Gauge theory

REVIEW of CLASSICAL ASPECTS

→ "LOCAL SYM" ⟷ REDUNDANCY IN DESCRIPTION

↳ the "GAUGE" (gauge choice) is unphysical

IN E&M: PICK IT TO BE CONVENIENT

(advanced: gives nontrivial behavior in GLOBAL PICTURE of FIELD THY AS A <u>FIBER BUNDLE</u> over spacetime)

<u>Non-Abelian gauge sym</u>

many ROTATION AXES ... so have indices

$$e^{i\theta^a T^a}$$

for us: HERMITIAN MATRICES

GROUP

GENERATORS (ALGEBRA)

eg $\frac{1}{2}\sigma^a$ for SU(2)

the T^a form AN <u>ALGEBRA</u> : ABSTRACT DESCRIPTION of HOW ROTATIONS ABOUT DIFF AXES DO NOT (in gen) COMMUTE.

$$[T^a, T^b] = \underbrace{f^{abc}}_{\text{totally antisym}} T^c$$

lin comb of GENERATORS

totally antisym ($= \varepsilon^{abc}$ for SU(2)

NB: the diff btwn T^a & $\frac{1}{2}\sigma^a$ is indices.
T is AN ABSTRACT OBJECT THAT OBEYS COMM. RELS

the <u>REPRESENTATION</u> of T^a is A MATRIX that acts on objects that are covariant.

fancy people may write $r(T^a) = \frac{1}{2}\sigma^a$
... we may be more prosaic.

nb. successive finite transf = a finite transf.

↳ follows from comm rel. & BCH

A few special REPS : ① fundamental & anti-fundamental

"defining rep"

for $SU(N)$, the N-DIM $\mathbb{C}$ vector.

② ADJOINT REP : the GENERATORS themselves are basis elements

for $X \in$ ALGEBRA of $SU(N)$ $(\mathcal{L}(SU(N)))$

$$\underbrace{T_{ab}(T_c)}_{\text{REP OF GENERATOR } T_c \text{ IN THE ADJOINT BASIS}} = \underset{\uparrow}{f_{acb}}$$

what the element is

↳ simply the ACTION of the PAULI MATRICES ON EACH OTHER

the GAUGE BOSONS THEMSELVES ARE IN the ADJOINT REP

matter is in any REP, for SIMPLICITY : fund/anti-fun.

RECALL: (USING OSBORN NOTATION)

for $U(1)$: $g(x) = e^{i\lambda(x)}$ "GAUGE"

$$A_\mu(x) \to A_\mu^g(x) = A_\mu(x) + i\partial_\mu \lambda(x)$$

for Non-Abelian: $g(x) = e^{i\theta^a(x)T^a}$

$$A_\mu(x) \to A_\mu^g(x) = g(x)^{-1} A_\mu g(x) + \underbrace{g(x)^{-1}\partial_\mu g(x)}_{\text{"MAURER CARTAN" ... elem of ALGEBRA}}$$

$$A_\mu = A_\mu^a T^a$$

RECALL: one way to motivate this

∂_μ IS PROBLEMATIC FOR LOCAL SYM

BECAUSE $\partial_\mu \Phi$ DOES NOT TRANSFORM

LIKE Φ UNDER THE LOCAL SYM.

SO INTRODUCE COVARIANT DERIVATIVE THAT DOES:

if $\Phi \to g(x)\Phi$

then $D_\mu \Phi \to g(x) D_\mu \Phi$

seems ad hoc, but this is what "moving along the MANIFOLD" REQUIRES in the SAME WAY that GR MANDATES A COVARIANT DERIV.

$$D_\mu = \partial_\mu + i A_\mu^a T^a$$

maybe has coupling

REP DEPENDS ON REP OF WHAT YOU'RE ACTING ON

cf. D_μ on v vs. e.

