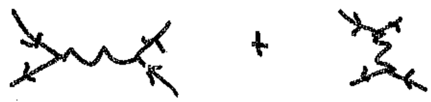
EXERCISE: (CREATED BY $a^\dagger_{\vec{p}}$)

- WHAT IS QFT? ← is it a theory of particles or of fields?

- in QFT I: you learned to "do this":



WHERE DO VERTICES COME FROM? ($\mathcal{L}$)

WHERE DO PROPAGATORS COME FROM?

$$\longrightarrow \sim \frac{i}{p^2 - M^2} \qquad - - - \sim \frac{i}{p^2 - m^2}$$

"inverting EOM"

... where does EOM come from?

($\delta S = 0$) ← where was $\boxed{S}$ in QFT I??

sure, $\mathcal{L}$ showed up...
BUT THERE ARE TIMES WHEN S MATTERS MORE.

this course:
- REDERIVE QFT, USING PATH INTEGRAL, complements canonical formalism
  - → why?
    - HELPS EXPLAIN PECULIARITIES OF HIGHER SPIN FIELDS
    - CONNECTS TO A MORE UNIVERSAL LANGUAGE OF THEORETICAL PHYS (eg statmech ⟷ QFT)
    - connects to nonpert/topo phenomena (not in this class)
- LOOPS: RENORMALIZATION
  wtf is it.
- GAUGE THY: wtf is GAUGE INV?
- ANOMALIES: wtf is THIS?

but to motivate what we're going to do, start w/ something (a priori) completely different.

CONSIDER THE GAUSSIAN:

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

mean

std dev.

this is as exptl as this class gets.

normalization ... because $g(x)dx$ IS A PROBABILITY DISTRIBUTION

$\int_a^b g(x)dx$ is $P(x \in [a,b])$

PROBABILITY DISTRIBUTION FUNCTIONS:

Moment : the n^{th} moment of a pdf $p(x)$ is

$$M_n \equiv \int_{-\infty}^{\infty} x^n \, p(x) \, dx = \langle x^n \rangle$$

think about: moment of inertia
angular momentum
dipole moment ...

a fancy way to mathematically write these out: GENERATING FUNCTION.

$$M(\xi) = \int e^{\xi x} \, p(x) \, dx$$

$$\left(1 + \xi x + \tfrac{1}{2}\xi^2 x^2 + \ldots\right)$$

$$= \underbrace{\langle 1 \rangle}_{M} + \xi \underbrace{\langle x \rangle}_{M_1} + \tfrac{1}{2}\xi^2 \underbrace{\langle x^2 \rangle}_{M_2} + \cdots$$

$$\Rightarrow M_n = \left.\frac{\partial^n M(\xi)}{\partial \xi^n}\right|_{\xi=0}$$

remark (useful for later)

cousin of GENERATING FUNCTION:
CHARACTERISTIC FUNCTION:

$$\phi(k) = \underbrace{\int_{-\infty}^{\infty} dx \; e^{ikx} \, p(x)}_{\equiv \tilde{p}(k)} = 1 + ikM_1 + \cdots$$

inverse fourier transf.
(USING P251 conventions)

$$\text{s.t.} \quad p(x) = \int dk \; \left(1 + ikM_1 + \frac{(ik)^2}{2!} M_2 \cdots\right) e^{-ikx}$$

from $p = \int dk \; \phi \, e^{-ikx}$

looks like an overall momentum conserving δ-function...

---

here's a cute trick: GAUSSIAN INTEGRAL

$$G = \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}x^2}$$ *how to evaluate?*

$$\begin{aligned} \Big[\; G^2 &= \int dx\,dy \; e^{-\frac{1}{2}(x^2+y^2)} \\ &= \int_0^{\infty} 2\pi \, r\,dr \; e^{-\frac{1}{2}r^2} \quad \leftarrow \quad w = \tfrac{1}{2}r^2 \quad dw = r\,dr \\ &= 2\pi \int_0^{\infty} dw \; e^{-w} \\ &= 2\pi \end{aligned}$$

$$\Rightarrow \boxed{G = \sqrt{G^2} = \sqrt{2\pi}}$$

RESCALING: $$\int dx \; e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}$$ chk: DIM ANALYSIS

HW: $$\int dx \; e^{-\frac{1}{2}ax^2 + \underset{\substack{\uparrow \\ \text{source}}}{Jx}} = \sqrt{\frac{2\pi}{a}} \, e^{J^2/2a}$$

$$\langle x \rangle = \frac{\delta}{\delta J} \sqrt{\frac{2\pi}{a}} \, e^{J^2/2a} \Big|_{J=0} = 0$$

$$\langle x^2 \rangle = \left(\frac{\delta}{\delta J}\right)^2 \sqrt{\frac{2\pi}{a}} \, e^{J^2/2a} \Big|_{J=0}$$

