BIG PICTURE

DIFF. OPERATOR: $\mathcal{O}$ eg $\mathcal{O} = \left(\frac{d}{dt}\right)^2 + \omega^2$

→ for GIVEN B.C., $\exists$ Green's function

$G(t, t')$

OBS. TIME, SOURCE TIME

can prove: $G(t,t') = G(t-t')$

argument: time translation invariance

such that:

SOURCE (eg. DRIVING FORCE)

if $\mathcal{O} f(t) = s(t)$

thing whose dynamics you want ("response")

then $$\boxed{f(t) = \int dt'\, G(t-t')\, s(t')}$$

dynamics given by overlap integral of Green's function w/ source.

ANALOG OF $\boxed{\underline{f} = \mathcal{O}^{-1}\underline{s}}$

$$\underline{f}^i = \sum_j (\mathcal{O}^{-1})^i{}_j\, \underline{s}^j$$

$G(t-t')$ IS DEFINED BY

$$\boxed{\mathcal{O}_t\, G(t-t') = \delta(t-t')}$$

not really physical (OFTEN A GOOD APPROX... BUT REALLY MUST BE INTEGRATED OVER.)

G is the response to a "unit" source → δ-function source so we care about solving for $G(t-t')$.

$$G(t-t') = \int dk\, e^{-ik(t-t')}\tilde{G}(k)$$

definition of Fourier transform

$$\text{then}\quad \mathcal{O}_t G(t-t') = \int dk\, e^{-ik(t-t')}\, \underbrace{P(k)}_{\text{polynomial}}\, \tilde{G}(k)$$

$$\delta(t-t') = \int dk\, e^{-ik(t-t')}\; 1$$

$$\boxed{P(k)\,\tilde{G}(k) = 1}$$

TRUE, BUT "CANCELING THE INTEGRAL ON BOTH SIDES" IS NOT A RIGOROUS STEP!!

$$\tilde{G}(k) = P(k)^{-1}$$

eg. for $\mathcal{O} = \left(\frac{d}{dt}\right)^2 + \omega^2$

$$P(k) = -k^2 + \omega^2$$

$$\boxed{\tilde{G}(k) = \frac{-1}{k^2-\omega^2}}$$

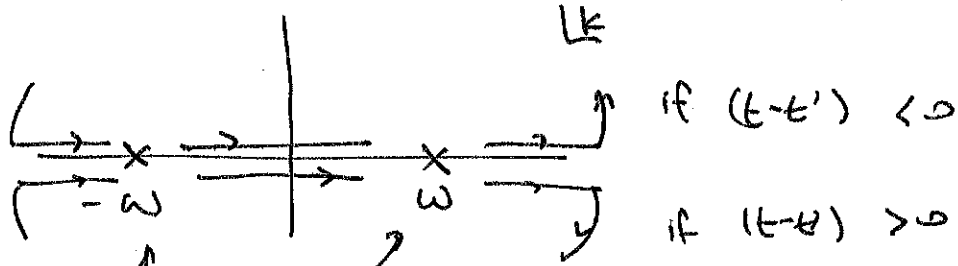
then we just plug in to Fourier transform:

$$G(t-t') = \int dk\, e^{-ik(t-t')}\, P(k)^{-1}$$

$$\text{eg} \quad = \underbrace{\int dk\, e^{-ik(t-t')}\, \frac{-1}{k^2-\omega^2}}_{\text{trick: } \mathbb{C} \text{ contour integral}}$$

$k = \mathrm{Re}\,k + i(\mathrm{Im}\,k)$ — DETERMINES CONVERGENCE

$$G(t-t') = \int dk \, \frac{-1}{2\pi} \, \frac{e^{-ik(t-t')}}{(k+\omega)(k-\omega)}$$



aw shit. poles on integ. contour

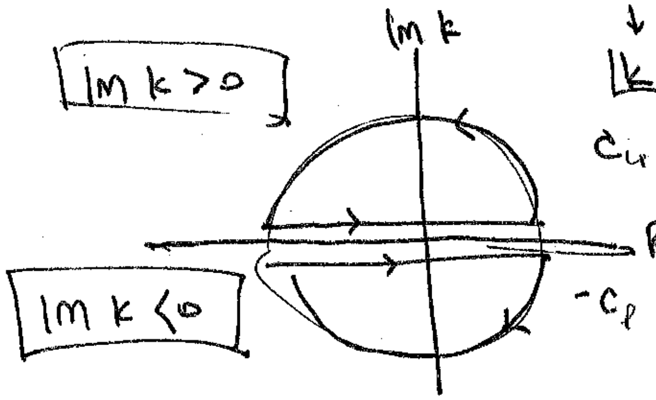
I HAVE ALTERED THE CONTOUR. PRAY THAT I DON'T ALTER IT ANY FURTHER.

