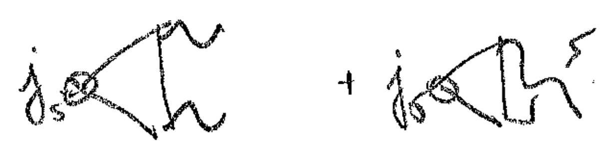
$$-i\Gamma^{\mu\nu\rho}(p_1,p_2,q) =$$





nb: $p_1 + p_2 = -q$

eg if $U(1)_V$ were gauged,



what the start of the calc looks like:

$$-i\Gamma^{\mu\nu\rho} = -\int d^4k \, \mathrm{tr}\left[\frac{i}{\not k_1}\gamma^\rho\gamma^5\frac{i}{\not k_3}\gamma^\nu\frac{i}{\not k_2}\gamma^\mu\right] + \left(\begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu\end{smallmatrix}\right)$$

$\Downarrow$ WORK

$$-i q_\rho \Gamma^{\mu\nu\rho} = \Delta_1^{\mu\nu} + \Delta_2^{\mu\nu}$$

$$\Delta_1^{\mu\nu} = i\int d^4k \, \mathrm{tr}\left[\frac{1}{\not k}\gamma^5\gamma^\nu\frac{1}{\not k+\not p_1}\gamma^\mu - \frac{1}{\not k_1+\not p_2}\gamma^5\gamma^\nu\frac{1}{\not k-\not q}\gamma^\mu\right]$$

$$\Delta_2^{\mu\nu} = \text{——}\left[-\frac{1}{\not k+\not p_1}\gamma^5\gamma^\nu\frac{1}{\not k-\not q}\gamma^\mu + \frac{1}{\not k}\gamma^5\gamma^\nu\frac{1}{\not k+\not p_2}\gamma^\mu\right]$$

shift $k \to k+p_2$ then the two terms in $\Delta_1^{\mu\nu}$ cancel!

similar shift for $\Delta_2^{\mu\nu}$!

isn't k a dummy variable, after all?

$\rightsquigarrow$ so it looks like $q_\rho\Gamma^{\mu\nu\rho} = 0 \to$ no anomaly

BUT: each term is divergent!

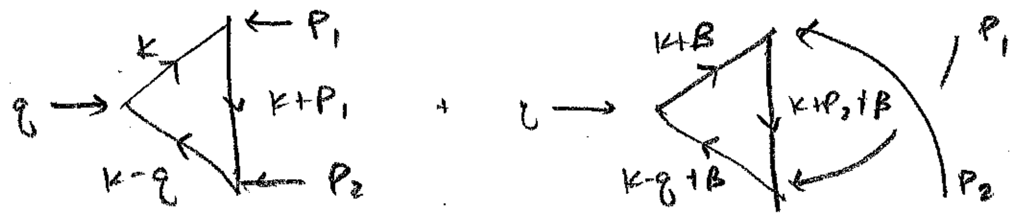
$\hookrightarrow$ turns out linear div. b/c of γ's for LINEAR DIV integral $\int d^4k\, f(k)$,

$$\int d^4k\, f(k)-f(k+a) = \underbrace{\int d^4k\, a^\mu \frac{\partial}{\partial k^\mu} f(k)} + \mathcal{O}(a^\mu a^\nu)$$

$$\int_{S^3_\infty} \frac{d\hat{k}_\mu}{(2\pi)^4}\, a^\mu |k|^3 f(k)$$

BOUNDARY TERM.

DIAGNOSING by PUTTING IN ARB. SHIFT:



shift by β,
see β dep later.

RESULT: using $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\varepsilon^{\mu\nu\rho\sigma}$

$$\boxed{-iq_\rho \Gamma^{\mu\nu\rho} = \frac{-1}{8\pi^2}\varepsilon^{\mu\nu\rho\sigma}\left[2(P_1)_\rho(P_2)_\sigma + (P_1+P_2)_\rho\beta_\sigma\right]} \quad (A)$$

SIMILARLY:

$$\left.\begin{aligned} -i(P_1)_\mu\Gamma^{\mu\nu\rho} &= \frac{1}{8\pi^2}\varepsilon^{\rho\nu\mu\sigma}(P_1)_\mu(\beta-P_2)_\sigma \\ -i(P_2)_\nu\Gamma^{\mu\nu\rho} &= \frac{1}{8\pi^2}\varepsilon^{\rho\mu\nu\sigma}(P_2)_\nu(\beta+P_1)_\sigma \end{aligned}\right] \quad (V)$$

β shows up everywhere.

