Questions?

PUTTING TOGETHER SOME THINGS

Mtg 9: $\langle f, g \rangle = \int dx\, f^* g$

MTG 6 (Monday last wk) (10/8)

$$\underline{\underline{A}}\, \underline{x} = \underline{y} \longrightarrow \underline{x} = \underline{\underline{A}}^{-1} \underline{y}$$

$$\mathcal{O}_x f(x) = g(x) \longrightarrow f(x) = \boxed{\mathcal{O}^{-1} g}$$

$$\underline{\underline{A}}^{-1} \underline{y} = A^{-1} y^1 |1\rangle + \underbrace{A^{-1} y^2 |2\rangle}_{\text{eg.}} + \cdots$$

$$= \left((A^{-1})^1{}_1\, y^1 + (A^{-1})^1{}_2\, y^2 + \cdots \right) |1\rangle$$

$$+ \left((A^{-1})^2{}_1\, y^1 + \cdots \right) |2\rangle$$

$$+ \cdots$$

So what is $\mathcal{O}^{-1} g$?

POSITION OF OBS.

POSITION OF SOURCE

$$\sum_j (A^{-1})^i{}_j\, y^j \leftarrow A^i{}_j\, |i\rangle\langle j|\, y^k |k\rangle$$

SUM OVER SOURCE

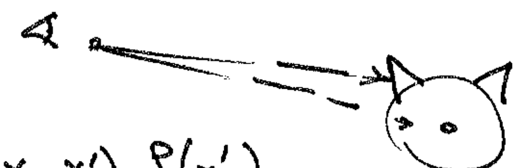
$$\Downarrow$$

OBS

$$\int dx'\, G(x, x')\, g(x')$$

SUM OVER SOURCE

OBS: To say $G = \mathcal{O}^{-1}$ is still a little glib ⋮ really $\int dx'\, G(x,x')$



$$\Phi = \int d^3\underline{x}'\, \underbrace{G(\underline{x}, \underline{x}')}_{-\frac{1}{4\pi}\frac{1}{|\underline{x}-\underline{x}'|}}\, \rho(\underline{x}')$$

$$\leftarrow (\nabla^2)^{-1}$$

GOAL: BE ABLE TO DO THIS $\forall\ \mathcal{O}$

WANT TO DIG DEEPER

$$A^{-1} \text{ def by } A^{-1}A = \mathbb{1} = \sum_i |i\rangle\langle i|$$

second time in this course where we "multiply by 1"

complete set of basis vectors ((true $\forall$ basis!))

What is the analog of this?

CLAIM: $\delta(x-x')$

$$\int_{-\infty}^{\infty} dx' \, \delta(x-x') = 1$$

and $\delta(x-x') = 0$ if $x' \neq x$

nb: this is a distribution, NOT a function. it only makes sense if it's being integrated over!

TRIVIAL eg: $\Theta = \mathbb{1}$, $\Theta f = g$

$$f(x) = \int dx' \, G(x,x') \, g(x') \leftarrow \equiv g(x)$$

$$\Rightarrow G(x,x') = \delta(x-x')$$

eg: LAST FRI: $\Theta = -\partial_x^2$ [0,1], DIRICH.

norm: $\langle f, f\rangle = 1$

$$f_n(x) = \sqrt{2} \sin(n\pi x)$$

$$\lambda_n = n^2\pi^2$$

same x

$$2\int_0^1 dx \sin(n\pi x) \sin(m\pi x) = \delta_{nm}$$

DIFF EIGVAL.

orthonormality

Restrict to <u>NICE</u> operators : $\mathcal{O} = \mathcal{O}^{\dagger}$

self-Adjoint / hermitian / "symmetric"

... typically what shows up in physics, anyway

<u>PLUS</u> : complete, orthonormal eigenfunc w/ $\mathbb{R}$ eigval

$$\mathbb{1} = |i\rangle\langle i| \qquad \langle i|j\rangle = \delta^i_j$$

$$\int (e^{*}(x))_n e^m(x)\, dx = \delta^m_n$$

$$\sum_n (e^{*}(x))_n e^n(y) = \delta(x-y)$$

hey ... that δ function... can we use this to construct <u>Green's function</u> ?

defined w/rt $\delta(x,y)$!

EIGEN DECOMPOSITION : (assume $\mathcal{O}$ nice)

$$\mathcal{O}\,\psi = s \quad \longleftarrow \quad \underset{\text{EIGENBASIS}}{\mathcal{O}|e_i\rangle = \lambda_i |e_i\rangle}$$

$$\mathcal{O}\,\psi^i|e_i\rangle = s^j|e_j\rangle$$

$$\sum_i \lambda_i \psi^i |e_i\rangle = s^j|e_j\rangle \xrightarrow{\langle e_k|} \psi^k = \frac{s^k}{\lambda_k} = \frac{\langle e_k|s\rangle}{\lambda_k}$$

completeness :

$$\sum_{n=1}^{\infty} \sqrt{2}\, \sin(n\pi x) \cdot \sqrt{2}\, \sin(n\pi y) = \delta(x-y)$$

DIFF POS.

