Green's functions in diverse dimensions
$\hookrightarrow$ qualitative discussion.

$$\left[\left(\frac{d}{dt}\right)^2 + \cdots\right] f(t) = s(t)$$

$$\Downarrow$$

$$\star \begin{cases} \left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \frac{\partial^2}{\partial \underline{x}^2}\right] \varphi(\underline{x},t) = \rho(\underline{x},t) \\ \left[\qquad '' \qquad\right] \underline{A}(\underline{x},t) = \underline{j}(\underline{x},t) \end{cases}$$

clearly: picking units where $c = \varepsilon_0 = \mu_0 = 1$ is useful.

PDE: many types of derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots$
LINEAR: $\mathcal{O}$ (vec) = (vec)

SPECIAL RELATIVITY (built in!)

$$A_\mu = (\varphi, \underbrace{A_1, A_2, A_3}_{\underline{A}})$$

$$j_\mu = (\rho, \underbrace{j_1, j_2, j_3}_{\underline{j}})$$

RECALL inner product for Minkowski space:

some vector B $\rightarrow$ $B^\mu B_\mu \doteq B^2 = (B^0)^2 - (B^1)^2 - (B^2)^2 - (B^3)^2$

s.t. $\partial^2 = \partial_t^{\,2} - \underline{\partial}_x^{\,2}$

$$\star \Rightarrow \boxed{\partial^2 A_\mu = j_\mu}$$

still 4 EQNS for 4 functions each of 4 variables so maybe in our lives we might want the GREEN'S FUNC for ∂^2

$$\boxed{\partial^2 G(x, x') = \underbrace{\delta^{(4)}(x-x')}_{= \delta(t-t')\,\delta(x-x') \ldots}}$$

$x^\mu, (x')^\mu$

how many eqns? $\longrightarrow$ one!

given source $j_\mu(x)$, solution to VAC. WAVE EQ is

$$A_\mu(x) = \int d^4x' \, \boxed{G(x, x')} \, j_\mu(x')$$

one func.

four eqns

$\boxed{\text{nb}}$: we haven't specified coordinates ... usually some hybrid of polar + cartesian

how to do it?
PRETTY MUCH THE SAME

$$\delta^{(4)}(x-x') = \int \text{đ}E \; e^{-iE(t-t')} \cdot \int \text{đ}k_x \; e^{+ik_x(x-x')} \cdot \int \text{đ}k_y \; e^{+ik_y(y-y')} \cdot \int \text{đ}k_z \; e^{+ik_z(z-z')}$$

CHOICES!! Convention of Fourier transform

choose wisely $\rightarrow$ manifest Lorentz invariance so we can write:

$$e^{-ik\cdot x} = e^{-i(Et - \underline{k}\cdot\underline{x})}$$

$(E, \underline{k})$ $\quad$ $(t, \underline{x})$

$$\left[\left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial \underline{x}}\right)^2\right] e^{-ik\cdot x}$$

Green's func:

SOURCE POS.

$$\partial^2 \int d^4k \, e^{-ik\cdot x} \tilde{G}(k, x') = \int d^4k \left(-E^2 + \underline{k}^2\right) \times e^{-ik\cdot x} \tilde{G}(k, x')$$

$dE\,dk_x\,dk_y\,dk_z$ $\quad$ (E, k_x, k_y, k_z)

$$= \int d^4k \, e^{-ik(x-x')}$$

$$\Rightarrow \tilde{G}(k, x') = \frac{-e^{ikx'}}{E^2 - \underline{k}^2}$$

$$\boxed{G(x, x') = \int d^4k \, \frac{-e^{-ik(x-x')}}{E^2 - \underline{k}^2}}$$

you can do this integral.
IT'S HARD the first time.
HELPS TO SEE IT DONE IN A BOOK.

USEFUL COORDS: hypercylindrical ← ALIGN k_z W/ $\underline{x}-\underline{x}'$

trivial

$$d^4k = dE \, d|\underline{k}| \, d\cos\theta \, d\varphi$$

$|\underline{k}|^2$ $\quad$ $(\underline{x}-\underline{x}')\cdot\underline{k} = |\underline{x}-\underline{x}'|\,|\underline{k}|\cos\theta$

SPHERICAL IN SPACE

<u>SKETCH</u> :

$$d\varphi \longrightarrow 2\pi$$

$d\cos\theta \longrightarrow$ also simple. write as

$$\int_{-1}^{1} d\cos\theta = \int_{-1}^{1} dc$$

c only shows up in

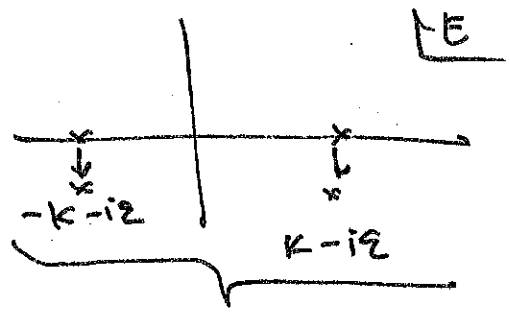
$$\underline{k}\cdot(\underline{x}-\underline{x}') = |\underline{k}|\,|\underline{x}-\underline{x}'|\,c = krc$$

