DOI: 10.1002/int.21974

WILEY

RESEARCH ARTICLE

Triangular norm-based cuts and possibility characteristics of triangular intuitionistic fuzzy numbers for decision making

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Abstract

Triangular intuitionistic fuzzy numbers (TIFNs) is one of the useful tools to manage the fuzziness and vagueness in expressing decision data and solving decision making problems. In this paper, triangular norm (t-norm) based cuts of TIFNs are developed to synthesize the membership and nonmembership functions in describing the cut sets, then the possibility characteristics of TIFNs, i.e., the possibility mean, the possibility variance, and the possibility mean-standard deviation ratio, are given. Thereby, on the ground of the possibility mean-standard deviation ratio, a ranking method of TIFNs is introduced. With these elements, an approach to multiple attributes decision making (MADM) is proposed and illustrated by a numerical example. It is shown that the approach to MADM comprehensively considers both the membership and nonmembership functions and can lead to objective and reasonable results.

KEYWORDS

triangular intuitionistic fuzzy numbers, t-norms, t-norm-based cuts, multiple attribute decision making

1 | INTRODUCTION

As an extension of fuzzy set by Zadeh,[1] intuitionistic fuzzy set (IFS) introduced by Atanassov can express more abundant and flexible information than fuzzy set in dealing with uncertainty and vagueness, due to the remarkable characteristic that a membership degree and a nonmembership degree are assigned for each element in the universe of IFS. Along the spirit of the ordinary interval-valued fuzzy set (IVFS), Atanassov and Gargov[2] further extended the IFS and developed the notation of interval-valued intuitionistic fuzzy set (IVIFS). Both IFS and IVIFS have been found to be useful in decision making, where human expertise and knowledge are inevitably noncrisp and not always reliable. As a special case of fuzzy set, a fuzzy number is extended to an intuitionistic fuzzy number(IFN), which have been confirmed to be a useful tool to suitably describe an ill-known quantity.

T-norms and t-conorms are of vital importance in fuzzy set theory and soft computing. The typical case is that these operators can be used to model the intersection and union of fuzzy sets. The pertaining results and the related applications on t-norms and t-conorms can be found in Refs. 3–6. As is known, the cut set of fuzzy sets, which is an essential notation in defuzzification and a bridge among the fuzzy sets and the crisp ones, are employed in many fields, such as decision making,[7] fuzzy reasoning,[8,9] and fuzzy analysis.[10–12] In the recent years, many researchers have developed various cut sets for IFSs,[13–17] especially those with respect to t-norms[13,14,16,17] and investigated some fundamental properties. However, until now, there are few works on the notations and applications of t-norm based cut sets for typical extensions of IFNs.

Notably, there are several typical extensions of IFNs, such as the triangular intuitionistic fuzzy numbers (TIFNs),[18–21] the trapezoidal intuitionistic fuzzy numbers (TrIFNs),[22,23] the interval-valued trapezoidal intuitionistic fuzzy numbers,[24,25] and the recently introduced I-fuzzy numbers,[26] in which the information in the form of those alternatives, given by decision makers in decision making, can be reflected exactly and expressed in multiple dimensions. It must be mentioned that, in the most recent, Yager[27,28] freed the restriction that the sum of the membership degree and nonmembership degree is bounded by one. He introduced the q-rung orthopair fuzzy sets,[28] which is also called Pythagorean fuzzy sets if $q = 2$.[27] Here we only concentrate on TIFNs. In the literature, many works are concentrated on the ranking methods of TIFNs, which are of vital importance in application, such as the ranking method based on the ratio of the value index to the ambiguity index,[20] the one based on the value index and ambiguity index,[29] and the one based on the ratio of the possibility mean to the possibility standard deviation.[30] However, among these methods, the membership function and nonmembership function are separately considered, then inevitably the ranking results depend only on some quantitative characteristics of membership function or nonmembership function. Thereby, the obtained results are biased and one sided.

To overcome the referred defects, in this paper we focus on the quantitative characteristics and ranking of TIFNs using triangular norms based cuts, taking account of both membership and nonmembership functions of a TIFN comprehensively. In detail, t-norm-based cuts of TIFNs are developed to synthesize the membership and nonmembership functions in describing the cut sets, then the possibility characteristics of TIFNs, i.e., the possibility mean, the possibility variance, and the possibility mean-standard deviation ratio, are given. Thereby, on the ground of the possibility mean-standard deviation ratio, a ranking method of TIFNs is introduced. With these elements, an approach to multiple attributes decision making (MADM) is proposed and illustrated by a numerical example.

The paper is unfolded as follows: In Section 2, we recall some results on t-norms and t-conorms. Section 3 presents the notations of TIFNs and new operation laws for the normalized TIFNs. In Section 4, possibility means and possibility variance of TIFN are defined in virtue of the triangular norms based cuts of TIFNs, then the possibility mean-standard deviation ratio is developed to rank TIFNs. In Section 5, an approach to MADM is developed and a numerical example is provided for illustration and comparison analysis. Finally, we end up with some concluding remarks.

2 | SOME CONCEPTS AND RESULTS ON t-NORMS AND t-CONORMS

In this part, we recall some concepts and results employed in the rest of the paper. For more details about t-norms and t-conorms, we recommend to refer to the monographs.[5,31]

Definition 2.1. [5,31] A binary function $T : [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm for short) if the following properties hold for all $x, y, z \in [0,1]$,

(1) (Commutativity) $T(x,y) = T(y,x)$;

(2) (Associativity) $T(x, T(y,z)) = T(T(x,y), z)$;

(3) (Monotonicity) $T(x,z) \leq T(y,z)$ if $x \leq y$;

(4) (Neutrality) $T(1,x) = x$.

Definition 2.2.[5,31] A binary function $S : [0,1]^2 \to [0,1]$ is called a triangular conorm (t-conorm for short) if (1)–(3) hold and $S(0,x) = x$ for all $x \in [0,1]$.

Obviously, t-norms and t-conorms only differ in the neutral element. t-norm and t-conorm are dual to each other in the sense that $T^*(x,y) = 1 - T(1-x, 1-y)$ is a t-conorm if T is a t-norm, whereas $S^*(x,y) = 1 - S(1-x, 1-y)$ is a t-norm if S is a t-conorm. The most referred t-norms are the product t-norm T_P given by $T_P(x,y) = xy$, and the Łukasiewicz t-norm T_L given by $T_L(x,y) = \max(x+y-1, 0)$; Dually, the most referred t-conorms are the product t-conorm S_P given by $S_P(x,y) = x + y - xy$, and the Łukasiewicz t-norm S_L given by $S_L(x,y) = \min(x+y, 1)$.

Since t-norm and t-conorm are dual to each other, then the notations about t-conorms are omitted here. The idempotents of a t-norm T are those x satisfying $T(x,x) = x$. The bounds 0 and 1 are trivial idempotents. A t-norm is called Archimedean if each sequence $(x)_n^T, n \in \mathbf{N}$ (the set of all natural numbers), where $(x)_n^T = T(\underbrace{x, x, \ldots, x}_{n \text{ times}})$, converges to $0 (n \to \infty)$. A continuous t-norm is Archimedean if and only if it has no idempotents in $(0,1)$. An Archimedean t-norm is called strict if $T(x,x) > 0$ for all $x > 0$. An Archimedean t-norm, which is not strict, is called nilpotent. The product t-norm T_P is strict, whereas the Łukasiewicz t-norm T_L is nilpotent. Dually, we have the dual notations about t-conorms. Archimedean t-norms and t-conorms can be represented by additive generators.

