A Note on Markman's Example

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This is a note on Markman's secant sheaves on abelian varieties, which will provide an example of semiregular sheaf on abelian 6-fold, and thus guarantee the Hodge conjecture of abelian 4-folds through a degeneration argument. The example is actually constructed on abelian threefold and use the doubling method to obtain a secant$^{\boxtimes 2}$ sheaf on abelian 6-fold. The materials is mainly base on Markman's paper [2], the construction of the example is the main results in Section §8 and will be the main inputs of §9.

The outline of section §8 is as follows, in section §8.1, the secant sheaf on an abelian variety of dimension 3 is constructed via non-hyperelliptic genus 3 and its Jacobian, the sheaf is the ideal of several translations of the image of the curve via the Abel-Jacobi map with a twist of the ample divisor. The number of the translations of the curve is chosen explicitly with respect to the number of the imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$, and the Chern character is computed in detail and has a close relation with the Chern character of the exponential $\exp(\sqrt{-d}\Theta)$, the number and the curves are chosen carefully to make such relation valid. Moreover, such sheaves can always be constructed in any dimension via non-hyperelliptic curves of any genus, while in higher dimensions, the coefficients are more subtle to choose and need more iterations on the coefficients. In addition, more Chern characters need to be computed in higher dimensions, and the computation is not trivial when the dimension rises.

In section §8.2 and the rest of the paper, the attention is focused on the $g = 3$ case, in this case, the deformation of the ideal sheaf F is computed via the indcution on the number of curves, and an isomorphism is obtained to related the same number of copies of the tangent spaces of the abelian variety as the number of curves and the global automorphism of the abelian variety. A detailed analysis of this isomorphism gives the information of the obstruction map of F, two inequality are established to obtain an exact computation of the dimension of the kernel of the obstruction map. One way is easy and follows directly from the injectivity of the deformation map, another way is more subtle, an detailed Yoneda computation is needed to show that the kernel is contained in only one curve, and we have full kwowledge of this case, and thus obtain the other inequality.
In section §8.3, we need to extend the sheaf on one abelian variety of dimension 3 to a external product of two such sheaves on the self product of one abelian variety. The kernel of the obstruction map of the product sheaf is computed with the information of the two sheaf on the two sides. Then we need to transfer the product sheaf to the dual pair of an abelian variety via the Orlov's equivalence. This equivalence transfers the product sheaf to a desired sheaf on the dual pair. The information of the kernel of the obstruction map of the image sheaf is also preserved via the equivalence.
In section §8.4, more details are visited, in particular, it is shown that if we choose the product sheaf carefully, then the diagonal deformation will be transfered to commutatative and gerby deformation of the dual pair, that is to say that no non commutatative deformation will appear. This is important and makes the situation always in the appropriate abelian variety cases. The dimension of the kernel is determined by one part of the product sheaf since the correct choice will make the two sides have the same kernel and the kernel will be sent to the same space and will not decline after the intersection, so the desired sheaf on the dual pair still has the required size of the kernel and this will make sure the deformation will extend in the whole Weil type locus and gives the desired Hodge type properties in all the moduli space.

The main text is a re-typesetting of Markman's paper and some typos are fixed, while some remarks are omitted. The colored boxes like this one are added by the author, which are used as detailed remainder and may contain mistakes, which are due to the author but no one else.

1. Construction of the secant sheaf as an ideal and twist by Theta divisor in genus 3 in detail and sketch in genus 4 and greater;
2. Computation of obstruction map of the secant sheaf and other required properties:
   (a) The deformation map of the ideal sheaf as an isomorphism to deformation of curve and the Jacobian;
   (b) The determination of the rank of the obstruction map by two inequality, first is done using the isomorphism in last step and a induction procedure;
   (c) The detailed explanation of the isomorphismin step 2 as decomposition of $n+1$ pieces which is called the Yoneda product;
   (d) The second inequality using the Yoneda product convey the kernel of one curve into the whole ideal;
3. Two doubling method and computation of required spaces using the known half dimensional results:
   (a) two pieces of the sheaf on the product and then the Orlov's equivalence on the dual pair;
   (b) Diagonal embedding of HT^2 of X into $X \times X$, and the Orlov's equivalence convey the diagonally embedded subspace to the commutatative and gerby deformations and thus the kernel of E contains a desired 9 dimensional subspace which wiil be projects onto the Weil type deformations. This will be crucial to the deformation of the sheaf to the whole Weil type locus.

Recall on Semiregularity

Recall on Atiyah class.

Let F be a coherent sheaf on M. Let $\mathcal{J} \subset \mathcal{O}_{M\times M}$ be the ideal of the diagonal. Consider the extension

$$0 \to \mathcal{J}/\mathcal{J}^2 \cong \Omega^1_M \to \mathcal{O}_{M\times M}/\mathcal{J}^2 \to \mathcal{O}_M \to 0,$$

let p_i be the projections $p_i : X \times X \to X$, tensoring with $p_1^*(F)$ gives

$$0 \to F \otimes \Omega^1_X \to p_1^*(F) \otimes \mathcal{O}_{X\times X}/\mathcal{J}^2 \to F \to 0$$

as an extension of $\mathcal{O}_M$-modules via p_{2*} so that it defines an element

$$at_F \in \mathrm{Ext}^1_M(F, F \otimes \Omega^1_X)$$

which is called the Atiyah class of F.

A universal method Let $\Delta_M \subset M \times M$ be the diagonal and let $\mathcal{M} \subset M \times M$ the first order infinitesimal neighborhood of Δ_M, which means that $\mathcal{M}$ is the subscheme of $M \times M$ with ideal sheaf $\mathcal{I}^2_{\Delta_M}$. Let $\pi_i : \mathcal{M} \to M$ be the two projections and let $\delta : M \to \mathcal{M}$ be the inclusion. Then we have the short exact sequence

$$0 \to \delta_*\Omega_M \to \mathcal{O}_{\mathcal{M}} \xrightarrow{\delta^*} \delta_*\mathcal{O}_M \to 0.$$

Its extension class is called the *universal Atiyah class* and is a morphism

$$at : \delta_*\mathcal{O}_M \to \delta_*\Omega_M[1]$$

in $D^b(M \times M)$, which is regarded as a natural transformation

$$at : id \to \Omega_M[1] \otimes (\bullet)$$

of endofunctors of $D^b(M)$.

Recall F is a coherent sheaf on a d-dimensional complex manifold M, the Atiyah class of F is $at_F \in \mathrm{Ext}^1(F, F \otimes \Omega^1_M)$, the q-th component σ_q of the semiregularity map

$$\sigma = (\sigma_0, \sigma_1, \cdots, \sigma_{d-2}) : \mathrm{Ext}^2(F, F) \to \prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M)$$

is the composition

$$\mathrm{Ext}^2(F, F) \xrightarrow{(at_E)^q/q!} \mathrm{Ext}^{q+2}(F, F \otimes \Omega^q_M) \xrightarrow{Tr} H^{q+2}(M, \Omega^q_M).$$

Semiregular commutative diagram

$$\begin{tikzcd}
H^1(M, TM) \arrow[rr, "at_F"] \arrow[dr, "\lrcorner ch(F)"'] & & \mathrm{Ext}^2(F, F) \arrow[dl, "\sigma"] \\
& \prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M) &
\end{tikzcd}$$

Conjeture:

Let $\pi : M \to S$ be a deformation of a smooth complex projective variety M_0 over s smooth germ $(S, 0)$ and set $M_s := \pi^{-1}(s)$ for $s \in S$. Assume that $\mathcal{B}$ is a semiregular rank r coherent sheaf over M_0, twisted by a cocycle with coefficients in μ_r, such that for all p the class $ch_p(\mathcal{B})$ extends to a horizontal section of $R^{2p}\pi_*\mathbb{Q}$, which belongs to the direct summand $R^p\pi_*\Omega^p_\pi$ under the Hodge decomposition.

Then $\mathcal{B}$ extends to a twisted coherent sheaf over $\pi^{-1}(U)$ for some open analytic neighborhood U of 0 in S.

In section §7, Markman proved this when $\pi : \mathcal{M} \to S$ is a family of connected abelian varieties.

1 Secant sheaves on abelian varieties

The examples of secant sheaves will be ideal sheaves twisted by a line bundle. The ideal sheaf will be of the form $F := \mathcal{I}_{\cup_{i=1}^{d+1} C_i}$ of $d+1$ translates of C_i of the Abel-Jacobi image of a non-hyperelliptic curve C of genus 3 in its Jacobian $X = \mathrm{Pic}^2(C)$.

Let C be a non-hyperelliptic curve of genus 3. Let $X := \mathrm{Pic}^2(C)$ and let $\Theta \subset X$ be the canonical divisor. The natural morphism $C^{(2)} \to \Theta$ is an isomorphism. Note that this morphism is given as $(c, c') \mapsto AJ(c) + AJ(c')$. The morphism is injective and Θ is smooth, by Riemann's Singularity Theorem, and so the morphism is an isomorphism by Zariski's Main Theorem.

Let $AJ : C \to \mathrm{Pic}^1(C)$ be the Abel-Jacobi morphism. Given a point $t \in \mathrm{Pic}^1(C)$, denote by $C_t \subset X$ the translate of $AJ(C)$ by t, which is $t_* AJ(C) = C_t$.

Denote by $[pt] \in H^6(X, \mathbb{Z})$ the class Poincaré dual to a point. Given a subvariety Z of X, denote by $[Z]$ the class in $H^*(X, \mathbb{Z})$ Poincaré dual to Z. Then $[\Theta]^3/3! = [pt]$ and $[C_t] = [\Theta]^2/2$ for any t by Poincaré's formula. Denote $[\Theta]$ by Θ for simplicity.

We now explain the degree choice of the Jacobian variety, actually, the choice is made such that the Serre dual of L is of the same degree of L and thus they all correspond to points on X, or in other words, we require the degree choice of the Jacobian variety X makes X invariant under the Serre duality which becomes an involution on X. Therefore as ω_C has degree $2g_C - 2$, L and $\omega_C \otimes L^{-1}$ have degrees k and $2g_C - 2 - k$, thus $k = g_C - 1$. In particular, when $g = 3$, we require $X = \mathrm{Pic}^2(C)$.

Recall some Poincaré's formulat, Let C be a genus g curve and X be its Jacobian which is a dimension g abelian variety. Let $C^{(k)}$ be the k-th symmetric product of C and the Abel-Jacobi map $AJ : C \to X$ extends to $C^{(k)} \to X$ as the sum of the images. Then let W_k be the class of the image of $C^{(k)}$, then the Theta divisor of X, which is the polarization of X is given by $W_{g-1} = \Theta$, and $W_k = \Theta^{g-k}/(g-k)!$

Let d be a positive integer. Let $\alpha = 1 - \frac{d}{2}\Theta^2$ and $\beta = \Theta - d[pt]$. α and β comes from the expansion of $\exp(\sqrt{-d}\Theta)$, which we recall as follows.

Therefore, α and β are the real part and imaginary part of $\exp(\sqrt{-d}\Theta)$ under the field $K = \mathbb{Q}(\sqrt{-d})$.

$$\begin{aligned}\exp(\sqrt{-d}\Theta) &= 1 + \sqrt{-d}\Theta + \frac{1}{2}(\sqrt{-d}\Theta)^2 + \frac{1}{3!}(\sqrt{-d}\Theta)^3 \\ &= 1 + \sqrt{-d}\Theta - \frac{d}{2}\Theta^2 - d\sqrt{-d}[pt] \\ &= (1 - \frac{d}{2}\Theta^2) + \sqrt{-d}(\Theta - d[pt]) \\ &= \alpha + \sqrt{-d}\beta.\end{aligned}$$

Recall that cup product with $\exp(\sqrt{-d}\Theta)$ is an automorphism of $S_K = H^*(X, K)$, which belong to the image of $m : \mathrm{Spin}(V_K) \to \mathrm{GL}(S_K)$, where m is the extension of $V \to \mathrm{End}(S)$ to $C(V) \to \mathrm{End}(S)$, and the action of $V = H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z})$ on S is given by left multiplication and contraction.

Now $1 \in H^0(X, \mathbb{Z})$ is an even pure spinor since the kernel is the maximal isotropic space $H^1(\hat{X}, \mathbb{Z})$, hence so is $\exp(\sqrt{-d}\Theta) = \exp(\sqrt{-d}\Theta) \cdot 1$ is also an even pure spinor.

