# Controllability and Stabilizability

- Point-to-Point Control and Controllability
- The Kalman Matrix and its Relevance
- Controllability Canonical Form (SI) and Normal Form (MI)
- Uncontrollable Modes and the Hautus-Test
- Stabilizability
- Open-loop and State-Feedback Control
- Pole-Placement and Stabilization

## Related Reading

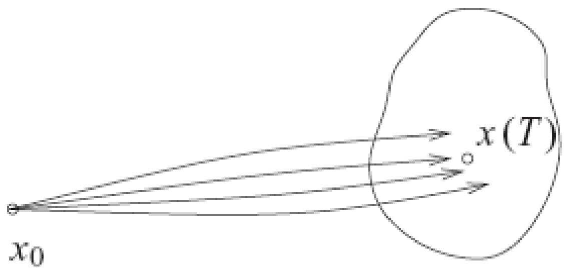
[AM]: Chapters 6.1-6.3 and [F]: Chapters 5.1-5.4, 6.1-6.3, 6.5

# Reachable Set

For a linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

and some given time-instant $T > 0$, we now want to analyze which states can be reached at time $T$ by choosing a suitable control function.



## Reachable Set

Recall that

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau$$

and observe that $e^{AT}x_0$ is not influenced by the control input. It hence suffices to understand which values can be reached with

$$\int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau.$$

This is actually the system response for $u(\cdot)$ and zero initial condition.

The **reachable set** $\mathcal{R}_T$ of $\dot{x} = Ax + Bu$ at time $T > 0$ is the set of all states $x(T)$ that can be reached from initial state zero by any control input.

This trajectory-based definition is due to Kalman. Just from its definition it is not easy to get a handle on the set $\mathcal{R}_T$.

# Recap: Three Facts from Linear Algebra

Let $M \in \mathbb{R}^{n \times p}$ be any rectangular matrix.

1. The **range space** of $M$ is the set of all linear combinations of the columns of $M$. This set is denoted by $R(M)$. It is actually a linear space and equals $\{Mx : \ x \in \mathbb{R}^p\}$.

2. The range space $R(M)$ is $\mathbb{R}^n$ if and only if $M$ has **full row rank**. This means that the rows of $M$ (or the columns of $M^T$) are linearly independent. This can e.g. be verified by showing that

$$z^T M = 0 \ \text{ implies } \ z = 0.$$

3. For **square** matrices $A \in \mathbb{R}^{n \times n}$ and $k \geq 0$, the $k$-th power $A^k$ is a linear combination of the powers up to order $n-1$:

$$I, \ \ A, \ \ A^2, \ \ \ldots, \ \ A^{n-2}, \ \ A^{n-1}.$$

   A proof is not difficult by using the Jordan form of $A$.

## An Observation

Let us choose any $u(\cdot)$. With the matrix exponential note that

$$x(T) = \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau = \lim_{N\to\infty} \underbrace{\int_0^T \sum_{k=0}^{N} \frac{1}{k!}[A(T-\tau)]^k Bu(\tau)\, d\tau}_{x_N}$$

an further observe that

$$x_N = \sum_{k=0}^{N} A^k B \int_0^T \frac{(T-\tau)^k}{k!} u(\tau)\, d\tau.$$

Hence $x_N$ is a linear combination of the columns of $B$, $AB$, ..., $A^N B$. Since each $A^n B$, $A^{n+1}B$, ..., $A^N B$ is a linear combination of the matrices $B$, $AB$, ..., $A^{n-1}B$ (slide 4), we conclude that

$$x_N \text{ is a linear combination of the columns of } B, AB, \ldots, A^{n-1}B.$$

This means $x_N \in R\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$ and with $N \to \infty$ hence

$$x(T) \in R\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}.$$

## Interpretation

This motivates the following definition.

The **controllability matrix** or **Kalman matrix** for the linear system $\dot{x} = Ax + Bu$ or the pair $(A, B)$ is defined by

$$K = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}.$$

In Matlab use `ctrb`.

We have actually proved that any state $x(T)$ that can be reached from zero is contained in the range space of $K$. In short this means

$$\mathcal{R}_T \subset R(K).$$

This is already good information since we can exclude states that, for sure, cannot be reached, namely those outside $R(K)$. However, the really exciting part is one of the most fundamental results in linear control theory: Actually we have **equality**. Let's prove that.

## An Auxiliary Result

The **controllability Gramian** of $(A, B)$ at time $T$ is defined as

$$W_T = \int_0^T e^{At} B B^T e^{A^T t}\, dt = \int_0^T e^{A(T-\tau)} B B^T e^{A^T (T-\tau)}\, d\tau \in \mathbb{R}^{n\times n}.$$

Remember that the elements of $e^{At} B B^T e^{A^T t}$ are linear combinations of terms of the form $t^k e^{\lambda t}$ - one can hence explicitly compute $W_T$.

$W_T$ is symmetric and positive semi-definite, and $R(W_T) = R(K)$.

**Proof (somewhat advanced).** $[B^T e^{A^T t}]^T = e^{At} B$ implies symmetry. If $z \in \mathbb{R}^n$ then $z^T W_T z = \int_0^T z^T e^{At} B B^T e^{A^T t} z\, dt = \int_0^T \|z^T e^{At} B\|^2\, dt \geq 0$ and hence $W_T$ is positive semi-definite. With some math skills we infer

$$z^T W_T = 0 \;\Leftrightarrow\; z^T W_T z = 0 \;\Leftrightarrow\; z^T e^{At} B = 0 \text{ for all } t \in [0, T] \;\Leftrightarrow$$
$$\Leftrightarrow\; z^T A^k B = 0 \text{ for all } k = 0, 1, 2, \ldots \;\Leftrightarrow\; z^T K = 0.$$

Hence the vectors orthogonal to $R(W_T)$ are exactly those which are orthogonal to $R(K)$. This implies $R(W_T) = R(K)$.

## Construction of Control Functions

Take **any** vector $x_f$ in the range space of $K$. As just seen $x_f$ can **always** be written as a linear combination of the columns of $W_T$, say, $x_f = W_T\alpha$ with $\alpha$ collecting the corresponding coefficients. With

$$u(\tau) = B^T e^{A^T(T-\tau)}\alpha$$

we then obtain

$$\int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau = \int_0^T e^{A(T-\tau)}BB^T e^{A^T(T-\tau)}\alpha\,d\tau = W_T\alpha = x_f.$$

In summary:

- The particular control function that is constructed with $\alpha \in \mathbb{R}^n$ satisfying $x_f = W_T\alpha$ hence steers $0$ into the state $x_f$ at time $T$.
- Since $x_f \in R(K)$ was chosen arbitrarily, we can indeed reach each vector in the range space of $K$ by a suitable control function.

