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Portfolio selection under uncertainty by the ordered modular average operator

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Abstract

In a world under uncertainty, the beliefs for the information underlie the behavioral style of portfolio decisions in portfolio management. In this work, we use the copula-based ordered modular averages (OMAs) in the calculation of the mean and variance of the assets' returns for portfolio selection to capture the beliefs of the investors and the departure of rationality in evaluation. Specially, the outcomes and the probability information in terms of the decumulative probabilities are jointly transformed using appropriate copulas while satisfying the stochastic dominance in the probability-sensitivity evaluation. In addition, the diversity of the underlying copulas facilitates the challenge of the diversity of investors with different beliefs for expectations. Consequently, the mean-variance model in this work using OMA with the decumulative probabilities can encode not only the decision makers' assessment of relative likelihoods but also the confidence attached to such assessment in the evaluation.

Keywords Aggregation operator · Portfolio selection · The mean-variance model · The ordered modular averages · The ordered weighted averages

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1 Introduction

The fundamental work of portfolio selection would date back to the mean-variance theory by Markowitz (1952), which provides a trade off between the return and the risk (characterized by variance) under economic conditions. Since its introduction, the theory has drew extensive attention of the scholars in the community and is admittedly referred to be the basis of the modern portfolio theory. Until now, there are a variety of portfolio optimization models (Wang and Xia 2012; Fang et al. 2008) available for portfolio and risk management for various efficiency in corresponding terminology.

Indeed, up to now, in view of the complexity and uncertainty of the realistic world, some non-probabilistic methodologies have emerged and became an eye-catching topic in dealing with the human-centric uncertainty in portfolio selection. In particular, fuzzy set theory (Fang et al. 2008; Zadeh 1965; Wang and Zhu 2002) and uncertainty theory (Liu 2002; Huang 2009, 2008) provide a well integration of quantitative and qualitative analysis, the knowledge of experts and the subjective attitude of the investors into the models of portfolio selection. Without probabilistic information, the subjective information regarding the attitudes of decision makers contributes to exploring the available information with uncertainty in the information aggregation. To this end, many scholars use OWA operator by Yager (1988), which can be considered to be a special case of the rank dependent utility (Quiggin 1982; Wakker 2010), to handle the subjective uncertainty in various fields (Yager 1998; Merigó et al. 2015).

In the very recent, Laengle et al. (2016) took an initial step toward the application of the well established OWA operator (Yager 1988) in portfolio selection. Specifically, they replaced the common weighted average operator with OWA operator in the calculation of mean and variance. By different OWA operators (Yager 1993), different expectations about the future can be embodied in portfolio selection. In effect, under uncertainty, where the probabilistic information is confused or unavailable, people react to each expectation quite differently due to the differences in the beliefs for a particular expectation. In the literature, many works focus on using probability sensitivity which underlies all nonexpected utility and has the advantage in incorporating people's confidence and beliefs in probability information for expectations. One direct intuition is to transform the separate outcome probabilities over OWA concavely or convexly, which, however, is shown to result in the violation of stochastic dominance (Wakker 2010). Yet, it is reasonable to use the decumulative probabilities (also known as good-news probabilities or ranks) (Quiggin 1982; Tversky and Kahneman 1992; Lopes and Oden 1999; Quiggin 2014) rather than the *separate* outcome probabilities in the probability-sensitivity evaluation of prospects. Moreover, there is a psychological reasoning and behavioral foundation (Wakker 2010) showing that it makes more sense to transform decumulative probability than to transform separate probability. Fortunately, the copula based OMA operators in Mesiar and Mesiarova-Zemankova (2011) provide an integrated way to accommodate such a probability sensitivity with decumulative probabilities. Therefore, in this work we attempt to employ OMA operator in the portfolio selection for entailing the departure of the rationality (i.e., the theory of expect utility) and encoding the investors' confidence and beliefs in probability information for their expectations. Further, through copula, OMA can offset the inflexibility in the generation of OWA and the diversity of OMA operators facilitates the challenge of the diversity of investors with heterogeneous beliefs and the generation of optimum portfolios tailored to the investors with specifical beliefs about the future performance of the assets in different market scenarios. By this token, the mean-variance model in this work by copula based OMA with the decumulative probabilities can encode not only the decision makers' preferences about the assessment of relative likelihoods but also the confidence attached to such assessment in the evaluation. Numerical examples show that using copula based OMA operators, we can get higher (or lower)-utility portfolios than those by the OWA-framework and the classical Markowitzian framework for the investors with different beliefs for expectations of market performance.

