LAST TIME

1. $$Z[J] = \int d^N q \; e^{-\frac{1}{2} q A q - V(q) + J q}$$

$\llcorner$ PARTITION FUNC or GENERATING FUNC

$$\langle q_i \cdots q_j \rangle = \frac{1}{\mathcal{N}} \underbrace{\left(\frac{\delta}{\delta J_i} \cdots \frac{\delta}{\delta J_j}\right)} \underbrace{Z[J]}\Big|_{J=0}$$

"SOLVING"

ext. sources @ J

$$Z = e^{\frac{1}{2} \frac{\delta}{\delta q} A^{-1} \frac{\delta}{\delta q}} \left[e^{-V(q)} e^{Jq} \right]$$

gives PROPAGATORS

gives VERTICES @ SAME LATTICE POINT (by locality)

GIVES ~~HITS~~ SOURCES: brings down $(q_i \cdots q_j)$ FOR $e^{\frac{1}{2} \delta/\delta q \, A^{-1} \, \delta/\delta q}$ to HIT.

2. then we looked at PI in QM

$$\langle q_{r+1} | e^{-i\hat{H}\delta t} | q_r \rangle = \sqrt{\frac{m}{2\pi i \, \delta t}} \; e^{\frac{i}{2} m \left(\frac{q_{r+1} - q_r}{\delta t}\right)^2 - iV(q_r)\delta t}$$

small, s.t.

$$e^{-i\left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)\delta t} \approx e^{-i\frac{\hat{p}^2}{2m}\delta t} e^{-iV(\hat{q})\delta t}$$

$$\langle q | e^{-i\hat{H}t} | q_0 \rangle = \left(\frac{m}{2\pi i \delta t}\right)^{N/2} \int \prod_r dq_r \; e^{i \text{---}}$$

$$\longrightarrow \int \mathcal{D}q \; e^{iS[q]}$$

WE USE $i\mathcal{L}$ VS. HAMILTONIAN.

factor of i? WICK ROT. SCHWARTZ: S is ANALYTIC IN q so take $\hbar \to 0$ m slightly $\mathbb{C}$ ok for classical limit.

Scalar Field Theory

$$Z[J] = \int \mathcal{D}\phi \; e^{iS[\phi] + i\int d^dx\, J(x)\,\phi(x)}$$

CLEARLY GENERALIZES $\sum_i J_i q_i$

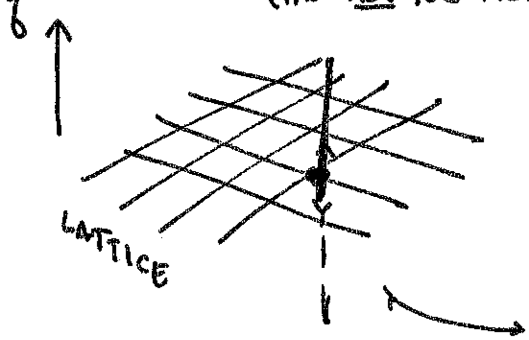
A MORE CAREFUL LOOK AT THE LATTICE → CONTINUUM

$$S = \int_0^T dt \sum_i \tfrac{1}{2} M \dot{q}_i^2 - V[q]$$

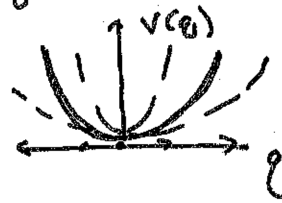
$$V[q] = \tfrac{1}{2}\sum_{\langle ij\rangle} K (q_i - q_j)^2$$

(nb: not the FIELD POT!)

btw: this is why HEP QFT gets worked up about renormalizability while cond-mat doesn't care so much.



each vertical line is a quantum H.O.



(THIS IS WHY WE LABEL AXES)

DEPENDING ON NEIGHBORS

field ≡ lattice of H.O. w/ quadratic nearest-neigh. ints.

