Expert Systems With Applications 133 (2019) 49–58

Contents lists available at ScienceDirect

Expert Systems With Applications

journal homepage: www.elsevier.com/locate/eswa

Portfolio selection with coherent Investor's expectations under uncertainty☆

Hong-Quan Li[a], Zhi-Hong Yi[b,*]

[a] College of Business, Hunan Normal University, China
[b] College of Modern Economics and Management, Jiangxi University of Finance and Economics, China

ARTICLE INFO

Article history:
Received 14 December 2018
Revised 18 April 2019
Accepted 8 May 2019
Available online 9 May 2019

Keywords:
Fuzzy portfolio selection
The mean-variance-skewness model
Coherence
Investor's expectations

ABSTRACT

Fuzzy portfolio selection is effective in coping with the uncertainty in financial decision making, in which investor's expectation plays an important role. In this paper, to capture the coherence of the investor's expectation we develop a new trapezoidal fuzzy numbers with an adaptive index, through which the membership degrees for favorable and unfavorable scenarios are transformed consistently to avoid the logical confusion. We also present the possibilistic expected mean, variance and skewness under the new measurement. Then, the new trapezoidal fuzzy numbers are employed in fuzzy mean-variance model and mean-variance-skewness model for optimal asset allocation. The validity and advantages of these models can be illustrated by the numerical examples in the end.

1. Introduction

Uncertainty exists universally in the world and one of the effective ways to cope with the uncertainty in academic community is to use fuzzy set theory (Zadeh, 1965), which has the advantages in expressing the vagueness and uncertainty by natural languages. Financial decision making is no exception. In the literature, fuzzy portfolio selection (Huang, 2009; Wang & Zhu, 2002) has drawn much attention from academia. Moreover, some advanced fuzzy methodologies, such as the fuzzy optimizations (Fang, Lai, & Wang, 2008; Gupta, Mehlawat, Inuiguchi, & Chandra, 2014), fuzzy decisions (Fang, Lai, & Wang, 2006), are incorporated in portfolio selection. With the upgrading models in the classical asset allocation, some pertaining methods within fuzzy frame are developed, such as the fuzzy Black-Litterman model (Fang, Bo, Zhao, & Wang, 2018), the fuzzy mean-variance-skewness model (Li, Guo, & Yu, 2015), the fuzzy mean-variance-skewness-semi-kurtosis model (Kamdem, Deffo, & Fono, 2012), the possibilistic mean-semivariance models (Liu & Zhang, 2013).

Under uncertainty, it is reasonable to assume the heterogeneity of investors' expectation. Following this train of thought, many researchers employ some parameterized fuzzy numbers in different forms for describing returns of assets to model the different expectations of investors in fuzzy portfolio selection. Kocadağlı and Keskin (2015) and Tsaur (2013) use distorted piecewise-linear fuzzy numbers to cover the different risk preferences of investors; In the work of Liu and Zhang (2018) and Jalota, Thakur, and Mittal (2017), L-R fuzzy numbers with power reference functions is used. However, some confusion in investor's expectation would be generated. Taking the work of Tsaur (2013) as an example, the membership functions are transformed convexly or concavely in a similar fashion to the utility functions. Consequently, for the convex case, higher membership degrees are assigned to both the favorable and unfavorable returns in a unitary pattern, which suggests an illogical and confused expectation of an investor. The same problem also exists for the concave case.

In this paper, we would guarantee the coherence of a specific investor's expectation. In other words, the investors with optimistic attitude tempt to assign higher membership degrees to favorable returns while lower membership degrees to unfavorable ones; for those with pessimism, it is the other way around. Therefore, in this work we construct a new membership function for a trapezoidal fuzzy number (called coherent trapezoidal fuzzy number) to embody the coherent investor's expectation in fuzzy quantification of the asset returns through an index. In detail, with the single index, we overestimate the membership degrees of the favorable returns and at the same time underestimate those

☆ The work is supported by National Natural Science Foundation of China under Grants 71473081 and 71871092, and Science and Technology Project of Jiangxi Provincial Educational Department of China under Grant GJJ151602, and Humanities and Social Sciences Projects for Universities in Jiangxi Province of China under Grant GL18212.

* Corresponding author.
E-mail addresses: hqli@hunnu.edu.cn (H.-Q. Li), yizhihong1206@163.com (Z.-H. Yi).

https://doi.org/10.1016/j.eswa.2019.05.008
0957-4174/

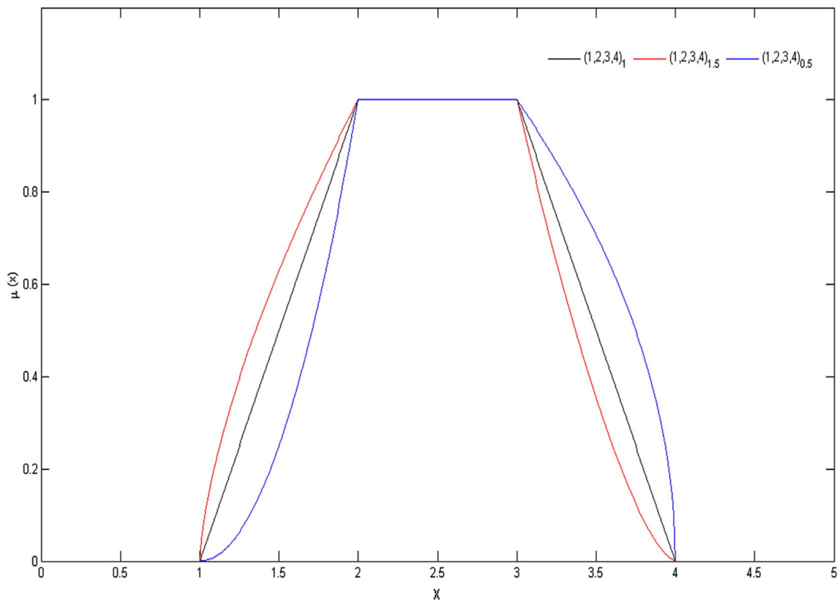

Fig. 1. The membership functions for coherent trapezoidal fuzzy numbers $(1, 2, 3, 4)_k$ with different indices $k(k = 0.5, 1, 1.5)$.

of the unfavorable returns by the appropriate transformations of piece-wise fuzzy numbers for the optimism, and reverse the transformation for the pessimistic. Consequently we propose the possibilistic mean-variance and mean-variance-skewness model for portfolio selection, in which the coherent investors' expectation can be reflected through the different transformations of the favorable and unfavorable parts of the underlying fuzzy numbers. Thus, taking account of the diversity of the investors' expectations, we can derive optimal portfolios well compatible with the specified expectation of an investor.

The paper is unfolded as follows: In Section 2, we introduce the coherent trapezoidal fuzzy numbers and some related notations. Section 3 presents the possibilistic expected mean-variance model and the possibilistic expected mean-variance-skewness model for portfolio selection. In Section 4 two numerical examples with real security dataset is provided to validate the constructed models for portfolio selection. Finally, we end up with some concluding remarks.

2. The coherent trapezoidal fuzzy numbers

As a special class of fuzzy sets, fuzzy numbers are used to model the ambiguous numbers like 'about m' or 'approximately n'. A fuzzy number (Dijkman, Haeringen, Zhao, & de Lange, 1983; Li et al., 2015) is formally interpreted as a normalized convex fuzzy set with its membership function being piecewise continuous. The typical ones are the triangular fuzzy numbers and the trapezoidal ones. Here we present the generalized trapezoidal fuzzy numbers, i.e., the coherent trapezoidal fuzzy numbers, to describe the asset returns for investments considering the different expectations of the investors through an adaptive index k.