Theorem 2.3.[5,31] *Let T and S be t-norm and t-conorm, respectively. Then the following statements hold.*

(1) *T is Archimedean if and only if there is a continuous strictly decreasing unary function $t : [0,1] \to [0, +\infty]$ with $t(1) = 0$, such that*

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))) \tag{1}$$

for all $x, y \in [0,1]$, where t is called the additive generator of T.

(2) *S is Archimedean if and only if there is a continuous strictly increasing unary function $s : [0,1] \to [0, +\infty]$ with $s(0) = 0$, such that*

$$S(x,y) = s^{-1}(\min(s(x) + s(y), s(1))) \tag{2}$$

for all $x, y \in [0,1]$, where s is called the additive generator of S.

In fact, if $t(0) < +\infty$ ($t(0) = +\infty$), then T is nilpotent (strict); if $s(1) < +\infty(s(1) = +\infty)$, then S is nilpotent (strict). Moreover, if T and S in the above theorem are dual, then it holds that $s(x) = t(1 - x)$. If T and S are strict, then Equations 1 and 2 reduce to $T(x, y) = t^{-1}(t(x) + t(y))$ and $S(x, y) = s^{-1}(s(x) + s(y))$, respectively. In the literature, different kinds of t-norms and t-conorms[3,4] are employed in many fields. In the rest of the paper, we will use strict t-norms and t-conorms. Here, for future use, we list some families of parameterized strict t-norms and t-conorms and their additive generators in the following.

The Schweizer–Sklar family of strict t-norms $T_\lambda^{SS} = \begin{cases} xy, & \lambda = 0; \\ (x^\lambda + y^\lambda - 1)^{\frac{1}{\lambda}}, & \lambda < 0; \end{cases}$ with its additive generator $t_\lambda^{SS}(x) = \begin{cases} -\ln x, & \lambda = 0; \\ \frac{1-x^\lambda}{\lambda}, & \lambda < 0; \end{cases}$ for $\lambda \in (-\infty, 0]$.

The Hamacher family of strict t-norms $T_\lambda^H(x, y) = \begin{cases} 0, & \lambda = x = y = 0; \\ \frac{xy}{\lambda + (1-\lambda)(x+y-xy)}, & \text{else}; \end{cases}$ with its additive generator $t_\lambda^F(x) = \begin{cases} \frac{1-x}{x}, & \lambda = 0; \\ \ln \frac{\lambda + (1-\lambda)x}{x}, & \lambda > 0; \end{cases}$ for $\lambda \in [0, +\infty)$. Note that $T_0^{SS} = T_1^H = T_P$, T_0^H is the Hamacher product and T_2^H is the Einstein t-norm.

By duality, the Schweizer–Sklar family of strict t-conorms T_λ^{SS} and Hamacher family of strict t-conorms S_λ^H can be easily obtained.

3 | OPERATION LAWS AND WEIGHTED AVERAGE OPERATOR FOR TIFNs

Definition 3.1.[20,29] A TIFN $\tilde{a} = ((\underline{a}, a, \overline{a}); w_{\tilde{a}}, u_{\tilde{a}})$ is an IFS on the set of real numbers $\boldsymbol{R}$, whose membership and nonmembership functions are respectively defined as

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - \underline{a}}{a - \underline{a}} w_{\tilde{a}}, & \underline{a} \le x < a; \\ w_{\tilde{a}}, & x = a; \\ \dfrac{\overline{a} - x}{\overline{a} - a} w_{\tilde{a}}, & a < x \le \overline{a}; \\ 0, & \text{else}; \end{cases}$$

and

$$\nu_{\tilde{a}}(x) = \begin{cases} \dfrac{a - x + (x - \underline{a})u_{\tilde{a}}}{a - \underline{a}}, & \underline{a} \le x < a; \\ u_{\tilde{a}}, & x = a; \\ \dfrac{x - a + (\overline{a} - x)u_{\tilde{a}}}{\overline{a} - a}, & a < x \le \overline{a}; \\ 1, & \text{else}; \end{cases}$$

where $w_{\tilde{a}}$ and $u_{\tilde{a}}$ are the maximum degree of membership and the minimum degree of nonmembership, respectively, such that $0 \le w_{\tilde{a}} \le 1, 0 \le u_{\tilde{a}} \le 1$ and $w_{\tilde{a}} + u_{\tilde{a}} \le 1$. $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - \nu_{\tilde{a}}(x)$ is called an intuitionistic fuzzy index of an element x in $\tilde{a}$.

A TIFN $\tilde{a} = ((\underline{a}, a, \overline{a}); w_{\tilde{a}}, u_{\tilde{a}})$ can be used to model an ill-known quantity "approximate" a, which is approximately equal to a. That is, the ill-known quantity "approximate a" is expressed using any value between $\underline{a}$ and $\overline{a}$ with different degrees of membership and degrees of non-membership. In other words, the most possible value is a with the degree $w_{\tilde{a}}$ of membership and the degree $u_{\tilde{a}}$ of nonmembership;

the pessimistic value is $\underline{a}$ with the degree 0 of membership and the degree 1 of nonmembership; the optimistic value is $\overline{a}$ with the degree 0 of membership and the degree 1 of nonmembership; other values are any $x \in (\underline{a}, \overline{a})$ with the pertaining membership degree $w_{\tilde{a}}(x)$ and non-membership degree $u_{\tilde{a}}(x)$. A TIFN is a generalization of triangular fuzzy number.

In decision making, when assessing the alternatives in the form of TIFNs, people mainly focus on the cost attributes and benefit attributes. The operation laws over TIFNs are mainly employed in aggregating information on different attributes, which are mostly measured in different dimensions. Therefore, the information to be handled must be nondimensionalized beforehand. In the literature, there are some known operation laws for TIFNs, which are important in aggregating information in the form of TIFNs. However, these operation laws ignore the different measurements in practice. In many existing works, to obtain nondimensional information, normalization transformations are made in advance by some known methods.[20,29,30] Then, it is enough to define operation laws only for normalized TIFNs. It is trivial that $\underline{a}, a, \overline{a} \in [0, 1]$ for a normalized TIFN $\tilde{a} = ((\underline{a}, a, \overline{a}), w_{\tilde{a}}, u_{\tilde{a}})$. Additionally, a normalized TIFN is also called a normal TIFN in the sequel.