(chk fac of 2)

$$\text{from:} \quad -\tfrac{1}{2}ax^2 + Jx = -\tfrac{1}{2}a \underbrace{(x - J/a)}_{y}{}^2 + J^2/2a$$

Generalize to many dim

$$\int\cdots\int dx_1\cdots dx_N \; e^{-\frac{1}{2}x^i A_{ij} x^j} = \sqrt{\frac{(2\pi)^n}{\det A}} \equiv \mathcal{N}$$

$\uparrow$ q^i $\uparrow$ q^j

why? DIAGONALIZE W/ A ROTATION
MEASURE IS UNCHANGED

BECOMES $\int dy_1\cdots dy_n \; e^{-\frac{1}{2}a_i y_i^2} = \prod_i \sqrt{\frac{2\pi}{a_i}}$

$\uparrow$ EIGENVALS

then add sources : N-vector of sources

$$\int \underbrace{dx_1\cdots dx_N}_{d^N x \to d^N q} \; e^{-\frac{1}{2}\underline{x}\cdot\underline{\underline{A}}\cdot\underline{x} + \underline{J}\cdot\underline{x}} = \sqrt{\frac{(2\pi)^n}{\det A}} \; e^{\frac{1}{2}\underline{J}\cdot\underline{\underline{A}}^{-1}\cdot\underline{J}} \qquad (*)$$

$\uparrow q$ $\uparrow q$ $\uparrow q$

now this part is important :
CAN ASK FOR <u>CORRELATION</u> FUNCTIONS -- LIKE MULTI-MOMENTS:

$$\langle x_i x_j \rangle = \frac{\int d^N x \; e^{-\frac{1}{2}\underline{x}\cdot\underline{\underline{A}}\cdot\underline{x}} \, x_i x_j}{\int d^N x \; e^{-\frac{1}{2}\underline{x}\cdot\underline{\underline{A}}\cdot\underline{x}}} = A^{-1}_{ij} \qquad (\because)$$

q_i q_j q q q q

denominator is just
a normalization, gives $\mathcal{N}$

Pf
<u>TRICK</u> : USE function w/ SOURCE $(*)$

$$\frac{\delta}{\delta J^i}\frac{\delta}{\delta J^j}(\text{LHS})\Big|_{J=0} = \mathcal{N}\,\langle x_i x_j\rangle$$

$q_i \; q_j$

$$\text{---}\,''\,\text{---}\quad (\text{RHS})\Big|_{J=0} = \mathcal{N}\,\frac{\delta}{\delta J^i}\frac{\delta}{\delta J^j}\, e^{\frac{1}{2}J^k (A^{-1})_{kl} J^l}$$

$$= \mathcal{N}\,(A^{-1})_{ij}$$

now recall P231: GREEN'S FUNCTIONS
SOLVE LINEAR DIFF EQ.

WHERE DO LINEAR EOM COME FROM?

QUADRATIC LAGRANGIANS

LINEAR IN ϕ

$$L = \phi\, \mathcal{O}\, \phi \quad \longrightarrow \quad \mathcal{O}\phi = 0$$

eg $(\partial/\partial t)^2$

SOME DIFF OP, CONSTR. BY SYMMETRY & SUBJ. TO DIM. ANALYSIS.

WHAT IS A SOURCE? $\mathcal{O}\phi = J$

WHERE DOES IT COME FROM?

$$L = \phi\, \mathcal{O}\, \phi - J\phi \longrightarrow \boxed{\mathcal{O}\phi = J}$$

Solution: $\boxed{\phi = \mathcal{O}^{-1} J}$

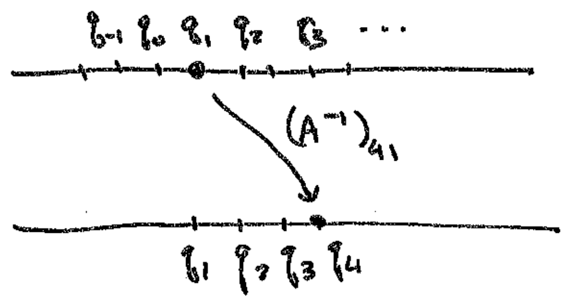
What is this?
GREEN'S FUNCTION
→ PROPAGATES INFORMATION

---

now go back to (☺, p.4)

$(A^{-1})_{ij}$ is the inverse of "OPERATOR"