Definition 1. The coherent trapezoidal fuzzy number, denoted by $A = (a, b, c, d)_k$ for short, is a fuzzy set with the $R \to [0, 1]$ membership function given by

$$\mu_A(x) = \begin{cases} \left(\dfrac{x-a}{b-a}\right)^{\frac{1}{k}}, & a < x \le b; \\ 1, & b < x \le c; \\ \left(\dfrac{d-x}{d-c}\right)^{k}, & c < x \le d; \\ 0, & else; \end{cases} \tag{1}$$

where k is a fixed positive real number. Moreover, if $b = c$, then the coherent trapezoidal fuzzy number is called a coherent triangular fuzzy number.

In describing the return of an asset by a coherent trapezoidal fuzzy number $A = (a, b, c, d)_k$ for financial decision making, the interval $[b, c]$ is considered as the most possible interval for the percentage of the return of the asset while the intervals $[a, b]$ and $[c, d]$ are respectively for the unfavorable and favorable returns with less possibility. Notably, the adaptive index k serves as the indications of the expectation for the investors. Obviously, if $k = 1$, then the coherent trapezoidal fuzzy numbers reduce to the usual trapezoidal fuzzy numbers, where the linearity in $[a, b]$ and $[c, d]$ means neutrality of investors. For the case $k > 1$, the coherent trapezoidal fuzzy number assigns relatively larger membership degrees to unfavorable returns in $[a, b]$ and smaller membership degrees to favorable returns in $[c, d]$, comparing with the corresponding trapezoidal fuzzy number. In view of the fact, $k > 1$ can be seen as an indication of the pessimistic expectation of the investors. Larger $k(>1)$ means more pessimism for the investors. Conversely, $k < 1$ is a demonstration of the optimistic expectation of the investors. For illustration, we present several instances for the coherent trapezoidal fuzzy numbers with different k in Fig. 1. By the specific index k, the expectations of the involved investors can be incorporated in their strategies for portfolio selection.

Remark 1. By contrast, Tsaur (2013) modeled the return of the assets by membership functions in the special form $\mu(x) = \begin{cases} 1-(\frac{r-x}{c})^k, & r-c \le x \le r; \\ 1, & x = r; \\ 1-(\frac{x-r}{d})^k, & r \le x \le r+d; \\ 0, & else, \end{cases}$ where the convexity, linearity and concavity of the left and right tail represent the pessimistic, neutral and optimistic expectations respectively through the adaptive value k. However, by the above equation the investors with pessimistic or optimistic expectations simultaneously assign overestimated or underestimated membership degree to both the favorable and unfavorable returns in a unitary pattern, which suggests an illogical and confused expectation.

In the work by Li et al. (2015), in order to uniformly present the numerical characteristics of a fuzzy number, they defined a possibility density function for a fuzzy number with membership function $\mu(x)$ as $\mu(x)|\mu'(x)|$. Here we formally present the possibility density function for a coherent trapezoidal fuzzy number.

Definition 2. The possibility density function $f(x)$ for the coherent trapezoidal fuzzy number $A=(a,b,c,d)_k$ by Eq. (1) is defined as

$$f(x)=\mu_A(x)|\mu_A'(x)|=\begin{cases}\left(\dfrac{x-a}{b-a}\right)^{\frac{2}{k}-1}\frac{1}{k(b-a)}, & a<x<b;\\ \left(\dfrac{d-x}{d-c}\right)^{2k-1}\frac{k}{d-c}, & c<x<d;\\ 0, & else;\end{cases} \tag{2}$$

As a counterpart of a probability density function for a random variable, the possibility density function $f(x)$ has the following properties:

1. $f(x)\geq 0$;
2. $\int_{-\infty}^{+\infty} f(x)dx=1$.

Consequently, the possibilistic expected mean EA and the possibilistic variance VA of a coherent trapezoidal fuzzy number $A=(a,b,c,d)_k$ can be defined respectively as follows.

Definition 3. For a coherent trapezoidal fuzzy number $A=(a,b,c,d)_k$ with the possibility density function $f(x)$ by Eq. (2), the possibilistic expected mean EA and the possibilistic variance VA are, respectively given by

$$EA=\int_{-\infty}^{+\infty} xf(x)dx;$$

$$VA=\int_{-\infty}^{+\infty}(x-EA)^2 f(x)dx.$$

By calculation,

$$EA=\frac{k}{2(k+2)}a+\left(\frac{1}{2}-\frac{k}{2(k+2)}\right)b+\left(\frac{1}{2}-\frac{1}{2(2k+1)}\right)c+\frac{1}{2(2k+1)}d, \tag{3}$$

which is a weighted average of a, b, c and d.

$$VA=\rho_1(b-a)^2+\rho_2(c-b)^2+\rho_3(d-c)^2+\rho_4(b-a)(c-b)+\rho_5(b-a)(d-c)+\rho_6(c-b)(d-c), \tag{4}$$

where $\rho_1=\frac{1}{4}+\frac{1}{2(k+1)}-\frac{1}{(k+2)^2}-\frac{1}{k+2}$, $\rho_2=\frac{1}{4}$, $\rho_3=\frac{1}{2k+1}-\frac{1}{2(k+1)}-\frac{1}{4(2k+1)^2}$, $\rho_4=\frac{1}{2}-\frac{1}{k+2}$, $\rho_5=\frac{1}{3(k+2)}-\frac{1}{6(2k+1)}$, $\rho_6=\frac{1}{2(2k+1)}$. Actually, VA is a nonstandard weighted average of the intercross products for the lengths of the intervals $[a, b]$, $[b, c]$ and $[c, d]$.

Specifically, for the possibilistic expected mean of a coherent trapezoidal fuzzy number $A=(a,b,c,d)_k$, if $k=1$, then

$$EA=\frac{1}{6}a+\frac{1}{3}b+\frac{1}{3}c+\frac{1}{6}d;$$

Further, if $k>1$, then in Eq. (3) the coefficient $\frac{k}{2(k+2)}>\frac{1}{6}$ for a, $\frac{1}{2}-\frac{k}{2(k+2)}<\frac{1}{3}$ for b, $\frac{1}{2}-\frac{1}{2(2k+1)}>\frac{1}{3}$ for c and $\frac{1}{2(2k+1)}<\frac{1}{6}$ for d; if $k<1$, some converse results can be obtained. That is to say, in the calculation of EA there is a tendency to weight more for the smaller numbers of a, b, c and d for the case $k>1$ and less for them for the case $k<1$ in comparison with the case $k=1$. This fact goes a further step to demonstrate that the cases $k>1$ and $k<1$ are representations of the pessimism and optimism of the investors, respectively. It can be easily seen that, for the extreme pessimistic investor($k\to\infty$) $EA=\frac{a+c}{2}$, while for the extreme optimistic one ($k\to 0$) $EA=\frac{b+d}{2}$. Here we refer to the work by Tsaur (2013), in which Eq. (18) tells us that the possibilistic mean value is independent of the index k if the fuzzy return is symmetric, i.e., $d-c=b-a$. In other words, the different expectations can not be demonstrated for this special case.