Let t_i, $1 \leq i \leq d+1$ be distinct points of $\mathrm{Pic}^1(C)$, and set $C_i = C_{t_i} = (t_i)_* AJ(C) = \tau_{t_i} AJ(C)$.

The first important lemma is a computation on the Chern character which will be shown to be Galois invariant.

Lemma 1. *[2, Lemma 8.1.1] Assume that C_i, $1 \leq i \leq d+1$, are pairwise disjoint. Then the following equality holds.*

$$ch(\mathcal{I}_{\cup_{i=1}^{d+1} C_i} \otimes \mathcal{O}_X(\Theta)) = 1 + \Theta - \frac{d}{2}\Theta^2 - d[pt] = \alpha + \beta.$$

Consequently, $ch(\mathcal{I}_{\cup_{i=1}^{d+1} C_i} \otimes \mathcal{O}_X(\Theta))$ and $ch(\mathcal{R}Hom(\mathcal{I}_{\cup_{i=1}^{d+1} C_i} \otimes \mathcal{O}_X(\Theta), \mathcal{O}_X))$ both belong to the secant line to the spinor variety through the pure spinor $\exp(\sqrt{-d}\Theta)$ and its complex conjugate $\exp(-\sqrt{-d}\Theta)$.

Proof. First note that $ch(\mathcal{O}_{C_i}) = \Theta^2/2 - 2[pt]$, since $\chi(\mathcal{O}_{C_i}) = -2$.

More explanations:

$$0 \to \mathcal{I}_{C_i} \to \mathcal{O}_X \to \mathcal{O}_{C_i} \to 0,$$

$c_1(\mathcal{O}_{C_i}) = [C_i] = \Theta^2/2$, $c_2(\mathcal{O}_{C_i}) = \chi(\mathcal{O}_{C_i}) = 1 - g_C = 1 - 3 = -2$.

Hence, given n disjoint translates C_i of $AJ(C)$, we get

$$ch(\mathcal{I}_{\cup_{i=1}^n C_i}) = 1 - n(\Theta^2/2 - 2[pt]) = 1 - \frac{n}{2}\Theta^2 + 2n[pt].$$

Then

$$\begin{aligned} ch(\mathcal{I}_{\cup_{i=1}^n C_i} \otimes \mathcal{O}_X(k\Theta)) &= (1 - \frac{n}{2}\Theta^2 + 2n[pt])(1 + k\Theta + \frac{k^2}{2}\Theta^2 + k^3[pt]) \\ &= 1 + k\Theta + \frac{k^2 - n}{2}\Theta^2 + (k^3 - 3kn + 2n)[pt]. \end{aligned}$$

Now we take $k = 1$ and $n = d + 1$, then $k^2 - n = -d$ and $k^3 - 3kn + 2n = 1 - (d + 1) = -d$, which gives the desired equality. $\square$

Example 1. [2, Example 8.1.3] When X is an abelian n-fold with $n \geq 4$. Then Lemma 1 can be generalized to produce secant sheaves with a secant line inducing complex multiplication by $\mathbb{Q}(\sqrt{-d})$ as follows.

In order to have higher dimensional secant sheaf, we need to choose the coefficients of all the translation of lower dimension images, from dimension 0 to $g - 2$, and the number of translation of each dimension along with the highest twist number of the Theta divisor.

Let C be a Brill-Noether generic curve of genus n and set $X := \mathrm{Pic}^{n-1}(C)$. Denote by W_k the Abel-Jacobi image of $C^{(k)}$ in $\mathrm{Pic}^k(C)$ and by $[W_k]$ the class of any translate of W_k in X. Then $[W_k] = \Theta^{g-k}/(g-k)!$ again by Poincaré' s formula. Let t_{jk} be generic points in $\mathrm{Pic}^{n-k-1}(C)$. Let the subscheme Z of X be the union

$$Z := \bigcup_{k=0}^{n-2} \bigcup_{j=1}^{a_k} \tau_{t_{jk}}(W_k).$$

Let $F = \mathcal{I}(a_{n-1}\Theta)$ be the tensor product of the ideal sheaf of Z with $\mathcal{O}_X(a_{n-1}\Theta)$. Let α be the real part of $\exp(\sqrt{-d}\Theta)$ and $\sqrt{d}\beta$ its imaginary part.

$$\alpha := 1 - d[W_{n-2}] + d^2[W_{n-4}] + \cdots + (-d)^{n/2}[pt]$$
$$\beta := \Theta - d[W_{n-3}] + \cdots$$

The integers a_k, $0 \leq k \leq n-1$ in the definition of Z should be chosen to satisfy the equation

$$ch(F) = \alpha + a_{n-1}\beta.$$

For example, when $n = 4$ we may assume only the irreducible components of Z, which are translates of W_2, intersect and every such pair intersects at $\Theta^4/4 = 4!/4 = 6$ points. Note that $\chi(\mathcal{O}_{W_1}) = \chi(\mathcal{O}_C) = 1 - g_C = 1 - 4 = -3$ and $\chi(W_2) = 3$.

$\chi(\mathcal{O}_{W_2})$ is computed as follows. First note that W_2 is isomorphic to $C^{(2)}$ when C is non-hyperelliptic, $h^{0,1}(C^{(2)}) = h^{0,1}(C) = 4$ and $h^{0,2}(C^{(2)}) = \binom{4}{2} = 6$ and thus $\chi(\mathcal{O}_{W_2}) = 1 - 4 + 6 = 3$.

Then we get

$$\begin{aligned}
ch(\mathcal{O}_{W_1}) &= \Theta^3/6 - 3[pt] \\
ch(\mathcal{O}_{W_2}) &= \Theta^2/2 - \Theta^3/3 + 3[pt] \text{ (This will be proved later)} \\
ch(\mathcal{O}_Z) &= ch(\bigcup_{k=0}^{n-2}\bigcup_{j=1}^{a_k} \tau_{t_{jk}}(W_k)) = \sum_{k=0}^{2} a_k ch(\mathcal{O}_{W_k}) - 6\binom{a_2}{2}[pt] \\
ch(\mathcal{I}_Z) &= 1 - \frac{a_2}{2}\Theta^2 + \left[\frac{2a_2 - a_1}{6}\right]\Theta^2 + \left[\frac{a_3^3 - 3a_3a_2 + 2a_2 - a_1}{6}\right]\Theta^3 \\
&\quad + \left[a_3^4 - 6a_2a_3^2 + 8a_2a_3 - 4a_1a_3 + 6\binom{a_2}{2} - 3a_2 + 3a_1 - a_0\right][pt] \\
\alpha + a_3\beta &= 1 + a_3\Theta - \frac{d}{2}\Theta^2 - \frac{da_3}{6}\Theta^3 + d^2[pt]
\end{aligned}$$

Comparing coefficients we get $a_2 = d + a_3^2$, $a_1 = 2(d + a_3^2)(1 - a_3)$,

$$a_0 = 6a_3^4 - 6a_3^3 + 8da_3^2 - 6da_3 + 2d^2.$$

We get a secant ideal sheaf tensored with $\mathcal{O}_X(a_3\Theta)$ for every choice of an integer $a_3 \leq 1$. If we choose $a_3 = 1$, then $a_2 = d+1$, $a_1 = 0$ and $a_0 = 2d(d+1)$. Once again we can choose Z to be invariant with respect to a subgroup of X of order $d+1$.

The following lemma is a second computation on the Chern character.

Lemma 2. *We have* $ch(\mathcal{O}_{W_2}) = \Theta^2/2 - \Theta^3/3 + 3[pt]$.

Proof. We have $ch(\mathcal{O}_{W_2}) = \Theta^2/2 + \lambda\Theta^3 + 3[pt]$ by the Poincaré's formula and the computation of $\chi(\mathcal{O}_{W_2})$. Now

$$\chi(\mathcal{O}_{W_2}(-\Theta)) = \int_X (\Theta^2/2 + \lambda\Theta^3 + 3[pt])(1 - \Theta + \Theta^2/2 - \Theta^3/6 + \Theta^4/24) = 9 - 24\lambda,$$

since $\dim H^{3,3}(X, \mathbb{Q})$ is the rank of the Nerón-Severi group of X, which is 1 for generic C.

This computation is based on the special case of C as a $(2,3)$ complete intersection in $\mathbb{P}^3$.

Now assume that C is the intersection of a smooth quadric Q and a cubic in $\mathbb{P}^3$. Note that $2g_C - 2 = \deg K_C = (-4+2+3)\cdot 2\cdot 3 = 6$, and thus $g_C = 4$. So C has two g^1_3's, associated to the two rulings of Q. Note that g^1_3 means a complete linear system of degree 3 and dimension 1, which is $h^0 - 1$. Let $q_1 + q_2 + q_3$ be a reduced divisor in one g^1_3, denote by $\mathcal{L} := \omega_C(-q_1 - q_2 - q_3)$ the other g^1_3, and let $p \in C\backslash\{q_1, q_2, q_3\}$ be a point, such that $|\mathcal{L}(-p)| = \{a+b\}$ for distinct points $a, b \in C\backslash\{p\}$.

We calculate the intersection $W_2 \cap \tau_{p-q_1-q_2}(\Theta)$. The intersection has cohomology class $\Theta^2/2\cdot\Theta = \Theta^3/2$. It consists of the union of three irreducible components of class $\Theta^3/6$ each. As a subset of $X = \mathrm{Pic}^2(C)$, the intersection consists of classes of effective divisors D on C of degree 2, such that $D = q_1 + q_2 - p$ is effective. One irreducible component is $C_1 := \tau_p(AJ(C))$.

The complement of C_1 in the intersection consists of divisors D, such that $h^0(\mathcal{O}_C(D + q_1 + q_2)) = 2$, the inequality $h^0(\mathcal{O}_C(D + q_1 + q_2)) \leq 2$ follows from the assumption that C is not hyperelliptic. Such divisors D satisfy

$$\omega_C(-D') \cong \mathcal{O}_C(D + q_1 + q_2),$$

for some effective divisor D'. So $D + D'$ belongs to $|\omega_C(-q_1 - q_2)|$. If q_3 is in the support of D, then $D = q_3 + q$, where q is any point of C, since $h^0(\mathcal{O}_C(q_1 + q_2 + q_3)) = 2$. Hence, a second irreducible component is $C_2 := \tau_{q_3}(AJ(C))$. The third irreducible component C_3 consists of D, such that $D + q \in |\mathcal{L}|$, for somme $q \in C$. So $C_3 = \tau_{\mathcal{L}}(-AJ(C))$, where $-AJ(C)$ is a curve in $\mathrm{Pic}^{-1}(C)$.

The curves C_1 and C_2 intersect at the point corresponding to the divisor $p + q_3$. The curves C_1 and C_3 intersect ate the two point $\{p + a, p + b\}$. Let P be the plane tangent to Q at q_3. Then $P \cap Q$ consists of two lines through q_3 and $P \cap C = q_1 + q_2 + 2q_3 + a' + b'$, for some points a', b' of C, which we may assume distinct, possibly by changing the choice of the divisor $q_1 + q_2 + q_3$. The curves C_2 and C_3 intersect at two points $\{q_3 + a', q_3 + b'\}$. We conclude that

$$\chi(\mathcal{O}_{W_2 \cap \tau_{p-q_1-q_2}(\Theta)}) = \chi(\mathcal{O}_{C_1}) + \chi(\mathcal{O}_{C_2}) + \chi(\mathcal{O}_{C_3}) - 5 = 3 \cdot (1 - 4) - 5 = -14.$$
$$\chi(\mathcal{O}_{W_2}(-\tau_{p-q_1-q_2}(\Theta))) = \chi(\mathcal{O}_{W_2}) - \chi(\mathcal{O}_{W_2 \cap \tau_{p-q_1-q_2}(\Theta)}) = 3 - (-14) = 17.$$

Thus we get $9 - 24\lambda = 17$ and $\lambda = -1/3$.

Detailed explanations: First note that $W_2 \cap \tau_{p-q_1-q_2}(\Theta)$, D is a component means that D is an effective divisor on C with degree 2 and $D = p - q_1 - q_2 + A$ with A a degree 3 effective divisor since Θ is image of $C^{(3)}$ when $g_C = 4$. Therefore we have $D + q_1 + q_2 - p > 0$. Now note that $\tau_p(AJ(C)) = C_1$ is of the form $p+x$ for $x \in C \cong AJ(C)$ and thus $p+x+q_1+q_2-p = q_1+q_2+x$ is effective, and thus C_1 is one irreducible component.