## Main Theorems on Controllability

The reachable set $\mathcal{R}_T$ is equal to the range space $R(K)$ of the Kalman matrix $K = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$.

Consequently and notably, $\mathcal{R}_T$ is a **subspace** of $\mathbb{R}^n$ and it is actually **independent from** $T$ as long as $T > 0$.

The particularly important case that all vectors in $\mathbb{R}^n$ are reachable deserves extra attention and has far-reaching consequences.

The linear system $\dot{x} = Ax + Bu$ or the pair $(A, B)$ is said to be **controllable** if $\mathcal{R}_T = \mathbb{R}^n$.

We immediately obtain the celebrated **Kalman-test** for controllability.

The system defined by $(A, B)$ is controllable if and only if the Kalman matrix $K = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$ has full row rank.

# Point-to-Point Control

If we try to reach $x_f \in \mathbb{R}^n$ from a **nonzero** initial condition $x_0 \in \mathbb{R}^n$ we need to find a control function such that

$$x_f = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau.$$

This just means that $x_f - e^{AT}x_0$ equals $\int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau$ and is hence reachable from zero. This in turn translates into $x_f - e^{AT}x_0 \in R(K)$.

The state $x(0) = x_0$ can be controlled into the state $x(T) = x_f$ ($T > 0$) if and only if

$$x_f - e^{AT}x_0 \text{ is contained in the range space of } K.$$

We have also seen how to **construct** suitable control functions.

For **controllable systems**, one can steer **any** initial state $x_0 \in \mathbb{R}^n$ at time $0$ to **any** final state $x_f \in \mathbb{R}^n$ at time $T > 0$.

# Example

Let's consider the example system on p.191 of [F]:

$$A = \begin{pmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix}.$$

The controllability matrix is

$$K = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -2 & 4 & -10 & 28 \\ 2 & -6 & 18 & -54 \\ -1 & 3 & -9 & 27 \end{pmatrix}.$$

It can be written as $K = LU$ (LU-factorization with `lu`) where

$$L = \begin{pmatrix} -0.5 & -0.5 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0.5 & -0.5 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 4 & -10 & 28 \\ 0 & -2 & 8 & -26 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

## Example

- Since the rank of $K$ is **two** we infer that $(A, B)$ is **not** controllable.
- The range space of $K$ is given by all linear combinations of the first **two** columns of $L$:

$$\begin{pmatrix} -0.5 \\ 1 \\ -1 \\ 0.5 \end{pmatrix}, \quad \begin{pmatrix} -0.5 \\ 0 \\ 1 \\ -0.5 \end{pmatrix}.$$

- Choose $u_{e1} = -1$ and $u_{e2} = 1$ and the corresponding state equilibria $x_{e1} = -A^{-1}Bu_{e1}$ and $x_{e2} = -A^{-1}Bu_{e2}$. With $T = 1$ we have

$$x_{e1} = \begin{pmatrix} -1 \\ 1.33 \\ -0.67 \\ 0.33 \end{pmatrix}, \quad x_{e2} = \begin{pmatrix} 1 \\ -1.33 \\ 0.67 \\ -0.33 \end{pmatrix}, \quad x_1 = x_{e2} - e^{AT} x_{e1} = \begin{pmatrix} 1.37 \\ -1.72 \\ 0.7 \\ -0.35 \end{pmatrix}.$$

Since the rank of $(K \ x_1)$ is **two**, $x_1$ is contained in the range space of $K$. Hence we can steer $x_{e1}$ at time $0$ to $x_{e2}$ at time $T = 1$.

# Example

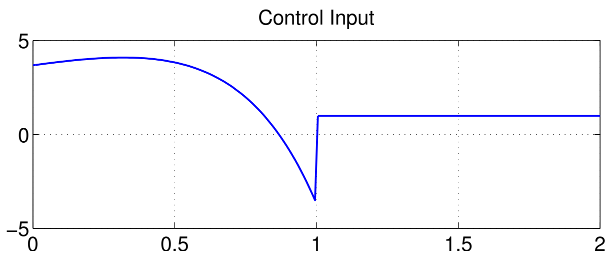
We compute

$$W_T = \begin{pmatrix} 0.43 & -0.68 & 0.49 & -0.25 \\ -0.68 & 1.09 & -0.82 & 0.41 \\ 0.49 & -0.82 & 0.67 & -0.33 \\ -0.25 & 0.41 & -0.33 & 0.17 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} 8.07 \\ -4.08 \\ -7.97 \\ 3.98 \end{pmatrix}$$

such that $W_T \alpha = x_1$. This leads to the control input function

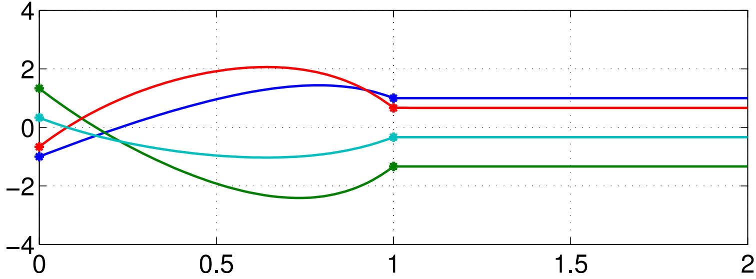
$$u(t) = B^T e^{A^T(T-t)} \alpha = 12.15\, e^{-1+t} - 15.84\, e^{-3+3\,t} \quad \text{for} \quad t \in [0, T].$$

Concatenate this input with the equilibrium input $u(t) = 1$ for $t \in [T, 2]$:



# Example

The system's state-response (with initial and target states) is




### Remarks

- We steer the system state from $x_{e1}$ to $x_{e2}$ and then keep it staying at the equilibrium $x_{e2}$ with the constant control input $u_{e2}$. Since $A$ is Hurwitz, the state does not drift away.
- For non-equilibrium states this is in general not possible!

