CORRECTION from LAST TIME:

- LINEAR TRANSFORMATION: $\underline{\underline{A}}(\alpha \underline{v} + \beta \underline{w}) = \alpha \underline{\underline{A}}\underline{v} + \beta \underline{\underline{B}}\underline{w}$
- in what sense is FOURIER SPACE the same as "histogram" space?

$$\underbrace{\text{span}(e^{ix}, e^{2ix}, \ldots)_{N\to\infty}}_{\text{things you get from series of sines + cosines}} = \underbrace{\text{span}(1, x, x^2, \ldots)_{N\to\infty}}_{\text{polynomials}}$$

nb $\neq$ space of all conceivable functions.

REVIEW: BRA-KET NOTATION

vector $|\underline{v}\rangle = v^i |i\rangle$

dual vector (row) $\langle \underline{w}| = w_j \langle j|$

} w/ $\langle j|i\rangle = \delta^j{}_i$

linear trans. $\underline{\underline{A}} = A^i{}_j \underbrace{|i\rangle\langle j|}$ ← tensor product

or $|i\rangle \otimes \langle j|$

two separate things

MULTILINEAR: eg: $B^i{}_{jk} \, |i\rangle \otimes \langle j| \otimes \langle k|$

(takes vector, spits out lin transf.
or takes 2 vectors, spits out vector
etc.

"GREEN's FUNCTIONS in LINEAR ALGEBRA"

$$A\,|\psi\rangle = |s\rangle \quad \leftarrow \quad \underline{\underline{A}}\,\underline{x} = \underline{y}$$

Dynamics, state, source

state → **EFFECT**

source → **CAUSE**

given a "disturbance" $|s\rangle$, what happens to $|\psi\rangle$?

$|\psi\rangle$ is s.t. $A|\psi\rangle = |s\rangle$ is true.

A encodes <u>physics</u>, independent of source.

<u>IMPLICIT</u>: the relation btwn $|s\rangle$ & $|\psi\rangle$ is <u>linear</u>

if $|\psi_0\rangle$ is the effect of $|s_0\rangle$, then

$$2|\psi_0\rangle \longleftrightarrow 2|s_0\rangle$$

this is <u>not</u> a truth in <u>nature</u>, it is just true of the kinds of dynamics that we can solve most conveniently

→ this is worth thinking about!
IF NATURE IS NOT LINEAR... WHY MAKE A BIG DEAL ABOUT LINEAR LIMIT?

If we <u>only</u> do linear part, what do we miss?

<u>SOLUTION</u>:

$$|\psi\rangle = A^{-1}|s\rangle$$

if you know A, you can probably calculate A^{-1}

... tedious, but doable

$A^{-1} A = \mathbb{1}_{N\times N}$

N^2 unknowns (A^{-1}); N^2 known (GIVEN) (A); N^2 known values ($\mathbb{1}$)

LHS : N^2 elements
RHS : N^2 elements

↳ can solve N^2 coupled linear sys of eqs.

A LIN. TRANSFORMATION ENCODES (most literally) WHAT IT DOES TO BASIS VECTORS.

$$\underline{\underline{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Big| \quad B^i{}_j \, |i\rangle\langle j|$$

means: $\underline{\underline{B}}\,|1\rangle = a|1\rangle + c|2\rangle$ ← $B^2{}_1$, e.g.

$\underline{\underline{B}}\,|2\rangle = b|1\rangle + d|2\rangle$

$$\Leftrightarrow \quad B\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \quad \& \quad B\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

Knowing the action on basis vectors ⟺ knowing the transformation

if I tell you: $(A^{-1})\,|1\rangle = x|1\rangle + y|2\rangle$

$(A^{-1})\,|2\rangle = z|1\rangle + w|2\rangle$

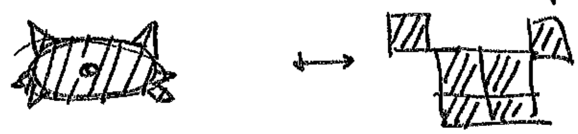
then you <u>know</u> what (A^{-1}) is.
ASSUMING $\mathbb{R}^2$ IS YOUR VECTOR SPACE.

if I give you (A^{-1}) AND s^1, s^2
then you know what $|\psi\rangle$ is.

$$|\psi\rangle = A^{-1}|s\rangle = A^{-1}\left(s^1|1\rangle + s^2|2\rangle\right)$$
$$= s^1 A^{-1}|1\rangle + s^2 A^{-1}|2\rangle$$
$$= s^1\left(x|1\rangle + y|2\rangle\right) + s^2\left(z|1\rangle + z|2\rangle\right)$$
$$= (s^1 x + s^2 z)|1\rangle + (s^1 y + s^2 z)|2\rangle$$

Physics example

BASIS "BLOCKS" OF UNIT CHARGE W/ SOME COEFFICIENT ρ.