<u>Remarks</u>

$$\underbrace{[D_\mu, D_\nu]}_{\substack{\text{commutator of}\\ \text{2 cov. deriv.}}} = \underbrace{F^a_{\mu\nu}}_{\substack{\text{field strength.}\\ \text{not a derivative op}}} T^a$$

for U(1), $F_{\mu\nu}$ is GAUGE INVARIANT. $\rightarrow$ $-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu}$ is kin. term.

for Non Abelian : $-\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu}$ IS A GOOD KINETIC TERM.

SUM OVER a

<u>to check</u> : this gives correct kinetic terms for the GAUGE BOSONS

($1/g^2$ is a convenient normaliz... but does <u>not</u> give canon. norm.)

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in coset space language, the <u>PHYSICAL</u> states live in A/G $\leftarrow$ "mod out" by gauge redundancy.

G $\leftarrow$ GAUGE SYM.

what could go wrong when I quantize?

<u>canonical</u> : $P_i = \frac{\delta L}{\delta \dot{q}^i}$ $\leftarrow$ q^i is $A_\mu(x)$

$$\frac{\delta L}{\delta \dot{A}_i} = -\frac{1}{g^2} F^{0i} \qquad \boxed{\frac{\delta L}{\delta \dot{A}_0} = 0}$$

$\uparrow$ no $\partial_0 A_0$ term in $F_{\mu\nu}F^{\mu\nu}$!

... BUT THEN THERE IS NO MOMENTUM CONJUGATE TO A_0!

↳ clearly not A DYNAMICAL VARIABLE

... in fact, EXISTS AS A $\mathcal{L}$ MULTIPLIER TO ENFORCE $D_\mu F^{0\mu} = 0$. ⟵ $\nabla \cdot \underline{E} = 0$ in vac.

hmm... what's going on?

to "DO PERT THY" we start w/ FREE THY, "QUADRATIC" / 'CLASSICAL'

which we can solve explicitly w/ GREEN'S FUNC.

↳ we invert the quadratic part of S.

$$Z \sim \int \mathcal{D}A \, e^{i \int d^4x \, A_\mu \mathcal{O}^{\mu\nu} A_\nu} + \ldots$$

but some of those CONFIGURATIONS are GAUGE EQUIVALENT

equivalent in the most tautological "OBVIOUS" way

... $S[A^g] = S[A]$

... DEGENERACY!

$\Rightarrow$ CANNOT INVERT $\mathcal{O}^{\mu\nu}$

IDEA: fix GAUGE REDUNDANCY — somehow — then "DO PATH INTEGRAL"

"finite integral" example

$\underline{x} \in \mathbb{R}^n$

SO(n) ROTATION

consider $\int d^n \underline{x} \; f(\underline{x})$ s.t. $f(x) = f(\underline{\underline{R}}\,\underline{x})$

then: since $d^n \underline{x} = d^n(R\underline{x})$, we know that REALLY the func depends on $r = |\underline{x}|$.

$$\int d^n x \; f(\underline{x}) = S_n \int_0^\infty dr \; r^{n-1} \hat{f}(r)$$

n-DIM sphere AREA

$$\left(S_n = \frac{2\sqrt{\pi^n}}{\Gamma(n/2)} \right)$$

"GAUGE FIXING" APPROACH:

PICK A REFERENCE DIRECTION:

$$\underbrace{\underline{x} \to \underline{x}_0}_{\text{gauge fix.}} = \underbrace{r\,(0, 0, \cdots, 0, 1)}_{\text{PROJECT onto 1D PROBLEM}}$$

x_n

[IMPOSE THIS w/ SOME fancy δ-function

$$\delta(\underbrace{F(x)}_{\text{gauge fixing func}}) \equiv \Theta(x_n) \prod_{i=1}^{n-1} \delta(x_i)$$

