# Tracking and Disturbance Rejection

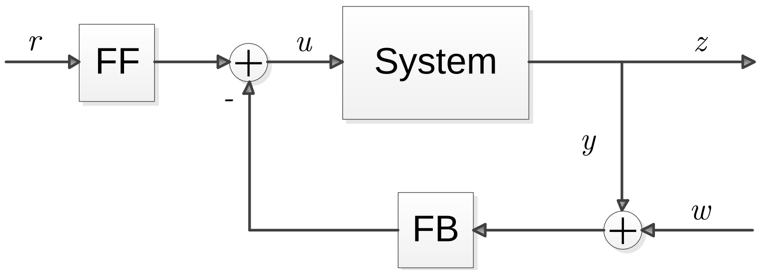
- Reference tracking by full-information and output-feedback
- A substantial generalization: The regulation problem
- The regulator equation
- Disturbance estimators
- Regulation by full-information and output-feedback
- Extension to general signal models
- Detailed discussion of motor example

## Related Reading

[AM]: Chapters 3 (examples), 6.4 [F]: Chapters 5.5, 6.4, 7.4, 8.5, 9.6

## Performance

So far we have concentrated a lot on stabilization. As you know from classical control, the two most important "next" tasks are **tracking of reference inputs** and the **rejection of disturbances**. A typical two-degrees-of-freedom configuration with a feedback controller FB and a feed-forward controller FF looks as follows:




The goal is to let $y$ track the reference $r$ irrespective of disturbance $w$. We mainly concentrate on **constant** reference and disturbance signals. However we also hint at generalization to e.g. sinusoidal signals or ramps.

## The Tracking Problem

Let's first look at tracking in somewhat more generality. Consider

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \\ z &= \tilde{C}x + \tilde{D}u. \end{aligned}$$

Next to $y$, the system output which is available for control, the output signal $z$ serves to impose a **performance specification**. Note that $z$ can - but need not - be just a copy of $y$ ($C = \tilde{C}$ and $D = \tilde{D}$)!

The goal is to design an asymptotically stabilizing controller for which $z$ asymptotically tracks all **constant** reference signals $r$:

$$\lim_{t \to \infty} [r - z(t)] = 0.$$

In order to be able to stabilize the system we assume from now on that

$(A, B)$ is stabilizable and $(A, C)$ is detectable.

# Tracking by Full-Information Feedback

Let us start with the assumption that the signals $x$ and $r$ are available for control. A linear static **full-information controller** is described by

$$u = -Fx + Gr$$

with to-be-chosen gain matrices $F$ and $G$. How should they be taken? If applying this controller to the system (and neglecting $y$) we arrive at

$$\dot{x} = (A - BF)x + BGr, \quad z = (\tilde{C} - \tilde{D}F)x + \tilde{D}Gr.$$

Since the system should be stabilized, we choose $F$ such that $A - BF$ is Hurwitz. Then the steady-state response of the controlled system is

$$z = [\tilde{D} - (\tilde{C} - \tilde{D}F)(A - BF)^{-1}B]Gr.$$

For asymptotic tracking we would like to guarantee that $z = r$ for all possible reference inputs $r$. This requires to take $G$ with

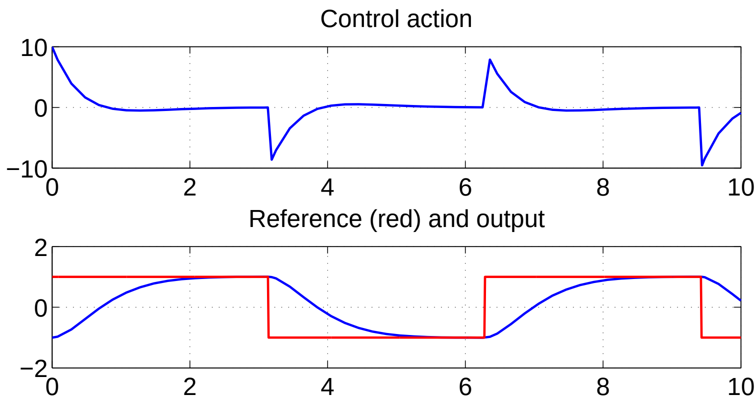
$$[\tilde{D} - (\tilde{C} - \tilde{D}F)(A - BF)^{-1}B]G = I. \qquad (\star)$$

## Motor Example

Let us consider the normalized model of a DC-motor ([F] p.20) with the applied voltage and the shaft-angle as its input and output:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = z = \begin{pmatrix} 1 & 0 \end{pmatrix} x, \quad x = \begin{pmatrix} \phi \\ \omega \end{pmatrix}.$$

With a state-feedback gain $F$ satisfying eig$(A - BF) = \{-2 \pm i\}$, the feedforward gain is $G = 5$. Simulated responses ($\phi(0) = -1$, $\omega(0) = 0$):



# Tracking by Output-Feedback

If $x$ is not measurable, we can choose $L$ such that $A - LC$ is Hurwitz and design the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{y} = C\hat{x} + Du$$

in order to asymptotically reconstruct the state. Then the separation principle motivates to control the system - with the gains $F$ and $G$ designed for full-information feedback - as

$$u = -F\hat{x} + Gr.$$

The constructed output-feedback controller (based on the measured signals $r$ and $y$) stabilizes the system and achieves tracking.

