$$\oint_C f(z)\,dz = \sum_{\substack{j \\ \text{(POLES in C)}}} 2\pi i \, \mathrm{Res}_f(z_i)$$

$$f(z) = \cdots + \frac{a_{-1}(z_i)}{z - z_0} + a_0(z_i) + \cdots$$

USUALLY WE'LL JUST HAVE SIMPLE POLES.

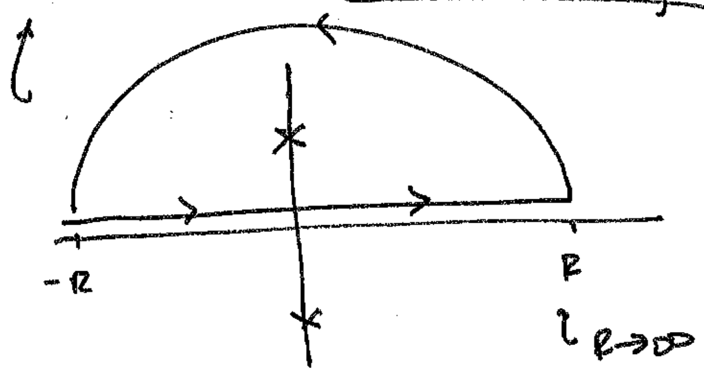
$$\mathrm{Res}_f(z_i) \underset{z \to z_i}{=} (z - z_i)\, f(z_i)$$

exercise: $\cot(z)$. SEE MTG 17 notes p.5.

LAST TIME:

a $\mathbb{R}$ integral we may want

$$\oint_C \frac{dz}{z^2+1} = \boxed{\int_{-\infty}^{\infty} \frac{dx}{x^2+1}} + \int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{R^2 e^{2i\theta}+1}$$

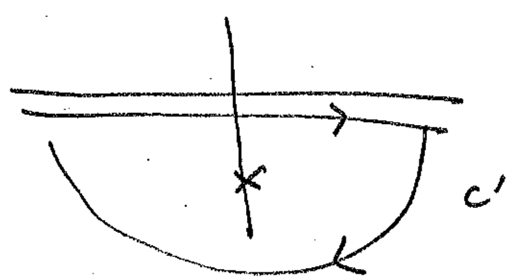


POLES: $z = \pm i$
Res : $1/\pm 2i$

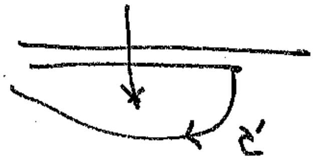
$$\oint_C \frac{z}{z^2+1} = \pm\pi$$

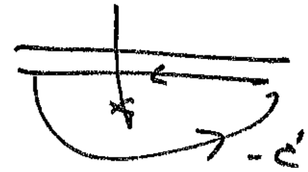
+ for C

① why not



?

then: 

= − 

CORRECT ORIENTATION

$$\oint_{-c'} \frac{dz}{z^2+1} = -\int_{-\infty}^{\infty} \frac{dx}{x^2+1} + \underbrace{\int_{\pi}^{2\pi} \frac{iRe^{i\theta}\,d\theta}{R^2 e^{2i\theta}+1}}_{\text{still small}}$$

b/c of order of int. limits

PICK UP OPPOSITE SIGN POLE

$2\pi i \,\mathrm{Res}_f(-i) = -\pi$

→ same conclusion

$$\boxed{-\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = -\pi}$$

(2) is "complexify" a valid mathematical operation? ← b/c physicists do it a <u>lot</u>.

in other words: is $x \to z$ the unique way to write a $\mathbb{C}$ function that reduces to some $f(x)$ on the $\mathbb{R}$ line?

<u>ANSW</u>: yes. reason: Analytic continuation

dom of g

dom of f

if f & g have domains of analyticity

then $(f-g)$ is analytic here ... and can extend Taylor exp

③ Are there "EDGE EFFECTS" ?

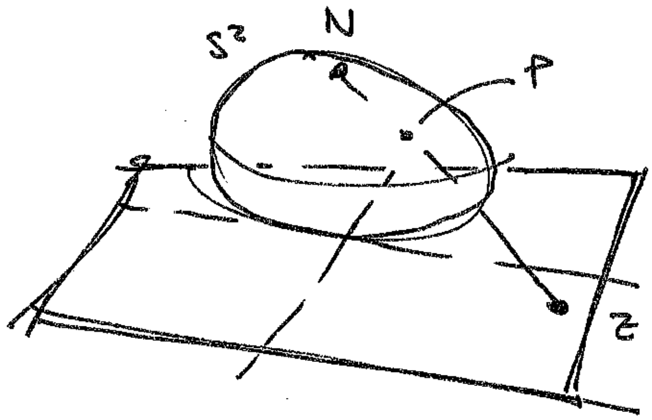
$$\lim_{R\to\infty} \int_0^{\pi} \frac{R i e^{i\theta}}{R^2 e^{2i\theta}+1}\, d\theta$$

DOES ORDER OF LIMIT MATTER?!

