# Blockwise empirical likelihood method for spatial dependent data

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**Abstract** Existing blockwise empirical likelihood (BEL) method blocks the observations or their analogues, which is proven useful under some dependent data settings. In this paper, we introduce a new BEL (NBEL) method by blocking the scoring functions under high dimensional cases. We study the construction of confidence regions for the parameters in spatial autoregressive models with spatial autoregressive disturbances (SARAR models) with high dimension of parameters by using the NBEL method. It is shown that the NBEL ratio statistics are asymptotically $\chi^2$-type distributed, which are used to obtain the NBEL based confidence regions for the parameters in SARAR models. A simulation study is conducted to compare the performances of the NBEL and the usual EL methods.

*Keywords:* SARAR model; empirical likelihood; confidence region; high-dimensional statistical inference;

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## 1. Introduction

We firstly outline the developments of the empirical likelihood (EL) method under **independent samples**. The likelihood method is undoubtedly one of the most popular methods in statistics. Owen (1988, 1990) introduced the EL as a general nonparametric methodology for creating likelihood-type inference. Many researchers have contributed to the EL method for various models under independent samples. For instance, Owen (1991) used the EL method to construct confidence regions for a linear model with independence errors, Kolaczyk (1994) popularized the EL method for generalized linear models, and Chen (1996) obtained the EL confidence intervals for a probability density function, among others. A prerequisite for the existence of the EL ratio statistics is that 0 must be inside the convex hull of the scoring function set. To address this problem, by adding an additional pseudo-observation, Chen et al. (2008) proposed an adjusted EL (AEL) method to guarantee that the convex hull of the scoring function set contains 0. A similar AEL method can also be found in Emerson and Owen (2009). Owen (2001) provided a good review on the developments of the EL method under independent data as well as some developments for dependent data. Chen and Van Keilegom (2009) reviewed EL methods for regression problems.

Scondly, we briefly review the developments of the EL method for **dependent data** including spatial data. The EL methodology has also been studied under dependent data by several scholars. Kitamura (1997) first proposed blockwise EL (BEL) method to construct confidence intervals for parameters with mixing samples. Zhang (2006) further studied the BEL method under associated samples. Chen and Wong (2009) investigated the BEL method for the quantiles of a population under mixing samples. Qin and Li (2011) proposed a new BEL method to construct confidence intervals for regression vectors in linear models with associated model errors. Other applications of the BEL method proposed in Qin an Li (2011) can be found in Qin et al. (2011), Lei and Qin (2011) and Li et al. (2012), among others. Nordman and Lahiri (2014) provided a comprehensive review on the EL method for dependent data. Spatial data are also dependent, which appear in many fields such as econometrics, epidemiology, environmental science, image analysis, oceanography and many others. The EL method applied to spatial data is developed by Jin and Lee (2019) and Qin (2021). However, the usual EL method fails in the cases when dimensions is large in comparison with the sample size. The extension of EL to high dimensional problems, especially for dependent data, is itself a challenging task.

Our study is motivated by the interest in dimension reduction based on the EL method for spatial models since there is no EL method used for dimension reduction in spatial models. We propose to use a new blockwise EL (NBEL) method to treat this issue in this article. Instead of the penalized EL method in Tang and Leng (2010) and Leng and Tang (2012) based on sparse models to reduce the dimension of the parameter vector and unlike the usual BEL method in Chang et al. (2015) that blocks the data to reduce the dependence of data, the NBEL method blocks the scoring functions used in the usual EL approach to produce lower dimensional scoring functions. Instead of existing EL methods applicable only to the case that the sample size is larger than the dimension of parameters, the EL method presented in this article also works when the sample size is lower than the dimension of parameters. This similar EL method has also been successfully applied to test whether the regression coefficients are equivalent to given values in high dimensional linear models in Peng et al (2014a) and Zeng (2016). For more relevant literatures, one can be referred to Peng et al (2014a, 2014b).

The research of the EL method based on dimension reduction for non-sparse models is still in its infancy, and there is no research work on the NBEL method for spatial autoregressive models with spatial autoregressive disturbances (SARAR models). The results in this article show that the NBEL ratio statistics are asymptotically $\chi^2$-type distributed unrelated to the dimension of parameters, which are used to obtain a NBEL-based confidence region for the parameters in SARAR models. A simulation study is conducted to compare the performances of the NBEL and the usual EL method. Simulation results show that the NBEL confidence regions perform better than the usual EL method, especially when the data dimension is larger than the sample size. However, the power of the NBEL test performs not as good as the usual EL test when the distance between the parameters and the true parameters is small (See Remark 1 for explanation). Nevertheless, the performance of the NBEL test is comparable with the usual EL test when the distance between the parameters and the true parameters is large.

In Section 2, the NBEL method is presented. Section 3 uses the NBEL method to construct confidence regions for SARAR models. Simulation results are stated in Section 4. All technical proofs are in Section 5.

## 2. NBEL method and estimating equations

In this section, we introduce the concept of estimation equation, state a conclusion on the usual EL method for estimation equations, and present a result on the NBEL method for estimation equations.

### 2.1 Estimation equations

Suppose that $X \in R^d$ is a population and $X_1, X_2, \cdots, X_n$ are the i.i.d. observations of $X$. We further assume that there are $r$ known functions $g_j(x, \theta), 1 \leq j \leq r$, such that

$$Eg_j(X, \theta) = 0, 1 \leq j \leq r,$$

where $\theta \in \Theta \subseteq R^p$.

### 2.2 Usual EL for estimation equations

Suppose that 0 is inside the convex hull of the $g(X_i, \theta), 1 \leq i \leq n$. Define the scoring function

$$g(x, \theta) = (g_1(x, \theta), g_2(x, \theta), \cdots, g_r(x, \theta))^\tau, x \in R^d, \theta \in \Theta,$$

and the EL statistic (e.g., Qin and Lawless, 1994):

$$\ell_E(\theta) = \sum_{i=1}^{n} \log\{1 + t^\tau(\theta) g(X_i, \theta)\},$$

where $t(\theta) \in R^r$ is the solution of the following equations:

$$\sum_{i=1}^{n} \frac{g(X_i, \theta)}{1 + t^\tau(\theta) g(X_i, \theta)} = 0.$$

Following Qin and Lawless (1994), one can obtain the following result which states the limiting distribution of $\ell_E(\theta)$.