For the possibilistic variance of a coherent trapezoidal fuzzy number $A=(a,b,c,d)_k$, if $k=1$, then

$$VA=\frac{1}{18}(b-a)^2+\frac{1}{4}(c-b)^2+\frac{1}{18}(d-c)^2+\frac{1}{6}(b-a)(c-b)+\frac{1}{18}(b-a)(d-c)+\frac{1}{6}(c-b)(d-c),$$

which is consistent with the results by Carlsson and Fullér (2001), Zhang and Wang (2007) and Li et al. (2015). In general, the equation to calculate the possibilistic variance seems to be quite complex. In spite of the fact, the particular cases for the interactions of the possibilistic variance and the underlying index k presented in Fig. 2 suggest that the possibilistic variances are closely related to the underlying indices and the selection of appropriate indices can effectively avoid the overestimation or underestimation of the possibilistic variances(or risk) the investors can bear in portfolio selection. To be specific, for the coherent trapezoidal fuzzy number $(1, 2, 3, 4)_k$, the possibilistic variance reaches its minimum when $k=1$. For this case, if only the usual trapezoidal fuzzy number (1,2,3,4) is employed in modelling the asset return without considering the adaptive index, then the possibilistic variance underestimates the risk for both the investors with the pessimism and those with optimism; for the coherent fuzzy number $(1, 2, 3, 5)_k$, if only the usual trapezoidal fuzzy number (1,2,3,5) is employed, then the possibilistic variance underestimates and overestimates the risks respectively for the investors with the pessimism and those with optimism. From the aspect of the modelling risk, the incorporation of the adaptive index in the coherent trapezoidal fuzzy numbers is of essential importance in fuzzy portfolio optimization.

Skewness is also an important characteristic for potential topics in portfolio selection and can be defined as follows.

Definition 4. For a coherent trapezoidal fuzzy number $A=(a,b,c,d)_k$ with the possibility density function $f(x)$ by Eq. (2), the possibilistic skewness SA is given by

$$SA=\int_{-\infty}^{+\infty}(x-EA)^3 f(x)dx.$$

By calculation, we find that SA is not continuous w.r.t. the adaptive index k. To be concrete, for the case $k\leq\frac{1}{2}$,

$$SA=\sigma_1(b-a)^3+\sigma_2(b-a)^2(c-b)+\sigma_3(b-a)^2(d-c)+\sigma_4(b-a)(c-b)^2+\sigma_5(b-a)(c-b)(d-c)+\sigma_6(b-a)(d-c)^2+\sigma_7(c-b)^3+\sigma_8(c-b)^2(d-c)+\sigma_9(c-b)(d-c)^2+\sigma_{10}(d-c)^3, \tag{5}$$

where

$$\sigma_1=\frac{k^3(3k^2+25k+58)}{16(3k+2)(k+1)(k+2)^3},$$

$$\sigma_2=\frac{3}{16}-\frac{6k+3}{4(k+1)(k+2)^2},$$

$$\sigma_3=\frac{3k^2(4k^2+5k+5)}{16(2k+1)(k+1)(k+2)^2},\ \sigma_4=\frac{3}{16}-\frac{3}{8(k+2)},$$

$$\sigma_5=\frac{3k(4k+1)}{8(2k+1)(k+2)},\ \sigma_6=\frac{3k(16k^3+24k^2+5k+1)}{16(2k+1)^2(k+1)(k+2)},$$

$$\sigma_7=\frac{1}{16},\ \sigma_8=\frac{3}{8}-\frac{3}{16(2k+1)},\ \sigma_9=\frac{3}{4}-\frac{24k^2+45k+9}{16(2k+1)^2(k+1)},$$

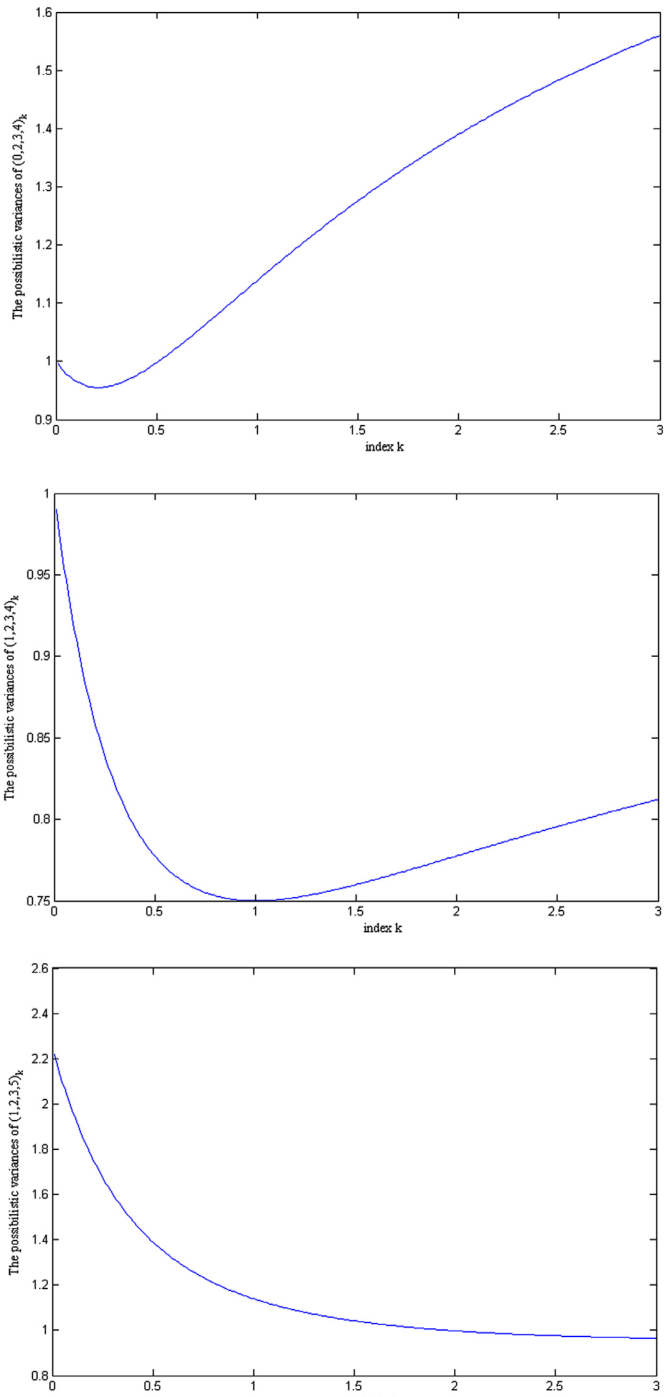

Fig. 2. The interactions of the possibilistic variance and the underlying index k for several coherent trapezoidal fuzzy numbers.

$$\sigma_{10} = \frac{1}{2} - \frac{96k^4 + 312k^3 + 394k^2 + 163k + 21}{16(2k+1)^3(2k+3)(k+1)};$$

for $k > \frac{1}{2}$,

$$\begin{aligned} SA = {} & \tau_1(b-a)^3 + \tau_2(d-c)^3 \\ & + \tau_3(b-a)^2(c-b) + \tau_4(b-a)(d-c)^2 \\ & + \tau_5(b-a)^2(d-c) + \tau_6(d-c)^2(c-b), \end{aligned} \tag{6}$$

where

$$\tau_1 = \frac{2}{(k+2)^3} + \frac{1}{3k+2} + \frac{3}{(k+2)^2} + \frac{3}{2(k+2)} - \frac{9}{4(k+1)},$$

$$\tau_2 = \frac{1}{4(2k+1)^3} - \frac{3}{2(2k+1)^2} + \frac{3}{2k+1} + \frac{3}{2(2k+3)} - \frac{9}{4(k+1)},$$

$$\tau_3 = \frac{3}{(k+2)^2} - \frac{3}{4(k+1)},$$

$$\tau_4 = \frac{3}{4(k+1)} - \frac{5}{6(2k+1)} + \frac{1}{4(2k+1)^2} - \frac{1}{3(k+2)},$$

$$\tau_5 = \frac{3}{4(k+1)} - \frac{1}{6(2k+1)} - \frac{2}{3(k+2)} - \frac{1}{(k+2)^2},$$

$$\tau_6 = \frac{3}{2(2k+1)} - \frac{3}{4(2k+1)^2} - \frac{3}{4(k+1)}.$$

Remark 2. In view of the jump discontinuity point $\frac{1}{2}$ of the possibilistic skewness and the fact that in portfolio selection the behavior of investing is smoothly affected by the expectation of the investor, the adaptive index k can not be too small($\leq \frac{1}{2}$) when there is a reference to possibilistic skewness in application.