The paper is structured as follows: in Sect. 2, we recall some notations and generating methods on OMA operator. Sect. 3 gives the definitions for the OMA based mean and variance of an asset return, and then presents the OMA based mean and variance for a portfolio. A real numerical example for portfolio selection with OMA operator is shown in Sect. 4. Finally, we end up with some concluding remarks.

2 Preliminaries

2.1 A review of OMAs

Aggregation functions (Beliakov et al. 2007) are fundamental and essential in information fusing. Note that an aggregation function A is a nondecreasing $[0, 1]^n \to [0, 1]$ mapping with $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$ for a fixed $n \geq 2$. In the sequel, an aggregation function A is called idempotent if it satisfies that $A(x, x, \ldots, x) = x$ for all $x \in [0, 1]$. The pair $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ is called comonotone if $(x_i - x_j)(y_i - y_j) \geq 0$. Yager (1988) introduced the ordered weighted average (OWA) as a symmetrization of arithmetic mean by incorporating a reordering step, which is useful in many applications.

Definition 1 (Yager 1988) The operator $\mathrm{OWA} : [0, 1]^n \to [0, 1]$ is given by

$$\mathrm{OWA}(r_1, r_2, \ldots, r_n) = \sum_{j=1}^{n} p_j b_j, \tag{1}$$

where b_j is the jth element of the descendingly ordered sequence of the r_is and $\sum_{i=1}^{n} p_i = 1$ with $p_i \in [0, 1]$ for each $i = 1, 2, \ldots, n$.

By genuinely extending of the OWA operator, the ordered modular average (OMA) (Mesiar and Mesiarova-Zemankova 2011) was developed using symmetrization from the modular average operator in the framework of aggregation operators.

Definition 2 (Mesiar and Mesiarova-Zemankova 2011) The operator $\mathrm{OMA} : [0, 1]^n \to [0, 1]$ is given by

$$\mathrm{OMA}(r_1, r_2, \ldots, r_n) = \sum_{j=1}^{n} f_j(b_j), \tag{2}$$

where b_j is the jth element of the descendingly ordered sequence of the r_is and $\sum_{i=1}^{n} f_i(x) = x$ with the weighting function $f_i : [0, 1] \to [0, 1]$ being nondecreasing for each $i = 1, 2, \ldots, n$.

Trivially, an OWA operator is necessarily an OMA operator. Further, the OWMax operator (Mesiar and Mesiarova-Zemankova 2011) is also an OMA operator. Similarly, an OWA operator can also be characterized in the sense that an OMA operator is exactly a symmetric, idempotent, comonotone modular aggregation function.

2.2 Copula based OMAs

In this part, OMA operators, generated by copula, are presented. An aggregation function $C : [0, 1]^n \to [0, 1]$ is called a copula (Klement et al. 2000; Nelsen 2007) if it is supermodular, i.e., $C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, and has a neutral element 1, i.e., $C(1, x) = C(x, 1) = x$. There are three basic copulas $\Pi(x, y) = xy$; $W(x, y) = \max(x + y - 1, 0)$ and $Min(x, y) = \min(x, y)$.

Recall that a capacity m on a discrete universe $X = \{1, 2, \ldots, n\}$ is a mapping $m : 2^X \to [0, 1]$ that is nondecreasing in the sense that $m(E) \leq m(F)$ if $E \subseteq F \subseteq X$, and satisfies that $m(\emptyset) = 0, m(X) = 1$. A capacity m is symmetric if there is a nonnegative weighting vector $(w_1, w_2, \ldots, w_n)^T$ with $\sum_{i=1}^{n} w_i = 1$, such that $m(E) = \sum_{i=1}^{|E|} w_i$. As a nonadditive set function that extends the standard probability, capacities specify the decision weighting information for the expect utility (Tversky and Kahneman 1992).

Let C be a fixed copula and m be a given capacity on X, then a (C, m)-based integral (Klement et al. 2004) is a mapping $C_m : [0, 1]^n \to [0, 1]$ given by

$$C_m(\mathbf{x}) = \sum_{i=1}^{n} [C(x_{\sigma(i)}, m(\Omega_i)) - C(x_{\sigma(i)}, m(\Omega_{i-1}))] \tag{3}$$

where $\Omega_i = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ and σ is a permutation of $\{1, 2, \ldots, n\}$ satisfying $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots, x_{\sigma(n)}$, i.e., $x_{\sigma(j)}$ is the jth element of descendingly ordered sequence, and the convention $\{\sigma(1), \sigma(0)\} = \emptyset$ is used. If $C = \Pi$ and $C = Min$, then the (C, m)-based integral reduces to the Choquet integral and Sugeno integral, respectively. Specially, if $C = \Pi$, then Eq. 3 becomes exactly the expect utility in the frame of the cumulative prospect theory (Tversky and Kahneman 1992) and $m(\Omega_i) - m(\Omega_{i-1})$, interpreted as a marginal weight of $x_{\sigma(i)}$, is designed to accommodate the normality of the weights. With the nonadditivity incorporated in the capacity, the fact that the nonadditive probability measure of the union of disjoint events is strictly greater than the sum of those for single evens, suggests that there is ambiguity about the relative likelihood of these disjoint events and thus there is a residual probability mass the individual does not know how to allocate among these events. Therefore, in terms of capacity the individual's assessment of the relative likelihoods and the beliefs attached to such assessment can be encoded.