SAME EIG. VAL

<u>SO</u>: a succinct definition of Green's func.

given a diff. operator, Θ_x, (eg $\underline{\nabla}^2$)

implicitly: function space w/ B.C.
(B.C. <u>part</u> of def!)

the Green's function, $G(x, x')$, is:

$$\int \Theta_x \, G(x, x') \, dx = \int \delta(x - x') \, dx$$

in approp. dim

$$\text{eg.} \int d^3x \, \underline{\nabla}_x^2 \frac{-1}{4\pi} \frac{1}{|\underline{x} - \underline{x}'|} = \int \delta^{(3)}(\underline{x} - \underline{x}') \, d^3\underline{x}$$

$$\sum_i \sum_j (A)^i{}_j (A^{-1})^j{}_k |i\rangle\langle k| = \delta^i{}_i |i\rangle\langle k|$$

again: SO WHAT ?

when source = 0, lots of ways to solve

<u>physics</u>: [ugly Θ] ψ = source

FOR <u>GENERAL</u> SOURCE...
WANT GENERAL PROCEDURE.

only need Θ to define $G(x, x')$

$$\varphi = \sum_i \frac{\langle e_i | s \rangle}{\lambda_i} | e_i \rangle$$

$\leftarrow e_i(x)$

$$\langle e_i | s \rangle = \int dy \; e_i^*(y) \, s(y)$$

$$= \int dy \underbrace{\sum_i \frac{e_i(x) \, e_i^*(y)}{\lambda_i}} \; s(y) \; dy$$

$G(x,y)$ AS A SUM OVER eigenfunctions

eg. $\underline{\nabla}^2$

$$G(x,y) = \underbrace{\frac{-1}{4\pi} \frac{1}{|\underline{x} - \underline{y}|}}_{\text{from physics}} \overset{?}{=} \underbrace{\sum_i \frac{e_i(x) \, e_i^*(y)}{\lambda_i}}_{\text{from analogy to LIN. ALG.}}$$

what are $e_i(x)$ for $\underline{\nabla}^2$?

$\hookrightarrow$ spherical harmonics (FOR USUAL B.C.)

RESULT:

$$G(\underline{x},\underline{x}') = \underbrace{\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}}_{\substack{\text{more indices} \\ \text{(not a big deal)}}} \frac{1}{2\ell+1} \underline{Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(\theta',\varphi')} \frac{r_<^{\ell}}{r_>^{\ell+1}}$$

$r_> = \max(r, r')$
$r_< = \min(r, r')$

$e_i(\underline{x})\, e_j^*(\underline{x}')$
structure!

here's the Idea: you are given ∇^2 operator from E&M. (FROM PHYSICS)

↳ "by hook or by crook", suppose you know that the eigenfunctions are something like $Y_{\ell m}$.

GIVEN ANY SOURCE $\rho(\underline{x})$, CAN WRITE A CLOSED EXPRESSION FOR ELECTROSTATIC POT.

$$\phi(\underline{x}) = \sum_{\ell,m} \int d^3\underline{x}' \frac{r_<^{\ell}}{r_>^{\ell+1}} \underbrace{\sum_{\ell,m} \frac{Y_{\ell,m}(\theta,\varphi) Y^*_{\ell,m}(\theta',\varphi')}{2\ell+1}} \rho(\underline{x}')$$

nb. has a special name: Legendre polynomial

$P_\ell(\hat{\underline{r}} \cdot \hat{\underline{r}}')$

limit: $r \gg r'$
OBS FAR FROM SOURCE

$$\phi(x) = \sum_{\ell,m} \frac{1}{2\ell+1} \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}} \underbrace{\int d^3\underline{x}' (r')^{\ell} Y^*_{\ell m}(\theta',\varphi') \rho(\underline{x}')}_{\substack{\text{PROP. OF SOURCE:} \\ \text{MULTIPOLE MOMENTS}}}$$

gives a hint why this is meaningful:
BECOMES A <u>TAYLOR EXPANSION</u> in a small quantity.

<u>WHERE WE'RE GOING</u>

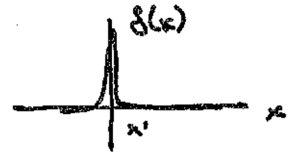
<u>want</u>: given $\mathcal{O}$ → what is $G(x, x')$?

3 main ways:

1. using eigenfunc & completeness
   ↳ gives series sol.
   NEED EIGENFUNC
   ... & hope for CONVERGENCE

2. patching

$$\mathcal{O}_x G(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}')$$



solve for $\underline{x} < \underline{x}'$, $\underline{x} > \underline{x}'$ } $\underbrace{\mathcal{O}_x G(x, x') = 0}_{\text{PRESUMABLY EASY TO SOLVE}}$

then <u>patch</u> solutions together consistently

& related

3. <u>FOURIER TRANSFORM</u> : convert DIFF EQ TO ALGEBRAIC EQ. in momentum space
   ... then go back to position space