<u>WANT</u>: when $t > t'$, ~~contour~~ ~~[illegible]~~ $G(t-t') \neq 0$

← contour encloses a pole

when $t < t'$, $G(t-t') = 0$

← satisfied if contour encloses no poles.



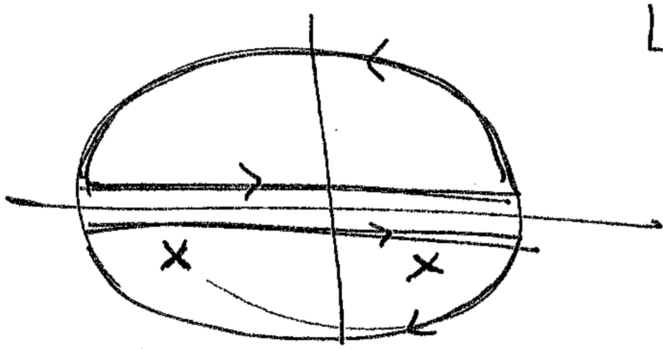
$\mathrm{Re}\,k + i\,\mathrm{Im}\,k$ ↓ Lk

C_u ← $e^{-ik(t-t')} \propto e^{-\mathrm{BIG}}$ if $t-t' < 0$ ← ACAUSAL.

$\mathrm{Im}\,k < 0$ ↓

$-C_l$ ← $e^{-ik(t-t')} \propto e^{-\mathrm{BIG}}$ if $(t-t') > 0$ ← CAUSAL

$\boxed{SD}$:



k

$\boxed{t-t' < 0}$ ADVANCED ×

$\boxed{t-t' > 0}$ RETARDED ✓

$$G_R(t-t') = \int dk \, \frac{-1}{2\pi} \, \frac{e^{-ik(t-t')}}{(k+\omega+i\varepsilon)(k-\omega+i\varepsilon)}$$

now you are fully equipped to do this.

~~$G_R(t-t')$~~

<u>REMARK</u> : REVERSE ENGINEERING :

$$G_R(t-t') = \int dk \, e^{-ik(t-t')} \, \frac{-1}{k^2 - \omega^2 - 2ik\varepsilon + O(\varepsilon^2)}$$

$$P_R(k) = -k^2 + 2ik\varepsilon + \omega^2$$

$$\mathcal{O}_R = \left(\frac{d}{dt}\right)^2 + 2\varepsilon \frac{d}{dt} + \omega^2$$

this is a <u>different</u> OPERATOR THAN THE ONE WE STARTED WITH. REDUCES IN the CASE $\varepsilon \to 0^+$ from POSITIVE DIR. ONLY!

LOOKS FAMILIAR?!! (analogous to dispersion relation — the absorptive part & diffracted part are related!)

✓ $\boxed{\text{CAUSALITY}}$

LET'S SOLVE for $G_R(t-t')$ for HARMONIC OSCILLATOR.

$$G_R(t-t') = \oint dk \cdots \; - \int_{ARC} dk \cdots$$

which arc? DEPENDS on SIGN of $t-t'$... BUT WE CHOOSE THE ARC s.t. THIS INTEGRAL IS ALWAYS ZERO.

$$= 2\pi i \sum_j \text{Res}(z_j)$$

enclosed poles of integrand

POLES: $\omega - i\varepsilon$ Res: $\frac{-1}{2\pi} \frac{e^{-i(\omega - i\varepsilon)(t-t')}}{2\omega}$