Generally, copula-based fuzzy integrals w.r.t. symmetric capacities are OMAs.

Lemma 1 (Mesiar and Mesiarova-Zemankova 2011) *Let C be a copula and m a symmetric capacity on X, then C_m by* Eq. 3 *is an OMA.*

By the above results, OMAs can be obtained on the basis of some copula and some symmetric capacity. In fact, the family of weighting functions $\{f_i\}_{1\leq i\leq n}$ in Eq. 2 can be structured by a nonnegative weighting or probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)^T$ with $\sum_{i=1}^{n} p_i = 1$ and an appropriate copula C. Specifically, the weighting functions can be constructed by

$$f_i(x) = \begin{cases} C(x, p_1), & i = 1 \\ C\left(x, \sum_{j=1}^{i} p_j\right) - C\left(x, \sum_{j=1}^{i-1} p_j\right), & i \in \{2, \ldots, n\}, \end{cases} \tag{4}$$

Compared with Eq. 3, the capacity in Eq. 4 is exactly the decumulative probabilities through which both the decision makers' assessment of relative likelihoods and the attached confidence are encoded. The value of the weighting function is interpreted as a marginal contribution of each outcome x in the utility. Specially, in Eq. 4, $f_1(x) = C(x, p_1)$ is the marginal contribution for extreme or favorable outcomes by the ordering steps in the operators. In order to present the heterogeneous beliefs in utility by the copula based OMA operator, we can resort to some monotonic parameterized copulas through which the marginal contribution for the extreme or favorable outcomes can be underestimated or overestimated by using appropriate parameter.

In the sequel, we restrict ourselves to use the Frank family of copulas, i.e., $C_\theta(x, y) = -\frac{1}{\theta}\ln(1 + \frac{(e^{-\theta x}-1)(e^{-\theta y}-1)}{e^{-\theta}-1})$ $(\theta \neq 0)$ which is pointwise strictly increasing w.r.t. the parameter θ, thus Eq. 4 becomes

$$f_i^{\theta}(x) = \begin{cases} -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta x}-1)(e^{-\theta p_1}-1)}{e^{-\theta}-1}\right), & i = 1 \\ \frac{1}{\theta}\ln \frac{(e^{-\theta}-1)+(e^{-\theta x}-1)\left(e^{-\theta\sum_{j=1}^{i-1} p_j}-1\right)}{(e^{-\theta}-1)+(e^{-\theta x}-1)\left(e^{-\theta\sum_{j=1}^{i} p_j}-1\right)}, & i \in \{2, \ldots, n\} \end{cases} \tag{5}$$

3 On the return and risk of an asset using OMA operators

In this part, the OMA based mean and variance of an asset are defined and then the ornesses for the OMA based operations are given.

Definition 3 Let r_i be the return of an asset at the state i $(i = 1, 2, \ldots, n)$, then the OMA based mean of the asset return, denoted by $E_{OMA}(r_1, r_2, \ldots, r_n)$ is defined by

$$E_{OMA}(r_1, r_2, \ldots, r_n) = \sum_{j=1}^{n} f_j(b_j), \tag{6}$$

where b_j is the jth element of the descendingly ordered sequence of r_is and f_j is the underlying weighting function for OMA (see Eq. 2).

Remark 1 As a special case of OMA based mean, OWA based one provides a simple linear and independent way for the incorporation of the separate subjective probabilities for the ordered inputs, by which the investors' expectations can be embodied. Actually, the OWA based mean directly considers the subjective or objective state probabilities as the decision weights without any transformations. Nevertheless, under uncertainty it is usually to differentiate between the two and derive the latter from some appropriate transformation of the former, like Choquet expect utility or rank dependent expect utility (Schmeidler 1989) where the decumulative probabilities. In addition, to incorporate the beliefs in expectation, the use of separate probabilities for transformation in evaluation of expected utility may easily leads to the violation of stochastic dominance, which however can be settled by using decumulative probabilities (Quiggin 1982, 2014) in the transformation. Moreover, there is a psychological and heuristic intuition (Wakker 2010) showing that it makes more sense to use decumulative probabilities than to use separate ones. Since that, by Eq. 4, the constructing method for the weighting functions provides a nonlinear and integrated way to incorporate the outcomes and decumulative probabilities by the pertaining copula, then it is natural and reasonable to use the copula based OMAs in the mean evaluation.