1st (n-1) COMPONENTS ZERO

n^{th} COMP > 0

UNDER SMALL ROTATION, $x_0 \to r(\theta_1, \dots, \theta_{n-1}, 1) + \mathcal{O}(\theta^2)$

$$\delta(F(R\underline{x})) = \Theta(r) \prod_{i=1}^{n-1} \delta(r\theta_i)$$

def $$\int_{SO(n)} \underbrace{d\mu(R)}_{\text{INV MEASURE OVER } SO(n)} \delta(F(R\underline{x})) \underbrace{M(\underline{x})}_{\text{COMPENSATES FOR CHOICE OF } F} = 1$$

$$d\mu(R) = d\mu(R') \prod_{s=1}^{n-1} d\theta_s \, (1 + \mathcal{O}(\theta))$$

nb $$\int_{SO(n)} d\mu(R) \, \delta(F(Rx_0)) = \int_{SO(n-1)} d\mu(R') \int \prod_{s=1}^{n-1} d\theta_s \, \delta(r\theta_s)$$

$$= \frac{V_{SO(n-1)}}{r^{n-1}}$$

SO DEFINE: $M(\underline{x}) = \dfrac{r^{n-1}}{V_{SO(n-1)}}$

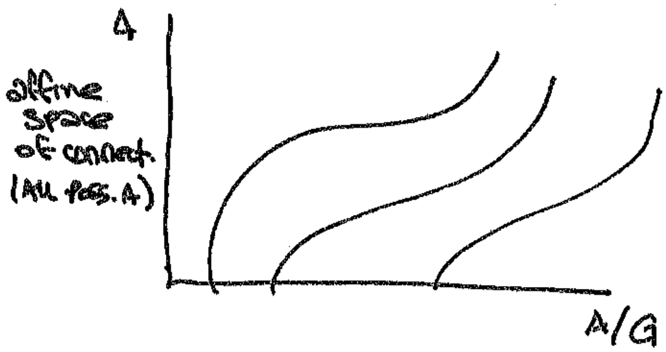
SO: $$\int d^n x \, f(\underline{x}) = \int d^n x \int_{SO(n)} d\mu(R) \, \delta(F(R\underline{x})) \, M(\underline{x}) \, f(\underline{x})$$

$$= \frac{1}{V_{SO(n-1)}} \int_{SO(n)} d\mu(R) \int d^n x \, \delta(F(R\underline{x})) \, r^{n-1} \boxed{f(\underline{x}_0)}$$

$$= \quad \text{———} \, '' \, \text{———} \quad \Theta(x_0) \prod_i \delta(x_i) \quad \text{———}$$

$$= \underbrace{\frac{V_{SO(n)}}{V_{SO(n-1)}}}_{S_n} \int dx_n \, r^{n-1} \, f(\underline{x}_0(r))$$

BACK to BIG PIC.



GAUGE ORBITS gen by G (foliate A)

$\leftarrow$ PHYSICAL CONFIG

$$\underset{\int dA\, e^{iS[A]}}{Z} = \underbrace{(\text{volume of } G)}_{\text{formally } \infty \text{ - ONE OVERALL PREFACTOR}} \underbrace{Z_{\text{phys}}}_{\text{path integral over PHYSICAL CONFIG}}$$

<u>OUR trick : Fadeev-Popov</u>

Gauge fixing via <u>LAGRANGE MULTIPLIER</u> in the action

TRICK: $1 = \int dg \; \det\left(\frac{\partial f[A^g]}{\delta g}\right) \; \delta\left(f[A^g]\right)$

