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Gaussian approximation of nonlinear statistics on the sphere

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A B S T R A C T

We show how it is possible to assess the rate of convergence in the Gaussian approximation of triangular arrays of U-statistics, built from wavelets coefficients evaluated on a spherical Poisson field of arbitrary dimension. For this purpose, we exploit the Stein–Malliavin approach introduced in the seminal paper by Peccati, Solé, Taqqu and Utzet (2011); we focus in particular on statistical applications covering evaluation of variance in non-parametric density estimation and Sobolev tests for uniformity.

1. Introduction and overview

1.1. Motivations

The purpose of this paper is to establish quantitative central limit theorems for some U-statistics on wavelets coefficients evaluated either on spherical Poisson fields or on a vector of independent and identically distributed (i.i.d.) observations with values on a sphere. These statistics are motivated by standard problems in statistical inference, such as evaluation of the variance in density estimations and Sobolev tests of uniformity of the underlying Poisson measure. Such problems are certainly very classical in statistical inference; however, we shall investigate their solution under circumstances which are somewhat non-standard, for a number of reasons. In particular, we will focus mainly on "high-frequency" procedures, where the scale to be investigated and the number of tests to be implemented are themselves a function of the number

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of observations available, according to rules to be discussed below; for these statistics, we shall establish quantitative central limit theorems by means of the so-called *Malliavin–Stein technique*. Such a technique will allow, for instance, to determine how many joint procedures can be run while maintaining a given level of accuracy in the Gaussian approximation for the sample distribution of the resulting statistics; as shown in Sections 1.4.2 and 1.4.3 below, a refined version of an argument contained in the classic paper by Dynkin and Mandelbaum [9] will allow us to extend our quantitative result (in a fully multidimensional setting) to the framework of U-statistics based on i.i.d. spherical observations.

As already mentioned, we shall assume that the domain of interest is the unit sphere $\mathbb{S}^q \subset \mathbb{R}^{q+1}$. The arguments we exploit can be extended to other compact manifolds, but we shall not pursue these generalizations here for brevity and simplicity; however, on the contrary of most of the existing literature, our procedures can also be easily adapted to cover "local" tests, i.e. the possibility that these spheres are only partially observable, as it is often the case for instance in astrophysical experiments, cf. for instance [37], see also the recent monograph [5] for several other applications of spherical data analysis.

Malliavin–Stein techniques for Poisson processes have recently drawn a lot of attention in the probabilistic literature, see for instance [6,20,21,26,28,32], as well as the textbooks [27] and [7] for background results on Gaussian approximations by means of Stein's method. As motivated above, our aim here is to apply and extend the now well-known results of [28,30] in order to deduce bounds that are well-adapted to the applications we mentioned; our principal motivation originates from the implementation of wavelet systems on the sphere in the framework of statistical analysis for Cosmic Rays data, as for instance in [16,19,34,36]. As noted in [8], under these circumstances, when more and more data become available, higher and higher frequencies (i.e., smaller and smaller scales) can be probed. We shall hence be concerned with sequences of Poisson fields, whose intensity grows monotonically; it is then possible to exploit local Normal approximations, where the rate of convergence to the asymptotic Gaussian distribution is related to the scale parameter of the corresponding wavelet transform in a natural and intuitive way. Similar arguments were earlier exploited for linear statistics in [8]; the proofs in the nonlinear case we consider here are considerably more complicated from the technical point of view, but remarkably the main qualitative conclusions go through unaltered.

1.2. U-statistics on the Poisson space

We will now recall a few basic definitions on Poisson random measures and Stein–Malliavin bounds; we refer for instance to [29,31,33] for more discussions and details. Assuming that we are working on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the following definition is standard:

Definition 1.1. Let $(\Theta, \mathcal{A}, \lambda)$ be a σ-finite measure space, and assume that λ has no atoms (that is, $\lambda(\{x\}) = 0$, for every $x \in \Theta$). A Poisson random measure on Θ with intensity measure (or control measure) λ is a collection of random variables $\{N(A) : A \in \mathcal{A}\}$, taking values in $\mathbb{Z}_+ \cup \{+\infty\}$, such that the following two properties hold:

1. $N(A)$ has Poisson distribution with mean $\lambda(A)$, for every $A \in \mathcal{A}$;
2. $N(A_1), \ldots, N(A_n)$ are independent whenever $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint.

In what follows, we shall consider a special case of Definition 1.1; more precisely, we take $\Theta = \mathbb{R}_+ \times \mathbb{S}^q$, with $\mathcal{A} = \mathcal{B}(\Theta)$, the class of Borel subsets of Θ. The symbol N indicates a Poisson random measure on Θ, with homogeneous intensity given by $\lambda = \rho \times \mu$. We shall take $\rho(ds) = R \cdot \ell(ds)$, where ℓ is the Lebesgue measure and $R > 0$ is a fixed parameter, in such a way that $\rho([0,t]) := R_t = R \cdot t$. Also, we assume that μ is a probability on $\mathbb{S}^q$ of the form $\mu(dx) = f(x)dx$, where f is a density on the sphere. Given such an object, we will denote by N_t $(t > 0)$ the Poisson measure on $(\mathbb{S}^q, \mathcal{B}(\mathbb{S}^q))$ given by

$$N_t(B) := N([0,t] \times B), \quad B \in \mathcal{B}(\mathbb{S}^q); \tag{1.1}$$

it is easy to verify that N_t has control $\mu_t := R_t\mu$.

We introduce the concept of star-contraction between an element of $L^2(\mu_t^p)$ and an element of $L^2(\mu_t^m)$. Contraction operators play a crucial role in the computation of expectations involving powers of functionals of the Poisson measure N_t. In what follows, we shall define these operators and discuss some of their basic properties.

Definition 1.2. The kernel $f \star_r^l g \in L^2\left(\mu_t^{p+m-r-l}\right)$, associated with the symmetric functions $f \in L^2(\mu_t^p)$ and $g \in L^2(\mu_t^m)$, where $p, m \geq 1$, $r = 1, \dots, p \wedge m$ and $l = 1, \dots, r$, is defined as follows:

$$\begin{aligned} &f \star_r^l g(\gamma_1, \dots, \gamma_{r-l}, t_1, \dots, t_{p-r}, s_1, \dots, s_{m-r}) \\ &= \int\limits_{(\mathbb{S}^q)^l} f(z_1, \dots, z_l, \gamma_1, \dots, \gamma_{r-l}, t_1, \dots, t_{p-r}) g(z_1, \dots, z_l, \gamma_1, \dots, \gamma_{r-l}, s_1, \dots, s_{m-r}) \mu_t^l(dz_1, \dots, dz_l). \end{aligned}$$

Roughly speaking, the star operator '$\star_r^l$' reduces the number of variables in the tensor product of f and g from $p+m$ to $p+m-r-l$: this operation is realized by first identifying r variables in f and g, and then by integrating out l among them. To deal with the case $l = 0$ for $r = 0, \dots, p \wedge q$, we set

$$f \star_r^0 g(\gamma_1, \dots, \gamma_r, t_1, \dots, t_{p-r}, s_1, \dots, s_{m-r}) = f(\gamma_1, \dots, \gamma_r, t_1, \dots, t_{p-r}) g(\gamma_1, \dots, \gamma_r, s_1, \dots, s_{m-r}),$$

and

$$f \star_0^0 g(t_1, \dots, t_p, s_1, \dots, s_m) = f \otimes g(t_1, \dots, t_p, s_1, \dots, s_m) = f(t_1, \dots, t_p) g(s_1, \dots, s_m).$$

The kernel $f \star_r^l g$ is called the *star-contraction of index* (r,l) between f and g.

Let us also review some standard distances between laws of random variables taking values in $\mathbb{R}^q$; the first two (Wasserstein and Kolmogorov distances) will be only used in the univariate case. Given a function $g \in \mathcal{C}(\mathbb{R}^q)$, we write

$$\|g\|_{Lip} = \sup_{x,y \in \mathbb{R}^q, x \neq y} \frac{|g(x) - g(y)|}{\|x - y\|_{\mathbb{R}^q}}.$$

If $g \in \mathcal{C}^2(\mathbb{R}^q)$, we set

$$M_2(g) = \sup_{x \in \mathbb{R}^q} \|\text{Hess } g(x)\|_{op},$$

where $\|\cdot\|_{op}$ indicates the operator norm.

Definition 1.3. The Wasserstein distance d_W, between the laws of two random vectors X, Y with values in $\mathbb{R}^q$ $(q \geq 1)$ and such that $\mathbb{E}\|X\|_{\mathbb{R}^q}, \mathbb{E}\|Y\|_{\mathbb{R}^q} < \infty$, is given by:

$$d_W(X,Y) = \sup_{g: \|g\|_{Lip} \leq 1} |\mathbb{E}[g(X)] - \mathbb{E}[g(Y)]|.$$

Definition 1.4. The Kolmogorov distance d_K, between the laws of two random variables X, Y with values in $\mathbb{R}$, is given by:

$$d_K(X,Y) = \sup_{z \in \mathbb{R}} |\mathbb{P}[X \leq z] - \mathbb{P}[Y \leq z]|.$$

Definition 1.5. The distance d_2 between the laws of two random vectors X, Y with values in $\mathbb{R}^q$ $(q \geq 1)$, such that $\mathbb{E}\,\|X\|_{\mathbb{R}^q}, \mathbb{E}\,\|Y\|_{\mathbb{R}^q} < \infty$, is given by:

$$d_2\,(X, Y) = \sup_{g \in \mathcal{H}} |\mathbb{E}\,[g\,(X)] - \mathbb{E}\,[g\,(Y)]|\,,$$

where $\mathcal{H}$ denotes the collection of all functions $g \in \mathcal{C}^2\,(\mathbb{R}^q)$ such that $\|g\|_{Lip} \leq 1$ and $M_2(g) \leq 1$.

The concept of a U-statistic was introduced in a seminal paper by Hoeffding [15], and since then it has become a central notion for statistical inference (see e.g., [35]). Since now, for $p \geq 2$, the class $L^p_s(\nu^k)$ indicates the subspace of $L^p(\nu^k)$ of functions which are ν^k – almost everywhere symmetric; let us recall a general definition, following [32].

Definition 1.6 *(U-statistics).* Consider a Poisson random measure N with control ν on $(A, \mathcal{A})$. Fix $k \geq 1$. A random variable F is called a *U-statistic of order k*, based on the Poisson measure N with control ν, if there exists a kernel $h \in L^1_s(\nu^k)$ such that

$$F = \sum_{(x_1,\ldots,x_k) \in N^k_{\neq}} h(x_1, \ldots, x_k), \tag{1.2}$$

where the symbol $N^k_{\neq}$ indicates the class of all k-dimensional vectors $(x_1, \ldots, x_k)$ such that $x_i \in N$ and $x_i \neq x_j$ for every $1 \leq i \neq j \leq k$.

As anticipated, in this paper we will focus, for all $t > 0$, U-statistics on the q-dimensional sphere $\mathbb{S}^q$ based on the Poisson measure N_t introduced in (1.1), corresponding to the case $A = \mathbb{S}^q$, with $\mathcal{A}$ the associated Borel σ field, and $\nu = \mu_t = R_t\mu$.

As discussed in the introduction, we shall consider two classical issues in the statistical analysis of Poisson processes, namely estimation of variance in density estimation and testing for uniformity of the governing measure. In these two cases, a common form of statistic is

$$U_j = \sum_{(z_1,z_2) \in N^2_{t\neq}} h_j(z_1, z_2), \tag{1.3}$$

where $h_j(\cdot,\cdot)$ is a kernel in the space $L^2\left(\mu_t^2\right)$.

