Observability, Separation, Realizations

- Observability: Definition, tests, canonical/normal forms
- Duality
- Observer design and detectability
- Separation principle
- Design of stabilizing output-feedback controllers
- Constructing realizations
- Minimal realizations and their properties

Related Reading

[AM]: Chapters 7.1-7.3 [F]: Chapters 7.1-7.3, 7.5, 8.1-8.3, 8.7

State Reconstruction

We have designed stabilizing state-feedback controllers for

$$\dot{x} = Ax + Bu.$$

An implementation requires that the full state is measurable.

Assuming availability of all states for control is unrealistic.

Let us hence suppose that we only have the output

$$y = Cx + Du$$

as information available that can be used by the controller.

Two - still rather vague - questions come to mind:

- If we only know u and y, is it possible to reconstruct x?
  This is the problem of state-reconstruction.
- Can one implement a controller with the reconstructed state?

State Reconstruction

Typically y has much fewer components than x. It is hence impossible to extract x from the measurements y instantaneously. However the system is dynamic, and we could try to collect the information in y over time in order to reconstruct x.

The linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$ is observable if, for any $T > 0$, it is possible to uniquely determine $x(t)$ for $t \in [0, T]$ based on knowledge of the input $u(t)$ and output $y(t)$ for $t \in [0, T]$.

Given $y(t)$ for $t \in [0, T]$, how can we possibly extract more information? One idea is to differentiate $y(t)$. In fact we indeed have at our disposal

$$y(t),\ \dot{y}(t),\ \ddot{y}(t),\ \dots\ , y^{(n-1)}(t).$$

We could use even higher derivatives but they do not provide additional information, as we will see.

State Reconstruction

Now observe that

$$Y(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}}_{W} x(t) + \mathcal{D} \underbrace{\begin{pmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix}}_{U(t)}.$$

W is called the **observability matrix** of the system or of (A, C).

If W has full column rank the equation $W^{+}W = I$ has a solution W^{+}. With any such matrix we can reconstruct $x(t)$ from $Y(t)$ and $U(t)$ as

$$x(t) = W^{+}Y(t) - W^{+}\mathcal{D}U(t).$$

This leads us to the following fundamental result.

The linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$ is observable if and only if its observability matrix W has full column rank.

Proof

If W has full column rank, reconstruction can be done as seen above.

Suppose that W does not have full column rank. This implies that

$$\begin{pmatrix} C^T & A^TC^T & \cdots & (A^{n-1})^TC^T \end{pmatrix} \text{ does not have full row rank.}$$

Due to the Hautus-test for controllability of (A^T, C^T) we infer that there exist $\lambda \in \mathbb{C}$ and a complex vector $v \neq 0$ with $v^* \begin{pmatrix} A^T - \lambda I & C^T \end{pmatrix} = 0$. Hence we infer $Av = \bar{\lambda} v$ and $Cv = 0$. Therefore $c(t) = e^{\bar{\lambda} t} v$ satisfies $\dot{c}(t) = Ac(t)$ and $Cc(t) = 0$ for all $t \geq 0$. Hence, with $u(t) = 0$, either $r(t) = \text{Re}[c(t)]$ or $s(t) = \text{Im}[c(t)]$ is a **nonzero** system trajectory for which the output is identically zero. Suppose $r(t) \neq 0$ for $t \in [0, T]$.

This prevents observability: Indeed let $y(t)$ be an output for $x(t)$, $u(t)$. Then both $x(t)$ and $x(t) + r(t)$ are **different** state-trajectories for the very same input/output pair $(u(t), y(t))$. Thus the state-trajectory is certainly not uniquely defined by the input- and output-trajectories.

Duality

Let us recognize, as just exploited, that

$$(A, C) \text{ is observable} \quad \Longleftrightarrow \quad (A^T, C^T) \text{ is controllable.}$$

Indeed the transposed observability matrix of (A, C) is just the Kalman matrix of (A^T, C^T). More generally, we can exploit everything that we already learnt about controllability of (A^T, C^T) and translate the related insights into facts about observability of the pair (A, C).

Translating questions about observability of (A, B, C, D) into the corresponding questions on controllability of (A^T, C^T, B^T, D^T) (or vice-versa) is called the **duality principle** in linear control.

This statement is brought to life through the examples considered next. Note that it saves a lot of work since we do not need to re-derive results about observability, but we just need to dualize the results of Lecture 3.

Hautus Test for Observability

As a first example let us formulate the Hautus-test for observability.

The pair (A, C) is observable if and only if

$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \text{ has full column rank for all } \lambda \in \mathbb{C}.$$

Equivalently, there exists no eigenvector $e \neq 0$ of A with $Ce = 0$

Proof. (A, C) is observable iff (A^T, C^T) is controllable iff (Hautus test for controllability) $\begin{pmatrix} A^T - \lambda I & C^T \end{pmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$; the row rank of a complex matrix is equal to the column rank of its transpose (without conjugation); hence the latter property is equivalent to $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ having full column rank for all $\lambda \in \mathbb{C}$. ■

We have now all the means to verify observability of (A, C) in practice.

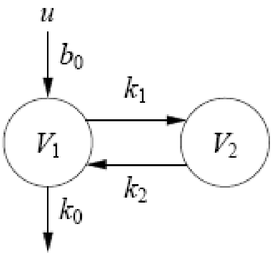
Example

Consider the two-compartment model (e.g. for drug admin, [AM] pp.85):

$$\dot{c} = \begin{pmatrix} -k_0 - k_1 & k_2 \\ k_1 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} c$$

$$k_0 > 0,\ k_1 > 0,\ k_2 > 0.$$



By measuring the concentration in the first compartment, it is indeed possible to reconstruct the not directly accessible concentration in the second compartment: Since $k_1 \neq 0$,

$$W = \begin{pmatrix} 1 & 0 \\ -k_0 - k_1 & k_2 \end{pmatrix} \quad \text{has full column rank (Kalman criterion).}$$

Similarly $Ae = \lambda e$ and $Ce = 0$ implies $e_1 = 0$ (due to $Ce = 0$) and then $e_2 = 0$ (first equation and $k_1 \neq 0$) - we can apply the Hautus criterion.

Unobservable Subspace and Modes

If the observability matrix does not have full column rank, it has a non-zero null-space. From slide 4 we infer that non-zero state-trajectories in this space are "swallowed" by W and "cannot be seen" at the output.

The null-space of W is called **unobservable subspace** of (A, C).