THEOREM 1. *Suppose that* $E\|g(X,\theta_0)\|^3 < \infty$ *and* $Cov(g(X,\theta_0))$ *is positive definite, where* $\|a\|$ *is the* $L_2$ *norm of the vector* $a \in R^r$ *and* $\theta_0$ *is the true value of* $\theta$*. Then for fixed* $p$ *and* $r$*, as* $n \to \infty$*,*

$$2\ell_E(\theta_0) \xrightarrow{d} \chi_r^2,$$

*where* $\chi_r^2$ *is a chi-squared distributed random variable with* $r$ *degrees of freedom.*

### 2.3 NBEL method for estimation equations

Instead of the usual block method which blocks the data, we block the scoring function as follows. Let $l$ is a positive number and $l \le r$ and let $s = [r/l]$, where $[a]$ stands for the integer part of $a$. Let

$$\tilde{g}_i(X,\theta) = \frac{1}{s} \sum_{j=(i-1)s+1}^{is} g_j(X,\theta), 1 \le i \le l,$$

and if $r > sl$

$$\tilde{g}_{l+1}(X,\theta) = \frac{1}{r-sl} \sum_{j=sl+1}^{r} g_j(X,\theta),$$

otherwise, $\tilde{g}_{l+1}(X,\theta)$ vanishes.

Let $l_0 = l+1$ if $r > sl$, otherwise $l_0 = l$. Define the block scoring function

$$\tilde{g}(x,\theta) = (\tilde{g}_1(x,\theta), \tilde{g}_2(x,\theta), \cdots, \tilde{g}_{l_0}(x,\theta))^\tau, x \in R^d, \theta \in \Theta,$$

and the NBEL statistic:

$$\tilde{\ell}_E(\theta) = \sum_{i=1}^{n} \log\{1 + \tilde{t}^\tau(\theta)\tilde{g}(X_i,\theta)\},$$

where $\tilde{t}(\theta) \in R^{l_0}$ is the solution of the following equations:

$$\sum_{i=1}^{n} \frac{\tilde{g}(X_i,\theta)}{1 + \tilde{t}^\tau(\theta)\tilde{g}(X_i,\theta)} = 0.$$

In fact, let $\mathbf{1_s}$ present the $s$-dimensional vector with 1 as its components and if $r = sl$ define

$$H = \begin{pmatrix} s^{-1}\mathbf{1}_\mathbf{s}^\tau & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & s^{-1}\mathbf{1}_\mathbf{s}^\tau & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & s^{-1}\mathbf{1}_\mathbf{s}^\tau \end{pmatrix}_{l \times r},$$

and if $r > sl$ one can similarly define the matrix $H$ as

$$H = \begin{pmatrix} s^{-1}\mathbf{1}_{\mathbf{s}}^{\tau} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & s^{-1}\mathbf{1}_{\mathbf{s}}^{\tau} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & s^{-1}\mathbf{1}_{\mathbf{s}}^{\tau} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & (r-sl)^{-1}\mathbf{1}_{\mathbf{r-sl}}^{\tau} \end{pmatrix}_{(l+1)\times r}.$$

Then

$$\tilde{g}(x, \theta) = Hg(x, \theta).$$

Based on Theorem 1, it is ready to obtain the limiting distribution of $\tilde{\ell}_E(\theta)$ presented in Theorem 2.

THEOREM 2. *Suppose that the conditions of Theorem 1 hold true. Then for fixed $p$ and $r$, as $n \to \infty$,*

$$2\tilde{\ell}_E(\theta_0) \xrightarrow{d} \chi^2_{l_0},$$

*where $\chi^2_{l_0}$ is a chi-squared distributed random variable with $l_0$ degrees of freedom.*

REMARK 1. *It is clear by the results in Qin and Lawless (1994), that the asymptotic efficiency of the NBEL estimator is less than or equal to that of the usual EL estimator, which provides an explanation that in the simulations the NBEL rejection rates under an alternative hypothesis perform not as good as the usual EL method when the distance between the parameters and the true parameters is relatively small.*

## 3. NBEL method for SARAR models

In this section, the NBEL method introduced in the previous section is used for SARAR models. We notice that the NBEL method relies on the estimation equations used in the usual EL method so that we first introduce the existing conclusion of the usual EL method on SARAR models before the asymptotic distribution based on the NBEL method for SARAR models is given. Although a different model is studied in this section, to simplify the use of notation we will use the same notations used in the previous section to present the usual EL and NBEL statistics.

### 3.1 Usual EL for SARAR models

In this article, the following spatial autoregressive model with spatial autoregressive disturbances (SARAR model) is investigated:

$$Y_n = \rho_1 W_n Y_n + X_n\beta + u_{(n)}, u_{(n)} = \rho_2 M_n u_{(n)} + \epsilon_{(n)}, \tag{1}$$

where $n$ is spatial sample size, $Y_n = (y_1, y_2, ..., y_n)^\tau$ is an $n \times 1$ vector of observations on the dependent variable, the matrix $X_n = (x_1, x_2, \cdots, x_n)^\tau$ with $x_i, i = 1, 2, \cdots, n$ is a $p \times 1$ exogenous vector of observations on the independent variable, $\beta$ is the $p \times 1$ vector of regression parameters, the scalar parameters $\rho_j, j = 1, 2,$ are spatial autoregressive coefficients with $|\rho_j| < 1, j = 1, 2,$ $W_n$ and $M_n$ are known $n \times n$ spatial weight matrices whose diagonal elements are zero, the disturbance vector $u_{(n)} = (u_1, u_2, \cdots, u_n)^\tau$ contains a spatially autocorrelated setting, the vector $\epsilon_{(n)} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)^\tau$ represents the random innovations which satisfies

$$E\epsilon_{(n)} = 0, Var(\epsilon_{(n)}) = \sigma^2 I_n.$$

The $\rho_1 W_n$ term is a spacial lag in the dependent variable and its coefficient represents the spatial influence due to neighbors realized dependent variable. The $\rho_2 M_n$ term is a spacial lag in the disturbances and its coefficient represents the spacial effect of unobservables on neighboring units. This model is introduced by Cliff and Ord (1973) and Anselin (1988). The usual EL method to construct the confidence region for the parameters in this model is proposed in Qin (2021). For the sake of conceptual integrity, we introduce the usual EL method in the following.