Remark 2 With Eq. 5, if θ converges to 0 ($\theta \to 0$), an OMA operator with such weighting functions reduces to an OWA, where the marginal contribution for extreme outcomes is neither underestimated or overestimated. Thus, the beliefs of the investors are neutral or indifferent. In virtue of the fact that (Klement et al. 2000) the Frank family of copulas is pointwise strictly increasing w.r.t. the parameter θ. Through the copulas, the marginal contribution for extreme outcomes is overestimated for $\theta > 0$, and underestimated for $\theta < 0$. Therefore, the optimistic and pessimistic beliefs can respectively incorporated. Moreover, the extent to which they are overestimated or underestimated is closely related to the absolute of the parameter θ. In virtue of the Frank copula based OMA with the parameter θ ($\theta > 0, \theta \to 0$ and $\theta < 0$), the models specified by Eqs. 8 and 9 have the advantage in expressing the heterogeneous beliefs (optimistic, neutral and pessimistic beliefs) of the investors in comparison with the OWA based framework of Laengle et al. (2016). Moreover, the beliefs of the investors can be reinforced or relaxed by the Frank copula with which the marginal contribution of the favorable outcomes can be underestimated or overestimated through the underlying parameter θ. Therefore, OMA based mean by the Frank copula facilitates the modeling of different beliefs of the assets' future performance in a more flexible and practical way.

By the OMA based mean, we can develop an OMA based variance to measure the risk of an asset analogously.

Definition 4 Let r_i be the return of an asset at the state i ($i = 1, 2, \ldots, n$), then the OMA based variance of the asset return is defined by

$$V_{OMA}(r_1, r_2, \ldots, r_n) = \sum_{j=1}^{n} f_j(t_j), \tag{7}$$

where $s_i = (r_i - E_{OMA})^2$ and t_j is the jth element of the ascendingly ordered sequence of s_is and f_j is the underlying weighting function in OMA (see Eq. 2).

Note that here an ascending order for the OMA operator in the above definition is used since a stabler return is much more preferable to a less stable one for most of the investors, thus the ascending order is consistently from the best scenario to the worse.

Now we focus on a portfolio with m individual assets and assume that r_{ik} is the return of the kth asset at the state i $(k = 1, 2, \ldots, m; i = 1, 2, \ldots, n)$. $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is the vector for the proportions of the allocation such that $\omega_k \in [0, 1]$, i.e., short selling is not allowed, and $\sum_{k=1}^m \omega_k = 1$. Then the return r_i^p of the portfolio at state i can be obtained by $r_i^p = \sum_{k=1}^m \omega_k r_{ik}$. It follows that the OMA based mean $E_{OMA}(r^p, \omega)$ of the portfolio can be calculated by

$$E_{OMA}(r^p, \omega) = \sum_{j=1}^{n} f_j(b_j^p),$$

where the f_js are the given weighting functions for the OMA operator and b_j^p is the jth element of the descendingly ordered sequence of the r_i^ps. Denote the square of deviation from the OMA based mean by $s_i^p = (r_i^p - E_{OMA}(r^p, \omega))^2$, then the OMA based variance is calculated by

$$V_{OMA}(r^p, \omega) = \mathrm{OMA}(s_1^p, s_2^p, \ldots, s_n^p) = \sum_{j=1}^{n} f_j(t_j^p),$$

where t_j^p is the jth element of the ascendingly ordered sequence of the s_i^ps.

Remark 3 If we assume $f_i(x) = p_i x$ for each i, i.e., $\theta \to 0$ in Eq. 5, then OMA based mean and variance reduce to the OWA based ones (Laengle et al. 2016), i.e., $E_{OWA}(r^p, \omega)$ and $V_{OWA}(r^p, \omega)$, respectively.

By the Markowitzian approach, one can maximize the expected return of a portfolio with a given degree of portfolio risk or minimize the expected risk with a specific expected return. By the OMA based mean and variance, we can get the following two models for portfolios selection.