Definition 3.2.[6] Let $\tilde{a}_i = ((\underline{a}_i, a_i, \overline{a}_i), w_{\tilde{a}_i}, u_{\tilde{a}_i})(i = 1, 2)$ be two normalized TIFNs and $\lambda \geq 0$, then the operation laws are defined as follows:

(1) $\tilde{a}_1 \bigoplus \tilde{a}_2 = ((\underline{a}_1 + \underline{a}_2 - \underline{a}_1\underline{a}_2, a_1 + a_2 - a_1a_2, \overline{a}_1 + \overline{a}_2 - \overline{a}_1\overline{a}_2), w_{\tilde{a}_1}w_{\tilde{a}_2}, u_{\tilde{a}_1} + u_{\tilde{a}_2} - u_{\tilde{a}_1}u_{\tilde{a}_2});$

(2) $\lambda\tilde{a}_1 = ((1 - (1 - \underline{a})^{\lambda}, 1 - (1 - a)^{\lambda}, 1 - (1 - \overline{a})^{\lambda}), w_{\tilde{a}_1}^{\lambda}, 1 - (1 - u_{\tilde{a}_1})^{\lambda}).$

It can be checked that the obtained TIFNs by the above operation laws are also normal. However, with some existing operation laws on the normalized TIFNs, we may obtain a TIFN that is not normal. Let $\tilde{a}_1 = ([0.4, 0.7, 0.8]; 0.6, 0.5), \tilde{a}_2 = ([0.4, 0.5, 0.6]; 0.4, 0.4)$ be two normal TIFNs, then

(1) According to the operations given in Ref. 29, it holds that $\tilde{a}_1 + \tilde{a}_2 = ([0.8, 1.2, 1.4]; 0.4, 0.5)$, which is not normal. Furthermore, it is shown[29] that the defined multiplication and division of two TIFNs are not TIFNs.

(2) In virtue of Definition 5 in Ref. 21, we have $\tilde{a}_1 + \tilde{a}_2 = ([0.8, 1.2, 1.4]; 0.5, 0.45)$, i.e., the obtained TIFN is not normal.

Definition 3.3. Assume that $\tilde{a}_j(j = 1, 2, \ldots, n)$ is a collection of TIFNs, then the weighted average operator WA_{ω} is defined as follows:

$$WA_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \bigoplus_{i=1}^{n} \omega_i\tilde{a}_i, \tag{3}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector such that $\omega_i \in [0, 1]$ for each $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$. In particular, if $\omega_i = \frac{1}{n}$ for each $i = 1, 2, \ldots, n$, then the weighted average operator WA_{ω} is reduced to the arithmetic average operator of TIFNs.

By calculation, it can be obtained that

$$WA_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\left(1 - \prod_{i=1}^{n}(1 - \underline{a}_i)^{\omega_i}, 1 - \prod_{i=1}^{n}(1 - a_i)^{\omega_i}, 1 - \prod_{i=1}^{n}(1 - \overline{a}_i)^{\omega_i}\right), \prod_{i=1}^{n} w_{\tilde{a}_i}^{\omega_i},\right.$$
$$\left. 1 - \prod_{i=1}^{n}(1 - u_{\tilde{a}_i})^{\omega_i}\right) \tag{4}$$

4 | t-NORM-BASED CUTS AND POSSIBILITY CHARACTERISTICS OF TRIANGULAR INTUITIONISTIC FUZZY NUMBERS

Cuts of IFSs, also known as level sets, are important for applications. For instance, in the process of defuzzification, cuts of IFSs are employed to transform an IFS into a crisp one. There are a variety of definitions for cuts of IFSs.[14–17,32,33] Among them, there is a consensus that a t-norm is a natural tool to model a conjunction of two statements on cuts. In the following, we introduce the cuts set for TIFNs on the basis of a t-norm, which is developed in a way that binds the membership and nonmembership functions by a t-norm.

Definition 4.1. For a t-norm T and a TIFN $\tilde{a} = ((\underline{a}, a, \overline{a}); w_{\tilde{a}}, u_{\tilde{a}})$, let $\alpha \in (0, T(w_{\tilde{a}}, 1-u_{\tilde{a}})]$, then T based $\alpha-$cut of $\tilde{a}$ is defined as

$$\tilde{a}_\alpha^T = \{x \in R \mid T(\mu_{\tilde{a}}(x), 1-\nu_{\tilde{a}}(x)) \geq \alpha\}.$$

For convenience, we denote $\tilde{a}_\alpha^T$ by $[\tilde{a}_\alpha^{T,l}, \tilde{a}_\alpha^{T,r}]$.

Remark 4.2. In Refs. 20,30 and 32 $(\alpha, \beta)-$cut set of a TIFN $\tilde{a}$ is defined as $\tilde{a}_{\alpha,\beta} = \{x \mid \mu_{\tilde{a}}(x) \geq \alpha, \nu_{\tilde{a}}(x) \leq \beta\}$ based on both the membership function and nonmembership function whereas the $\alpha-$cut set $\tilde{a}_\alpha$ and $\beta-$cut set $\tilde{a}_\beta$ are, respectively, given by $\tilde{a}_\alpha = \{x \mid \mu_{\tilde{a}}(x) \geq \alpha\}$ and $\tilde{a}_\beta = \{x \mid \nu_{\tilde{a}}(x) \leq \beta\}$. Here the T-based $\alpha-$cut $\tilde{a}_\alpha^T$ is defined on the basis of the synthesization of the membership function and nonmembership function. If $T = \min$, then $\tilde{a}_\alpha^T = \tilde{a}_{\alpha,1-\alpha}$. Thereby, the T based $\alpha-$cut $\tilde{a}_\alpha^T$ is an extension of the $(\alpha, \beta)-$cut set $\tilde{a}_{\alpha,\beta}$. In the following, we list some cases when T is a strict t-norm. If $T = T_\lambda^{SS}$ for $\lambda \in (-\infty, 0]$, then $\tilde{a}_\alpha^T = [\underline{a} + (\frac{1+\alpha^\lambda}{w_{\tilde{a}}^\lambda + (1-u_{\tilde{a}})}^\lambda)^{\frac{1}{\lambda}}(a - \underline{a}), \overline{a} + (\frac{1+\alpha^\lambda}{w_{\tilde{a}}^\lambda + (1-u_{\tilde{a}})}^\lambda)^{\frac{1}{\lambda}}(\overline{a} - a)]$. In particular, for $\lambda = 0$, i.e., $T = T_P$, it can be obtained that

$$\tilde{a}_\alpha^T = \left[\underline{a} + \sqrt{\frac{\alpha}{w_{\tilde{a}}(1-u_{\tilde{a}})}}(a - \underline{a}), \overline{a} - \sqrt{\frac{\alpha}{w_{\tilde{a}}(1-u_{\tilde{a}})}}(\overline{a} - a)\right]$$

denoted by $\tilde{a}_\alpha^P = [\tilde{a}_\alpha^{P,l}, \tilde{a}_\alpha^{P,r}]$ for short. If $T = T_\lambda^H$ for $\lambda \in [0, \infty)$, then

$$\tilde{a}_\alpha^T = \left[\underline{a} + \frac{a - \underline{a}}{\frac{(\lambda-1)w_{\tilde{a}}}{2\lambda} + \frac{(\lambda-1)(1-u_{\tilde{a}})}{2\lambda} + \sqrt{\frac{(1-\lambda)^2 w_{\tilde{a}}^2}{4\lambda^2} - \frac{2(1-\lambda)^2 w_{\tilde{a}}(1-u_{\tilde{a}})}{4\lambda^2} + \frac{(1-\lambda)^2(1-u_{\tilde{a}})^2}{4\lambda^2} + \frac{\lambda+(1-\lambda)\alpha}{\lambda^2\alpha} w_{\tilde{a}}(1-u_{\tilde{a}})}},\right.$$

