Questions?

<u>SO FAR:</u> VECTORS (KET) $\in V$ vector space

ROW VECTORS (BRA) $\in V^*: \underbrace{V \to \#}$

linear func. of V.

<u>NOW:</u> what is transpose/adjoint/Hermitian conjugate?

↳ way of turning vectors into row vec. & vice versa.

(gives a sense of there being only one type of object)

TO UNDERSTAND THIS, WE NEED TO INTRODUCE AN ADDITIONAL MATHEMATICAL OBJECT:

dot product / inner product / metric

$$\underline{v} \cdot \underline{w} \qquad \underbrace{\langle \underline{v}, \underline{w} \rangle} \qquad g(\underline{v}, \underline{w})$$

$$\langle \underline{v} | \underline{w} \rangle \leftarrow \text{suggestive!}$$

<u>PROPERTIES:</u> $V \times V \to \#$ takes 2 vecs, spits out #

bilinear $\begin{cases} g(\underline{v}+\underline{v}', \underline{w}) = g(\underline{v},\underline{w}) + g(\underline{v}',\underline{w}) \\ g(\underline{v}, \underline{w}+\underline{w}') = g(\underline{v},\underline{w}) + g(\underline{v},\underline{w}') \end{cases}$

[conjugate] symmetric $g(\underline{v}, \underline{w}) = g(\underline{w}, \underline{v})^*$

simplest eg: <u>EUCLIDEAN SPACE</u> in <u>CARTESIAN COORDS</u>

$$\underline{v} \cdot \underline{w} = \langle \underline{v}, \underline{w} \rangle = v^1 w^1 + v^2 w^2 + v^3 w^3 + \cdots$$

<u>SPECIFIC</u> to this example

more generally (I WILL SWAP btwn $\cdot$, $\langle , \rangle$, g TO MAKE AS CLEAR AS POSS.)

$$g(\underline{v}, \underline{w}) = g(v^1\underline{e}_1 + v^2\underline{e}_2 + \cdots, w^1\underline{e}_1 + w^2\underline{e}_2 + \cdots)$$
$$= v^1 w^1 g(\underline{e}_1, \underline{e}_1) + v^1 w^2 g(\underline{e}_1, \underline{e}_2) + \cdots$$
$$+ v^2 w^1 g(\underline{e}_2, \underline{e}_1) + v^2 w^2 g(\underline{e}_2, \underline{e}_2) + \cdots$$
$$\vdots$$

FOR <u>ORTHONORMAL BASES</u> (nice bases)

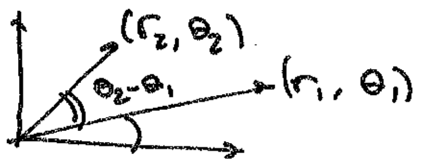
$$g(\underline{e}_i, \underline{e}_j) = \pm \delta^i_j$$

<u>signature</u> (has to do w/ 'space' vs. 'time')

but not true in general
EG: POLAR COORDINATES, CURVED SPACES } not our concern (see P208)

there are a lot of subtleties here
$\hookrightarrow$ eg how does this make sense for polar coordinates?

END UP GOING INTO SOME DIFF. GEOMETRY
hint: related to the idea that "<u>POSITION VECTORS</u> DO NOT EXIST"



$$\langle \underline{v}_1, \underline{v}_2 \rangle = r_1 r_2 \cos(\theta_2 - \theta_1)$$

think about this. :)

IN INDICES:

$$g(\underline{v}, \underline{w}) = v^i g_{ij} w^j$$

$\quad\quad\quad\quad\quad\quad\quad\quad \llcorner\, g(\underline{e}_i, \underline{e}_j)$

LOOKS ALMOST LIKE MATRIX MULTIPLICATION.

<u>OBSERVE</u>: $(v^i g_{ij})$ is an object w/ <u>one</u> lower index

$$\left(\ \boxed{\quad\quad}\ \right)_j$$

ROW VECTOR!
takes vec, gives #

"metric w/ a vector pre-loaded"

$g(\underline{v}, \_\_)$ is a machine that takes a vector & spits out #

PUT IT HERE

$g(\underline{v}, \underline{w})$

so the metric takes vectors & turns them into row vectors.

CAN ALSO DERIVE INVERSE METRIC.

$\hookrightarrow$ shortcut: $g_{ij}\, g^{jk} = \pm \delta^k{}_i$

$\llcorner$ DEP on your vector space

(SPACE VS SPACETIME)

in general, we work w/ COMPLEX functions.