Based on an eigenvalue λ for which we find an eigenvector e of A such that $Ce = 0$, we actually constructed such a state-trajectory on slide 5. These eigenvalues carry a special name.

Any $\lambda \in \mathbb{C}$ for which $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ does not have full column rank is called an **unobservable mode** of (A, C).

By using a duality argument it is now very simple to nicely exhibit the unobservable subspace and the unobservable modes as follows.

Observability Normal Form

There exists a state-coordinate change $z = Tx$ (T invertible) that transforms $\dot{x} = Ax + Bu$, $y = Cx + Du$ into

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u = \tilde{A}z + \tilde{B}u,$$

$$y = \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Du = \tilde{C}z + Du$$

such that (A_{11}, C_1) is observable.

Learn to read these equations more explicitly as

$$\dot{z}_1 = A_{11}z_1 + B_1u, \quad \dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2u, \quad y = C_1z_1 + Du.$$

We infer that z_1 and hence also y are **not influenced** by z_2. For example a modification of the initial condition $z_2(0)$ cannot be observed in y.

The unobservable subspace of $(\tilde{A}, \tilde{C})$ is $\{(0, z_2) : \ z_2 \in \mathbb{R}^{\dim z_2}\}$ and its unobservable modes are just the eigenvalues of A_{22}.

Proofs

First Theorem. Just transform (A^T, C^T, B^T, D^T) into the controllability normal form on slide 3-23 and transpose the matrices. **Note:** One can show that T^{-1} can be chosen as $S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$ that is non-singular and such that the columns of S_2 form a basis of the null-space of W.

Second Theorem. The observability matrix $\tilde{W}$ of $(\tilde{A}, \tilde{C})$ has the rows

$$\begin{pmatrix} C_1 & 0 \end{pmatrix}, \begin{pmatrix} C_1 A_{11} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} C_1 A_{11}^{n-1} & 0 \end{pmatrix}.$$

Now observe that $\tilde{W} z = 0$ if and only if z_2 is arbitrary and z_1 satisfies

$$C_1 z_1 = 0,\ C_1 A_{11} z_1 = 0,\ \ldots,\ C_1 A_{11}^{n-1} z_1 = 0. \qquad (\star)$$

Since (A_{11}, C_1) is observable $(\star)$ holds if and only if $z_1 = 0$.

Since $\begin{pmatrix} A_{11} - \lambda I \\ C_1 \end{pmatrix}$ has full column rank, $\begin{pmatrix} A_{11} - \lambda I & 0 \\ A_{21} & A_{22} - \lambda I \\ C_1 & 0 \end{pmatrix}$ can only loose column rank if $A_{22} - \lambda I$ is not invertible.

Example

Let's consider the example system on p.191 of [F]:

$$A = \begin{pmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 7 & 6 & 4 & 2 \end{pmatrix}.$$

The observability matrix is

$$W = \begin{pmatrix} 7 & 6 & 4 & 2 \\ -10 & -9 & -6 & -3 \\ 16 & 15 & 10 & 5 \\ -28 & -27 & -18 & -9 \end{pmatrix}$$

and can be written as $W = LU$ (LU-factorization with `lu`) where

$$L = \begin{pmatrix} -0.25 & 1 & 0 & 0 \\ 0.36 & -0.86 & 1 & 0 \\ -0.57 & 0.57 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} -28 & -27 & -18 & -9 \\ 0 & -0.75 & -0.5 & -0.25 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

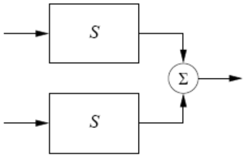
Example

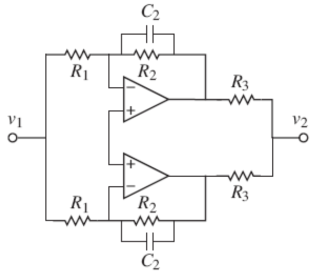
If we define $S = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.67 & -0.33 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ the last columns form a basis of the null space of U and hence of W, and the overall matrix is by construction non-singular. Let's check that it transforms (A, C) into the observability normal form:

$$\left(\begin{array}{c|c} S^{-1}AS & S^{-1}B \\ \hline CS & 0 \end{array}\right) = \left(\begin{array}{rrrr|r} 2 & 3 & \color{red}{0} & \color{red}{0} & 1 \\ -4 & -5 & \color{red}{0} & \color{red}{0} & -1 \\ -2 & -2 & -2.67 & 0.67 & 2 \\ -2 & -2 & -0.67 & -4.33 & -1 \\ \hline 7 & 6 & \color{red}{0} & \color{red}{0} & 0 \end{array}\right).$$

$\left(\begin{pmatrix} 2 & 3 \\ -4 & -5 \end{pmatrix}, \begin{pmatrix} 7 & 6 \end{pmatrix}\right)$ is observable; due to the red zeros this is indeed the desired observability normal form. Hence we infer that (A, C) has the unobservable modes eig $\begin{pmatrix} -2.67 & 0.67 \\ -0.67 & -4.33 \end{pmatrix} = \{-3, -4\}$.

Unobservable Systems





Again, there are many reasons for unobservability. For example if we interconnect two identical observable systems (A_S, B_S, C_S, D_S) as in the block-diagram, we get $\dot{x} = Ax + Bu$, $y = Cx + Du$ with

$$A = \begin{pmatrix} A_S & 0 \\ 0 & A_S \end{pmatrix},\ B = \begin{pmatrix} B_S & 0 \\ 0 & B_S \end{pmatrix},\ C = \begin{pmatrix} C_S & C_S \end{pmatrix},\ D = \begin{pmatrix} D_S & D_S \end{pmatrix}.$$

The observability matrix of (A, C) has two identical block columns and, hence, cannot have full column rank. If A_S has the dimension n, the unobservable subspace of (A, C) actually equals $\left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : x \in \mathbb{R}^n \right\}$.

Observable Canonical Form

Let us consider a system with a **single output** only.

