Homework 3b: Complex Analysis

1   Practice with complex integration

Remark: This was originally posed in Homework 2B problem 3. If you already did it, please note "already submitted." If you did not complete it—for example, because we didn't get to the material in our discussions—then please complete them as part of this homework.

1.1   This is not a closed loop

The goal of this problem is to review line integrals in a 2D space. Perform the integral

$$\int_{i}^{1+i} dz\, z \tag{1.1}$$

with respect to the path that connects $z = i$ and $z = i + 1$ by a straight line segment. This is *not* a closed loop, there's no magic formula for this. Just parameterize the path by writing z as a function of a real parameter. Then write out dz in terms of the parameter. Do the integral.

1.2   Contour integrals

Calculate the following integrals using the residue theorem:

(a) $\int_{0}^{\infty} \frac{x^2}{x^4 + 16} dx$

(b) $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx$

(c) $\int_{0}^{\infty} \frac{1}{x^6 + 1} dx$ using a contour including the real line and a large semicircle in the complex plane

(d) $\int_{0}^{\infty} \frac{1}{x^6 + 1} dx$ using a contour that encloses the 'pizza slice' wedge between $\theta = 0$ and $\theta = \pi/3$.

Hint: for (d), the line integral along the diagonal is proportional to the integral you want.

2 How to find Residues

This problem is based on Boas (page 599) and Appel (section 4.5d). We saw that a **meromorphic** (analytic up to poles) function has a Laurent series expansion about a point z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \ . \tag{2.1}$$

The a_{-1} term has special significance and is known as the **residue** of f at z_0, $\text{Res}(f, z_0)$. When we have an explicit Laurent expansion about z_0, identifying the residue is a matter of reading off the $(z - z_0)^{-1}$ coefficient. Alternately, when z_0 is a simple pole the function can be written as

$$f(z) = \frac{F(z)}{(z - z_0)} \ , \tag{2.2}$$

where $F(z)$ is analytic at z_0. In this case, the residue is $\text{Res}(f, z_0) = F(z_0)$. This leads to the sometimes useful guide: If $f(z_0)$ is not finite but $(z - z_0) f(z)$ is finite, then

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) \ . \tag{2.3}$$

What do we do if $(z - z_0) f(z)$ is *not* finite? For example, what if both a_{-1} and a_{-2} were non-zero? How does determine the residue (a_1) in such a case? Explain why the following algorithm works: Find the positive integer m such that $F_m(z) = (z - z_0)^m f(z)$ is finite at $z = z_0$, then the residue is

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} F_m(z) \Big|_{z=z_0} \ . \tag{2.4}$$

3 A Fourier transform refresher

Fourier transforms are annoying because there are a few choices that one has to make to establish conventions. The convention that we will use is:

$$f(x) = \int \frac{dk}{2\pi} e^{-ikx} \widetilde{f}(k) \qquad\qquad \widetilde{f}(k) = \int dx \, e^{ikx} f(x) \ . \tag{3.1}$$

In this convention, the (2π) comes with the dk, so I will often write $đk = dk/(2\pi)$.

3.1 Fourier decomposition of $\delta(x)$

What is the Fourier transform of $\delta(x)$? In other words, find $\widetilde{\delta}(k)$ in

$$\delta(x) = \int đk \, e^{ikx} \widetilde{\delta}(k) \ . \tag{3.2}$$

What about $\delta(x - x_0)$?

3.2 Other conventions

A general convention for the Fourier transform is:

$$f(x) = |B|^{1/2} \int \frac{dk}{\sqrt{(2\pi)^{1+A}}} \, e^{-iBkx} \, \widetilde{f}(k) \qquad \widetilde{f}(k) = |B|^{1/2} \int \frac{dx}{\sqrt{(2\pi)^{1-A}}} \, e^{iBkx} f(x) \; . \qquad (3.3)$$

Our conventions correspond to $B = 1$ and $A = 1$. Show that in this general form, the inverse Fourier transform of a Fourier transform is simply the original function.