Let $A_n(\rho_1) = I_n - \rho_1 W_n$, $B_n(\rho_2) = I_n - \rho_2 M_n$ and suppose that $A_n(\rho_1)$ and $B_n(\rho_2)$ are nonsingular. Then (1) can be written as

$$Y_n = A_n^{-1}(\rho_1) X_n \beta + A_n^{-1}(\rho_1) B_n^{-1}(\rho_2) \epsilon_{(n)}.$$

At this moment, suppose that $\epsilon_{(n)}$ is normally distributed, which is firstly used to derive the EL statistic only and not employed in our main results. Then the log-likelihood function based on the response vector $Y_n$ is

$$L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 + \log|A_n(\rho_1)| + \log|B_n(\rho_2)| - \frac{1}{2\sigma^2}\epsilon_{(n)}^\tau \epsilon_{(n)},$$

where $\epsilon_{(n)} = B_n(\rho_2)\{A_n(\rho_1)Y_n - X_n\beta\}$. Let $G_n = B_n(\rho_2) W_n A_n^{-1}(\rho_1) B_n^{-1}(\rho_2), H_n = M_n B_n^{-1}(\rho_2), \tilde{G}_n = \frac{1}{2}(G_n + G_n^\tau)$ and $\tilde{H}_n = \frac{1}{2}(H_n + H_n^\tau)$. It can be shown that (e.g.

Anselin, 1988, p. 74-75)

$$\partial L/\partial\beta = \frac{1}{\sigma^2} X_n^\tau B_n^\tau(\rho_2)\epsilon_{(n)},$$

$$\partial L/\partial\rho_1 = \frac{1}{\sigma^2}\{B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta\}^\tau\epsilon_{(n)} + \frac{1}{\sigma^2}\{\epsilon_{(n)}^\tau\tilde{G}_n\epsilon_{(n)} - \sigma^2 tr(\tilde{G}_n)\},$$

$$\partial L/\partial\rho_2 = \frac{1}{\sigma^2}\{\epsilon_{(n)}^\tau\tilde{H}_n\epsilon_{(n)} - \sigma^2 tr(\tilde{H}_n)\},$$

$$\partial L/\partial\sigma^2 = \frac{1}{2\sigma^4}\{\epsilon_{(n)}^\tau\epsilon_{(n)} - n\sigma^2\}.$$

Letting above derivatives be 0, we obtain the following estimating equations:

$$X_n^\tau B_n^\tau(\rho_2)\epsilon_{(n)} = 0, \tag{2}$$

$$\{B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta\}^\tau\epsilon_{(n)} + \{\epsilon_{(n)}^\tau\tilde{G}_n\epsilon_{(n)} - \sigma^2 tr(\tilde{G}_n)\} = 0, \tag{3}$$

$$\epsilon_{(n)}^\tau\tilde{H}_n\epsilon_{(n)} - \sigma^2 tr(\tilde{H}_n) = 0, \tag{4}$$

$$\epsilon_{(n)}^\tau\epsilon_{(n)} - n\sigma^2 = 0. \tag{5}$$

Let $\tilde{g}_{ij}$, $\tilde{h}_{ij}$, $b_i$ and $s_i$ denote the $(i,j)$ element of the matrix $\tilde{G}_n$, the $(i,j)$ element of the matrix $\tilde{H}_n$, the $i$-th column of the matrix $X_n^\tau B_n^\tau(\rho_2)$ and $i$-th component of the vector $B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta$, respectively, and adapt the convention that any sum with an upper index of less than one is zero. To deal with the quadratic forms in (2) and (3), define the $\sigma$-fields: $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_i = \sigma(\epsilon_1, \epsilon_2, \cdots, \epsilon_i), 1 \le i \le n$. Let

$$\tilde{Y}_{in} = \tilde{g}_{ii}(\epsilon_i^2 - \sigma^2) + 2\epsilon_i\sum_{j=1}^{i-1}\tilde{g}_{ij}\epsilon_j, \quad \tilde{Z}_{in} = \tilde{h}_{ii}(\epsilon_i^2 - \sigma^2) + 2\epsilon_i\sum_{j=1}^{i-1}\tilde{h}_{ij}\epsilon_j. \tag{6}$$

Then $\mathcal{F}_{i-1} \subseteq \mathcal{F}_i$, $\tilde{Y}_{in}$ is $\mathcal{F}_i$-measurable and $E(\tilde{Y}_{in}|\mathcal{F}_{i-1}) = 0$. Thus $\{\tilde{Y}_{in}, \mathcal{F}_i, 1 \le i \le n\}$ and $\{\tilde{Z}_{in}, \mathcal{F}_i, 1 \le i \le n\}$ form two martingale difference arrays and

$$\epsilon_{(n)}^\tau\tilde{G}_n\epsilon_{(n)} - \sigma^2 tr(\tilde{G}_n) = \sum_{i=1}^n \tilde{Y}_{in}, \quad \epsilon_{(n)}^\tau\tilde{H}_n\epsilon_{(n)} - \sigma^2 tr(\tilde{H}_n) = \sum_{i=1}^n \tilde{Z}_{in}. \tag{7}$$

Based on (2) to (7), we can get the scoring function:

$$\omega_i(\theta) = \begin{pmatrix} b_i\epsilon_i \\ \tilde{g}_{ii}(\epsilon_i^2 - \sigma^2) + 2\epsilon_i\sum_{j=1}^{i-1}\tilde{g}_{ij}\epsilon_j + s_i\epsilon_i \\ \tilde{h}_{ii}(\epsilon_i^2 - \sigma^2) + 2\epsilon_i\sum_{j=1}^{i-1}\tilde{h}_{ij}\epsilon_j \\ \epsilon_i^2 - \sigma^2 \end{pmatrix}_{(p+3)\times 1},$$

where $\epsilon_i$ is the $i$-th component of $\epsilon_{(n)} = B_n(\rho_2)\{A_n(\rho_1)Y_n - X_n\beta\}$.