# Computing Range-Spaces

Given any matrix $M \in \mathbb{R}^{n\times p}$ it is often required to determine **linearly independent columns** that span the range space of $M$.

Here is one procedure:

- Compute an LU-factorization $M = LU$ (Gauss-elimination).
- Then $L$ is square and **invertible**. $U$ has the structure

  $U = \begin{pmatrix} U_1 \\ 0 \end{pmatrix}$ and the rows of $U_1 \in \mathbb{R}^{r\times p}$ are linearly independent.

  The number $r$ of rows of $U_1$ is the **rank** of $M$.
- We can compatibly partition the columns of $L$ as

  $$L = \begin{pmatrix} L_1 & L_2 \end{pmatrix} \text{ where } L_1 \text{ has } r \text{ columns.}$$

  The columns of $L_1$ are linearly independent and span $R(M)$.

In short: $R(L_1) = R(M)$ and $L_1$ has full column rank.

## State-Coordinate Change

Recall that a state-coordinate change $z = Tx$ (with invertible $T$) for

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

leads to the transformed system

$$\begin{aligned} \dot{z} &= \tilde{A}z + \tilde{B}u \\ y &= \tilde{C}z + \tilde{D}u \end{aligned} \quad \text{with} \quad \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{pmatrix}.$$

It is often extremely helpful to find a suitable state-coordinate change such that the transformed system has a "nice description".

This is even more relevant since many system theoretic properties or objects **do not change** under state-coordinate changes, or it is easy to see how they should be transformed.

The Kalman matrices $K$ of $(A, B)$ and $\tilde{K}$ of $(\tilde{A}, \tilde{B})$ are related as $\tilde{K} = TK$. Therefore controllability is invariant under state-coordinate change.

## Single-Input Systems

As seen in Lecture 1 we often encounter single-input systems

$$\dot{x} = \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u = \tilde{A}x + \tilde{B}u.$$

The Kalman matrix $\tilde{K}$ of $(\tilde{A}, \tilde{B})$ is square and equals

$$\begin{pmatrix} 1 & -\alpha_1 & \star & \cdots & \star \\ 0 & 1 & -\alpha_1 & \cdots & \star \\ 0 & 0 & 1 & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{which is invertible.}$$

This leads to the following important result.

Irrespective of $\alpha_1, \ldots, \alpha_n$ the pair $(\tilde{A}, \tilde{B})$ is always controllable.

## Controllable Canonical Form

The matrix $\tilde{A}$ on slide 17 is said to be in **companion form** and $\tilde{B}$ is the **first standard unit vector**. Systems with such a description are said to be in **controllable canonical form**.

This terminology is justified since **any** controllable system with a single input can always be brought into this form by a state-coordinate change:

If $\dot{x} = Ax + Bu$ has only one input $(m = 1)$ and is controllable, there exists a state-coordinate change such that $\dot{z} = [TAT^{-1}]z + [TB]u$ is in controllable canonical form.

If $\tilde{A} = TAT^{-1}$ and $\tilde{A}$ is as on slide 17 recall that

$$\det(\lambda I - A) = \det(\lambda I - \tilde{A}) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n.$$

Hence the controllability canonical form of $(A, B)$ is uniquely determined by the coefficients of the characteristic polynomial of $A$.

# Constructive Proof

We need to find the columns of $T^{-1} = S = \begin{pmatrix} S_1 & \cdots & S_n \end{pmatrix}$ such that

$$B = S \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad AS = S \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \qquad (\star)$$

With the first $n$ relations we can recursively solve for the columns as

$$S_1 = B, \; S_2 = (A + \alpha_1 I)B, \; S_3 = (A^2 + \alpha_1 A + \alpha_2 I)B, \ldots,$$

$$S_n = (A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-1} I)B.$$

The very last equation reads as $AS_n = -\alpha_n S_1$ and is satisfied by the Cayley-Hamilton theorem. Hence the constructed $S$ fulfills $(\star)$.

Clearly $S = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} T_\alpha$ with an upper triangular matrix $T_\alpha$ having ones on the diagonal (slide 40); hence $\det(T_\alpha) \neq 0$. Since $\det(K) \neq 0$ (because $(A, B)$ is controllable), $S$ is invertible.

## Example

Let's change $B$ for the example on slide 11 to $B = \begin{pmatrix} 1 & -2.1 & 2 & -1 \end{pmatrix}^T$. Then $(A, B)$ is controllable. The characteristic polynomial of $A$ is

$$\lambda^4 + 10\lambda^3 + 35\lambda^2 + 50\lambda + 24.$$

We then obtain as in the proof the matrix

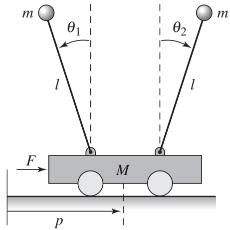
$$S = \begin{pmatrix} 1 & 8.7 & 23.9 & 20.4 \\ -2.1 & -16.7 & -40.8 & -30.4 \\ 2 & 14.2 & 29.2 & 17 \\ -1 & -6.8 & -13.4 & -7.6 \end{pmatrix}.$$

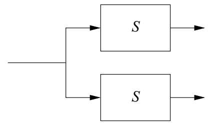
And indeed

$$S^{-1}AS = \begin{pmatrix} -10 & -35 & -50 & -24 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S^{-1}B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $B = \begin{pmatrix} 1 & -2+\epsilon & 2 & -1 \end{pmatrix}^T$, then $T = S^{-1}$ becomes more and more ill-conditioned when $\epsilon$ approaches zero.