1. **The minimization of OMA based variance.**

$$\min_{\omega} V_{OMA}(r^p, \omega) \text{ subject to } \begin{cases} E_{OMA}(r^p, \omega) = E_0; \\ \sum_{k=1}^{m} \omega_k = 1 (\omega_k \in [0, 1]). \end{cases} \tag{8}$$

2. **The maximization of OMA based mean.**

$$\max_{\omega} E_{OMA}(r^p, \omega) \text{ subject to } \begin{cases} V_{OMA}(r^p, \omega) = V_0; \\ \sum_{k=1}^{m} \omega_k = 1 (\omega_k \in [0, 1]). \end{cases} \tag{9}$$

In the above-mentioned models, the objective functions and the underlying constraints are dependent on OMA based mean or variance, which can facilitate the expression of the heterogeneous beliefs of the investors through the weighting functions for the OMA specified by Eq. 5. Moreover, just as stated in Remark 1, the use of decumulative probabilities in the underlying OMA operators intuitionistically makes more sense than that of the separate ones for OWA operators.

One of the most important attributes in decision making for portfolio selection is the beliefs of the investors, i.e., the optimistic belief, the neutral belief and the pessimistic belief. To incorporate the heterogeneous beliefs in portfolio selection, the central work is to provide an appropriate measurement of the optimistic degree of investors. Usually, the expectation effect for an aggregation operator is indicated by so-called orness indices (Laengle et al. 2016). By orness, one can easily measure the degree of optimism or pessimism of the decision maker. In the work of Laengle et al. (2016), in view of the unbalanced input arguments, Laengle et al. respectively presented two ameliorated ornesses for OWA based mean and variance, which are given by

$$\alpha^*(\mathbf{p}, \mathbf{r}) = \sum_{j=1}^{n} p_j \left(\frac{b_j - b_n}{b_1 - b_n} \right), \tag{10}$$

and

$$\alpha^*(\mathbf{p}_{asc}, \mathbf{r}) = \sum_{j=1}^{n} p_j \left(\frac{s_j - s_n}{s_1 - s_n} \right), \tag{11}$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is the input arguments to be aggregated and b_j and s_j are respectively the j-th element of the descendingly and ascendingly ordered sequence of the r_is.

Accordingly, we introduce the orness of the weighting functions vector for the OMA based mean and variance separately to reflect the beliefs of the investors in the portfolio decisions.

Definition 5 Let the OMA be given by Eq. 2 with the weighting function $f_i : [0, 1] \to [0, 1]$ being nondecreasing for each $i = 1, 2, \ldots, n$ and satisfying $\sum_{i=1}^{n} f_i(x) = x$, then the orness of the weighting functions vector $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$ with the input arguments $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ for OMA based mean and variance are respectively defined by

$$\alpha_*^m(\mathbf{f}, \mathbf{r}) = \sum_{j=1}^{n} \frac{\int_0^1 f_j(x)dx}{\int_0^1 \sum_{i=1}^{n} f_i(x)dx} \cdot \frac{b_j - b_n}{b_1 - b_n}; \tag{12}$$

$$\alpha_*^v(\mathbf{f}, \mathbf{r}) = \sum_{j=1}^{n} \frac{\int_0^1 f_j(x)dx}{\int_0^1 \sum_{i=1}^{n} f_i(x)dx} \cdot \frac{s_j - s_n}{s_1 - s_n} \tag{13}$$

where b_j and s_j are respectively the jth element of the descendingly and ascendingly ordered sequence of the r_is.

In fact, the above two equations can be respectively simplified to

$$\alpha_*^m(\mathbf{f}, \mathbf{r}) = \sum_{j=1}^{n} 2\int_0^1 f_i(x)dx \cdot \frac{b_j - b_n}{b_1 - b_n}. \tag{14}$$

$$\alpha_*^v(\mathbf{f}, \mathbf{r}) = \sum_{j=1}^{n} 2\int_0^1 f_i(x)dx \cdot \frac{s_j - s_n}{s_1 - s_n}. \tag{15}$$

$\alpha_*^m(\mathbf{f}, \mathbf{r})$ is designated as an indicating index of degree of optimism for the return level of an asset while $\alpha_*^v(\mathbf{f}, \mathbf{r})$ is that for the volatility of the asset return.

Moreover, if $f_i(x) = p_i x$ for each $i \in \{1, 2, \ldots, n\}$, then Eqs. 12 and 13 reduce to Eqs. 10 and 11, respectively. Therefore, the ornesses of underlying weighting functions of OMA generalize the orness of the underlying weights of OWA by Eqs. 10 and 11. It can be easily obtained that $0 \le \alpha_*^m(\mathbf{f}, \mathbf{r}) \le 1$ and $0 \le \alpha_*^v(\mathbf{f}, \mathbf{r}) \le 1$.