DICTIONARY:

$$q_i \longrightarrow q(x)$$

$$(q_i - q_j) \longrightarrow \frac{\partial^2 q(x)}{\partial x^2}\,\Delta x^2$$

$$\sum_i \longrightarrow \int d^Dx\, \frac{1}{(\Delta x)^D}$$

not the "mass of the field" → $M \longrightarrow \rho\,(\Delta x)^D$ ← DENS × VOL.

$$S = \int_0^T dt \int \frac{d^Dx}{(\Delta x)^D}\Big[\tfrac{1}{2}\rho(\Delta x)^D \frac{\partial^2 q}{\partial t^2} - \underbrace{\tfrac{1}{2}(\Delta x)^2 K \frac{\partial^2 q}{\partial x^2}}_{V[q]}\Big]$$

$$= \int dt\, d^Dx\; \tfrac{1}{2}\Big[\rho\,\partial_t^2 q - \frac{K}{(\Delta x)^{D-2}}\,\underline{\partial}^2 q\Big]$$

$$\longrightarrow \int \underbrace{d^dx}_{d=D+1}\; \tfrac{1}{2}\,(\underbrace{\partial_t^2 - c^2\partial_x^2}_{=\partial^\mu\partial_\mu})\,\varphi^2 \;\leftarrow\; \varphi \equiv q/\sqrt{\rho}$$

← SPACETIME!

$$Z[J] = \int \mathcal{D}\varphi \; e^{iS + i\int d^dx \, J\phi}$$

$$S = \int d^dx \; \tfrac{1}{2} \left(\varphi \partial^2 \varphi - V(\varphi) \right)$$

lec 2

$$= e^{\frac{i}{2}\int d^dx \, d^dy \, \frac{\delta}{\delta\phi(x)} \Delta(x-y) \frac{\delta}{\delta\phi(y)}}$$

$$\times \left[e^{i\int d^dx \, [-V[\phi(x)]]} \; e^{i\int d^dx \, J(x)\phi(x)} \right] \Big|_{\phi=0}$$

same trick as prev. lec.:

$$g\left(i\frac{\delta}{\delta J}\right) f(iJ) = f\left(\frac{\delta}{\delta\phi}\right) g(\phi) \, e^{i\int d^dx \, \phi(x) J(x)}$$

CORRELATION FUNCTIONS:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \left(\frac{-i\delta}{\delta J(x_1)} \cdots \frac{-i\delta}{\delta J(x_n)} \right) Z[J] \Big|_{J=0}$$

eg for $V=0$ (*free theory*)

$$\langle \phi(x) \cdots \phi(x_4) \rangle = \left(\frac{-i\delta}{\delta J_1} \cdots \right) e^{\frac{i}{2}\int \frac{\delta}{\delta\phi_i} \Delta_{ij} \frac{\delta}{\delta\phi_j}} \left[e^{i\int J_i \phi_i} \right]_{\phi=0}$$

$$= \underbrace{e^{\frac{i}{2}\int \frac{\delta}{\delta\phi} \Delta \frac{\delta}{\delta\phi}}}_{\text{HOW MANY WAYS CAN THIS EXPANSION OF VARIATIONS ...}} \; \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{HIT THIS SET OF VARIABLES}} \Big|_{\phi=0}$$

disconnected

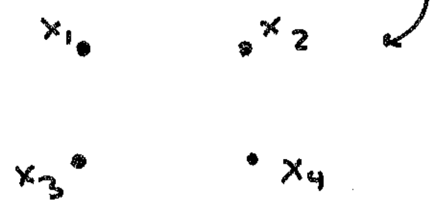
full result $\longrightarrow$? no other terms.