# Uncontrollable Systems




Uncontrollability can have many reasons. One situation occurs if two identical controllable systems $\dot{x}_S = A_S x_S + B_S u$ are driven by one input:

$$\dot{x} = Ax + Bu \;\text{ with }\; A = \begin{pmatrix} A_S & 0 \\ 0 & A_S \end{pmatrix}, \quad B = \begin{pmatrix} B_S \\ B_S \end{pmatrix}.$$

The Kalman matrix of $(A, B)$ cannot have full row rank since it equals

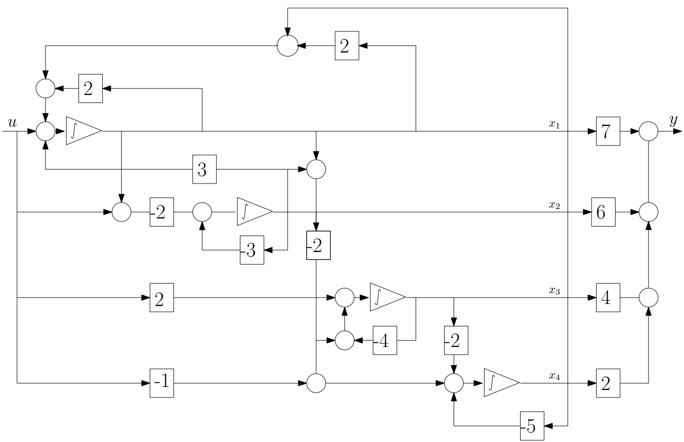
$$\begin{pmatrix} B_S & A_S B_S & \cdots & A_S^{n-1} B_S \\ B_S & A_S B_S & \cdots & A_S^{n-1} B_S \end{pmatrix}.$$

The reachable set of $(A, B)$ actually equals $\left\{ \begin{pmatrix} x \\ x \end{pmatrix} \; : \; x \in \mathbb{R}^n \right\}$.

## Uncontrollable Systems

By interconnecting controllable systems (parallel, series, feedback), controllability may or may not be destroyed.

The matrices on slide 11 results from a state-space description of the following interconnection, that consists of controllable subsystems:



## Uncontrollable Systems

We have seen on slide 15 that we can construct a square and invertible matrix $S \in \mathbb{R}^{n\times n}$ whose first $n_1$ columns span the range space of $K$:

$$S = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \in \mathbb{R}^{n\times(n_1+n_2)} \ \text{ with } \ R(S_1) = R(K) \ \ (n_1 = \mathrm{rank}(K)).$$

If using $S$ as a state-coordinate change for $\dot{x} = Ax + Bu$, we arrive at the following particular structure and properties:

$$\tilde{A} = S^{-1}AS = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad \tilde{B} = S^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

with $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$ and $B_1 \in \mathbb{R}^{n_1\times m}$. Moreover

$$(A_{11}, B_1) \ \text{ is controllable.}$$

We will see that this decomposition into a “controllable subsystem” and dynamics that cannot be influenced by $u$ is extremely insightful.

## Proof of Properties

Since the columns of $S_1$ are linear combinations of those of $K$, the same holds for $AS_1$ by the last property on slide 4. Hence the columns of $AS_1$ are in $R(K)$ and thus also in $R(S_1)$. This implies $AS_1 = S_1 A_{11}$ for some square matrix $A_{11}$ . The columns of $B$ are also in $R(K)$ such that there must exist a matrix $B_1$ with $B = S_1 B_1$. Hence

$$AS_1 = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} A_{11} \\ 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} B_1 \\ 0 \end{pmatrix}.$$

Left-multiplication with $S^{-1}$ leads to the special structure of $(\tilde{A}, \tilde{B})$.

Further the Kalman matrix $\tilde{K}$ of $(\tilde{A}, \tilde{B})$ is

$$\begin{pmatrix} B_1 & A_{11}B_1 & A_{11}^2 B_1 & \cdots & A_{11}^{n-1} B_1 \\ {\color{red}0} & {\color{red}0} & {\color{red}0} & \cdots & {\color{red}0} \end{pmatrix}.$$

Since the rank of $\tilde{K}$ is $n_1 = \text{rank}(K)$ and since the last $n - n_1$ rows are zero, the rank of $\begin{pmatrix} B_1 & A_{11}B_1 & A_{11}^2 B_1 & \cdots & A_{11}^{n-1} B_1 \end{pmatrix}$ must be $n_1$. Therefore $(A_{11}, B_1)$ is controllable.

# Controllability Normal Form

There exists a state coordinate change which transforms the linear system $\dot{x} = Ax + Bu$ into

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u$$

such that $(A_{11}, B_1)$ is controllable. In Matlab use `ctrbf`.

You should learn to read these equations actually as

$$\dot{z}_1 = A_{11} z_1 + A_{12} z_2 + B_1 u, \quad \dot{z}_2 = A_{22} z_2.$$

Hence the evolution of $z_2(t)$ **cannot** be influenced by the control input.

The eigenvalues of $A_{22}$ are called **uncontrollable modes** of $(A, B)$.

The terminology should remind us of the fact that these eigenvalues of the matrix $\tilde{A}$ cannot be modified by control. More later.

## Controllability Normal Form

Recall that we actually have $z_2(t) = e^{A_{22}t} z_2^0$. For any such trajectory we infer from the variation-of-constants formula that

$$z_1(t) = e^{A_{11}t} z_1^0 + \int_0^t e^{A_{11}(t-\tau)} A_{12} z_2(\tau)\, d\tau + \int_0^t e^{A_{11}(t-\tau)} B_1 u(\tau)\, d\tau.$$

or somewhat more explicitly

$$z_1(t) = e^{A_{11}t} \left( z_1^0 + \left[ \int_0^t e^{-A_{11}\tau} A_{12} e^{A_{22}\tau}\, d\tau \right] z_2^0 \right) + \int_0^t e^{A_{11}(t-\tau)} B_1 u(\tau)\, d\tau.$$

This formula allows to argue as on slide 10 that the state $z_1$ can be controlled from any initial point $z_1^0$ at time $0$ to any final point $z_1^f$ at time $T > 0$ (despite the extra "perturbation term" $A_{12} z_2(t)$).

Recall that the original state-trajectory is obtained as $x(t) = Sz(t)$. In summary, we provided an illustration of how to read and argue in terms of the controllability normal form of a linear control system.

# Example

Let's come back to the example of slide 11. The matrix $S = L$ from the LU-factorization of $K$ is actually a transformation matrix that can be used. Indeed one can check that

$$S^{-1}AS = \left(\begin{array}{rr|rr} -2 & 1 & -2 & 0 \\ 1 & -2 & -4 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & -5 \end{array}\right), \quad S^{-1}B = \left(\begin{array}{r} -2 \\ 0 \\ \hline 0 \\ 0 \end{array}\right).$$

Obviously (check!) $\left(\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}\right)$ is controllable.