In the sequel, we only focus on the case $k > \frac{1}{2}$, for which various expectations of investors can be covered. For this case, we can find that the possibilistic skewness SA is decreasing w.r.t the underlying index k from the interactions of the possibilistic skewness and the underlying index k for the three typical coherent trapezoidal fuzzy numbers(See Fig. 3.). For the coherent trapezoidal fuzzy numbers, the larger index k results in lower skewness, which is consistent with the interactions of the figures and the index k(See Fig. 1.). Note that if the index $k = 1$ in Eq. (6), then some known results by Li et al. (2015) can be obtained. In particular, if A is a symmetric trapezoidal fuzzy number, i.e., $k = 1$ and $b - a = d - c$, then $SA = 0$; if A is a triangular fuzzy number, i.e., $k = 1$ and $b = c$, then $SA = \frac{19(d-c)^3 - 19(b-a)^3 + 15(b-a)(d-c)^2 - 15(b-a)^2(d-c)}{1080}$.

In what follows, for the convenience of computation, we also propose a counterpart of probabilistic distribution function for the coherent trapezoidal fuzzy number $A = (a, b, c, d)_k$.

Definition 5. The possibility distribution function $F(x)$ for the coherent trapezoidal fuzzy number $A = (a, b, c, d)_k$ by Eq. (1) is defined as

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & x \leq a; \\ \frac{1}{2}\left(\frac{x-a}{b-a}\right)^{\frac{2}{k}}, & a < x \leq b; \\ \frac{1}{2}, & b < x \leq c; \\ 1 - \frac{1}{2}\left(\frac{d-x}{d-c}\right)^{2k}, & c < x \leq d; \\ 1, & x > d; \end{cases}$$

In fact, the possibilistic expected mean EA of a coherent trapezoidal fuzzy number can also be obtained by

$$EA = a + \int_a^d (1 - F(x))dx.$$

And the possibilistic expected mean VA can be calculated by

$$VA = (d - EA)^2 - 2\int_a^b (x - EA)F(x)dx.$$

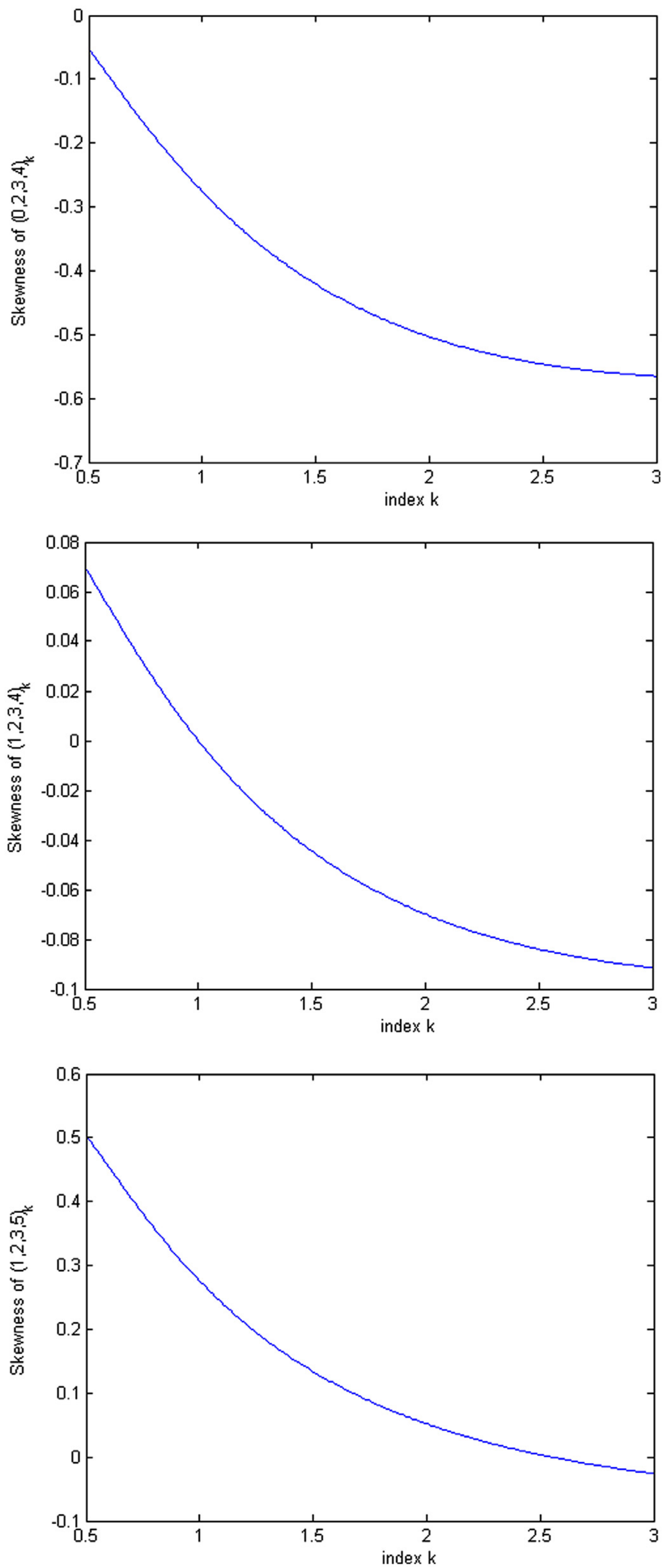

Fig. 3. The interactions of the possibilistic skewness and the underlying index k for several coherent trapezoidal fuzzy numbers.

Some operations for the coherent trapezoidal fuzzy numbers can be defined similarly like those for the usual fuzzy numbers.

Definition 6. Let $A_i = (a_i, b_i, c_i, d_i)_k (i = 1, 2)$ be the coherent trapezoidal fuzzy numbers with the index k and $x \geq 0$. Then the addition and the scalar multiplication are respectively defined as follows:

1) $A_1 \oplus A_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)_k$;
2) $xA_1 = (xa_1, xb_1, xc_1, xd_1)_k$.

It is supposed that short sale is not allowed in this work, hence here we restrict ourselves to the case $x \geq 0$ for the scalar multiplication. By Eqs. (3)–(6), the following properties can be obtained.

$$E(A_1 \oplus A_2) = E(A_1) + E(A_2);$$

$$V(xA_1) = x^2 V(A_1);$$

$$S(xA_1) = x^3 S(A_1).$$

In general $V(A_1 \oplus A_2) \neq V(A_1) + V(A_2)$, which is analogous to the traditional property of variance on probabilistic random variables in the presence of covariance.

3. Portfolio selection by the coherent trapezoidal fuzzy numbers

In this part, analogous to Markowitzian methodology, the possibilistic mean-variance methodology is proposed; and then the possibilistic expected mean-variance-skewness methodology is developed. In the sequel, we assume that the returns of assets are described by the coherent trapezoidal fuzzy numbers in the form of $A = (a, b, c, d)_k$.