IF WE TURN ON $V[\phi]$ ← subject to locality

(NB: this V is local ... the $V[\delta]$ for the field was nearest neighbor!!)

$$\langle \phi(x_1)\cdots\rangle = \left(\frac{-i\delta}{\delta J(x_1)}\cdots\right) e^{\frac{i}{2}\int \frac{\delta}{\delta\phi_i}\Delta_{ij}\frac{\delta}{\delta\phi_j}}\left[\underline{e^{-i\int d^dx\, V[\phi]}}\; e^{i\int d^dx\, J\phi}\right]$$

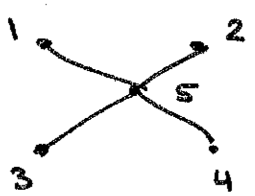
1. still hits the sources to bring down $\phi(x)\cdots$

2. 

3. BUT NOW IN ADDITION TO $=$ $||$ $\times$ each term in a Taylor exp of $V[\phi]$ introduces new vertices

↳ new spacetime point that admits some number of connections

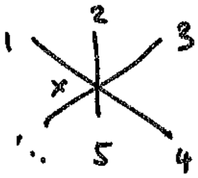
eg if $V[\phi] = \lambda\phi(x)^4$

$\int d^dx_5$ 

← integrating over all internal positions

FEYNMAN RULES IN POSITION SPACE

$= i\Delta(x-y)$ ← propagator



$= -i\left(\frac{\delta}{\delta\phi}\right)^n V[\phi]\Big|_{\phi=0}$ & integrate d^dx

$= iJ(x)$

ext line

MOMENTUM SPACE

$$\langle \phi(p_1)\cdots\phi(p_n)\rangle = \int d^dx_1\cdots d^dx_n\, e^{i(p_1\cdot x_1+\cdots)}\langle\phi(x_1)\cdots\rangle$$

DEPENDS ON DIFFERENCES of x, eg $(x_1 - x_2)$

If we write as $X = \frac{1}{\sqrt{n}}\sum_i x_i$

& ORTHOGONAL VARIABLES (eg $\frac{1}{\sqrt{2}}(x_1 - x_2)$)

then: $\int d^dx_1\cdots e^{i(p_1x_1\cdots)} = \int d^dX\cdots e^{iP\cdot X}$

total momentum

& we can do the d^dX integral since $\langle\phi(x_1)\cdots\rangle$ is independent of "center"

$\rightarrow$ gives $\delta^{(d)}\left(\sum_i^n p_i\right)$

conservation of total momentum

OR: from HOMOGENEITY OF SPACETIME

$$i\Delta(x-y) = \int \underbrace{\frac{d^d p}{(2\pi)^d}}_{đ^d p} e^{ip\cdot(x-y)} \frac{i}{p^2-m^2 \underset{\uparrow}{+i\varepsilon}}$$

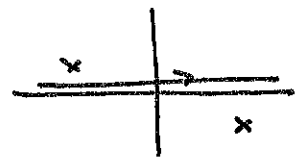
FEYNMAN PROPAGATOR

CHECK SIGN

POLES:

REMEMBER THIS FROM 231?

$$\frac{i}{[E-(p^2+m^2-i\varepsilon)][E-(-p^2-m+i\varepsilon)]}$$

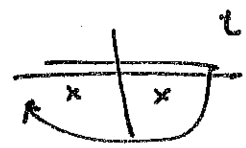


HARMONIC OSC. GREEN'S FUNC

$$G(t) = \int dE \frac{-e^{-iEt}}{E^2-\omega^2}$$

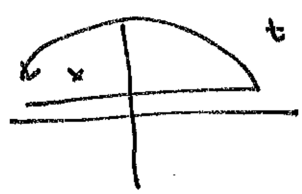
POLES ON CONTOUR!

ADVANCED
(acausal)



RETARDED
(classical)

FEYNMAN:



$e^{-i\hat{H}t}$

$$G_F(t>0) = \frac{i}{2\omega} e^{-i\omega t}$$

$$G_F(t<0) = \underbrace{\frac{-i}{2\omega}}_{\text{PHASE}} e^{-i\underset{\uparrow}{(-\omega)t}}$$

$\omega(-t)$

POS E, 'BKWD' IN TIME.

