# Panel Test

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# 1 Model

$I_\kappa$ is an identity matrix of dimension $\kappa$, $J_\kappa$ is a matrix of ones of dimension $\kappa$, $\mathbf{1}_\kappa$ is a vector of ones of dimension $\kappa$, $\otimes$ denotes the Kronecker product.

Consider the following panel data regression model,

$$y_{it} = x'_{it}\beta + u_{it}, \quad i = 1, 2, ..., N, \; t = 1, 2, ..., T, \tag{1}$$

where $y_{it}$ is the observation on the $i$-th region for the $t$-th time period, $x_{it}$ denotes the $k \times 1$ vector of observations on the non-stochastic regressors and $u_{it}$ is the regression disturbance. The disturbance vector of (1) is assumed to have random region effects as well as spatially autocorrelated residual disturbances, i.e.,

$$u_t = \mu + \varepsilon_t \tag{2}$$

with

$$\varepsilon_t = \lambda W \varepsilon_t + v_t, \tag{3}$$

where $u'_t = (u_{1t}, u_{2t}, ..., u_{Nt})$, $\varepsilon'_t = (\varepsilon_{1t}, \varepsilon_{2t}, ..., \varepsilon_{Nt})$ and $\mu' = (\mu_1, \mu_2, ..., \mu_N)$ denote the vector of random region effects which are assumed to be IIN(0, $\sigma^2_\mu$). $\lambda$ satisfes $|\lambda| < 1$. $W$ is known $N \times N$ spatial weight matrix. $v'_t = (v_{1t}, v_{2t}, ..., v_{Nt})$, where $v_{it}$ is i.i.d over $i$ and $t$ and is assumed to be N(0, $\sigma^2_v$). One can rewrite (3) as

$$\varepsilon_t = (I_N - \lambda W)^{-1} v_t = B^{-1} v_t, \tag{4}$$

where $B = I_N - \lambda W$. Then (2) can be written as

$$u_t = \mu + B^{-1} v_t, \tag{5}$$

Denoting $y_t = (y_{1t}, y_{2t}, ..., y_{Nt})'$, and $x_t = (x_{1t}, x_{2t}, ..., x_{Nt})'$, the model (1) has the following reduced-form representation,

$$y_t = x_t \beta + u_t, \quad t = 1, 2, ..., T, \tag{6}$$

We continue with the model (5)-(6). With $t = 1, 2, ..., T$, model (5)-(6) can be written into a matrix form as follows:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{pmatrix} \beta + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{pmatrix}$$

with

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{pmatrix} = \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix} + \begin{pmatrix} B^{-1} & 0 & \cdots & 0 \\ 0 & B^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_T \end{pmatrix}$$

or

$$y = X\beta + u, \tag{7}$$

with

$$u = (\mathbf{1}_T \otimes I_N)\mu + (I_T \otimes B^{-1})v, \tag{8}$$

where $y' = (y_1', y_2', ..., y_T')$, $X' = (x_1', x_2', ..., x_T')$, $u' = (u_1', u_2', ..., u_T')$, and $v' = (v_1', v_2', ..., v_T')$.

# 2 LR Test

To clearly show the model in this report, the model (7)-(8) is rewritten as follows

$$y = X\beta + u,\ u = (\mathbf{1}_T \otimes I_N)\mu + (I_T \otimes B^{-1})v.$$

This variance–covariance matrix can be written as

$$\Omega_u = \sigma_v^2 \Sigma_u$$

where $\Sigma_u = \bar{J}_T \otimes [T\phi I_N + (B'B)^{-1}] + E_T \otimes (B'B)^{-1}$, with $\bar{J}_T = J_T/T$, $\phi = \sigma_\mu^2/\sigma_v^2$, $E_T = I_T - \bar{J}_T$. $\Sigma_u^{-1}$ is given by

$$\Sigma_u^{-1} = \bar{J}_T \otimes [T\phi I_N + (B'B)^{-1}]^{-1} + E_T \otimes (B'B). \tag{9}$$

Under the assumption of normality, the log-likelihood function for this model was derived as

$$\begin{aligned} L &= -\frac{NT}{2}\log(2\pi\sigma_v^2) - \frac{1}{2}\log|\Sigma_u| - \frac{1}{2\sigma_v^2}u'\Sigma_u^{-1}u \\ &= -\frac{NT}{2}\log(2\pi\sigma_v^2) - \frac{1}{2}\log|T\phi I_N + (B'B)^{-1}| + \frac{(T-1)}{2}\log|B'B| - \frac{1}{2\sigma_v^2}u'\Sigma_u^{-1}u \end{aligned} \tag{10}$$

with $u = y - X\beta$.

Baltagi et al. (2003) derived the lagrange multiplier (LM) tests and the corresponding likelihood ratio (LR) tests for this model. Here, we redo the LR test under the hypotheses for $H_0^a : \lambda = \sigma_\mu^2 = 0$ and the alternative $H_1^a$ is that at least one component is not zero.