1.3. Spherical needlets

In this section we will provide a short overview about the construction of needlet frames over the q-dimensional sphere; further details can be found in [24,25], see also [2,3,11,13,23] and [22], Chapter 10. From now on, we will use the simplified notation $L^2\,(dz) = L^2\,(\mathbb{S}^q, dz)$ to denote the space of square-integrable functions with respect to Lebesgue measure on the sphere. It is well-known result that the following decomposition holds:

$$L^2\,(dz) = \oplus_{\ell=0}^{\infty}\,(\mathcal{H}_\ell)\,,$$

where $\mathcal{H}_\ell$ is the restriction to $\mathbb{S}^q$ of the homogeneous polynomials of degree ℓ on $\mathbb{R}^{q+1}$, for which an orthonormal basis is provided by the system of spherical harmonics $\{Y_{\ell,m} : \mathbb{S}^q \mapsto \mathbb{C}\}_{m=1,\ldots,d_{\ell,q}}$ of degree ℓ, with dimension

$$d_{\ell,q} = \frac{\ell + \eta_q}{\eta_q}\binom{\ell + 2\eta_q - 1}{\ell}, \eta_q = (q-1)\,/2.$$

Given any real-valued $f \in L^2(dz)$, the orthogonal projector over $\mathcal{H}_\ell$ is provided in spherical coordinates by the kernel operator

$$P_{\ell,q} f(z) = \sum_{m=1}^{d_{\ell,q}} a_{\ell m} Y_{\ell,m}(z), \quad z \in \mathbb{S}^q, \quad a_{\ell m} = \int_{\mathbb{S}^q} \overline{Y}_{\ell,m}(z) f(z)\, dz \in \mathbb{C},$$

see for instance [38]. For $z_1, z_2 \in \mathbb{S}^q$, the kernel associated to the projector $P_{\ell,q}$ is given by

$$P_{\ell,q}(z_1, z_2) = \sum_{m=1}^{d_{\ell,q}} \overline{Y}_{\ell,m}(z_1) Y_{\ell,m}(z_2) = \frac{\ell + \eta_q}{\eta_q \omega_q} \mathcal{C}_\ell^{(\eta_q)}(\langle z_1, z_2 \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product over $\mathbb{R}^{q+1}$, $\mathcal{C}_\ell^{(\eta_q)}$ denotes the Gegenbauer polynomial of degree ℓ with parameter η_q, (see for instance [39]), and ω_q is the measure of the surface of the q-dimensional sphere, namely

$$\omega_q = \frac{2\pi^{\frac{q+1}{2}}}{\Gamma\left(\frac{q+1}{2}\right)}.$$

We write $\mathcal{K}_\ell = \oplus_{i=0}^{\ell} \mathcal{H}_i$ for the linear space of polynomials with degree smaller or equal than ℓ; as showed in [24], for every integer $\ell = 1, 2, \ldots$ there exists a finite set of cubature points $\{\xi\} \in \mathcal{Q}_l \subset \mathbb{S}^q$ and corresponding weights $\{\lambda_\xi\}$, such that, for $f \in \mathcal{K}_\ell$

$$\int_{\mathbb{S}^q} f(x)\, dx = \sum_{\{\xi\} \in \mathcal{Q}_l} \lambda_\xi f(\xi).$$

Now let us fix a parameter $B > 1$; we will denote by $\{\xi_{jk}\}_{k=1,\ldots,K_j} = \mathcal{Q}_{[2B^{j+1}]}$, and $\{\lambda_{jk}\}_{k=1,\ldots,K_j}$ the set of cubature points and weights associated to the resolution level j: we recall that $\lambda_{jk} \approx B^{-qj}$ and $K_j \approx B^{qj}$, where $a \approx b$ indicates that there exist two positive constants c_1, c_2 s.t. $c_1 b \le a \le c_2 b$. Consider a real-valued function b on $(0, \infty)$, such that (i) $b(\cdot)$ has compact support in $\left[B^{-1}, B\right]$; (ii) $b \in C^\infty((0,\infty))$; (iii) for every $\ell \ge 1$ $\sum_{j=0}^{\infty} b^2\left(\ell / B^j\right) = 1$. The set of spherical needlets is then defined as

$$\psi_{jk}(z) = \sqrt{\lambda_{jk}} \sum_{\ell \in [B^{j-1}, B^{j+1}]} b\left(\frac{\ell}{B^j}\right) \frac{\ell + \eta_q}{\eta_q \omega_q} \mathcal{C}_\ell^{(\eta_q)}(\langle z, \xi_{jk} \rangle), z \in \mathbb{S}^q. \tag{1.4}$$

The needlet coefficients (of index j, k) are defined as follows

$$\beta_{jk} := \int_{\mathbb{S}^q} f(z) \psi_{jk}(z)\, dz,$$

and the following reconstruction formula holds, in the L^2 sense:

$$f(z) = \sum_{j,k} \beta_{jk} \psi_{jk}(z), z \in \mathbb{S}^q.$$

The following localization property was established by [24] (see also [11,13]): for any positive integer τ, there exists $\kappa_\tau > 0$ such that for any j, k and $z \in \mathbb{S}^q$

$$|\psi_{jk}(z)| \le \frac{\kappa_\tau B^{\frac{q}{2} j}}{\left(1 + B^{\frac{q}{2} j} d(z, \xi_{jk})\right)^\tau}, \tag{1.5}$$

where $d(\cdot,\cdot)$ is the geodesic distance on the sphere (i.e. for $q=2$ and $z_1,z_2\in\mathbb{S}^q$, $d(z_1,z_2)=\arccos(z_1,z_2)$). Following [25], from this localization result, the following bounds on the L^p-norms hold:

$$c_p B^{jq\left(\frac{p}{2}-1\right)} \le \|\psi_{jk}\|^p_{L^p(\mathbb{S}^q)} \le C_p B^{jq\left(\frac{p}{2}-1\right)}. \tag{1.6}$$

Remark 1.1. In the sequel, for $z_1,z_2\in\mathbb{S}^q$, we shall also meet functions $\psi_j^{(s)}(\cdot,\cdot)$ given by

$$\psi_j^{(s)}(z_1,z_2)=B^{-\frac{q}{2}j}\sum_{\ell} b^s\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}\left(\langle z_1,z_2\rangle\right). \tag{1.7}$$

It is immediate to see that for any integer $s>0$, the function $b^s(\cdot)$ is compactly supported, nonnegative and it belongs to the space $C^\infty((0,\infty))$: therefore, following the same arguments to establish the localization property in [24] and [12], it can be shown that, for any $\tau>2$, there exist a constant $C_\tau>$ such that

$$\left|\psi_j^{(s)}(z_1,z_2)\right| \le \frac{C_\tau B^{\frac{q}{2}j}}{\left(1+B^{\frac{q}{2}j}d\left(\langle z_1,z_2\rangle\right)\right)^\tau},$$

and hence

$$\psi_j^{(s)}(z_1,z_2)=O_j\left(B^{\frac{q}{2}j}\right).$$

1.4. Statement of the main results

1.4.1. Poissonized case

Throughout this paper, we shall assume that the function f in the governing Poisson measure is bounded and bounded away from zero, e.g. $m\le f(z)\le M$ for some $m,M>0$ for all $z\in\mathbb{S}^q$. Let us now consider first the vector of U-statistics

$$U_j^{(1)}(t)=\left(U_{jk_1}^{(1)}(t),\ldots,U_{jk_d}^{(1)}(t)\right) \tag{1.8}$$

where for any $k=k_i$, $i=1,\ldots,d$, we have

$$U_{jk}^{(1)}(t)=\frac{1}{u!}\sum_{(z_1,z_2)\in N^2_{t\neq}}\left(\psi_{jk}(z_1)-\psi_{jk}(z_2)\right)^u, \tag{1.9}$$

and the needlet functions $\psi_{jk_i}(\cdot)$ are given by (1.4), for some fixed locations $\{\xi_{k_i}\}\in\mathbb{S}^q$, $i=1,\ldots,d$. Observe that (1.9) has the form of (1.3) where, for $z_1,z_2\in\mathbb{S}^q$, the kernel $h_j(\cdot,\cdot)$ is defined as

$$h_j(z_1,z_2)\equiv h_{jk;u}(z_1,z_2):=\frac{1}{u!}\left(\psi_{jk}(z_1)-\psi_{jk}(z_2)\right)^u. \tag{1.10}$$

In the case $u=2$, (1.9) provides the sample variance of the (de-Poissonized) random variables $\psi_{jk}(X)$ (up to a normalization factor), where $X\in\mathbb{S}^q$ has density $f(\cdot)$, while for $u=3$, it provides skewness estimator. More precisely, for $u=2$ it is a standard exercise to show that

$$\frac{1}{R_t^2}\mathbb{E}\left[U_{jk}^{(1)}(t)\right]=\int_{\mathbb{S}^q}\psi_{jk}^2(z)f(z)\,dz-\left(\int_{\mathbb{S}^q}\psi_{jk}(z)f(z)dz\right)^2,$$

i.e., $U_{jk}^{(1)}(t)$ provides an unbiased estimator for the variance of $\psi_{jk}(X)$.

As a second application, we shall consider so-called Sobolev tests of uniformity, i.e. testing the null hypothesis that the function f is constant over the sphere (that is, $f(z) = \frac{1}{\omega_q}$, $z \in \mathbb{S}^q$), as discussed for instance by [17,18], see also [14]; in these references, the corresponding statistics are built out of Fourier basis over manifolds, and in the case of the sphere they would take the following form (compare [18], p. 1247 and following):

$$\sum_{\ell_1 \ell_2} a_{\ell_1} a_{\ell_2} \left(\sum_{(z_1,z_2)\in N^2_{t\neq}} \sum_{m=1}^{d_{\ell,q}} \overline{Y}_{\ell,m}(z_1) Y_{\ell,m}(z_2) \right). \tag{1.11}$$

Here, $\{a_\ell\}$ is a square-summable sequence introduced to combine the statistics evaluated at different multipoles ℓ into a single value; actually the procedure discussed by [18] is slightly different as it includes in the sum also the diagonal terms $z_1 = z_2$, but it is simple to show that after centering the two alternatives are asymptotically equivalent. The integral of spherical harmonics with respect to the uniform measure is obviously zero, so (1.11) provides a natural statistic to test uniformity.

Our proposal exploits the same idea, with two modifications: we consider a needlet-frame, rather than a Fourier dictionary, and we manage to provide asymptotic behavior also for the single summands, rather than for the combined statistics. More precisely, we advocate the usage of following vector of U-statistics

$$U^{(2)}_{j_1,\dots,j_d}(t) = \left(U^{(2)}_{j_1}(t), \dots, U^{(2)}_{j_d}(t) \right),$$

where for any $j = j_i$, $i = 1, \dots, d$, we have

$$U^{(2)}_{j}(t) = \sum_{(z_1,z_2)\in N^2_{t\neq}} \sum_{k} \psi_{jk}(z_1) \psi_{jk}(z_2). \tag{1.12}$$

Observe that (1.12) has the form of (1.3) where the kernel $h_j(\cdot,\cdot)$ is defined as

$$h_j(z_1,z_2) := \sum_{k} \psi_{jk}(z_1) \psi_{jk}(z_2).$$

As for the spherical harmonics, it is readily seen that under the null hypothesis of uniformity, $f(z) = \omega_q^{-1}$, $z \in \mathbb{S}^q$, we have

$$\int_{\mathbb{S}^q} \psi_{jk}(z) f(z)\, dz = \frac{1}{\omega_q} \int_{\mathbb{S}^q} \psi_{jk}(z)\, dz = 0;$$

more generally, this integral is zero for all functions $f(\cdot)$ that are band-limited, i.e. when $f \in \left\{ \oplus_{\ell=0}^{B^{j'}} \mathcal{H}_\ell \right\}$, $j' < j-1$; in this sense, needlets provide a natural building block to implement a Sobolev test of uniformity, and investigating the full vector of statistics $U^{(2)}_{j_1,\dots,j_d}(t)$ seems to provide more information than combining the components into a single value.

As argued below, due to the real-domain localization properties of the needlet frames both (1.8) and (1.12) are feasible and asymptotically justifiable in circumstances where the sphere $\mathbb{S}^q$ is only partially observed, a situation which takes place very often in practice. This means, for instance, that it may be feasible to test for uniformity of the function $f(\cdot)$ even from observations which cover a fraction of the sky, as it is the case for most astrophysical experiments [16].

Before stating our main results, additional notation is needed. For any given random vector $X = (X_1, \dots, X_d)$, we denote by $\widetilde{X}$ the normalized counterpart of X given by

$$\widetilde{X} := \left(\frac{X_1 - \mathbb{E}(X_1)}{\sqrt{\operatorname{Var}(X_1)}}, \ldots, \frac{X_d - \mathbb{E}(X_d)}{\sqrt{\operatorname{Var}(X_d)}} \right).$$

Furthermore, let Z_d be a centered d-dimensional Gaussian vector with the covariance matrix corresponding to the d-dimensional identity matrix. Our first result covers the statistics defined in (1.8):

Theorem 1.1. *Let $U_j^{(1)}(t)$ be given by (1.8). Then, for any $\tau > 2$,*

$$d_2\left(\widetilde{U_j^{(1)}}(t), Z_d\right) = O\left(B^{\frac{q}{2}j} R_t^{-\frac{1}{2}} + B^{-\frac{q}{2}j\tau} + R_t^{-\frac{1}{2}} B^{j\frac{q}{2}(u-1)}\right).$$

Therefore, taking $j = j(t)$ such that $j(t) \underset{t\to\infty}{\to} \infty$ and $B^{\frac{q}{2}j(t)} R_t^{-\frac{1}{2}} = o_t(1)$, it follows that

$$\widetilde{U_j^{(1)}}(t) \to_d Z_d.$$

An analogous result can be obtained for $d = 1$ as follows (see also Remark 1.2).

Corollary 1.1. *For any given f and for any j, k,*

$$d_W\left(\widetilde{U_{jk}^{(1)}}(t), \mathcal{N}(0,1)\right) = O\left(B^{j\frac{q}{2}} R_t^{-\frac{1}{2}}\right)$$

and

$$d_K\left(\widetilde{U_{jk}^{(1)}}(t), \mathcal{N}(0,1)\right) = O\left(B^{j\frac{q}{2}} R_t^{-\frac{1}{2}}\right).$$

Our second main result covers the statistics defined by (1.12):

Theorem 1.2. *Let $U_{j_1,\ldots,j_d}^{(2)}(t) = \left(U_{j_1}^{(2)}(t), \ldots, U_{j_d}^{(2)}(t)\right)$ be a d-dimensional vector where the components are of the form (1.12), being N_t a homogeneous spherical Poisson process, with $|j_i - j_{i'}| \geq 2$ for all $i \neq i'$. Then,*

$$d_2\left(\widetilde{U_{j_1,\ldots,j_d}^{(2)}}(t), Z_d\right) = O\left(\max_{j_1,\ldots,j_d}\left(B^{qj_i} R_t^{-1} + B^{-\frac{q}{2}j_i} + R_t^{-\frac{1}{2}}\right)\right).$$

Therefore, for $i = 1, \ldots, d$, taking $j_i = j_i(t)$ such that $j_i(t) \underset{t\to\infty}{\to} \infty$ and $B^{qj_i(t)} R_t^{-1} = o_t(1)$, it follows that

$$\widetilde{U_{j_1,\ldots,j_d}^{(2)}}(t) \to_d Z_d.$$

As in the previous case, for $d = 1$, we obtain the following result (see Remark 1.2).