→ GREEN'S FUNCTION
→ PROPAGATOR



what is A ? ⟵ almost ALWAYS $\boxed{\partial^2}$

$$\left(\frac{d}{dt}\right)^2 \quad \underbrace{-\left(\frac{\partial}{\partial \underline{x}}\right)^2}_{\substack{x \text{ takes} \\ \text{place of} \\ \text{index}}} \quad - m^2$$

↑ CONSTANT, but recall MASSES are quadratic in L

DISCRETE LIMIT:

$$A \sim \begin{pmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & & & \end{pmatrix} \qquad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \end{pmatrix}$$

REMEMBER - this is the discretized $(d/dx)^2$

continuum limit

$\boxed{q_i \longrightarrow q(x)}$ $\quad$ $\boxed{A \rightarrow (\partial/\partial x)^2}$

↑ q becomes a <u>field</u>.

(by the way - this <u>is</u> statistical field theory)

$(A^{-1})_{ij}$ PROPAGATES INFO

↳ becomes $\frac{i}{p^2 - m^2}$ in momentum space

or $\sim \frac{1}{4\pi r} e^{-mr}$ in position space

NOW GO BACK TO $(*)$

def: $Z_0 = \int d^N q \; e^{-\frac{1}{2} q \underline{A} q} = \sqrt{\frac{(2\pi)^N}{\det A}} = \mathcal{N}$

$$Z[J] = \int \text{———''———} \; e^{Jq}$$

$$= e^{-\frac{1}{2} J (\underline{\underline{A^{-1}}}) \underline{J}} \cdot \mathcal{N}$$

EXPECTATION VALUES:

$$\langle q_i \rangle = \frac{1}{\mathcal{N}} \int d^N q \; q_i \; e^{-\frac{1}{2} q \underline{A} q}$$

$$= \frac{1}{Z_0} \frac{\delta Z[J]}{\delta q_i} \Big|_{J=0}$$

$$\Rightarrow \boxed{\langle f(q) \rangle = \frac{1}{Z_0} f\left(\frac{\delta}{\delta q}\right) Z[J] \Big|_{J=0}}$$

now add "interactions" — eg $\sum_i \lambda q_i^4$

$$Z_V[J] = \int d^N q \; e^{-\frac{1}{2} q A q + Jq - V(q)}$$

← not quad. (non-linear!)

$$= \int d^N q \; e^{-V\left(\frac{\delta}{\delta J}\right)} \left[e^{-\frac{1}{2} q A q + Jq} \right]$$

$$= e^{-V(\delta/\delta J)} \int d^N q \; [\cdots]$$

$$= e^{-V(\delta/\delta J)} \left[\mathcal{N} \; e^{-\frac{1}{2} J (\underline{\underline{A^{-1}}}) J} \right]$$

EXPAND THIS IN PERT. EXP.

eg $V(q) = \sum_i \lambda (q_i)^4$ ← not $(\sum q_i)^4$!!

ASSUME λ IS SMALL
(check dimensions ...)

$$Z_v[J] = Z[J] + \lambda \left(-\sum_i \left(\frac{\delta}{\delta J_i}\right)^4 \right) \mathcal{N} e^{-\frac{1}{2} J \underline{A}^{-1} J} + \dots$$

NOW SUPPOSE I WANT $\langle q_1 q_2 q_3 q_4 \rangle$

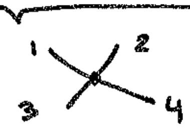
↳ correlation of 4 points
BIG IF ALL 4 POINTS ARE NONZERO W/ SAME SIGN IN SAME REGION OF pdf.

$$\underbrace{\langle q_1 q_2 q_3 q_4 \rangle}_{\text{ASSUME ALL DISTINCT}} = \frac{1}{\mathcal{N}} \int d^N q \; q_1 q_2 q_3 q_4 \, e^{-\frac{1}{2} q A q - V(q)}$$

$$= \frac{1}{Z_0} \underbrace{\frac{\delta^4 Z_v[J]}{\delta q_1^4}}_{\delta_{q_1} \delta_{q_2} \delta_{q_3} \delta_{q_4} Z_v[J]} \Big|_{J=0}$$

$$= \frac{1}{Z_0} \delta_{q_1} \cdots \delta_{q_4} \left(Z[J] - \lambda \sum_i \delta^4_{q_i} \mathcal{N} e^{-\frac{1}{2} J A^{-1} J} + \dots \right)$$

$$- \lambda \sum_i \underbrace{\delta_{q_1} \cdots \delta_{q_4} \delta^4_{q_i} e^{-\frac{1}{2} J A^{-1} J}}$$