Moreover the uncontrollable modes of $(A, B)$ are given by

$$\text{eig}\begin{pmatrix} -1 & 1 \\ -3 & -5 \end{pmatrix} = \{-4, -2\}.$$

The other eigenvalues of $A$ are

$$\text{eig}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = \{-1, -3\}.$$

# Hautus-Test for Controllability

If $\lambda$ is an eigenvalue of $A$ then $A - \lambda I$ loses rank. Hence there exists a non-zero complex vector $e$ with

$$e^*(A - \lambda I) = 0 \;\; \text{where} \;\; e^* = \bar{e}^T.$$

Since this reads as $e^* A = \lambda e^*$ we call $e$ a **left-eigenvector** of $A$. In terms of such vectors controllability can be characterized as follows.

The pair $(A, B)$ is controllable if and only if every left-eigenvector $e$ of the matrix $A$ satisfies $e^* B \neq 0$. Equivalently, the matrix

$$\begin{pmatrix} A - \lambda I & B \end{pmatrix} \;\; \text{has full row rank for all} \;\; \lambda \in \mathbb{C}.$$

This so-called **Hautus-test** for controllability has many **equivalent** reformulations. For example, testing uncontrollability requires to search for a left-eigenvector $e$ of $A$ with $e^* B = 0$. Equivalently check whether

$$\begin{pmatrix} A - \lambda I & B \end{pmatrix} \;\; \text{loses row rank for **some** eigenvalue } \lambda \text{ of } A.$$

## Proof

Suppose there exists some $e \neq 0$ with $e^* A = \lambda e^*$ and $e^* B = 0$. Then

$$e^* K = e^* \left( B \;\; AB \;\; \cdots \;\; A^{n-1} B \right) = \left( e^* B \;\; \lambda e^* B \;\; \cdots \;\; \lambda^{n-1} e^* B \right) = 0.$$

Hence $K$ does not have full row rank and thus $(A, B)$ is not controllable.

Conversely suppose $(A, B)$ is not controllable. As on slide 25 we can transform this pair into to the controllability normal form $(\tilde{A}, \tilde{B})$ where $A_{22}$ is not the empty matrix. We can hence determine some $\lambda \in \mathbb{C}$ and $e_2 \neq 0$ with $e_2^*(A_{22} - \lambda I) = 0$. Let us then define $\tilde{e} = \left( 0 \;\; e_2^* \right)^*$. Then

$$\tilde{e}^* \left( \tilde{A} - \lambda I \,\middle|\, \tilde{B} \right) = \left( 0 \;\; e_2^* \right) \left( \begin{array}{cc|c} A_{11} - \lambda I & A_{12} & B_1 \\ \color{red}{0} & A_{22} - \lambda I & \color{red}{0} \end{array} \right) = 0.$$

With $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TB$ we get

$$\tilde{e}^* T \left( A - \lambda I \,\middle|\, B \right) \operatorname{diag}(T^{-1}, I) = 0.$$

Since $\tilde{e}^* T \neq 0$ and since $\operatorname{diag}(T^{-1}, I)$ is invertible we can conclude that $\left( A - \lambda I \,\middle|\, B \right)$ does not have full row rank.

## Example

For $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and with $e^* = \begin{pmatrix} p_1 & p_2 \end{pmatrix}$ consider

$$\begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \lambda p_1 & \lambda p_2 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 0.$$

The last relation implies $p_1 = 0$. Then the first relation reads as

$$\begin{pmatrix} 0 & p_2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & \lambda p_2 \end{pmatrix}.$$

The first equation implies $p_2 = 0$. Hence $(A, B)$ is controllable.

With **Matlab** one can compute the pair-wise different eigenvalues $\lambda_1, \ldots, \lambda_p$ of $A$. If all the matrices

$$\begin{pmatrix} A - \lambda_l I & B \end{pmatrix}, \quad l = 1, \ldots, p,$$

have full row rank then $(A, B)$ is controllable. If at least one of them loses rank then $(A, B)$ is not controllable. Uncontrollable are exactly those eigenvalues $\lambda_l$ for which rank-loss occurs.

## Summary

Every system $\dot{x} = Ax + Bu$ can be transformed by state-coordinate change into the controllability normal form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u, \quad (A_{11}, B_1) \text{ is controllable.}$$

- Controllability of $(A_{11}, B_1)$ means that $\begin{pmatrix} B_1 & A_{11}B_1 & \cdots & A_{11}^{n-1}B_1 \end{pmatrix}$ has full row rank. Equivalently, $\begin{pmatrix} A_{11} - \lambda I & B_1 \end{pmatrix}$ has full row rank for all complex numbers $\lambda \in \mathbb{C}$.

- The matrix $\begin{pmatrix} A - \lambda I & B \end{pmatrix}$ loses rank at $\lambda \in \mathbb{C}$ if and only if

$$\left(\begin{array}{cc|c} A_{11} - \lambda I & A_{12} & B_1 \\ 0 & A_{22} - \lambda I & 0 \end{array}\right) \quad \text{loses rank at} \quad \lambda \in \mathbb{C}$$

  if and only if $\lambda \in \mathbb{C}$ is an eigenvalue of $A_{22}$. Hence the uncontrollable modes of $(A, B)$ are exactly those complex numbers $\lambda \in \mathbb{C}$ for which $\begin{pmatrix} A - \lambda I & B \end{pmatrix}$ loses rank.

## Stabilizability

Controllability implies that each initial state of the linear system

$$\dot{x} = Ax + Bu$$

can be steered to $0$ in a finite time-interval. If we only require this to happen **asymptotically** for $t \to \infty$, we arrive at the following concept.

The linear system is **stabilizable** if for each initial state $x_0 \in \mathbb{R}^n$ there exists a control input $u(t)$ for $t \geq 0$ such that the solution of $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$ satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Construction of such control functions (for each $x_0$) solves the task of stabilization, which is fundamental in control.

**Controllable systems are stabilizable**: Steer the system to zero at any time $T > 0$ and keep it there with $u(t) = 0$ for $t \geq T$.

## Hautus-Test for Stabilizability

The system $\dot{x} = Ax + Bu$ is stabilizable iff all uncontrollable modes are contained in the open left-half complex plane. Equivalently

$$\begin{pmatrix} A - \lambda I & B \end{pmatrix} \text{ has full row rank for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}(\lambda) \geq 0.$$

We can assume without loss of generality that the system is transformed into the controllability normal form slide 25. Then the formulated property is equivalent to the fact that $A_{22}$ is Hurwitz.