3.1. Possibilistic expected mean-variance model

As the descriptions of the mean returns and risks of asset returns by the coherent trapezoidal fuzzy numbers, the possibilistic expected mean and variance for the coherent trapezoidal fuzzy numbers are counterparts of the usual expected mean and variance of asset returns in Markowitzian mean-variance methodology. Therefore, following Markowitzian expected mean-variance methodology, in order to obtain an optimized portfolio the possibilistic expected mean can be maximized given the upper bound of the risk the investor can bear, i.e., the possibilistic variance of the portfolio. Specifically, the possibilistic expected mean-variance model for portfolio selection by the coherent trapezoidal fuzzy numbers can be structured as follows:

$$\begin{cases} \max E(x_1A_1 \oplus x_2A_2 \oplus \ldots \oplus x_nA_n) \\ \quad \text{subject to:} \\ \qquad V(x_1A_1 \oplus x_2A_2 \oplus \ldots \oplus x_nA_n) \leq v; \\ \qquad \sum_{i=1}^{n} x_i = 1; \\ \qquad 0 \leq x_i \leq 1, i = 1, 2, \ldots, n, \end{cases} \tag{7}$$

where $A_i = (a_i, b_i, c_i, d_i)_k (i = 1, 2, \ldots, n)$ is the coherent trapezoidal fuzzy numbers used to model asset returns, x_i is the associated proportion, v is the upper bound for the possibilistic variance and short sell is not allowed. By Eqs. (3) and (4), Eq. (7) can be rewritten as follows.

$$\max \frac{k}{2(k+2)} \sum_{i=1}^{n} x_i a_i + \left(\frac{1}{2} - \frac{k}{2(k+2)}\right) \sum_{i=1}^{n} x_i b_i$$

$$+ \left(\frac{1}{2} - \frac{1}{2(2k+1)}\right) \sum_{i=1}^{n} x_i c_i + \frac{1}{2(2k+1)} \sum_{i=1}^{n} x_i d_i$$

subject to:

$$\begin{cases} \rho_1\left(\sum_{i=1}^{n} x_i(b_i-a_i)\right)^2 + \rho_2(\sum_{i=1}^{n} x_i(c_i-b_i))^2 \\ +\rho_3\left(\sum_{i=1}^{n} x_i(d_i-c_i)\right)^2 + \rho_4\left(\sum_{i=1}^{n} x_i(b_i-a_i)\right)\left(\sum_{i=1}^{n} x_i(c_i-b_i)\right) \\ +\rho_5\left(\sum_{i=1}^{n} x_i(b_i-a_i)\right)\left(\sum_{i=1}^{n} x_i(d_i-c_i)\right) \\ +\rho_6\left(\sum_{i=1}^{n} x_i(c_i-b_i)\right)\left(\sum_{i=1}^{n} x_i(d_i-c_i)\right) \le \nu; \\ \sum_{i=1}^{n} x_i = 1; 0 \le x_i \le 1, i = 1, 2, \ldots, n, \end{cases} \tag{8}$$

where $\rho_i (i = 1, 2, \ldots, 6)$ are those coefficients in Eq. (4).

In the aforementioned model for portfolio selection, different expectations of the investors can be demonstrated by choosing different indices k. In detail, for the investors with pessimism, it is recommended to use $k > 1$ in the model according to the justifications presented in Section 2; for the investors with neutrality expectation and optimism, $k = 1$ and $k < 1$ can be respectively used.

Similarly, the possibilistic expected mean-variance model by Eq. (7) can be restructured by minimizing the possibilistic variance given the lower bound of the possibilistic expected mean. In detail,

$$\begin{cases} \min V(x_1A_1 \oplus x_2A_2 \oplus \ldots \oplus x_nA_n) \\ \quad \text{subject to:} \\ \quad\quad E(x_1A_1 \oplus x_2A_2 \oplus \ldots \oplus x_nA_n) \ge \alpha; \\ \quad\quad \sum_{i=1}^{n} x_i = 1; \\ \quad\quad 0 \le x_i \le 1, i = 1, 2, \ldots, n, \end{cases} \tag{9}$$

where α is a specific lower bound of the possibilistic expected mean. Furthermore, both of the models by Eqs. (7) and (9) with the polynomial objective functions and constrains can be analytically solved.

3.2. Possibilistic expected mean-variance-skewness model

In the classic methodology, given some specific mean and variance of a portfolio return, investors have a preference of higher skewness (i.e., the right skewed return), therefore, the mean-variance-skewness model for portfolio selection was formulated and was investigated in the non-probabilistic framework. Here we propose the possibilistic expected mean-variance-skewness model by the coherent trapezoidal fuzzy numbers. In detail,

$$\begin{cases} \max S(x_1A_1 \oplus x_2A_2 \oplus \ldots \oplus x_nA_n) \\ \quad \text{subject to:} \\ \quad\quad E(x_1A_1 \oplus x_2A_2 \oplus \ldots \oplus x_nA_n) \ge \alpha; \\ \quad\quad V(x_1A_1 \oplus x_2A_2 \oplus \ldots \oplus x_nA_n) \le \nu; \\ \quad\quad \sum_{i=1}^{n} x_i = 1; \\ \quad\quad 0 \le x_i \le 1,\ i = 1, 2, \ldots, n, \end{cases} \tag{10}$$

By Eqs. (3), (4) and (6), Eq. (10) can be rewritten as follows.

$$\begin{aligned} \max\ & \tau_1\left(\sum_{i=1}^{n} x_ib_i - \sum_{i=1}^{n} x_ia_i\right)^3 + \tau_2\left(\sum_{i=1}^{n} x_id_i - \sum_{i=1}^{n} x_ic_i\right)^3 \\ & + \tau_3\left(\sum_{i=1}^{n} x_ib_i - \sum_{i=1}^{n} x_ia_i\right)^2\left(\sum_{i=1}^{n} x_ic_i - \sum_{i=1}^{n} x_ib_i\right) \\ & + \tau_4\left(\sum_{i=1}^{n} x_ib_i - \sum_{i=1}^{n} x_ia_i\right)\left(\sum_{i=1}^{n} x_id_i - \sum_{i=1}^{n} x_ic_i\right)^2 \\ & + \tau_5\left(\sum_{i=1}^{n} x_ib_i - \sum_{i=1}^{n} x_ia_i\right)^2\left(\sum_{i=1}^{n} x_id_i - \sum_{i=1}^{n} x_ic_i\right) \\ & + \tau_6\left(\sum_{i=1}^{n} x_id_i - \sum_{i=1}^{n} x_ic_i\right)^2\left(\sum_{i=1}^{n} x_ic_i - \sum_{i=1}^{n} x_ib_i\right) \end{aligned}$$

subject to:

$$\begin{cases} \frac{k}{2(k+2)} \sum_{i=1}^{n} x_ia_i + \left(\frac{1}{2} - \frac{k}{2(k+2)}\right) \sum_{i=1}^{n} x_ib_i + \left(\frac{1}{2} - \frac{1}{2(2k+1)}\right) \sum_{i=1}^{n} x_ic_i \\ \quad + \frac{1}{2(2k+1)} \sum_{i=1}^{n} x_id_i \ge \alpha; \\ \rho_1\left(\sum_{i=1}^{n} x_i(b_i-a_i)\right)^2 + \rho_2\left(\sum_{i=1}^{n} x_i(c_i-b_i)\right)^2 \\ +\rho_3\left(\sum_{i=1}^{n} x_i(d_i-c_i)\right)^2 + \rho_4\left(\sum_{i=1}^{n} x_i(b_i-a_i)\right)\left(\sum_{i=1}^{n} x_i(c_i-b_i)\right) \\ \rho_5\left(\sum_{i=1}^{n} x_i(b_i-a_i)\right)\left(\sum_{i=1}^{n} x_i(d_i-c_i)\right) \\ +\rho_6\left(\sum_{i=1}^{n} x_i(c_i-b_i)\right)\left(\sum_{i=1}^{n} x_i(d_i-c_i)\right) \le \nu; \\ \sum_{i=1}^{n} x_i = 1; \\ 0 \le x_i \le 1, i = 1, 2, \ldots, n, \end{cases} \tag{11}$$

The model by Eq. (10) can be transformed by the optimizations of the possibilistic mean or variance given the prescribed lower or upper bound of the remaining two characteristics. These models can be analytically solved in virtue of their polynomial objective functions and constrains. Note that for this case we have to only consider the case $k > \frac{1}{2}$ as a result of Remark 2.

