ANNOUNCE: Ian OH TUE 1-3 pm (3004 S)

Index cards: many responses: "quality research"

WHAT DOES THAT MEAN TO YOU?

how will people decide if your research is quality

... bigger & more immediate things to discuss on monday pm.

follow up from last time

BRIAN'S QUESTION: $D^n \underline{f} = 0$

eg: $f(x) = ax^2 + bx + c$

I can make this into a "discrete" vector space

$$\underline{f}_P = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

POLYNOMIAL BASIS

where: $\left.\begin{aligned} \underline{e}_{(1)} &= x^2 \\ \underline{e}_{(2)} &= x \\ \underline{e}_{(3)} &= 1 \end{aligned}\right\}$ is this an orthonormal basis?

(we'll talk about that

clearly, $\left(\frac{d}{dx}\right)^3 f(x) = 0$

$$D \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$$

$$D^2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} ; \quad D^3 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

BUT MANY OTHER BASES. A NICE ONE THAT WE WILL USE IS THE FOURIER BASIS & IT'S RELATIVES.

ie momentum space

eg

$$\underline{e}_{(1)} = \sin(x)$$
$$\underline{e}_{(2)} = \sin(2x)$$
$$\vdots$$

$$\Longleftrightarrow \quad D^n \underline{e}_{(1)} \neq 0$$

LINEAR TRANSFORMATIONS

$\lambda\ \omega$ just #'s

$$t(\lambda \underline{\underline{A}} + \omega \underline{\underline{B}})\underline{v} = \lambda \underline{\underline{A}}\,\underline{v} + \omega \underline{\underline{B}}\,\underline{v}$$

Q: what is an example of a transformation that is not linear?

eg common misconception that

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial x}\right)^2 \right] f(x,t) = g(x,t)$$

is somehow NOT linear.

A KEY POINT: LINEAR TRANSF. ACT ON BASIS VECTORS

$$\underline{V} = v^i\, \underline{e}_{(i)}$$

carries all the vector-ness

just #

WHAT IS A BASIS OF LINEAR TRANSFORMATIONS ?

does this even make sense ?!

go back to row vectors / 1-forms / ...

$\underline{v}^T$: object that takes vector, spits out number.

this has a basis.

sometimes written $\underline{e}^{T(i)}$ or $\underline{e}^{*(i)}$ or $\underset{\sim}{e}^{(i)}$

let me use, for this lecture:

$ᗧ^i$ ← PACMAN

$$\underline{w}^T = (w_1, w_2, \ldots) = \sum_i w_i ᗧ^i$$

WHERE $ᗧ^i$ IS DEFINED TO EAT $\underline{e}_{(i)}$ & SPIT OUT 1.

$$ᗧ^j \underline{e}_{(i)} = \delta^j_i = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

$$ᗧ^1 \underline{e}_{(2)} = 0$$

$$ᗧ^5 \underline{e}_{(5)} = 1$$

$$\underline{w}^T \underline{v} = (w_1 \beta^1 + w_2 \beta^2 + \cdots)(v^1 \underline{e}_1 + \cdots)$$

no dot!
(this is $\underline{w} \cdot \underline{v}$)

$$= w_1 v^1 \overbrace{\beta^1 \underline{e}_1}^{=1} + w_1 v^2 \overbrace{\beta^1 \underline{e}_2}^{=0} + \cdots$$
$$+ w_2 v^1 \underbrace{\beta^2 \underline{e}_1}_{=0} + w_2 v^2 \underbrace{\beta^2 \underline{e}_2}_{=1} + \cdots$$

$$= w_1 v^1 + w_2 v^2 + \cdots$$

so we get $\underset{\text{"}w_i\text{"}}{\underline{w}^T} \underset{\text{"}v^j\text{"}}{\underline{v}} = \sum_i w_i v^i$

BRA-KET VERSION

$$\beta^i \longrightarrow \langle i | \qquad \underline{e}_j \longrightarrow | j \rangle$$

$$\langle i | j \rangle = \delta^i_j$$

$$\left.\begin{aligned} |v\rangle &= \sum_j v^j |j\rangle \\ \langle w| &= \sum_i w_i \langle i| \end{aligned}\right\} \langle v | w \rangle = w_i v^i$$

MATRIX (linear transformation)

$w_i \langle i|$ is a machine that takes a vector & spits out #

LIN TRANSF: A MACHINE THAT TAKES A VECTOR & SPITS OUT A VECTOR

oh. we have a basis for this

$\underline{e}_k \cdots$  i

eats a vector (nonzero for vector w/ nonzero $\underline{e}_i$)

POOPS OUT A VECTOR! gives one unit in the $\underline{e}_k$ direction.

in bra-ket notation: BASIS IS

$$|k\rangle\langle i|$$

$$\underline{\underline{A}} = A^k{}_i \, |k\rangle\langle i|$$

implicit sum over i & k

$$\underline{\underline{A}}\ \underline{V} = \left(\underbrace{A^i{}_k}\ |i\rangle\langle k|\right)\left(\underbrace{V^j}\ |j\rangle\right)$$

just numbers, pull them out

$$= \sum_{i,j,k} A^i{}_k V^j\ |i\rangle \underbrace{\langle k|j\rangle}_{\delta^k_j}$$

$$= \sum_{i,j} A^i{}_j V^j\ |i\rangle$$

$$= \sum_i \left(\sum_j A^i{}_j V^j\right)\ |i\rangle$$

what we had before.
the i^{th} element of $(\underline{A}\,\underline{V})$

$$\begin{pmatrix} A^1{}_1 & A^1{}_2 & \cdots \\ \vdots & & \\ A^i{}_1 & A^i{}_2 & \cdots \end{pmatrix}\begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ A^i{}_1 V^1 + A^i{}_2 V^2 + \cdots \end{pmatrix}$$

<u>think</u> : BASIS for <u>Multilinear transf</u>
that takes a vector & spits out
a linear transformation?

WHY ARE WE DOING THIS?

LINEAR TRANSF. ACT ON BASIS

things that carry vector nature

IN A NICE BASIS, LIN TRANSF ACTS SIMPLY.

$$\underline{\underline{A}}\, \underbrace{\underline{\xi}_{(i)}}_{\text{BASIS OF EIGENVECTORS}} = \lambda_i \underline{\xi}_{(i)}$$

$$\underline{V} = v^i \underline{\xi}_{(i)}$$

coeff. in eigenbasis

$$\underline{\underline{A}}\, \underline{V} = \sum_i \underbrace{\lambda_i V^i}_{} \underline{\xi}_{(i)}$$

just rescale these coeff. w/ eigenvals.

HOW WE'LL USE THIS:

e.g. $\underline{\underline{A}} = \partial^2$

A NICE BASIS: $\xi_{(k)} = \sin(kx)$

$$\underline{\underline{A}}\, \underline{\xi}_{(k)} = \underbrace{-k^2}_{\lambda_k} \underline{\xi}_{(k)}$$