**Proof.** If $A_{22}$ is Hurwitz we infer that $z_2(t) \to 0$ for $t \to \infty$ (irrespective of the initial condition) and that $z_1(t)$ can be steered exactly to zero and kept there (since $(A_{11}, B_1)$ is controllable).

If $A_{22}$ is not Hurwitz, we find an initial condition $z_2(0)$ such that $z_2(t)$ does not converge to $0$ for $t \to \infty$. This behavior cannot be influenced by the control input. Hence the system is not stabilizable.

## Examples

- The system on slide 11 is not controllable but stabilizable. This just follows from the fact that the uncontrollable modes are $\{-4, -2\}$.

- If $A$ is Hurwitz then $\dot{x} = Ax + Bu$ is stabilizable. $u(t) = 0$ proves it.

- If $\mathrm{eig}(A) = \{\lambda_1, \ldots, \lambda_p\}$ then check whether $\begin{pmatrix} A - \lambda_l I & B \end{pmatrix}$ has full row rank for all $\lambda_l$ with **non-negative real part**.

- For $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and with $e^* = \begin{pmatrix} p_1 & p_2 \end{pmatrix}$ consider

$$\begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \lambda p_1 & \lambda p_2 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

  The last equation shows $p_2 = -p_1$. The first relation then implies

$$p_1 \begin{pmatrix} 1 & -1 \end{pmatrix} = p_1 \lambda \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

  This holds for $p_1 = 1$ and $\lambda = 1$. Hence $(A, B)$ is **not** stabilizable.

# Open-Loop Control

So-far we have discussed so-called **open-loop control** strategies. They are realized through an a priori given time-function $u(t)$ for $t \geq 0$ with which the system is steered. The controlled system is described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

Here are some disadvantages of this approach:

- Different initial conditions require their individual choice of control functions that fulfill the respective task. The control functions need to be "manually" adjusted to the respective initial state.
- Future unforeseen events are not dealt with. Such strategies are pre-planned and do not take situation into account in which the system "does not behave as expected". They are inherently non-robust.

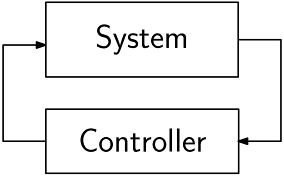
This motivates to look for alternative control strategies.

# Feedback Control

A **feedback controller** receives information **from** the system, processes this information and generates a control signal that goes **back** into the system in order to actuate it.

The control action is hence adjusted "automatically" on the basis of on-line measured information about the system.

Such a feedback mechanism is often illustrated by the block-diagram




A controller should be seen as a to-be-constructed dynamical system such that the **feedback interconnection** with the given to-be-controlled system obeys a certain desired specification.

## State-Feedback Control

In a particularly simple but very important case it is assumed that the whole state of the system can be measured on-line and that the control law is just a (static) gain.

A **linear state-feedback** controller with gain $F$ is defined as

$$u = -Fx.$$

For a linear system $\dot{x} = Ax + Bu$ this control law leads to the **controlled** or **closed-loop** system

$$\dot{x} = Ax - BFx = (A - BF)x.$$

- The controller actually **changes the system dynamics** from open-loop $\dot{x} = Ax$ to closed-loop $\dot{x} = (A - BF)x$.
- Another interpretation: At time $t$ it takes the measured $x(t)$ and generates the control action as $u(t) = -Fx(t)$ by linear combinations.

## Pole-Placement

After this introduction let us now investigate how the dynamics of

$$\dot{x} = (A - BF)x$$

can be influenced by suitable choices of $F$. For this purpose remember from Lecture 2 that the modes of the system, the eigenvalues of $A - BF$, shape the dynamic system behavior.

It is a very surprising fact that controllable systems allow to assign these modes arbitrarily. This is the celebrated **pole-placement theorem**.

Let $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ be controllable. If the complex numbers $\lambda_1, \ldots, \lambda_n$ (with possible repetitions) are located symmetrically with respect to the real axis, there exists a real matrix $F \in \mathbb{R}^{m\times n}$ such that the set of eigenvalues of $A - BF$ exactly equals $\{\lambda_1, \ldots, \lambda_n\}$.

The Matlab command for achieving pole-placement is `place`.

## Constructive Proof for Single-Input Systems

If $(A,B)$ is controllable we can find an invertible $T$ such that $(\tilde{A},\tilde{B}) = (TAT^{-1},TB)$ admits the controllable canonical form on slide 17. With $\tilde{F} = \begin{pmatrix} \tilde{f}_1 & \cdots & \tilde{f}_n \end{pmatrix}$ we then infer that

$$\tilde{A}-\tilde{B}\tilde{F} = \begin{pmatrix} -\alpha_1-\tilde{f}_1 & -\alpha_2-\tilde{f}_2 & -\alpha_3-\tilde{f}_3 & \cdots & -\alpha_n-\tilde{f}_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

which has the characteristic polynomial

$$\det(sI-(\tilde{A}-\tilde{B}\tilde{F})) = s^n+(\alpha_1+\tilde{f}_1)s^{n-1}+\cdots+(\alpha_{n-1}+\tilde{f}_{n-1})s+(\alpha_n+\tilde{f}_n).$$

Hence $\tilde{F}$ can be used to assign the coefficients of this polynomial arbitrarily. In particular we can make sure that its zeros are $\lambda_1,\ldots,\lambda_n$. For $F=\tilde{F}T$ we infer $T(A-BF)T^{-1} = TAT^{-1}-TBFT^{-1} = \tilde{A}-\tilde{B}\tilde{F}$. Hence the gain $F$ does the job for the original system.

## An Explicit Formula

If $\alpha_1, \ldots, \alpha_n$ are the coefficients of the characteristic polynomial of $A$, we infer from slide 19 that

$$T^{-1} = S = \underbrace{\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}}_{K} \underbrace{\begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \cdots & \alpha_{n-2} \\ 0 & 0 & 1 & \cdots & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}}_{T_\alpha}$$

does transform $(A, B)$ into the controllability canonical form. Hence:

The coefficients of the characteristic polynomial of $A - BF$ are $\alpha_1 + \tilde{f}_1, \ldots, \alpha_n + \tilde{f}_n$ if we choose

$$F = \begin{pmatrix} \tilde{f}_1 & \cdots & \tilde{f}_n \end{pmatrix} [KT_\alpha]^{-1}.$$

**Warning**: This Bass-Gura formula or the alternative Ackerman formula are, typically, numerically not very reliable!

## Uncontrollable Systems

Any $(A, B)$ can be transformed into the normal form $(\tilde{A}, \tilde{B})$ on slide 25.