We conclude that **the separation principle holds**: We can combine a full-information controller with an observer to obtain an output-feedback controller that achieves the desired task.

# Proof

The dynamics of the estimation error $\tilde{x} = x - \hat{x}$ are described by

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - L(y - \hat{y}) = (A - LC)\tilde{x}.$$

Since $u = -Fx + F\tilde{x} + Gr$, the controlled system admits the description

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A - BF & BF \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} BG \\ 0 \end{pmatrix} r,$$

$$z = \begin{pmatrix} \tilde{C} - \tilde{D}F & \tilde{D}F \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \tilde{D}Gr.$$

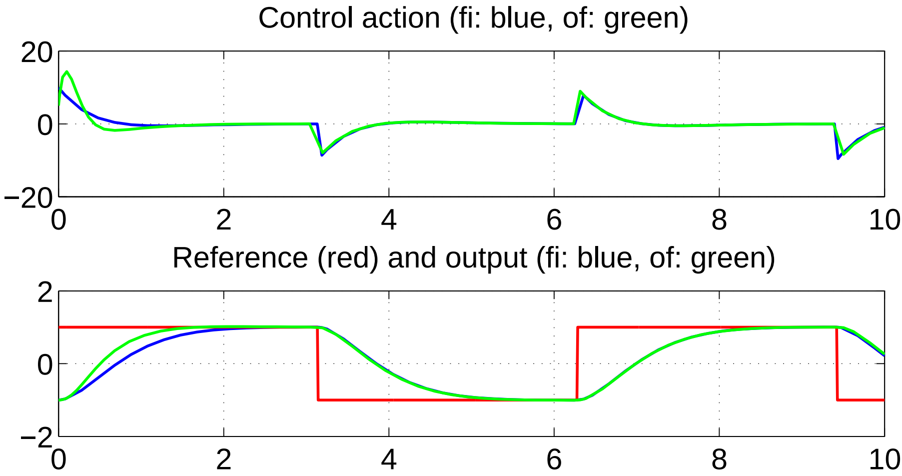
Since $A - BF$ and $A - LC$ are Hurwitz, the closed-loop system is asymptotically stable. Then note that the reference input does not drive the estimation error dynamics! Therefore the steady-state value of the output is, as for full-information feedback, still simply given by

$$[\tilde{D} - (\tilde{C} - \tilde{D}F)(A - BF)^{-1}B]Gr.$$

Due to the choice of $G$ this indeed equals $r$.

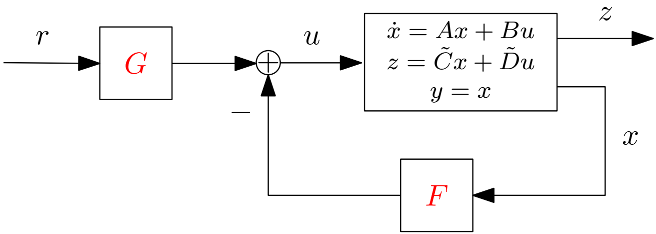
# Motor Example

Placing the estimation error dynamics eigenvalues at $-8 \pm 4i$ leads to the following response for output-feedback control:

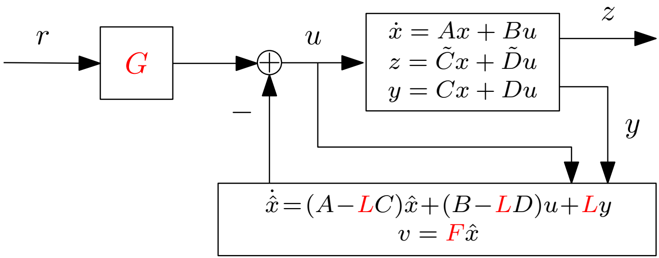


# Employed Implementations

Full-information control:




Output-Feedback Control:



# Matlab Code

```
A=[0 1;0 -1];B=[0;1];Ch=[1 0];Dh=0;C=Ch;;D=Dh;x0=[-1;0];
p=[-2+i;-2-i];F=place(A,B,p);G=1/(Dh-(Ch-Dh*F)*inv(A-B*F)*B);
L=place(A',C',4*p)';
%observer: inputs u,y
ob=ss(A-L*C,[B-L*D L],F,0);xc0=[0;0];
%system for simulation: inputs r,u  outputs u,z,r,u,y
z=[0 0];sy=ss(A,[z' B],[z;Ch;[z;z];C],[0 1;0 Dh;eye(2);0 D]);
%controller: inputs r u y
co=[G -ob];cl=lft(sy,co); %nice command for closing loops!
t=linspace(0,10,1e3)';r=square(t);[y,to]=lsim(cl,r,t,[x0;xc0])
figure(1);np=2;subplot(np,1,1);plot(to,y(:,1));grid on;
title('Control action')
subplot(np,1,2);plot(to,y(:,2),t,r,'r');grid on;
title('Reference (red) and output');
```

## Some Reflections

We obtained a recipe for designing tracking controllers. However, we have not yet gotten any insights under which conditions the procedure is successful at all.