$$\int_{-1}^{1} dc\; e^{-iE\Delta t}\, e^{i\overset{krc}{\cancel{|\underline{k}||\Delta\underline{x}|c}}}$$

$$= e^{-iE\Delta t}\cdot \underbrace{\frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)}$$

$$G(x,x') = \frac{2\pi}{(2\pi)^4}\int d\overset{|\underline{k}|}{k}\, dE\; k^2 \times \boxed{} \times \frac{-e^{-iE\Delta t}}{E^2-k^2}$$

$$= \frac{1}{8\pi^3\underset{r}{|\Delta\underline{x}|}}\int_0^\infty dk\;\frac{k}{i}\left(e^{ikr}-e^{-ikr}\right)\underbrace{\int dE\;\frac{-e^{-iE\Delta t}}{E^2-k^2}}_{\text{familiar!}}$$



$$\int dE\ldots = \frac{-\pi i}{k}\left(e^{ik\Delta t} - e^{-ik\Delta t}\right)\Theta(\Delta t)$$

$$G(x,x') = \frac{1}{8\pi^2 r}\int_0^\infty dk\,\Theta(\Delta t)\left[\,e^{ik(r+\Delta t)} - e^{ik(r-\Delta t)} - e^{ik(-r+\Delta t)} + e^{ik(-r-\Delta t)}\right]$$

k is POSITIVE

$e^{+ik(r-\Delta t)}$ w/ negative k

$e^{+ik(r+\Delta t)}$ w/ neg. k

$$= \frac{1}{8\pi^2 r}\int_{-\infty}^{\infty} dk\,\Theta(\Delta t)\left[\,e^{ik(r+\Delta t)} - e^{ik(r-\Delta t)}\right]$$

nb. $\int dk\, e^{ikx} = \delta(x) \times 2\pi$

$$= \frac{\Theta(\Delta t)}{4\pi r}\left[\,\delta(r-\Delta t) - \delta(r+\Delta t)\right]$$

famous 4π

$r > 0$ radial coord

$\Delta t > 0$ by causality

so this $= 0$

CHARGED CAT GREENS F

$$= \boxed{\frac{1}{4\pi r}\,\Theta(\Delta t)\,\delta(r-\Delta t)}$$

$c\Delta t$

BLIP

things to think, about.

dimension of G?

$[G] = 2$ ← from $\frac{1}{r}\delta(r\ldots)$

another way to see it:

∂^2 is always $[\partial^2] = 2$

$\partial^2 G = \delta^{(d)}(x-x')$

2 DIM d

$\boxed{\text{DIM } d-2}$

so in 3D, GREEN'S FUNCTION is DIFFERENT!

↳ in 3 spatial: $G = \frac{1}{4\pi r}$ ✓

in 2+1 : G "totally" different!

↳ result:

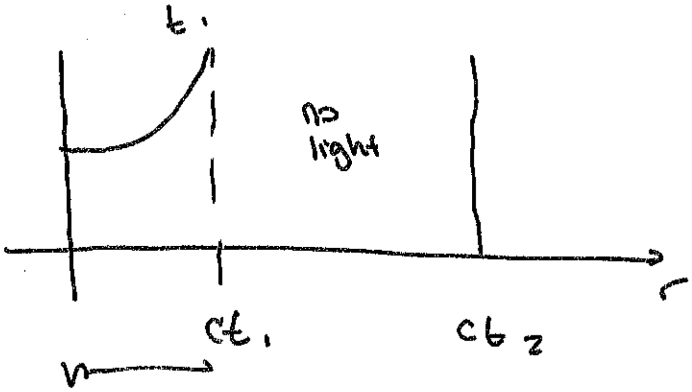
$$G = \frac{1}{2\pi}\sqrt{\frac{1}{(\Delta t)^2 - r^2}}\ \Theta(r - \Delta t)$$

compare to 3D w/ WAVE FRONT!!

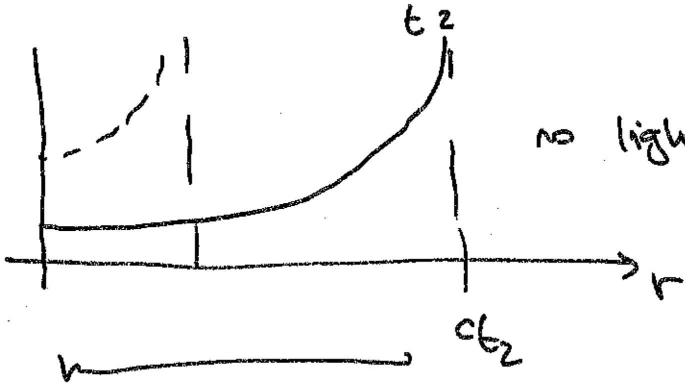
nb: 2D is very special.

2+1 DIM

flash of light
@ $t=0$, $r=0$



this entire volume is illuminated @ t_1 !!
(not just surface)



BIGGER VOL. ILLUMINATED
↳ BUT DIMMER (cons. of E)