Corollary 1.2. *For any given f and for any j,*

$$d_W\left(\widetilde{U_j^{(2)}}(t), \mathcal{N}(0,1)\right) = O\left(B^{qj} R_t^{-1} + B^{-\frac{q}{2}j} + R_t^{-\frac{1}{2}}\right)$$

and

$$d_K\left(\widetilde{U_j^{(2)}}(t), \mathcal{N}(0,1)\right) = O\left(B^{\frac{3q}{2}j} R_t^{-\frac{3}{2}} + B^{-\frac{3}{4}qj} + R_t^{-\frac{3}{4}}\right).$$

Remark 1.2. In Corollaries 1.1 and 1.2, the bound on the Kolmogorov distance is a direct consequence of [10, Theorem 4.1] and the proofs of these two results are omitted for the sake of brevity. In 1.2, the bound on the Kolmogorov distance is a consequence of using the sharper one-dimensional bound from [10, Theorem 4.1] instead of the multidimensional bounds used in the proof of Theorem 1.2.

Remark 1.3. The results in both the theorems can be easily generalized to cover the case where the dimension d grows itself with the "time" parameter t. The details are completely analogous to those given in related circumstances by [8], and hence are omitted here for brevity's sake.

Remark 1.4. The rates in Theorems 1.1 and 1.2 are actually different and indeed the proofs of these results are based on unrelated arguments. In particular, for Theorem 1.1, we shall show that the asymptotic behavior is governed by a stochastic integral belonging to the first Wiener chaos, while for Theorem 1.2, the dominant term is a double stochastic integral with respect to the underlying Poisson random measure. The condition $B^{\frac{q}{2}j_i(t)}R_t^{-\frac{1}{2}} = o_t(1)$ should be interpreted as the requirement that "the effective sample size" diverges to infinity, as argued in related circumstances by [8].

Remark 1.5. The components of the vector $\widetilde{U_j^{(1)}}(t)$ in Theorem 1.1 are related to needlets evaluated at the same scale j, but around different locations $(\xi_{k_1}, \dots, \xi_{k_d})$ on the sphere; on the other hand, the components of the vector $\widetilde{U_{j_1,\dots,j_d}^{(2)}}(t)$ in Theorem 1.2 are evaluated on the full sphere at different frequencies $(j_1, \dots, j_d)$. Both results are formulated under the assumption that the sphere is fully observable. This is done, however, only for notational simplicity: as mentioned earlier, exploiting the localization properties of the needlet construction it is simple to modify the statements for the case where these statistics are evaluated only on subsets of the sphere, the same convergence rates in the quantitative central limit theorems remaining valid up to constants. The arguments are completely analogous to those exploited for instance in [2], Section 7 in a Gaussian environment, and they are omitted here for brevity's sake.

Remark 1.6. In Theorem 1.2 the assumption that $|j_i - j_{i'}| \geq 2$ ensures that the limiting covariance matrix is exactly diagonal; relaxing this assumption makes the statement notationally more complicated but does not require any new ideas for the proofs.

Remark 1.7. The expression for the variance in Theorem 1.2 can be provided in a very explicit analytic form, for any fixed value of j. Moreover we also have the asymptotic convergence, for $j, t \to \infty$

$$\frac{1}{B^{qj}R_t^2}\mathrm{Var}\left[\widetilde{U_j^{(2)}}(t)\right] \to 2\gamma_q, \quad \text{where} \quad \gamma_q := \frac{1}{\eta_q \omega_q^2}\frac{1}{(q-2)!}\int_{1/B}^{B} b^4(u)u^{q-1}du. \tag{1.13}$$

1.4.2. A quantitative de-Poissonization lemma

In what follows, we shall show that the explicit bounds stated in Theorem 1.1 and Theorem 1.2 can be extended, at the cost of an additional factor, to the case of U-statistics based on a vector of i.i.d. observations, rather than on a Poisson measure. Our main tool in order to achieve this task is a new quantitative version of an argument taken from the fundamental paper by Dynkin and Mandelbaum [9], that we shall state in the general framework of U-statistics of arbitrary order. Note that one could alternatively deal with the one-dimensional case by using general Berry–Esseen bounds for U-statistics (see e.g. [4]); however we believe that our approach (which has independent interest) is more adapted in to directly study multi-dimensional probabilistic approximations. In the statement of the forthcoming Lemma 1.1, we shall work within the following framework:

- $X = \{X_i : i \geq 1\}$ is a sequence of i.i.d. random variables with values in some measurable space $(E, \mathcal{E})$;
- Let $m \geq 1$ be a fixed integer: we write $\{h_n : n \geq 1\}$ to indicate a sequence of jointly measurable symmetric kernels $h_n : E^m \to \mathbb{R}$ such that $\mathbb{E}[h_n(X_1, \ldots, X_m)] = 0$ and $\mathbb{E}[h_n(X_1, \ldots, X_m)^2] < \infty$;
- $\{N_n : n \geq 1\}$ is a sequence of Poisson random variables independent of X, such that N_n has a Poisson distribution with mean n for every n;
- For every $n \geq m$, the symbol U_n denotes the Poissonized U-statistic

$$U_n = \sum_{1 \leq i_1, \ldots, i_m \leq N_n} h_n(X_{i_1}, \ldots, X_{i_m}),$$

  where the sum runs over all m-ples $(i_1, \ldots, i_m)$ such that $i_j \neq i_k$ for $j \neq k$;
- For every $n \geq 1$, the symbol U'_n denotes the classical U-statistic

$$U'_n = \sum_{1 \leq i_1, \ldots, i_m \leq n} h_n(X_{i_1}, \ldots, X_{i_m}),$$

  where, as before, the sum runs over all m-ples $(i_1, \ldots, i_m)$ such that $i_j \neq i_k$ for $j \neq k$. It is easily checked that $\mathbb{E}[U_n] = \mathbb{E}[U'_n] = 0$.

Lemma 1.1 *(Quantitative de-Poissonization lemma). Assume that $\mathbb{E}[U_n^2] \to 1$ as $n \to \infty$. Then $\mathbb{E}[U_n'^2] \to 1$ as $n \to \infty$ and*

$$\mathbb{E}[(U_n - U'_n)^2] = O(n^{-1/2}), \quad n \to \infty.$$

1.4.3. Applications to needlet-based U-statistics of i.i.d. observations

A direct application of Lemma 1.1 leads to de-Poissonized versions of the main results of the paper. In what follows, we shall denote by $\{X_i : i \geq 1\}$ a sequence of i.i.d. random variables with values in $\mathbb{S}^q$, whose common distribution has a density f with respect to the Lebesgue measure. As before, we assume that $0 < m \leq f(z) \leq M < \infty$, for every $z \in \mathbb{S}^q$.

We start with an extension of Theorem 1.1. For every n, we write

$$U_j^{(1)}(n)' = \left(U_{jk_1}^{(1)}(n)', \ldots, U_{jk_d}^{(1)}(n)'\right)$$

where, for $k = k_i$, $i = 1, \ldots, d$,

$$U_{jk}^{(1)}(n)' = \frac{1}{u!} \sum_{1 \leq i_1 \neq i_2 \leq n} (\psi_{jk}(X_{i_1}) - \psi_{jk}(X_{i_2}))^u,$$

that is, each $U_{jk}^{(1)}(n)'$ is obtained from (1.9) (in the case $t = n$) by replacing the Poisson measure on the sphere $A \mapsto N([0, n] \times A)$ with the random measure $A \mapsto \sum_{i=1}^n \delta_{X_i}(A)$, where δ_x stands for the Dirac mass at x.

Theorem 1.3. *Under the above notation and assumptions, for any $\tau > 2$,*

$$d_2\left(\widetilde{U_j^{(1)}}(n)', Z_d\right) = O\left(B^{\frac{q}{2}j} n^{-\frac{1}{2}} + B^{-\frac{q}{2}j\tau} + n^{-1/2} B^{j\frac{q}{2}(u-1)} + n^{-1/4}\right).$$

Therefore, taking $j = j(n)$ such that $j(n) \underset{n\to\infty}{\to} \infty$ and $B^{\frac{q}{2}j(n)} n^{-\frac{1}{2}} = o_n(1)$, it follows that

$$\widetilde{U_j^{(1)}}(n)' \to_d Z_d.$$

Analogously to the notation introduced above, we shall write, for every n,

$$U^{(2)}_{j_1,\dots,j_d}(n)' = \left(U^{(2)}_{j_1}(n)',\dots,U^{(2)}_{j_d}(n)'\right)$$

where, for $j = j_a$, $a = 1,\dots,d$,

$$U^{(2)}_{j}(n)' = \sum_{1\le i_1\ne i_2\le n}\sum_k \psi_{jk}(X_{i_1})\psi_{jk}(X_{i_2}).$$

The following result extends Theorem 1.3.

Theorem 1.4. *Under the above notation and assumptions, assume in addition that $|j_i - j_{i'}| \ge 2$ for all $i \ne i'$. Then,*

$$d_2\left(\widetilde{U^{(2)}_{j_1,\dots,j_d}}(n)', Z_d\right) = O\left(\max_{j_1,\dots,j_d}\left(B^{qj_i}n^{-1} + B^{-\frac{q}{2}j_i} + n^{-\frac{1}{2}} + n^{-1/4}\right)\right).$$

Therefore, for $i = 1,\dots,d$, taking $j_i = j_i(n)$ such that $j_i(n) \underset{n\to\infty}{\to} \infty$ and $B^{qj_i(n)}n^{-1} = o_n(1)$, it follows that

$$\widetilde{U^{(2)}_{j_1,\dots,j_d}}(n)' \to_d Z_d.$$

Remark 1.8. The presence of the additional term $n^{-1/4}$ in the bound of Theorems 1.4 and 1.3 yields a phase transition in the convergence to the normal distribution. Indeed, taking the example of Theorem 1.4, depending on how fast $j(n)$ grows to infinity, the rate of convergence could be given either by $n^{-1/4}$, by $n^{-1/2}B^{\frac{q}{2}j(n)}$ or by $B^{-\frac{q}{2}j(n)}$ (depending on the relationship between n and $B^{qj(n)}$). Similarly, the rate of convergence in Theorem 1.3 can be given either by $n^{-1/4}$, by $n^{-1/2}B^{j(n)\frac{q}{2}(u-1)}$ or by $B^{-\frac{q}{2}j(n)\tau}$ (depending on the more intricate relationship between n, $j(n)$, τ and u).

1.5. Plan of the paper

The plan of this paper is as follows: the proofs for Theorems 1.1, 1.2 are collected in Sections 2 and 3 respectively. Section 4 deals with the proof of Lemma 1.1. Auxiliary results are collected in Section 5, which is divided into four parts, the first devoted to background results on Stein–Malliavin approximations in a Poisson environment, the second concerned with some functional inequalities for needlet kernels, the third and fourth devoted to specific computations for the two main theorems.

2. Proof of Theorem 1.1

Let us define $G_n(j) := \int_{\mathbb{S}^q} \psi^n_{jk}(z) f(z)\,dz$ and note that, from (1.6)

$$|G_n(j)| = O\left(B^{jq\left(\frac{n}{2}-1\right)}\right). \tag{2.1}$$

Example 2.1. Assuming that f is constant, $G_1(j) = 0$ and $G_2(j) = \|\psi_{jk}\|^2_{L^2(\mu_t)}$.

Recall that we are considering the process $U^{(1)}_j(t) = \left(U^{(1)}_{jk_1}(t),\dots,U^{(1)}_{jk_d}(t)\right)$, whose components, for $k = k_1,\dots,k_d$, are given by

$$U_{jk}^{(1)}(t) = \frac{1}{u!} \sum_{(z_1,z_2)\in N_{t\neq}^2} (\psi_{jk}(z_1) - \psi_{jk}(z_2))^u.$$

From Lemma 5.6, we have

$$\mathbb{E}\left[U_{jk}^{(1)}(t)\right] = R_t^2 \Gamma_1(j) \quad \text{and} \quad \operatorname{Var}\left[U_{jk}^{(1)}(t)\right] = 4R_t^3\Gamma_{21}(j) + 2R_t^2\Gamma_{22}(j),$$

where

$$\Gamma_1(j) = \frac{1}{u!}\sum_{r=0}^{u}\binom{u}{r} G_{u-r}(j)\, G_r(j), \tag{2.2}$$

$$\Gamma_{21}(j) = \frac{1}{(u!)^2}\sum_{q,r=0}^{u}\binom{u}{q}\binom{u}{r} G_q(j) G_r(j) G_{2u-(q+r)}(j), \tag{2.3}$$

$$\Gamma_{22}(j) = \frac{1}{(u!)^2}\sum_{q,r=0}^{u}\binom{u}{q}\binom{u}{r} G_{q+r}(j)\, G_{2u-(q+r)}(j). \tag{2.4}$$

Remark 2.1. Note that $\Gamma_{2i}(j)$, $i = 1, 2$, provides the variance of the components of order i in the Wiener chaos decomposition of $U_{jk}^{(1)}(t)$ (see Lemma 5.6).

