ANOMALY

BRST IS A POWERFUL TOOL FOR PROVING THE "NICENESS" OF GAUGE THEORIES

Renormalizability

Conservation of current @ quantum level (WARD IDENTITIES)
SLAVNOV/TAYLOR

We've introduced the tool. Doing the checks are now technical, but standard, procedures.

Let's instead move on to: WHAT COULD POSSIBLY GO WRONG?

SYMMETRY (global or gauge): under transf. of fields:

classical : $\delta S = 0$ (weaker condition)

quantum : $\delta Z = 0$ (stronger condition)

WHAT'S THE DIFFERENCE?

$$\int \underbrace{\mathcal{D}(\text{fields})}\, e^{iS[\text{fields}]}$$

this part.

We know this: integration measure may shift $\supset$ Jacobian.

Manifestation : $\partial \cdot j \neq 0$ "quantumly"

implication: many (exp: $\pi^0 \to \gamma\gamma$

thy: η' MESON MASS

Θ_{YM})

INGREDIENTS: CHIRAL GAUGE THEORY (at least in 4D)

fermions SPIN-1/2

GAUGE FIELD SHOWS UP IN non-invariance of measure (curious)

two CHIRALITIES

ψ_L^{α} & $\psi_{R\dot{\alpha}}$

Distinct (matched) reps of LORENTZ (Poincaré) GROUP.

NB GAUGE FIELD CAN TALK TO ψ_L & ψ_R DIFFERENTLY.

familiar DIRAC FERMION REP IS:

$$\underbrace{\Psi_a}_{\substack{1,2,\ 3,4 \\ \alpha=1,2 \quad \dot{\alpha}=1,2}} = \begin{pmatrix} \psi_L^{\alpha} \\ \psi_{R\dot{\alpha}} \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} & (\sigma^\mu)^{\alpha}{}_{\dot{\alpha}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}}{}_{\alpha} & \end{pmatrix}$$

Before jumping into 4D, lets look at 2D* where it's simpler → Holstein Am.J.P 61 142 ('93)

in 2D MINKOWSKI SPACETIME

$$\varepsilon^{\mu\nu} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \quad \leftarrow \text{ vs } \varepsilon^{\mu\nu\rho\sigma} \text{ in 4D}$$

$$\left.\begin{aligned} \gamma^0 &= \sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \gamma^1 &= i\sigma^2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \end{aligned}\right] \text{ GIVES WRONG METRIC...} \quad \boxed{\begin{aligned} \gamma^0 &= \sigma^2 \\ \gamma^1 &= i\sigma^1 \end{aligned}} \text{ (Roskin)}$$

$$\rightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

*: in 3D (eg for cond mat) anomaly structure is very different or 10D...

looks like ψ_+ & ψ_- are <u>separately</u> symmetric.

so expect <u>another current</u> to be conserved

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

CLASSICAL

indeed, classically, $\partial^\mu j_{5\mu} = 0$.

CONVENIENT: $j^5_\mu = \epsilon_{\mu\nu} j^\nu$ =

WHAT ACTUALLY HAPPENS:

$$\mathcal{L}_{\text{CLASSIC}} \longrightarrow \mathcal{L}_{\text{QUANT.}} = \underbrace{\bar{\psi}' i \not{\partial} \psi'}_{\substack{\text{eh?!} \\ \text{NON-INTERACTING}}} - \tfrac{1}{4} F_{..}F^{..} \underbrace{- \tfrac{e^2}{2\pi} A^2}_{\substack{\text{MASSIVE} \\ \text{PHOTON!}}}$$

$$m_\gamma^2 = e^2/\pi$$

further

$$\partial^\mu j^5_\mu = -\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \neq 0$$

{ axial symmetry is <u>not</u> a valid sym!