$$\left.\overline{a} - \frac{\overline{a} - a}{\frac{(\lambda-1)w_{\tilde{a}}}{2\lambda} + \frac{(\lambda-1)(1-u_{\tilde{a}})}{2\lambda} + \sqrt{\frac{(1-\lambda)^2 w_{\tilde{a}}^2}{4\lambda^2} - \frac{2(1-\lambda)^2 w_{\tilde{a}}(1-u_{\tilde{a}})}{4\lambda^2} + \frac{(1-\lambda)^2(1-u_{\tilde{a}})^2}{4\lambda^2} + \frac{\lambda+(1-\lambda)\alpha}{\lambda^2\alpha} w_{\tilde{a}}(1-u_{\tilde{a}})}}\right],$$

which reduces to $\tilde{a}_\alpha^P$ if $\lambda = 1$ and

$$\tilde{a}_\alpha^T = \left[\underline{a} + \frac{\alpha(w_{\tilde{a}} + (1-u_{\tilde{a}}))}{(1+\alpha)w_{\tilde{a}}(1-u_{\tilde{a}})}(a - \underline{a}), \overline{a} - \frac{\alpha(w_{\tilde{a}} + (1-u_{\tilde{a}}))}{(1+\alpha)w_{\tilde{a}}(1-u_{\tilde{a}})}(\overline{a} - a)\right]$$

if $\lambda = 0$, denoted by $\tilde{a}_\alpha^H = [\tilde{a}_\alpha^{H,l}, \tilde{a}_\alpha^{H,r}]$ for short.

Definition 4.3. Let $\tilde{a}_\alpha^T = [\tilde{a}_\alpha^{T,l}, \tilde{a}_\alpha^{T,r}]$ be T-based $\alpha-$cut of $\tilde{a}$, Then the f weighted lower and upper possibility means of the TIFN $\tilde{a}$ are, respectively, defined as

$$\underline{m}^T(\tilde{a}) = \int_0^{T(w_{\tilde{a}},1-u_{\tilde{a}})} f\left(Pos\left(\tilde{a} \le \tilde{a}_\alpha^{T,l}\right)\right)\tilde{a}_\alpha^{T,l} d\alpha$$

and

$$\overline{m}^T(\tilde{a}) = \int_0^{T(w_{\tilde{a}},1-u_{\tilde{a}})} f\left(Pos\left(\tilde{a} \ge \tilde{a}_\alpha^{T,r}\right)\right)\tilde{a}_\alpha^{T,r} d\alpha,$$

where $f : [0, T(w_{\tilde{a}}, 1 - u_{\tilde{a}})] \to R$ is a nonnegative, monotone increasing weighting function such that $f(0) = 0$ and $\int_0^{T(w_{\tilde{a}},1-u_{\tilde{a}})} f(x)dx = T(w_{\tilde{a}}, 1 - u_{\tilde{a}})$, Pos means possibility and $Pos(\tilde{a} \le \tilde{a}_\alpha^{T,l}) = \sup \{T(\mu_{\tilde{a}}(x), 1 - \nu_{\tilde{a}}(x)) \mid x \le \tilde{a}_\alpha^{T,l}\} = \alpha$ and $Pos(\tilde{a} \ge \tilde{a}_\alpha^{T,r}) = \sup\{T(\mu_{\tilde{a}}(x), 1 - \nu_{\tilde{a}}(x)) \mid x \ge \tilde{a}_\alpha^{T,r}\} = \alpha$

The weighting function f measures the importance of T-based α-cut set of TrIFN $\tilde{a}$ and usually has the forms as $f(x) = \frac{(n+1)x^n}{(T(w_{\tilde{a}},1-u_{\tilde{a}}))^n}$, where n is any positive integer.

As is shown in Ref. 32, $(\alpha, \beta)-$cut set $\tilde{a}_{\alpha,\beta}$ is the intersection of $\alpha-$cut set $\tilde{a}_\alpha$ and $\beta-$cut set $\tilde{a}_\beta$, which rely only on the membership function and non-membership function, respectively. Hence, it is enough to consider $\alpha-$cut set $\tilde{a}_\alpha$ and $\beta-$cut set $\tilde{a}_\beta$ when defining the possibility means. In detail, in Refs. 20 and 30, the lower and upper possibility means of membership function and nonmembership function are, respectively, defined and then the lower and upper possibility means can be obtained by synthesizing. Just on the contrary, when defining f weighted lower and upper possibility means of a TIFN in Definition 4.3, the synthesization of membership function and nonmembership function takes place in the formulation of cut sets in advance.

Definition 4.4. For a TIFN $\tilde{a}$, the f weighted possibility mean of the TIFN $\tilde{a}$ is defined as

$$m^T(\tilde{a}) = \frac{\underline{m}^T(\tilde{a}) + \overline{m}^T(\tilde{a})}{2}.$$

For the case $T = T_P$, if $n = 1$, then $\underline{m}^T(\tilde{a}) = \frac{\underline{a}+4a}{5} w_{\tilde{a}}(1 - u_{\tilde{a}})$ and $\overline{m}^T(\tilde{a}) = \frac{\overline{a}+4a}{5} w_{\tilde{a}}(1 - u_{\tilde{a}})$, thus

$$m^T(\tilde{a}) = \frac{\underline{a} + 8a + \overline{a}}{10} w_{\tilde{a}}(1 - u_{\tilde{a}});$$

if $n = 2$, then $\underline{m}^T(\tilde{a}) = \frac{\underline{a}+6a}{7} w_{\tilde{a}}^2(1 - u_{\tilde{a}})^2$ and $\overline{m}^T(\tilde{a}) = \frac{\overline{a}+6a}{7} w_{\tilde{a}}^2(1 - u_{\tilde{a}})^2$, thus

$$m^T(\tilde{a}) = \frac{\underline{a} + 12a + \overline{a}}{14} w_{\tilde{a}}^2(1 - u_{\tilde{a}})^2.$$

For the case $T = T_0^H$, if $n = 1$, then

$$\underline{m}^T(\tilde{a}) = \frac{\underline{a} + 4a}{5} \frac{w_{\tilde{a}}(1 - u_{\tilde{a}})}{w_{\tilde{a}} + (1 - u_{\tilde{a}}) - w_{\tilde{a}}(1 - u_{\tilde{a}})}$$

and

$$\overline{m}^T(\tilde{a}) = \frac{\overline{a} + 4a}{5} \frac{w_{\tilde{a}}(1 - u_{\tilde{a}})}{w_{\tilde{a}} + (1 - u_{\tilde{a}}) - w_{\tilde{a}}(1 - u_{\tilde{a}})},$$

thus

$$m^T(\tilde{a}) = \frac{\underline{a} + 8a + \overline{a}}{10} \frac{w_{\tilde{a}}(1 - u_{\tilde{a}})}{w_{\tilde{a}} + (1 - u_{\tilde{a}}) - w_{\tilde{a}}(1 - u_{\tilde{a}})};$$

if $n = 2$, then

$$\underline{m}^T(\tilde{a}) = \frac{\underline{a} + 6a}{7} \frac{w_{\tilde{a}}^2(1 - u_{\tilde{a}})^2}{(w_{\tilde{a}} + (1 - u_{\tilde{a}}) - w_{\tilde{a}}(1 - u_{\tilde{a}}))^2}$$

and

$$\overline{m}^T(\tilde{a}) = \frac{\overline{a} + 6a}{7} \frac{w_{\tilde{a}}^2(1 - u_{\tilde{a}})^2}{(w_{\tilde{a}} + (1 - u_{\tilde{a}}) - w_{\tilde{a}}(1 - u_{\tilde{a}}))^2},$$

thus

$$m^T(\tilde{a}) = \frac{\underline{a} + 12a + \overline{a}}{14} \frac{w_{\tilde{a}}^2(1 - u_{\tilde{a}})^2}{(w_{\tilde{a}} + (1 - u_{\tilde{a}}) - w_{\tilde{a}}(1 - u_{\tilde{a}}))^2}.$$

For simplicity, in the sequel we only use $n = 1$ for the weighting function $f(x)$ and, respectively, denote the possibility means by $m^P(\tilde{a})$ and $m^H(\tilde{a})$ for the aforementioned cases.