Qin (2021) defined the following EL ration statistic for $\theta = (\beta^\tau, \rho_1, \rho_2, \sigma^2)^\tau \in R^{p+3}$:

$$L_n(\theta) = \sup\left\{\prod_{i=1}^n (n\hat{p}_i) : \hat{p}_i \geq 0, \sum_{i=1}^n \hat{p}_i = 1, \sum_{i=1}^n \hat{p}_i \omega_i(\theta) = 0\right\}.$$

Following Owen (1990), we have

$$\ell_n(\theta) \hat{=} -2\log L_n(\theta) = 2\sum_{i=1}^n \log\{1 + \hat{\lambda}^\tau(\theta)\omega_i(\theta)\},$$

where $\hat{\lambda}(\theta) \in R^{p+3}$ is the solution of the following equation:

$$\frac{1}{n}\sum_{i=1}^n \frac{\omega_i(\theta)}{1 + \hat{\lambda}^\tau(\theta)\omega_i(\theta)} = 0.$$

Let $\mu_j = E(\epsilon_1^j), j = 3, 4$, Use $Vec(diagA)$ to denote the vector formed by the diagonal elements of a matrix $A$ and use $\|a\|$ to denote the $L_2$-norm of a vector $a$. To obtain the asymptotical distribution of $\tilde{\ell}_n(\theta)$, we need following assumptions.

A1. $\{\epsilon_i, 1 \leq i \leq n\}$ are independent and identically distributed random variables with mean 0, variance $\sigma^2$ and $E|\epsilon_1|^{4+\eta_1} < \infty$ for some $\eta_1 > 0$.

A2. $W_n, M_n, A_n^{-1}(\rho_1), B_n^{-1}(\rho_2)$ and $\{x_i\}$ be as described above. They satisfy the following conditions:

(i) The row and column sums of $W_n, M_n, A_n^{-1}(\rho_1)$ and $B_n^{-1}(\rho_2)$ are uniformly bounded in absolute value,

(ii) $\{x_i\}, i = 1, 2, ..., n$ are uniformly bounded.

A3. There is a constants $c_j > 0, j = 1, 2$, such that $0 < c_1 \leq \lambda_{min}(n^{-1}\Sigma_{p+3}) \leq \lambda_{max}(n^{-1}\Sigma_{p+3}) \leq c_2 < \infty$, where $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues of a matrix $A$, respectively,

$$\Sigma_{p+3} = \Sigma_{p+3}^\tau = Cov\left\{\sum_{i=1}^n \omega_i(\theta)\right\} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} \\ \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44} \end{pmatrix}, \quad (8)$$

where

$$
\begin{aligned}
\Sigma_{11} &= \sigma^2\{B_n(\rho_2)X_n\}'B_n(\rho_2)X_n, \\
\Sigma_{12} &= \sigma^2\{B_n(\rho_2)X_n\}'B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta + \mu_3\{B_n(\rho_2)X_n\}'Vec(diag\tilde{G}_n), \\
\Sigma_{13} &= \mu_3\{B_n(\rho_2)X_n\}'Vec(diag\tilde{H}_n), \quad \Sigma_{14} = \mu_3\{B_n(\rho_2)X_n\}'\mathbf{1}_n, \\
\Sigma_{22} &= 2\sigma^4tr(\tilde{G}_n{}^2) + \sigma^2\{B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta\}'B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta \\
&\quad +(\mu_4 - 3\sigma^4)\|Vec(diag\tilde{G}_n)\|^2 + 2\mu_3\{B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta\ \}'Vec(diag\tilde{G}_n), \\
\Sigma_{23} &= 2\sigma^4tr(\tilde{G}_n\tilde{H}_n) + (\mu_4 - 3\sigma^4)Vec'(diag\tilde{G}_n)Vec(diag\tilde{H}_n) \\
&\quad +\mu_3\{B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta\ \}'Vec(diag\tilde{H}_n), \\
\Sigma_{24} &= (\mu_4 - \sigma^4)tr(\tilde{G}_n) + \mu_3\{B_n(\rho_2)W_nA_n^{-1}(\rho_1)X_n\beta\ \}'\mathbf{1}_n, \\
\Sigma_{33} &= 2\sigma^4tr(\tilde{H}_n{}^2)(\mu_4 - 3\sigma^4)\|Vec(diag\tilde{H}_n)\|^2, \\
\Sigma_{34} &= (\mu_4 - \sigma^4)tr(\tilde{H}_n), \quad \Sigma_{44} = n(\mu_4 - \sigma^4).
\end{aligned}
$$

We now state the main results in Qin (2021).

THEOREM 3. *Suppose that conditions A1-A3 hold. Let $\theta_0$ be the true value of $\theta$. As $n \to \infty$,*

$$\ell_n(\theta_0) \xrightarrow{d} \chi^2_{p+3},$$

*where $\chi^2_{p+3}$ is a chi-squared distributed random variable with $p+3$ degrees of freedom.*

### 3.2 NBEL method for SARAR models

Following the NBEL method proposed in Section 2.3, we define the NBEL scoring functions for the SARSAR model as follows. Let $l$ is a positive number and $l \le p+3$ and let $s = [(p+3)/l]$, where $[a]$ stands for the integer part of $a$. If $p+3 = sl$, define

$$
H = \begin{pmatrix}
s^{-1}\mathbf{1}_\mathbf{s}^\tau & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & s^{-1}\mathbf{1}_\mathbf{s}^\tau & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & s^{-1}\mathbf{1}_\mathbf{s}^\tau
\end{pmatrix}_{l\times(p+3)},
$$

and if $p+3 > sl$, define

$$H = \begin{pmatrix} s^{-1}\mathbf{1}_{\mathbf{s}}^{\tau} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & s^{-1}\mathbf{1}_{\mathbf{s}}^{\tau} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & s^{-1}\mathbf{1}_{\mathbf{s}}^{\tau} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & (p+3-sl)^{-1}\mathbf{1}_{\mathbf{p+3-sl}}^{\tau} \end{pmatrix}_{(l+1)\times(p+3)}.$$

Let

$$\widetilde{\omega}_i(\theta) = H\omega_i(\theta), 1 \le i \le n.$$

Then define the NBEL statistic for the SARSAR model:

$$\tilde{\ell}_n(\theta) = \sum_{i=1}^{n} \log\{1 + \tilde{t}^{\tau}(\theta)\widetilde{\omega}_i(\theta)\},$$

where $\tilde{t}(\theta)$ is the solution of the following equations:

$$\sum_{i=1}^{n} \frac{\widetilde{\omega}_i(\theta)}{1 + \tilde{t}^{\tau}(\theta)\widetilde{\omega}_i(\theta)} = 0.$$

Let $l_0 = l+1$ if $p+3 > sl$, otherwise $l_0 = l$. We now state the main results.