Here are some of the questions that will be sequentially addressed in the remaining part of the lecture:

- What are the precise conditions for the existence of $G$? Is it depending on the choice of $F$ whether we can construct $G$?
- Is it possible to design tracking controllers even if $r$ is not measurable? E.g., sometimes only the tracking error $y = z - r$ can be measured.
- In addition to tracking, can we also reject disturbances?
- Can one track other than constant signals? Interesting practical cases are ramps or sinusoidal signals of a specified frequency.

## Solvability Condition

Note that $[\tilde{D} - (\tilde{C} - \tilde{D}F)(A - BF)^{-1}B]G = I$ can be re-written with $\Pi = -(A - BF)^{-1}BG$ as $(\tilde{C} - \tilde{D}F)\Pi + \tilde{D}G = I$. We arrive at

$$(A - BF)\Pi + BG = 0 \;\text{ and }\; (\tilde{C} - \tilde{D}F)\Pi + \tilde{D}G = I$$

which is easily re-arranged to

$$A\Pi + B(G - F\Pi) = 0 \;\text{ and }\; \tilde{C}\Pi + \tilde{D}(G - F\Pi) - I = 0.$$

With $\Gamma = G - F\Pi$ we finally arrive at the so-called **regulator equation**

$$\boxed{\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} + \begin{pmatrix} 0 \\ -I \end{pmatrix} = 0.}$$

The existence of a solution to this equation is the **precise condition** for the existence of a tracking controller! We already proved necessity.

Moreover, if $\Pi$ and $\Gamma$ are solutions and $F$ renders $A - BF$ Hurwitz, set $G = \Gamma + F\Pi$ and reverse the arguments to see $(\star)$. (Proves sufficiency.)

# A Leap Forward: The Regulation Problem

Let us now **substantially** generalize the discussed scenario. Consider

$$\begin{aligned} \dot{x} &= Ax + Bu + B_d d, \\ y &= Cx + Du + D_d d, \\ e &= \tilde{C}x + \tilde{D}u + \tilde{D}_d d \end{aligned}$$

where $y$ comprises all signals that are available for control, while $e$ is a performance output; $(A, B)$ is stabilizable and $(A, C)$ is detectable.

The signal $d$ is a **constant** (generalized) disturbance. The controller should asymptotically stabilize the system and achieve **regulation**:

$$\lim_{t \to \infty} e(t) = 0.$$

We also say that $d$ is asymptotically rejected from $e$ or suppressed at $e$.

Note that $B_d$, $D_d$, $\tilde{D}_d$ determine how the disturbance affects the state and the two outputs. This allows to model cases in which $d$ is a "real" disturbance that influences the system but is not available for control. But $d$ can as well represent a reference signal. Let's consider examples.

# Reference Tracking

The reference tracking problem for the system on slide 3 is subsumed to the regulation problem for the following system description:

$$\begin{aligned}
\dot{x} &= Ax + Bu + 0d, \\
y &= \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} D \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ I \end{pmatrix} d, \\
e &= Cx + Du + (-I)d.
\end{aligned}$$

Here $d$ is interpreted as a reference signal (as $r$ on slide 2). Observe that

- $d$ does not directly affect the system state.
- The measured output is a stacking of $Cx + Du$ and $d$. It could as well consist of other linear combinations of $Cx + Du$ and $d$.
- The output $e$ is the tracking error $Cx + Du - d$. Clearly $Cx + Du$ asymptotically tracks $d$ iff $e(t) \to 0$ for $t \to \infty$.

## Disturbance Rejection

If $d$ is a "real" disturbance that is not directly measurable, one would use $D_d = 0$. In case that $d$ does not affect the system-state directly we have $B_d = 0$. If $d$ is a load disturbance that acts on the regulated variable additively, we take $\tilde{D}_d = I$.

Many useful variations on this theme are conceivable. For example if $d$ is a process and measurement disturbance that affects the state as $B_d d$ and the measurement output (noise) as $D_d d$, we use the configuration

$$\begin{aligned} \dot{x} &= Ax + Bu + B_d d, \\ y &= Cx + Du + D_d d, \\ e &= \tilde{C}x + \tilde{D}u + 0d. \end{aligned}$$

Pure "measurement noise" would correspond to $B_d = 0$ and $D_d = I$. If only the first component of $y$ is noisy we use $D_d = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$.

## Reference Tracking and Disturbance Rejection

One can easily combine reference tracking and disturbance rejection. For example the configuration on slide 2 corresponds to

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} D \\ 0 \end{pmatrix} u + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix}, \\ e &= Cx + Du + \begin{pmatrix} 0 & -I \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix} \end{aligned}$$

where we view $d = \begin{pmatrix} w \\ r \end{pmatrix}$ as the generalized disturbance.

In summary, if we are able to solve the regulation problem on slide 13, we actually provide a solution to a whole variety of in itself relevant more special control problems.

Seemingly different design problems are handled in a unified framework.

# Main Result: Nominal Regulation

It's a beautiful coincidence that, despite this substantially higher level of generality, the previous solution approach extends without much effort.