If $\tilde{F}$ is a feedback gain for the transformed system, its columns can be partitioned as those of $\tilde{A}$ into $\tilde{F} = \begin{pmatrix} \tilde{F}_1 & \tilde{F}_2 \end{pmatrix}$. Then $u = -\tilde{F}z$ leads to the closed-loop system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} - B_1\tilde{F}_1 & A_{12} - B_1\tilde{F}_2 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

- Since $(A_{11}, B_1)$ is controllable, the modes of $A_{11} - B_1\tilde{F}_1$ can be "arbitrarily" assigned. We can hence always choose $\tilde{F}_1$ to put all the eigenvalues of $A_{11} - B_1\tilde{F}_1$ into the open left-half complex plane.
- The modes of $A_{22}$ can in no way be influenced by state-feedback control. This motivates again the name of these modes on slide 25.

## Stabilization by State-Feedback

We can hence infer for uncontrollable $(A, B)$ that there will always be modes of $A - BF$ that are fixed and cannot be moved by $F$.

Second, we conclude that stabilizable systems can actually be stabilized by state-feedback control.

The system $\dot{x} = Ax + Bu$ is stabilizable if and only if there exists some matrix $F$ such that $\dot{x} = (A - BF)x$ is asymptotically stable ($A - BF$ is Hurwitz).

Indeed, choose $\tilde{F}_1$ such that $A_{11} - B_1\tilde{F}_1$ is Hurwitz and take $\tilde{F}_2$ arbitrary. Then $\tilde{A} - \tilde{B}\tilde{F}$ is Hurwitz. With $S$ as on slide 23 we define

$$F = \tilde{F}S^{-1}$$

to get $A - BF = [S\tilde{A}S^{-1}] - [S\tilde{B}]F = S(\tilde{A} - \tilde{B}\tilde{F})S^{-1}$ which is Hurwitz.

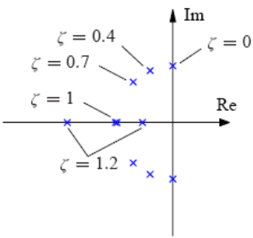
## Dominant Modes

The **damping ratio** of an eigenvalue $\lambda$ of $A$ is defined as

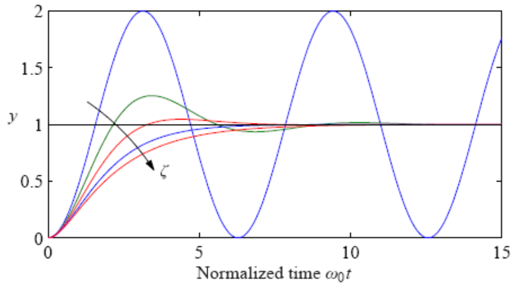
$$\zeta = -\frac{\mathrm{Re}(\lambda)}{|\lambda|}.$$

A pair of eigenvalues $\lambda, \bar{\lambda}$ is **dominant** if its damping ratio is smallest among those of all eigenvalues of $A$.

$e^{At}$ is often (but not necessarily) mainly determined by the dominant mode of $A$. Recall the classical facts for $2^{\text{nd}}$ order systems [AM, 6.3]:




(a) Eigenvalues




(b) Step responses

## Where to Place the Eigenvalues?

This question does not have a simple answer.

- The formula on slide 40 reveals: The less we move the coefficients of the characteristic polynomial (and hence the modes), the smaller the coefficients (gains) of $F$.

  The precise quantitative relation (sensitivity) is determined by $[KT_\alpha]^{-1}$. However this relation is not easy to use/interpret in practice.

- The role of dominant modes leads to the following design recipe:
  - Choose a $2^{\text{nd}}$ order system with desired dynamics
  - Place two eigenvalues at the two poles of this system
  - Choose all other eigenvalues to be faster (to render them less dominant) but not too fast (to avoid too large control action)
  - Assign these modes and evaluate by dynamic simulation

  Typically this process has to be **iterated** to achieve good designs.

# Example

With the data of [AM, p. 189] we linearize the Segway in the upright position (zero input). This leads to

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 6.41 & -1.8\,10^{-3} & -0.1\,10^{-3} \\ 0 & 7.21 & -0.8\,10^{-3} & -0.1\,10^{-3} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 18.4\,10^{-3} \\ 8.2\,10^{-3} \end{pmatrix}$$

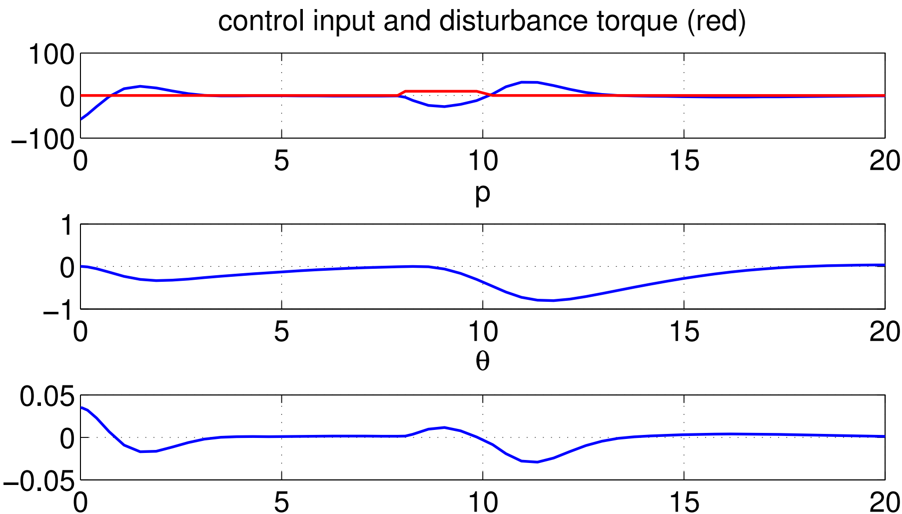
with eig$(A) = \begin{pmatrix} 0 & 2.68 & -2.69 & -1.1\,10^{-3} \end{pmatrix}$. This equilibrium is unstable (as we already saw in simulations). Let's stabilize by $u = -Fx$.