THEOREM 4. *Suppose that conditions A1-A3 hold. Let $\theta_0$ be the true value of $\theta$. As $n \to \infty$,*

$$\tilde{\ell}_n(\theta_0) \xrightarrow{d} \chi^2_{l_0},$$

*where $\chi^2_{l_0}$ is a chi-squared distributed random variable with $l_0$ degrees of freedom.*

Let $z_\alpha(l_0)$ satisfy $P(\chi^2_{l_0} \ge z_\alpha(l_0)) = \alpha$ for $0 < \alpha < 1$. It follows from Theorem 4 that an NBEL-based confidence intervals (CIs) for $\theta$ with asymptotically correct coverage probability (CP) $1-\alpha$ can be constructed as

$$\{\theta : \tilde{\ell}_n(\theta) \le z_\alpha(l_0)\}. \tag{9}$$

## 4. Simulations

According to Qin (2021), let $z_\alpha(p+3)$ satisfy $P(\chi^2_{p+3} \geq z_\alpha(p+3)) = \alpha$ for $0 < \alpha < 1$, and the usual EL-based CI for $\theta$ is given by equation (10)

$$\{\theta : \ell_n(\theta) \leq z_\alpha(p+3)\}. \tag{10}$$

We conducted a small simulation study to compare the finite sample performances of the confidence regions based on the usual EL and the NBEL methods with confidence level $1 - \alpha = 0.95$. In the simulations, we take $l = 1$, i.e. there is only one block for the scoring functions. One can download R codes related to this article at https://github.com/Tang-Jay/NBEL.

In the simulations, we used the model: $Y_n = \rho_1 W_n Y_n + X_n \beta_0 + u_{(n)}, u_{(n)} = \rho_2 M_n u_{(n)} + \epsilon_{(n)}$, with $(\rho_1, \rho_2) = (0.85, 0.15)$, $\beta_0 = \mathbf{1}_p$, and $\{X_i\} \sim N(\mathbf{0}, I_p)$, where $\mathbf{1}_p$ denotes a $p \times 1$ vector of ones and $I_p$ denotes a $p \times p$ identity matrix. To make $p$ and $n$ increase simultaneously, we took $p = [3n^{index}]$, where $[x]$ is the integer part of $x$.

For the contiguity weight matrix $W_n = (w_{ij})$, we took $w_{ij} = 1$ if spatial units $i$ and $j$ are neighbours by queen contiguity rule (namely, they share common border or vertex), $w_{ij} = 0$ otherwise (Anselin, 1988, p. 18). We considered some ideal cases of spatial units: $n = m \times m$ and $M_n = W_n$.

**Experiment 1**

Denote the true value of $\theta$ as $\theta_0' = (\beta_0', \rho_1, \rho_2, \sigma_0^2)$, and then let $m = 10, 20, 30$ and $index = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. In addition, $\epsilon_i's$ are taken from $N(0,1)$, $N(0, 0.75)$, $t(5)$ and $\chi^2_4 - 4$, respectively, and $\sigma_0^2$ are chosen as 1, 0.75, 2.5, 8, respectively. We report the proportion of $\tilde{\ell}_n(\theta_0) \leq z_{0.05}(1)$ and $\ell_n(\theta_0) \leq z_{0.05}(p+3)$ respectively in our $2,000$ simulations. We use the mark '-' to denote the invalid result. The results of simulations are reported in Table 1.

**Experiment 2**

Denote $\theta' = (\beta', \rho_1, \rho_2, \sigma^2)$, and let $m = 10, 15, 20, 25, 30$, $index = 0, 0.1, 0.2$,$0.3, 0.4, 0.5$, $(\beta, \sigma^2)' = (\beta_0, \sigma_0^2)' + \mathbf{\Delta 1}'_{p+1}$, $\Delta = 0, 0.1, 1, 2$, $\epsilon_i \sim N(0,1)$. We report the proportion of $\tilde{\ell}_n(\theta) \geq z_{0.05}(1)$ and $\ell_n(\theta) \geq z_{0.05}(p+3)$, respectively in our 2,000 simulations. The results of simulations are reported in Table 2.

**Experimental Analysis**

Table 1 shows that the confidence regions based on EL behave well with coverage probabilities very close to the nominal level 0.95, as $n$ increases, only when $index \leq 0.2$. When $0.2 < index \leq 0.5$, the coverage probabilities of the confidence regions based on EL fall to the range [0.022, 0.928] for $N(0,1)$ distribution , [0.020, 0.936] for $N(0,0.75)$, [0.011, 0.906] for $t(5)$ and [0.010, 0.902] for $\chi^2$ distribution, which are far from the nominal level 0.95, as $p$ increases, and when $index > 0.5$, the EL method is invalid.

Table 1 also shows that the confidence regions based on NBEL behave well with coverage probabilities very close to the nominal level 0.95, as $n$ increases, whether $index \leq 0.2$ or not. When $0.2 < index \leq 0.5$, the coverage probabilities of the confidence regions based on NBEL fall to the range [0.938, 0.952] for $N(0,1)$ distribution, [0.937, 0.957] for $N(0,0.75)$, [0.918, 0.948] for $t(5)$ and [0.919, 0.952] for $\chi^2$ distribution, which are close to the nominal level 0.95, as $p$ increases, and even if $index > 0.5$, the NBEL method is still valid.

Table 2 shows that the rejection rates based on usual EL and NBEL tests both behave well with coverage probabilities very close to 1, as $\|\theta_0 - \theta\|$ increases. However, the NBEL rejection regions perform not as good as EL method when the distance between the alternatives and the null hypothesis is small.

When $\epsilon_i \sim N(0,1)$, we compared the sample quantiles of $\ell_n(\theta_0)$ with the quantiles of $\chi^2(p+3)$ shown in Figure 1 and the sample quantiles of $\tilde{\ell}_n(\theta_0)$ with the quantiles of $\chi^2(1)$ in Figure 2. Based on the simulation results, in low dimension case as $n$ increases, all EL sample distributions and the theoretic distributions agree well. However, when the dimension is high, all the EL sample distributions fit the theoretic distributions poor while all the NBEL sample distributions agree the theoretic distributions well when the dimension is growing.