$\sim \frac{1}{Re^{i\theta}}$ ... but what about "small θ" ?

No. denominator always has a piece that goes like R^2

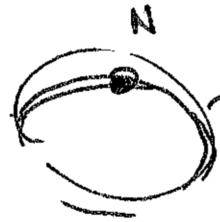
DEEPER ANSWER (for culture) RIEMANN SPHERE



each point in $\mathbb{C}$ maps to one point on S^2 by stereographic projection

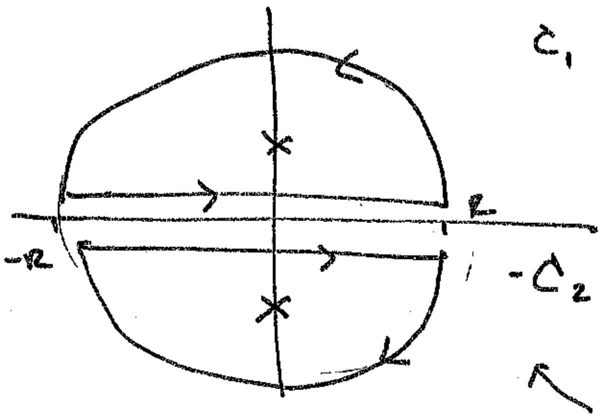
(nb: S^2 has one "extra point" $\longrightarrow \infty$!)

all ∞'s are the same ... sense of topology



a particular meridian is $\int_{-\infty}^{\infty} dx$ } no "curvy piece" eg. $\underbrace{\int_{-\infty}^{\infty} \frac{2\cos x}{x^2+1}\,dx}_{\text{want this}} \xrightarrow{\mathbb{C}} \oint_c \frac{e^{iz}+e^{-iz}}{(z+i)(z-i)}\,dz$

need a contour



$\boxed{\text{DOES IT MATTER?}}$

(unlike last eg, it does!)

no I can easily evaluate EITHER. only one will give me $\mathbb{R}$ int I want.

How to decide? Here's the nuance

as $R \to \infty$, integrand:

$$f(z)\,dz \sim \frac{e^{iz}+e^{-iz}}{R^2 e^{\cdots}}\, iRe^{\cdots}$$

$\sim 1/R$

exponents beat power laws

$$e^{iz} = e^{i(\underbrace{R\cos\theta + iR\sin\theta}_{\text{write in terms of angular var that we'll do arc integral with}})}$$

$$\underset{\uparrow \atop iRe^{i\theta}}{e^{\pm iz}\,dz} = iR \underbrace{\boxed{e^{\mp R\sin\theta}}}_{\text{exponential suppression if } \mp\sin\theta < 0} e^{\pm iR\cos\theta + i\theta}\, d\theta$$

exponential suppression <u>if</u> $\mp\sin\theta < 0$

$\sin\theta > 0$

$\boxed{\text{im part}}$

$\boxed{180}$: we're going to have to break it up:

☆ $$\oint_{C_A} \frac{e^{iz}}{(z+i)(z-i)}\,dz + \oint_{C_B} \frac{e^{-iz}}{(z+i)(z-i)}\,dz$$

$\sin\theta > 0 \rightarrow C_1$ $\quad\quad$ $\sin\theta < 0 \rightarrow -C_2$

© $$= \left.\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx + \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+1}\,dx \right\} \text{ orig integrand}$$

☼ $$\left. + \int_{\substack{\text{UPPER}\\\text{ARC}}} \frac{iRe^{-R\sin\theta}\ldots}{R^2\ldots}\,d\theta + \int_{\substack{\text{LOWER}\\\text{ARC}}} \frac{iRe^{+R\sin\theta}\ldots}{R^2}\,d\theta \right\} = 0$$

$$\oint_{C_1} \frac{e^{iz}}{(z+i)(z-i)}\,dz = 2\pi i \times \frac{e^{-1}}{2i}$$

POLE @ $+i$

$$\oint_{C_2} \frac{e^{-iz}}{(z+i)(z-i)}\,dz = -2\pi i \times \frac{e^{-1}}{-2i}$$

POLE @ $-i$

minus sign from orientation

so $\bigstar$ = $\boxed{\frac{2\pi}{e} = \int_{-\infty}^{\infty} \frac{2\cos x}{x^2+1}\,dx}$

So what?

$$G(x) = \int_{-\infty}^{\infty} dk\, \frac{e^{ikx}}{(k^2-m^2)}$$ ← for eg

this is exactly what we just evaluated!