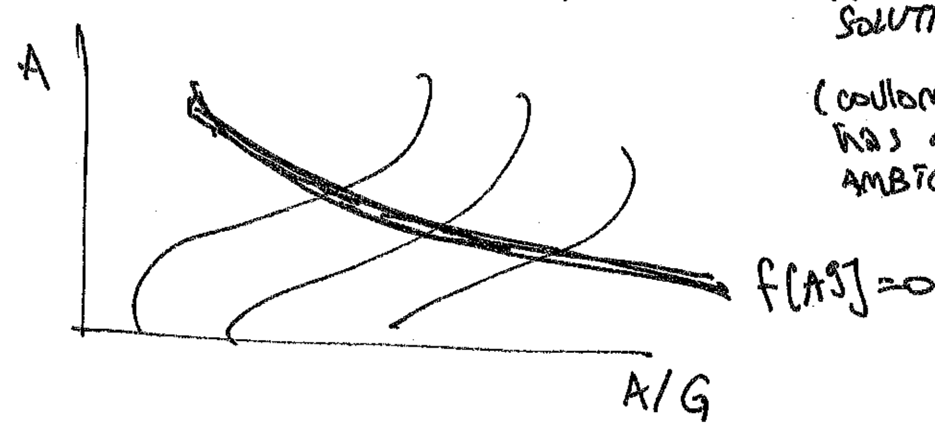
int over GAUGE ORBITS

Jacobian $= \Delta_{FP}$

$f[A^g]=0$ is GAUGE FIXING CONDITION

MUST HAVE UNIQUE SOLUTION

(coulomb gauge has a Gribov AMBIGUITY)



ARGUMENT:

$$Z = \int \mathcal{D}A \, \underbrace{dg \, \Delta_{FP}[A^g] \, \delta(F[A^g])}_{\text{multiply by one}} \, e^{iS[A]}$$

$$= \underbrace{\int dg \int \mathcal{D}A}_{} \, \Delta_{FP}[A] \, \delta[F[A]) \, e^{iS[A]}$$

change ORDER: $\mathcal{D}A$, $\delta[A]$, dg ARE GAUGE INVT.

$$= \boxed{\int dg} \; Z_{phys}$$

volume of phase space

done? no. $\int \mathcal{D}A \, \Delta_{FP}[A] \, \delta[F[A]] \, e^{iS[A]}$

does not look like a partition func.

Send the $\Delta_{FP} \, \delta[F]$ into exponential.

What we conventionally do:

Restrict to a convenient set of GAUGE FIXING conditions:

$$\underbrace{f^a[A] - b^a(x)}_{F[A]} = 0$$

then integrate over a GAUSSIAN weight for $b(x)$

$$\int db \, e^{-\frac{i}{2\xi} \int dx \, b^a(x) b^a(x)}$$

throw out overall normaliz.

$$Z = (\mathrm{vol}\, G) \int \mathcal{D}b \; e^{-\frac{i}{2\xi}\int d^4x\, b^a(x) b^a(x)} \int \mathcal{D}A \; \Delta_{FP}\, \delta(f-b)\, e^{iS}$$

evaluate db over δ-fnc.

$$= (\mathrm{vol}\, G) \int \mathcal{D}A \; \underbrace{\Delta_{FP}[A]} \; e^{iS[A] - \frac{i}{2\xi}\int d^4x\, f^a[A] f^a[A]}$$

$$= \det \frac{\delta f[A^g]}{\delta g}\Big|_{g=1} = \det \frac{\delta f[A^\Lambda]}{\delta \Lambda}\Big|_{\Lambda=0}$$

$$g = 1 + \Lambda$$

now a reminder:

FERMIONIC GAUSSIAN INTEGRALS ARE PROPORTIONAL TO DETERMINANTS

↳ cf BOSONIC INTEGRALS ARE PROPORTIONAL TO <u>inverse</u> power of DETERMINANT.

⇒ REVIEW THIS IF YOU HAVE FORGOTTEN

this only has to do w/ commuting vs anti-commuting variables.

$$\det \frac{\delta f}{\delta \Lambda} = \int \underbrace{\mathcal{D}\bar{c}\, \mathcal{D}c} \; e^{-i\int d^4x\, \bar{c}\, \frac{\delta f}{\delta \Lambda} c}$$

INDEPENDENT, ADJOINT SCALAR FIELDS w/ <u>GRASSMAN</u>, STATISTICS (anticommuting)

... represent <u>NEGATIVE</u> DOF.

↳ <u>GHOST</u> FIELDS.