Remark 2.2. Note in particular that using the notation for (compensated) Poisson random measure introduced in Subsection 5.1, for $u = 2$ we have that

$$\begin{aligned}
&\frac{1}{2}\sum_{z_1,z_2} (\psi_{jk}(z_1) - \psi_{jk}(z_2))^2 = \frac{1}{2}\int_{(\mathbb{S}^q)^2} (\psi_{jk}(z_1) - \psi_{jk}(z_2))^2 N_t(dz_1)\, N_t(dz_2) \\
&= \frac{1}{2}\int_{(\mathbb{S}^q)^2} (\psi_{jk}(z_1) - \psi_{jk}(z_2))^2 \widehat{N}_t(dz_1)\, \widehat{N}_t(dz_2) + \int_{(\mathbb{S}^q)^2} (\psi_{jk}(z_1) - \psi_{jk}(z_2))^2 \mu_t(dz_1)\, \widehat{N}_t(dz_2) \\
&\quad + \frac{1}{2}\int_{(\mathbb{S}^q)^2} (\psi_{jk}(z_1) - \psi_{jk}(z_2))^2 \mu_t(dz_1)\, \mu_t(dz_2)
\end{aligned}$$

so that

$$\mathbb{E}\left[\frac{1}{2}\sum_{z_1,z_2}(\psi_{jk}(z_1) - \psi_{jk}(z_2))^2\right] = R_t^2\left[\int_{\mathbb{S}^q}\psi_{jk}(z)^2 f(z)\,dz - \left(\int_{\mathbb{S}^q}\psi_{jk}(z) f(z)\,dz\right)^2\right].$$

From now we will consider the centered and asymptotically normalized counterpart of $U_{jk}^{(1)}(t)$, given by

$$\widetilde{U_{jk}^{(1)}}(t) = I_2\left(\widetilde{h}_{jk,t}\right) + I_1\left(\widetilde{g}_{jk,t}\right),$$

where

$$L^2\left(\mu_t^2\right) \ni \widetilde{h}_{jk,t}(z_1,z_2) := (z_1,z_2) \mapsto \frac{\frac{1}{u!}\sum_{r=0}^{u}\binom{u}{r}\psi_{jk}^{u-r}(z_1)\,\psi_{jk}^{r}(z_2)}{\sqrt{4R_t^3\Gamma_{21}(j) + 2R_t^2\Gamma_{22}(j)}},$$

$$L^2\left(\mu_t\right) \ni \widetilde{g}_{jk,t}(z) := z \mapsto \frac{\frac{2}{u!}\int_{\mathbb{S}^q}\sum_{r=0}^{u}\binom{u}{r}\psi_{jk}^{u-r}(z)\,\psi_{jk}^{r}(y)\,\mu_t(dy)}{\sqrt{4R_t^3\Gamma_{21}(j) + 2R_t^2\Gamma_{22}(j)}}.$$

Let $Z_d \sim \mathcal{N}_d(0, I)$ be a centered standard d-dimensional Gaussian vector and consider the following random vector

$$\widetilde{U_j^{(1)}}(t) = \left(\widetilde{U_{jk_1}^{(1)}}(t), \ldots, \widetilde{U_{jk_d}^{(1)}}(t)\right) = \left(I_1(\widetilde{g}_{jk_1,t}) + I_2\left(\widetilde{h}_{jk_1,t}\right), \ldots, I_1(\widetilde{g}_{jk_d,t}) + I_2\left(\widetilde{h}_{jk_d,t}\right)\right).$$

Our strategy to prove Theorem 1.1 will be based on two steps: we shall bound the distance between the U-statistics we consider and an approximating stochastic integral in the first Wiener chaos, and then bound the probability distance between the latter and the limiting Gaussian distribution. In particular, we shall focus on the distance

$$d_2\left(\widetilde{U_j^{(1)}}(t), Z_d\right) = d_2(I_{1,j}(t) + I_{2,j}(t), Z_d),$$

where

$$I_{1,j}(t) = (I_1(\widetilde{g}_{jk_1,t}), \ldots, I_1(\widetilde{g}_{jk_d,t})) \quad \text{and} \quad I_{2,j}(t) = \left(I_2\left(\widetilde{h}_{jk_1,t}\right), \ldots, I_2\left(\widetilde{h}_{jk_d,t}\right)\right).$$

Applying the triangle inequality, we obtain

$$\begin{aligned} d_2\left(\widetilde{U_j^{(1)}}(t), Z_d\right) &\leq d_2\left(\widetilde{U_j^{(1)}}(t), I_{1,j}(t)\right) + d_2(I_{1,j}(t), Z_d) \\ &\leq \sqrt{\mathbb{E}\left[\left\|\widetilde{U_j^{(1)}}(t) - I_{1,j}(t)\right\|_{\mathbb{R}^d}^2\right]} + d_2(I_{1,j}(t), Z_d) = \mathbb{E}\left[\|I_{2,j}(t)\|_{\mathbb{R}^d}^2\right] + d_2(I_{1,j}(t), Z_d). \end{aligned}$$

Our task is then to study the asymptotic behavior of these two summands. Consider now the d-dimensional random vector $\widetilde{U_j^{(1)}}(t)$, where $d \leq K_j$ is fixed: following Proposition 5.4, it holds that

$$\lim_{t\to\infty} \mathbb{E}[I_1(\tilde{g}_{jk_1,t}) I_1(\tilde{g}_{jk_2,t})] = \delta_{k_1}^{k_2}.$$

By using Lemma 5.7, we have the existence of a positive constant $\sigma \in \mathbb{R}$ such that

$$\sqrt{4R_t\Gamma_{21}(j) + 2\Gamma_{22}(j)} \geq R_t^{\frac{1}{2}} B^{j\frac{q}{2}(u-1)}\sigma,$$

which will allow us to prove that

$$d_2\left(\widetilde{U_j^{(1)}}(t), Z_d\right) \leq \left(\frac{2d^{\frac{1}{2}} C_{2u} M^{\frac{1}{2}}}{R_t^{\frac{1}{2}} B^{j\frac{q}{2}(u-1)}\sigma} + \frac{dC_\tau M B^{-\frac{q}{2}j\tau}}{mc_2^2 \left(1 + \inf_{k_1\neq k_2} d(\xi_{k_1}, \xi_{k_2})\right)^{2\tau}} + d\frac{\sqrt{2\pi}}{8}\frac{C_{3u}M^4}{c_2^3 m}\frac{B^{\frac{q}{2}j}}{R_t^{\frac{1}{2}}}\right).$$

Indeed,

$$\begin{aligned} \|I_d - \Sigma_{j,t}\|_{H.S.} &\leq \sqrt{\sum_{k_1\neq k_2=1}^{d} \mathbb{E}^2[I_1(\tilde{g}_{jk_1,t}) I_1(\tilde{g}_{jk_2,t})]} \leq d \sup_{k_1\neq k_2=1,\ldots,d} \frac{C_{\tau,M,u,\sigma}}{B^{j\frac{q}{2}(u-1)}\sigma^2 \left(1 + B^{\frac{q}{2}j} d(\xi_{k_1}, \xi_{k_2})\right)^\tau} \\ &\leq d \sup_{k_1\neq k_2=1,\ldots,d} \frac{1}{mc_2^2} \frac{C_{\tau,M,u,\sigma}}{\left(1 + \inf_{k_1\neq k_2=1,\ldots,d} B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^\tau} := A_1(t). \end{aligned}$$

On the other hand

$$\sqrt{\mathbb{E}\left[\left\|I_{2,j}\right\|_{\mathbb{R}^d}^2\right]} \leq \sqrt{2\sum_{k=1}^{d}\left\|\widetilde{h}_{jk,t}\right\|_{L^2(\mu_t^2)}^2} \leq \frac{2C_{2u}M^{\frac{1}{2}}d^{\frac{1}{2}}}{R_t^{\frac{1}{2}}B^{j\frac{q}{2}(u-1)}\sigma}\left(1+o\left(\frac{1}{R_t^{\frac{1}{2}}}\right)\right) := A_2(t).$$

Finally,

$$\begin{aligned} d_2\left(\widetilde{U_j^{(1)}}(t), Z_d\right) &\leq A_1(t) + A_2(t) + \frac{\sqrt{2\pi}}{8}\sum_{k_1,k_2,k_3=1}^{d}\int_{\mathbb{S}^q}\left|\widetilde{g}_{jk_1,t}(z)\right|\left|\widetilde{g}_{jk_2,t}(z)\right|\left|\widetilde{g}_{jk_3,t}(z)\right|\mu_t(dz) \\ &\leq A_1(t) + A_2(t) + \frac{\sqrt{2\pi}}{8}\frac{M}{R_t^{\frac{1}{2}}B^{j\frac{3q}{2}(u-1)}\sigma^3}\sum_{k_1,k_2,k_3=1}^{d}\int_{\mathbb{S}^q}\left|\psi_{jk_1}^u(z)\right|\left|\psi_{jk_2}^u(z)\right|\left|\psi_{jk_3}^u(z)\right|dz. \end{aligned}$$

By Lemma 5.3, we have

$$d_2\left(\widetilde{U_j^{(1)}}(t), Z_d\right) \leq A_1(t) + A_2(t) + d\frac{\sqrt{2\pi}}{8}\frac{C_{3u}M^4}{c_2^3 m}\frac{B^{\frac{q}{2}j}}{R_t^{\frac{1}{2}}},$$

as claimed.

3. Proof of Theorem 1.2

In this section, our purpose is to study the statistic

$$U_{j_1,\ldots,j_d}^{(2)}(t) = \left(U_{j_1}^{(2)}(t), \ldots, U_{j_d}^{(2)}(t)\right)$$

where for any $j = j_i$, $i = 1, \ldots, d$,

$$U_j^{(2)}(t) = \sum_{(z_1,z_2)\in N_{t\neq}^2}\sum_k \psi_{jk}(z_1)\psi_{jk}(z_2).$$

Recall that here we are focusing on Sobolev tests of uniformity, and hence we are assuming the Poisson governing measure is given by

$$\mathbb{E}\left[N_t(dz)\right] = \mu_t(dz) = R_t f(z)\,dz = \frac{R_t}{\omega_q}dz. \tag{3.1}$$

This yields

$$\int_{\mathbb{S}^q}\psi_{j_ik}(z)\mu_t(dz) = \frac{R_t}{\omega_q}\int_{\mathbb{S}^q}\psi_{j_ik}(z)\,dz = 0.$$

Using this fact along with Proposition 5.1, we have

$$U_{j_i}^{(2)}(t) = I_2\left(\sum_k \psi_{j_ik}\otimes\psi_{j_ik}\right).$$

Let $\gamma_{j,q}$ be given by

$$\gamma_{j,q} := \left(\omega_q B^{qj}\right)^{-1}\sum_{\ell=B^{j-1}}^{B^{j+1}} b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\binom{\ell+q-2}{\ell}. \tag{3.2}$$

Using the isometry property of multiple Wiener–Itô integrals, we have that, for any $j = j_i$, $i = 1, \ldots, d$,

$$\operatorname{Var}\left(U_j^{(2)}(t)\right) = \mathbb{E}\left[I_2^2\left(\sum_k \psi_{jk} \otimes \psi_{jk}\right)\right] = 2\left\|\sum_k \psi_{jk} \otimes \psi_{jk}\right\|_{L^2(\mu_t^2)}^2 = 2R_t^2 B^{qj}\gamma_{j,q} \tag{3.3}$$

by Lemmas 5.8 and 5.9. We can hence focus on the normalized statistics

$$\widetilde{U_j^{(2)}}(t) := \frac{U_j^{(2)}(t)}{R_t B^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}} = I_2\left(\widetilde{h}_{j_i,t}\right),$$

where

$$L^2\left(\mu_t^2\right) \ni \widetilde{h}_{j,t} \colon (z_1, z_2) \mapsto \frac{\sum_k \psi_{jk}(z_1)\,\psi_{jk}(z_2)}{R_t B^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}}.$$

Observe that, by construction, $\widetilde{U_j^{(2)}}(t)$ is normalized. Therefore, in order to prove Theorem 1.2, it remains to check that $\widetilde{h}_{j,t}$ satisfies the five conditions in Proposition 5.3, for all $j = j_i, i = 1, \ldots, d$. Before doing so, notice that the kernel $\widetilde{h}_{j,t}$ can be rewritten as

$$\widetilde{h}_{j,t}(z_1, z_2) = \left(R_t B^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}\right)^{-1} \sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{\ell + \eta_q}{\eta_q \omega_q}\mathcal{C}_\ell^{(\eta_q)}\left(\langle z_1, z_2\rangle\right),$$

as pointed out in Lemma 5.4. Condition 1 in Proposition 5.3 is hence automatically satisfied by construction. Recall that the star-contraction operators that are going to be used in what follows are defined in Definition 1.2.

