Homework 5a: Damped Harmonic Oscillator Example

Course: Physics 231, *Methods of Theoretical Physics* (2018)
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Let's go back to the damped harmonic oscillator from homework 4:

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t) \ . \tag{0.1}$$

We would like to find the solution to the forced, damped spring with some specific force function $F(t)$. In homework 4, you showed that the Green's function is:

$$G(t-t') = \frac{e^{-\gamma(t-t')}\sin\left[\sqrt{\omega_0^2-\gamma^2}\,(t-t')\right]}{\sqrt{\omega_0^2-\gamma^2}} \ . \tag{0.2}$$

Caveat emptor: there may be overall minus sign errors in this formula or in lecture.

The full solution of this system is $x(t)$, given by

$$x(t) = Ax_1(t) + Bx_2(t) + \int_{t_1}^{t_2} \frac{e^{-\gamma(t-t')}\sin\left[\sqrt{\omega_0^2-\gamma^2}(t-t')\right]F(t')}{\sqrt{\omega_0^2-\gamma^2}} dt' \ , \tag{0.3}$$

where $x_{1,2}(t)$ are solutions to the *homogeneous* differential equation, that is (0.1) with $F(t) = 0$. We may take the upper limit of integration to be $t_2 = t$ (why?). The lower limit, t_1 can be taken to be the time at which the driving force is applied. (Confirm for yourself that this is "obvious.")

1 Initial Conditions

The coefficients A and B are determined by the initial conditions of the problem. Even without knowing the precise form of $x_{1,2}(t)$, suppose we know that $x(t_1) = 0$. What are the values of A and B?

2 Exponentially falling blip

For simplicity[1], set $t_1 = 0$. Suppose that the time-dependent driving force is

$$F(t) = F_0 e^{-\alpha t} \ . \tag{2.1}$$

Assume the initial conditions above. Solve (0.3) by performing the integral. No need to do anything fancy like a contour integral. You may find it useful to write the sine as a sum of exponentials. Show that the solution is

$$x(t) = \frac{F_0}{\sqrt{\omega_0^2-\gamma^2}} \frac{\sin\left[\sqrt{\omega_0^2-\gamma^2}t - \delta\right]}{\sqrt{\omega_0^2+\alpha^2-2\alpha\gamma}} e^{-\gamma t} + \frac{F_0}{\omega_0^2+\alpha^2-2\alpha\gamma} e^{-\alpha t} \ . \tag{2.2}$$

[1] Convince yourself that this is essentially a shift in coordinates and does not change any physics.

Here we've defined

$$\tan\delta = \frac{\sqrt{\omega_0^2-\gamma^2}}{\alpha-\gamma} \ . \tag{2.3}$$

If you don't believe this solution, check it by plugging into (0.1). Write out the limiting form when the damping goes to zero, $\gamma \to 0$. Show that in the limit of negligible damping and for 'late times', that

$$x(t) = \frac{F_0}{\omega_0}\frac{\sin(\omega_0 t-\delta)}{\sqrt{\omega_0^2+\alpha^2}} \ . \tag{2.4}$$

What does 'late times' mean in this context? Identify the quantity with dimension T (time) that you can use to define 'late.' What does this mean physically? Explain what's happening as one of the terms in (2.2) vanishes.

3 Energy of the system

If the system initially has zero energy—confirm that this is consistent with the boundary conditions that set $A = B = 0$—show that the energy of the system at late times is

$$E = \frac{F_0^2}{2(\omega_0^2+\alpha^2)} \ . \tag{3.1}$$

HINT: recall that the energy for the system is $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2$.