Reminders

Gaussian int: $\int dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

Grassmann int: $\int \underbrace{d\bar{\theta} d\theta}\, e^{-\bar{\theta} b \theta} = b$

$= d^2\theta \, (1 - \bar{\theta} b \theta)$

$= d\bar{\theta}\, d\theta \, (+\theta \bar{\theta} b)$

Generalized to $\int d^n x\, e^{-\frac{1}{2} \underline{x} \underline{\underline{A}} \underline{x}} = \sqrt{\frac{(2\pi)^n}{\det A}}$

$$\int d^{2n}\theta \, e^{-\bar{\underline{\theta}} \underline{\underline{B}} \underline{\theta}} = \det B$$

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so at this point :

$$Z = (\text{Vol } G) \int \mathcal{D}A\, \mathcal{D}c\, \mathcal{D}\bar{c}\; e^{i \int d^4x\, \mathcal{L}_{YM} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{GF} + \mathcal{L}_{FP} + \text{sources}}$$

$\cdot\, \mathcal{D}(\text{matter})$

$-\frac{1}{2\xi} f^a[A] f^a[A]$ $\qquad\qquad$ $-\bar{c}\, \frac{\delta f}{\delta \Lambda}\, c$

eg $f^a(A) = \partial_\mu A^{\mu a}$

Peskin (9.53)

$$1 = \int \underbrace{\mathcal{D}\alpha(x)}_{\text{VOL of GAUGE ORBIT}} \, \delta[G(A^\alpha)] \; \det \frac{\delta G(A^\alpha)}{\delta \alpha}$$

$$A^\alpha = A_\mu + \frac{1}{e}\partial_\mu \alpha(x)$$

cf. $$\boxed{1 = \prod_i da_i \; \delta^{(n)}(g(\underline{a})) \; \det\left(\frac{\partial g_i}{\partial a_j}\right)}$$

eg. LOREN(T)Z GAUGE: $G(A) = \partial^\mu A_\mu = 0$

$$G(A^\alpha) = \partial^\mu A_\mu + \frac{1}{e}\partial^2 \alpha(x)$$

$$\frac{\delta G}{\delta \alpha} = \det\left(\partial^2/e\right)$$

Non Abelian (16.28)

$$(A^\alpha)^a_\mu T^a = e^{i\alpha^a T^a}\left[A^b_\mu T^b + \frac{i}{g}\partial_\mu\right]e^{-i\alpha^c T^c}$$

$$= A^a_\mu + \frac{1}{g} D_\mu \alpha^a \qquad \text{(infinitesimal)}$$

<u>then</u> : $$\frac{\delta G[A^\alpha]}{\delta \alpha} = \frac{1}{g}\partial^\mu D_\mu$$

$$\det(-\text{''}) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\; e^{i\int d^4x\, \bar{c}(-\partial^\mu D_\mu)c}$$

$$G[A] = \partial^\mu A_\mu(x) - \omega(x)$$

generalize Lorenz gauge

(same $\delta G[A^\alpha]/\delta\alpha$ no matter $\omega(x)$)

$$\int \mathcal{D}A\, e^{iS[A]} = \det \frac{\delta G[A^\alpha]}{\delta\alpha} \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\, \delta(G[A^\alpha])$$

now take LIN COMB OF WEIGHTS $\omega(x)$:
GAUSSIAN DIST :

$$\mathcal{N}(\xi) \int \mathcal{D}\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}} \det \frac{\delta G[A^\alpha]}{\delta\alpha} \int \mathcal{D}\alpha$$

$$\times \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))$$

$$= \mathcal{N}(\xi) \det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right) \int \mathcal{D}\alpha$$

$$\times \int \mathcal{D}A\, e^{iS[A]} \exp\left[-i\int d^4x\, \underbrace{\frac{1}{2\xi}(\partial^\mu A_\mu)^2}\right]$$

new effective term to $\mathcal{L}$.