Now we consider the other components, let D be one of the component in the complement, note that $h^0(\mathcal{O}_C(D+q_1+q_2)) = 2$, first $\deg \mathcal{O}_C(D+q_1+q_2) = 4$ and $\chi(\mathcal{O}_C(D+q_1+q_2)) = 4+(1-4) = 1$. Note that p is a base point of $\mathcal{O}_C(D+q_1+q_2)$ and $D+q_1+q_2-p$ is effective of degree 2 by construction. Therefore $h^0(\mathcal{O}_C(D+q_1+q_2)) = h^0(\mathcal{O}_C(D+q_1+q_2-p)) + 1 \geq 1+1 = 2$. Now we show that $h^0(\mathcal{O}_C(D+q_1+q_2)) \leq 2$, note that $h^1(\mathcal{O}_C(D+q_1+q_2)) = h^0(\omega_C(-D-q_1-q_2))$. $\omega_C(-D-q_1-q_2)$ is of degree $2g_C-2-4 = 2$, thus as C is non-hyperelliptic, $h^0(\omega_C(-D-q_1-q_2)) \neq 2$, then $h^1(\mathcal{O}_C(D+q_1+q_2)) = h^0(\omega_C(-D-q_1-q_2)) \leq 1$. and thus $h^0(\mathcal{O}_C(D+q_1+q_2)) \leq 2$ and thus $h^0(\mathcal{O}_C(D+q_1+q_2)) = 2$.

Note that such D satisfy $\omega_C(-D') \cong \mathcal{O}_C(D+q_1+q_2)$ for some effective divisor D'. $D+D'$ belongs to $|\omega_C(-q_1-q_2)|$. Then notice that $D = q_3 + q$ for any $q \in C$ will satisfy this, since $q_1+q_2+q_3$ is a g_3^1 by assumption. We now find a second irreducible component $C_2 = \tau_{q_3}(AJ(C))$.

For the third component, note that $D+q \in |\mathcal{L}|$ for some $q \in C$. $\omega_C(-D') \cong \mathcal{O}_C(D+q_1+q_2)$ gives $\mathcal{L} = \omega_C(-q_1-q_2-q_3) \cong \mathcal{O}_C(D+D'-q_3)$, $\mathcal{L}$ is effective and if q_3 is not in the support of D, then q_3 must be in the support of D' and thus $\mathcal{L} = D+q$ for some $q \in C$, and $C_3 = \tau_L(-AJ(C))$, which is $D = \mathcal{L} - q$.

$\square$

2 Computation on obstruction map

Back to abelian threefold case. In the remaining part of the section, it is focused on the abelian threefold cases and corresponded two kinds of abelian sixfolds, doubling and dual pair.

Set $F_1 = \mathcal{I}_{\cup_{i=1}^{d+1} C_i} \otimes \mathcal{O}_X(\Theta)$. Let $P = \mathrm{span}\{\alpha, \beta\}$ be the rational $\mathbb{Q}(\sqrt{-d})$-secant plane. Note that $(\alpha, \beta)_S = \int_X \tau(\alpha) \cup \beta = \int_X \alpha \cup \beta = \int_X (1 - \frac{d}{2}\Theta^2)(\Theta - d[pt]) = -\frac{d}{2}\Theta^3 - d = -\frac{d}{2}3! - d = -4d \neq 0$. This satisfies [2, Assumption 2.4.1] which requires that P is non-isotropic. Let h be an ample class in the rank 1 subgroup of $H^2(X \times \hat{X}, \mathbb{Z})^{\mathrm{Spin}(V)_P}$, such an ample class exists due to [2, Proposition 2.4.4].

Proposition 1. *The rank of* $\Phi(F_1 \boxtimes F_1)$ *is non-zero. The* $\mathrm{Spin}(V)_P$*-invariant classes* h^3 *and* $\kappa_3(\Phi(F_1 \boxtimes F_1))$ *are linearly independent.*

Proof. Use notation of [2, Proposition 6.4.1], we have

$$ch(\Phi(F_1 \boxtimes F_1)) = \phi(ch(F_1) \boxtimes ch(F_1)) = \phi'(ch(F_1) \boxtimes \tau(ch(F_1))).$$

Recall that

$$ch(F_1) = F_1 = \mathcal{I}_{\cup_{i=1}^{d+1} C_i} \otimes \mathcal{O}_X(\Theta) = \alpha + \beta,$$

and

$$\lambda_1 = \alpha + \sqrt{-d}\beta.$$

Set $\lambda_1 = \exp(\sqrt{-d}\Theta)$ and $\lambda_2 = \overline{\lambda}_1$, so that

$$ch(F_1) = \frac{1}{2}\left[(\lambda_1 + \lambda_2) + \frac{1}{\sqrt{-d}}(\lambda_1 - \lambda_2)\right] = \frac{1}{2\sqrt{-d}}[(1 + \sqrt{-d})\lambda_1 + (-1 + \sqrt{-d})\lambda_2]$$

Now, τ interchange λ_1 and λ_2.

$$\begin{aligned} ch(F_1) \boxtimes \tau(ch(F_1)) &= \frac{-1}{4d}\left[[(1 + \sqrt{-d})\lambda_1 + (-1 + \sqrt{-d})\lambda_2] \boxtimes [(-1 + \sqrt{-d})\lambda_1 + (1 + \sqrt{-d})\lambda_2]\right] \\ &= \frac{d+1}{4d}[\lambda_1 \boxtimes \lambda_1 + \lambda_2 \boxtimes \lambda_2] + \frac{d-1}{4d}[\lambda_1 \boxtimes \lambda_2 + \lambda_2 \boxtimes \lambda_1] + \frac{\sqrt{-d}}{2d}[\lambda_2 \boxtimes \lambda_1 - \lambda_1 \boxtimes \lambda_2] \end{aligned}$$

The rank of $\Phi(F_1 \boxtimes F_1)$ is non-zero, since the coefficients of $[\lambda_2 \boxtimes \lambda_1 - \lambda_1 \boxtimes \lambda_2]$ is non-zero.

Note that this follows from the fact that $[\lambda_2 \boxtimes \lambda_1 - \lambda_1 \boxtimes \lambda_2]$ has weight 0 as we are in the odd case, which means that it is irreducible in the whole cohomology ring and not in any small pieces and thus must be non torison.

We prove that the classes $\kappa_3(\Phi(F_1 \boxtimes F_1))$ and h^3 are linearly independent. By contradiction, assume otherwise, then $\kappa(\Phi(F_1 \boxtimes F_1))$ belongs to the subring generated by powers of h and is thus $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-$\rho$-invariant, by [2, Lemma 2.2.7]. Hence, $ch(\Phi(F_1 \boxtimes F_1))$ is $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-$\rho'$-invariant. It follows that $ch(F_1) \boxtimes \tau(ch(F_1))$ is invariant under $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$ by the $\mathrm{Spin}(V_K)$-ρ'-equivariance of $\phi \circ (id \otimes \tau)$. But the last two summands $\lambda_1 \boxtimes \lambda_2 + \lambda_2 \boxtimes \lambda_1$ and $\lambda_2 \boxtimes \lambda_1 - \lambda_1 \boxtimes \lambda_2$ in the formula above are $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-invariant, while the first summand is a scalar multiple of $\lambda_1 \boxtimes \lambda_1 + \lambda_2 \boxtimes \lambda_2$ and is not a $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-invariant. Hence $ch(F_1) \boxtimes \tau(ch(F_1))$ is not $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$-invariant, since the coefficient of $\lambda_1 \boxtimes \lambda_1 + \lambda_2 \boxtimes \lambda_2$ is non zero. A contradiction. $\square$

The above proposition actually says that the sheaf constructed use the ideal of curves provides the desired Hodge-Weil class which is not the classes generated by divisors.

The following is the main discussion of the remaining of the section, which provide the crucial isomorphism in the computation of obstruction. This should be considered as the second important statement of this section.

Set $F = \mathcal{I}_{\cup_{i=1}^n C_i}$. Let $U \subset (\mathrm{Pic}^0(C))^n$ be the Zariski open subset of points $(\ell_1, \cdots, \ell_n)$, such that translating each C_i by ℓ_i results in n disjoint curves. We have a natural morphism from U to the Hilbert scheme of X. Explicitly, let Z_i be the product $X^{i-1} \times C_i \times X^{n-i}$, for $1 \le i \le n$. Then F is the pullback of the ideal of the union $\cup_{i=1}^n Z_i$ via the diagonal embedding of X in X^n. $\mathrm{Pic}^0(C)^n$ acts on X^n introducing the desired map from the set U to the Hilbert scheme of X. The symmetric group $\mathfrak{S}_n$ acts freely on U as follows. If $(t_1, t_2, \cdots, t_n) \in \mathrm{Pic}^1(C)^n$ translates $C \times \cdots \times C$ to $C_1 \times C_2 \times \cdots \times C_n$, then U is invariant with respect to the action on $\mathrm{Pic}^1(X)^n$ and the action on U is fixed point free. The connected component of F in the moduli space of simple sheaves on X contains a smooth subscheme is isomorphic to $(U/\mathfrak{S}_n) \times \mathrm{Pic}^0(X)$ obtained by translating the C_i's and tensoring F by line bundles. Hence, we have a canonical injective homomorphism

$$H^0(X, TX)^n \oplus H^1(X, \mathcal{O}_X) \to \mathrm{Ext}^1(F, F)$$

and $\dim \mathrm{Ext}^1(F, F) \ge 3n + 3$.

We explain this injective homomorphism in more details.
First $h^0(X,TX) = h^0(X,\Omega_X^2) = h^{2,0} = 3(3-1)/2 = 3$ and $h^1(X,\mathcal{O}_X) = \binom{3}{1} = 3$ which gives $\dim \mathrm{Ext}^1(F,F) \geq 3n+3$.
The canonical injective homomorphism is the differential of $(U/\mathfrak{S}_n) \times \mathrm{Pic}^0(X)$ which is isomorphic to subscheme contained in the component containing F. We check the three terms appeared in the map as follows:

- Note that $U/\mathfrak{S}_n$ is a free quotient, and $\mathrm{Pic}^0(C)$ is just X and differentiate at 0 gives an element in $H^0(X,TX)$,
- $\mathrm{Pic}^0(X)$ is an component of $H^1(X,\mathcal{O}_X^*)$, differentiate at e gives $H^1(X,\mathcal{O}_X)$, note that $\mathbb{G}_m$ differentiates to $\mathbb{G}_a$,
- and in the Hilbert scheme of X, the tangent space at F is clearly $\mathrm{Ext}^1(F,F)$.

Lemma 3. *The homomorphism is an isomorphism. Consequently,* $\dim \mathrm{Ext}^1(F,F) = 3n+3$.

$$H^0(X,TX)^n \oplus H^1(X,\mathcal{O}_X) \xrightarrow{\cong} \mathrm{Ext}^1(F,F)$$

Proof. It suffices to prove $\dim \mathrm{Ext}^1(F,F) \leq 3n+3$. Set $\mathcal{I}_k = \mathcal{I}_{\cup_{i=1}^k C_i}$. We have the short exact sequence

$$0 \to \mathcal{I}_k \to \mathcal{I}_{k-1} \to \mathcal{O}_{C_n} \to 0$$

and the long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{I}_n,\mathcal{I}_n) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{I}_n,\mathcal{I}_{n-1}) \xrightarrow{0} \mathrm{Hom}(\mathcal{I}_n,\mathcal{O}_{C_n}) \to \mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_n) \to \mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_{n-1})$$

We have the isomorphism $\mathrm{Hom}(\mathcal{I}_n,\mathcal{O}_{C_n}) \cong H^0(C_n, N_{C_n/X})$. The quotient of $H^0(C_n,N_{C_n/X})$ by $H^0(C_n,TX_{|C_n})$ is the kernel of the differential of the Torelli map $H^1(C_n,TC_n) \to H^1(C_n,TX_{|C_n}) \cong H^{0,1}(C_n) \otimes H^{0,1}(C_n)$, and the latter is injective for our non-hyperelliptic curve C_n, by Noether's theorem. Hence $H^0(C_n,N_{C_n/X})$ is 3-dimensional.