$-\omega - i\varepsilon$ Res: $\frac{-1}{2\pi} \frac{e^{-i(-\omega - i\varepsilon)(t-t')}}{-2\omega}$

$$= -i\left(\frac{e^{-i\omega\Delta t}}{2\omega} - \frac{e^{i\omega\Delta t}}{2\omega}\right)$$

$$= -i\,\frac{1}{2\omega}\cdot(-2i\sin\omega\Delta t)$$

$e^{i\theta}$
$-e^{-i\theta}$
$= 2i\sin\theta$

$$= \boxed{-\frac{\sin\omega\Delta t}{\omega}} \quad \text{if } \Delta t > 0$$

$$= \boxed{0} \quad \text{if } \Delta t < 0$$

PS: check overall phase. I may have made sign error(s)

<u>DAMPED H.O.</u>

DAMPING term

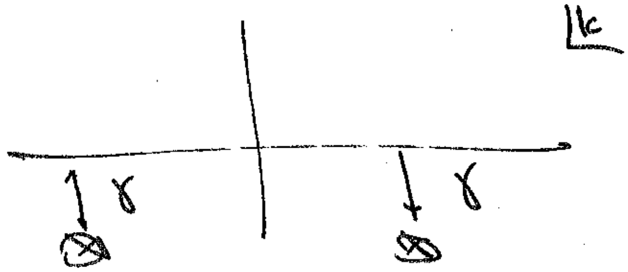
$$\underbrace{\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega^2 x(t)}_{\Theta\, x(t)} = F(t)$$

$$\Theta = \left(\frac{d}{dt}\right)^2 + \underbrace{2\gamma\frac{d}{dt}}_{} + \omega^2 \qquad (\text{cf. p.4 !!})$$

we now know that this pushes poles <u>DOWN</u>
when $\gamma > 0$

[$\gamma < 0$ is <u>unphysical</u>.]

γ finite: no choice of pole pushing



$$G(t-t') = \int dk\, e^{-ik(t-t')} \frac{-1}{k^2 + 2ik\gamma + \omega^2}$$

complete the square.

fyi: $$k_\pm = \frac{-2i\gamma \pm \sqrt{-4\gamma^2 + 4\omega^2}}{2} = \pm \underbrace{\sqrt{\omega^2-\gamma^2}} - i\gamma$$

Real if $\omega^2 > \gamma^2$

but what if $\omega^2 < \gamma^2$?

... then it's a pretty shitty oscillator :

(DAMPING is more important than oscillations.)

& so it doesn't make sense to look for a characteristic oscillation frequency !

imaginary $k_\pm \longleftrightarrow$ YOUR SYSTEM IS REALLY MORE of A DECAYING SYSTEM than OSCILLATING.

$$G(t-t') = \int dk \, \frac{1}{2\pi} \frac{-e^{-ik\Delta t}}{(k-k_+)(k-k_-)}$$

$$= 2\pi i \sum_j \operatorname{Res}(z_j)$$ SAME TRICK AS BEFORE

$$= 2\pi i \cdot \left[\frac{1}{2\pi} \frac{-e^{-ik_+\Delta t}}{\underbrace{k_+ - k_-}_{=2\sqrt{\omega^2-\gamma^2}}} + \frac{1}{2\pi} \frac{-e^{-ik_-\Delta t}}{\underbrace{k_- - k_+}_{-2\sqrt{\omega^2-\gamma^2}}} \right]$$

$$e^{-ik_\pm \Delta t} = e^{-\gamma \Delta t} \, e^{\mp i\sqrt{\omega^2-\gamma^2}\,\Delta t}$$

$$\underbrace{G(t-t')}_{\substack{\text{ASSUMING} \\ t-t'>0}} = e^{-\gamma \Delta t} \underbrace{\frac{1}{\sqrt{\omega^2-\gamma^2}} \underbrace{\frac{-i}{2}\left(e^{-i\sqrt{\;}\Delta t} - e^{i\sqrt{\;}\Delta t}\right)}_{-\sin\left(\sqrt{\omega^2-\gamma^2}\,\Delta t\right)}}_{\text{REDUCES TO SHO when } \gamma \to 0}$$

DAMPING term
AS Δt GETS BIGGER,
$G(t-t')$ GETS SMALLER.

$$G(\underbrace{t-t'}_{=\Delta t}) = e^{-\gamma \Delta t} \frac{-1}{\sqrt{\omega^2-\gamma^2}} \sin\left(\sqrt{\omega^2-\gamma^2}\,\Delta t\right) \Theta(\Delta t)$$

every GREEN'S FUNCTION IN physics is some variant of this.