4. Illustrative examples

In this section, in order to verify the validity of the proposed models for portfolio selection by the coherent trapezoidal fuzzy numbers, we provide some numerical examples on the allocation of the assets in a portfolio. Specifically, we consider portfolio selections for the investments on global stock indices and US multiple-class strategy, respectively.

4.1. On global stock indices

Firstly, we focus on a portfolio consisting of some securities available for investors in the world, including CSI 300 Index, SSE 180, Nikkei 225, EURONEXT 100 INDEX, NDX.GI, FTSE 100 Index, Xetra DAX, S&P 500 Index, CAC 40 Index, SX5P.GI. The data set is the weekly historical return percentages of these securities from Jan. 6th 2006 to Dec. 9th 2016, which are extracted from Wind Info. In order to quantify the weekly returns of the referred securities by the coherent trapezoidal fuzzy numbers in the form of $(q_{0.1}, q_{0.4}, q_{0.6}, q_{0.9})_k$, in which the intervals are specified by the sample quantiles q_α of the history returns by the method of Vercher and Bermudez (2013) and $k > 0$ is the given index, an indication of the expectations of the involved investors. Consequently, we can derive the returns of the securities by the coherent trapezoidal fuzzy numbers, listed in Table 1. Considering the different expectations

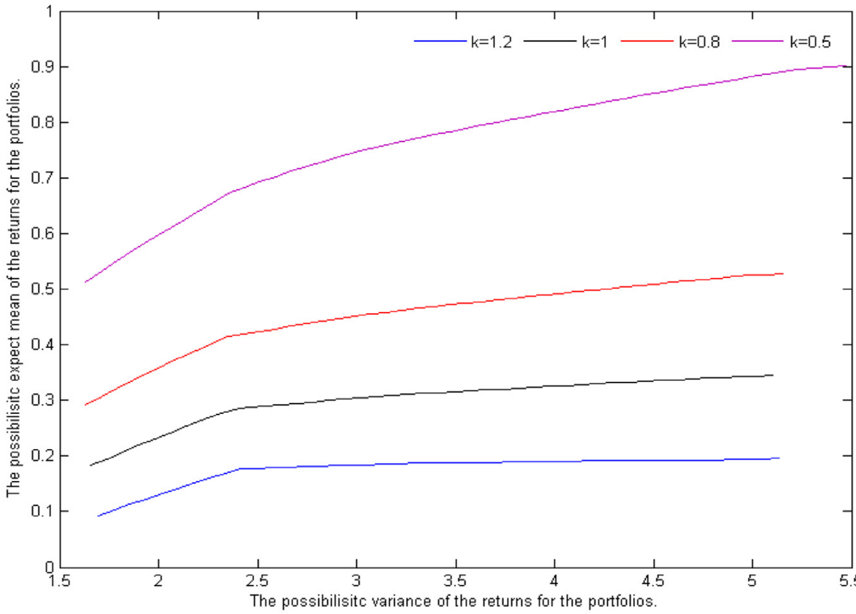

Fig. 4. The efficient frontiers by the possibilistic expected mean and variance using the coherent trapezoidal fuzzy numbers with different indices k which embody the expectations of the investors.

Table 1
The returns of the referred securities by the coherent trapezoidal fuzzy numbers with the adaptive index k.

Securities	Percentage of return
CSI 300 Index(S1)	$(-4.1620, -0.5500, 1.1810, 4.9700)_k$
SSE 180(S2)	$(-4.1460, -0.4910, 1.1600, 4.8680)_k$
Nikkei 225(S3)	$(-3.5640, -0.4540, 1.0300, 3.6240)_k$
EURONEXT 100 INDEX(S4)	$(-3.3100, -0.3400, 0.8910, 3.2300)_k$
NDX.GI(S5)	$(-3.0080, -0.1800, 0.8700, 3.3300)_k$
FTSE 100 Index(S6)	$(-2.5700, -0.3000, 0.6700, 2.7300)_k$
Xetra DAX(S7)	$(-3.5940, -0.1720, 1.0800, 3.6240)_k$
S&P 500 Index(S8)	$(-2.6340, -0.1710, 0.6920, 2.6880)_k$
CAC 40 Index(S9)	$(-3.6640, -0.4400, 0.9310, 3.4340)_k$
SX5P.GI(S10)	$(-2.9580, -0.3610, 0.6900, 2.9040)_k$

of the investors, we can use different values for k. Note that $k < 1$ means the optimism of the investors, $k = 1$ means neutrality and $k > 1$ means the pessimism of the investors.

4.1.1. For the possibilistic expected mean-variance model

For illustration, we will refer to the cases $k = 0.5, 0.8, 1$ and 1.2. In the following, to obtain the numerical results of the abovementioned models, we will use *fmincon* by the algorithm '*active-set*' in the Matlab(R2012a) optimization tools. Consequently, the model by Eq. (8) can be solved and thus the efficient frontiers by the possibilistic expected mean and variance can be obtained. Some illustrative results are presented in Fig. 4 and Table 2.

It can be seen from Fig. 4 that, under the same level of risk by possibilistic variance less returns of the portfolio can be obtained with the larger index k. Namely, the investors with more pessimism(i.e., larger index k) have less returns of the portfolio in the frame of the possibilistic mean and variance by the coherent trapezoidal fuzzy numbers. It can be seen that, the efficient frontiers in Fig. 4 with smaller indices are much more abrupt than those for the larger ones,which means the optimistic would get much favorable mean-risk profile in the fuzzy frame. Notably, each of these frontiers suggests that, by accepting a slightly higher variance, the investor can achieve a much higher return. Moreover, by Table 2 we can derive that, for the case that the upper bound v for the possibilistic variance changes from 2 to 3, the increments of the maximized possibilistic expected means of the returns for the cases $k = 0.5, 0.8, 1$ and 1.2 are respectively 0.1504, 0.0936, 0.0703, 0.0538; a similar result can be obtained for the case v changes from 3 to 4. Namely, the investors with

Table 2
The summary of some portfolios generated by the possibilistic expected mean and variance method by the coherent trapezoidal fuzzy numbers, where m is the maximized possibilistic expected mean of the return under the upper bound v of the possibilistic variance.

v	k	m	(S1)	(S2)	(S3)	(S4)	(S5)	(S6)	(S7)	(S8)	(S9)	(S10)
2	0.5	0.5975	0	0	0	0	0.5138	0	0	0.4862	0	0
	0.8	0.3580	0	0	0	0	0.5388	0	0	0.4612	0	0
	1	0.2335	0	0	0	0	0.5033	0	0	0.4967	0	0
	1.2	0.1301	0	0	0	0	0.4493	0	0	0.5507	0	0
3	0.5	0.7479	0	0.0004	0	0	0	0	0.9996	0	0	0
	0.8	0.4516	0	0	0	0	0.0684	0	0.9316	0	0	0
	1	0.3038	0	0	0	0	0.1613	0	0.8387	0	0	0
	1.2	0.1839	0	0	0	0	0.2579	0	0.7421	0	0	0
4	0.5	0.8194	0	0.4831	0	0	0	0	0.5169	0	0	0
	0.8	0.4913	0	0.5190	0	0	0	0	0.4810	0	0	0
	1	0.3257	0	0.5044	0	0	0	0	0.4956	0	0	0
	1.2	0.1904	0.4315	0	0	0	0	0	0.5685	0	0	0

Table 3
The capital allocations of the optimal portfolios by Eq. (10) with $\alpha = 0.11$ and $\nu = 3$ for investors with different expectations. Note that the case $k = 1$ is just the results by Li et al. (2015) and in this work means neutrality of the investors' expectation.