The **regulator equation** now just reads as

$$\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} + \begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix} = 0.$$

We then arrive at one of the main results of this course.

If there exists a solution to the regulator equation then one can construct a stabilizing controller that achieves regulation.

Solvability of the regulator equation is also a necessary condition!

A **full-information controller** controller $u = -Fx + Gd$ is designed as earlier: Choose $F$ such that $A - BF$ is Hurwitz and set $G = \Gamma + F\Pi$.

## Proof for Full-Information Feedback

The controlled system is

$$\begin{aligned} \dot{x} &= (A - BF)x + BGd + B_d d, \\ e &= (\tilde{C} - \tilde{D}F)x + \tilde{D}Gd + \tilde{D}_d d. \end{aligned}$$

**Asymptotic stability**: If $d = 0$ then $\lim_{t\to\infty} x(t) = 0$ for any $x(0)$.

To show regulation let us perform the transformation $\xi = x - \Pi d$. Due to $\dot{d} = 0$ and $x = \xi + \Pi d$ the closed-loop system admits the description

$$\begin{aligned} \dot{\xi} &= (A - BF)\xi + \underbrace{[(A - BF)\Pi + BG + B_d]}_{A\Pi + B\Gamma + B_d = 0} d = (A - BF)\xi, \\ e &= (\tilde{C} - \tilde{D}F)\xi + \underbrace{[(\tilde{C} - \tilde{D}F)\Pi + \tilde{D}G + \tilde{D}_d]}_{\tilde{C}\Pi + \tilde{D}\Gamma + \tilde{D}_d = 0} d = (\tilde{C} - \tilde{D}F)\xi. \end{aligned}$$

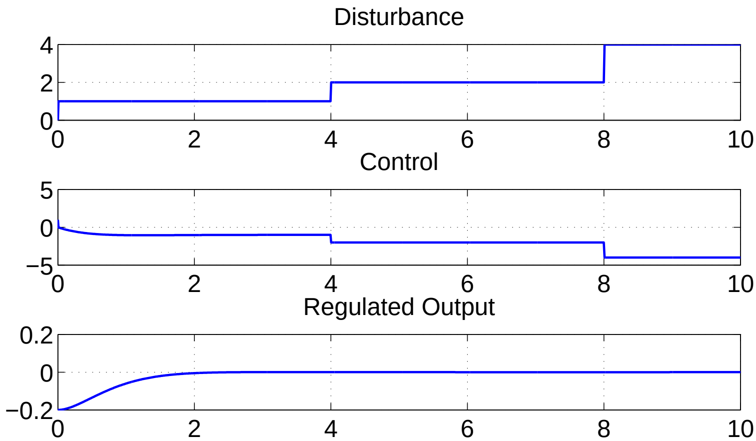
**Regulation**: $\lim_{t\to\infty} e(t) = 0$ for any constant disturbance $d$ and $x(0)$.

# Motor Example

Let the motor on slide 5 be affected by a torque disturbance:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} d, \quad y = e = \begin{pmatrix} 1 & 0 \end{pmatrix} x, \quad \begin{pmatrix} \phi(0) \\ \omega(0) \end{pmatrix} = \begin{pmatrix} -.2 \\ 0 \end{pmatrix}.$$

A full information controller $u = -{\color{red}F}x + {\color{red}G}d$ acts as follows:




However this controller cannot be implemented - only $y$ is measured.

# Disturbance Estimators

Suppose we are only allowed to use $y$ for control. Again, we intend to exploit the separation principle for designing a regulator. This requires to construct an observer that asymptotically reconstructs both the system state $x$ and the disturbance signal $d$ from $y$. For this purpose we include the **model for the considered class of disturbances** $\dot{d} = 0$ in the system dynamics. If we momentarily "neglect" $e$ we get

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{d} \end{pmatrix} &= \underbrace{\begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix}}_{A_e} \begin{pmatrix} x \\ d \end{pmatrix} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{B_e} u, \\ y &= \underbrace{\begin{pmatrix} C & D_d \end{pmatrix}}_{C_e} \begin{pmatrix} x \\ d \end{pmatrix} + Du. \end{aligned}$$

The existence of an observer that reconstructs the state of this so-called **extended system** requires the following hypothesis:

$(A_e, C_e)$ is detectable.

## Disturbance Estimators

By the Hautus-test we need to check whether

$$\begin{pmatrix} A-\lambda I & B_d \\ 0 & -\lambda I \\ C & D_d \end{pmatrix} \quad \text{has full column rank if } \operatorname{Re}(\lambda) \geq 0.$$

