So far: ANALYTIC → nice → DIFFERENTIABLE → TAYL

$\partial_x u = \partial_y v$; $\partial_y u = -\partial_x v$

$f = f(z)$

too nice

$$\oint_c f(z)\, dz = 0$$

if f IS ANALYTIC IN THE ENCLOSED REGION.

MEROMORPHIC → nice enough, not too nice

ANALYTIC UP TO SINGULAR POINTS.
IDEA: just avoid those points
↳ like introducing a topology

mesomorphic

$$\oint_c f(z)\, dz = \sum_{\substack{j:\text{POLES}\\ \text{ENCL.}\\ \text{BY } C}} 2\pi i \ \mathrm{Res}_f(z_j)$$

position of pole

rather than Taylor expansion, LAURENT EXP

$$f(z) = \underbrace{\cdots \frac{a_{-2}(z_0)}{(z-z_0)^2} + \boxed{\frac{a_{-1}(z_0)}{z-z_0}}}_{} + \underbrace{a_0 + a_1(z-z_0) + \cdots}_{\text{TAYLOR EXPANSION}}$$

terms that encode non-analytic behavior

$$\mathrm{Res}_f(z_j) = a_{-1}(z_j)$$

have to expand about z_j

sketch "PROOF" (motivation)

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz$$

C enclosing z_0

$$\boxed{\mathrm{Res}_{\left(\frac{f(z)}{z-z_0}\right)}(z_0)}$$

MAPS TO

$z(\theta) = z_0 + \varepsilon e^{i\theta}$

$dz = i\varepsilon e^{i\theta}$

what if integrand were

$b \in \mathbb{Z}$

$f(z) \cdot (z-z_0)^b \qquad b > -1 \quad ?$

or $\quad f(z) \cdot (z-z_0)^b \qquad b < -1 \quad ?$

then: $\int_0^{2\pi} \underbrace{f(z)}_{\approx f(z_0)} \, i\,(\varepsilon e^{i\theta})^{1+b}$

$= \boxed{0 \quad \text{if} \quad b \neq -1}$

coef. when exp about z_0

so if $\displaystyle g(z) = \sum_{n=-\infty}^{\infty} \overline{a_n(z_0)} \cdot (z-z_0)^n$

only $n=-1$ term survives $\oint_C g(z)\, dz$

C enclosing z_0

z_0 that gives the RESIDUE :

$$\mathrm{Res}_g\, z_0 = a_{-1}(z_0)$$

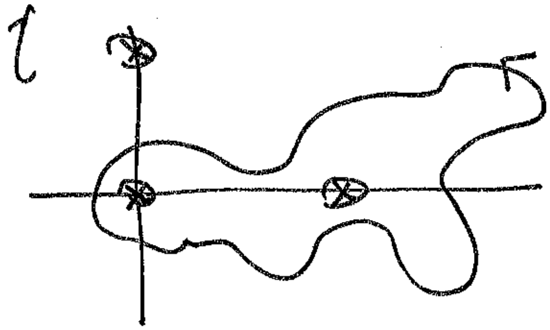
EXAMPLES

$$f(z) = \frac{1}{z(z-1)(z+i)}$$

POLES: $z = 0, \quad +1, \quad -i$

$$\mathrm{Res}_f(0) = \frac{1}{(-1)(i)} \qquad \mathrm{Res}_f(1) = \frac{1}{1+i} \qquad \mathrm{Res}_f(-i) = \frac{1}{-i(-i-1)}$$

$$\oint_C f(z)\,dz = 2\pi i \left(\mathrm{Res}_f(0) + \mathrm{Res}_f(1) \right)$$



$$= 2\pi i \left(i + \frac{1}{1+i} \right)$$

$$= 2\pi i \left(\frac{i-1+1}{1+i} \right)$$

$$= \boxed{\frac{-2\pi}{1+i}}$$

if my arithmetic is correct

WHAT ABOUT: $f(z) = \dfrac{1}{(z-2i)^2}$ ← 2^{ND} ⊖ POLE (ALEX'S Q)

naive: no simple pole → $\mathrm{Res}_f(2i) = 0$

... correct!