QM $H = \frac{1}{2m} P^2 + V(q)$

$$i \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle \Rightarrow |\psi(t)\rangle = e^{-i\hat{H}t} |\psi_0\rangle$$

POSITION STATE: $\hat{q}(t) \, |q,t\rangle = q \, |q,t\rangle$

S.T. WAVEFUNCTION IS $\psi(q,t) = \langle q | \psi(t) \rangle$

$\&$ $\hat{H} \rightarrow -\frac{1}{2m} \frac{d^2}{dq^2} + V(q)$ ACTING ON $\psi(q,t)$

PATH INTEGRAL:

$$\psi(q,t) = \langle q | \; e^{-i\hat{H}t} \; | \psi_0 \rangle$$

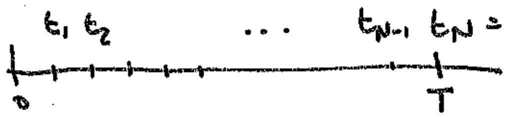
insert $\int dq_0 \, |q_0\rangle\langle q_0| = \mathbb{1}$

$$= \int dq_0 \, \underbrace{K(q, q_0; t)}_{} \, \psi(q_0, 0)$$

GREEN'S FUNCTION

$K = \langle q | e^{-i\hat{H}t} | q_0 \rangle$

IN FACT, BREAK UP INTO LITTLE TIME SLICES:



$$e^{-i\hat{H}T} = e^{-i\hat{H}(t_N - t_{N-1})} \, e^{-i\hat{H}(t_{N-1} - t_{N-2})} \cdots e^{-i\hat{H}(t_1 - 0)}$$

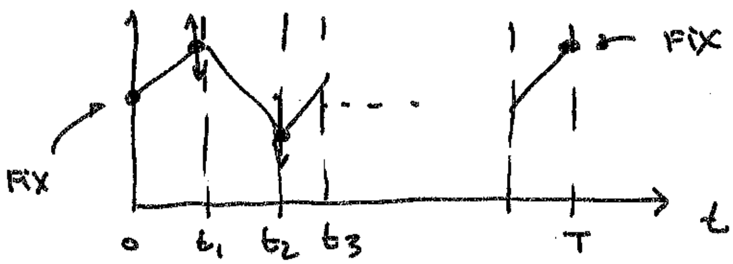
INSERT COMPLETE SETS OF STATES

$$K(q, q_0 ; T) = \int d^N q \prod_r \langle q_{r+1} | e^{-i\hat{H}\underbrace{(t_{r+1} - t_r)}_{\delta t}} | q_r \rangle$$

finite time

what are we integrating?

ALL POSSIBLE INTERMEDIATE STATES



integrate over "abacus positions"

EVALUATE FOR SMALL δt IN $\boxed{V = 0 \text{ LIMIT}}$

$$K_0(q, q' ; t) = \langle q | e^{-i\hat{p}^2/2m\, t} | q' \rangle$$

USE $\langle q | p \rangle = e^{ipq}$

$\langle p | q' \rangle = e^{-ipq'}$

will insert

$\mathbb{1} = \int dp\, |p\rangle\langle p|$

$$K_0(q, q' ; t) = \int dp\, \langle q | e^{-i\frac{\hat{p}^2}{2m}t} | p \rangle \langle p | q' \rangle$$

$$= \int dp\, e^{-i\frac{p^2}{2m}t}\, \underline{e^{ip(q-q')}}$$

Gaussian

$$= \sqrt{\frac{m}{2\pi i t}}\, e^{i\frac{m(q-q')^2}{2t}}$$

in $t \to 0$ limit we recover δ

Restoring $V(q)$

↳ note that $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$ $\quad [\hat{q}, \hat{p}] \neq 0$

BUT: $e^{\varepsilon(\hat{A}+\hat{B})} = e^{\epsilon \hat{A}} e^{\epsilon \hat{B}} (1 + O(\varepsilon^2))$

this is why we slice up into small δt

$$\langle q_{r+1} | e^{-i\hat{H}\delta t} | q_r \rangle = \langle q_{r+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} e^{-iV(\hat{q})\delta t} | q_r \rangle$$

$$= \sqrt{\frac{m}{2\pi i \delta t}}\, e^{\frac{i}{2} m \left(\frac{q_{r+1}-q_r}{\delta t}\right)^2 - iV(q_r)\delta t}$$

minus!

or N+1?

$$K(q, q_0; T) = \left(\frac{m}{2\pi i \delta t}\right)^{N/2} \int \prod_r e^{i \text{———}}$$

$$\equiv \int \mathcal{D}q \; e^{iS[q]}$$

↳ see OSBORN §1.