3.3 Higher dimensions

In two Euclidean dimensions, the Fourier transform is

$$f(x, y) = \int \frac{dk_x}{2\pi} \int \frac{dk_y}{2\pi} \, e^{-ik_x x} e^{-ik_y y} \widetilde{f}(k_x, k_y) \; . \qquad (3.4)$$

This can be rewritten in terms of a position 2-vector[1] $\mathbf{x}$ and corresponding momentum 2-vector $\mathbf{k}$,

$$f(\mathbf{x}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \, e^{-i\mathbf{k}\cdot\mathbf{y}} \widetilde{f}(\mathbf{k}) \; . \qquad (3.5)$$

Observe that the exponential is built out of the rotationally-invariant scalar quantity, $\mathbf{k} \cdot \mathbf{y}$. When we deal with *partial* differential equations, we'll need to Fourier transform in multiple dimensions. Sometimes we'll have to Fourier transform in both time and space. It is conventional to choose signs so that we may write this as

$$f(x, t) = \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \, e^{+ikx} e^{-i\omega t} \widetilde{f}(k, \omega) \; . \qquad (3.6)$$

Briefly comment why this is a good idea from two points of view:

1. The idea that we are expanding about a basis of traveling plane waves.

2. Lorentz invariance, in case we want our expressions to respect special relativity.

4 Integral representation of the step function

[CAHILL, Problem 5.32] The step function is defined by $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$. Show that this is equivalent to

$$\Theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ixz}}{z - i\varepsilon} dz \; . \qquad (4.1)$$

HINT: What does the $-i\epsilon$ mean? Compare this to the advanced and retarded Green's functions that we explored in Lecture 15.

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[1]Mathematicians laugh at us when we say 'position vector.'

5 Evaluating an NSF GRFP proposal

There is a sample NSF Graduate Research Fellowship proposal posted on this course's `iLearn` page under Course Materials. The proposal is from a recent Caltech graduate student who won the fellowship and has shared these materials to help other graduate students who are applying in the future. Please read the two essays (personal statement and research statement). Please answer the following:

1. What did you like about the research statement? What did you find difficult to read?

2. What did you like about the personal statement? What did you find difficult to read?

3. Think about the visual format of the document. Given that an NSF reviewer reads dozens of these at a time, how did the formatting help/hurt the statements? Consider the spacing, use of **bold** text, sectioning, etc.

4. The research statement is tricky because you have to show competency in a field that you are just entering. Comment on how much progress the author has made on the proposed study at the time that the proposal was written. What kind of tasks have already been complete (e.g. discussions with faculty, literature reviews, ...) and what remains to be done (e.g. has a data analysis pipeline already been developed?).

5. Comment on how the tone and purpose of the personal statement in contrast to the research statement. What parts of the personal narrative did the author think it was important to highlight for the reviewer?

Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

1 Laurent Theorem

We shall prove the Laurent Theorem for the expansion about z_0 of a function $f(z)$ in a region where it is meromorphic:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \qquad\qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \ , \qquad (1.1)$$

where C is a contour that loops once counter-clockwise around z_0.

Consider the contour below (image from Cahill, *Physical Mathematics*), enclosing an annular region that includes the point z without any poles (asterisks).
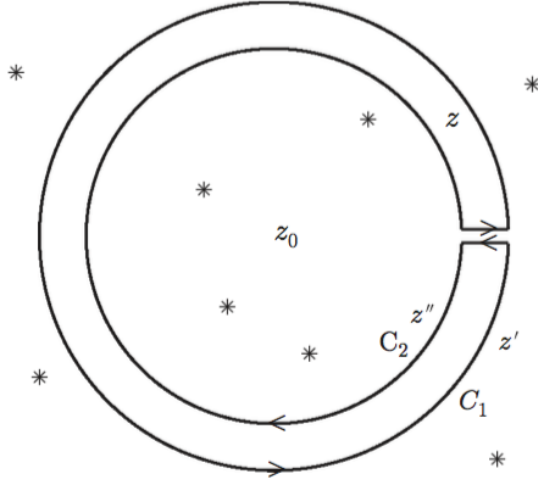
Both the outer and inner contours, C_1 and C_2, encircle z_0. Use the **Cauchy theorem**,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z_0} dw , \tag{1.2}$$

where C encloses a region in which $f(z)$ is *analytic*. Taking $C = C_1 - C_2$, the contour shown above, one has

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')}{z'' - z} dz'' . \tag{1.3}$$

Consider the following quantities:

$$r(z') = \frac{z - z_0}{z' - z_0} \qquad\qquad R(z'') = \frac{z - z_0}{z'' - z_0} . \tag{1.4}$$

Note that $|r(z)| < 1$ and $|1/R(z)| < 1$. Write (1.3) as

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{(z' - z_0)\,[1 - r(z')]} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')}{(z - z_0)\,[1 - 1/R(z'')]} dz'' . \tag{1.5}$$

Use the series expansion

$$\frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n , \tag{1.6}$$

for $|s| < 1$. Argue that we may now deform C_1 and C_2 to an intermediate contour C. Show that the result of all this proves (1.2).