In what follows, given a fixed parameter θ of the parameterized copula C_θ and an initial generating weighting vector $(p_1, p_2, \ldots, p_n)$, the family of weighting functions can be generated by Eq. 4. In particular, when handling the OMA based mean and variance, the ornesses of the weighting functions $\alpha_*^m(\mathbf{f}, \mathbf{r})$ and $\alpha_*^v(\mathbf{f}, \mathbf{r})$, a measurement of the attitude of an investor on the mean and variance of the return for a portfolio, can be identified. Thereby, in such a portfolio selection by OMA, there is no obstacles to see the compatibility of the portfolio with the specifical attitude of an investor. In general, we can choose an appropriate parameter of the underlying copula to fit the specifical attitude with the reference to the ornesses of the weighting functions.

4 Numerical example for portfolio selection with OMA operator

In this section, we present an example to show the feasibility and validity of the OMA based mean-variance model in the optimal allocation of the assets in a portfolio. In detail, we consider a portfolio for international investment consisting of indices from main economies of the world, in particular, the FTSE-100 in UK, the S&P-500 in USA and the SSE Constituent Index (SEE180) in China and rely on the historical monthly returns[1] from January 2006 to September 2016, a sample with 129 realizations ($i = 1, 2, \ldots, 129.$). The monthly return for the three indices ($k = 1, 2, 3$) are denoted by r_{ik} hereafter.

In financial decision making, the style of the behavior and psychological factors of the investors are of essential importance. The beliefs of the investor to the expectation of the market performance directly affect the performance of the related portfolios. In what follows, we respectively present the related results of the portfolio selection for the international investors with different beliefs for different market scenarios. Specifically, we respectively use $\theta > 0$, $\theta \to 0$ and $\theta < 0$ for the underlying Frank copula C_θ to obtain different OMAs for portfolio selections and thus reflect the optimistic belief, neutral belief and pessimistic belief of the investors. Accordingly, we will use

[1] The data is extracted from WIND (www.wind.com.cn).

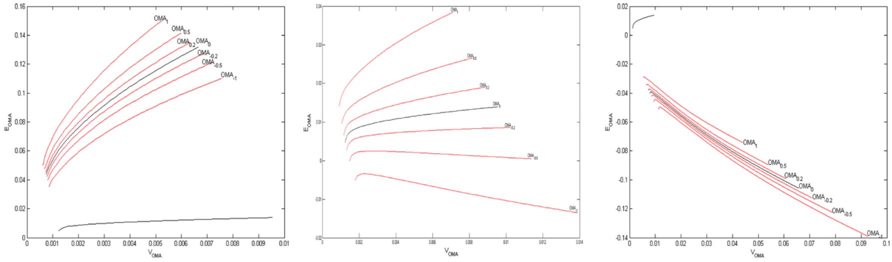

Fig. 1 The OMA/OWA-efficient frontiers with different parameters θ for the investors in expectations of a bull (left), balance (middle) and bear (right) market respectively. OMA$_\theta$ means the efficient frontier is obtained by the OMA operator generated by the Frank copula with parameter θ. OMA$_0$-efficient frontier is the OWA-efficient frontier

the weighting functions $f_i^\theta(x)$ in Eq. 5 and rely upon Eq. 9 to generate the efficient frontiers, which is called OMA-efficient frontiers. In fact, Eq. 9 reduces to

$$\max_{\omega} \frac{1}{\theta}\left(-\ln\left(1+\frac{(e^{-\theta b_1^p}-1)(e^{-\theta p_1}-1)}{e^{-\theta}-1}\right)+\sum_{j=2}^{129}\ln\frac{(e^{-\theta}-1)+(e^{-\theta b_j^p}-1)(e^{-\theta\sum_{i=1}^{j-1}p_i}-1)}{(e^{-\theta}-1)+(e^{-\theta b_j^p}-1)(e^{-\theta\sum_{i=1}^{j}p_i}-1)}\right)$$

$$s.t.\begin{cases}\frac{1}{\theta}\left(-\ln\left(1+\frac{(e^{-\theta t_1^p}-1)(e^{-\theta p_1}-1)}{e^{-\theta}-1}\right)+\sum_{j=2}^{129}\ln\frac{(e^{-\theta}-1)+(e^{-\theta t_j^p}-1)(e^{-\theta\sum_{i=1}^{j-1}p_i}-1)}{(e^{-\theta}-1)+(e^{-\theta t_j^p}-1)(e^{-\theta\sum_{i=1}^{j}p_i}-1)}\right)=V_0\\ \omega_1+\omega_2+\omega_3=1(\omega_1,\omega_2,\omega_3\in[0,1])\end{cases} \tag{16}$$

where $b_j^p(t_j^p)$ is the jth element of the descendingly (ascendingly) ordered sequence of $r_i^p = \sum_{k=1}^3 \omega_k r_{ik}(s_i^p = (r_i^p - \sum_{j=1}^{129} f_j^\theta(b_j^p))^2)$ and $(p_1, p_2, \ldots, p_{129})$ is the generating probability vector.