In conclusion, our simulation results recommend NBEL method when the dimension is high.

## 5. Proofs

Lemma 1. *Suppose that Assumptions A1-A3 are satisfied, then as $n \to \infty$,*

$$\max_{1 \leq i \leq n} \|\omega_i(\theta)\| = o_p(n^{1/2}), \tag{11}$$

Table 1: Coverage probabilities of the EL and NBEL confidence regions

| $n$ | $p$ | $\epsilon_i \sim N(0,1)$ | | $\epsilon_i \sim N(0,0.75)$ | | $\epsilon_i \sim t(5)$ | | $\epsilon_i \sim \chi^2(4)-4$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | EL | NBEL | EL | NBEL | EL | NBEL | EL | NBEL |
| 100 | 3 | 0.861 | 0.948 | 0.872 | 0.940 | 0.778 | 0.904 | 0.775 | 0.910 |
| | 5 | 0.824 | 0.944 | 0.824 | 0.941 | 0.720 | 0.931 | 0.730 | 0.920 |
| | 8 | 0.740 | 0.950 | 0.718 | 0.957 | 0.591 | 0.918 | 0.624 | 0.919 |
| | 12 | 0.548 | 0.943 | 0.545 | 0.937 | 0.392 | 0.928 | 0.452 | 0.919 |
| | 19 | 0.250 | 0.940 | 0.266 | 0.938 | 0.132 | 0.930 | 0.182 | 0.920 |
| | 30 | 0.022 | 0.941 | 0.020 | 0.940 | 0.011 | 0.928 | 0.010 | 0.929 |
| | 48 | - | 0.946 | - | 0.943 | - | 0.931 | - | 0.928 |
| | 75 | - | 0.949 | - | 0.945 | - | 0.928 | - | 0.944 |
| | 119 | - | 0.948 | - | 0.945 | - | 0.933 | - | 0.933 |
| | 189 | - | 0.945 | - | 0.940 | - | 0.936 | - | 0.943 |
| 400 | 3 | 0.931 | 0.950 | 0.929 | 0.950 | 0.895 | 0.929 | 0.907 | 0.941 |
| | 5 | 0.929 | 0.949 | 0.925 | 0.936 | 0.891 | 0.937 | 0.897 | 0.940 |
| | 10 | 0.905 | 0.952 | 0.911 | 0.956 | 0.856 | 0.936 | 0.850 | 0.945 |
| | 18 | 0.856 | 0.942 | 0.850 | 0.949 | 0.723 | 0.945 | 0.740 | 0.941 |
| | 33 | 0.655 | 0.938 | 0.655 | 0.952 | 0.459 | 0.948 | 0.476 | 0.950 |
| | 60 | 0.163 | 0.944 | 0.186 | 0.946 | 0.061 | 0.943 | 0.088 | 0.939 |
| | 109 | - | 0.951 | - | 0.950 | - | 0.948 | - | 0.959 |
| | 199 | - | 0.951 | - | 0.947 | - | 0.941 | - | 0.940 |
| | 362 | - | 0.957 | - | 0.947 | - | 0.946 | - | 0.938 |
| | 659 | - | 0.941 | - | 0.950 | - | 0.946 | - | 0.948 |
| 900 | 3 | 0.951 | 0.948 | 0.935 | 0.948 | 0.921 | 0.936 | 0.931 | 0.945 |
| | 6 | 0.942 | 0.946 | 0.941 | 0.955 | 0.915 | 0.946 | 0.922 | 0.949 |
| | 12 | 0.928 | 0.945 | 0.936 | 0.940 | 0.906 | 0.945 | 0.902 | 0.946 |
| | 23 | 0.905 | 0.948 | 0.909 | 0.948 | 0.840 | 0.937 | 0.849 | 0.942 |
| | 46 | 0.792 | 0.952 | 0.798 | 0.956 | 0.624 | 0.945 | 0.625 | 0.952 |
| | 90 | 0.353 | 0.946 | 0.337 | 0.951 | 0.118 | 0.942 | 0.172 | 0.949 |
| | 178 | - | 0.938 | - | 0.951 | - | 0.948 | - | 0.948 |
| | 351 | - | 0.950 | - | 0.948 | - | 0.954 | - | 0.955 |
| | 693 | - | 0.955 | - | 0.946 | - | 0.942 | - | 0.957 |
| | 1368 | - | 0.945 | - | 0.947 | - | 0.953 | - | 0.953 |