↳ so we often write "row vector" as Hermitian conjugate ⟵ transpose & complex conjugate

$\langle x| \equiv |x\rangle^{\dagger}$

$= \langle x, \_\_ \rangle$

so that $\langle x | y \rangle = \langle x, y \rangle$

BRA KET 2 KETS

~~eg. SPECIAL RELATIVITY~~

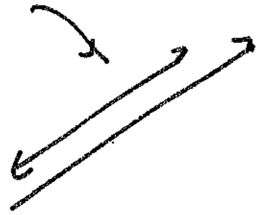
remark: $\langle x | x \rangle = \| \underline{x} \|^2$, norm

"length of a vector"

↳ usually relevant b/c it is preserved

⟶ IDEA OF SYMMETRY, CONSERVED QUAN

$\| \underline{x} \|$ $\underline{x}$



invariant under rotations

$|\underline{v}|^2 = (v^1)^2 + (v^2)^2 + \dots$

EUCLID

eg <u>RELATIVITY</u> $p_\mu = (E, \vec{p})$

$$\|p\|^2 = E^2 - \vec{p}^2 = m^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

(note units)

WHAT ABOUT <u>function space</u>?

DIFF OPERATOR $\mathcal{O} = p_0(x) + p_1(x)\frac{d}{dx} + p_2(x)\left(\frac{d}{dx}\right)^2 + \cdots$

call this a "<u>formal</u>" OPERATOR

almost like our lin. tranformations.

except we should specify <u>SPACE</u> & <u>BOUNDARY COND</u>

↳ then call it a "<u>concrete</u>" operator.

LET'S NOT MAKE A BIG DEAL ABOUT BC TODAY.
(eg assume periodic or DIRICHLET)

<u>INNER PRODUCT</u>? (you may guess from QM)

given functions $f(x)$ & $g(x)$

$$\langle f, g \rangle = \int f^*(x)\, g(x)\, dx$$

more general: $\langle f, g \rangle_w = \int dx \; w(x) \, f^*(x) \, g(x)$

weight

(SHOWS UP IF YOU USE WEIRD COORDS, FOR EXAMPLE)

So WHAT?

now that we have "matrices" $\mathcal{O}$ that act on "vectors" $|f\rangle$ & an inner product $\langle f, g \rangle$

CAN TALK ABOUT $\boxed{\mathcal{O}^\dagger}$

$\mathcal{O}|f\rangle$ is a vector $|\mathcal{O}f\rangle$ w/ row vector

$$\langle \mathcal{O}f| = \langle f|\mathcal{O}^\dagger$$

ok. this is like

$$\begin{cases} \underline{v}' = \underline{\underline{A}}\,\underline{v} \\ \underline{v}' \cdot \underline{w} = \underline{v}'^T \underline{w} = \underline{v}'^T \underline{\underline{A}}^T \underline{w} \end{cases}$$

in QM: $T \to \dagger$.

SPECIAL OPERATORS: HERMITIAN / SELF-ADJOINT

$\hookrightarrow \mathcal{O}^\dagger = \mathcal{O}$

<u>why?</u> $\longrightarrow$ <u>Real</u> EIGENVALUES ($\in \mathcal{O}$)

HERMITIAN : $\mathbb{R}$ eigval

COMPLETE BASIS of ORTHOG EIGENVECTORS

even gives a natural basis!

eg. $\Theta = -\partial_x^2$

DOMAIN : $[0,1]$ w/ $f(0) = f(1) = 0$

(L^2, square integrable functions)

INNER PROD : $\langle f, g \rangle = \int_0^1 dx\, f^* g$

is $\Theta^\dagger = \Theta$?

$$\langle f, \Theta g \rangle \overset{!}{=} \langle \Theta f, g \rangle$$

$$-\int dx\, f^* (\partial^2 g) \qquad\qquad -\int dx\, (\partial^2 f)^* g$$

$$= \underbrace{-(\partial f)^* g \Big|_0^1}_{=0} + \int (\partial f)^* \partial g\, dx$$

$$= \underbrace{+ f^* \partial^2 g \Big|_0^1}_{=0} - \int f^* (\partial^2 g)\, dx$$

$$= -\int dx\, f^* (\partial^2 g)$$

i think this is from Stone & Goldbart

EIGENVECTORS (you already know)

$$f_n(x) = \sin(n\pi x)$$

$$\lambda_n = n^2\pi^2$$

$$\int_0^1 \sin(n\pi x)\,\sin(m\pi x)\,dx = \frac{\delta_{nm}}{2}$$

have to normalize

& we know from Fourier theory that this spans "nice" L^2 functions over $[0,1]$ w/ DIRICHLET BC.