Detailed explanations: First note that $\mathrm{Hom}(\mathcal{I}_n,\mathcal{I}_{n-1}) \xrightarrow{0} \mathrm{Hom}(\mathcal{I}_n,\mathcal{O}_{C_n})$, this is due to $\mathcal{O}_{C_n}$ is support on low dimension, but $\mathcal{I}_n,\mathcal{I}_{n-1}$ are locally free sheaves.
The isomorphism $\mathrm{Hom}(\mathcal{I}_n,\mathcal{O}_{C_n}) \cong H^0(C_n, N_{C_n/X})$ is due to the fact that the kernel of $\mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_n) \to \mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_{n-1})$ is the deformation of C_n in X, and is thus $H^0(C_n, N_{C_n/X})$.
The quotient of $H^0(C_n, N_{C_n/X})$ by $H^0(C_n, TX_{|C_n})$ is the kernel of the differential of the Torelli map $H^1(C_n, TC_n) \to H^1(C_n, TX_{|C_n}) \cong H^{0,1}(C_n) \otimes H^{0,1}(C_n)$, since

$$0 \to TC_n \to TX_{|C_n} \to N_{C_n/X} \to 0$$

gives

$$0 \to H^0(C_n, TC_n) \to H^0(C_n, TX_{|C_n}) \to H^0(C_n, N_{C_n/X}) \to H^1(C_n, TC_n) \to H^1(C_n, TX_{|C_n}),$$

note that for a curve $H^0(C_n, TC_n) = H^0(C_n, \Omega^{-1}_{C_n})$, and $\deg(-K) = 2 - 2g_C = 2 - 2*3 = -4 < 0$, which means no global section, and thus

$$0 \to H^0(C_n, TX_{|C_n}) \to H^0(C_n, N_{C_n/X}) \to H^1(C_n, TC_n) \to H^1(C_n, TX_{|C_n}),$$

then note that $H^1(C_n, TC_n) \to H^1(C_n, TX_{|C_n})$ is the differential of the Torelli map, and injective for non-hyperelliptic curve. Recall that $H^1(C_n, TC_n) = H^1(C_n, \Omega^{-1}_{C_n}) = H^0(C_n, \Omega^{\otimes 2}_{C_n}) = 2(2*3-2)+(1-3) = 6$, $H^1(C_n, TX_{|C_n}) \cong H^{0,1}(C_n) \otimes H^{0,1}(C_n) = 3*3 = 9$.
Therefore $H^0(C_n, TX_{|C_n}) \to H^0(C_n, N_{C_n/X})$ is an isomorphism, and $H^0(C_n, TX_{|C_n}) = 3$ as $H^0(X, TX) = H^3(X, \Omega^2_X) = h^{2,3} = h^{2,0} = 3(3-1)/2 = 3$ and hence $H^0(N_{C_n/X}) = 3$.

It remains to prove the inequality $\dim \mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_{n-1}) \leq 3n$. We will prove by induction on n.
When $n = 1$, $\mathrm{Ext}^1(\mathcal{I}_n,\mathcal{I}_{n-1}) \cong H^2(\mathcal{I}_1)^* \cong H^2(X, \mathcal{I}_{C_1})^*$ is 3-dimensional.
Indeed, we have more generally the short exact sequence

$$0 \to \frac{\oplus_{i=1}^n H^1(\mathcal{O}_{C_i})}{H^1(X)} \to H^2(\mathcal{I}_n) \to H^2(\mathcal{O}_X) \to 0,$$

obtained from the long exact sequence of cohomology associated to the short exact sequence $0 \to \mathcal{I}_n \to$

$\mathcal{O}_X \to \oplus_{i=1}^n \mathcal{O}_{C_i} \to 0$ using the injectivity of $H^1(\mathcal{O}_X) \to \oplus_{i=1}^n H^1(C_i, \mathcal{O}_{C_i})$.

$$0 \to H^1(\mathcal{O}_X) \to \oplus_{i=1}^n H^1(\mathcal{O}_{C_i}) \to H^2(\mathcal{I}_n) \to H^2(\mathcal{O}_X) \to \oplus_{i=1}^n H^2(\mathcal{O}_{C_i}),$$

note that $H^2(\mathcal{O}_{C_i}) = 0$ by dimension, thus

$$0 \to H^1(\mathcal{O}_X) \to \oplus_{i=1}^n H^1(\mathcal{O}_{C_i}) \to H^2(\mathcal{I}_n) \to H^2(\mathcal{O}_X) \to 0,$$

then $h^2(\mathcal{I}_1) = 3 - \binom{3}{1} + \binom{3}{2} = 3$.

Assume that $n \geq 2$ and $\dim \operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{I}_{n-2}) = 3(n-1)$. Then $\dim \operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{I}_{n-1}) = 3n$.

Part of the induction, actually.

Consider the long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}(\mathcal{I}_{n-1}, \mathcal{I}_{n-1}) &\xrightarrow{\cong} \operatorname{Hom}(\mathcal{I}_{n-1}, \mathcal{O}_{C_n}) \\ \xrightarrow{0} \operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{I}_n) \longrightarrow \operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{I}_{n-1}) &\longrightarrow \operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{O}_{C_n}) \\ \xrightarrow{\xi} \operatorname{Ext}^2(\mathcal{I}_{n-1}, \mathcal{I}_n) \longrightarrow \operatorname{Ext}^2(\mathcal{I}_{n-1}, \mathcal{I}_{n-1}) &\longrightarrow 0. \end{aligned}$$

The sheaves $\mathcal{E}xt^i(\mathcal{I}_{n-1}, \mathcal{O}_{C_n})$ vanish for $i > 0$.

A possible explanation is that the later is torsion and the former is locally free.

Hence, the local to global spectral sequence computing $\operatorname{Ext}^i(\mathcal{I}_{n-1}, \mathcal{O}_{C_n})$ degenerates at the E_2 term. We get the vanishing of $\operatorname{Ext}^i(\mathcal{I}_{n-1}, \mathcal{O}_{C_n})$ for $i > 1$ and the isomorphism $\operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{O}_{C_n}) \cong H^1(\mathcal{O}_{C_n})$ and the latter is 3 -dimensional. Furthermore the composition

$$H^1(\mathcal{O}_X) \cong H^1(\mathcal{E}nd(\mathcal{I}_{n-1}, \mathcal{I}_{n-1})) \to \operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{I}_{n-1}) \to \operatorname{Ext}^1(\mathcal{I}_{n-1}, \mathcal{O}_{C_n}) \cong H^1(\mathcal{O}_{C_n})$$

is surjective. Hence the connecting homomorphism ξ vanishes and

$$\dim \mathrm{Ext}^1(\mathcal{I}_n, \mathcal{I}_{n-1}) = \dim \mathrm{Ext}^2(\mathcal{I}_{n-1}, \mathcal{I}_n) = \dim \mathrm{Ext}^2(\mathcal{I}_{n-1}, \mathcal{I}_{n-1}) = 3n,$$

where the first equality is by Serre's duality, the second by the vanishing of ξ, and the third was established above via the induction hypothesis. □

The following discussion will start the computation of the obstruction. Recall the two spaces HH and HT, HH is discussed in the following and $HT^k(X) = \oplus_{p+q=k} H^p(X, \wedge^q TX)$.

Let $\Delta_X \subset X \times X$ be the diagonal. A class in the Hochschild cohomology $HH^i(X) = \mathrm{Hom}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}[i])$ corresponds to a natural transformation from the identity endofunctor of $D^b(X)$ to its shift by i. The evaluation homomorphism

$$ev_F : HH^*(X) \to \mathrm{Ext}^*(F, F)$$

is a graded algebra homomorphism. Denote by $ev_F^i : HH^i(X) \to \mathrm{Ext}^i(F, F)$ the restriction of ev_F to $HH^i(X)$.

The second Hochschild cohomology $HH^2(X)$ parametrizes first order deformations of $D^b(X)$. Let

$$ob_F : HH^2(X) \to \mathrm{Ext}^2(F, F)$$

be the homomorphism ev_F^2. The kernel of ob_F parametrizes those deformations along which F deforms to first order. Set $F = \mathcal{I}_{\cup_{i=1}^n C_i}$, where $n \geq 1$. Denote by $ob_F : HT^2(X) \to \mathrm{Ext}^2(F, F)$ also the composition of ob_F with the HKR isomorphism $HT^2(X) = H^2(\mathcal{O}_X) \oplus H^1(TX) \oplus H^0(\wedge^2 TX) \cong HH^2(X)$. Then ob_F is given by contraction with the exponential Atiyah class $\exp(at_E)$.

Semiregular commutative diagram

$$\begin{tikzcd} H^1(M,TM) \arrow[rr, "at_F"] \arrow[dr, "\lrcorner ch(F)"] & & \mathrm{Ext}^2(F,F) \arrow[dl, "\sigma"] \\ & \prod_{q=0}^{d-2} H^{q+2}(M,\Omega_M^q) & \end{tikzcd}$$

can be extended to a diagram starting with the whole Hochschild cohomology

$$\begin{tikzcd} HT^2(M) \arrow[rr, "ob_E"] \arrow[dr, "\lrcorner ch(E)"] & & \mathrm{Ext}^2(E,E) \arrow[dl, "\sigma"] \\ & \prod_{q=0}^{d-2} H^{q+2}(M,\Omega_M^q) & \end{tikzcd}$$

Lemma 4. $\mathrm{rank}(ob_F) \geq 6$.

Proof. Consider the contraction homomorphism

$$H^2(\mathcal{O}_X) \oplus H^1(TX) \oplus H^0(\wedge^2 TX) \xrightarrow{\lrcorner ch(F)} H^2(\mathcal{O}_X) \oplus H^3(\Omega_X^1).$$

It restricts to the first summand as an isomorphism onto the first summand of the codomain, as $ch_0(F) = 1$, and to the second summand as an isomorphism onto the second summand of the codomain, as $ch_2(F) = \frac{-n}{2}\Theta^2$. Hence the above homomorphism is surjective and so its kernel has codimension 6. The kernel of ob_F is contained in the kernel of the above homomorphism, hence ob_F has rank ≥ 6.

Note that $h^2(\mathcal{O}_X) = h^{0,2} = 3(3-1)/2 = 3$,
$h^1(TX) = h^2(X, \Omega_X^2) = h^{2,2} = (3(3-1)/2)^2 = 9$,
$h^0(\wedge^2 TX) = 3(3-1)/2 = 3$,
$h^3(\Omega_X^1) = h^{1,3} = h^{0,2} = 3$.

□

Lemma 5. *If the kernels of ob_E and $\lrcorner ch(E)$ in $HT^2(M)$ are equal and ob_E is surjective, then E is semiregular.*

Proof. The hypothesis imply that there exists a unique map $\sigma' : \mathrm{Ext}^2(E,E) \to \prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M)$, such that $\lrcorner ch(E) = \sigma' \circ ob_E$. The equality $\sigma = \sigma'$ follows from the commutativity of the diagram. □

Remark 1. If we drop the assumption that ob_E is surjective, we still conclude that the semiregularity map restricts to the image of the obstruction map as an injective map.

Note that the hypothesis are invariant under derived equivalences. If $\Phi : D^b(M) \to B^b(M')$ is an equivalence of derived categories, E satisfies the hypothesis, and $\Phi(E)$ is represented by a coherent sheaf E', then E' satisfies the hypothesis and is thus semiregular as well. The space $\prod_{q=0}^{d-2} H^{q+2}(M, \Omega^q_M)$ in the diagrams is the graded summand $H\Omega_{-2}(M)$ of the Hochschild homology $HH_*(M)$. The equivalence Φ induces isomorphisms $\Phi : \mathrm{Ext}^2(E,E) \to \mathrm{Ext}^2(E',E')$ and $\Phi_* : H\Omega_{-2}(M) \to H\Omega_{-2}(M')$. The hypothesis imply that the semiregularity σ' of E' is the conjugate of the semiregularity map σ of E, $\sigma' \circ \Phi = \Phi_* \circ \sigma$. It is natural to expect that the latter equality holds, more generally, without the hypotheses.

Lemma 6. *(1) The algebra* $\mathrm{Ext}^*(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ *is generated by* $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$.