· so: each internal line: $\xrightarrow{p}$

$$\frac{i}{p^2 - m^2 + i\varepsilon}$$ ⟵ btw: in wrong metric:

$$\frac{-i}{p^2 + m^2 - i\varepsilon} = \frac{i}{-(p^2 - m^2) + i\varepsilon}$$

(peel off $đ^d p$)

(it's actually not too bad to convert)

· for EACH VERTEX

$$-i \left(\frac{\delta}{\delta\phi}\right)^n V[\phi]\Big|_{\phi=0} \;\underline{(2\pi)^d \, \delta^{(d)}(\Sigma p_i)}$$

$$\cancel{-i \int d^d x \, V[\phi] = -i \int d^d x \, đ^d p \, e^{ip\cdot x} \, V[\phi]}$$

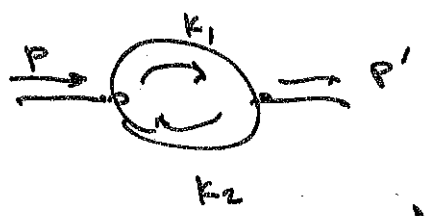
$$\cancel{\text{eg: } \int d^d x_1 \, d^d x_2 \, d^d x_3 \; J_1 \phi_1 \, J_2 \phi_2 \, J_3 \phi_3 \; e^{i p_1 x_1} e^{i p_2 x_2} e^{i p_3 x_3} \ldots}$$

from $đ^d p \, e^{ip\cdot(x-y)}$ of propagators.

· INTEGRATE OVER ALL INTERNAL MOMENTA (ALL VERTEX POS.)

$$\frac{d^d k_{int}}{(2\pi)^d}$$ ⤳ most of these are killed by $\delta^{(d)}(\Sigma p_i)$ in vertices

BUT for loop DIAGRAMS, not enough δ-functions & left w/ overall loop momentum integral.



$$\left.\begin{aligned} p + k_2 &= k_1 \\ k_1 - p' &= k_2 \end{aligned}\right\} \boxed{p - p' = 0}$$ BUT k_1 UNDETERMINED

int. over 'size' of loop → hi p ⟷ small scale

<u>Skimming over</u> : (1) $L = I - V + 1$

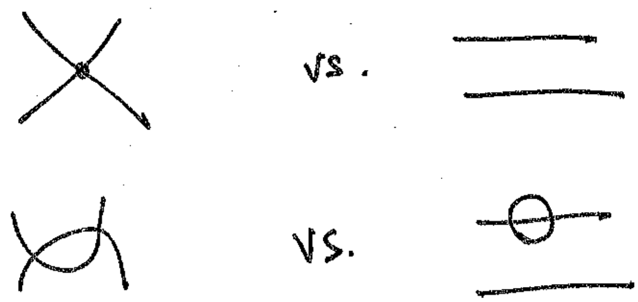
loop ↗ int lines ↑ vertices ↑

(2) symmetry factors

(3) calculating DIAGRAMS

↳ we have REDERIVED P23DA.

<u>REMARK</u> : we care about <u>connected</u> GRAPHS.



CONNECTED DISCONNECTED

turns out: these factorize according to their connected subgraphs.

NOW WE NEED A BLOB CONVENTION :

 $= \sum_{\text{connected}}$ 

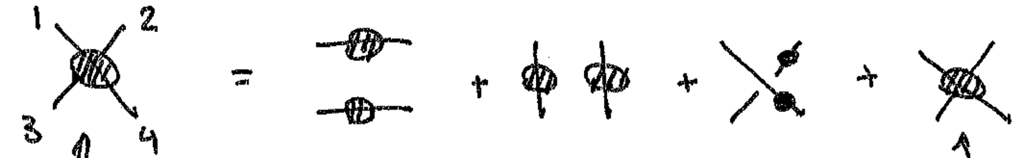
$\langle \cdots \rangle_c$

all allowed graphs s.t. ▨ is connected.