- Choose the eigenvalue $-1 \pm 1.7i$ (damping $0.5$) with a rise-time of about $1.2s$ to achieve a fast mode for stabilizing the pendulum.
- Choose $-0.35 \pm 0.35i$ (damping $0.7$) with a rise-time of about $6s$ to stabilize the cart.

With `place` we compute $F = \begin{pmatrix} -12.5 & 1.6\,10^{3} & -41.3 & 423 \end{pmatrix}$.

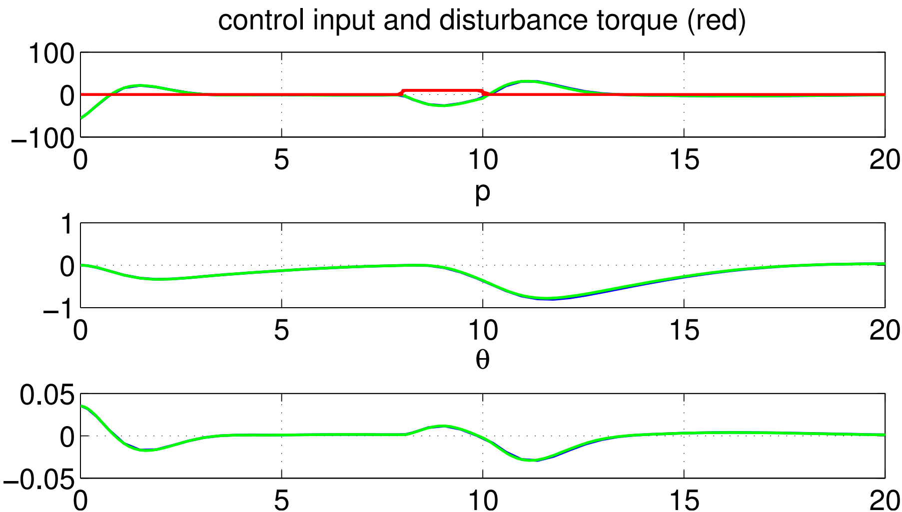
# Example

Response of linear controlled system to non-zero initial condition and disturbance torque on tip of pendulum:



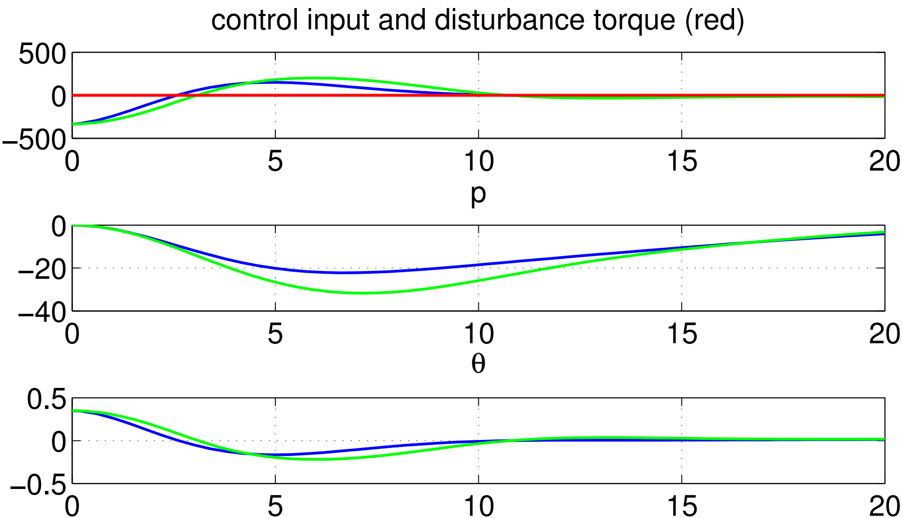
## Example

Comparison with (green) response of **nonlinear** system with state-feedback controller implemented as $u = u_e - F(x - x_e)$:



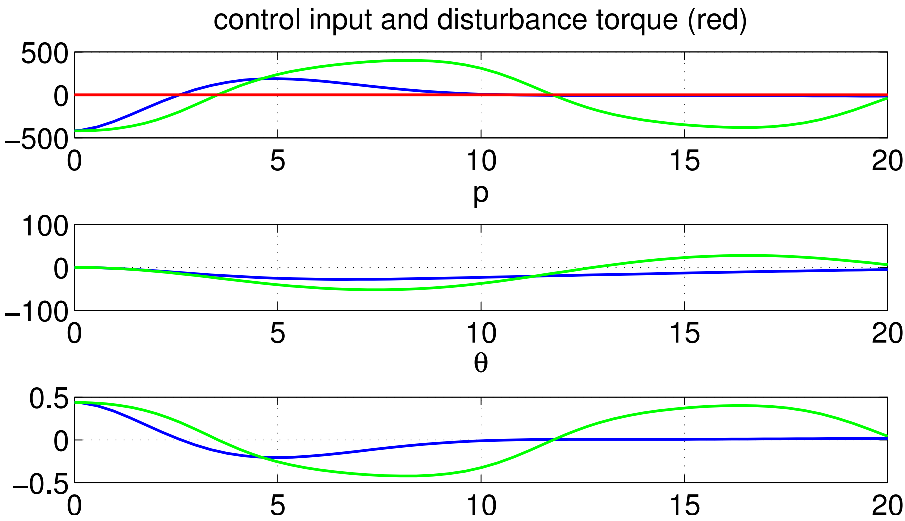
# Example

Slow down eigenvalues (to one-third of original natural frequency) and increase initial condition:



# Example

For even larger initial conditions we get instability of nonlinear system:



## Equivalent statements for controllability (summary)

- $(A, B)$ controllable
- $K$ is full row rank (for single input systems: $\det(K) \neq 0$)
- $\begin{pmatrix} A - \lambda I & B \end{pmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$
  (or for all eigenvalues $\lambda$ of $A$)
- $e^*B \neq 0$ for every left eigenvector $e^*$ of $A$
  (i.e., $e^* \neq 0$ s.t. $e^*A = \lambda e^*$)
- Eigenvalues of $(A - BF)$ can be freely assigned by $F$

## Covered in Lecture 3

- Controllability
  trajectory definition, Kalman criterion
- Uncontrollable Systems
  normal forms, uncontrollable modes, Hautus test
- Stabilization
  open-loop control, motivation of feedback-control
- State-feedback
  pole placement, stabilization, shaping the dynamic response