In what follows, we provide three market scenarios in which the investors have different expectations (i.e., bull market, balance market and bear market), which are represented by different generating probability vectors $(p_1, p_2, \ldots, p_{129})$. By Eq. (16), the optimal $\omega = (\omega_1, \omega_2, \omega_3)$ is derived for each V_0, thus the OMA-efficient frontier can be generated. In particular, the numerical results for each case are obtained by Matlab R2012a(Win7).

4.1 The investors in an expectation of bull market.

Generally, the investors in the bull market are more confident in the historical states with higher returns. Here, we use the 80/20 rule. Namely we assume that 80% of the probability mass are assigned to the 20% states (26 months) with the best returns and the rest to the other 80% states (103 months). For convenience, we structure an initial generating probability vector with the ith element is $p_i = \frac{0.8\lambda^{i-1}(1-\lambda)}{1-\lambda^{26}}$ $(i = 1, 2, \ldots, 26, \lambda = 0.95)$ and each element of the rest is $p_i = \frac{0.2}{103}$ $(27 \le i \le 129)$. Then by Eq. 16, the OMA-efficient frontiers for different θ can be obtained (see the left figure in Fig. 1).

Table 1 The summary of the minimum-risk portfolios with different parameters for the investors in bull market

The parameter θ	The proportions for the allocation	E_{OMA}	V^1_{OMA}	α^m_*	α^v_*
-1	(0.6003, 0.2461, 0.1535)	0.0354	0.0009	0.7880	0.6076
-0.5	(0.5257, 0.2980, 0.1763)	0.0401	0.0008	0.7789	0.6188
-0.2	(0.5209, 0.3023, 0.1768)	0.0425	0.0008	0.7825	0.6237
$\to 0$	(0.5104, 0.3175, 0.1720)	0.0440	0.0007	0.7872	0.6234
0.2	(0.5259, 0.3024, 0.1717)	0.0455	0.0007	0.7909	0.6285
0.5	(0.5514, 0.2838, 0.1648)	0.0476	0.0007	0.7999	0.6335
1	(0.6085, 0.2523, 0.1392)	0.0504	0.0006	0.8243	0.6378

[1] The fact that the OMA based variance is non-increasing w.r.t. the parameter θ can be more directly seen from the left endpoints in Fig. 1

Obviously, we can find the dominance of the OMA-efficient frontiers with different parameters θ. In particular, OMA_0 (i.e., $\theta \to 0$) is an OWA operator and thus the OMA_0-efficient frontier is trivially the OWA-efficient frontier. It can be easily seen that the OMA-efficient frontiers with the larger parameter θ are dominant over these with the smaller ones. Namely, the different choice of the parameter θ as an indicator of beliefs provides a quite different OMA based mean-variance profile of the portfolio. The OMA operator with larger parameters can provide a higher utility for the selection of the portfolio. In particular, in these obtained minimum-risk portfolios with different parameters θ, the underlying OMA based mean is non-decreasing w.r.t. the parameter θ and the underlying OMA based variance is non-increasing w.r.t. the parameter θ (see Table 1). Thus, the investors have much more flexibility in the choice of portfolios tailored to their beliefs by adjusting the parameter θ to a compatible degree with the reference to the ornesses for the OMA based mean and variance, even at the cost of the complexity of the computation in the process.

In such an expectation for bull market from the distribution of probability mass of the states, the OWA based model is dominant over the classical one by Markowitz from the obtained efficient frontiers. In a world of uncertainty, different people may show different beliefs or confidence degrees for the expectation. They can have deeply (optimistically, $\theta > 0$) or indifferently ($\theta = 0$), or conflictingly (pessimistically, $\theta < 0$) cherished beliefs for the expectation. Thus, the parameter θ can capture the difference of beliefs in the same expectation. Eventually, the differences in beliefs lead to different utility profiles in the obtained OMA efficient frontiers.

4.2 The investors in an expectation of balance market.

In this case, assume the probability mass for each states with higher or lower returns is equally allocated, i.e., $p_i = \frac{1}{129}$ ($i = 1, 2, \ldots, 129$). Compared with the left figure in Fig. 1, these OMA-efficient frontiers in in the middle on the whole has smoother slopes due to the market scenario due to the different expectations. In other words, the investors in the expectation for balance market get less marginal returns w.r.t. the risks in the OMA framework than those for the bull market. In this case, using appropriate parameters, we can also get a wider range of portfolios with favorable or unfavorable utility. In particular, from the fact that here OMA_0-efficient frontier is not only an OWA-efficient frontier but also the classical Markowitzian efficient frontier, we can find that these OMA-efficient frontiers with the positive parameters are in dominant position over the OWA-efficient frontier or the classical Markowitzian one. Nevertheless, in the expectation of a balance market, the OMA based framework can also provides a wide variety of dominant efficient portfolios if they hold an optimistic belief for the expectation or technically choose larger positive parameters to overestimate the favorable outcomes. As is seen in the mid-figure in Fig. 1, if the investors are pessimistic in the belief of the expectation, some inefficient portfolios may obtained and thus it is reasonable for them to keep minimum variance portfolios or keep their money in their safes conservatively.