(this is to <u>ALL ORDER</u>)

↑ <u>no</u> pert thy!

"obviously"



connected

all (not connected) ← need better notation

maybe $\boxed{/\!/\!/}$ for not nec. connected?

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \langle \phi_1 \phi_2 \rangle_c \langle \phi_3 \phi_4 \rangle + \cdots$$
$$+ \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_c$$

$\phi(x_1)$

$$(-i)^n \left(\frac{\delta}{\delta J_1} \cdots \right) Z[J] \Big|_{J=0}$$

CLAIM : WRITE $Z[J] \equiv e^{iW[J]}$ ← defines W

then $W[J]$ is the generating function of connected amplitudes (!!)

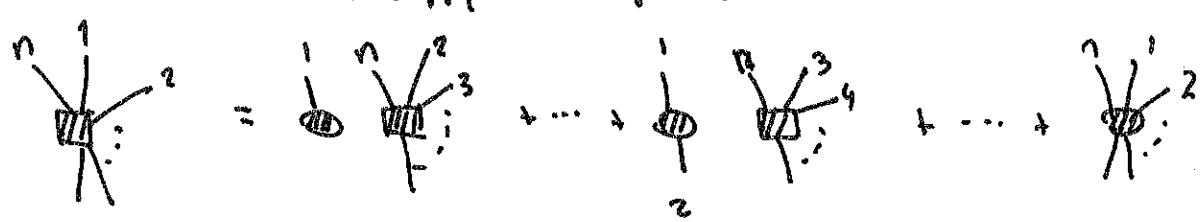
$$(-i)^{n-1} \left(\frac{\delta}{\delta J_1} \cdots \right) W[J] \Big|_{J=0} = \langle \phi_1 \cdots \phi_n \rangle_{\text{conn.}}$$

check

Pf/ DIRECT SUBSTITUTION.

$$\langle \phi_1 \cdots \phi_n \rangle = \sum_r \sum_{\{i\}} \langle \phi_1 \cdots \phi_{i_r} \rangle_c \langle \phi_{i_{r+1}} \cdots \phi_n \rangle$$

I'm sloppy w/ range of r, i



ASSUMING W GENERATES CONN GRAPHS, SHOW $Z = e^{iW}$

$$\left(\frac{\delta}{\delta J_1}\cdots\right) Z\Big|_{J=0} = i \sum_r \sum_{\{i\}} \left(\frac{\delta}{\delta J_1}\cdots\right) W\Big|_{J=0} \left(\frac{\delta}{\delta J_{i_{r+1}}}\cdots\right) Z\Big|_{J=0}$$

combinatorics (see H.O.)

$$= i\left(\frac{\delta}{\delta J_2}\cdots\right)\left[\left(\frac{\delta}{\delta J_1} W\right) Z\right]_{J=0} \qquad *$$

$$\Rightarrow \boxed{\frac{\delta}{\delta J(x)} Z[J] = i\left(\frac{\delta}{\delta J} W[J]\right) Z[J]} \qquad ☺$$

(*) ARE TERMS OF TAYLOR EXP OF ☺, SO ☺ IS TRUE

$\Rightarrow$ solution to ☺ is

$$\boxed{Z[J] = e^{iW[J]}}$$

- connected graphs have overall momentum conservation

<u>nb</u>: $\int d^N q \, e^{-\frac{1}{2} q A q} = e^{\frac{1}{2} J A^{-1} J}$ (free)

<u>EVIDENTLY</u>: $W = -\frac{1}{2}\int d^4x \, d^4y \, J(x)\, \Delta(x-y)\, J(y)$

See: SREDNICKI §9, next: §21

RESULT: $Z[J] = \Sigma$ all diagrams $\leftarrow$ $Z[J]\big|_{J=0}$ = bubbles

$\propto e^{\Sigma \text{ all connected diagrams}}$