Definition 4.5. For a TIFN $\tilde{a} = ((\underline{a}, a, \overline{a}); w_{\tilde{a}}, u_{\tilde{a}})$, the f-weighted possibility variance of the TIFN $\tilde{a}$ is defined as

$$V^T(\tilde{a}) = \int_0^{T(w_{\tilde{a}},1-u_{\tilde{a}})} f\left(Pos\left(\tilde{a} \le \tilde{a}_\alpha^{T,l}\right)\right)\left(\frac{\tilde{a}_\alpha^{T,l} + \tilde{a}_\alpha^{T,r}}{2} - \tilde{a}_\alpha^{T,l}\right)^2 d\alpha$$

$$+ \int_0^{T(w_{\tilde{a}},1-u_{\tilde{a}})} f\left(Pos\left(\tilde{a} \ge \tilde{a}_\alpha^{T,r}\right)\right)$$

$$\left(\frac{\tilde{a}_\alpha^{T,l} + \tilde{a}_\alpha^{T,r}}{2} - \tilde{a}_\alpha^{T,r}\right)^2 d\alpha = \frac{1}{2}\int_0^{T(w_{\tilde{a}},1-u_{\tilde{a}})} f(\alpha)(\tilde{a}_\alpha^{T,r} - \tilde{a}_\alpha^{T,l})^2 d\alpha,$$

where f is the same as that in Definition 4.3.

Let $T = T_P$, then $V^T(\tilde{a}) = \frac{1}{2}\int_0^{w_{\tilde{a}}(1-u_{\tilde{a}})} \frac{2\alpha}{w_{\tilde{a}}(1-u_{\tilde{a}})}(\tilde{a}_\alpha^{P,r} - \tilde{a}_\alpha^{P,l})^2 d\alpha = \frac{1}{3}w_{\tilde{a}}(1 - u_{\tilde{a}})(\overline{a} - \underline{a})^2$, denoted by $V^P(\tilde{a})$. Let $T = T_0^H$, then $V^T(\tilde{a}) = \frac{1}{2}\int_0^{T_0^H(w_{\tilde{a}},1-u_{\tilde{a}})} \frac{2\alpha}{T_0^H(w_{\tilde{a}},1-u_{\tilde{a}})}(\tilde{a}_\alpha^{H,r} - \tilde{a}_\alpha^{H,l})^2 d\alpha = \int_0^{T_0^H(w_{\tilde{a}},1-u_{\tilde{a}})} \frac{\alpha}{T_0^H(w_{\tilde{a}},1-u_{\tilde{a}})}[\frac{\alpha(w_{\tilde{a}}+(1-u_{\tilde{a}}))}{(1+\alpha)w_{\tilde{a}}(1-u_{\tilde{a}})}(\overline{a} - \underline{a})]^2 d\alpha = (\frac{1}{2}\frac{w_{\tilde{a}}(1-u_{\tilde{a}})}{w_{\tilde{a}}+(1-u_{\tilde{a}})-w_{\tilde{a}}(1-u_{\tilde{a}})} + \frac{w_{\tilde{a}}(1-u_{\tilde{a}})}{w_{\tilde{a}}+(1-u_{\tilde{a}})} + 3\frac{w_{\tilde{a}}+(1-u_{\tilde{a}})-w_{\tilde{a}}(1-u_{\tilde{a}})}{w_{\tilde{a}}(1-u_{\tilde{a}})} \ln \frac{w_{\tilde{a}}+(1-u_{\tilde{a}})}{w_{\tilde{a}}+(1-u_{\tilde{a}})-w_{\tilde{a}}(1-u_{\tilde{a}})} - 3) \frac{(w_{\tilde{a}}+1-u_{\tilde{a}})^2}{w_{\tilde{a}}^2(1-u_{\tilde{a}})^2}(\overline{a} - \underline{a})^2$, denoted by $V^H(\tilde{a})$.

Definition 4.6. For a TIFN $\tilde{a} = ((\underline{a}, a, \overline{a}); w_{\tilde{a}}, u_{\tilde{a}})$, the f-weighted possibility standard deviation of the TIFN $\tilde{a}$ is defined as

$$D^T(\tilde{a}) = \sqrt{V^T(\tilde{a})}.$$

It can be obtained that $D^P(\tilde{a}) = \sqrt{V^P(\tilde{a})} = \sqrt{\frac{1}{3}w_{\tilde{a}}(1 - u_{\tilde{a}})}(\overline{a} - \underline{a})$ and $D^H(\tilde{a}) = \sqrt{V^H(\tilde{a})}$

$$= \sqrt{\frac{1}{2}\frac{w_{\tilde a}(1-u_{\tilde a})}{w_{\tilde a}+(1-u_{\tilde a})-w_{\tilde a}(1-u_{\tilde a})} + \frac{w_{\tilde a}(1-u_{\tilde a})}{w_{\tilde a}+(1-u_{\tilde a})} + 3\frac{w_{\tilde a}+(1-u_{\tilde a})-w_{\tilde a}(1-u_{\tilde a})}{w_{\tilde a}(1-u_{\tilde a})}\ln\frac{w_{\tilde a}+(1-u_{\tilde a})}{w_{\tilde a}+(1-u_{\tilde a})-w_{\tilde a}(1-u_{\tilde a})} - 3\frac{(w_{\tilde a}+1-u_{\tilde a})}{w_{\tilde a}(1-u_{\tilde a})}}(\overline{a}-\underline{a}).$$

Definition 4.7. For a TIFN $\tilde a = ((\underline{a}, a, \overline{a}); w_{\tilde a}, u_{\tilde a})$, the f-weighted possibility mean-standard deviation ratio (ratio for short) of the TIFN $\tilde a$ is defined as

$$R^T(\tilde a) = \frac{m^T(\tilde a)}{D^T(\tilde a)}. \tag{5}$$

Let $T = T_P$ and $T = T_0^H$, then the following ratio of a TIFN $\tilde a$ can be obtained, respectively.