MORE SUBTLE :

$$f(z) = \frac{z}{(z-2i)^2}$$

$$\mathrm{Res}_f(2i) \neq 0 \quad (!)$$

HOW TO SEE: $f(z)$ is not a LAURENT EXP. even though it captures pole structure.

$$f(z) = \frac{h(z)}{(z-z_0)^2} \leftarrow \underbrace{h(z_0) + h'(z_0)(z-z_0)}_{= z_0 + (z-z_0)}$$

$$= \frac{z_0}{(z-z_0)^2} + \underbrace{\frac{1}{(z-z_0)}}$$

} this is a LAURENT EXPANSION

$$\boxed{\mathrm{Res}_f(z_0) = 1}$$

you can generalize this in HW.

eg. $f(z) = \cot(z)$ find $\mathrm{Res}_f(0)$

$$\frac{\cos(z)}{\sin(z)} \begin{array}{l} \leftarrow 1+\cdots \\ \leftarrow z+\cdots \end{array}$$

simple pole @ 0

JUST PLUG IT

$$\mathrm{Res}_f(0) = \lim_{z\to 0} \frac{z\cos z}{\sin z} = \cos 0 \lim_{z\to 0} \frac{z}{\sin z} = \boxed{1}$$

eg. $f(z) = \cot^2 z$ find $\mathrm{Res}_f(0)$

$$\frac{(1 - z^2/2 + \cdots)^2}{(z - z^3/3! + \cdots)^2} \sim \frac{1}{z^2} + O(1)$$

or: $$\sim \frac{\cos^2 z}{z^2} = \frac{\cos^2 0}{z^2} + \underbrace{\frac{z(-2\cos(0)\sin(0))}{z^2}}_{\text{RESIDUE TERM B/C } \sin(0) = 0} + \cdots$$

$$\boxed{\mathrm{Res}_f(0) = 0}$$

eg. $f(z) = z\cot^2 z$ find $\mathrm{Res}_f(0)$

$$\sim \frac{z\cot^2 z}{z^2} = \underbrace{\frac{\cos^2 0}{z}}_{} + \cdots$$

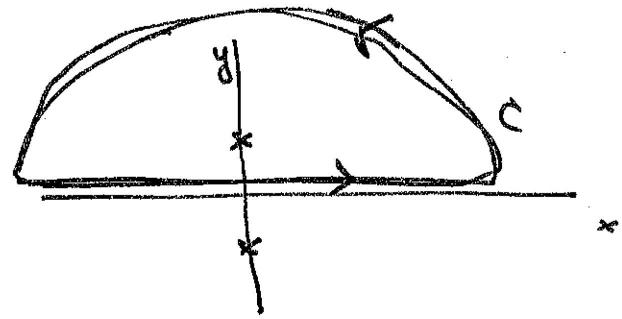
$$\boxed{\mathrm{Res}_f(0) = 1}$$

Something useful : $\mathbb{R}$ integrals

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = ?$$

LET'S TRY SOMETHING RELATED:

$$\oint_C \frac{dz}{z^2+1}$$



POLES @ $z = \pm i$

$$= 2\pi i \underbrace{\mathrm{Res}_f(i)}_{\frac{1}{2i}} = \boxed{\pi}$$

BUT: CAN BREAK APART:

$$\oint_C dz = \underbrace{\int_{-R}^{R} dx}_{R\to\infty} + \underbrace{\int_0^{\pi} d\theta}_{\substack{\text{along} \\ r=R}}$$

← $z = Re^{i\theta}$

$$\underset{\pi}{\underset{\parallel}{\oint_C f(z)\, dz}} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} + \underbrace{\int_0^{\pi} \frac{iRe^{i\theta}\, d\theta}{R^2 e^{2i\theta}+1}}_{\sim \frac{d\theta}{R} \to 0}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi}$$

OTHER NOTES:
See LEC 14 FROM 2019