If $\dot{x} = Ax + Bu$, $y = Cx + Du$ has one output (y scalar) and is observable, there exists a coordinate change $z = Tx$ (T invertible) that transforms it into

$$\dot{z} = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} u = \tilde{A}z + \tilde{B}u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} z + du = \tilde{C}z + du.$$

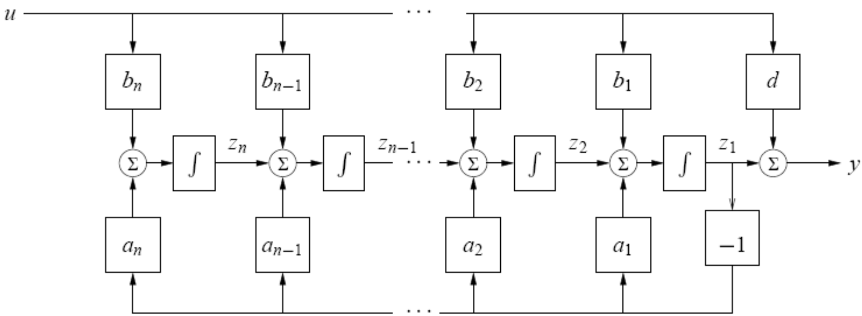
As in Lecture 3, $a_1, \ldots, a_n$ are uniquely determined by the coefficients of the characteristic polynomial of A:

$$\det(\lambda I - A) = \det(\lambda I - \tilde{A}) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

Block-Diagram

The proof is immediate by dualization. Transform (A^T, C^T, B^T, D^T) into the controllable canonical form and transpose both the system matrices and the transformation matrix.

A system in observable canonical form can be depicted as



Summary of Duality: $(A, B, C, D) \leftrightarrow (A^T, C^T, B^T, D^T)$

Controllability	**Observability**
$K = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$ full row rk	$W = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$ full column rk
Range space K = Controllable subspace	Null space W = Unobservable subspace
$\begin{pmatrix} A - \lambda I & B \end{pmatrix}$ full row rk $\forall \lambda \in \mathbb{C}$	$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ full column rk $\forall \lambda \in \mathbb{C}$
λ with rk drop: Uncontrollable Modes	λ with rk drop: Unobservable Modes
Normal Forms	
$\left(\begin{array}{cc\|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right)$, (A_{11}, B_1) controllable	$\left(\begin{array}{cc\|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0 & D \end{array}\right)$, (A_{11}, C_1) observable
SISO: $\left(\begin{array}{cccc\|c} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_n & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ \hline \gamma_1 & \gamma_2 & \cdots & \gamma_n & D \end{array}\right)$	SISO: $\left(\begin{array}{cccc\|c} -\alpha_1 & 1 & \cdots & 0 & \beta_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} & \cdots & 0 & 1 & \beta_{n-1} \\ -\alpha_n & \cdots & 0 & 0 & \beta_n \\ \hline 1 & 0 & \cdots & 0 & D \end{array}\right)$

Observers

Instantaneous reconstruction of the state as on slide 4 is not practical, since one should avoid differentiation of signals (noise!) and since the observability matrix might be ill-conditioned.

This motivates to rather try to reconstruct the state asymptotically, by a linear dynamical system that has as its inputs the signals u and y. This leads us to the following fundamental concept.

An **observer** for the linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$ is the dynamical system

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{y} = C\hat{x} + Du.$$

that is specified by choosing an **observer gain** $L \in \mathbb{R}^{n \times k}$.

An observer is a **copy** of the original system with a correction term $L(y - \hat{y})$ that serves to drive the **estimated state** $\hat{x}$ towards x in case that the measured output y deviates from the estimated output $\hat{y}$.

Observers

How should we choose L? Actually we would like the **estimation error**

$$\tilde{x} = x - \hat{x}$$

to converge quickly to zero. Let us hence determine its dynamics:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - [A\hat{x} + Bu + L(y - \hat{y})] =$$

$$A\tilde{x} - L(Cx + Du - C\hat{x} - Du) = (A - LC)\tilde{x}.$$

The **error dynamics** is described by $\dot{\tilde{x}} = (A - LC)\tilde{x}$.

Hence L should render $A - LC$ Hurwitz such that $\lim_{t\to\infty} \tilde{x}(t) = 0$. The speed and the "character" of the response (such as the overshoot) is determined by the eigenvalues of $A - LC$ and by $e^{(A-LC)t}$.

For observable (A, C) the eigenvalues of $A - LC$ can be placed arbitrarily in $\mathbb{C}$ if they are located symmetrically w.r.t. the real axis.

Detectability

Indeed if (A, C) is observable then (A^T, C^T) is controllable; hence we can place the eigenvalues of $A^T - C^T F$ to the desired locations; then $A - LC$ has exactly the same eigenvalues for $L = F^T$. Use `place`.

It suffices to just place the eigenvalues into the open left half-plane. This leads to the following concept.

The system $\dot{x} = Ax + Bu$, $y = Cx + Du$ or the pair (A, C) is said to be **detectable** if there exists some matrix L (of compatible dimension) such that $A - LC$ is Hurwitz.

Hautus-Test: (A, C) is detectable iff all its unobservable modes are in the open left half-plane. Equivalently:

$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.

Detectability

Proof of Hautus test. Since $A-LC$ is Hurwitz iff $A^T-C^TL^T$ is Hurwitz, detectability of (A,C) is equivalent to stabilizability of (A^T,C^T). Now just transpose the Hautus-test for stabilizability.

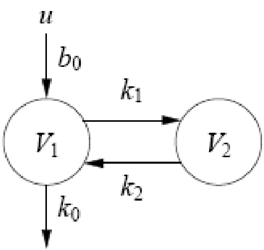
There do exist as well various trajectory-based characterization of detectability (often used as definitions); let us formulate one as follows.

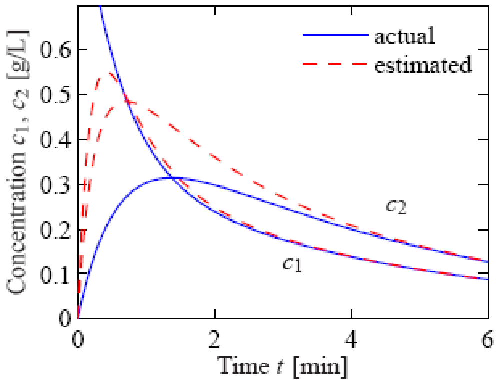
The system $\dot{x}=Ax+Bu$, $y=Cx+Du$ is detectable exactly when $u(t)=0$ and $y(t)=0$ for $t\geq 0$ imply $\lim_{t\to\infty} x(t)=0$.

