Homework 3a: **Analytic functions are *too* nice**

Course: Physics 231, *Methods of Theoretical Physics* (2019)
Instructor: Professor Flip Tanedo (flip.tanedo@ucr.edu)

Due by: Wed, October 30

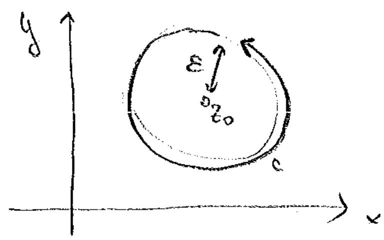
Cauchy's Integral Formula is the powerful statement that the integral of an analytic function, $f(z)$ over any closed loop C, is zero:

$$\oint_C dz\ f(z) = 0\ . \tag{0.1}$$

In other words, analytic functions are not only nice, they are *too nice*—their integrals don't contain any information! In this mini-homework we prove the integral formula in two ways: a *little tiny* circle and a *little tiny* square.

1 Little tiny circle

We show that the Cauchy integral formula (0.1) holds for the a small, counter-clockwise circular path of radius ϵ around the point z_0:



1.1 Parameterize the path

Parameterize the path in terms of the polar angle θ:

$$z(\theta) = z_0 + \epsilon e^{i\theta}\ . \tag{1.1}$$

Write the left-hand side of (0.1) as a definite integral by filling in the right-hand side of the following:

$$\oint_C dz\ f(z) = \int_0^{2\pi} d\theta\ f\left(z(\theta)\right)(\cdots)\ . \tag{1.2}$$

Fill in the facotr "$\cdots$".

1.2 Leading terms

Taylor expand $f(z)$ about $z = z_0$. Keep the zeroth and first order terms and plug them into the $d\theta$ integral. Write the two terms in the integrand as explicit functions of ϵ and θ:

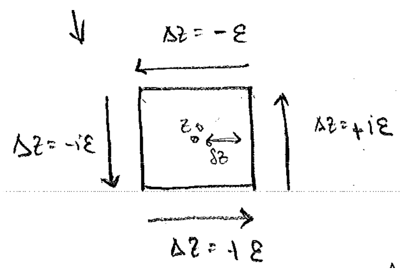
$$\oint_C dz\ f(z) = \int_0^{2\pi} d\theta\ [\text{ function of only } \epsilon \text{ and } \theta\ ] \tag{1.3}$$

1.3 Argue that the integral vanishes

Argue that the $d\theta$ integral indeed vanishes. Did you have to use $\epsilon \to 0$?

2 Little tiny square(s)

Now show that the Cauchy integral formula (0.1) holds for the a small, counter-clockwise square path of length ϵ around the point z_0:



2.1 Parameterize the path, leading terms

Decompose the path into four straight line pieces, $\Delta z_1, \cdots, \Delta z_4$. For each leg, do a Taylor expansion and keep the zeroth and first order terms. Write the integral as

$$\oint_C dz\, f(z) = f(z_0)\,[\cdots] + f'(z_0)\,[\cdots]\ , \tag{2.1}$$

where you fill in the "$\cdots$".

2.2 Argue that the integral vanishes

Show that both of the "$\cdots$" terms vanish. HINT: it may be useful to think of $dz = \Delta z$ in the integral and $(z - z_0) = \delta z$ in the Taylor expansion as two separate quantities that depend on ϵ.

3 Where did we use analyticity?

Why don't these proofs hold for *any* complex function? Where did we use the fact that the function $f(z)$ is *analytic*? HINT: it was important that our contours were "little tiny" shapes.

Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

1 Spot the error

What's wrong with the following sequence of steps:

$$e^{2\pi i} = 1 \tag{1.1}$$

$$\left(e^{2\pi i}\right)^{2\pi i} = 1^{2\pi i} = 1 \tag{1.2}$$

$$\left(e^{-2\pi}\right)^{2} = 1 \tag{1.3}$$

$$e^{-4\pi} = 1\ . \tag{1.4}$$

But since $e^x = 1 \Rightarrow x = 0$, this means that $\pi = 0$.