(2) The homomorphism $ev_{\mathcal{I}_{C_j}} : HT^*(X) \to \mathrm{Ext}^*(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ *is surjective and its kernel is the annihilator* $\mathrm{ann}(ch(\mathcal{I}_{C_j}))$ *of the Chern character* $1 - \frac{1}{2}\Theta^2 + 2[pt]$ *of* $\mathcal{I}_{C_j}$.

$$0 \to \mathrm{ann}(ch(\mathcal{I}_{C_j})) \cap HT^2(X) \to \begin{array}{c} H^2(\mathcal{O}_X) \\ \oplus \\ H^1(TX) \\ \oplus \\ H^0(\wedge^2 TX) \end{array} \xrightarrow{\begin{pmatrix} 1 & 0 & -\Theta^2/2 \\ 0 & -\Theta^2/2 & 2[pt] \end{pmatrix}} \begin{array}{c} H^2(\mathcal{O}_X) \\ \oplus \\ H^3(\Omega^1) \end{array} \to 0.$$

(3) The sheaf $\mathcal{I}_{C_j}$ *is semiregular.*

Proof. (1) This is the case $n = 1$. In this case $\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ is 6-dimensional, which is $3n + 3 = 3 * 1 + 3 = 6$, by Lemma. Note that $\mathrm{rank}(ob_F) \geq 6$, which means that $ob_{\mathcal{I}_{C_j}}$ is surjective. Now $HT^*(X)$ is generated by $HT^1(X)$ and $ev_{\mathcal{I}_{C_j}}$ is an algebra homomorphism. Hence the surjectivity of $ev^2_{\mathcal{I}_{C_j}} = ob_{\mathcal{I}_{C_j}}$ implies that the Yoneda product $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \otimes \mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \to \mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ is surjective. The surjectivity of $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \otimes \mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) \to \mathrm{Ext}^3(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ follows from Serre's duality.

(2) The homomorphism $ev^1_{\mathcal{I}_{C_j}} : HT^1(X) \to \mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ is an isomorphism.

$\dim \mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}) = 3 * 1 + 3 = 6$, $\dim HT^1(X) = H^1(\mathcal{O}_X) \oplus H^0(TX) = 3 + 3 = 6$.

Hence, by (1), $ev_{\mathcal{I}_{C_j}}$ is surjective. The inclusion $\mathrm{Ker}(ev_{\mathcal{I}_{C_j}}) \subset \mathrm{ann}(ch(\mathcal{I}_{C_j}))$ follows from [1]. Both ideals are graded (homogeneous). Indeed, the homomorphism $ev_{\mathcal{I}_{C_j}}$ is graded, by definition, and contraction with $ch(\mathcal{I}_{C_j})$ maps $HT^k(X)$ to $\oplus_{q-p=k} H^{p,q}(X)$. The graded summands of $\mathrm{ann}(ch(\mathcal{I}_{C_j}))$ in $HT^0(X)$ and $HT^1(X)$ vanish. The equality of $\mathrm{Ker}(ev^2_{\mathcal{I}_{C_j}})$ and the graded summand of $\mathrm{ann}(ch(\mathcal{I}_{C_j}))$ in $HT^2(X)$ follows from the proof of $\mathrm{rank}(ob_F) \geq 6$. The graded summands of both ideals in $HT^3(X)$ have codimension 1, since $\mathrm{Ext}^3(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ and $H^3(\mathcal{O}_X)$ are both one-dimensional, and the summand $H^3(\mathcal{O}_X)$ of $HT^3(X)$ surjects onto both. Hence the inclusion $\mathrm{Ker}(ev_{\mathcal{I}_{C_j}}) \subset \mathrm{ann}(ch(\mathcal{I}_{C_j}))$ implies the equality of the graded summands in $HT^3(X)$ of both ideals. $HT^k(X)$ is contained in both ideals for $k > 3$ for degree reason.

(3) Follows from (2) and previous lemma.

□

Recall the isomorphism

$$H^0(X, TX)^n \oplus H^1(X, \mathcal{O}_X) \to \mathrm{Ext}^1(F, F),$$

the following will make more use of this isomorphism and decompose it into $n+1$ pieces to obtain the corresponded decomposition of $\mathrm{Ext}^1(F, F)$ and also $\mathrm{Ext}^2(F, F)$ by the perfect pairing between them.

The proof of Lemma 3 identifies the i-th direct summand in the domain with $H^0(C_i, N_{C_i/X})$, $1 \leq i \leq n$.

Note that $H^0(C_i, N_{C_i/X}) \cong H^0(C_i, TX_{|C_i}) \cong H^0(X, TX)$.

Denote this direct summand by $\tilde{E}^1_i$ and set $\tilde{E}^1_0 = H^1(X, \mathcal{O}_X)$, so that the domain is $\oplus_{i=0}^n \tilde{E}^1_i$. Note that each $\tilde{E}^1_i$ is 3-dimensional.

We denote by E_i^1 the image of $\tilde{E}_i^1$ via the isomorphism.

Thus E_i^1 are contained in $\mathrm{Ext}^1(F,F)$.

The Yoneda product $\mathrm{Ext}^1(F,F)\otimes\mathrm{Ext}^2(F,F)\to\mathrm{Ext}^3(F,F)$ is a perfect pairing. Let E_i^2, $1\leq i\leq n$ be the subspace of $\mathrm{Ext}^2(F,F)$ annihilating $E_0^1\oplus\oplus_{j=1,j\neq i}^n E_i^1$. Let E_0^2 be the image of $H^2(\mathcal{O}_X)$ in $\mathrm{Ext}^2(F,F)$. Then E_i^2 is 3-dimensional, for $1\leq i\leq n$. The Yoneda product restricts to $E_0^1\otimes E_0^2\to\mathrm{Ext}^3(F,F)$ as a perfect pairing. Indeed, the algebra homomorphism

$$\iota:\mathrm{Ext}^*(\mathcal{O}_X,\mathcal{O}_X)\to\mathrm{Ext}^*(F,F)$$

is injective, since it composes with the trace linear homomorphism $tr:\mathrm{Ext}^*(F,F)\to\mathrm{Ext}^*(\mathcal{O}_X,\mathcal{O}_X)$ to the identity of $\mathrm{Ext}^*(\mathcal{O}_X,\mathcal{O}_X)$, and the equality $\mathrm{rank}(F)=1$. We get the direct sum decomposition

$$\mathrm{Ext}^2(F,F)=\oplus_{i=0}^n E_i^2.$$

When considering different ideal sheaves $\mathcal{I}_{\cup_{i=1}^n C_i}$, we will denote E_j^i by $E_j^i(\mathcal{I}_{\cup_{i=1}^n C_i})$.

We have the isomorphism $F=\mathcal{I}_{\cup_{i=1}^n C_i}\cong\otimes_{i=1}^n\mathcal{I}_{C_i}$, hence the functor of tensoring with $\otimes_{i=1,i\neq j}^n\mathcal{I}_{C_i}$ induces an algebra homomorphism

$$e_j:\mathrm{Ext}^*(\mathcal{I}_{C_j},\mathcal{I}_{C_j})\to\mathrm{Ext}^*(F,F).$$

The ring structure of the Yoneda algebra $\mathrm{Ext}^*(F,F)$ is determined by the above Lemma and the following Lemma.

Lemma 7. *(1) The Yoneda product maps $E_i^1\otimes E_j^1$ to 0 in $\mathrm{Ext}^2(F,F)$ if $i\neq j$ and $1\leq i,j\leq n$.*

(2) The Yoneda product $\mathrm{Ext}^1(F,F)\otimes\mathrm{Ext}^1(F,F)\to\mathrm{Ext}^2(F,F)$ is anti-symmetric.

(3) The Yoneda product maps $(E_0^1\oplus E_i^1)\otimes(E_0^1\oplus E_i^1)$ surjectively onto $E_0^2\oplus E_i^2$ for all $1\leq i\leq n$. In particular, the algebra $\mathrm{Ext}^*(F,F)$ is generated by $\mathrm{Ext}^1(F,F)$.

(4) The image of e_j is $\mathrm{Hom}(F,F)\oplus(E_0^1\oplus E_j^1)\oplus(E_0^2\oplus E_j^2)\oplus\mathrm{Ext}^3(F,F)$.

(5) The homomorphism e_j maps $E_1^d(\mathcal{I}_{C_j})$ isomorphically onto $E_j^d(F)$ for $d = 1, 2$.

Proof. (1) Let ξ_i be a section of $H^0(N_{C_i/X})$ and $\tilde{\xi}_i$ the corresponding class in E_i^1. Set $R = \mathbb{C}[\epsilon_1, \epsilon_2]/\langle \epsilon_1^2, \epsilon_2^2, \epsilon_1\epsilon_2 \rangle$. The product $\tilde{\xi}_i \circ \tilde{\xi}_j$ vanishes, if and only if there exists a deformation of $\mathcal{I}_{\cup_{i=1}^n C_i}$ by an ideal over $X \times \mathrm{Spec}(R)$, which restricts to the first order deformation along ξ_i over $\mathrm{Spec}(\mathbb{C}[\epsilon_1, \epsilon_2]/\langle \epsilon_1^2, \epsilon_2 \rangle)$ and to the first order deformation along ξ_j over $\mathrm{Spec}(\mathbb{C}[\epsilon_1, \epsilon_2]/\langle \epsilon_1, \epsilon_2^2 \rangle)$. Assume that $i \neq j$. Let $\mathcal{F}$ be the ideal sheaf over $X \times \mathrm{Spec}(R)$ consisting of elements locally of the form $f_0 + f_1\epsilon_1 + f_2\epsilon_2$, where $f_0 \in F = \mathcal{I}_{\cup_{k=1}^n C_k}$, $f_1 \in \mathcal{I}_{\cup_{k=1, k\neq i}^n C_k}$, $f_2 \in \mathcal{I}_{\cup_{k=1, k\neq j}^n C_k}$,

$$(f_1)_{|C_i} + df_0(\xi_i) = 0, \text{ and } (f_2)_{|C_j} + df_0(\xi_j) = 0.$$

One easily checks that $\mathcal{F}$ is indeed an ideal and $\mathcal{F}$ clearly restricts to the ideals of the two desired first order deformations.

(2) The anti-symmetry would follow from that of $(E_0^1 \oplus E_i^1) \otimes (E_0^1 \oplus E_i^1) \to \mathrm{Ext}^2(F, F)$ by (1). Now $E_0^1 \oplus E_i^1$ is equal to $e_i(\mathrm{Ext}^1(\mathcal{I}_{C_i}, \mathcal{I}_{C_i}))$, e_i is an algebra homomorphism, and $\mathrm{Ext}^1(\mathcal{I}_{C_i}, \mathcal{I}_{C_i}) \otimes \mathrm{Ext}^1(\mathcal{I}_{C_i}, \mathcal{I}_{C_i}) \to \mathrm{Ext}^2(\mathcal{I}_{C_i}, \mathcal{I}_{C_i})$ is anti-symmetric, by the anti-symmetry of the product $HT^1(X) \otimes HT^1(X) \to HT^2(X)$ and the surjectivity of $ev_{\mathcal{I}_{C_i}}$.

(3) We prove first that the image of $(E_0^1 \oplus E_i^1) \otimes (E_0^1 \oplus E_i^1)$ is contained in $E_0^2 \oplus E_i^2$. It suffices to prove that the image of $E_i^1 \otimes \mathrm{Ext}^1(F, F)$ is contained in $E_0^2 \oplus E_i^2$. This follows from part (1) of the lemma as $E_i^1 \otimes \mathrm{Ext}^1(F, F)$ annihilates E_j^1 for all $j \neq i$, anti-symmetric is used. Surjectivity would follow from the above Lemma.

(4) We know that e_j maps $\mathrm{Ext}^k(\mathcal{I}_{C_i}, \mathcal{I}_{C_i})$ onto $\mathrm{Ext}^k(F, F)$, for $k = 0$ and $k = 3$. We also know that $e_j(\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})) = E_0^1 \oplus E_j^1$. It remains to show that $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})) = E_0^2 \oplus E_j^2$. Clearly, $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}))$ is contained in the image of $(E_0^1 \oplus E_j^1) \otimes (E_0^1 \oplus E_j^1)$ as e_j is an algebra homomorphism. Hence $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j}))$ is contained in $E_0^2 \oplus E_j^2$, by part (3). Now the fact that the restriction of e_j is injective on $\mathrm{Ext}^1(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ implies that it is also injective on $\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$, since the pairing induced by the Yoneda product of their images in $\mathrm{Ext}^*(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ pulls back to the perfect pairing of the Yoneda product in $\mathrm{Ext}^*(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$, as e_j induces an isomorphism of $\mathrm{Ext}^3(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})$ with $\mathrm{Ext}^3(F, F)$. The equality $e_j(\mathrm{Ext}^2(\mathcal{I}_{C_j}, \mathcal{I}_{C_j})) = E_0^2 \oplus E_j^2$ follows for dimension reasons.