$$R^P(\tilde a) = \frac{m^P(\tilde a)}{D^P(\tilde a)} = \frac{\frac{\underline{a}+8a+\overline{a}}{10} w_{\tilde a}(1-u_{\tilde a})}{\sqrt{\frac{1}{3}w_{\tilde a}(1-u_{\tilde a})}(\overline{a}-\underline{a})} = \frac{\sqrt{3w_{\tilde a}(1-u_{\tilde a})}(\underline{a}+8a+\overline{a})}{10(\overline{a}-\underline{a})}$$

$$R^H(\tilde a) = \frac{m^H(\tilde a)}{D^H(\tilde a)}$$

$$= \frac{\frac{w_{\tilde a}(1-u_{\tilde a})}{w_{\tilde a}+(1-u_{\tilde a})-w_{\tilde a}(1-u_{\tilde a})}(\underline{a}+8a+\overline{a})}{10\sqrt{\frac{1}{2}\frac{w_{\tilde a}(1-u_{\tilde a})}{w_{\tilde a}+(1-u_{\tilde a})-w_{\tilde a}(1-u_{\tilde a})} + \frac{w_{\tilde a}(1-u_{\tilde a})}{w_{\tilde a}+(1-u_{\tilde a})} + 3\frac{w_{\tilde a}+(1-u_{\tilde a})-w_{\tilde a}(1-u_{\tilde a})}{w_{\tilde a}(1-u_{\tilde a})}\ln\frac{w_{\tilde a}+(1-u_{\tilde a})}{w_{\tilde a}+(1-u_{\tilde a})-w_{\tilde a}(1-u_{\tilde a})} - 3\frac{(w_{\tilde a}+1-u_{\tilde a})}{w_{\tilde a}(1-u_{\tilde a})}}(\overline{a}-\underline{a})}$$

For simplicity, we only use $R^P(\tilde a)$ for demonstrations in the sequel.

Definition 4.8. Let $\tilde a_i = ((\underline{a}_i, a_i, \overline{a}_i); w_{\tilde a_i}, u_{\tilde a_i})(i = 1, 2)$ be two TIFNs, then the ratio-based ranking of $\tilde a_1$ and $\tilde a_2$ can be summarized as follows:

(i) If $R^T(\tilde a_1) < R^T(\tilde a_2)$, then $\tilde a_1$ is smaller than $\tilde a_2$, denoted by $\tilde a_1 \prec \tilde a_2$;

(ii) If $R^T(\tilde a_1) = R^T(\tilde a_2)$, then $\tilde a_1$ and $\tilde a_2$ represent the same information, denoted by $\tilde a_1 \approx \tilde a_2$.

There are some known ranking methods for TIFNs in the literature, such as the methods in Refs. 29 and 30. In Ref. 29, to aggregate the characteristics of the membership and nonmembership functions, Li developed a ranking method in virtue of the ratio of the value index to the ambiguity index for a TIFN $\tilde a$ by incorporating the decision maker's preference information. Specifically, in the aggregating process, a weight is used to demonstrate the preference of uncertain or positive feelings of decision makers. Surely, the ranking results depend heavily on the attitudes of the decision makers and thus may lead to some subjective results practically, which can be seen from the different ranking orders of the three candidates in Section 5.1 in Ref. 29 caused by the diversity in weights. The ranking method by Wan and Li[30] is based on the both the ratios of the possibility mean to the possibility standard deviation for membership and nonmembership functions, inevitably the ranking result relies only one of them. Thereby, from this aspect the obtained result in Ref. 30 is one-sided and biased. Depending on the synthesization of membership and nonmembership functions in virtue of the t-norm-based cuts, we can get the ranking order more comprehensively. For instance, let $\tilde a = ((0.2, 0.4, 0.6); 0.4, 0.6)$ and $\tilde b = ((0.3, 0.35, 0.7); 0.4, 0.6)$ be two TIFNs, i.e., two triangular fuzzy numbers, then according to the definition of the ratios of the possibility mean to the possibility standard deviation for membership and nonmembership functions by Wan and Li,[30] it can be obtained by calculation that $R(\tilde a_\alpha) = R(\tilde b_\alpha) = 1.9596$, $R(\tilde a_\beta) = 2.6128$ and $R(\tilde b_\beta) = 2.8169$, then $\tilde a \prec \tilde b$ by the method of Wan and Li,[30] which is not consistent with the fact that both the maximum degree of membership and the minimum degree of nonmembership of the TIFNs are the same and the most possible values satisfy 0.4 > 0.35. Whereas, here by the ranking method in Definition 4.8, since $R^P(\tilde{a}) = 0.4$ and $R^P(\tilde{b}) = 0.38$, it follows that $\tilde{b} \prec \tilde{a}$, which seems more reasonable. Additionally, owing to the independence on the preference of decision makers, the result is more objective, compared with the method in Ref. 29.

5 | APPROACH TO MADM USING TIFNs

5.1 | Approach to MADM using TIFNs

In general, MADM problems relate to a set of alternatives $A = \{A_1, A_2, \dots, A_m\}$ and a set of attributes set $\{a_1, a_2, \dots, a_n\}$, and aim to rank the alternatives on the attributes or find the most desired alternative. Here we assess the alternatives on attributes using TIFNs. Specifically, the assessment of an alternative A_i on an attribute a_j is given by a TIFN $\tilde{a}_{ij} = ((\underline{a}_{ij} a_{ij}, \overline{a}_{ij}), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}})$. Then a MADM problem using TIFNs can be expressed by a TIFN decision matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$. As is known, attributes can be classified into two types: benefit attributes and cost attributes. To uniform the different measurements of attributes, the TIFN decision matrix $\tilde{A}$ must be normalized to $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$, where $\tilde{r}_{ij} = ((\underline{r}_{ij}, r_{ij}, \overline{r}_{ij}), w_{\tilde{r}_{ij}}, u_{\tilde{r}_{ij}})$ and $w_{\tilde{r}_{ij}} = w_{\tilde{a}_{ij}}, u_{\tilde{r}_{ij}} = u_{\tilde{a}_{ij}}$. Here the normalized methods for benefit attributes and cost attributes are, respectively, based on the following two equations:

$$\underline{r}_{ij} = \frac{\underline{a}_{ij}}{\sqrt{\sum_{i=1}^{m} \overline{a}_{ij}^2}}, \quad r_{ij} = \frac{a_{ij}}{\sqrt{\sum_{i=1}^{m} a_{ij}^2}}, \quad \overline{r}_{ij} = \frac{\overline{a}_{ij}}{\sqrt{\sum_{i=1}^{m} \underline{a}_{ij}^2}} \tag{6}$$

and

$$\underline{r}_{ij} = \frac{\frac{1}{\overline{a}_{ij}}}{\sqrt{\sum_{i=1}^{m} \frac{1}{\underline{a}_{ij}^2}}}, \quad r_{ij} = \frac{\frac{1}{a_{ij}}}{\sqrt{\sum_{i=1}^{m} \frac{1}{a_{ij}^2}}}, \quad \overline{r}_{ij} = \frac{\frac{1}{\underline{a}_{ij}}}{\sqrt{\sum_{i=1}^{m} \frac{1}{\overline{a}_{ij}^2}}} \tag{7}$$

In process, the weighting vector is used to demonstrate the different importance of different attributes. Denote a weighting vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, where $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$ for all $i = 1, 2, \dots, n$. Assume $\Lambda = \{\omega = (\omega_1, \omega_2, \dots, \omega_n)^T \mid \sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0 (i = 1, 2, \dots, n)\}$. Practically, the information of attribute weights is incomplete and has several different forms of structures. Usually, these incomplete weight information structures may consist of several sets of the following basic sets:[34]