CAN MAKE $(V)=0$ IF $\boxed{\beta = P_2-P_1}$ $\Rightarrow -iq_\rho\Gamma^{\mu\nu\rho} = \underbrace{-\frac{\varepsilon^{\mu\nu\rho\sigma}(P_1)_\rho(P_2)_\sigma}{2\pi^2}}_{\text{ANOMALY}}$

req, eg if this is gauged

Remarks

1. FUJIKAWA DERIVATION of ANOMALY had no "choice" of β
   ↳ WE MADE THIS CHOICE WHEN WE PICKED A GAUGE-INVT REGULATOR

2. DIFF. REGULATORS VIOLATE DIFF. COMB. OF SYM. USUALLY AXIAL SUFFERS.
   ↳ PAULI-VILLARS INTRODUCES MASSIVE FERMION

$$\Lambda \underbrace{\bar{\psi}_L \psi_R}_{\text{not axial-invt}}$$

MASS term BREAKS $U(1)_A$ BY A LOT.

3. GRAVITY: COUPLE A SPINOR TO CURVED SPACE

$$D_\mu \psi^\alpha = \partial_\mu \psi^\alpha + \frac{1}{2} \omega_\mu^{ab} (M_{ab})^\alpha{}_\beta \psi^\beta$$

GRAVITY AS A GAUGE THEORY { SPIN CONNECTION

$= \frac{1}{4}[\gamma^a, \gamma^b]$

$$D_\mu j^\mu_A = \frac{-1}{384\pi^2} \cdot \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}{}^{\lambda\tau}$$

gravitational contr. to anomaly

ATIYAH - SINGER interp. of ANOMALY

$i\not{D}\phi_n = \lambda_n \phi_n$ ← EIGENSPINOR

$\lambda_n \in \mathbb{R}$ ($i\not{D}$ is HERMITIAN)

nb: $\gamma^5\phi_n$ is EIGENSPINOR w/ EIG. VAL $(-\lambda_n)$
from $\{\gamma^5, \gamma^\mu\} = 0$

EIGFUN. w/ DIFF EIG. VAL ARE ORTHOGONAL

... what about ZERO MODE? $\lambda_0 = 0$

LET $n_\pm$ BE NUMBER OF 0-MODES OF $i\not{D}$
WITH γ^5 EIGENVAL ± 1.

DEF $\boxed{\mathrm{Ind}(i\not{D}) = n_+ - n_-}$

index

NOW RECALL FUJIKAWA: $\mathcal{D}\psi\mathcal{D}\bar{\psi}' = \left(\frac{1}{\det(\mathbb{1}+C)}\right)^2 \mathcal{D}\psi\mathcal{D}\bar{\psi}$

$$\det(\mathbb{1}+C) = e^{\Sigma \int d^2x\, \alpha(x) \Sigma \phi_n^\dagger \gamma^5 \phi_n}$$

in 2D: $\Sigma \phi_n^\dagger \gamma^5 \phi_n = \frac{-e}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu}$

4D: $\quad$ — " — $\quad = \frac{e^2}{32\pi^2} \varepsilon^{\cdots} F \cdot F \cdot$

BUT: $\int d^4x \sum_n \phi_n^\dagger (\gamma^5 \phi_n) = n_+ - n_-$

orthogonal when $\lambda_n \neq 0$
w/ r.t d^4x

so this is sum over 0 modes

$$\Rightarrow \text{Ind}\,(i\not{D}) = \frac{e^2}{32\pi^2}\int d^4x\, \varepsilon^{\cdots} F_{\cdot\cdot} F_{\cdot\cdot}$$

$$= \Delta Q_A = \int dt \int_{\text{space}} \dot{j}^0_A$$

change in axial charge; DISCRETE

<u>REMARK</u>: ANOMALY IS NOT PERT. CORRECTED

would not be integer.

<u>REMARK</u>: for ABELIAN GAUGE: RHS only 'on' WHEN FIELD IS 'ON' (BG field)

for <u>NON-ABELIAN</u>, HAVE non-trivial <u>VACUUM</u>

↳ instanton configurations

(compact 2D QED mocked that up)