Sketch of Proof. Let the system be transformed into observability normal form on slide 10. With $u(t)=0$ and $y(t)=0$, we actually infer $\dot{z}_1(t)=A_{11}z_1(t)$, $y(t)=C_1z_1(t)=0$ for $t\geq 0$; since (A_{11},C_1) is observable this is equivalent to $z_1(t)=0$ for $t\geq 0$. Hence only $z_2(t)$ is nontrivial and equals $z_2(t)=e^{A_{22}t}z_2(0)$. Therefore, the whole state $z(t)$ converges to zero for $t\to\infty$ iff A_{22} is Hurwitz.

Example

The design of an observer for the compartment model on slide 8 leads to the following simulation results:





Based on measuring $y = c_1$, the observer nicely achieves the task of asymptotically reconstructing the unmeasured concentration c_2.

An Evaluation

Here's a quote from [AM] which emphasizes the relevance of observers:

> "The problem of observability is one that has many important applications, even outside feedback systems. If a system is observable, then there are no hidden dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs.
>
> As we shall see, the problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a virtual sensor that gives information about variables that are not measured directly. The process of reconciling signals from many sensors with mathematical models is also called sensor fusion."

Separation Principle

We have seen in Lecture 3 how to stabilize a system by static state-feedback, which requires that all states of the system are measured on-line (at all times). In this lecture we learnt how to asymptotically reconstruct the system state from a measured output.

It was a tremendously influential idea to merge these techniques.

For the stabilizable and detectable linear time-invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

design F and L such that $A - BF$ and $A - LC$ are Hurwitz. Then $u = -Fx$ stabilizes the system, and the observer with gain L generates a state-estimate $\hat{x}$ which asymptotically reconstructs x. The key idea is to replace the unavailable x by the available $\hat{x}$ for control:

$$u = -F\hat{x}.$$

Separation Principle

This leads to the **observer-based output-feedback** controller

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{y} = C\hat{x} + Du, \quad u = -F\hat{x}$$

that is determined by the design parameters F and L.

The following two versions are obviously equivalent implementations:

$$\dot{\hat{x}} = (A - LC)\hat{x} + (B - LD)u + Ly, \quad u = -F\hat{x};$$

$$\dot{\hat{x}} = (A - LC - BF + LDF)\hat{x} + Ly, \quad u = -F\hat{x}.$$

If we interconnect the constructed controller with $\dot{x} = Ax + Bu$, $y = Cx + Du$ we arrive at the **closed-loop** system description

$$\begin{aligned} \dot{x} &= Ax - BF\hat{x} \\ \dot{\hat{x}} &= (A - LC - BF)\hat{x} + LCx \end{aligned}$$

which is **asymptotically stable**.

Proof

The closed-loop system as derived above reads as

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BF \\ LC & A - LC - BF \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

Let us perform the coordinate change

$$\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = T \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

A very elementary calculation shows

$$T \begin{pmatrix} A & -BF \\ LC & A - LC - BF \end{pmatrix} T^{-1} = \begin{pmatrix} A - BF & BF \\ 0 & A - LC \end{pmatrix}.$$

This leads to the closed-loop system description

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A - BF & BF \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \mathcal{A} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

whose state is x and the estimation error $\tilde{x} = x - \hat{x}$. (This description can as well be obtained by just writing down the differential equations for x and $\tilde{x}$.) It remains to observe that $\mathcal{A}$ is Hurwitz.

Summary

The resulting design procedure can be described as follows:

- Test whether (A, B) is stabilizable and (A, C) is detectable.
  If **no**, one can show that no linear stabilizing controller can exist.
- If **yes**, one can determine F and L such that $A - BF$ and $A - LC$ are Hurwitz.
- The observer-based controller leads to a closed-loop system whose eigenvalues are $\text{eig}(A - BF) \cup \text{eig}(A - LC)$.
- If (A, B) is controllable and (A, C) is observable, one can even place the closed-loop eigenvalues (symmetrically) to arbitrary locations.

In view of the independent construction of F and L for state-feedback stabilization and observer design, the proposed controller is said to be based on the **separation principle**.

How to Choose the Observer Gain?

- The open-loop eigenvalues of the system and the controller, A, $A - LC - BF + LDF$, and the closed-loop eigenvalues of

$$\dot{x} = (A - BF)x + BF\tilde{x}, \quad \dot{\tilde{x}} = (A - LC)\tilde{x}.$$

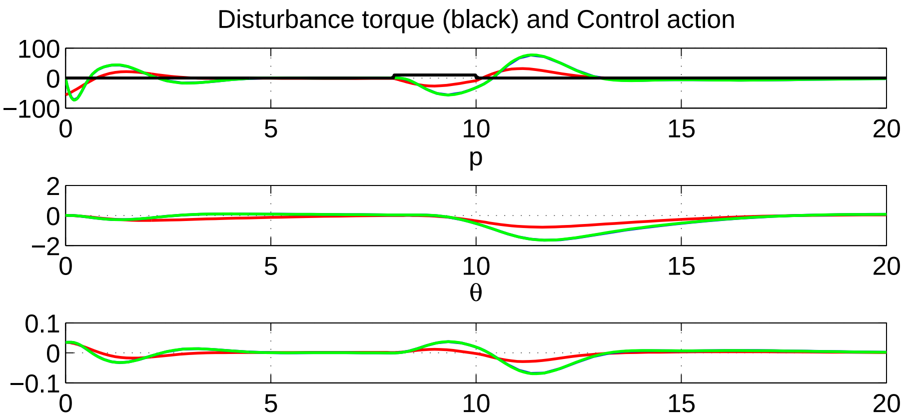
  can all be different. In particular it might happen that the controller itself is unstable. Then care has to be taken in implementing such a controller in practice (start-up, fault-prevention)!

- The structure of the closed-loop system nicely displays the influence of a non-zero initial estimation error $\tilde{x}(0) = x(0) - \hat{x}(0)$ onto the dynamics of x via the term $BF\tilde{x}$; it's visible as a transient response.

- It also illustrates that we wish to use "fast" eigenvalues for the error dynamics $A - LC$. However, they should not be taken too fast since measurement noise (not modeled here) might then be amplified. Moreover large observer gains can adversely influence robustness.