Table 2: Frequencies of rejection of the EL and NBEL tests

| $n$ | $p$ | $\Delta = 0$ | | $\Delta = 0.1$ | | $\Delta = 1$ | | $\Delta = 2$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | EL | NBEL | EL | NBEL | EL | NBEL | EL | NBEL |
| 100 | 3 | 0.146 | 0.058 | 0.205 | 0.091 | 0.994 | 0.083 | 1.000 | 0.976 |
| | 5 | 0.180 | 0.062 | 0.256 | 0.091 | 1.000 | 0.427 | 1.000 | 0.999 |
| | 8 | 0.282 | 0.066 | 0.325 | 0.082 | 1.000 | 0.732 | 1.000 | 1.000 |
| | 12 | 0.453 | 0.053 | 0.475 | 0.082 | 1.000 | 0.851 | 1.000 | 1.000 |
| | 19 | 0.755 | 0.058 | 0.726 | 0.065 | 1.000 | 0.891 | 1.000 | 1.000 |
| | 30 | 0.979 | 0.046 | 0.968 | 0.091 | 1.000 | 0.945 | 1.000 | 1.000 |
| 225 | 3 | 0.079 | 0.050 | 0.140 | 0.094 | 1.000 | 0.358 | 1.000 | 1.000 |
| | 5 | 0.097 | 0.044 | 0.129 | 0.089 | 1.000 | 0.907 | 1.000 | 1.000 |
| | 9 | 0.114 | 0.063 | 0.181 | 0.099 | 1.000 | 0.994 | 1.000 | 1.000 |
| | 15 | 0.220 | 0.060 | 0.263 | 0.100 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 26 | 0.513 | 0.057 | 0.525 | 0.100 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 45 | 0.925 | 0.045 | 0.911 | 0.108 | 1.000 | 1.000 | 1.000 | 1.000 |
| 400 | 3 | 0.072 | 0.061 | 0.146 | 0.141 | 1.000 | 0.735 | 1.000 | 1.000 |
| | 5 | 0.062 | 0.061 | 0.160 | 0.166 | 1.000 | 0.999 | 1.000 | 1.000 |
| | 10 | 0.092 | 0.047 | 0.143 | 0.117 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 18 | 0.137 | 0.046 | 0.271 | 0.141 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 33 | 0.342 | 0.048 | 0.586 | 0.142 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 60 | 0.807 | 0.050 | 0.984 | 0.177 | 1.000 | 1.000 | 1.000 | 1.000 |
| 625 | 3 | 0.059 | 0.056 | 0.152 | 0.191 | 1.000 | 0.926 | 1.000 | 1.000 |
| | 6 | 0.060 | 0.058 | 0.134 | 0.184 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 11 | 0.076 | 0.052 | 0.151 | 0.168 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 21 | 0.120 | 0.054 | 0.351 | 0.170 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 39 | 0.253 | 0.050 | 0.857 | 0.194 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 75 | 0.740 | 0.056 | 1.000 | 0.234 | 1.000 | 1.000 | 1.000 | 1.000 |
| 900 | 3 | 0.049 | 0.058 | 0.212 | 0.226 | 1.000 | 0.989 | 1.000 | 1.000 |
| | 6 | 0.050 | 0.055 | 0.168 | 0.221 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 12 | 0.061 | 0.055 | 0.216 | 0.224 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 23 | 0.101 | 0.046 | 0.552 | 0.235 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 46 | 0.221 | 0.059 | 0.994 | 0.272 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 90 | 0.654 | 0.049 | 1.000 | 0.334 | 1.000 | 1.000 | 1.000 | 1.000 |

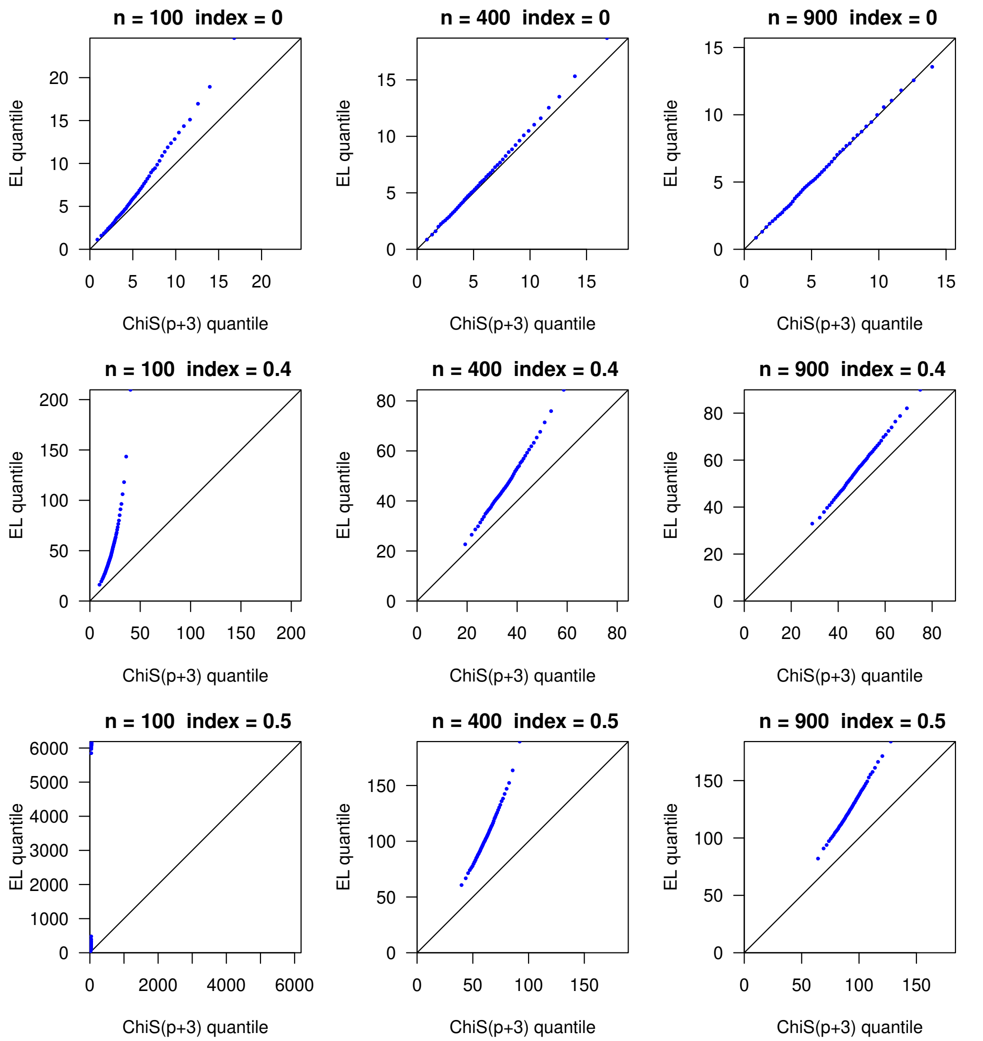


Figure 1: Q-Q plots of $\ell_n(\theta_0)$ and $\chi^2_{p+3}$ with $\epsilon_i \sim N(0,1)$