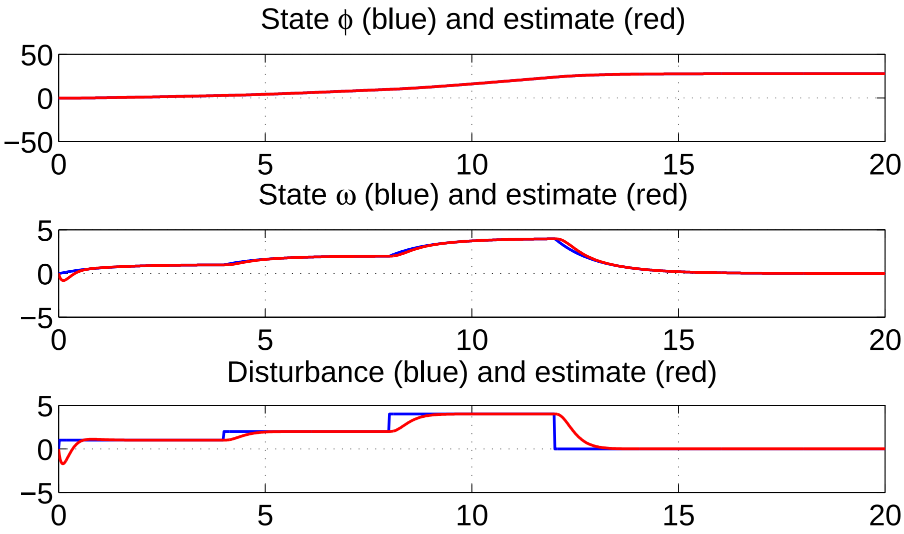
Since $(A, C)$ is detectable this holds if and only if

$$\boxed{\begin{pmatrix} A & B_d \\ C & D_d \end{pmatrix} \quad \text{has full column rank.}}$$

- For the reference tracking configuration on slide 14 this is true.
- For disturbance rejection as on slide 15 the rank condition has to be checked. If $B_d = 0$ and $D_d = I$, this means that $A$ is non-singular; the open-loop system should not have an eigenvalue at zero.
- The same condition emerges for the combination on slide 16.

## Motor Example

Let us design a state- and disturbance estimator for the model on slide 19. The detectability hypothesis is satisfied. A pole-placing observer for the extended system with eigenvalues $\{-5, -6, -7\}$ leads to



## Output Feedback Control

As motivated on slide 20 we design a regulator as follows: Check whether

- $(A, B)$ is stabilizable and $\left( \begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C & D_d \end{pmatrix} \right)$ is detectable;
- the regulator equation $\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} + \begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix} = 0$ is solvable.

If the answers are yes then choose

- $F$, $L$ such that $A - BF$, $\begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix} - L \begin{pmatrix} C & D_d \end{pmatrix}$ are Hurwitz;
- $G = \Gamma + F\Pi$ where $\Pi$, $\Gamma$ satisfy the regulator equation.

Then the following controller solves the regulation problem:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{pmatrix} = \begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + L(y - \hat{y}),$$

$$u = \begin{pmatrix} -F & G \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} C & D_d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + Du.$$

## Proof: Output Feedback Control

If we merge $\dot{d} = 0$ with the system dynamics we get the extended system

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{d} \end{pmatrix} &= \underbrace{\begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix}}_{A_e} \underbrace{\begin{pmatrix} x \\ d \end{pmatrix}}_{x_e} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{B_e} u, \\
y &= \underbrace{\begin{pmatrix} C & D_d \end{pmatrix}}_{C_e} \begin{pmatrix} x \\ d \end{pmatrix} + Du, \\
e &= \underbrace{\begin{pmatrix} \tilde{C} & \tilde{D}_d \end{pmatrix}}_{\tilde{C}_e} \begin{pmatrix} x \\ d \end{pmatrix} + \tilde{D}u.
\end{aligned}$$

The constructed observer-based controller can then be written as

$$\dot{\hat{x}}_e = A_e \hat{x}_e + B_e u + L(y - \hat{y}), \quad \hat{y} = C_e \hat{x}_e + Du, \quad u = -F_e \hat{x}_e$$

where we made use of the abbreviations

$$\hat{x}_e = \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} \quad \text{and} \quad F_e = \begin{pmatrix} F & -G \end{pmatrix}.$$

It is now routine to compute the closed-loop dynamics.

## Proof: Output Feedback Control

With the states $x_e$ and $\tilde{x}_e = x_e - \hat{x}_e$ they are described by

$$\begin{pmatrix} \dot{x}_e \\ \dot{\tilde{x}}_e \end{pmatrix} = \begin{pmatrix} A_e - B_e F_e & B_e F_e \\ 0 & A_e - L C_e \end{pmatrix} \begin{pmatrix} x_e \\ \tilde{x}_e \end{pmatrix}, \; e = \begin{pmatrix} \tilde{C}_e - \tilde{D} F_e & \tilde{D} F_e \end{pmatrix} \begin{pmatrix} x_e \\ \tilde{x}_e \end{pmatrix}.$$

Note that $\dot{x}_e = \ldots$ and $e = \ldots$ read more explicitly as

$$\begin{pmatrix} \dot{x} \\ \dot{d} \end{pmatrix} = \begin{pmatrix} A - BF & B_d + BG \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix} + \begin{pmatrix} B F_e \\ 0 \end{pmatrix} \tilde{x}_e,$$

$$e = \begin{pmatrix} \tilde{C} - \tilde{D} F & \tilde{D}_d + \tilde{D} G \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix} + (\tilde{D} F_e) \tilde{x}_e.$$