For Condition 2, following Lemma 5.5, we obtain

$$\begin{aligned}\left\|\widetilde{h}_{j,t}\right\|_{L^4\left(\mu_t^{\otimes 2}\right)}^4 &= \left(R_t B^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}\right)^{-4} B^{2qj} \int_{(\mathbb{S}^q)^2}\left(\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{\ell + \eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}\left(\langle z_1, z_2\rangle\right)B^{-\frac{q}{2}j}\right)^4 \mu_t^{\otimes 2}(dz_1, dz_2) \\ &= O\left(R_t^{-2}B^{2qj}\right).\end{aligned}$$

Recall that, using equation (22.4.2) in [1],

$$\mathcal{C}_\ell^{(\eta_q)}(1) = \binom{\ell + 2\eta_q - 1}{\ell} = \binom{\ell + q - 2}{\ell}. \tag{3.4}$$

Now for Condition 3, as

$$\begin{aligned}\left(\widetilde{h}_{j,t} \star_1^1 \widetilde{h}_{j,t}\right)(z_1, z_2) &= \left(R_t B^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}\right)^{-2} R_t \int_{\mathbb{S}^q}\sum_{\ell_1,\ell_2}\left(\prod_{i=1,2} b^2\left(\frac{\ell_i}{B^j}\right)\frac{\ell_i + \eta_q}{\eta_q\omega_q}\mathcal{C}_{\ell_i}^{(\eta_q)}\left(\langle z_i, a\rangle\right)\right)da \\ &= \frac{1}{R_t}\left(\sqrt{2\gamma_{j,q}}B^{\frac{q}{2}j}\right)^{-2}\sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell + \eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}\left(\langle z_1, z_2\rangle\right),\end{aligned}$$

we obtain, using Lemma 5.5 again,

$$\left\|\widetilde{h}_{j,t}\star_1^1\widetilde{h}_{j,t}\right\|^2_{L^2\left(\mu_t^{\otimes 2}\right)}=\left(4R_t^2B^{2qj}\gamma_{j,q}^2\right)^{-1}\int\limits_{(\mathbb{S}^q)^2}\left(\sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}\left(\langle z_1,z_2\rangle\right)\right)^2\mu_t\left(dz_1\right)\mu_t\left(dz_2\right)$$

$$=\left(4B^{2qj}\gamma_{j,q}^2\right)^{-1}\frac{B^{qj}}{\omega_q^2}\int\limits_{\mathbb{S}^q}\sum_\ell b^8\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}\left(1\right)B^{-qj}dz$$

$$=O\left(B^{-qj}\right).$$

For Condition 4, we start by observing that

$$\left(\widetilde{h}_{j,t}\star_2^1\widetilde{h}_{j,t}\right)(z)=R_t\left(R_tB^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}\right)^{-2}\int\limits_{\mathbb{S}^q}\sum_{\ell_1,\ell_2}\left(\prod_{i=1,2}b^2\left(\frac{\ell_i}{B^j}\right)\frac{\ell_i+\eta_q}{\eta_q\omega_q}\mathcal{C}_{\ell_i}^{(\eta_q)}\left(\langle z,a\rangle\right)\right)da$$

$$=R_t^{-1}\left(B^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}\right)^{-2}\sum_{\ell=B^{j-1}}^{B^{j+1}}b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\binom{\ell+q-2}{\ell}.$$

Therefore,

$$\left\|\widetilde{h}_{j,t}\star_2^1\widetilde{h}_{j,t}\right\|^2_{L^2(\mu_t)}=R_t^{-2}\left(B^{\frac{q}{2}j}\sqrt{2\gamma_{j,q}}\right)^{-4}B^{2qj}\left(\sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}B^{-qj}\binom{\ell+q-2}{\ell}\right)^2\int\limits_{\mathbb{S}^q}\mu_t\left(dz\right)$$

$$=O\left(R_t^{-1}\right).$$

For the fifth and last condition, let $1\le i_1\ne i_2\le d$. We clearly have

$$\left\langle\widetilde{h}_{j_{i_1},t},\widetilde{h}_{j_{i_2},t}\right\rangle_{L^2\left(\mu_t^{\otimes 2}\right)}=0 \tag{3.5}$$

as by assumption we have $|j_{i_2}-j_{i_1}|>1$ and hence, it is enough to exploit the orthogonality properties of Gegenbauer polynomials. Gathering all these estimates together along with the fact that, by (3.5), the two first sums appearing in the bound (5.2) of Proposition 5.3 vanish, yields

$$d_2\left(\widetilde{U_{j_1,\dots,j_d}^{(2)}}\left(t\right),Z_d\right)=O\left(\max_{j\in\{j_1,\dots,j_d\}}\left(B^{qj}R_t^{-1}+B^{-\frac{q}{2}j}+R_t^{-\frac{1}{2}}\right)\right).$$

The proof of Theorem 1.2 is hence concluded.

4. Proof of Lemma 1.1

Using the classical theory of Hoeffding decompositions as in [6, Section 3.6] and [9], we infer that there exist nonnegative constants $u(n,l)\in[0,\infty)$, $1\le l\le m\le n$ such that

$$\mathrm{Var}(U_n)=\sum_{l=1}^m\frac{n^l}{l!}u(n,l)\quad\text{and}\quad\mathrm{Var}(U_n')=\sum_{l=1}^m\binom{n}{l}u(n,l),$$

from which we deduce immediately that $\mathrm{Var}(U_n')\to 1$ and also that each mapping $n\mapsto\binom{n}{l}u(n,l)$, $l=1,\dots,m$, is necessarily bounded. We will now adopt the usual falling factorial notation, namely: $n_{[l]}=n(n-1)\cdots(n-l+1)$. Reasoning as in [9, p. 745], one infers that

$$\mathbb{E}[(U_n - U_n')^2] = \sum_{l=1}^{m} \binom{n}{l} u(n,l) \left\{ 1 + \frac{n^l}{n_{[l]}} - 2b(n,l) \right\},$$

where $b(n,l) := e^{-n} \sum_{p=0}^{\infty} \frac{n^p}{p!} \binom{n \wedge p}{l} \binom{n}{l}^{-1}$. Since $n^l / n_{[l]} - 1 = O(1/n)$, the conclusion is achieved once we show that $1 - b(n,l) = O(1/\sqrt{n})$, $l = 1, \dots, m$. Elementary computations yield that

$$1 - b(n,l) = e^{-n} \sum_{p=n-l+1}^{n} \frac{n^p}{p!} + \left(1 - \frac{n^l}{n_{[l]}}\right) e^{-n} \sum_{p=0}^{n-l} \frac{n^p}{p!} = e^{-n} \sum_{p=n-l+1}^{n} \frac{n^p}{p!} + O\left(\frac{1}{n}\right).$$

By virtue of a standard application of Stirling's formula one has that, for $l = 1, \dots, m$,

$$e^{-n} \sum_{p=n-l+1}^{n} \frac{n^p}{p!} \sim n^{-1/2},$$

and the desired conclusion follows at once.

5. Auxiliary results

Fix a Poisson measure N_t with control μ_t, $t > 0$. Consider an integer $i \geq 1$ as well as a symmetric kernel $f \in L^2(\mu_t^i)$: we shall denote by $I_i(f)$ the usual Wiener–Itô integral of order i, of f with respect to N_t. See for instance [29, Chapter 5] for a detailed discussion of this concept.

5.1. Gaussian approximations using Stein–Malliavin methods

The following crucial fact is proved by Reitzner & Schulte in [32, Lemma 3.5 and Theorem 3.6]; let ν be a Poisson point process on the measure space $(Z, \mathcal{B}(Z), \nu)$ where Z is a Borel space and ν is a σ-finite nonatomic Borel measure.

Proposition 5.1. *Consider a kernel $h \in L_s^1(\nu^k)$ such that the corresponding U-statistic F in (1.2) is square-integrable. Then, h is necessarily square-integrable, and F admits a chaotic decomposition of the form*

$$F = \int_{Z^k} h(x_1, \dots, x_k)\, d\nu^k + \sum_{i=1}^{\infty} I_i(h_i),$$

with

$$h_i(x_1, \dots, x_i) = \binom{k}{i} \int_{Z^{k-i}} h(x_1, \dots, x_i, x_{i+1}, \dots, x_k) d\nu^{k-i}, \quad (x_1, \dots, x_i) \in Z^i,$$

for $1 \leq i \leq k$, and $h_i = 0$ for $i > k$. In particular, $h = h_k$ and the projection h_i is in $L_s^{1,2}(\nu^i)$ for each $1 \leq i \leq k$.

In our case, we have $\nu = N_t$ and $Z = \mathbb{S}^q$. We need also to recall two upper bounds involving random variables living in the *first Wiener chaos* associated to the Poisson measure N. The first bound was proved in [28], and concerns normal approximations in dimension 1 with respect to the Wasserstein distance. The second bound appears in [30], and provides estimates for multidimensional normal approximations with respect to the distance d_2. Both bounds are obtained by means of a combination of the Malliavin calculus of variations and the Stein's method for probabilistic approximations. In what follows, we shall use the symbols $N(f)$ and $\hat{N}(f)$, respectively, to denote the Wiener–Itô integrals of f with respect to N and with respect to the *compensated Poisson measure*

$$\hat{N}(A) = N(A) - \mu(A), \quad A \in \mathcal{B}(\Theta),$$

where we use the convention $N(A) - \mu(A) = \infty$ whenever $\mu(A) = \infty$ (recall that μ is σ-finite). We shall consider Wiener–Itô integrals of functions f having the form $f = \mathbf{1}_{[0,t]} \times h$, where $t > 0$ and $h \in L^2(\mathbb{S}^q, \mu_t) \cap L^1(\mathbb{S}^q, \mu_t)$. For a function f of this type we simply write

$$N(f) = N(\mathbf{1}_{[0,t]} \times h) := N_t(h), \quad \text{and} \quad \hat{N}(f) = \hat{N}(\mathbf{1}_{[0,t]} \times h) := \hat{N}_t(h).$$

Proposition 5.2 *(Gaussian approximations in the linear regime). (See [28,30].) Let $h \in L^2(\mu_t)$, let $Z \sim \mathcal{N}(0,1)$ and fix $t > 0$. Then, the following bound holds:*

$$d_W(\hat{N}_t(h), Z) \le \left|1 - \|h\|^2_{L^2(\mu_t)}\right| + \int_{\mathbb{S}^q} |h(z)|^3 \mu_t(dz).$$

For a fixed integer $d \ge 1$, let $Z_d \sim \mathcal{N}_d(0, \Sigma)$ where Σ is a positive definite covariance matrix and let

$$F_t = (F_{t,1}, \dots, F_{t,d}) = \left(\hat{N}_t(h_{t,1}), \dots, \hat{N}_t(h_{t,d})\right)$$

be a collection of d-dimensional random vectors such that $h_{t,a} \in L^2(\nu)$. If we call Γ_t the covariance matrix of F_t, then

$$\begin{aligned} d_2(F_t, Z_d) &\le \left\|\Sigma^{-1}\right\|_{op} \|\Sigma\|^{\frac{1}{2}}_{op} \|\Sigma - \Gamma_t\|_{H.S.} \\ &\quad + \frac{\sqrt{2\pi}}{8} \left\|\Sigma^{-1}\right\|^{\frac{3}{2}}_{op} \|\Sigma\|_{op} \sum_{i,j,k=1}^{d} \int_{\mathbb{S}^q} |h_{t,i}(z)|\,|h_{t,j}(z)|\,|h_{t,k}(x)|\,\mu_t(dx) \\ &\le \left\|\Sigma^{-1}\right\|_{op} \|\Sigma\|^{\frac{1}{2}}_{op} \|\Sigma - \Gamma_t\|_{H.S.} + \frac{d^2\sqrt{2\pi}}{8} \left\|\Sigma^{-1}\right\|^{\frac{3}{2}}_{op} \|\Sigma\|_{op} \sum_{i=1}^{d} \int_{\mathbb{S}^q} |h_{t,i}(x)|^3 \mu_t(dx), \end{aligned} \tag{5.1}$$

where $\|\cdot\|_{op}$ and $\|\cdot\|_{H.S.}$ stand, respectively, for the operator and Hilbert–Schmidt norms.

Remark 5.1. The estimate (5.1) is a direct consequence of Theorem 3.3 in [30].

Proposition 5.3 *(Gaussian approximations in the quadratic regime). (See Proposition 6.1 in [30].) Let $d = d_1 + d_2$, with $d_1, d_2 \ge 1$ two integers and let $Z_d \sim \mathcal{N}(0, I_d)$. Assume that*

$$F_j = (F_{j,1}, \dots, F_{j,d}) := \left(I_1(g_{j,1}), \dots, I_1(g_{j,d_1}), I_2(h_{j,1}), \dots, I_2(h_{j,d_2})\right),$$

where the symmetric kernels $g_{j,1}, \dots, g_{j,d_1}, h_{j,1}, \dots, h_{j,d_2}$ satisfy the following conditions: for $k = 1, \dots, d_1$, $g_{j,k} \in L^2(\mu_t) \cap L^3(\mu_t)$; for $k = 1, \dots, d_2, h_{j,k} \in L^1(\mu_t^2) \cap L^2(\mu_t)$ is such that (a) for $1 \le k_1, k_2 \le d_2$, $h_{j,k_1} \star_2^1 h_{j,k_2} \in L^2(\mu_t)$, (b) $h_{j,k} \in L^4(\mu_t^2)$, (c) $|h_{j,k_1}| \star_2^1 |h_{j,k_2}|$, $|h_{j,k_1}| \star_2^0 |h_{j,k_2}|$ and $|h_{j,k_1}| \star_1^0 |h_{j,k_2}|$ are well defined and finite for every value of their arguments (the general definition of the star-contraction kernels are given in Definition 1.2) and (d) it holds that, for $1 \le k_1, k_2 \le d_2$,

$$\int_{\mathbb{S}^q} \sqrt{\int_{\mathbb{S}^q} h^2_{j,k_1}(z_1, z_2)\, h^2_{j,k_2}(z_1, z_2)\, \mu_t(dz_2) \mu_t(dz_1)} < \infty.$$

Assume that the following five conditions hold:

1. *for* $k = 1, \ldots, d_1$, $\|g_{j,k}\|_{L^4(\mu_t)} \to 0$;
2. *for* $k = 1, \ldots, d_2$, $\|h_{j,k}\|_{L^4(\mu_t^2)} \to 0$;
3. *for* $1 \le k \le d_2$, $\left\|h_{j,k} \star_1^1 h_{j,k}\right\|_{L^2(\mu_t^2)} \to 0$, *where* $\left(h_{j,k} \star_1^1 h_{j,k}\right)(z_1, z_2) = \int_{\mathbb{S}^q} h_{j,k}(z_1, z_3)\, h_{j,k}(z_2, z_3)\, \mu_t(dz_3)$;
4. *for* $1 \le k \le d_2$, $\left\|h_{j,k} \star_2^1 h_{j,k}\right\|_{L^2(\mu_t)} \to 0$, *where* $\left(h_{j,k} \star_2^1 h_{j,k}\right)(z) = \int_{\mathbb{S}^q} h_{j,k}^2(z, a)\, \mu_t(da)$;
5. *for* $1 \le k_1, k_2 \le d$, $\mathbb{E}\left[F_{j,k_1} F_{j,k_2}\right] \to \delta_{k_1}^{k_2}$.

