# Homework 3b: **Complex Analysis**

Course: Physics 231, *Methods of Theoretical Physics* (2019)
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Due by: Mon, November 11

## 1 How to find Residues

This problem is based on Boas (page 599) and Appel (section 4.5d). We saw that a **meromorphic** (analytic up to poles) function has a Laurent series expansion about a point $z_0$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \ . \tag{1.1}$$

The $a_{-1}$ term has special significance and is known as the **residue** of $f$ at $z_0$, $\text{Res}(f, z_0)$. When we have an explicit Laurent expansion about $z_0$, identifying the residue is a matter of reading off the $(z-z_0)^{-1}$ coefficient. Alternately, when $z_0$ is a simple pole the function can be written as

$$f(z) = \frac{F(z)}{(z-z_0)} \ , \tag{1.2}$$

where $F(z)$ is analytic at $z_0$. In this case, the residue is $\text{Res}(f, z_0) = F(z_0)$. This leads to the sometimes useful guide: If $f(z_0)$ is not finite but $(z-z_0)\, f(z)$ is finite, then

$$\text{Res}(f, z_0) = \lim_{z\to z_0} (z-z_0)\, f(z) \ . \tag{1.3}$$

What do we do if $(z-z_0)\, f(z)$ is *not* finite? For example, what if both $a_{-1}$ and $a_{-2}$ were non-zero? How does determine the residue $(a_1)$ in such a case? Explain why the following algorithm works: Find the positive integer $m$ such that $F_m(z) = (z-z_0)^m f(z)$ is finite at $z = z_0$, then the residue is

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} F_m(z)\Big|_{z=z_0} \ . \tag{1.4}$$

## 2 A Fourier transform refresher

Fourier transforms are annoying because there are a few choices that one has to make to establish conventions. The convention that we will use is:

$$f(x) = \int \frac{dk}{2\pi} e^{-ikx}\, \widetilde{f}(k) \qquad\qquad \widetilde{f}(k) = \int dx\, e^{ikx}\, f(x) \ . \tag{2.1}$$

In this convention, the $(2\pi)$ comes with the $dk$, so I will often write $đk = dk/(2\pi)$.

### 2.1 Fourier decomposition of $\delta(x)$

What is the Fourier transform of $\delta(x)$? In other words, find $\widetilde{\delta}(k)$ in

$$\delta(x) = \int đk\, e^{ikx}\, \widetilde{\delta}(k) \ . \tag{2.2}$$

What about $\delta(x-x_0)$?

### 2.2 Other conventions

A general convention for the Fourier transform is:

$$f(x) = |B|^{1/2} \int \frac{dk}{\sqrt{(2\pi)^{1+A}}}\, e^{-iBkx} \widetilde{f}(k) \qquad \widetilde{f}(k) = |B|^{1/2} \int \frac{dx}{\sqrt{(2\pi)^{1-A}}}\, e^{iBkx} f(x)\ . \tag{2.3}$$

Our conventions correspond to $B = 1$ and $A = 1$. Show that in this general form, the inverse Fourier transform of a Fourier transform is simply the original function.

### 2.3 Higher dimensions

In two Euclidean dimensions, the Fourier transform is

$$f(x, y) = \int \frac{dk_x}{2\pi} \int \frac{dk_y}{2\pi}\, e^{-ik_x x} e^{-ik_y y} \widetilde{f}(k_x, k_y)\ . \tag{2.4}$$

This can be rewritten in terms of a position 2-vector[1] $\mathbf{x}$ and corresponding momentum 2-vector $\mathbf{k}$ ,

$$f(\mathbf{x}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e^{-i\mathbf{k}\cdot\mathbf{y}} \widetilde{f}(\mathbf{k})\ . \tag{2.5}$$

Observe that the exponential is built out of the rotationally-invariant scalar quantity, $\mathbf{k}\cdot\mathbf{y}$. When we deal with *partial* differential equations, we'll need to Fourier transform in multiple dimensions. Sometimes we'll have to Fourier transform in both time and space. It is conventional to choose signs so that we may write this as

$$f(x, t) = \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi}\, e^{+ikx} e^{-i\omega t} \widetilde{f}(k, \omega)\ . \tag{2.6}$$

Briefly comment why this is a good idea from two points of view:

1. The idea that we are expanding about a basis of traveling plane waves.
2. Lorentz invariance, in case we want our expressions to respect special relativity.