Firstly, we obtain the estimations of the unrestricted log-likelihood function using the method of scoring. Let $\hat{\sigma}_v^2$, $\hat{\phi}$, $\hat{\lambda}$ and $\hat{\beta}$ denote the unrestricted maximum likelihood estimators (MLEs) and let $\hat{B} = I_N - \hat{\lambda} W$ and $\hat{u} = y - X'\hat{\beta}$, and then, by (10), the unrestricted maximum log-likelihood function is given by

$$L_U = -\frac{NT}{2}\log(2\pi\hat{\sigma_v^2}) - \frac{1}{2}\log|T\hat{\phi}I_N + (\hat{B}'\hat{B})^{-1}| + (T-1)\log|\hat{B}| - \frac{1}{2\hat{\sigma_v^2}}\hat{u}'\hat{\Sigma}_u^{-1}\hat{u},$$

where $\hat{\Sigma}_u^{-1}$ is obtained from (9) with $\hat{\beta}$ replacing $\beta$ and $\hat{\phi}$ replacing $\phi$.

Secondly, we obtain the estimations of the restricted log-likelihood function. Under the null hypothesis $H_0^a$, the restricted MLE of $\beta$ is $\widetilde{\beta}_{OLS}$, and $\widetilde{\beta}_{OLS} = (X'X)^{-1}X'Y$, so that $\widetilde{u} = y - X'\widetilde{\beta}_{OLS}$ and $\widetilde{\sigma}_v^2 = \widetilde{u}'\widetilde{u}/NT$. Therefore, the restricted maximum log-likelihood function under $H_0^a$ is given by

$$L_R = -\frac{NT}{2}\log(2\pi\widetilde{\sigma}_v^2) - \frac{1}{2\widetilde{\sigma_v^2}}\widetilde{u}'\widetilde{u}.$$

Finally, the LR test statistic for $H_0^a$ is given by

$$LR = 2(L_U - L_R) \tag{11}$$

and this should be asymptotically distributed as a mixture of $\chi^2$ given in (12) under the null hypothesis $H_0^a$, $\chi_m^2$ has a mixed $\chi^2$-distribution:

$$\chi_m^2 \sim \frac{1}{4}\chi^2(0) + \frac{1}{2}\chi^2(1) + \frac{1}{4}\chi^2(2) \tag{12}$$

The critical values for the mixed $\chi_m^2$ are 7.289, 4.321 and 2.952 for $\alpha = 0.01$ ,0.05 and 0.1, respectively.

# 3 Simulation

We conducted a small simulation study to compare the finite sample performances of the critial regions based on LR tests at $\alpha = 0.05$ level, and report the proportion of LR $> \chi_m^2$ in our 2,000 simulations. The results of simulations are reported in Tables 1-2.

In the simulations, we used the model as Baltagi et al. (2003):

$$y_{it} = \beta_1 + x_{it}'\beta_2 + u_{it}, i = 1, \cdots, N, t = 1, \cdots, T,$$

where $\beta_1 = 5$, $\beta_2 = 0.5$ and $x_{it} = 0.1t + 0.5x_{i,t-1} + z_{it}$ with $z_{it} \sim U(-0.5, 0.5)$ and $x_{i0} = 5 + 10z_{i0}$. $W$ is a queen type weight matrix. For the disturbances, $u_{it} = \mu_i + \varepsilon_{it}$, $\varepsilon_{it} = \lambda\sum_{j=1}^N w_{ij}\varepsilon_{it} + v_{it}$

with $\mu_i$ ∼IIN(0, $\sigma_\mu^2$) and $v_{it}$ ∼N(0, $\sigma_v^2$). We fix $\sigma_\mu^2 + \sigma_v^2 = 20$ and let $\rho = \sigma_\mu^2/(\sigma_\mu^2 + \sigma_v^2)$ vary over the set (0, 0.2, 0.5). The spatial autocorrelation factor $\lambda$ is varied over a positive range from 0 to 0.9 by increments of 0.1. Two values for $N$ = 25 and 49, and two values for $T$ =3 and 7 are chosen.