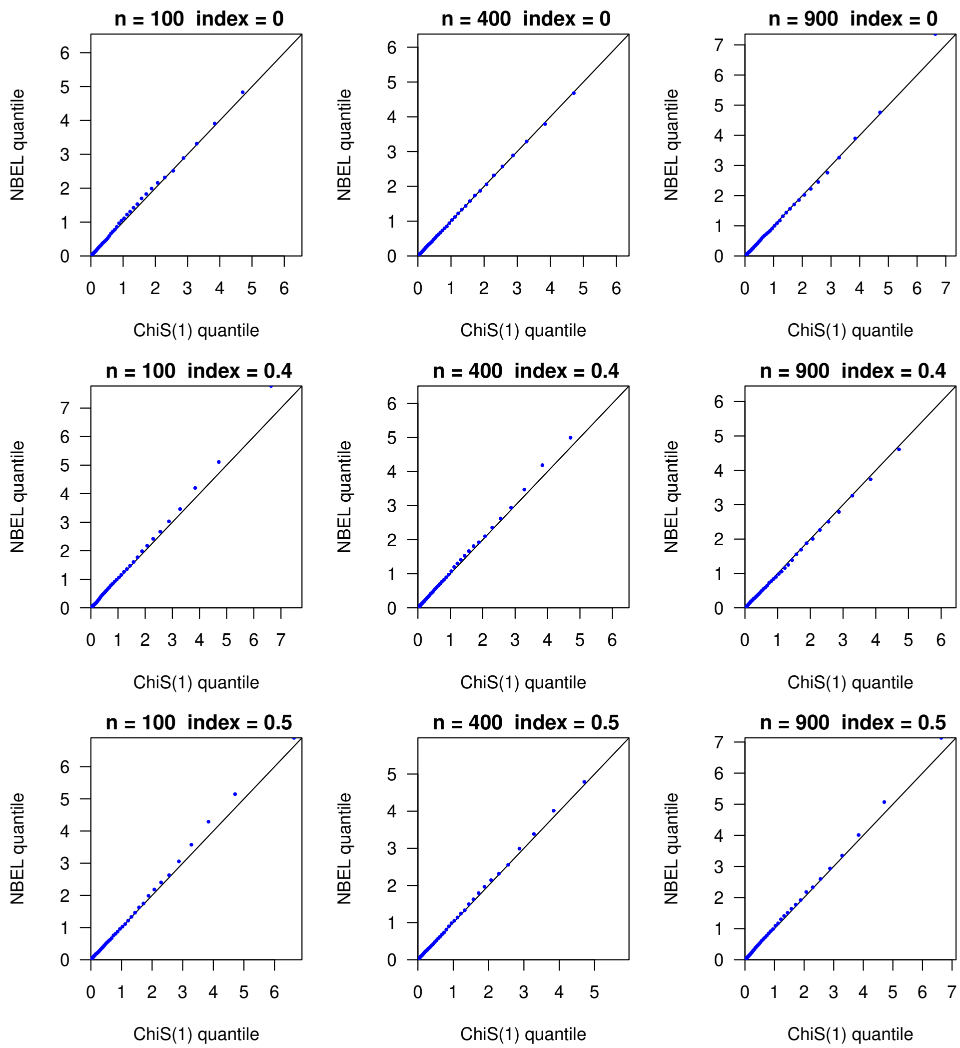


Figure 2: Q-Q plots of $\tilde{\ell}_n(\theta_0)$ and $\chi_1^2$ with $\epsilon_i \sim N(0,1)$

$$\Sigma_{p+3}^{-1/2}\sum_{i=1}^{n}\omega_i(\theta)\xrightarrow{d}N(0,1), \tag{12}$$

$$n^{-1}\sum_{i=1}^{n}\omega_i(\theta)\omega_i^{\tau}(\theta)=n^{-1}\Sigma_{p+3}+o_p(1), \tag{13}$$

$$\sum_{i=1}^{n}\|\omega_i(\theta)\|^3=O_p(n), \tag{14}$$

*where $\Sigma_{p+3}$ is given in (8).*

**Proof.** See Lemma 3 in Qin (2021).

LEMMA 2. *Suppose that Assumptions A1-A3 are satisfied, then as $n\to\infty$,*

$$Z_n=\max_{1\le i\le n}\|\tilde{\omega}_i(\theta)\|=o_p(n^{1/2}), \tag{15}$$

$$\sum_{i=1}^{n}\tilde{\omega}_i(\theta)\xrightarrow{d}N(\mathbf{0}_{l_0},H\Sigma_{p+3}H^{\tau}), \tag{16}$$

$$n^{-1}\sum_{i=1}^{n}\tilde{\omega}_i(\theta)\tilde{\omega}_i^{\tau}(\theta)=n^{-1}H\Sigma_{p+3}H^{\tau}+o_p(1), \tag{17}$$

$$\sum_{i=1}^{n}\|\tilde{\omega}_i(\theta)\|^3=O_p(n), \tag{18}$$

*where $\Sigma_{p+3}$ is given in (8) and $\mathbf{0}_{l_0}$ presents the $l_0$-dimensional vector with 0 as its components.*

**Proof.** If $p+3=sl$, by $\tilde{\omega}_i(\theta)=H\omega_i(\theta)$, we have

$$\|\tilde{\omega}_i(\theta)\|=\|H\omega_i(\theta)\|=(\omega_i^{\tau}(\theta)H^{\tau}H\omega_i(\theta))^{1/2}\le\left(\lambda_{max}(H^{\tau}H)\|\omega_i(\theta)\|^2\right)^{1/2}\le C\|\omega_i(\theta)\|,$$

where $\lambda_{max}(H^{\tau}H)$ denotes the maximum eigenvalues of a matrix $H^{\tau}H$. Further,

$$\sum_{i=1}^{n}\|\tilde{\omega}_i(\theta)\|^3=\sum_{i=1}^{n}\|H\omega_i(\theta)\|^3\le C\sum_{i=1}^{n}\|\omega_i(\theta)\|^3.$$

If $p+3>sl$, the above equation still holds, and the proof is analogous. Combining with (11) and (14), we have $\max_{1\le i\le n}\|\tilde{\omega}_i(\theta)\|=o_p(n^{1/2})$ and $\sum_{i=1}^{n}\|\tilde{\omega}_i(\theta)\|^3=O_p(n)$. (15) and (18) is proved.

Obviously,

$$\sum_{i=1}^{n} \tilde{\omega}_i(\theta) = H \sum_{i=1}^{n} \omega_i(\theta),$$
$$n^{-1} \sum_{i=1}^{n} \tilde{\omega}_i(\theta) \tilde{\omega}_i^{\tau}(\theta) = H n^{-1} \sum_{i=1}^{n} \omega_i(\theta) \omega_i^{\tau}(\theta) H^{\tau}.$$

Using (12) and (13), we obtain (16) and (17).

**Proof of Theorem 4** The proof is analogous to the proof of Theorem 1 in Qin (2021).

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