Securities	(S1)	(S2)	(S3)	(S4)	(S5)	(S6)	(S7)	(S8)	(S9)	(S10)
$k = 0.7$	0.4280	0	0	0	0	0.5720	0	0	0	0
$k = 1$	0.4382	0	0	0	0	0.5618	0	0	0	0
$k = 1.3$	0	0	0	0	0.8141	0.1859	0	0	0	0

more pessimism derive less marginal returns w.r.t. the risks in the frame of the possibilistic mean and variance by the coherent trapezoidal fuzzy numbers. Further, if we propose a sharpe-like ratio: $\frac{\text{possibilistic expect mean}-\text{rate for free riks}}{\sqrt{\text{possibilistic variance}}}$, called possibilistic sharpe ratio for short. By calculation on the basis of the data set in Table 2, it holds that the possibilistic sharpe ratio is nonincreasing w.r.t the index k under the specified upper bound of the possibilistic variance.

By the above justifications, it can be concluded that the different expectations of investors can be demonstrated by the incorporation of the index in the fuzzy portfolio selection by possibilistic mean-variance frame by coherent trapezoidal fuzzy numbers.

4.1.2. For thepossibilistic expected mean-variance-skewness model

In this subsection, we will use $k = 1.3, 1,$ and 0.7 to respectively represent the pessimism, neutrality and optimism of the investors. By Eq. (11) with the lower bound $\alpha = 0.11$ for the possibilistic expected mean and the upper bound $\nu = 3$ for the possibilistic variance, we can derive the optimal portfolios for investors with different expectations. The corresponding results are listed in Table 3. The obtained maximized possibilistic skewnesses are respectively $-0.5416, -0.0342, 0.5620$ for the cases $k = 1.3, 1,$ and 0.7. Namely, the optimal maximized possibilistic skewness is closely related to the expectation of the investors and nonincreasing w.r.t. the adaptive index k.

It must be mentioned that the mean-variance-skewness model in fuzzy frame by Li et al. (2015), which facilitates the optimal investing under situations with complexity and uncertainty, is just the case $k = 1$ for the model. Without specifying any differences in expectations of the investors, the model by Li et al.actually focuses on the neutral expectation of investors from our perspective. Therefore, the mean-variance-skewness model in this work can be advantaged by taking account of the heterogeneous expectations of the investors and the obtained optimal portfolios can be exactly compatible with the investors with specific expectations.

Table 4
The returns of six US financial assets by the coherent trapezoidal fuzzy numbers with index k.

Securities	Percentage of return
A1: SPY(ETF)	$(-2.1340, -0.0500, 0.7010, 2.4420)_k$
A2: US10Y Bond	$(-5.2600, -1.4640, 0.6800, 6.0760)_k$
A3: S&P 500 VIX Futures	$(-12.0580, -3.4730, 1.5630, 13.7700)_k$
A4: Crude Oil WTI Futures	$(-4.7740, -0.6330, 1.0140, 5.2840)_k$
A5: S&P 500 Futures	$(-2.1960, 0, 0.7200, 2.4660)_k$
A6: Copper Futures	$(-3.5420, -0.7540, 0.5710, 3.9660)_k$

4.2. On US multi-class investment

In the last subsection, the models are tested in the asset allocation in global investment on the main stock indices. To further check the effectiveness in multiple-class strategy, we check the portfolio selection by these models in an US market investment on a basket of A1: SPY(ETF), A2: US10Y Bond, A3: S&P 500 VIX Futures, A4: Crude Oil WTI Futures, A5: S&P 500 Futures, A6: Copper Futures. The weekly return data, extracted from investing.com, ranges from May 17th 2009 to Feb. 24th 2019 with 511 observations. The returns represented by coherent trapezoidal fuzzy numbers are presented in Table 4. By the possibilistic expected mean-variance model and the mean-variance-skewness model,

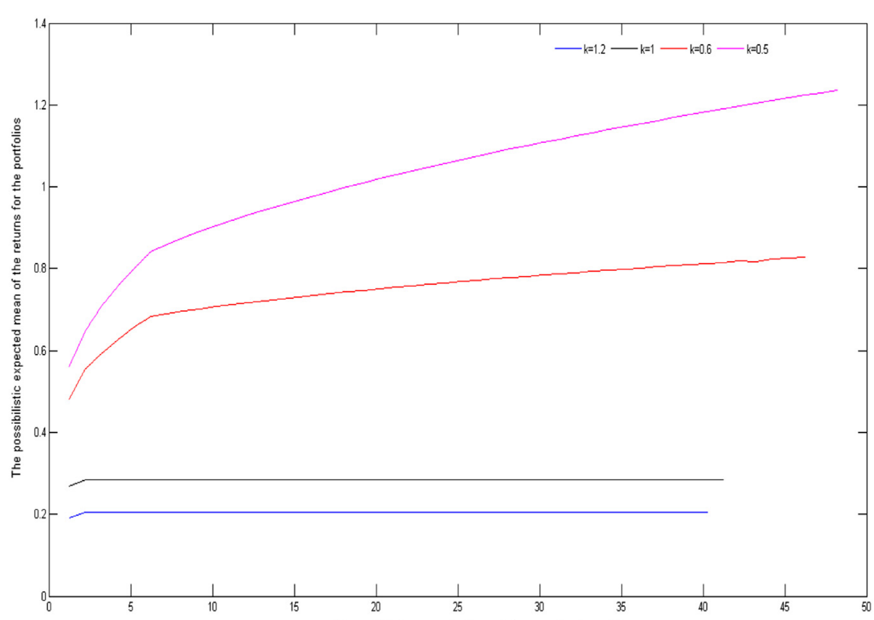

Fig. 5. The efficient frontiers by possibilistic expected mean and variance for the US investment with multiple-asset-class strategy.

Table 5
The summary of some portfolios generated by the possibilistic expected mean and variance method by the coherent trapezoidal fuzzy numbers, where m is the maximized possibilistic expected mean of the return under the upper bound v of the possibilistic variance.

v	k	m	A1	A2	A3	A4	A5	A6
5	0.6	0.6513	0	0	0	0.8226	0.1774	0
	0.8	0.4141	0	0	0	0.8424	0.1576	0
	1	0.285	0	0	0	0	1	0
	1.2	0.2050	0	0	0	0	1	0
10	0.6	0.7061	0	0	0.1575	0.8425	0	0
	0.8	0.4201	0	0	0	1	0	0
	1	0.285	0	0	0	0	1	0
	1.2	0.2050	0	0	0	0	1	0
15	0.6	0.7299	0	0	0.3214	0.6786	0	0
	0.8	0.4201	0	0	0	1	0	0
	1	0.285	0	0	0	0	1	0
	1.2	0.2050	0	0	0	0	1	0

Table 6
The capital allocations of the optimal portfolios by Eq. (10) with $\alpha = 0.15$ and $v = 15$ for investors with different indices k.

Securities	A1	A2	A3	A4	A5	A6
$k = 0.6$	0	0.7664	0.2336	0	0	0
$k = 0.7$	0	0.7488	0.2512	0	0	0
$k = 0.8$	0.1036	0.5812	0.3152	0	0	0
$k = 0.9$	0.1054	0	0.3847	0.51	0	0
$k = 1$	0	0	0.1101	0.8899	0	0
$k = 1.1$	0.8272	0.1728	0	0	0	0
$k = 1.2$	0.8666	0	0	0	0	0.1334

the optimal allocations can be specifically obtained for given conditions.