Example: Segway

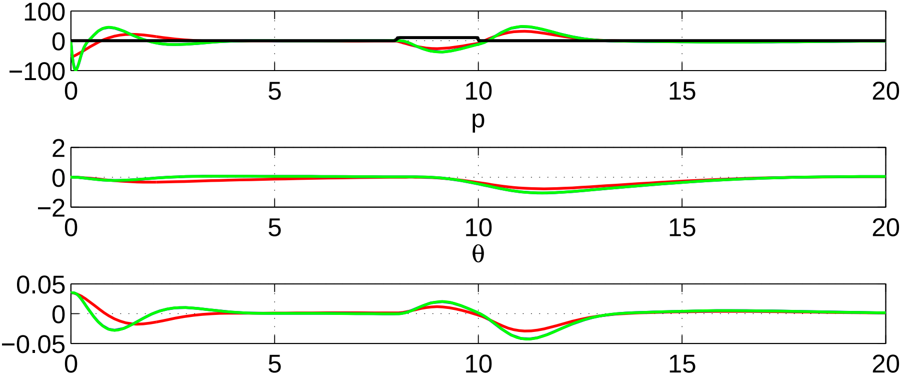
Based on slide 3-45 design an output-feedback controller with observer error-dynamics eigenvalue locations equal to multiples of those used for designing the state-feedback gain. We assume that only p and θ (and not their derivatives) are measured. With the error dynamics eigenvalues $-4 \pm 6.93i, -1.4 \pm 1.43i$ we get



sf (red), of with linear model (blue), of with non-linear model (green)

Example: Segway

"Speeding up" the error dynamics to $-8 \pm 13.86i, -2.8 \pm 2.86i$ leads to output-feedback responses closer to those for state-feedback synthesis:



If "speeding up" the error dynamics, the blue/green responses come closer and closer to the red curves. **However**, this results in high-gain feedback controllers that amplify measurement noise!

How to Analyze Sensivity to Noise?

If the measured output is affected by a noise signal v, the system and controller are described, with $A_c = (A - LC - BF + LDF)$, as

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx + Du + v \end{cases} \quad \text{and} \quad \begin{cases} \dot{\hat{x}} &= A_c\hat{x} + Ly, \\ u &= -F\hat{x}. \end{cases}$$

With the system's transfer matrix $G(s) = C(sI - A)^{-1}B + D$, the controller transfer matrix $K(s) = F(sI - A_c)^{-1}L$ and if assuming zero initial conditions, this reads in the Laplace domain as

$$\hat{y}(s) = G(s)\hat{u}(s) + \hat{v}(s) \quad \text{and} \quad \hat{u}(s) = -K(s)\hat{y}(s)$$

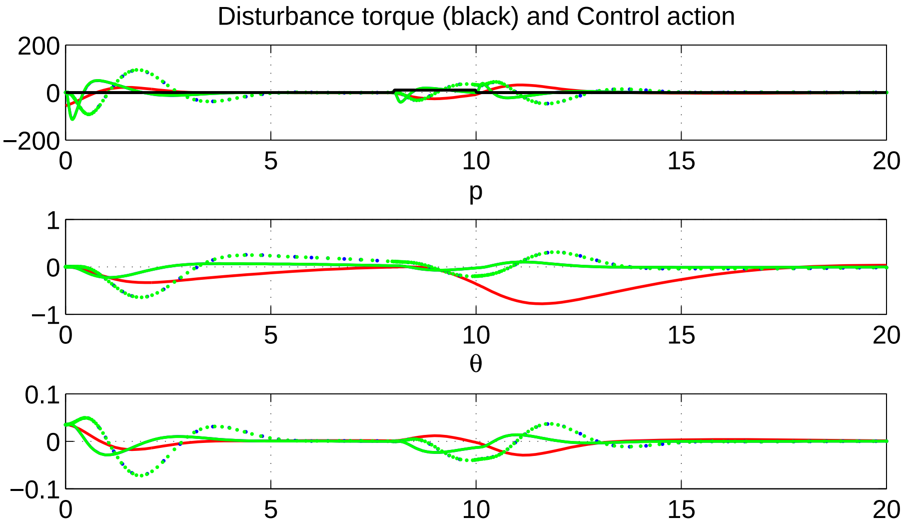
(with apologies for the ambiguous use of the notation $\hat{}$!) We obtain

$$\hat{u}(s) = -K(s)(I + G(s)K(s))^{-1}\hat{v}(s).$$

If this transfer matrix is SISO, we can hence analyze the influence of the measurement noise onto the control input in closed-loop by considering the Bode-magnitude plot of $-K(s)(I + G(s)K(s))^{-1}$.

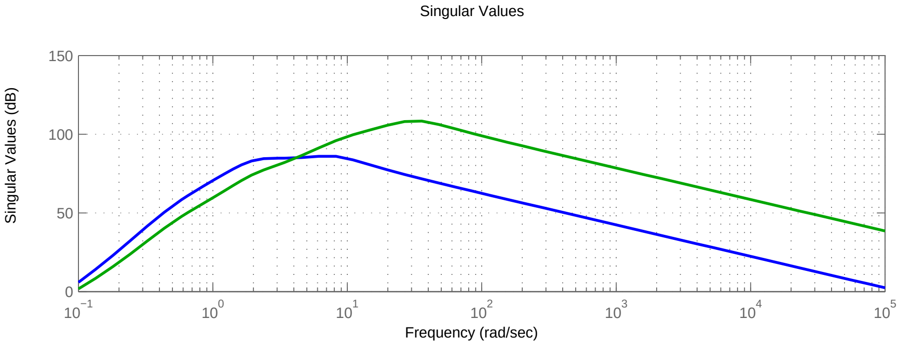
Example

Suppose only p is measured. Compare the designs for error dynamics eigenvalues $-4 \pm 6.9i$, $-1.4 \pm 1.4i$ (dotted) and $-16 \pm 27.7i$, $-5.6 \pm 5.7i$. The state-feedback responses (red) are now **not** approximated any more:



Example

Moreover the magnitude plot of the frequency response from v to u in closed-loop shows that measurement noise is considerably amplified at u for faster error dynamics: (blue:ev -4; green: ev -16).



By means of this example we have identified an important trade-off in controller synthesis: Improved performance comes at the expense of higher noise amplification. There are many more!

Realizations

Remember the realization problem from Lecture 2.

For a given $k \times m$-matrix $G(s)$ of proper rational functions, compute $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times m}$ such that

$$G(s) = C(sI - A)^{-1}B + D.$$

It is easy to determine D. Just recall that $(sI - A)^{-1}$ is strictly proper; therefore $\lim_{\omega\to\infty} C(i\omega I - A)^{-1}B = 0$; hence we need to choose

$$D = \lim_{\omega\to\infty} G(i\omega).$$

If having determined D it is required to realize the strictly proper transfer matrix $G(s) - D$ as $C(sI - A)^{-1}B$. Hence we need to actually construct realizations for strictly proper transfer matrices $G(s)$.