(1) $\Lambda_1 = \{\omega \in \Lambda \mid \omega_i \geq \xi\omega_j, \omega_i \geq \varepsilon \textit{ for some } i \textit{ and } j\}$, where $\xi, \varepsilon > 0$ are given constants;

(2) $\Lambda_2 = \{\omega \in \Lambda \mid \beta \geq \omega_i - \omega_j \geq \alpha, \omega_i \geq \varepsilon \textit{ for some } i \textit{ and } j\}$, where $\beta > \alpha > 0$ are given constants;

(3) $\Lambda_3 = \{\omega \in \Lambda \mid \gamma \geq \omega_i \geq \eta, \omega_i \geq \varepsilon \textit{ for some } i\}$, where $\gamma > \eta > 0$ are given constants;

(4) $\Lambda_4 = \{\omega \in \Lambda \mid \omega_i - \omega_j \geq \omega_k - \omega_l, \omega_i \geq \varepsilon \textit{ for some } i, j, k \textit{ and } l\}$, where $\varepsilon > 0$ is a given constant.

In this paper, the MADM problem is to select the best alternative from the finite alternative set A according to the matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ and the information structure Ω.

For each alternative, the bigger the possibility mean to standard deviation–based ratios on the overall attribute value of alternative, the larger possibility of the alternative being the best alternative. Thereby, it follows that the bigger the possibility mean to standard deviation–based ratios, the better

TABLE 1 Decision matrix in the form of TIFNs

	a_1	a_2	a_3	a_4	a_5
A_1	((5.7, 7.7, 9.3); 0.7, 0.2)	((5, 7, 9); 0.6, 0.3)	((5.7, 7.7, 9); 0.8, 0.1)	((8.33, 9.67, 10); 0.6, 0.4)	((3, 5, 7); 0.6, 0.3)
A_2	((6.5, 8.6, 10); 0.4, 0.5)	((8, 9, 10); 0.6, 0.3)	((8.3, 9.7, 10); 0.7, 0.2)	((8, 9, 10); 0.6, 0.3)	((7, 9, 10); 0.6, 0.2)
A_3	((6.5, 8.2, 9.3); 0.8, 0.1)	((7, 9, 10); 0.7, 0.2)	((7, 9, 10); 0.5, 0.2)	((6, 8, 9); 0.6, 0.2)	((6.3, 8.3, 9.7); 0.7, 0.2)

the alternative. Therefore, the reasonable attribute weighting vector ω should be obtained such that all the possibility mean to standard deviation–based ratios of alternatives could be as big as possible. The weighting vector ω can be obtained by solving the mathematical programming models constructed as follows:

$$\max R^T(\tilde{a}_i) \quad \text{s.t. } \omega_i = (\omega_{1i}, \omega_{2i}, \dots, \omega_{ni})^T \in \Omega. \tag{8}$$

where $\tilde{a}_i = WA_{\omega_i}(\tilde{a}_{1i}, \tilde{a}_{1i}, \dots, \tilde{a}_{ni})$, i.e., the overall attribute value of alternative A_i. By Equation 8, ω_i^* and the maximum value of the objective function, denoted by R_i^*, can be obtained. Note that, since Equation 8 is nonlinear, then (ω_i^*, R_i^*) is a local optimal solution of Equation 8 given some appropriate initial value ω_i^0.

To sum up, the ratio-based approach to MADM using TIFNs can be summarized as follows:

S1: Normalize the decision matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ by Equations 6 and 7.

S2: Obtain (ω_i^*, R_i^*) by solving Equation 8 for each $i \in \{1, 2, \dots, m\}$;

S3: According to the maximum values of the objective functions, i.e., the maximum ratios of $\tilde{a}_i$ for alternative A_i $(i = 1, 2, \dots, m)$, the order of the alternatives can be obtained by Definition 4.8.

5.2 | Illustrative example

In the following, we refer to the personnel selection problem, which is adapted from Refs. 20,29 and 30 In detail, a company plans to employ a system analyst. After basic screening, there are three candidates (i.e., alternatives) A_1, A_2, and A_3 for further evaluation. The committee of the selection assesses the three candidates from the aspect of emotional steadiness a_1, communication skills a_2, personality a_3, work experience a_4, and self-confidence a_5. The five attributes are all benefit attributes. The assessments of the candidates with respect to attributes are given as in Table 1. Assume the preference information structure of the decision makers on the attributes $\{a_1, a_2, \dots, a_5\}$ can be expressed by the set as follows:

$$\Omega = \{\omega_i = (\omega_{1i}, \omega_{2i}, \dots, \omega_{5i})^T \mid \omega_{2i} \geq \omega_{1i}, 0.05 \leq \omega_{1i} - \omega_{3i} \leq 0.2, 0.15 \leq \omega_{3i} \leq 0.3,$$
$$\omega_{4i} - \omega_{2i} \geq \omega_{1i} - \omega_{3i}, \omega_{4i} \geq 2\omega_{5i}, 0.1 \leq \omega_{5i} \leq 0.5\}$$

According to Equation 6, the normalized TIFN decision matrix is obtained as in Table 2. Take the alternative A_1 as an example, by Equation 4

$$\begin{aligned}\tilde{a}_1 &= AW_{\omega_1}(\tilde{a}_{11}, \tilde{a}_{21}, \dots, \tilde{a}_{51})\\ &= ((1 - 0.9791^{\omega_{11}} 0.9822^{\omega_{21}} 0.9797^{\omega_{31}} 0.9704^{\omega_{41}} 0.9877^{\omega_{51}},\end{aligned}$$

TABLE 2 The normalized decision matrix.

	A_1	A_2	A_3
a_1	((0.0209, 0.0384, 0.0795); 0.7, 0.2)	((0.0238, 0.0429, 0.0855); 0.4, 0.5)	((0.0238, 0.0409, 0.2669); 0.8, 0.1)
a_2	((0.0178, 0.0332, 0.0652); 0.6, 0.3)	((0.0285, 0.0427, 0.0725); 0.6, 0.3)	((0.0249, 0.0427, 0.0725); 0.7, 0.2)
a_3	((0.0203, 0.0329, 0.0598); 0.8, 0.1)	((0.0295, 0.0414, 0.0665); 0.7, 0.2)	((0.0249, 0.0384, 0.0665); 0.5, 0.2)
a_4	((0.0296, 0.0405, 0.0590); 0.6, 0.4)	((0.0285, 0.0377, 0.0590); 0.6, 0.3)	((0.0214, 0.0335, 0.0531); 0.6, 0.2)
a_5	((0.0123, 0.0286, 0.0717); 0.6, 0.3)	((0.0288, 0.0515, 0.1024); 0.6, 0.2)	((0.0259, 0.0475, 0.0993); 0.7, 0.2)

$$1 - 0.9616^{\omega_{11}} 0.9668^{\omega_{21}} 0.9671^{\omega_{31}} 0.9595^{\omega_{41}} 0.9714^{\omega_{51}}, 1 - 0.9205^{\omega_{11}} 0.9348^{\omega_{21}} 0.9402^{\omega_{31}}$$

$$0.9410^{\omega_{41}} 0.9283^{\omega_{51}}), 0.7^{\omega_{11}} 0.6^{\omega_{21}} 0.8^{\omega_{31}} 0.6^{\omega_{41}} 0.6^{\omega_{51}}, 1 - 0.8^{\omega_{11}} 0.7^{\omega_{21}} 0.9^{\omega_{31}} 0.6^{\omega_{41}} 0.7^{\omega_{51}}).$$