表 1: LR test for $H_0^a$, $\lambda = \sigma_\mu^2 = 0$

| $N, T$ | $\lambda$ | $\rho = 0$ | | $\rho = 0.2$ | | $\rho = 0.5$ | |
|---|---|---|---|---|---|---|---|
| | | Baltagi | Tang | Baltagi | Tang | Baltagi | Tang |
| 25, 3 | 0.0 | 0.064 | 0.071 | 0.372 | 0.412 | 0.958 | 0.976 |
| | 0.1 | 0.066 | 0.100 | 0.380 | 0.421 | 0.967 | 0.969 |
| | 0.2 | 0.140 | 0.180 | 0.411 | 0.459 | 0.968 | 0.969 |
| | 0.3 | 0.282 | 0.316 | 0.519 | 0.564 | 0.981 | 0.977 |
| | 0.4 | 0.514 | 0.533 | 0.650 | 0.670 | 0.984 | 0.987 |
| | 0.5 | 0.743 | 0.740 | 0.806 | 0.845 | 0.992 | 0.989 |
| | 0.6 | 0.891 | 0.908 | 0.923 | 0.946 | 0.995 | 0.999 |
| | 0.7 | 0.974 | 0.977 | 0.980 | 0.985 | 1.000 | 0.999 |
| | 0.8 | 0.997 | 1.000 | 0.996 | 1.000 | 0.999 | 1.000 |
| | 0.9 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 25, 7 | 0.0 | 0.056 | 0.074 | 0.876 | 0.902 | 1 | 1.000 |
| | 0.1 | 0.117 | 0.107 | 0.904 | 0.910 | 1 | 1.000 |
| | 0.2 | 0.317 | 0.340 | 0.932 | 0.945 | 1 | 0.999 |
| | 0.3 | 0.646 | 0.670 | 0.951 | 0.973 | 1 | 1.000 |
| | 0.4 | 0.883 | 0.894 | 0.989 | 0.992 | 1 | 1.000 |
| | 0.5 | 0.987 | 0.989 | 0.999 | 0.999 | 1 | 1.000 |
| | 0.6 | 0.999 | 1.000 | 1.000 | 1.000 | 1 | 1.000 |
| | 0.7 | | 1.000 | | 1.000 | | 1.000 |
| | 0.8 | | 1.000 | | 1.000 | | 1.000 |
| | 0.9 | | 1.000 | | 1.000 | | 1.000 |

Simulation results show that the frequency of rejections in Baltagi et al. (2003) is larger than Tang and Zou. Some blank means no data in Baltagi et al. (2003).

表 2: LR test for $H_0^a$, $\lambda = \sigma_\mu^2 = 0$

| $N, T$ | $\lambda$ | $\rho = 0$ | | $\rho = 0.2$ | | $\rho = 0.5$ | |
|---|---|---|---|---|---|---|---|
| | | Baltagi | Tang | Baltagi | Tang | Baltagi | Tang |
| 49, 3 | 0.0 | 0.057 | 0.035 | 0.622 | 0.599 | 1.000 | 0.998 |
| | 0.1 | 0.100 | 0.042 | 0.636 | 0.598 | 1.000 | 0.999 |
| | 0.2 | 0.232 | 0.075 | 0.716 | 0.635 | 1.000 | 1.000 |
| | 0.3 | 0.517 | 0.119 | 0.804 | 0.685 | 1.000 | 1.000 |
| | 0.4 | 0.782 | 0.235 | 0.901 | 0.762 | 1.000 | 1.000 |
| | 0.5 | 0.953 | 0.382 | 0.973 | 0.820 | 1.000 | 1.000 |
| | 0.6 | 0.993 | 0.600 | 0.995 | 0.894 | 1.000 | 0.999 |
| | 0.7 | 1.000 | 0.790 | 1.000 | 0.958 | 1.000 | 0.997 |
| | 0.8 | | 0.907 | | 0.982 | | 0.985 |
| | 0.9 | | 0.906 | | 0.979 | | 0.996 |
| 49, 7 | 0.0 | 0.056 | 0.025 | 0.995 | 0.992 | 1.000 | 1.000 |
| | 0.1 | 0.153 | 0.054 | 0.995 | 0.992 | 1.000 | 1.000 |
| | 0.2 | 0.536 | 0.198 | 0.996 | 0.996 | 1.000 | 1.000 |
| | 0.3 | 0.899 | 0.503 | 0.999 | 0.997 | 1.000 | 1.000 |
| | 0.4 | 0.994 | 0.777 | 1.000 | 0.998 | 1.000 | 1.000 |
| | 0.5 | 1.000 | 0.926 | 1.000 | 0.997 | 1.000 | 1.000 |
| | 0.6 | 1.000 | 0.992 | 1.000 | 0.998 | 1.000 | 1.000 |
| | 0.7 | | 0.999 | | 1.000 | | 1.000 |
| | 0.8 | | 1.000 | | 1.000 | | 1.000 |
| | 0.9 | | 0.963 | | 1.000 | | 1.000 |

# 4 EL Test

$\ell$ is the likelihood function, $L$ is the log-likelihood function (10):

$$LR = -2\log\frac{\ell_R}{\ell_U} = 2(L_U - L_R) \tag{13}$$

$\tilde{\ell}$ is the empirical likelihood, $\tilde{\ell}(b) = \sup_{\{p_i\}_{i=1}^n} \prod_{i=1}^n p_i$, $\tilde{L}(b)$ is the empirical log-likelihood:

$$EL = -2\log\frac{\sup_{b\in\Omega_0}\tilde{L}(b)}{\sup_{b\in\Omega_0+\Omega_1}\tilde{L}(b)} = 2(\tilde{L}_{\Omega_0+\Omega_1} - \tilde{L}_{\Omega_0}) \tag{14}$$

$$EL = -2\log\frac{\sup_{b\in\Omega_0}\tilde{L}(b)}{\sup_{b\in\Omega_0+\Omega_1}\tilde{L}(b)} = -2\log\frac{\sup_{b\in\Omega_0}\tilde{L}(b)}{\sup_{b\in\mathbf{R}}\tilde{L}(b)} I(\hat{b}\in\Omega_1) \tag{15}$$