How does this happen?

$$\psi \to U\psi$$

↑ for AXIAL: $U = e^{ie\theta(x)\gamma^5}$

CAN CHOOSE $\theta(x)$ s.t. $\partial_\mu \theta(x) = A_\mu(x)$

then $\bar\psi i D\!\!\!/ \psi = \underbrace{\bar\psi' i \partial\!\!\!/ \psi'}_{\text{free theory}}$

but: $\mathcal{D}\psi \mathcal{D}\bar\psi = \mathcal{D}\psi' \mathcal{D}\bar\psi' \boxed{J}$

turns out: $J \to e^{-i\int d^2x \, \frac{e^2}{2\pi} A_\mu A^\mu}$

→ we can see this by doing a small chiral rotation

see: P&S p. 664 - 668 ... just go to 2D (EASIER!)

$$\psi \to \psi' = (1 + i\alpha(x)\gamma^5)\psi$$
$$\bar\psi \to \bar\psi' = \bar\psi(1 + i\alpha(x)\gamma^5)$$

$$\bar\psi' i \partial\!\!\!/ \psi' = \bar\psi i \partial\!\!\!/ \psi \underbrace{- \partial_\mu \alpha(x) \bar\psi \gamma^\mu \gamma^5 \psi}$$
$$+ \alpha(x) \partial_\mu (\underbrace{\bar\psi \gamma^\mu \gamma^5 \psi}_{j_5})$$

so far: classical

nb: $\begin{cases} \text{2 component spinors (no sense of L or R ... just fermion)} \\ \text{2 component vectors} \end{cases}$

different reps.

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - \frac{1}{4}F_{..}F^{..} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

$\not{D}$: $\partial_\mu + ieA_\mu$

EOM $\quad i\not{D}\psi = 0$

$\quad \partial^2 A_\mu = -ej_\mu \qquad j_\mu = i\bar{\psi}\gamma^\mu\psi$

(using Lorenz gauge $\partial \cdot A = 0$)

current cons: $\partial^\mu j_\mu = i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = i\bar{\psi}\overleftarrow{\not{\partial}}\psi + i\bar{\psi}\overrightarrow{\not{\partial}}\psi$

there is an analog of $\gamma^5 = \gamma^0\gamma^1 = \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

(anticommutes w/ γ^μ's.)

can "see" sym:

$$\bar{\psi}\not{D}\psi = \underbrace{(\psi_+^\dagger \ \ \psi_-^\dagger)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}_{\bar{\psi} = \psi^\dagger\gamma^0}\begin{pmatrix} & D_t + D_x \\ D_t - D_x & \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$$= \psi_+^\dagger(D_t - D_x)\psi_+ + \psi_-^\dagger(D_t + D_x)\psi_-$$

(sometimes called left-mover & right mover)

MORE CAREFULLY — take care w/ MEASURE
EXPAND ψ IN EIGENSTATES OF $\not{D}$

↳ def L + R EIGENVECTORS OF $\not{D}$

$$i\not{D}\,\phi_m = \lambda_m \phi_m$$

$$\hat{\phi}_m (i\overleftarrow{\not{D}}) = \underline{-\lambda_m} \hat{\phi}_m$$ ← def by int by parts (equiv: HERMITICITY of $i\not{D}$)

$$\lambda^2 = k^2 = (k^0)^2 - (k^x)^2$$

then: $\psi = \sum_m a_m \underbrace{\phi_m(x)}$

↑ anticom. ↖ BASIS (COMMUTING)

$$\bar{\psi} = \sum_m \hat{a}_m \hat{\phi}_m(x)$$

and $\mathcal{D}\psi\,\mathcal{D}\bar{\psi} = \prod_{mn} da_m\, d\hat{a}_n$

WANT $\mathcal{D}\psi'\,\mathcal{D}\bar{\psi}'$ W/RT $a_m, \hat{a}_n$ ← transformed