Assume $T = T_P$, thereby the mathematical programming model can be obtained as follows:

$$\begin{aligned}\max\{ \boldsymbol{R}^T(\tilde{a}_1) = {} & \sqrt{3 \times 0.56^{\omega_{11}} 0.42^{\omega_{21}} 0.72^{\omega_{31}} 0.36^{\omega_{41}} 0.42^{\omega_{51}}}(10 - 0.9791^{\omega_{11}} 0.9822^{\omega_{21}} 0.9797^{\omega_{31}} \\ & 0.9704^{\omega_{41}} 0.9877^{\omega_{51}} - 8 \times 0.9616^{\omega_{11}} 0.9668^{\omega_{21}} 0.9671^{\omega_{31}} 0.9595^{\omega_{41}} 0.9714^{\omega_{51}} \\ & -0.9205^{\omega_{11}} 0.9348^{\omega_{21}} 0.9402^{\omega_{31}} 0.9410^{\omega_{41}} 0.9283^{\omega_{51}})/[10 \times (0.9791^{\omega_{11}} 0.9822^{\omega_{21}} \\ & 0.9797^{\omega_{31}} 0.9704^{\omega_{41}} 0.9877^{\omega_{51}} - 0.9205^{\omega_{11}} 0.9348^{\omega_{21}} 0.9402^{\omega_{31}} 0.9410^{\omega_{41}} 0.9283^{\omega_{51}})]\}\end{aligned}$$

$$\text{s.t.} \begin{cases} \omega_{21} \geq \omega_{11}, \\ 0.05 \leq \omega_{11} - \omega_{31} \leq 0.2, \\ 0.15 \leq \omega_{31} \leq 0.3, \\ \omega_{41} - \omega_{21} \geq \omega_{11} - \omega_{31}, \\ \omega_{41} \geq 2\omega_{51}, \\ 0.1 \leq \omega_{51} \leq 0.5. \end{cases}$$

Let the initial value be $\omega_1^0 = (0.225, 0.225, 0.15, 0.3, 0.1)^T$, then we can get

$$(\omega_1^*, \boldsymbol{R}_1^*) = ((0.2, 0.2, 0.15, 0.35, 0.1)^T, 1.0220)$$

with Matlab R2012a. Similarly, we can get $(\omega_2^*, \boldsymbol{R}_2^*) = ((0.2, 0.2, 0.15, 0.35, 0.1)^T, 1.0374)$ and $(\omega_3^*, \boldsymbol{R}_3^*) = ((0.2, 0.25, 0.15, 0.3, 0.1)^T, 0.6511)$. Since $\boldsymbol{R}_2^* > \boldsymbol{R}_1^* > \boldsymbol{R}_3^*$, then $\tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_3$. Namely, the best alternative is A_2.

5.3 | Comparison with the existing methods

There are many known approaches to MADM using TIFNs. Here we list the ranking results of the referred problem using these methods in Table 3 for comparison.

It can be found that the based characteristics of TIFNs in Refs. 20,29 and 30 consider the membership function and nonmembership function separately, then the ranking methods depend only on the membership functions or nonmembership functions. For instance, the lexicographic ranking method, based on the ratios of the possibility mean to the possibility standard deviation of membership and nonmembership functions for TIFNs, relies only on the membership functions or nonmembership functions

TABLE 3 The results using different methods.

Methods	The ranking results	the underlying characteristics
Li[20]	$\begin{cases} A_1 \succ A_3 \succ A_2, & \lambda \in [0, 0.1899]; \\ A_3 \succ A_1 \succ A_2, & \lambda \in (0.1899, 0.9667); \\ A_3 \succ A_2 \succ A_1, & \lambda \in [0.9667, 1]. \end{cases}$	the ratio of the value index to the ambiguity index
Li et al[29]	$\begin{cases} A_3 \succ A_1 \succ A_2, \ \lambda \in [0, 0.793]; \\ A_1 \succ A_3 \succ A_2, \ \lambda \in (0.793, 1]. \end{cases}$	the value-index and ambiguity-index
Wan and Li[30]	$A_3 \succ A_1 \succ A_2$	the ratio of the possibility mean to the possibility standard deviation
In this paper	$A_2 \succ A_1 \succ A_3$	the ratio of the possibility mean to the possibility standard deviation using t-norm based cuts

*Here λ represents the decision maker's preference information.

in practice. Moreover, in the approach to MADM, the biobjective mathematical programming models are solved lexicographically. It can be seen from the fact that the ranking results of the numerical example only depend on the ratios of the membership function. Hence the results in Ref. 30 are biased and one sided. In Refs. 20 and 29, by incorporating the decision maker's preference information, the membership and nonmembership functions for TIFNs are synthesized in the characteristics of TIFNs. However, these ranking results depend heavily on the preference information of the decision makers. From Table 3, quite different results could be obtained if different values of λ are considered. Thereby, these results in Refs. 20 and 29 are sensitive to preference information of the decision makers. In other words, these results are subjective.

In this paper, in the calculation of the characteristics of TIFNs, the membership function and the nonmembership function are comprehensively considered taking advantage of the t-norm-based cuts. Moreover, the ranking results are independent of the preference information of the decision makers. To sum up, the result in this paper, which is quite different from these by the other methods, is much more reasonable and objective. Additionally, in Ref. 35, TIFNs are transformed to interval numbers. Since the ranking of interval numbers remains difficult. However, the proposed method of this paper can directly rank TIFNs by real numbers. Therefore, the proposed method is easier to be implemented than that in Ref. 35.

6 | CONCLUDING REMARKS

In this paper, triangular norm–based cuts of TIFNs are developed to synthesize the membership and nonmembership functions in describing the cut sets, then the possibility characteristics of TIFNs, i.e., the possibility mean, the possibility variance, and the possibility mean-standard deviation ratio, are given. On the basis of the possibility mean-standard deviation ratio, a ranking method of TIFNs is introduced. With these elements, an approach to MADM is proposed and illustrated by a numerical example. It is shown that the approach to MADM comprehensively considers both the membership and nonmembership functions and is independent of the preference information of the decision makers and can lead to objective and reasonable results.

ACKNOWLEDGMENTS

The first author is very grateful to Prof. Feng Qin of Jiangxi Normal University and Prof. Shuping Wan of Jiangxi University of Finance and Economics for their kindness and help. The work is supported by National Natural Science Foundation of China under Grants 71473081 and Hunan Provincial Innovation Foundation For Postgraduate under Grant CX2017B171, Science and Technology Project of Jiangxi Provincial Educational Department of China under Grant GJJ151602.

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How to cite this article: Yi Z-H, Li H-Q. Triangular norm-based cuts and possibility characteristics of triangular intuitionistic fuzzy numbers for decision making. *Int J Intell Syst*. 2018;33:1165–1179. https://doi.org/10.1002/int.21974