4.3 The investors in an expectation of bear market.

In this part, we use the 20/80 rule, which is the opposite of the case in Sect. 4.1. Specifically, the initial generating probability vector is obtained by ascending reordering of the vector used in Sect. 4.1. That is $p_i = \begin{cases} \frac{0.2}{103}, & 1 \le i \le 103; \\ \frac{0.8\lambda^{129-i}(1-\lambda)}{1-\lambda^{26}}, & 104 \le i \le 129. \end{cases}$ Similarly, we can obtain the OMA-efficient frontiers with different parameters (see the right figure in Fig. 1).

Notably, the above mentioned properties for the minimum risk portfolios still hold. However, some unrealistic OMA efficient frontiers (with the negative yields and the dissatisfaction of nondecreasingness) are obtained with the same parameters to those in Sects. 4.1 and 4.2. In the expectation of the bear market, in general, even the overestimation of the favorable outcomes (the investors with great optimism) can not compensate the negative influences in the expectation of the worse market scenario. Therefore, in such a case, irrespective of the different beliefs for their expectations, people tend to keep their money in the bank or other safes as a hedging for preserving monetary value.

In effect, as a special case of OMA frame, the OWA framework based mean-variance models can capture the different expectations for the market scenarios solely with indifferent beliefs. From the OWA efficient frontiers for each scenario denoted by OMA_0 in Fig. 1, it can be seen that, the OWA framework cannot demonstrate the difference in beliefs in each scenario. However, under uncertainty, it is natural that investors may have quite different beliefs or confidence degrees in an expectation of a particular market scenario. In the Frank copula based OMA framework, specifically, the parameter θ can capture the heterogeneity of beliefs for each expectation. That is to say, the investors may have deeply (optimistically, $\theta > 0$) or indifferently ($\theta = 0$), or conflictingly (pessimistically, $\theta < 0$) cherished beliefs for each expectation. Eventually, the differences in beliefs lead to different utility profiles in the obtained OMA efficient frontiers. To sum up, the OMA framework practically provides a methodology to incorporate the heterogeneous beliefs for each expectation in a world of uncertainty.

5 Concluding remarks

In general, people behave differently when there are some expectations of the world. Even, under uncertainty, people react to each expectation most diversely due to their heterogeneous beliefs of a particular expectation. In order to incorporate the diversity of the investors' beliefs for the expectation of the market in portfolio selection under uncertainty, we are motivated to rely on the probability sensitivity evaluation through the joint transformation of decumulative probabilities and outcomes. To be specific, instead of the ordinary weighted averages, copula based OMAs are employed in the calculation of the mean and variance of the assets' returns, i.e., the so called OMA based mean and variance for the returns, so as to avoid the violation of stochastic dominance while covering the psychological and heuristic intuition to use decumulative probabilities. Through copula based OMAs, we can provide an integrated way to incorporate the cumulative probabilities and the outcomes in the utility. Specially, the marginal contribution of the extreme outcome can be overestimated or underestimated by the adaptive parameter of the underlying copula, which can effectively model the different effect of heterogeneous beliefs of the investors for the future assets' performance. Therefore, in contrast with the OWA, OMA structured by copula and decumulative probabilities can encode not only the decision makers assessment of relative likelihoods but also the confidence attached to such assessment in the evaluation.

By the experimental results in the numerical example, it can be guaranteed that the heterogeneous beliefs for the future performance in the OMA based framework intrinsically affect the profile of the portfolio. Indeed, the obtained OMA-efficient frontiers, which reduce to the OWA-efficient frontiers or the Markowitz-efficient frontier in some particular case, have the property of dominance w.r.t. the underlying parameter θ (an indicator of the beliefs of the investors) of the copula, through which the marginal contribution of the favorable outcome is incorporated in an overestimated or underestimated way. The numerical results show that, for each expectation of the natural state, the higher degree of the optimistic belief in the expectation, the larger marginal return in the OMA based framework can be obtained.

Thus, in virtue of the Frank copula based OMA, by some appropriately choice of the parameter θ, the investors have much more flexibility in the choice of portfolios tailored to their beliefs with the reference to the ornesses for the OMA based mean and variance, even at the cost of the complexity of the computation in the process.

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