CALL IT $|\underline{x}'\rangle$ ← position of block

charged cat ⟷ discretized charged cat

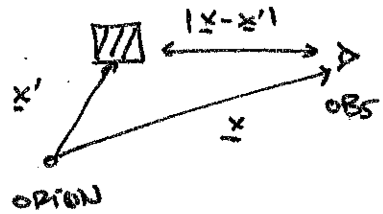
for simplicity, assume each block has unit "charge" $\rho_0\, dV$

<u>ELECTROSTATICS</u>: given source distribution, want the electrostatic potential

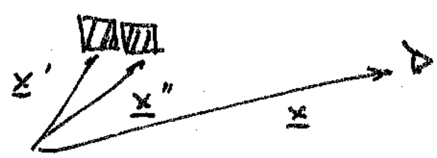
$$\nabla^2 \Phi(x) = -\rho(x)$$

← $\varepsilon_0 = 1$ in my units

BUT I KNOW THAT THE POTENTIAL FROM A UNIT CHARGE IS



$$\Phi(\underline{x}) = \frac{-1}{4\pi}\frac{1}{|\underline{x}-\underline{x}'|}$$



$$\Phi(x) = \frac{-1}{4\pi}\left(\frac{1}{|\underline{x}-\underline{x}'|} + \frac{1}{|\underline{x}-\underline{x}''|}\right)$$

evidently: $\Phi(x) = \underbrace{-(\nabla^2)^{-1}}_{\text{GREEN'S FUNCTION}}\rho$

<u>linear</u>: $\Phi(x) = -(\nabla^2)^{-1}(\rho_1 + \rho_2)$

of $\sum_i \psi^i |i\rangle = \sum_{i,j} \underbrace{(A^{-1})^i{}_j}_{|i\rangle\langle j|} \sum_k S^k |k\rangle$

$\sum_k \langle j|k\rangle = \delta^k_j$

$= \sum_{i,j} (A^{-1})^i{}_j S^j |i\rangle$

$\Downarrow$ $\langle l|$ on both sides

$$\boxed{\psi^l = \sum_j (A^{-1})^l{}_j S^j}$$

in continuum, <u>POSITIONS</u> are the indices

$$\phi(x) = \sum_{\underline{x}'} \underbrace{\frac{-1}{4\pi} \frac{1}{|\underline{x} - \underline{x}'|}}_{\equiv G(\underline{x}, \underline{x}')} \underbrace{\rho(\underline{x}')}_{} dV$$

analog of S^j

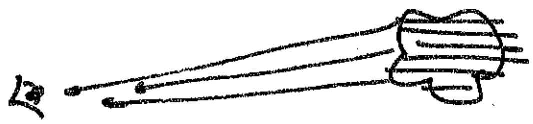
$\rho(\underline{x}') = \rho_0$ if "on"
0 otherwise

$$\rightarrow \int d^3\underline{x}' \; G(\underline{x}, \underline{x}') \, \rho(\underline{x}')$$

$$\sim \phi^l = \sum_j G^l{}_j \, \rho^j$$

(A^{-1}) where $A = \nabla^2$

G is the Green's function. ALSO OFTEN CALLED A <u>PROPAGATOR</u>.



$$\sum_{\underline{x}'} G(\underline{x}, \underline{x}')\, \rho(\underline{x}')$$

<u>PROPAGATES</u> information ρ at $\underline{x}'$ to the observation point $\underline{x}$.

⇒ in the case where the information propagation is linear.

DOUBLE THE $\rho(\underline{x}')$, double the contribution from that term in the Green's function sum.

We've written this down in <u>some</u> basis

... but $|\psi\rangle = \underbrace{(A^{-1})}_{G} |s\rangle$

is true in <u>any</u> basis.

(eg position space, momentum space, spherical harmonics, fourier, ...)