We can hence extract the closed-loop interconnection of the **original system** and our controller by just canceling $\dot{d} = 0$. This leads to

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}}_e \end{pmatrix} = \begin{pmatrix} A - BF & B F_e \\ 0 & A_e - L C_e \end{pmatrix} \begin{pmatrix} x \\ \tilde{x}_e \end{pmatrix} + \begin{pmatrix} B_d + BG \\ 0 \end{pmatrix} d,$$

$$e = \begin{pmatrix} \tilde{C} - \tilde{D} F & \tilde{D} F_e \end{pmatrix} \begin{pmatrix} x \\ \tilde{x}_e \end{pmatrix} + (\tilde{D}_d + \tilde{D} G) d.$$

## Proof: Output Feedback Control

**Asymptotic stability**: Suppose $d = 0$. Since $A - BF$ and $A_e - LC_e$ are Hurwitz, we infer for arbitrary $x(0)$ and $\tilde{x}_e(0)$ that

$$\lim_{t\to\infty} \begin{pmatrix} x(t) \\ \tilde{x}_e(t) \end{pmatrix} = 0.$$

**Regulation**: Suppose that $d$ is an arbitrary constant disturbance signal. Since $\dot{\tilde{x}}_e = (A_e - LC_e)\tilde{x}_e$ we then still infer that

$$\lim_{t\to\infty} \tilde{x}_e(t) = 0.$$

Hence the controller acts as a state and disturbance estimator in closed-loop (which could be highly relevant for diagnosis purposes)! Now note

$$\begin{aligned} \dot{x} &= (A - BF)x + (B_d + BG)d + (BF_e)\tilde{x}_e, \\ e &= (\tilde{C} - \tilde{D}F)x + (\tilde{D}_d + \tilde{D}G)d + (\tilde{D}F_e)\tilde{x}_e. \end{aligned}$$

If $\tilde{x}_e(t) = 0$, we already proved (full-information problem) that $e(t) \to 0$ for $t \to \infty$. This property holds even if only $\tilde{x}_e(t) \to 0$ for $t \to \infty$.

# Proof: Output Feedback Control

**Lemma.** Let $A$ be Hurwitz and suppose that $\lim_{t\to\infty} u(t) = 0$. Then all solutions of $\dot{x} = Ax + Bu$ satisfy $\lim_{t\to\infty} x(t) = 0$.

**Proof (somewhat advanced).** Let $\epsilon > 0$. Since $A$ is Hurwitz, there exist constants $K$ and $\alpha > 0$ with $\|e^{A\sigma}\| \leq Ke^{-\alpha\sigma}$ for $\sigma \geq 0$. Choose $T \geq 0$ such that $\|Bu(\tau)\| \leq \frac{\epsilon}{\alpha K}$ for $\tau \geq T$. Hence for all $t \geq T$:

$$\left\| \int_T^t e^{A(t-\tau)} Bu(\tau)\, d\tau \right\| \leq \frac{\epsilon}{\alpha K} \int_T^t Ke^{-\alpha(t-\tau)}\, d\tau \leq \epsilon[1 - e^{-\alpha(t-T)}] \leq \epsilon.$$

With the variation of constants formula we infer for $t \geq T$ and $t \to \infty$:

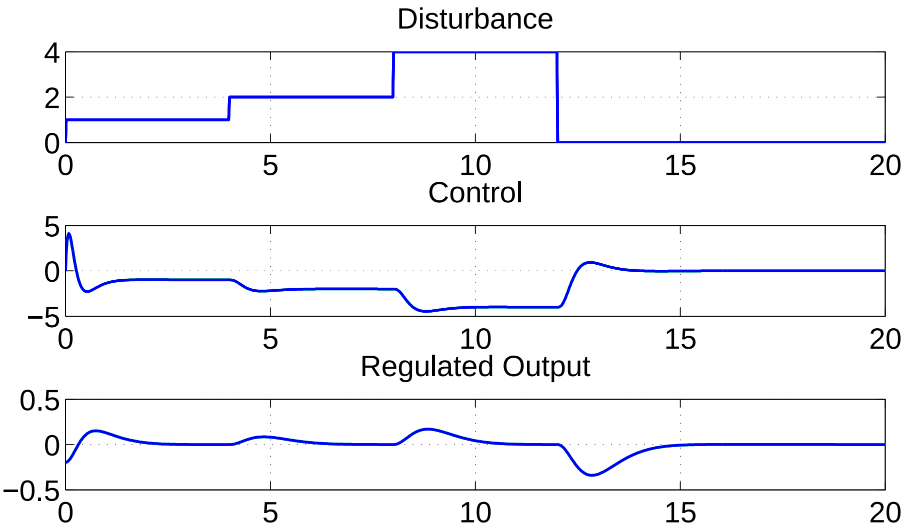
$$\|x(t)\| = \left\| e^{At}x(0) + \int_0^T e^{A(t-\tau)} Bu(\tau)\, d\tau + \int_T^t e^{A(t-\tau)} Bu(\tau)\, d\tau \right\| \leq$$

$$\leq \left\| e^{At} \left[ x(0) + \int_0^T e^{-A\tau} Bu(\tau)\, d\tau \right] \right\| + \epsilon \to \epsilon.$$

Since $\epsilon > 0$ was arbitrary we can infer $\lim_{t\to\infty} x(t) = 0$.