Then F_j *converges in distribution to* Z_d *and*

$$\begin{aligned} d_2(F_j, Z_d) \le \frac{1}{2} \Bigg(& \sum_{k_1,k_2=1}^{d_1} \left(\delta_{k_1}^{k_2} - \langle g_{j,k_1}, g_{j,k_2}\rangle_{L^2(\mu_t)}\right)^2 + \sum_{k_1,k_2=1}^{d_2} \left(\delta_{k_1}^{k_2} - \langle h_{j,k_1}, h_{j,k_2}\rangle_{L^2(\mu_t^2)}\right)^2 \\ & + \sum_{k_1,k_2=1}^{d_2} \left(4\left\|h_{j,k_1} \star_2^1 h_{j,k_2}\right\|_{L^2(\mu_t)}^2 + 8\left\|h_{j,k_1} \star_1^1 h_{j,k_1}\right\|_{L^2(\mu_t^2)}^2\right) \\ & + 5\sum_{k_1=1}^{d_1}\sum_{k_2=1}^{d_2} \left\|g_{j,k_1} \star_1^1 h_{j,k_2}\right\|_{L^2(\mu_t)}^2 \Bigg)^{\frac{1}{2}} + d_1^2 \sum_{k=1}^{d_1} \|g_{j,k}\|_{L^3(\mu_t)}^3 \\ & + 8d_2^2 \sum_{k=1}^{d_2} \|h_{j,k}\|_{L^2(\mu_t^2)} \left(\|h_{j,k}\|_{L^4(\mu_t^2)}^2 + \sqrt{2}\left\|h_{j,k} \star_2^1 h_{j,k}\right\|_{L^2(\mu_t)}\right). \end{aligned} \tag{5.2}$$

Remark 5.2. Observe that, as opposed to the statement of [30, Proposition 6.1], it is the norm $\left\|h_{j,k} \star_2^1 h_{j,k}\right\|_{L^2(\mu_t)}$ that appears in the last part of the above bound instead of the norm $\left\|h_{j,k} \star_1^0 h_{j,k}\right\|_{L^2(\mu_t^3)}$. This comes from the fact that these two norms are equal in the case of kernels of two variables (which is easily verified by a standard Fubini argument – see e.g. [28, Remark 4.3]) and hence we used the former instead of the latter in order to avoid having yet another contraction norm to study in the applications of this result.

5.2. Functional inequalities for needlet kernels

We present here some functional inequalities which are necessary for our main arguments. The first lemma is basically a consequence of (1.6). For $z_i \in \mathbb{S}^q$, $i = 1, \ldots, D$, let

$$\Omega = \left\{ v_1, \ldots, v_D \colon v_i \in \mathbb{Z}_+, \forall i = 1, \ldots, D, \sum_{i=1}^D v_i = V \right\};$$

$$L_V(z_1, z_2, \ldots, z_D) = \sum_{\{v_1,\ldots,v_D\} \in \Omega} c_{v_1,\ldots,v_D} \prod_{i=1}^D \psi_{jk}(z_i)^{v_i}.$$

Lemma 5.1. *For* C_{v_i} *as defined in* (1.6) *and denoting by* δ_0^v *the Kronecker delta function, it holds that*

$$\begin{aligned} & \left| \int_{(\mathbb{S}^q)^D} L_V(z_1, z_2, \ldots, z_D)\, \mu^{\otimes D}(dz_1, dz_2, \ldots, dz_D) \right| \\ & \le M^D \sum_{\{v_1,\ldots,v_D\} \in \Omega} c_{v_1,\ldots,v_D} \left(\prod_{i=1}^D C_{v_i}\right) B^{jq\left(\frac{1}{2}V - D + \sum_{i=1}^D \delta_0^{v_i}\right)}. \end{aligned}$$

Proof. Easy calculations lead to

$$\left|\int_{(\mathbb{S}^q)^D} L_V(z_1,z_2,\ldots,z_D)\,\mu^{\otimes D}(dz_1,dz_2,\ldots,dz_D)\right| \le \int_{(\mathbb{S}^q)^D} |L_V(z_1,z_2,\ldots,z_D)|\,\mu^{\otimes D}(dz_1,dz_2,\ldots,dz_D)$$

$$\le \sum_{\{v_1,\ldots,v_D\}\in\Omega} c_{v_1,\ldots,v_D}\prod_{i=1}^{D}\int_{\mathbb{S}^q}|\psi_{jk}(z_i)|^{v_i}\,\mu(dz_i) \le M^D\sum_{\{v_1,\ldots,v_D\}\in\Omega} c_{v_1,\ldots,v_D}\prod_{i=1}^{D}\int_{\mathbb{S}^q}|\psi_{jk}(z_i)|^{v_i}\,dz_i.$$

For any i, it follows from (1.6) that

$$\int_{\mathbb{S}^q}|\psi_{jk}(z_i)|^{v_i}\,dz_i \le C_{v_i}B^{j\left[q\left(\frac{v_i}{2}-1\right)+q\delta^0_{v_i}\right]}.$$

Therefore we obtain

$$\left|\int_{(\mathbb{S}^q)^D} L_V(z_1,z_2,\ldots,z_D)\,\mu^{\otimes D}(dz_1,dz_2,\ldots,dz_D)\right|$$

$$\le M^D\sum_{\{v_1,\ldots,v_D\}\in\Omega} c_{v_1,\ldots,v_D}\left(\prod_{i=1}^{D}C_{v_i}\right)B^{qj\sum_{i=1}^{D}\left[\left(\frac{v_i}{2}-1\right)+\delta^0_{v_i}\right]}$$

$$= M^D\sum_{\{v_1,\ldots,v_D\}\in\Omega} c_{v_1,\ldots,v_D}\left(\prod_{i=1}^{D}C_{v_i}\right)B^{qj\left(\frac{1}{2}V-D+\sum_{i=1}^{D}\delta^0_{v_i}\right)}. \qquad \square$$

Now, for any set $\{v_1,\ldots,v_D\}\in\Omega$ write $Z_{\{v_1,\ldots,v_D\}} := \sharp\{i\colon v_i=0,\ \nu_i\in\{v_1,\ldots,v_D\}\}$ and $Z_0 = \max_{\{v_1,\ldots,v_D\}\in\Omega} Z_{\{v_1,\ldots,v_D\}}$.

Corollary 5.1. *There exists $C'>0$ such that*

$$\left|\int_{(\mathbb{S}^q)^D} L_V(z_1,z_2,\ldots,z_D)\,\mu^{\otimes D}(dz_1,dz_2,\ldots,dz_D)\right| \le C'B^{jq\left(\frac{1}{2}V-D+Z_0\right)}.$$

Proof. We have

$$\left|\int_{(\mathbb{S}^q)^D} L_V(z_1,z_2,\ldots,z_D)\,\mu^{\otimes D}(dz_1,dz_2,\ldots,dz_D)\right|$$

$$\le M^D\sum_{\{v_1,\ldots,v_D\}\in\Omega} c_{v_1,\ldots,v_D}\left(\prod_{i=1}^{D}C_{v_i}\right)B^{qj\left(\frac{1}{2}V-D+\sum_{i=1}^{D}\delta^0_{v_i}\right)}$$

$$\le C'M^DB^{qj\left(\frac{1}{2}V-D+\sum_{i=1}^{D}\delta^0_{v_i}\right)} \le C'M^DB^{jq\left(\frac{1}{2}V-D+Z_0\right)}. \qquad \square$$

Lemma 5.2. *For any j, $k_1\ne k_2=1,\ldots,K_j$, $\tau\ge 2$, $n_1,n_2>1$, we have*

$$\int_{\mathbb{S}^q}\left|\psi^{n_1}_{jk_1}(z)\,\psi^{n_2}_{jk_2}(z)\right|\mu(dz) \le C_{\tau,M,n_1,n_2}B^{\left(\frac{(n_1+n_2)}{2}-1\right)qj}\left(\frac{1}{\left(1+B^{\frac{q}{2}j}d\left(\xi_{jk_1},\xi_{jk_2}\right)\right)^{\min(n_1,n_2)\tau}}\right).$$

Proof. As in Lemma 5.1 in [8] and Lemmas 4, 6 in [25], we split the sphere into two regions

$$S_1 = \{z \in \mathbb{S}^q \colon d(z, \xi_{jk_2}) > d(\xi_{jk_1}, \xi_{jk_2})/2\} \quad \text{and} \quad S_2 = \{z \in \mathbb{S}^q \colon d(z, \xi_{jk_1}) > d(\xi_{jk_1}, \xi_{jk_2})/2\},$$

so that $\mathbb{S}^q \subseteq S_1 \cup S_2$. On the other hand, we have by (1.5) that there exists $\tau > 2$ such that

$$\begin{aligned}
\int_{\mathbb{S}^q} \left|\psi_{jk_1}^{n_1}(z)\,\psi_{jk_2}^{n_2}(z)\right| \mu(dz) &\le \kappa_\tau^{n_1+n_2} M \int_{\mathbb{S}^q} \frac{B^{n_1 \frac{q}{2} j}}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_1})\right)^{n_1\tau}} \frac{B^{n_2 \frac{q}{2} j}}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_2})\right)^{n_2\tau}} dz \\
&\le \kappa_\tau^{n_1+n_2} M \left[\int_{S_1} \frac{B^{n_1 \frac{q}{2} j}}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_1})\right)^{n_1\tau}} \frac{B^{n_2 \frac{q}{2} j}}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_2})\right)^{n_2\tau}} dz \right. \\
&\quad \left. + \int_{S_2} \frac{B^{n_1 \frac{q}{2} j}}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_1})\right)^{n_1\tau}} \frac{B^{n_2 \frac{q}{2} j}}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_2})\right)^{n_2\tau}} dz \right].
\end{aligned}$$

Now, observe that

$$\begin{aligned}
& B^{\frac{(n_1+n_2)}{2} qj} \int_{S_1} \frac{1}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_1})\right)^{n_1\tau}} \frac{1}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_2})\right)^{n_2\tau}} dz \\
&\le \frac{B^{\frac{(n_1+n_2)}{2} qj}}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{n_2\tau}} \int_{S_1} \frac{1}{\left(1 + B^{\frac{q}{2}j} d(z, \xi_{jk_1})\right)^{n_1\tau}} dz \\
&\le \frac{2\pi B^{\frac{(n_1+n_2)}{2} qj}}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{n_2\tau}} \int_0^\pi \frac{\sin\theta}{\left(1 + B^{\frac{q}{2}j}\theta\right)^{n_1\tau}} d\theta \\
&\le \frac{2\pi B^{\frac{(n_1+n_2)}{2} qj} B^{-qj}}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{n_2\tau}} \left(\int_0^\infty \frac{y\,dy}{1 + y^{n_1\tau}} \right) \\
&\le \frac{2\pi B^{\frac{(n_1+n_2)}{2} qj} B^{-qj}}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{n_2\tau}} \left(\int_0^1 y\,dy + \int_1^\infty y^{1-n_1\tau} dy \right) \le \frac{2\pi C B^{\frac{(n_1+n_2)}{2} qj} B^{-qj}}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{n_2\tau}}.
\end{aligned}$$

The same result is obtained for S_2, so that

$$\begin{aligned}
& \int_{\mathbb{S}^q} \psi_{jk_1}^{n_1}(z)\,\psi_{jk_2}^{n_2}(z)\,\mu(dz) \\
&\le C_{\tau,M,n_1,n_2} B^{\left(\frac{(n_1+n_2)}{2} - 1\right) qj} \left(\frac{1}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{n_2\tau}} + \frac{1}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{n_2\tau}} \right) \\
&\le C_{\tau,M,n_1,n_2} B^{\left(\frac{(n_1+n_2)}{2} - 1\right) qj} \left(\frac{1}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{\min(n_1,n_2)\tau}} \right),
\end{aligned}$$

as claimed. □

Lemma 5.3. *It holds that*

$$\sum_{k_1,k_2,k_3=1}^{d} \int_{\mathbb{S}^q} \left|\psi_{jk_1}^u(z)\right| \left|\psi_{jk_2}^u(z)\right| \left|\psi_{jk_3}^u(z)\right| dz \le d C_u \kappa_t''' B^{q\left(\frac{3}{2}u - 1\right) j}.$$

Proof. Following similar arguments to those in [8], we have

$$\sum_{k_1,k_2,k_3=1}^{d}\int_{\mathbb{S}^q}\left|\psi_{jk_1}^u(z)\right|\left|\psi_{jk_2}^u(z)\right|\left|\psi_{jk_3}^u(z)\right|dz\le C\sum_{\lambda}\int_{\mathcal{B}(\xi_{j\lambda},B^{-qj})}\left(\sum_{k=1}^{d}\left|\psi_{jk}^u(z)\right|\right)^3dz,$$

so that there exists $\tau>2$ such that

$$\sum_{k=1}^{d}\left|\psi_{jk}^u(z)\right|\le\sum_{k=1}^{d}\frac{\kappa_\tau^uB^{\frac{q}{2}uj}}{\left(1+B^{\frac{d}{2}j}d(\xi_{jk},z)\right)^{u\tau}}\le\kappa_\tau^uB^{\frac{q}{2}uj}+\sum_{k:\xi_{jk}\notin\mathcal{B}(\xi_{j\lambda},B^{-qj})}^{d}\frac{\kappa_\tau^uB^{\frac{q}{2}uj}}{\left(B^{\frac{q}{2}j}d(\xi_{jk},\xi_{j\lambda})\right)^{u\tau}}.$$