For the possibilistic expected mean-variance model, similarly the efficient frontiers by possibilistic expected mean and variance can be derived for specific indice k. Like the case for global indices investment, a similar pattern of the efficient frontiers (see Fig. 5) to that for different index ($k = 0.5, 0.6, 1, 1.2.$) can be seen. That is, the smaller index k(more optimism) brings more favorable investment profile in the possibilistic expected mean-variance framework. Moreover, for this example, the results show that extreme position are common for the less optimist or pessimistic investors($k = 0.8, 1, 1.2$). In Table 5, if only the neutrality of expectation is considered(i.e., $k = 1$), these investors mainly hold an extreme position of just one asset, i.e., S&P 500 Futures; while, considering the heterogeneous expectations with different indices k in the paper, the portfolios for the more optimistic (such as $k = 0.6$) are diversified to some degree. The results are much reasonable and consistent with practical concerns. Therefore, from this angle, the introduction of adaptive indices practically helps cover the concern on the heterogeneous expectations of the investors. For the mean-variance-skewness model, we can get similar results (See Table 6.) as these in the last subsection. The obtained maximized possibilistic skewnesses is also shown to be decreasing with respect to the index k. Namely, the more pessimistic, the less maximized possibilistic skewnesses can be obtained.

In fact, the standard mean-variance model is established under the controversial rational expectation hypothesis that all the investors in the overall market have a uniformly conclusion on the distribution of the pertaining asset returns. Still, under uncertainty many fuzzy models follows the same intuition and would consider neutrality of the investors' expectations($k = 1$) in their fuzzy formulations(Li et al., 2015; Zhang, Wang, Chen, & Nie, 2007). Here, the heterogeneous and coherent expectations of investors in fuzzy portfolio selection are considered through the adaptive index, and consequently the portfolios compatible with the specific expectations can be obtained by pertaining models. The above numerical results by the above tables show that, these models based on the coherent trapezoidal fuzzy numbers provide a valid and advantageous way to cope with the heterogeneous and coherent expectations of investors in asset allocation.

5. Concluding remarks

In this work, to avoid logical confusion in modelling a specific investor's expectation, we propose a generalization of the usual trapezoidal fuzzy number with an adaptive index, called the coherent trapezoidal fuzzy number, which can overestimate the membership degrees for the favorable returns and meanwhile underestimate those for the unfavorable returns for the optimistic investors by appropriate transformation, and reverse the transformation for the pessimistic. It turns out that the possibilistic expected mean, variance and skewness are closely interacted the adaptive index. Comparing with the work by Tsaur (2013) which transforms the membership function in a convex or concave way similar to utility functions, here the coherent expectations underlying a coherent trapezoidal fuzzy number are much more apparently embodied in the characteristics. Then the possibilistic expected mean-variance model and the possibilistic expected mean-variance-skewness model for portfolio selection are developed in an attempt to incorporate the different expectations of investors through the coherent trapezoidal fuzzy numbers in modelling asset returns. For the case that the investors are neutral, these models reduce to the known methods by Zhang et al. (2007) and Li et al. (2015). The given numerical examples validate these models for application for an international investment and a US multiple-class investment. Further, the investors' expectations can be reflected in the obtained results. However, we have to note the underdiversification of the obtained portfolios, which may inherit some unreasonable properties (Black & Litterman, 1992) of the classical Markowitzian model when short sale is not allowed. Just as well, the Black-Litterman model was developed to settle the problem. Thus, in the future, we will focus on the Black-Litterman model (Black & Litterman, 1992; Fang et al., 2018) by the coherent trapezoidal fuzzy number.

Credit authorship contribution statement

Hong-Quan Li: Conceptualization, Writing - review & editing, Supervision, Funding acquisition. **Zhi-Hong Yi:** Formal analysis, Data curation, Methodology, Writing - original draft preparation, Software, Validation.

Acknowledgement

We are grateful to the participants in 2017 Annual Symposium(Guangzhou) of the Decision Science Branch of Operations Research Society of China for their helpful comments and suggestions. Special thanks go to the E.I.C. and the anonymous referees for their careful work and penetrating comments on the improvement of the manuscript.

Statements on conflict of interest

We declare that we have no conflicts of interest to this work and do not have any commercial or associative interest that represents a conflict of interest in connection with the work.

References

Black, F., & Litterman, R. (1992). Global portfolio optimization. *Financial Analysts Journal, 48*, 28–43.
Carlsson, C., & Fullér, R. (2001). On possibilistic mean value and variance of fuzzy numbers. *Fuzzy Sets and Systems, 122*, 315–326.

Dijkman, J. G., Haeringen, H. V., Zhao, D., & de Lange, S. J. (1983). Fuzzy numbers. *Journal of Mathematical Analysis and Applications, 92*, 301–341.
Fang, Y., Bo, L., Zhao, D., & Wang, S. (2018). Fuzzy views on Black-Litterman Portfolio selection model. *Journal of Systems Science and Complexity, 31*, 975–987.
Fang, Y., Lai, K. K., & Wang, S. (2008). *Fuzzy portfolio optimization: Theory and methods*. Springer Science & Business Media.
Fang, Y., Lai, K. K., & Wang, S. Y. (2006). Portfolio rebalancing model with transaction costs based on fuzzy decision theory. *European Journal of Operational Research, 175*, 879–893.
Gupta, P., Mehlawat, M. K., Inuiguchi, M., & Chandra, S. (2014). *Fuzzy portfolio optimization: Advances in hybrid multi-criteria methodologies*. Berlin Heidelberg: Springer- Verlag.
Huang, X. (2009). A review of credibilistic portfolio selection. *Fuzzy Optimization and Decision Making, 8*, 263–281.
Jalota, H., Thakur, M., & Mittal, G. (2017). Modelling and constructing membership function for uncertain portfolio parameters: A credibilistic framework. *Expert Systems with Applications, 71*, 40–56.
Kamdem, J. S., Deffo, C. T., & Fono, L. A. (2012). Moments and semi-moments for fuzzy portfolio selection. *Insurance: Mathematics and Economics, 51*, 517–530.
Kocadağlı, O., & Keskin, R. (2015). A novel portfolio selection model based on fuzzy goal programming with different importance and priorities. *Expert Systems with Applications, 42*, 6898–6912.
Li, X., Guo, S., & Yu, L. (2015). Skewness of fuzzy numbers and its applications in portfolio selection. *IEEE Transactions on Fuzzy Systems, 23*, 2135–2143.
Liu, Y. J., & Zhang, W. G. (2013). Fuzzy portfolio optimization model under real constraints. *Insurance: Mathematics and Economics, 53*, 704–711.
Liu, Y. J., & Zhang, W. G. (2018). Fuzzy portfolio selection model with real features and different decision behaviors. *Fuzzy Optimization and Decision Making, 17*, 317–336.
Tsaur, R. C. (2013). Fuzzy portfolio model with different investor risk attitudes. *European Journal of Operational Research, 227*, 385–390.
Vercher, E., & Bermudez, J. D. (2013). A possibilistic mean-downside risk-skewness model for efficient portfolio selection. *IEEE Transactions on Fuzzy Systems, 21*, 585–595.
Wang, S., & Zhu, S. (2002). On fuzzy portfolio selection problems. *Fuzzy Optimization and Decision Making, 1*, 361–377.
Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*, 338–353.
Zhang, W. G., & Wang, Y. L. (2007). Notes on possibilistic variances of fuzzy numbers. *Applied Mathematics Letters, 20*, 1167–1173.
Zhang, W. G., Wang, Y. L., Chen, Z. P., & Nie, Z. K. (2007). Possibilistic mean–variance models and efficient frontiers for portfolio selection problem. *Information Sciences, 177*, 2787–2801.