Recall: The problem is equivalent to representing an impulse response matrix $H(t)$ of linear combination of terms $t^k e^{\lambda t}$ as $H(t) = Ce^{At}B$.

Realizations of Transfer Functions (SISO)

If $g(s)$ is a proper transfer function, we can represent it as

$$g(s) = \frac{\beta_1 s^{n-1} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n} + d.$$

It is a matter of direct verification that one can then either choose

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccccc|c} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \hline \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n & d \end{array}\right)$$

controllable canonical realization

or

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccccc|c} -\alpha_1 & 1 & 0 & \cdots & 0 & \beta_1 \\ -\alpha_2 & 0 & 1 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} & 0 & \cdots & 0 & 1 & \beta_{n-1} \\ -\alpha_n & 0 & 0 & \cdots & 0 & \beta_n \\ \hline 1 & 0 & 0 & \cdots & 0 & d \end{array}\right)$$

observable canonical realization

Example

This is an illustrative example:

```
nu=[1 2 3 4 5];de=[6 7 8 9 10];n=length(de);
A=[-de;eye(n-1) zeros(n-1,1)];B=zeros(n,1);B(1)=1;C=nu;
g=tf(ss(A,B,C,0))

Transfer function:
    s^4 + 2 s^3 + 3 s^2 + 4 s + 5
--------------------------------------
s^5 + 6 s^4 + 7 s^3 + 8 s^2 + 9 s + 10
```

You should check for yourself that `ss(g)` does **not** lead back to the realization we started out with.

An Illustrative Observation

Suppose that $g(s)$ equals $g_1(s)g_2(s)$, the product of

$$g_1(s) = \frac{\beta_1^1 s^{n-1} + \cdots + \beta_n^1}{s^n + \alpha_1^1 s^{n-1} + \cdots + \alpha_n^1} + d^1, \quad g_2(s) = \frac{\beta_1^2 s^{n-1} + \cdots + \beta_n^2}{s^n + \alpha_1^2 s^{n-1} + \cdots + \alpha_n^2} + d^2.$$

We can then realize $g_1(s)$ and $g_2(s)$ and compute the realization of $g_1(s)g_2(s)$ according to the series interconnection of state-space systems (slide 39). With the controllable canonical realizations we get

$$\left(\begin{array}{cccc|cccc|c}
-\alpha_1^1 & -\alpha_2^1 & \cdots & -\alpha_n^1 & \beta_1^2 & \beta_2^2 & \cdots & \beta_n^2 & d^2 \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\alpha_1^2 & -\alpha_2^2 & \cdots & -\alpha_n^2 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
\hline
\beta_1^1 & \beta_2^1 & \cdots & \beta_n^1 & d^1\beta_1^2 & d^1\beta_2^2 & \cdots & d^1\beta_n^2 & d^1 d^2
\end{array}\right).$$

The procedure extends to general products $g_1(s) \cdots g_m(s)$.

Example

```
g1=ss([-1 -2;1 0],[1;0],[3 4],0);
g2=ss([-5 -6;1 0],[1;0],[7 8],0);
sy=ss(g1)*ss(g2)
a =                                b =
        x1   x2   x3   x4                 u1
   x1   -1   -2    7    8             x1    0
   x2    1    0    0    0             x2    0
   x3    0    0   -5   -6             x3    1
   x4    0    0    1    0             x4    0
c =                               d =
        x1   x2   x3   x4                u1
   y1    3    4    0    0            y1    0
```

Check `zpk(sy)` and `ss(zpk(sy))` and explain what you see!

Transfer Matrices

Remember that we discussed the parallel and series interconnection of systems with a state-space representation (Lecture 1) and a transfer matrix representation (Lecture 2). On slide 37 we already exploited the following immediate consequences for realizations.

If $G_1(s)$, $G_2(s)$ have realizations (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2) then $G_1(s)G_2(s)$ and $G_1(s) + G_2(s)$ have the realizations

$$\left(\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right).$$

This leads to an idea how to construct some realization of a transfer **matrix** $G(s)$: Write it as a sum or product of SISO transfer functions and real matrices, realize each of these, and apply the above formulas!

This can be done in many different (efficient and inefficient) ways!

A Generic Procedure

Indeed a general transfer matrix $G(s)$ reads as

$$G(s) = \begin{pmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ \vdots & \ddots & \vdots \\ g_{k1}(s) & \cdots & g_{km}(s) \end{pmatrix} \quad \text{with transfer functions} \quad g_{\nu\mu}(s).$$

This can be written as

$$\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} g_{11}(s) \begin{pmatrix} 1 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} g_{km}(s) \begin{pmatrix} 0 & \cdots & 1 \end{pmatrix}.$$

Let us choose realizations of $g_{11}(s)$, ..., $g_{km}(s)$ (having the state-matrices $A_{11}, ..., A_{km}$); we can then easily build one for the terms in the sum (each of which is a series interconnection of a tall static SIMO system, a SISO transfer function and a fat static MISO system); then we can write down a realization of the sum.

The resulting realization has the state-matrix $\mathrm{diag}(A_{11}, \ldots, A_{km})$.

Example

Consider

$$G(s) = \begin{pmatrix} \frac{1}{(s+1)^2} + \frac{1}{s+2} & \frac{1}{s+1} \\ \frac{1}{s} + \frac{1}{(s+2)^2} & \frac{1}{s} \end{pmatrix} = \begin{pmatrix} \frac{s^2+3s+3}{s^3+4s^2+5s+2} & \frac{1}{s+1} \\ \frac{s^2+5s+4}{s^3+4s^2+4s} & \frac{1}{s} \end{pmatrix}.$$

The procedure leads to

$$\left(\begin{array}{c||c} A & B \\ \hline\hline C & D \end{array}\right) = \left(\begin{array}{ccc|ccc|c|c||cc} -4 & -5 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline\hline 1 & 3 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 4 & 0 & 1 & 0 & 0 \end{array}\right)$$

with a state-matrix of dimension 8.