For $\xi_{jk}\notin\mathcal{B}\left(\xi_{j\lambda},B^{-qj}\right)$, $z\notin\mathcal{B}\left(\xi_{j\lambda},B^{-qj}\right)$, using the triangle inequality yields $d(\xi_{jk},\xi_{j\lambda})+d(\xi_{jk},z)\ge d(z,\xi_{j\lambda})$. Using the fact that $d(\xi_{jk},\xi_{j\lambda})\ge d(\xi_{jk},z)$ and $2d(\xi_{jk},\xi_{j\lambda})\ge d(z,\xi_{j\lambda})$, we obtain

$$\begin{aligned}&\sum_{k:\xi_{jk}\notin\mathcal{B}(\xi_{j\lambda},B^{-qj})}^{d}\frac{\kappa_\tau^uB^{\frac{q}{2}uj}}{\left(B^{\frac{q}{2}j}d(\xi_{jk},\xi_{j\lambda})\right)^{u\tau}}\\&\le\sum_{k:\xi_{jk}\notin\mathcal{B}(\xi_{j\lambda},B^{-qj})}^{d}\frac{1}{meas\left(\mathcal{B}\left(\xi_{j\lambda},B^{-qj}\right)\right)}\int_{\mathcal{B}(\xi_{j\lambda},B^{-qj})}\frac{\kappa_\tau^uB^{\frac{q}{2}uj}}{\left(B^{\frac{q}{2}j}d(\xi_{jk},\xi_{j\lambda})\right)^{u\tau}}dz\\&\le\kappa_\tau'B^{\frac{q}{2}uj},\end{aligned}$$

where we applied [2, Lemma 6]. We obtain $\sum_k^d\left|\psi_{jk}^u(z)\right|\le\kappa_t''B^{\frac{q}{2}uj}$ uniformly over $z\in\mathbb{S}^q$. Finally, in order to have

$$\int_{\mathbb{S}^q}\left(\sum_{k}^{d}\left|\psi_{jk}^u(z)\right|\right)^3dz\le dC_u\kappa_t'''B^{\left(\frac{3}{2}u-1\right)qj},$$

it is enough to check that

$$\sum_{k_1,k_2,k_3=1}^{d}\int_{\mathbb{S}^q}\left(\left|\psi_{jk_1}^u(z)\right|\left|\psi_{jk_2}^u(z)\right|\left|\psi_{jk_3}^u(z)\right|\right)dx\le C_u\kappa_t'''B^{quj}\sum_{k=1}^{d}\int_{\mathbb{S}^q}\left|\psi_{jk}(z)\right|^udz\le dC_u\kappa_t'''B^{\frac{3}{2}quj},$$

by using 1.6. □

Lemma 5.4. *For $z_1,z_2\in\mathbb{S}^q$, it holds that*

$$\sum_k\psi_{jk}(z_1)\psi_{jk}(z_2)=\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}(\langle z_1,z_2\rangle).$$

Proof. First observe that

$$\sum_k\psi_{jk}(z_1)\psi_{jk}(z_2)=\sum_k\sum_{\ell_1,\ell_2}\left(\prod_{i=1,2}b\left(\frac{\ell_i}{B^j}\right)\frac{\ell_i+\eta_q}{\eta_q\omega_q}\mathcal{C}_{\ell_i}^{(\eta_q)}(\langle z_i,\xi_{jk}\rangle)\right)\lambda_{jk}.$$

Using the cubature formula over the sphere (see [24]) along with the self-reproducing property of the Gegenbauer polynomials (see for instance [39]), we have

$$\sum_{k}^{(\eta_q)} \mathcal{C}_{\ell_1}^{(\eta_q)}\left(\langle z_1, \xi_{jk}\rangle\right) \mathcal{C}_{\ell_2}^{(\eta_q)}\left(\langle z_2, \xi_{jk}\rangle\right) \lambda_{jk} = \int_{\mathbb{S}^q} \mathcal{C}_{\ell_1}^{(\eta_q)}\left(\langle z_1, x\rangle\right) \mathcal{C}_{\ell_2}^{(\eta_q)}\left(\langle z_2, x\rangle\right) dx$$
$$= \left(\frac{\ell_1+\eta_q}{\eta_q \omega_q}\right)^{-1} \mathcal{C}_{\ell_1}^{(\eta_q)}\left(\langle z_1, z_2\rangle\right) \delta_{\ell_1}^{\ell_2},$$

where δ_y^x is the Kronecker delta function. The statement follows immediately. □

Lemma 5.5. *For any $s \in \mathbb{N}$*

$$\int_{(\mathbb{S}^q)^2} \left(B^{-\frac{qj}{2}} \sum_{\ell} b^s\left(\frac{\ell}{B^j}\right) \frac{\ell+\eta_q}{\eta_q\omega_q} \mathcal{C}_{\ell}^{(\eta_q)}\left(\langle z_1, z_2\rangle\right)\right)^n dz_2 dz_1 = O\left(B^{jq(n-2)}\right). \tag{5.3}$$

Proof. For any $s \in \mathbb{N}$, observe that the integrand in (5.3) behaves as the n-th power of $\psi_j^{(s)}(z)$ defined in (1.7), as stated in Remark 1.1. Hence we have

$$\int_{(\mathbb{S}^q)^2} \left(B^{-\frac{qj}{2}} \sum_{\ell} b^s\left(\frac{\ell}{B^j}\right) \frac{\ell+\eta_q}{\eta_q\omega_q} \mathcal{C}_{\ell}^{(\eta_q)}\left(\langle z_1, z_2\rangle\right)\right)^n dz_2 dz_1 = O\left(\left\|\psi_j^{(s)}\right\|_{L^n(dz)}^n\right) = O\left(B^{jq(n-2)}\right),$$

as an immediate consequence of Formula (3.12) in [25]. □

5.3. Auxiliary results related to the proof of Theorem 1.1

Lemma 5.6. *For any j, k, let $U_{jk}^{(1)}(t)$ be given by (1.9) and, let $\Gamma_1(j)$, $\Gamma_{21}(j)$ and $\Gamma_{22}(j)$ be given respectively by (2.2), (2.3) and (2.4). It holds that*

$$\mathbb{E}\left[U_{jk}^{(1)}(t)\right] = R_t^2 \Gamma_1(j) \quad \textit{and} \quad \operatorname{Var}\left(U_{jk}^{(1)}(t)\right) = 4R_t^3\Gamma_{21}(j) + 2R_t^2\Gamma_{22}(j).$$

Proof. We can easily observe that

$$\mathbb{E}\left[U_{jk}^{(1)}(t)\right] = \int_{(\mathbb{S}^q)^2} \frac{1}{u!} \sum_{r=0}^{u} \binom{u}{r} \psi_{jk}^{u-r}(x)\, \psi_{jk}^{r}(y)\, \mu_t^2(dx, dy) = \frac{R_t^2}{u!} \sum_{r=0}^{u} \binom{u}{r} G_{u-r}(j)\, G_r(j) = R_t^2 \Gamma_1(j).$$

On the other hand,

$$\operatorname{Var}\left(U_j^{(1)}(t)\right) = \|g_{jk}\|_{L^2(\mu_t)}^2 + 2\|h_{jk}\|_{L^2(\mu_t^2)}^2,$$

where

$$L^2\left(\mu_t^2\right) \ni h_{jk}\colon (z_1, z_2) \mapsto \frac{1}{u!} \sum_{r=0}^{u} \binom{u}{r} \psi_{jk}^{u-r}(z_1)\, \psi_{jk}^{r}(z_2) \quad \text{and} \quad L^2(\mu_t) \ni g_{jk}\colon z \mapsto 2\int_{\mathbb{S}^q} h_{jk}(z, a)\, \mu_t(da).$$

We hence have

$$\|g_{jk}\|_{L^2(\mu_t)}^2 = 4R_t^3 \int_{\mathbb{S}^q} \left(\frac{1}{u!} \sum_{r=0}^{u} \binom{u}{r} \psi_{jk}^{u-r}(z)\, G_r(j)\right)^2 \mu(dz)$$

$$= \frac{4R_t^3}{(u!)^2} \sum_{s,r=0}^{u} \binom{u}{s}\binom{u}{r} G_s(j)\, G_r(j) \int_{\mathbb{S}^q} \psi_{jk}^{2u-(s+r)}(z)\, \mu(dz)$$

$$= \frac{4R_t^3}{(u!)^2} \sum_{s,r=0}^{u} \binom{u}{s}\binom{u}{r} G_s(j)\, G_r(j)\, G_{2u-(s+r)}(j) = 4R_t^3 \Gamma_{21}(j)$$

and

$$\|h_{jk}\|^2_{L^2(\mu_t^2)} = \int_{(\mathbb{S}^q)^2} \left(\frac{1}{u!} \sum_{r=0}^{u} \binom{u}{r} \psi_{jk}^{u-r}(z_1)\, \psi_{jk}^{r}(z_2) \right)^2 \mu_t^2(dz_1, dz_2)$$

$$= \frac{R_t^2}{(u!)} \sum_{s,r=0}^{u} \binom{u}{s}\binom{u}{r} G_{s+r}(j)\, G_{2u-(s+r)}(j)$$

$$= R_t^2 \Gamma_{22}(j) . \qquad \square$$

Lemma 5.7. *Let $\Gamma_1(j)$, $\Gamma_{21}(j)$ and $\Gamma_{22}(j)$ be given respectively by (2.2), (2.3) and (2.4). Then, there exist $c_{u,m}, C_{u,m}, c^{(1)}_{2u,m}, C^{(1)}_{2u,M}, c^{(2)}_{2u,m}, C^{(2)}_{2u,M} > 0$ such that*

$$c_{u,m} B^{jq(\frac{u}{2}-1)} (1 + o_j(1)) \le |\Gamma_1(j)| \le C_{u,M} B^{jq(\frac{u}{2}-1)} (1 + o_j(1)),$$
$$c^{(1)}_{2u,m} B^{j\frac{q}{2}(u-1)} (1 + o_j(1)) \le |\Gamma_{21}(j)| \le C^{(1)}_{2u,M} B^{j\frac{q}{2}(u-1)} (1 + o_j(1)),$$
$$c^{(2)}_{2u,m} B^{j\frac{q}{2}(u-1)} (1 + o_j(1)) \le |\Gamma_{22}(j)| \le C^{(2)}_{2u,M} B^{j\frac{q}{2}(u-1)} (1 + o_j(1)).$$

Proof. From (2.1), we have that

$$|\Gamma_1(j)| = \left| \sum_{i=0}^{u} \binom{u}{i} G_{u-i}(j)\, G_i(j) \right| \le \sum_{i=0}^{u} \binom{u}{i} |G_{u-i}(j) G_i(j)| .$$

Using Corollary 5.1, we obtain

$$|\Gamma_1(j)| \le C_u G_u(j) (1 + o_j(1)) \le C_{u,M} B^{jq(\frac{u}{2}-1)} (1 + o_j(1)),$$

and likewise

$$|\Gamma_1(j)| \ge \left(2 |G_u(j)| - \left| \sum_{r=1}^{u-1} \binom{u}{r} |G_{u-r}(j)\, G_r(j)| \right| \right) \ge 2mc_u B^{jq(\frac{u}{2}-1)} (1 + o_j(1)),$$

because, using Corollary 5.1, the leading terms in both the sums correspond to the symmetric $i = 0$ and $i = u$. Likewise, it holds that

$$|\Gamma_{21}(j)| \le G_{2u}(j) (1 + o_j(G_{2u}(j))) \le C^{(1)}_{2u,M} B^{jq(u-1)} (1 + o_j(1))$$

and

$$|\Gamma_{21}(j)| \ge c^{(1)}_{2u,m} B^{jq(u-1)} (1 + o_j(1)) .$$

Finally,

$$c^{(2)}_{2u,m} B^{jq(u-1)} (1 + o_j(1)) \le |\Gamma_{22}(j)| \le C^{(2)}_{2u,M} B^{jq(u-1)} (1 + o_j(1)),$$

as claimed. $\square$

Proposition 5.4. *Let $\Sigma_{j,t} = \{\Sigma_{j,t}(k_1,k_2) : k_1,k_2 = 1,\dots,d\}$ be a $d \times d$ positive definite matrix such that*

$$\Sigma_{j,t}(k_1,k_2) = \mathbb{E}\left[I_1(\tilde{g}_{jk_1,t}), I_1(\tilde{g}_{jk_2})\right] = \langle \tilde{g}_{jk_1}, \tilde{g}_{jk_2} \rangle_{L^2(\mu_t)}.$$

Then, there exists a constant $C_{\tau,M,u,\sigma}$ such that

$$\Sigma_{j,t}(k_1,k_2) - \delta_{k_1}^{k_2} \leq \frac{C_{\tau,M,u,\sigma}}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{\tau}}.$$