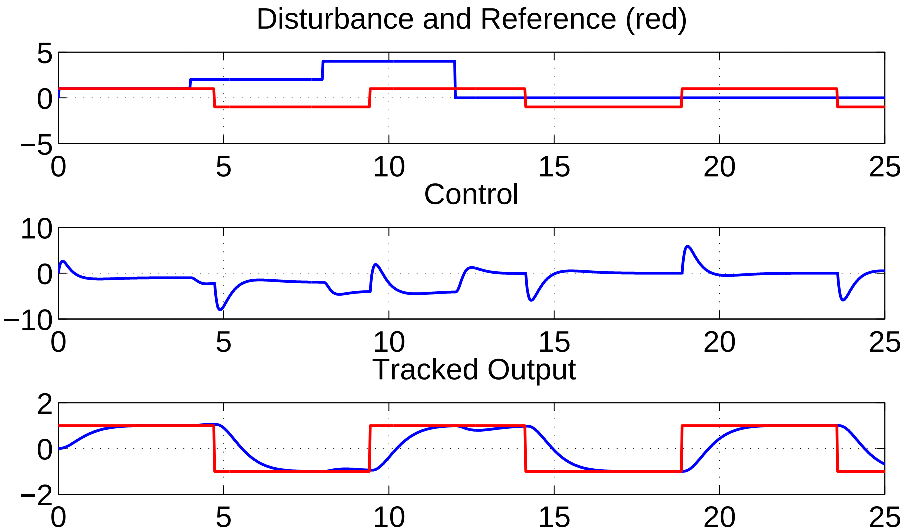
# Motor Example

Let us design an output-feedback controller for the motor example on slide 22 that rejects the torque disturbance from $e = \phi$. All hypotheses are satisfied. A regulating controller leads to the following results:



# Motor Example

Suppose that, in addition to disturbance rejection, $\phi$ should as well track a reference signal. This can be subsumed to our problem and, still, all hypotheses are satisfied. We obtain the following simulation results:



## Matlab Code

```
A=[0 1;0 -1];B=[0;1];Bd=[0 0;1 0];
Ch=[1 0];Dh=0;Dhd=[0 -1];
C=[1 0;0 0];D=[0;0];Dd=[0 0;0 1];
X=-inv([A B;Ch Dh])*[Bd;Dhd];Pi=X(1:2,:);Ga=X(3:end,:);
p=[-2+i;-2-i];F=place(A,B,p);G=Ga+F*Pi;
Ae=[A Bd;zeros(2,4)];Be=[B;zeros(2,1)];Ce=[C Dd];Fe=[F -G];
L=place(Ae',Ce',[-10 -11 -12 -13])';
z=[0 0];sys=ss(A,[Bd B],[Ch;z;C],[z Dh;z 1;Dd D]);
co=ss(Ae-L*Ce-Be*Fe+L*D*Fe,L,Fe,0);
cl=lft(sys,-co)
t=linspace(0,25,1e3)';
d=1*(t>0&t<4)+2*(t>4&t<8)+4*(t>8&t<12)+0*(t>12);
r=square(t/1.5);[y,to]=lsim(cl,[d r],t);
```

## Signal Models

The generalized disturbance signal was assumed to be constant. This set of signals is equal to the solution set of the the differential equation

$$\dot{d} = 0.$$

This system can thus be viewed as a **signal model** or **signal generator**.

This suggests immediate generalizations. For example sinusoidal signals of frequency $\omega$ are generated by

$$\dot{d} = Sd \text{ with } S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \text{ since } e^{St} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

Even more generally, we can work with block-diagonal matrices

$$S = \text{diag}\left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \omega_s \\ -\omega_s & 0 \end{pmatrix} \right)$$

in order to model constants, ramps and sinusoids of frequencies $\omega_1, \ldots, \omega_s$.

## Extension of Design Procedure

In our development we can replace $\dot{d} = 0$ all throughout by $\dot{d} = Sd$ with $S$ having all its **modes in the closed right half-plane**. This allows to design stabilizing controllers that achieve regulation for all signals

$$e^{St}s_0 \text{ with arbitrary } s_0 \in \mathbb{R}^{\dim(S)}.$$

- The regulator equation and $A_e$ for the extended system now read as

$$\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} - \begin{pmatrix} \Pi \\ 0 \end{pmatrix} S + \begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix} = 0, \quad A_e = \begin{pmatrix} A & B_d \\ 0 & S \end{pmatrix}.$$

  Anything else is unaltered and proofs require hardly any modification.

- For $y = e$ it suffices to check that $(A, C)$ is detectable if $(A_e, C_e)$ is not. $L$ then needs to be chosen such that $A_e - LC_e$ is "maximally Hurwitz" - its only eigenvalues in the closed right half-plane are the unobservable modes of $(A_e, C_e)$. Such an $L$ is easily found with the observability normal form. Proofs are more tricky, though.