Variations

There are **many variants** that typically lead to realizations with a smaller-dimensioned state-matrix. One could e.g. first find the least common denominators $d_1(s), \ldots, d_m(s)$ of each row and write $G(s)$ as

$$G(s) = \begin{pmatrix} \frac{n_{11}(s)}{d_1(s)} & \cdots & \frac{n_{1m}(s)}{d_1(s)} \\ \vdots & \ddots & \vdots \\ \frac{n_{k1}(s)}{d_k(s)} & \cdots & \frac{n_{km}(s)}{d_k(s)} \end{pmatrix}.$$

If we then realize $\frac{n_{\nu\mu}(s)}{d_\nu(s)}$ by $\left(\begin{array}{c|c} A_\nu & B_{\nu\mu} \\ \hline C_\nu & D_{\nu\mu} \end{array}\right)$ (in observable canonical form with a common C_ν independent of μ) then $G(s)$ admits the realization

$$\left(\begin{array}{ccc|ccc} A_1 & \cdots & 0 & B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & A_k & B_{k1} & \cdots & B_{km} \\ \hline C_1 & \cdots & 0 & D_{11} & \cdots & D_{1m} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & C_k & D_{k1} & \cdots & D_{km} \end{array}\right).$$

Example

In our example we can write $G(s) = \begin{pmatrix} \frac{s^2+3s+3}{(s+1)^2(s+2)} & \frac{(s+1)(s+2)}{(s+1)^2(s+2)} \\ \frac{s^2+5s+4}{(s+2)^2 s} & \frac{(s+2)^2}{(s+2)^2 s} \end{pmatrix}$ and

get the realization

$$\left(\begin{array}{ccc|ccc||cc} -4 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -5 & 0 & 1 & 0 & 0 & 0 & 3 & 3 \\ -2 & 0 & 0 & 0 & 0 & 0 & 3 & 2 \\ \hline 0 & 0 & 0 & -4 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 \\ \hline\hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

with a state-matrix of dimension 6.

Note that a special case of the operations on slide 39 is "stacking":

$$\begin{pmatrix} G_1(s) \\ G_2(s) \end{pmatrix} \leftarrow \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{array}\right), \quad \begin{pmatrix} G_1(s) \; G_2(s) \end{pmatrix} \leftarrow \left(\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_2 & D_1 & D_2 \end{array}\right).$$

Minimal Realizations

We have seen many ways to realize $G(s)$. In this fashion we have proved that there **always exist realizations** of proper transfer matrices.

However the realization matrices are **highly non-unique** and even the dimension of the state-matrix can vary. Let us finally discuss how we get order into this "chaos". Let us start with the following observation.

For a given proper transfer matrix $G(s)$ there exist realizations (A, B, C, D) for which A is of **minimal possible dimension**. Any such (A, B, C, D) is said to be a **minimal realization** of $G(s)$.

This leads us to the following three key questions:

- How can we construct minimal realizations?
- How can we detect whether a realization is minimal?
- How do minimal realizations differ from each other?

Construction

Given $G(s)$ suppose that (A, B, C, D) is **any** realization. Since a state-coordinate change does not modify the transfer matrix, we can assume that (A, B) is in controllability normal form:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} A_1 & A_{12} & B_1 \\ 0 & A_2 & 0 \\ \hline C_1 & C_2 & D \end{array}\right) \quad \text{with controllable} \quad (A_1, B_1).$$

Now observe that

$$C(sI - A)^{-1}B = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} sI - A_1 & -A_{12} \\ 0 & sI - A_2 \end{pmatrix}^{-1} \begin{pmatrix} B_1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} (sI - A_1)^{-1} & * \\ 0 & * \end{pmatrix} \begin{pmatrix} B_1 \\ 0 \end{pmatrix} = C_1(sI - A_1)^{-1}B_1.$$

Hence $G(s) = C_1(sI - A_1)^{-1}B_1 + D$ is another realization whose state-matrix dimension is reduced by exactly the dimension of A_2.

We can also say: The uncontrollable modes of (A, B) can be canceled.

Construction

Let us continue in case that (A_1, C_1) is not observable. Yet another state-coordinate change allows to assume

$$\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D \end{array}\right) = \left(\begin{array}{cc|c} A_{11} & 0 & B_{11} \\ A_{21} & A_{22} & B_{21} \\ \hline C_{11} & 0 & D \end{array}\right) \quad \text{with observable} \quad (A_{11}, C_{11}).$$

As above it is easy to see that

$$C_1(sI - A_1)^{-1}B_1 + D = C_{11}(sI - A_{11})^{-1}B_{11} + D.$$

$G(s) = C_{11}(sI - A_{11})^{-1}B_{11} + D$ is yet another realization whose state-matrix dimension is further reduced by the dimension of A_{22}.

We can also say: The unobservable modes of (A_1, C_1) can be canceled. Note that the Kalman matrix of (A_1, B_1) has full row rank and reads as

$$\begin{pmatrix} B_{11} & A_{11}B_{11} & \cdots & A_{11}^{\dim(A_1)-1}B_{11} \\ * & * & \cdots & * \end{pmatrix}.$$

Therefore (A_{11}, B_{11}) is controllable! Let's summarize on the next slide.

Key Results on Minimal Realizations

Starting from an arbitrary realization $G(s) = C(sI - A)^{-1}B + D$ one can systematically construct (e.g. with `minreal`) a new realization

$$G(s) = C_r(sI - A_r)^{-1}B_r + D_r \text{ with } \dim(A_r) \le \dim(A)$$

such that (A_r, B_r) is controllable and (A_r, C_r) is observable.

It remains to show that the constructed realization is indeed minimal.

A realization $G(s) = C(sI - A)^{-1}B + D$ is minimal iff (A, B) is controllable and (A, C) is observable. If $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is another minimal realization of $G(s)$, the realizations are related by a state-coordinate change: There exists an invertible matrix T such that

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{pmatrix}.$$

This answers all the three questions raised on slide 44.

Proof

We first prove the relation of minimal realizations. Hence suppose that

$$C(sI-A)^{-1}B=\tilde{C}(sI-\tilde{A})^{-1}\tilde{B} \text{ or equivalently } Ce^{At}B=\tilde{C}e^{\tilde{A}t}\tilde{B}$$

while (A,B), $(\tilde{A},\tilde{B})$ are controllable and (A,C), $(\tilde{A},\tilde{C})$ are observable.