Therefore, as $j := j(t) \underset{t\to\infty}{\to} \infty$,

$$\lim_{t\to\infty} \Sigma_{j,t}(k_1,k_2) = \delta_{k_1}^{k_2}.$$

Proof. Following [30], we have that for $1 \leq k_1, k_2 \leq d$,

$$\begin{aligned}
\langle \widetilde{g}_{jk_1}, \widetilde{g}_{jk_2} \rangle_{L^2(\mu_t)} &\leq \frac{1}{R_t^3 \sigma^2 B^{jq(u-1)}} \int_{\mathbb{S}^q} \left(\int_{\mathbb{S}^q} h_{jk_1}(z_1,z_2)\, \mu_t(dz_2) \int_{\mathbb{S}^q} h_{jk_2}(z_1,z_3)\, \mu_t(dz_3) \right) \mu_t(dz_1) \\
&= \frac{1}{\sigma^2 B^{jq(u-1)}} \left[\int_{\mathbb{S}^q} \left(\sum_{i_1=0}^{u} \binom{u}{i_1} \psi_{jk_1}^{u-i_1}(z)\, G_{i_1}(j) \right) \left(\sum_{i_2=0}^{u} \binom{u}{i_2} \psi_{jk_2}^{u-i_2}(z)\, G_{i_2}(j) \right) \mu(dz) \right] \\
&\leq \frac{M}{\sigma^2 B^{jq(u-1)}} \sum_{i_1=0}^{u} \sum_{i_2=0}^{u} \binom{u}{i_1}\binom{u}{i_2} G_{i_1}(j)\, G_{i_2}(j) \int_{\mathbb{S}^q} \psi_{jk_1}^{u-i_1}(z)\, \psi_{jk_2}^{u-i_2}(z)\, dz \\
&\leq C_{\tau,M,u,\sigma} \frac{B^{(\frac{2u}{2}-1)qj}}{B^{jq(u-1)}} \left(\frac{1}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{u\tau}} \right) \\
&\leq C_{\tau,M,u,\sigma} \left(\frac{1}{\left(1 + B^{\frac{q}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{u\tau}} \right).
\end{aligned}$$

From Lemma 5.2, we hence have

$$\left| \langle \widetilde{g}_{ju}, \widetilde{g}_{jv} \rangle_{L^2(\mu_t)} \right| \leq \frac{C_{\tau,M,u,\sigma}}{\left(1 + B^{\frac{d}{2}j} d(\xi_{jk_1}, \xi_{jk_2})\right)^{u\tau}},$$

as claimed. □

5.4. Auxiliary results related to the proof of Theorem 1.2

Lemma 5.8. *Let $\gamma_{j,q}$ be given by (3.2). We have that*

$$\left\| \sum_k \psi_{jk} \otimes \psi_{jk} \right\|_{L^2(\mu_t^2)}^2 = \gamma_{j,q} R_t^2 B^{qj}.$$

Proof. In view of (3.1), we have

$$\left\| \sum_k \psi_{jk} \otimes \psi_{jk} \right\|_{L^2(\mu_t^2)}^2 = \sum_{k_1,k_2} \left(\int_{\mathbb{S}^q} \psi_{jk_1}(z)\, \psi_{jk_2}(z)\, \mu_t(dz) \right)^2 = \left(\omega_j^{-1} R_t\right)^2 \sum_{k_1,k_2} \left(\int_{\mathbb{S}^q} \psi_{jk_1}(z)\, \psi_{jk_2}(z)\, dz \right)^2.$$

Hence, from Lemma 5.4 we obtain

$$\begin{aligned}
\sum_{k_1,k_2}\left(\int_{\mathbb{S}^q}\psi_{jk_1}(z)\,\psi_{jk_2}(z)\,dz\right)^2 &= \sum_{k_1,k_2}\int_{(\mathbb{S}^q)^2}\psi_{jk_1}(z_1)\,\psi_{jk_1}(z_2)\,\psi_{jk_2}(z_1)\,\psi_{jk_2}(z_2)\,dz_1dz_2 \\
&= \int_{(\mathbb{S}^q)^2}\left(\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}(\langle z_1,z_2\rangle)\right)^2 dz_1dz_2 \\
&= \sum_{\ell_1,\ell_2}\prod_{i=1,2}\left(b^2\left(\frac{\ell_i}{B^j}\right)\frac{\ell_i+\eta_q}{\eta_q\omega_q}\right)\int_{(\mathbb{S}^q)^2}\mathcal{C}_{\ell_1}^{(\eta_q)}(\langle z_1,z_2\rangle)\,\mathcal{C}_{\ell_2}^{(\eta_q)}(\langle z_1,z_2\rangle)\,dz_1dz_2 \\
&= \sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}(1)\int_{\mathbb{S}^q}dz.
\end{aligned}$$

Using (3.4), we obtain

$$\left\|\sum_k \psi_{jk}\otimes\psi_{jk}\right\|^2_{L^2(\mu_t^2)} = R_t^2\omega_q^{-1}\sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\binom{\ell+2\eta_q-1}{\ell} = R_t^2 B^{qj}\gamma_{j,q}. \qquad \square$$

Lemma 5.9. *Let $\gamma_{j,q}$ be given by (3.2). Then, there exist positive constants $c_1, c_2 > 0$ such that, for all $j > 0$*

$$c_1 \le \gamma_{j,q} \le c_2. \tag{5.4}$$

Moreover, as $j\to\infty$, $\gamma_{j,q}\to\gamma_q$, where γ_q is given by (1.13).

Proof. The inequality (5.4) is easily proved by rewriting $\gamma_{j,q}$ in the framework of the Remark 1.1 and using Lemma 5.5. Indeed, for Lemma 5.4 and considering (3.4)

$$\gamma_{j,q} = \left(\omega_q B^{qj}\right)^{-1}\sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\mathcal{C}_\ell^{(\eta_q)}(1) = \left(\eta_q\omega_q^2\right)^{-1}\frac{1}{B^j}\sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{B^j}\frac{\mathcal{C}_\ell^{(\eta_q)}(1)}{B^{j(q-2)}}.$$

Now, for all the values of j, we have that

$$c_1' = \frac{B^{-(q-2)}}{(q-2)!} \le \frac{\mathcal{C}_\ell^{(\eta_q)}(1)}{B^{j(q-2)}} = \frac{1}{B^{j(q-2)}}\binom{\ell+q-2}{\ell} \le \frac{B^{q-2}}{(q-2)!} + o_j(1) = c_2'. \tag{5.5}$$

Likewise,

$$B^{-1} + O\left(B^{-j}\right) \le \frac{\ell+\eta_q}{B^j} \le B + O\left(B^{-j}\right), \tag{5.6}$$

and

$$c_1'' \le \frac{1}{B^j}\sum_\ell b^4\left(\frac{\ell}{B^j}\right) \le c_2''. \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we obtain (5.4). On the other hand, up to factors of smaller order, (3.2) is a Riemann sum of the integral in (1.13), so that

$$\lim_{j\to\infty}\left(\omega_q B^{qj}\right)^{-1}\sum_\ell b^4\left(\frac{\ell}{B^j}\right)\frac{\ell+\eta_q}{\eta_q\omega_q}\binom{\ell+q-2}{\ell} = \gamma_q. \qquad \square$$

References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Natl. Bur. Stand., Appl. Math. Ser., vol. 55, U.S. Government Printing Office, Washington, D.C., 1964.

[2] P. Baldi, G. Kerkyacharian, D. Marinucci, D. Picard, Asymptotics for spherical needlets, Ann. Statist. 37 (2009) 1150–1171.

[3] P. Baldi, G. Kerkyacharian, D. Marinucci, D. Picard, Adaptive density estimation for directional data using needlets, Ann. Statist. 37 (2009) 3362–3395.

[4] V. Bentkus, B.Y. Jing, W. Zhou, On normal approximations to U-statistics, Ann. Probab. 37 (2009) 2174–2199.

[5] A. Bhattacharya, R. Bhattacharya, Nonparametric Inference on Manifolds: With Applications to Shape Spaces, Inst. Math. Stat. Monogr., vol. 2, Cambridge University Press, 2012.

[6] S. Bourguin, G. Peccati, Portmanteau inequalities on the Poisson space: mixed regimes and multidimensional clustering, Electron. J. Probab. 19 (2014), 42 pp.

[7] L.H.Y. Chen, L. Goldstein, Q.M. Shao, Normal Approximation by Stein's Method, Probab. Appl., Springer-Verlag, 2011.

[8] C. Durastanti, D. Marinucci, G. Peccati, Normal approximations for wavelet coefficients on spherical Poisson fields, J. Math. Anal. Appl. 409 (2014) 212–227.

[9] E.B. Dynkin, A. Mandelbaum, Symmetric statistics, Poisson point processes, and multiple Wiener integrals, Ann. Statist. 11 (1983) 739–745.

[10] P. Eichelsbacher, C. Thaele, New Berry–Esseen bounds for non-linear functionals of Poisson random measures, Electron. J. Probab. 19 (2014), 25 pp.

[11] D. Geller, A. Mayeli, Continuous wavelets on compact manifolds, Math. Z. 262 (2009) 895–927.

[12] D. Geller, A. Mayeli, Nearly tight frames and space-frequency analysis on compact manifolds, Math. Z. 263 (2009) 235–264.

[13] D. Geller, I.Z. Pesenson, Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds, J. Geom. Anal. 21 (2011) 334–371.

[14] M.E. Giné, Invariant tests for uniformity on compact Riemannian manifolds based on Sobolev norms, Ann. Statist. 3 (1975) 1243–1266.

[15] V. Hoeffding, A class of statistics with asymptotically normal distribution, Ann. Math. Stat. 19 (1948) 293–325.

[16] R. Iuppa, G. Di Sciascio, F.K. Hansen, D. Marinucci, R. Santonico, Cosmic-ray anisotropies observed by the ARGO-YBJ experiment, Nucl. Instrum. Methods Phys. Res., Sect. A 692 (2012) 160–164.

[17] P.E. Jupp, Sobolev tests of goodness of fit of distributions on compact Riemannian manifolds, Ann. Statist. 33 (2005) 2957–2966.

[18] P.E. Jupp, Data-driven Sobolev tests of uniformity on compact Riemannian manifolds, Ann. Statist. 36 (2008) 1246–1260.

[19] G. Kerkyacharian, T.M. Pham Ngoc, D. Picard, Localized spherical deconvolution, Ann. Statist. 39 (2011) 1042–1068.

[20] R. Lachièze-Rey, G. Peccati, Fine Gaussian fluctuations on the Poisson space II: rescaled kernels, marked processes and geometric U-statistics, Stochastic Process. Appl. 123 (2013) 4186–4218.

[21] G. Last, M. Penrose, M. Schulte, C. Thäle, Moments and central limit theorems for some multivariate Poisson functionals, Adv. in Appl. Probab. 46 (2014) 348–364.

[22] D. Marinucci, G. Peccati, Random Fields on the Sphere. Representation, Limit Theorems and Cosmological Applications, London Math. Soc. Lecture Note Ser., vol. 389, Cambridge University Press, 2011.

[23] D. Marinucci, D. Pietrobon, A. Balbi, P. Baldi, P. Cabella, G. Kerkyacharian, P. Natoli, D. Picard, N. Vittorio, Spherical needlets for CMB data analysis, Mon. Not. R. Astron. Soc. 383 (2008) 539–545.

[24] F.J. Narcowich, P. Petrushev, J.D. Ward, Localized tight frames on spheres, SIAM J. Math. Anal. 38 (2006) 574–594.

[25] F.J. Narcowich, P. Petrushev, J.D. Ward, Decomposition of Besov and Triebel–Lizorkin spaces on the sphere, J. Funct. Anal. 238 (2006) 530–564.

[26] I. Nourdin, G. Peccati, Stein's method on Wiener chaos, Probab. Theory Related Fields 145 (2009) 75–118.

[27] I. Nourdin, G. Peccati, Normal approximations with Malliavin calculus, in: From Stein's Method to Universality, in: Cambridge Tracts in Math., vol. 192, Cambridge University Press, 2012.

[28] G. Peccati, J.L. Solé, M.S. Taqqu, F. Utzet, Stein's method and normal approximation of Poisson functionals, Ann. Probab. 38 (2010) 443–478.

[29] G. Peccati, M.S. Taqqu, Wiener Chaos: Moments, Cumulants and Diagrams. A Survey with Computer Implementation, Bocconi Springer Ser., vol. 1, Bocconi University Press, 2011.

[30] G. Peccati, C. Zheng, Multi-dimensional Gaussian fluctuations on the Poisson space, Electron. J. Probab. 15 (2010) 1487–1527.

[31] N. Privault, Stochastic Analysis in Discrete and Continuous Settings with Normal Martingales, Lecture Notes in Math., Springer-Verlag, 1982.

[32] M. Reitzner, M. Schulte, Central limit theorems for U-statistics of Poisson point processes, Ann. Probab. 41 (2013) 3879–3909.

[33] R. Schneider, W. Weil, Stochastic and Integral Geometry, Probab. Appl., Springer-Verlag, 2008.

[34] S. Scodeller, F.K. Hansen, D. Marinucci, Detection of new point sources in WMAP 7 year data using internal templates and needlets, Astrophys. J. 753 (2012), 46 pp.

[35] P.K. Sen, Introduction to Hoeffding (1948) A class of statistics with asymptotically normal distribution, in: Breakthroughs in Statistics, vol. 1, 1992, pp. 299–307.

[36] J.L. Starck, J.M. Fadili, S. Digel, B. Zhang, J. Chiang, Source detection using a 3D sparse representation: application to the Fermi gamma-ray space telescope, Astronom. Astrophys. 504 (2009) 641–652.

[37] J.L. Starck, F. Murtagh, J.M. Fadili, Sparse Image and Signal Processing. Wavelets, Curvelets, Morphological Diversity, Cambridge University Press, 2010.

[38] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Math. Ser., vol. 32, Princeton University Press, 1971.
[39] G. Szego, Orthogonal Polynomials, fourth edition, Amer. Math. Soc. Colloq. Publ., vol. 23, American Mathematical Society, 1975.