(5) The statement is clear for $d = 1$. For $d = 2$ it follows from part (4) and the fact that e_j is an $H^*(\mathcal{O}_X)$-algebra homomorphism.

□

The following Proposition on the exact rank of the obstruction map is crucial in the later proof of semiregularity and construction of semiregular reflexive sheaf. The proof is already done in one side, and the other side will follow from the above properties of the Yoneda product. This should be considered the third important statement of this section.

Proposition 2. $\operatorname{rank}(ob_F) = 6$.

Proof. We already know that $\operatorname{rank}(ob_F) \geq 6$. It remains to prove that $\operatorname{rank}(ob_F) \leq 6$. If $n = 1$, then $\dim \operatorname{Ext}^2(F, F) = 6$ and so $\operatorname{rank}(ob_F) = 6$.

Denote by $ev_F : HT^*(X) \to \operatorname{Ext}^*(F, F)$ also the composition with the HKR isomorphism $HT^*(X) \cong HH^*(X)$. The HKR isomorphism is an $H^*(\mathcal{O}_X)$-algebra isomorphism, since the Todd class of X vanishes. Hence, the latter ev_F is an $H^*(\mathcal{O}_X)$-algebra homomorphism.

An element $\xi \in HT^1(X)$ decomposes uniquely as the sum $\xi' + \xi''$ with $\xi' \in H^1(\mathcal{O}_X)$ and $\xi'' \in H^0(TX)$. We have the following equalities:

$$ev_F(\xi') = e_j(ev_{\mathcal{I}_{C_j}}(\xi')),$$
$$ev_F(\xi'') = \sum_{j=1}^{n} e_j(ev_{\mathcal{I}_{C_j}}(\xi'')).$$

Note that $e_j \circ ev_{\mathcal{I}_{C_j}}$ is a composition of $H^*(\mathcal{O}_X)$-algebra homomorphisms.

If $j \neq k$, then $e_j(ev_{\mathcal{I}_{C_j}}(\xi_1''))e_k(ev_{\mathcal{I}_{C_k}}(\xi_2'')) = 0$, for every two elements $\xi_1, \xi_2 \in HT^1(X)$ by the structure of Yoneda product.

$$
\begin{aligned}
ev_F(\xi_1\xi_2) &= \left[ev_F(\xi_1') + \sum_{j=1}^{n} e_j(ev_{\mathcal{I}_{C_j}}(\xi_1''))\right]\left[ev_F(\xi_2') + \sum_{j=1}^{n} e_j(ev_{\mathcal{I}_{C_j}}(\xi_2''))\right] \\
&= ev_F(\xi_1'\xi_2') + \sum_{j=1}^{n} e_j(ev_{\mathcal{I}_{C_j}}(\xi_1'\xi_2'' + \xi_1''\xi_2' + \xi_1''\xi_2'')) \\
&= \sum_{j=1}^{n} e_j\left(ev_{\mathcal{I}_{C_j}}\left(\frac{1}{n}\xi_1'\xi_2' + \xi_1'\xi_2'' + \xi_1''\xi_2' + \xi_1''\xi_2''\right)\right).
\end{aligned}
$$

The algebra $HT^*(X)$ is generated $HT^1(X)$. The element $\xi_1'\xi_2'$ belongs to $H^2(\mathcal{O}_X)$, while the element $\xi_1'\xi_2''+\xi_1''\xi_2'+\xi_1''\xi_2''$ belongs to $H^1(TX)\oplus H^0(\wedge^2 TX)$. Thus, the two equations holds also for $\xi \in HT^2(X)$ under the decomposition $\xi = \xi' + \xi''$, with $\xi' \in H^2(\mathcal{O}_X)$ and $\xi'' \in H^1(TX) \oplus H^0(\wedge^2 TX)$.

Note that the previous two equations concern HT^1 and this proves for HT^2.

Let $\tau_{ij} : D^b(X) \to D^b(X)$ be the autoequivalence induced by the translation automorphism mapping C_j to C_i. This autoequivalence acts trivially on $HT^*(X)$ and $ev_{\mathcal{I}_{C_i}} = \tau_{ij} \circ ev_{\mathcal{I}_{C_j}}$. Furthermore, e_j is injective. Hence the kernel of the composition $e_j \circ ev_{\mathcal{I}_{C_j}} : HT^*(X) \to \mathrm{Ext}^*(F, F)$ is independent of j.

The injectivity is the part (4) of the last lemma.

Let $\gamma_n : HT^2(X) \to HT^2(X)$ be the automorphism multiplying the direct summand $H^2(\mathcal{O}_X)$ by n and acting as the identity on $H^1(TX) \oplus H^0(\wedge^2 TX)$. We see that γ_n maps the kernel of $e_j \circ ev_{\mathcal{I}_{C_j}} : HT^*(X) \to \mathrm{Ext}^*(F, F)$ into that of ob_F.

This is the computation of the previous paragraph in this proof, which is to say γ_n inverts the element of the kernel into normal element.

Now e_j is injective and $ev^2_{\mathcal{I}_{C_j}} = ob_{\mathcal{I}_{C_j}}$ has rank 6, by the case $n = 1$. Hence $\mathrm{rank}(ob_F) \leq \mathrm{rank}(ob_{\mathcal{I}_{C_j}}) = 6$. □

The following Lemma will show that the obstruction is not affected by the ample line bundle Θ and only depends on the ideal sheaves, which is a consequence of a general derived equivalence will preserve the kernel of the obstruction map if it is known to be the annihilator of some Chern character.

Lemma 8. *Let $\Phi : D^b(A) \to D^b(B)$ be an equivalence of derived categories of two abelian varieties and F an object of $D^b(A)$. Assume that the kernel of $ob_F : HT^2(A) \to \mathrm{Hom}(F, F[2])$ is equal to the subspce annihilating $ch(F)$. Then the kernel of $ob_{\Phi(F)}$ is equal to the subspace annihilating $ch(\Phi(F))$.*

Proof.

$$\begin{tikzcd} H^*(A,\mathbb{C}) \arrow[d, "\Phi^H"] & HT^2(A) \arrow[l, "ch(F)"'] \arrow[r, "\exp(at_F)"'] \arrow[d, "\Phi^{HT}"] & \mathrm{Hom}(F,F[2]) \arrow[d, "\Phi"] \\ H^*(B,\mathbb{C}) & HT^2(B) \arrow[l, "ch(E)"] \arrow[r, "\exp(at_E)"'] & \mathrm{Hom}(E,E[2]). \end{tikzcd}$$

The commutative diagram where the vertical arrows are isomorphisms and the kernels of the two horizontal arrows in the top row are equal. Hence, the same if for the bottom row. □

Remark 2. The above Lemma holds for more general projective varieties, replacing $ch(F)$ by the Mukai vector $v(F) := ch(F)td(X)^{1/2}$ and factoring the action of $HT^*(X)$ on its module $H^*(X, \mathbb{C})$ by the Duflo operator $D : HT^*(X) \to HT^*(X)$, given by $D(\alpha) = td(X)^{1/2} \lrcorner \alpha$. For abelian varieties $td(X)^{1/2} = 1$.

Corollary 1. *Set $F = \mathcal{I}_{\cup_{i=1}^n C_i}(\Theta)$. The kernel of ob_F is equal to the kernel of the homomorphism $(\bullet)\lrcorner ch(F) : HT^2(X) \to H^*(X, \mathbb{C})$ of contraction with the Chern character of F.*

Proof. By the above Lemma, it suffices to prove the statement for $F' = \mathcal{I}_{\cup_{i=1}^n C_i}$ as F is the image of F'' by the autoequivalence of tensoring by Θ. Now $ch(\mathcal{I}_{\cup_{i=1}^n C_i}) = 1 - \frac{n}{2}\Theta^2 + 2n[pt]$. The kernel of $ob_{F'}$ is contained in the kernel K of $\lrcorner ch(F')$. The kernel of $ob_{F'}$ is 9-dimensional. It suffices to show that $\dim K \leq 9$. Assume that $\dim K > 9$. Then the intersection $K \cap [H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX)]$ would be non-trivial. We claim that $K \cap [H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX)] = 0$. Indeed, contraction with $ch(F')$ induces the homomorphism

$$H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX) \to H^2(\mathcal{O}_X) \oplus H^3(\Omega^1_X)$$

with upper triangular matrix $\begin{pmatrix} 1 & -\frac{n}{2}\Theta^2 \\ 0 & 2n[pt] \end{pmatrix}$, which is invertible.

This matrix is clearly part of $\begin{pmatrix} 1 & 0 & -\Theta^2/2 \\ 0 & -\Theta^2/2 & 2[pt] \end{pmatrix}$ restricted to the $1,3$ columns and the product of n times.

$\square$

3 Secant sheaves over the product

Secant$^{\boxtimes 2}$-sheaves over $X \times \hat{X}$ with a 9-dimensional space of unobstructed commutative-gerby deformation. Conisder a pair F_1 and F_2, then the secant sheaf on the product will be $E = \Phi(\pi_1^* F_1 \otimes \pi_2^* F_2)$ with special choice of the curves.

Let π_i, $i = 1, 2$ be the projections from $X \times X$ to X. Denote by at_F the Atiyah class of F. The Atiyah class $at_{\pi_1^* F} \in \mathrm{Ext}^1(\pi_1^* F, (\pi_1^* F) \otimes \Omega^1_{X \times X})$ of $\pi_1^* F$ is equal to the pushforward of $\pi_1^* at_F \in \mathrm{Ext}^1(\pi_1^* F, \pi_1^*(F \otimes \Omega^1_X))$ via the inclusion of $\pi_1^* \Omega^1_X$ as a direct summand in $\Omega^1_{X \times X}$. Let F_1 and F_2 be the sheaves as $F_1 = \mathcal{I}_{\cup_{i=1}^{d+1} C_i}(\Theta)$ and $F_2 = \mathcal{I}_{\cup_{i=1}^{d+1} \Sigma_i}(\Theta)$, where C_i are translates of $AJ(C)$ and Σ_i are translates of $-AJ(C)$. The Atiyah class of $\pi_1^* F_1 \otimes \pi_2^* F_2$ satisfies

$$at_{\pi_1^* F_1 \otimes \pi_2^* F_2} = at_{\pi_1^* F_1} \otimes 1 + 1 \otimes at_{\pi_2^* F_2}.$$

The Künneth decomposition of $\mathrm{Ext}^2(\pi_1^* F_1 \otimes \pi_2^* F_2, \pi_1^* F_1 \otimes \pi_2^* F_2)$ is the direct sum

$$[\mathrm{Ext}^2(F_1, F_1) \otimes \mathrm{Ext}^0(F_2, F_2)] \oplus [\mathrm{Ext}^0(F_1, F_1) \otimes \mathrm{Ext}^2(F_2, F_2)] \oplus [\mathrm{Ext}^1(F_1, F_1) \otimes \mathrm{Ext}^1(F_2, F_2)].$$

We have the direct sum decomposition of $HT^2(X \times X)$

$$\begin{aligned} HT^2(X \times X) &= \pi_1^* HT^2(X) \otimes \pi_2^* HT^0(X) \\ &\oplus \pi_1^* HT^0(X) \otimes \pi_2^* HT^2(X) \\ &\oplus \pi_1^* HT^1(X) \otimes \pi_2^* HT^1(X) \end{aligned}$$

Note that $\exp(at_F) : H^1(\mathcal{O}_X) \oplus H^0(TX) \to \mathrm{Ext}^1(F,F)$ is injective as $\exp(at_F) : H^1(\mathcal{O}_X) \oplus H^0(TX)^n \to \mathrm{Ext}^1(F,F)$ is an isomorphism for F defined by n curves.

The obstruction map $ob_F : HT^2(X) \to \mathrm{Ext}^2(F,F)$ is the restriction to $HT^2(X)$ of the algebra homomorphism

$$(\bullet) \lrcorner \exp(at_F) : HT^*(X) \to \mathrm{Ext}^*(F,F).$$

We see that the obstruction map $ob_{\pi_1^* F_1 \otimes \pi_2^* F_2}$ maps the summand in the i-th row above into the i-th summand of $\mathrm{Ext}^2(\pi_1^* F_1 \otimes \pi_2^* F_2, \pi_1^* F_1 \otimes \pi_2^* F_2)$ in the decomposition above and $ob_{\pi_1^* F_1 \otimes \pi_2^* F_2}$ restricts as an injective homomorphism to the third summand.