Since the two impulse responses are identical, their derivatives at zero coincide. This implies that the so-called Markov-parameters of the two realizations are identical:

$$CA^{\nu}B=\tilde{C}\tilde{A}^{\nu}\tilde{B} \text{ for all } \nu=0,1,2,\ldots. \qquad (\star)$$

Let W, K and $\tilde{W}$, $\tilde{K}$ denote the observability and Kalman matrices of the two realizations. Then the matrices WK, WAK and $\tilde{W}\tilde{K}$, $\tilde{W}\tilde{A}\tilde{K}$ are built of exactly those blocks that appear in $(\star)$. We conclude

$$WK=\tilde{W}\tilde{K} \text{ and } WAK=\tilde{W}\tilde{A}\tilde{K}.$$

Recall that W, $\tilde{W}$ and K, $\tilde{K}$ have full column and row rank respectively.

Proof

The rank properties imply that we can solve the equations $I = KK^+$ and $\tilde{W}^+\tilde{W} = I$ for K^+ and $\tilde{W}^+$ respectively (i.e. pseudo-inverses).

If we right-multiply $WK = \tilde{W}\tilde{K}$ with K^+ we obtain with $T := \tilde{K}K^+$:

$$W = \tilde{W}T.$$

Since W has full column rank, the same is true for T; since T must be square it is invertible. Moreover we also get $C = \tilde{C}T$ or $\boxed{\tilde{C} = CT^{-1}}$.

Left-multiplying $WK = \tilde{W}\tilde{K}$ with $\tilde{W}^+$ leads to $\tilde{W}^+WK = \tilde{K}$ and right-multiplication with K^+ implies $\tilde{W}^+W = \tilde{K}K^+ = T$; hence

$$TK = \tilde{K}.$$

In turn this shows $\boxed{TB = \tilde{B}}$.

Now exploit $WAK = \tilde{W}\tilde{A}\tilde{K}$ to infer $\tilde{W}TAK = \tilde{W}\tilde{A}TK$, and again by left-, right-multiplying $\tilde{W}^+$, K^+ further $TA = \tilde{A}T$ or $\boxed{\tilde{A} = TAT^{-1}}$.

Proof

This proves that two different controllable and observable realizations of $G(s)$ are related by a state-coordinate change. It remains to show:

If a realization of $G(s)$ is controllable and observable, it is minimal.

Proof. Otherwise we could construct a second realization with a strictly smaller-dimensioned state-matrix; this second realization can be assumed to be controllable and observable (first theorem on slide 47); hence the original and the newly constructed realizations are related by a state-coordinate change; since they have state-matrices of different dimensions, this is a contradiction.

If a realization of $G(s)$ is **not** controllable or **not** observable, it is **not** minimal.

Proof. We have seen how to reduce uncontrollable realizations to ones with smaller dimensioned state-matrices; duality allows to conclude the same for unobservable realizations.

Example

The realization on slide 43 is **not** minimal. A little reflection reveals that it must be observable. However it is not controllable. Since the Kalman matrix has rank 5, we can reduce the size of the realization by one and construct a minimal realization that has a state-matrix of dimension 5.

`minreal` generates such a realization. Please note, however, that an application of `minreal` might not be successful since Matlab has to numerically decide which parts of the state-matrices to cancel!

In Exercise 3-3 we actually constructed the following realization:

$$A = \left(\begin{array}{rr|rr|r} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

It is controllable and observable and hence minimal.

Relation to Poles of Transfer Matrices

We have seen that uncontrollable and unobservable modes of state-space realizations are canceled if computing the transfer matrices and do not appear as poles. Cancelation does not happen for minimal realizations.

Let $G(s)$ have the minimal realization (A, B, C, D). Then the set of eigenvalues of A is equal to the set of poles of $G(s)$. The dimension of A is called the **McMillan degree** of the transfer matrix $G(s)$.

If a realization is stabilizable and detectable, cancelation can only occur for uncontrollable or unobservable modes in the open left half-plane. Hence the following corollary holds.

If $G(s) = C(sI - A)^{-1}B + D$ and A is Hurwitz then $G(s)$ is stable. Conversely, if $G(s)$ is stable and the realization is stabilizable and detectable then A is Hurwitz.

Example

The realization

$$A = \left(\begin{array}{cc|cc|c} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha \end{array}\right), \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

has the stable transfer matrix

$$G(s) = \begin{pmatrix} \frac{s^2+3s+3}{s^3+4s^2+5s+2} & \frac{1}{s+1} \\ \frac{1}{s^3+4s^2+4s} & 0 \end{pmatrix},$$

irrespective of $\alpha \in \mathbb{R}$.

- If $\alpha < 0$ the realization is stabilizable and detectable. Since $G(s)$ is stable the theorem implies that the state-matrix of the realization must be Hurwitz. This is indeed true.
- If $\alpha \geq 0$ the state-matrix of the realization is **not** Hurwitz, but $G(s)$ is still stable. This is a case of unstable pole-zero cancelation.

Example (Somewhat Advanced)

We reconsider the example on slide 12 (p.191 of [F]) with a **smarter** choice of S: Let S_3 be a basis of null$(W) \cap R(K)$; extend to a basis $(S_3\ S_4)$ of null(W) and to a basis $(S_1\ S_3)$ of $R(K)$; then extend to a non-singular matrix $S = (S_1\ S_2\ S_3\ S_4)$. We get for S and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$:

$$\begin{pmatrix} 0.74 & 0.63 & 0 & 0 \\ -0.62 & 0.63 & -0.29 & -0.44 \\ -0.25 & 0.42 & 0.58 & 0.22 \\ 0.12 & 0.21 & -0.29 & 0.87 \end{pmatrix}, \quad \left(\begin{array}{rrrr|r} -1 & 7.34 & 0 & 0 & 1.35 \\ 0 & -2 & 0 & 0 & 0 \\ 0.85 & -0.48 & -3 & 1.51 & 4.04 \\ 0 & -5.73 & 0 & -4 & 0 \\ \hline 0.74 & 10.22 & 0 & 0 & 0 \end{array}\right).$$

The controllable and unobservable subspaces then equal

$$\{(z_1, 0, z_3, 0) : \ z_1, z_3 \text{ free}\}, \quad \{(0, 0, z_3, z_4) : \ z_3, z_4 \text{ free}\}$$

and the corresponding modes are $\{-1, -3\}$ and $\{-3, -4\}$.

We built the so-called **Kalman decomposition** of the system.

Covered in Lecture 5

- Observability
  observers, observability, detectability, unobservable modes, duality
  observer design
- Separation Principle
  Output feedback synthesis for stability and response shaping
- Realization Minimality
  relevance, loss of minimality, reduction,
  relation to poles of transfer matrices