AFFECTS KINETIC TERM

$$S = \int d^4x \; \underbrace{-\tfrac{1}{4}(F_{\mu\nu})^2}$$

$$\tfrac{1}{2} A_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k)\left[-k^2 g^{\mu\nu} + \underbrace{k^\mu k^\nu}\right]\tilde{A}_\nu(-k)$$

nb: when $\tilde{A}_\mu(k) = k_\mu \alpha(k)$,
$S = 0$; $e^{iS} = 1$
for any $\alpha(k)$
... no GAUSSIAN DAMP.

$$\underbrace{\mathcal{O}_{\mu\nu}}_{\text{singular}} G^{\nu\rho}(x-y) = i\,\delta_\mu{}^\rho\, \delta^{(4)}(x-y)$$

With the ξ-term:

$$\left(-k^2 g^{\mu\nu} + \left(1-\tfrac{1}{\xi}\right)k^\mu k^\nu\right) G_\nu{}^\rho(k) = i\,\delta^{\mu\rho}$$

$$\Rightarrow \boxed{G^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]}$$

check:

$$\left(-k^2 g^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu\right)\left(g_{\mu\nu} - (1-\xi)\tfrac{1}{k^2}k_\mu k_\nu\right)$$

$$= -4k^2 + (1-\xi)k^2 + (1-\xi^{-1})k^2 - \underbrace{(1-\xi^{-1})(1-\xi)}_{1-\xi^{-1}-\xi+1} k^2$$

Ryder: (7.20)

$$I = \int dx\, dy \; e^{-(x^2+y^2)}$$

$$= \underbrace{\int d\theta}_{\substack{2\pi \\ \text{"gauge space"}}} \int dr\; r\; e^{-r^2}$$

projects to $\theta = 0$

$$= \int d\theta \; \underbrace{\int dr\, d\theta'\, r\; e^{-r^2}\, \delta(\theta')}_{\text{trivial}}$$

however: we can integrate over a path of non-zero θ ... $f(x)$ is not mut!

$$f(\theta) = \underbrace{y \cos\theta - x \sin\theta}_{\text{same path: } \frac{y}{x} = \frac{\sin\theta}{\cos\theta}} = 0$$

Use: $$\delta(f(\theta)) = \sum_{i} \frac{1}{|df(\theta_i)/d\theta|}\, \delta(\theta - \theta_i)$$

zeros of f

gives: $\theta_1 = \tan^{-1}(y/x)$ $\qquad \theta_2 = \pi + \tan^{-1}(y/x)$

$$\left.\frac{df}{d\theta}\right|_{\theta_1,\theta_2} = -r = -\sqrt{x^2+y^2}$$

$$\delta(f(\theta)) = \frac{1}{r}\left(\delta(\theta-\theta_1) + \delta(\theta-\theta_2)\right)$$

$$\int \delta(f(\theta))\, d\theta = \frac{2}{r} \equiv \Delta(r)^{-1}$$

$f(\theta)$ is simply a rotation from the y-axis

$$\left.\begin{aligned} y' &= y\cos\theta - x\sin\theta \\ x' &= x\cos\theta + y\sin\theta \end{aligned}\right\} \quad x^2+y^2 = (x')^2+(y')^2 = r^2$$

then: $\Delta(r)\int \delta(f(\theta))\,d\theta = 1 = \Delta(r')\int \delta(y)\,d\theta$

$$I = \underbrace{\int d\theta}_{\text{overall mult}} \underbrace{\int dx'\,dy'\,e^{-(x'^2+y'^2)}\,\Delta\left(\sqrt{x'^2+y'^2}\right)\,\delta(y')}_{\text{INDEP OF }\theta}$$

further:

$$\Delta(r)^{-1} = \int \delta(f(\theta))\,d\theta$$

$$= \int \delta(f(\theta)) \det\left|\frac{d\theta}{df}\right| df$$

$$= \det\left|\frac{d\theta}{df}\right|\Big|_{f=0}$$

$$\rightarrow \Delta(r) = \det\left|{}^{df}/_{d\theta}\right|\Big|_{f=0}$$

$$= \int d\theta \int d^2x'\, f(\underline{x}')\, \det\frac{df}{d\theta}\Big|_0\, \delta(y')$$