This is to say that the obstruction map respects the decomposition as

$$\begin{aligned} HT^2(X \times X) &\longrightarrow \mathrm{Ext}^2(\pi_1^* F_1 \otimes \pi_2^* F_2, \pi_1^* F_1 \otimes \pi_2^* F_2) \\ \pi_1^* HT^2(X) \otimes \pi_2^* HT^0(X) &\longrightarrow \mathrm{Ext}^2(F_1, F_1) \otimes \mathrm{Ext}^0(F_2, F_2) \\ \pi_1^* HT^0(X) \otimes \pi_2^* HT^2(X) &\longrightarrow \mathrm{Ext}^0(F_1, F_1) \otimes \mathrm{Ext}^2(F_2, F_2) \\ \pi_1^* HT^1(X) \otimes \pi_2^* HT^1(X) &\longrightarrow \mathrm{Ext}^1(F_1, F_1) \otimes \mathrm{Ext}^1(F_2, F_2) \end{aligned}$$

We get

$$\ker(ob_{\pi_1^* F_1 \otimes \pi_2^* F_2}) = \left[\pi_1^* \ker(ob_{F_1}) \otimes \pi_2^* H^0(\mathcal{O}_X)\right] \oplus \left[\pi_1^* H^0(\mathcal{O}_X) \otimes \pi_2^* \ker(ob_{F_2})\right].$$

Note that the third component is injective as we recalled, and the kernel in the first two components clearly appear in one side as the obstruction map, or the atiyah class has the corresponded component 1.

Let $\Phi : D^b(X \times X) \to D^b(X \times \hat{X})$ be the Orlov's equivalence and set

$$E = \Phi(\pi_1^* F_1 \otimes \pi_2^* F_2)$$

This E will be the starting point of the next section of Markman's paper.

Lemma 9. *(1) The kernel of $ob_{\pi_1^* F_1 \otimes \pi_2^* F_2}$ is equal to the subspace of $HT^2(X \times X)$ annihilating $ch(\pi_1^* F_1 \otimes \pi_2^* F_2)$.*

(2) The kernel of ob_E is equal to the subspace of $HT^2(X \times \hat{X})$ annihilating $ch(E)$.

Proof. (1) Let Z be the subspace of $HT^2(X \times X)$ annihilating $ch(\pi_1 * F_1 \otimes \pi_2^* F_2)$. The inclusion $\ker(ob_{\pi_1^* F_1 \otimes \pi_2^* F_2}) \subset Z$ follows from [1]. Note that

$$\ker(ob_{\pi_1^* F_1 \otimes \pi_2^* F_2}) = \left[\pi_1^* \ker(ob_{F_1}) \otimes \pi_2^* H^0(\mathcal{O}_X)\right] \oplus \left[\pi_1^* H^0(\mathcal{O}_X) \otimes \pi_2^* \ker(ob_{F_2})\right],$$

implies that $\ker(ob_{\pi_1^* F_1 \otimes \pi_2^* F_2})$ is contained in

$$\left[\pi_1^* HT^2(X) \otimes \pi_2^* HT^0(X)\right] \oplus \left[\pi_1^* HT^0(X) \otimes \pi_2^* HT^2(X)\right].$$

We have seen that the kernel of ob_{F_i} is equal to the subspace of $HT^2(X)$ annihilating $ch(F_i)$ under the action of $HT^*(X)$ on its module $H^*(X, \mathbb{C})$. In order to prove the inclusion $Z \subset \ker(ob_{\pi_1^* F_1 \otimes \pi_2^* F_2})$, it suffices to prove that Z is contained in

$$\left[\pi_1^* HT^2(X) \otimes \pi_2^* HT^0(X)\right] \oplus \left[\pi_1^* HT^0(X) \otimes \pi_2^* HT^2(X)\right],$$

by the decomposition of the Atiyah class of $\pi_1^* F_1 \otimes \pi_2^* F_2$. The homomorphism

$$(\pi_1^* ch(F_1) \cup \pi_2^* ch(F_2)) \lrcorner (\bullet) : HT^2(X \times X) \to H^*(X \times X, \mathbb{C})$$

maps the third summand $\pi_1^* HT^1(X) \otimes \pi_2^* HT^1(X)$ in the decomposition to a subspace of $H^*(X \times X, \mathbb{C})$ intersecting trivially the sum of the images of the other two summands. Indeed, the first summand is mapped into $\pi_1^* H^*(X, \mathbb{C}) \otimes \pi_2^* ch(F_2)$, the second into $\pi_1^* ch(F_1) \otimes \pi_2^* H^*(X, \mathbb{C})$ and every element in the direct sum of the latter two is the sum of classes of the form $\pi_1^* \alpha \cup \pi_2^* \beta$, where either α or β is a Hodge class. On the other hand, the image of $\pi_1^* HT^1(X) \otimes \pi_2^* HT^1(X)$ is contained in the subspace

$$\pi_1^* \left[\oplus_{q-p=1} H^{p,q}(X)\right] \otimes \pi_2^* \left[\oplus_{q-p=1} H^{p,q}(X)\right].$$

In order to prove that Z is contained in $[\pi_1^* HT^2(X) \otimes \pi_2^* HT^0(X)] \oplus [\pi_1^* HT^0(X) \otimes \pi_2^* HT^2(X)]$, it suffices to prove that the homomorphism $ch(F_i) \lrcorner (\bullet) : HT^1(X) \to H^*(X, \mathbb{C})$ is injective. Indeed, multiplication by $1 = \mathrm{rank}(F_i)$ induces an injective homomorphism from the subspace $H^1(\mathcal{O}_X)$ of $HT^1(X)$ to the subspace $H^1(\mathcal{O}_X)$ of $H^1(X, \mathbb{C})$ and contraction with $ch(F_i) = -\frac{n}{2}\Theta^2$ induces an injective homomorphism from $H^0(TX) \oplus H^{1,2}(X)$.

(2) Apply derived equivalence to $A = X \times X$, $B = X \times \hat{X}$, and $F = \pi_1^* F_1 \otimes \pi_2^* F_2$.

□

4 More on Orlov's isomorphism and the diagonal deformations

Orlov's isomorphism $\Phi^{HT} : HT^2(X \times X) \to HT^2(X \times \hat{X})$ maps diagonal deformations to commutative-gergy ones.

This whole section is the computation to show that the image of $HT^2(X)$ diagonally embedded in $HT^2(X \times X)$ contains no components form the $H^0(\wedge^2 T)$ factor and thus is still commutative and inside the algebraic geometry framework.

Then it is shown that the kernel of the obstruction of the sheaf E still has a subspace with desired dimension which does not drop as we use two pieces of F.

In section §9, Markman will show that this 9 dimensional subspace will projects onto the space tanget to the moduli space of Weil types, which is of dimension $n^2 = 3^2 = 9$.

The algebra $HT^*(X)$ acts on its module $H^*(X, \mathbb{C}) = \oplus H^{p,q}(X)$ and embeds in $\mathrm{End}(H^*(X, \mathbb{C}))$.

Given $\alpha \in HT^*(X)$ denote by $e_\alpha \in \mathrm{End}(H^*(X,\mathbb{C}))$ the corresponding endomorphism. Let τ be the involution. If α is an element of $H^i(\wedge^j TX)$ and x is a class in $H^k(X)$, then $e_\alpha(x)$ belongs to $H^{k+i-j}(X,\mathbb{C})$. Set $t = i - j$. We have

$$(\tau \circ e_\alpha \circ \tau)(x) = (-1)^{\frac{(k+t)(k+t-1)}{2}}(-1)^{\frac{k(k-1)}{2}} e_\alpha(x) = (-1)^{kt+\frac{t(t-1)}{2}} e_\alpha(x).$$

In particular, for k even, we have $(\tau \circ e_\alpha \circ \tau)(x) = (-1)^{\frac{t(t-1)}{2}} e_\alpha(x)$, in particular,

$$(\tau \circ e_\alpha \circ \tau)(ch(F)) = (-1)^{\frac{t(t-1)}{2}} e_\alpha(ch(F)).$$

Let $(\bullet)^* : HT^*(X) \to HT^*(X)$ act on $H^i(\wedge^j TX)$ by multiplication by $(-1)^{\frac{(i-j)(i-j-1)}{2}}$. We get

$$(\tau \circ e_\alpha \circ \tau)(ch(F)) = e_{\alpha^*}(ch(F)).$$

Recall that $\tau(ch(F)) = ch(F^\vee)$. In particular, e_{α^*} annihilates $ch(F^\vee)$, if and only if e_α annihilates $ch(F)$, For $(\alpha, \beta, \gamma) \in HT^2(X) = H^2(\mathcal{O}_X) \oplus H^1(TX) \oplus H^0(\wedge^2 TX)$ we have $(\alpha, \beta, \gamma)^* = (-\alpha, \beta, -\gamma)$.

The diagonal embedding of $HT^2(X)$ in $HT^2(X \times X)$ is given by $\alpha \mapsto \pi_1^*(\alpha) + \pi_2^*(\alpha)$. We let the involution $id \otimes (\bullet)^*$ act on $HT^*(X \times X)$ via the Künneth decomposition of the latter.

Given an equivalence $F : D^b(X) \to D^b(Y)$ of derived categories of two smooth projective varieties X and Y we get the graded ring isomorphism $F^{HT} : HT^*(X) \to HT^*(Y)$. The summand $HT^2(X)$ parametrizes first order deformations of $D^b(X)$ associated to first order deformations of the abelian category of coherent sheaves on X. The summand $HT^1(X)$ is the Lie algebra of the identity component of $\mathrm{Aut}(D^b(X))$, and F^{HT} restricts to the differential of the isomorphism induced by conjugation by F. If $F = f_*$, for an isomorphism $f : X \to Y$, then f_*^{HT} restricts to the summands $H^1(X, \mathcal{O}_X)$ and $H^0(X, TX)$ of $HT^1(X)$ as the homomorphism induced by the direct image functor composed with isomorphism induced by the natural sheaf isomorphisms $f_*\mathcal{O}_X \to \mathcal{O}_Y$ and $df : f_*TX \to TY$. When X is an abelian variety, the equivalence $\Phi_{\mathcal{P}} : D^b(\hat{X}) \to D^b(X)$ with Fourier-Mukai kernel the Poincaré line bundle $\mathcal{P}$ conjugates autoequivalences associated to translation automorphisms to autoequivalences associated with tensorization by line bundles in Pic^0. Hence, $\Phi_{\mathcal{P}}^{HT} : HT^1(\hat{X}) \to HT^1(X)$ maps the Lie subalgebra $H^0(T\hat{X})$ of the subgroup $\hat{X}$ of translations of $\hat{X}$ to the Lie subalgebra $H^1(\mathcal{O}_X)$ of the subgroup $\mathrm{Pic}^0(X)$ and it maps $H^1(\mathcal{O}_{\hat{X}})$ to $H^0(TX)$.

Lemma 10. *The composition* $\Phi^{HT} \circ (id \otimes (\bullet)^*) : HT^2(X \times X) \to HT^2(X \times \hat{X})$ *maps the diagonal* *embedding of $HT^2(X)$ into $H^1(T[X \times \hat{X}]) \oplus H^2(\mathcal{O}_{X\times\hat{X}})$. The image is the direct sum of the graphs of the following three homomorphisms:*

(1) The graph in $\pi_1^* H^1(TX) \oplus \pi_2^*(T\hat{X})$ of the isomorphism $\Psi^{HT}_{\mathcal{P}^{-1}[n]} : H^1(TX) \to H^1(T\hat{X})$.

(2) The graph in $\pi_1^* H^2(\mathcal{O}_X) \oplus H^1(T[X \times \hat{X}])$ of the homomorphism

$$\begin{aligned} &H^2(\mathcal{O}_X) \to H^1(T[X \times \hat{X}]) \\ &\quad \eta_1 \wedge \eta_2 \mapsto \pi_1^*(\eta_1) \wedge \pi_2^*(\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta_2)) - \pi_1^*(\eta_2) \wedge \pi_2^*(\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta_1)), \end{aligned}$$

where $\eta_i \in H^1(\mathcal{O}_X)$ for $i = 1, 2$.