## 3 Integral representation of the step function

[Cahill, Problem 5.32] The step function is defined by $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$. Show that this is equivalent to

$$\Theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ixz}}{z - i\varepsilon} dz\ . \tag{3.1}$$

Hint: What does the $-i\epsilon$ mean? Compare this to the advanced and retarded Green's functions that we explored in Lecture 15.

[1] Mathematicians laugh at us when we say 'position vector.'

# Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

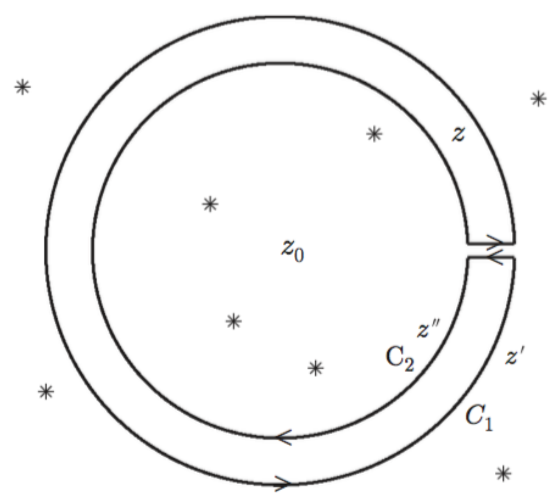
## 1 Laurent Theorem

*This problem is suggested.* We shall prove the Laurent Theorem for the expansion about $z_0$ of a function $f(z)$ in a region where it is meromorphic:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad\qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \ , \tag{1.1}$$

where $C$ is a contour that loops once counter-clockwise around $z_0$.

Consider the contour below (image from Cahill, *Physical Mathematics*), enclosing an annular region that includes the point $z$ without any poles (asterisks).




Both the outer and inner contours, $C_1$ and $C_2$, encircle $z_0$. Use the **Cauchy theorem**,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z_0} dw \ , \tag{1.2}$$

where $C$ encloses a region in which $f(z)$ is *analytic*. Taking $C = C_1 - C_2$, the contour shown above, one has

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z'')}{z'' - z} dz'' \ . \tag{1.3}$$

Consider the following quantities:

$$r(z') = \frac{z - z_0}{z' - z_0} \qquad\qquad R(z'') = \frac{z - z_0}{z'' - z_0} \ . \tag{1.4}$$

Note that $|r(z)| < 1$ and $|1/R(z)| < 1$. Write (1.3) as

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{(z' - z_0)\left[1 - r(z')\right]} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')}{(z - z_0)\left[1 - 1/R(z'')\right]} dz'' \ . \tag{1.5}$$

Use the series expansion

$$\frac{1}{1-s} = \sum_{n=0}^{\infty} s^n \ , \tag{1.6}$$

for $|s| < 1$. Argue that we may now deform $C_1$ and $C_2$ to an intermediate contour $C$. Show that the result of all this proves (1.2).

## 2 A bit of conformal mapping

The complex function

$$f(z) = \frac{az + b}{cz + d} \qquad\qquad ad - bc \neq 0 \tag{2.1}$$

shows up in a few places in physics. Personally I'm most interested in the context of electromagnetic duality. Transformations of this form map one theory of electromagnetism[2] to a physically equivalent theory of electromagnetism described by completely different sources and couplings. Remarkably, transformations of this type can exchange weakly and strongly coupled descriptions of physics. This is related to an idea called $S$-duality that you find in stringy-wingy theories. Show the following features:

1. The composition of two function of that form produces another function of the same form. In other words, if

$$f_i(z) = \frac{a_i z + b_i}{c_i z + d_i} \qquad\qquad a_i d_i - b_i c_i \neq 0 \ , \tag{2.2}$$

   then the composition is also

$$f_1\left(f_2(z)\right) = \frac{a_{12} z + b_{12}}{c_{12} z + d_{12}} \qquad\qquad a_{12} d_{12} - b_{12} c_{12} \neq 0 \ , \tag{2.3}$$

   for appropriate coefficients.

2. The transformation maps the complex plane minus the point $-d/c$ onto the complex plane minus the point $a/c$.

3. Transformations of this type map circles and lines onto other circles and lines.

*Answer*: See *Complex Analysis* by Bak and Newman, chapter 13; it's available free through the UCR library. This is also a good occasion to check that you know how to use the VPN to access digital resources off campus.

[2] A U(1) gauge theory. The relevant case is when $a, b, c, d \in \mathbb{Z}$. See, e.g. *Magnetic Monopoles* by Shnir.