USE ORTHOGONALITY OF BASIS $\langle \phi_m | \phi_n' \rangle$

$$a_m' = \sum_n \int d^4x\, \phi_m^\dagger(x)\left[1 + i\alpha(x)\gamma^5\right]\phi_n(x)\, a_n$$

$$= \sum_n (\delta_{mn} + C_{mn})\, a_n$$

↖ some matrix from $i\alpha\gamma^5$ term

$$\Rightarrow \mathcal{D}\psi'\,\mathcal{D}\bar{\psi}' = \left(\frac{1}{\det(\mathbb{1} + C)}\right)^2 \mathcal{D}\psi\,\mathcal{D}\bar{\psi}$$

$$\det(\mathbb{1}+C) = e^{\operatorname{tr}\log(1+C)} \leftarrow \log(1+C) \approx C$$

for infinitesimal ~~gauge~~ transf.

$$= e^{\sum_n C_{nn}}$$

$$i\int d^2x\ \alpha(x) \underbrace{\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)}_{\text{looks like } \operatorname{tr}\gamma^5 = 0\ !}$$

but this is infinite...
and not "just some prefactor" of Z
that gets modded out!

NEED to REGULATE :

$$\sum \phi_n^\dagger(x)\gamma^5\phi_n(x) = \lim_{M\to\infty}\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)\underbrace{e^{\lambda_n^2/M^2}}_{\text{regulate}}$$

λ_n^2 negative upon wrot

$$= \lim_{M\to\infty}\sum_n \phi_n^\dagger(x)\gamma^5 e^{+(i\not{D})^2/M^2}\phi_n(x)$$

by def of λ_n

$$= \lim_{M\to\infty}\langle x|\operatorname{tr}[\gamma^5 e^{(i\not{D})^2/M^2}]|x\rangle$$

position

DIRAC INDICES

what is $(i\not{D})^2$? $\quad (iD)_\mu (iD)_\nu \gamma^\mu \gamma^\nu$

use $\gamma^\mu \gamma^\nu = \underbrace{\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}}_{\eta^{\mu\nu}} + \underbrace{\tfrac{1}{2}[\gamma^\mu, \gamma^\nu]}_{+\frac{1}{2}\sigma^3 \text{''} = \frac{1}{2}\gamma^5}$

$$(i\not{D})^2 = -D^2 \;\cancel{=}\; \underbrace{-\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]}\;\cancel{A_\mu A_\nu}\,(\partial + ieA)_\mu(\partial+ieA)_\nu$$

$$+ \frac{e}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} \qquad \sim e\gamma^5 F_{\mu\nu}$$

$$\cancel{\text{ext}}\ \gamma^5\gamma = \lim_{M\to\infty} \langle x|\,\mathrm{tr}\left[\gamma^5\, e^{\frac{-D^2 + \frac{e}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}{M^2}}\right]|x\rangle$$

trace over $\gamma^5 = 0$
trace over $(\gamma^5)^2 = 1$

exp to LIN O

$$= \lim_{M\to\infty} \cancel{\langle x|}\ \mathrm{tr}\left[\gamma^5 \frac{1}{2}\frac{e}{M^2}\gamma^5 \overset{F_{12}}{E}\right]\cancel{|x\rangle}$$

$$\times \langle x|\, e^{-\partial^2/M^2}\,|x\rangle$$

ignore BG field

$$\langle x|e^{-\partial^2/M^2}|x\rangle = \lim_{x\to y}\int d^2k\; e^{-ik\cdot(x-y)}\, e^{k^2/M^2}$$

$$= i\int d^2k_E\, e^{-k_E^2/M^2}$$

$$= i\,\frac{M^2}{(2\pi)^2} \leftarrow \text{CHK FACTORS OF 2}$$

DIRAC

$$\Sigma \phi_n^\dagger \gamma^5 \phi_n = \lim_{M\to\infty} \frac{i e}{(2\pi)^2} M^2 \, \mathrm{tr}\left[\cancel{\gamma^5} \tfrac{1}{2} \frac{e}{M^2} \cancel{\gamma^5} E\right]$$

CHK!

$$= \frac{-ie}{4\pi} E \leftarrow \tfrac{1}{2}\varepsilon^{\mu\nu} F_{\mu\nu}$$

sign?