(3) The graph in $\pi_1^* H^2(\mathcal{O}_{\hat{X}}) \oplus H^1(T[X \times \hat{X}])$ of the homomorphism

$$\begin{aligned} &H^2(\mathcal{O}_{\hat{X}}) \to H^1(T[X \times \hat{X}]) \\ &\quad \eta_1 \wedge \eta_2 \mapsto -\pi_1^*(\Phi^{HT}_{\mathcal{P}}(\eta_1)) \wedge \pi_2^*(\eta_2) + \pi_1^*(\Phi^{HT}_{\mathcal{P}}(\eta_2)) \wedge \pi_2^*(\eta_1), \end{aligned}$$

where $\eta_i \in H^1(\mathcal{O}_{\hat{X}})$ for $i = 1, 2$.

In particular, the image of $HT^2(X)$ in $HT^2(X \times \hat{X})$ projects injectively into the direct summand $H^1(T[X \times \hat{X}])$.

Proof. $\Phi^{HT} : HT^*(X \times X) \to HT^*(X \times \hat{X})$ is a graded ring isomorphism. $HT^*(X \times X)$ is generated by $HT^1(X \times X)$. The isomorphism $\Phi^{HT}_{\mathcal{P}} : D^b(\hat{X}) \to D^b(X)$, associated to the Poincaré line bundle, maps $H^1(\mathcal{O}_X)$ to $H^0(TX)$ and $H^0(T\hat{X})$ to $H^1(\mathcal{O}_X)$. Hence, its inverse $\Psi^{HT}_{\mathcal{P}^{-1}[n]}$ maps $H^0(TX)$ to $H^1(\mathcal{O}_X)$ and $H^1(\mathcal{O}_X)$ to $H^0(TX)$.

The ismorphism $(\mu^{-1})_* = \mu^* : D^b(X \times X) \to D^b(X \times X)$ induces the isomorphism

$$(\mu^{-1})^{HT}_* : HT^1(X \times X) \to HT^1(X \times X),$$

where $\mu^{-1}(x, y) = (x - y, y)$.

Given $\xi_1, \xi_2 \in H^0(TX)$, we have

$$\begin{aligned}(\mu^{-1})_*^{HT}(\pi_1^*(\xi_1 \wedge \xi_2)) &= \pi_1^*(\xi_1 \wedge \xi_2).\\ (\mu^{-1})_*^{HT}(\pi_2^*(\xi_1 \wedge \xi_2)) &= [-\pi_1^*\xi_1 + \pi_2^*\xi_1] \wedge [-\pi_1^*\xi_2 + \pi_2^*\xi_2]\\ &= \pi_1^*(\xi_1 \wedge \xi_2) - \pi_1^*\xi_1 \wedge \pi_2^*\xi_2 + \pi_1^*\xi_2 \wedge \pi_2^*\xi_1 + \pi_2^*(\xi_1 \wedge \xi_2).\end{aligned}$$

On the other hand, given $\eta \in H^1(X, \mathcal{O}_X)$,

$$\begin{aligned}\mu^*(\pi_1^*(\eta)) &= (\pi_1 \circ \eta)^*(\eta) = \pi_1^*\eta + \pi_2^*\eta,\\ \mu^*(\pi_2^*(\eta)) &= \pi_2^*\eta.\end{aligned}$$

All the computation is the calculation of this multiplication map.

Hence, given $\xi \in H^0(X, TX)$ and $\eta_1, \eta_2 \in H^1(X, \mathcal{O}_X)$,

$$\begin{aligned}\mu^*(\pi_1^*(\eta_1 \wedge \eta_2)) &= \pi_1^*(\eta_1 \wedge \eta_2) + \pi_1^*\eta_1 \wedge \pi_2^*\eta_2 - \pi_1^*\eta_2 \wedge \pi_2^*\eta_1 + \pi_2^*(\eta_1 \wedge \eta_2),\\ \mu^*(\pi_1^*(\xi \wedge \eta)) &= \pi_1^*\xi \wedge (\pi_1^*\eta + \pi_2^*\eta).\\ \mu^*(\pi_2^*(\xi \wedge \eta)) &= [-\pi_1^*\xi + \pi_2^*\xi] \wedge \pi_2^*\eta = -(\pi_1^*\xi \wedge \pi_2^*\eta) + \pi_2^*(\xi \wedge \eta).\end{aligned}$$

Given $\xi_i \in H^0(TX)$, set $\eta_i = \Psi^{HT}_{\mathcal{P}^{-1}[n]}(\xi_i) \in H^1(\mathcal{O}_{\hat{X}})$, $i = 1, 2$. $(\mu^{-1})_*^{HT}(\pi_2^*(\xi_1 \wedge \xi_2))$ is sent via $(1 \boxtimes \Psi_{\mathcal{P}^{-1}[n]})^{HT}$ to

$$\Phi^{HT}(\pi_2^*(\xi_1 \wedge \xi_2)) = \pi_1^*(\xi_1 \wedge \xi_2) - \pi_1^*\xi_1 \wedge \pi_2^*\eta_2 + \pi_1^*\xi_2 \wedge \pi_2^*\eta_1 + \pi_2^*(\eta_1 \wedge \eta_2).$$

On the other hand, $\Phi^{HT}(\pi_1^*(\xi_1 \wedge \xi_2)) = \pi_1^*(\xi_1 \wedge \xi_2)$. Hence,

$$\Phi^{HT}(-\pi_1^*(\xi_1 \wedge \xi_2) + \pi_2^*(\xi_1 \wedge \xi_2)) = -\pi_1^*\xi_1 \wedge \pi_2^*\eta_2 + \pi_1^*\xi_2 \wedge \pi_2^*\eta_1 + \pi_2^*(\eta_1 \wedge \eta_2).$$

The non-commutatative first order deformation $-\pi_1^*(\xi_1 \wedge \xi_2) + \pi_2^*(\xi_1 \wedge \xi_2)$ of $X \times X$ in $H^0(\wedge^2 T(X \times X))$ is mapped to a commutative-gerby deformation of $X \times \hat{X}$.

Let $\eta'_i \in H^1(\mathcal{O}_X)$ and $\xi'_i \in H^0(T\hat{X})$ satisfy $\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta'_i) = \xi'_i$, $i = 1, 2$. Then

$$\Phi^{HT}(\pi_1^*(\eta'_1 \wedge \eta'_2)) = \pi_1^*(\eta'_1 \wedge \eta'_2) + \pi_1^*\eta'_1 \wedge \pi_2^*\xi'_2 - \pi_1^*\eta'_2 \wedge \pi_2^*\xi'_1 + \pi_2^*(\xi'_1 \wedge \xi'_2).$$
$$\Phi^{HT}(\pi_2^*(\eta'_1 \wedge \eta'_2)) = \pi_2^*(\eta'_1 \wedge \eta'_2).$$

Hence
$$\Phi^{HT}(\pi_1^*(\eta'_1 \wedge \eta'_2) - \pi_2^*(\eta'_1 \wedge \eta'_2)) = \pi_1^*(\eta'_1 \wedge \eta'_2) + \pi_1^*\eta'_1 \wedge \pi_2^*\xi'_2 - \pi_1^*\eta'_2 \wedge \pi_2^*\xi'_1.$$

Let $\eta' \in H^1(\mathcal{O}_X)$ and $\xi' \in H^0(T\hat{X})$ satisfy $\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\eta') = \xi'$. Let $\xi \in H^0(TX)$ and $\eta \in H^1(\mathcal{O}_{\hat{X}})$ satisfy $\Psi^{HT}_{\mathcal{P}^{-1}[n]}(\xi) = \eta$. Then

$$\Phi^{HT}(\pi_1^*(\xi \wedge \eta')) = \pi_1^*(\xi \wedge \eta') + \pi_1^*\xi \wedge \pi_2^*\xi'.$$
$$\Phi^{HT}(\pi_2^*(\xi \wedge \eta')) = -\pi_1^*\xi \wedge \pi_2^*\xi' + \pi_2^*(\eta \wedge \xi').$$
$$\Phi^{HT}(\pi_1^*(\xi \wedge \eta') + \pi_2^*(\xi \wedge \eta')) = \pi_1^*(\xi \wedge \eta') + \pi_2^*(\eta \wedge \xi').$$

So Φ^{HT} maps the diagonal commutative first order deformations of $X \times X$ to commutative first order deformations of $X \times \hat{X}$. Furthermore, it maps the anti-diagonal non-commutative and gerby deformations of $X \times X$ to commutative and gerby deformation of $X \times \hat{X}$. □

Let F_1, F_2 and $E = \Phi(\pi_1^* F_1 \otimes \pi_2^* F_2)$. Note that $ch(F_1) = ch(F_2)$. Hence $\ker(ob_{F_1}) = \ker(ob_{F_2})$.

The following statement should be considered the fourth important statement of this section. This statement will show that the deformation space of the secant sheaf constructed in this section will be able to projects onto the Weil type deformations.

Corollary 2. *The isomorphism* $\Phi^{HT} \circ (id \otimes (\bullet)^*) : HT^2(X \times X) \to HT^2(X \times \hat{X})$ *maps the diagonal embedding of the* 9*-dimensional subspace* $\ker(ob_{F_1})$ *of* $HT^2(X)$ *to a* 9*-dimensional subspace of* $H^1(T[X \times \hat{X}]) \oplus H^2(\mathcal{O}_{X \times \hat{X}})$ *of commutative and gerby deformations in the kernel of* ob_E*.*

Proof. Step 1: We claim that $\ker(ob_{F_i}) = \ker(ob_{F_i^\vee})$. The kernel of ob_{F_i} is the annihilator of $ch(F_i)$, which is the kernel of the following homomorphism.

$$\begin{matrix} H^2(\mathcal{O}_X) \\ \oplus \\ H^1(TX) \\ \oplus \\ H^0(\wedge^2 TX) \end{matrix} \xrightarrow{\begin{pmatrix} 1 & 0 & -(d/2)\Theta^2 \\ 0 & -(d/2)\Theta^2 & -(d/6)\Theta^3 \end{pmatrix}} \begin{matrix} H^2(\mathcal{O}_X) \\ \oplus \\ H^3(\Omega^1) \end{matrix}.$$

The subspace annihilating $ch(F_i^\vee)$ is the kernel of the homomorphism obtained by replacing the above matrix by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -(d/2)\Theta^2 \\ 0 & -(d/2)\Theta^2 & -(d/6)\Theta^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence, it suffices to show that the kernel is the direct sum of the subspace of $HT^1(X)$ annihilating Θ and the subspace of $H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX)$ in the kernel of $(1, -\frac{d}{2}\Theta^2)$. Indeed, both are 9-dimensional, and so it suffices to prove the inclusion of this direct sum in the subspace annihilating $ch(F_i)$. This follows from the fact that (i) the subspace of $HT^1(X)$ annihilating Θ is equal to the subspace annihilating Θ^2, and the subspace of $H^2(\mathcal{O}_X) \oplus H^0(\wedge^2 TX)$ in the kernel of $(1, -\frac{d}{2}\Theta^2)$ is equal to the subspace in the kernel of $(\Theta, -\frac{d}{3}\Theta^3)$. Fact (i) is easy to verify. Fact (ii) follows from the identity $\xi \lrcorner c\Theta^n = (-1)^i nc(\xi \lrcorner \Theta)\Theta^{n-1}$, for $\xi \in H^0(TX)$ and $c \in H^i(\mathcal{O}_X)$, $i \geq 0$, and the observation that $c(\xi \lrcorner \Theta)$ belongs to $H^{i+1}(\mathcal{O}_X)$. Indeed, the identity implies that the homomorphism $(\Theta, -\frac{d}{3}\Theta^3)$ is he composition

$$\begin{matrix} H^2(\mathcal{O}_X) \\ \oplus \\ H^0(\wedge^2 TX) \end{matrix} \xrightarrow{\begin{pmatrix} 1 & -(d/2)\Theta^2 \end{pmatrix}} H^2(\mathcal{O}_X) \xrightarrow{\Theta \cup} H^3(\Omega^1_X)$$

and the cup product with Θ is an injective homomorphism.

Step 2: If e_α annihilates $ch(F_1)$, then it annihilates $ch(F_2)$ and also $ch(F_2^\vee)$, by Step 1, and so e_{α^*}

annihilates $ch(F_2)$. Hence, $e_{\pi_1^*(\alpha)+\pi_2^*(\alpha^*)}$ annihilates $ch(F_1 \boxtimes F_2)$. It follows that $e_{\Phi^{HT}(\pi_1^*(\alpha)+\pi_2^*(\alpha^*))}$ annihilates $ch(E)$. Thus the statement follows from the above lemmas, in particular, the kernel of ob_E is equal to the subspace of $HT^2(X \times \hat{X})$ annihilating $ch(E)$.

$\square$

References

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