(b) $\det(1+C) = e^{\int \Sigma_n \alpha(x) \phi_n^\dagger \gamma^5 \phi_n \, d^2x}$

$$= e^{-i\int d^2x\, \alpha(x) \frac{e}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu}}$$

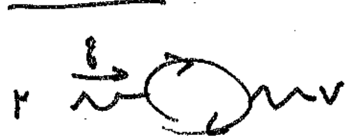
thus

$$Z' = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, e^{i\int d^2x \, \bar{\psi}(i\cancel{D})\psi + \alpha(x)\left[\partial\cdot j_5 + \frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}\right]}$$

VARY W/RT $\alpha(x)$ $\rightarrow$ $\partial\cdot j_5 = -\frac{e}{2\pi}\varepsilon^{\cdot\cdot}F_{\cdot\cdot} \neq 0$!!

usual procedure,
right?

REMARK



$$= i\Pi^{\mu\nu}(q)$$

$$= -i\underbrace{(q^2 g^{\mu\nu} - q^\mu q^\nu)}_{\text{"transverse"}} \frac{2e^2}{(4\pi)^{d/2}} \overset{\text{DIRAC}}{\mathrm{tr}(\mathbb{1})}$$

$$\times \underbrace{\int_0^1 dx\, x(1-x) \frac{\Gamma(2-d/2)}{(-x(1-x)q^2)^{2-d/2}}}_{\textit{finite}}$$

$$= i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{e^2}{\pi}$$

gives PHOTON MASS!

$$m_\gamma^2 = \frac{e^2}{\pi}$$

Alternative (& insightful) picture:

CHIRAL SOLUTIONS TO DIRAC: $H = i\not{\partial}$

$\psi_+ = \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}$ $E = k - eA$

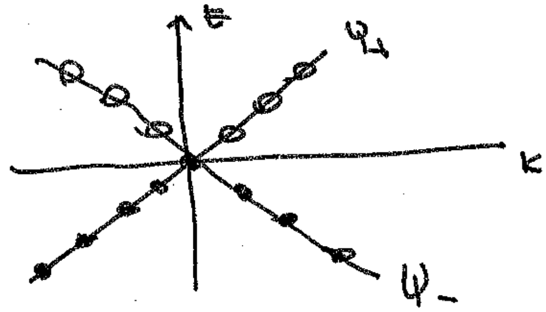
$\psi_- = \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}$ $E = -k + eA$

think of this as <u>DIRAC SEA</u>

for $A=0$, get <u>DISPERSION</u> $(E(k))$

$\psi_+ : E = k$ $\psi_- : E = -k$

E can be ±; assume neg E states are filled

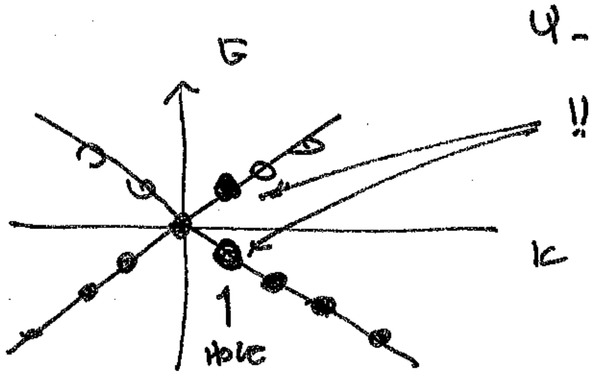


FILLED VACUUM STATES (antipart) holes

NOW CONSIDER $A = 0 \rightarrow A = \varepsilon$

then VACUUM states : $\psi_+ : k < e\varepsilon$ ← INCREASED ENERGY BY $e\varepsilon$

$\psi_- : k > e\varepsilon$ ← DECREASED ENERGY BY $e\varepsilon$



⇒ net PRODUCTION OF + CHIRALITY PARTICLES − CHIRAL ANTIPART.

$$\Delta\chi = 2\int_{-e\varepsilon}^{e\varepsilon} \frac{dk}{2\pi} = \frac{e\varepsilon}{2\pi}$$

CHANGE IN VACUUM STATE FROM EXT FIELD.