## Extension of Design Procedure

As motivated on slide 20 we design a regulator as follows: Check whether

- $(A, B)$ is stabilizable and $\left(\begin{pmatrix} A & B_d \\ 0 & S \end{pmatrix}, \begin{pmatrix} C & D_d \end{pmatrix}\right)$ is detectable;
- the equation $\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} - \begin{pmatrix} \Pi \\ 0 \end{pmatrix} S + \begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix} = 0$ is solvable.

If the answers are yes then choose

- $F$, $L$ such that $A - BF$, $\begin{pmatrix} A & B_d \\ 0 & S \end{pmatrix} - L \begin{pmatrix} C & D_d \end{pmatrix}$ are Hurwitz;
- $G = \Gamma + F\Pi$ where $\Pi$, $\Gamma$ satisfy the regulator equation.

Then the following controller is a regulator for all $d$ with $\dot{d} = Sd$:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{pmatrix} = \begin{pmatrix} A & B_d \\ 0 & S \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + L(y - \hat{y}),$$

$$u = \begin{pmatrix} -F & G \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} C & D_d \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + Du.$$

## Motor Example

Suppose that the disturbance is still constant but that the reference is

$$\alpha\cos(t) + \beta\sin(t) \quad \text{with arbitrary unknown} \quad \alpha, \beta \in \mathbb{R}.$$

This motivates to choose the signal generator

$$S = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right) \quad \text{since} \quad e^{St} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{array}\right).$$

Disturbances are all solutions of $\dot{d} = Sd$. Then $d_1(t)$ is the constant torque disturbance and $d_2(t)$ is the reference signal. Since $d_3(t)$ is irrelevant, we are lead to the following system used for design:

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} d, \\
y &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} d, \\
e &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & -1 & 0 \end{pmatrix} d.
\end{aligned}$$

## Motor Example

Clearly $(A,B) = \left( \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ is controllable and

$$(A_e, C_e) = \left( \left( \begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array} \right), \left( \begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \right)$$ is observable.

The regulator equation is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} - \begin{pmatrix} \Pi \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = 0$$

and has the solution

$$\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Hence we are sure that we can design a regulator!

# Motor Example

We take $F$ and $L$ with $\text{eig}(A - BF) = \{-2 \pm i\}$ and $\text{eig}(A_e - LC_e) = \{-5, -6, -7, -8, -9\}$. (These have not been chosen very carefully!)

We obtain the controller

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{pmatrix} = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right) \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 19.95 & -1.43 \\ 122.53 & -17.65 \\ 313.9 & -61.83 \\ -0.15 & 14.05 \\ -1.09 & 47.38 \end{pmatrix} (y - \hat{y}),$$

$$u = -\begin{pmatrix} 5 & 3 & 1 & -4 & -4 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix}.$$

For simulations this controller is interconnected with the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} d_1,$$

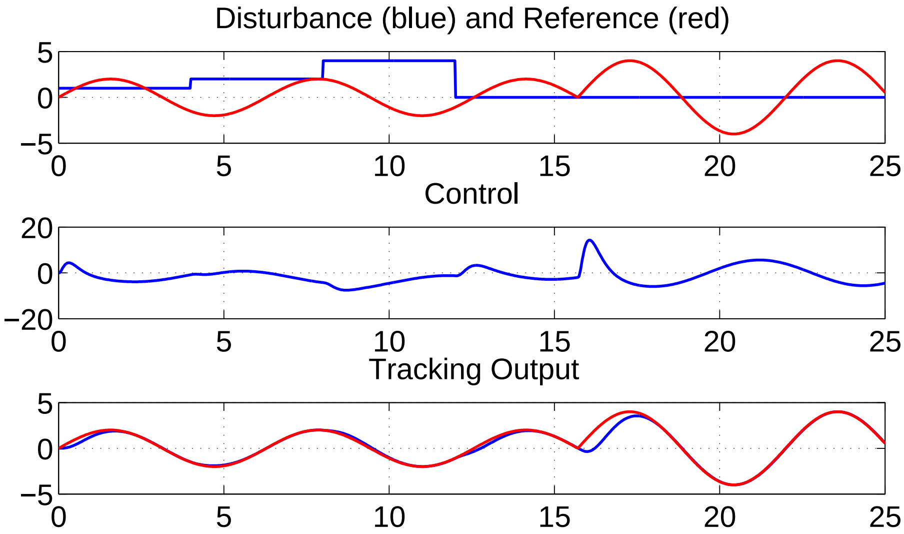
$$y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} d_2, \quad z = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

## How Did I solve the Regulator Equation?

```
%solve regulator equation with Kronecker calculus
M=[A B;Ch Dh];
n=size(A,1);n1=size(M,1)-n;n2=size(M,2)-n;
As=kron(eye(3),M);
As=As-kron(S',[eye(n) zeros(n,n2);zeros(n1,n) zeros(n1,n2)]);
bs=-vec([Bd;Dhd]);
x=As\bs;
X=reshape(x,3,3);
Pi=X(1:2,:);
Ga=X(3,:);
%check whether indeed satisfied:
M*[Pi;Ga]-[Pi;zeros(1,3)]*S+[Bd;Dhd]
```

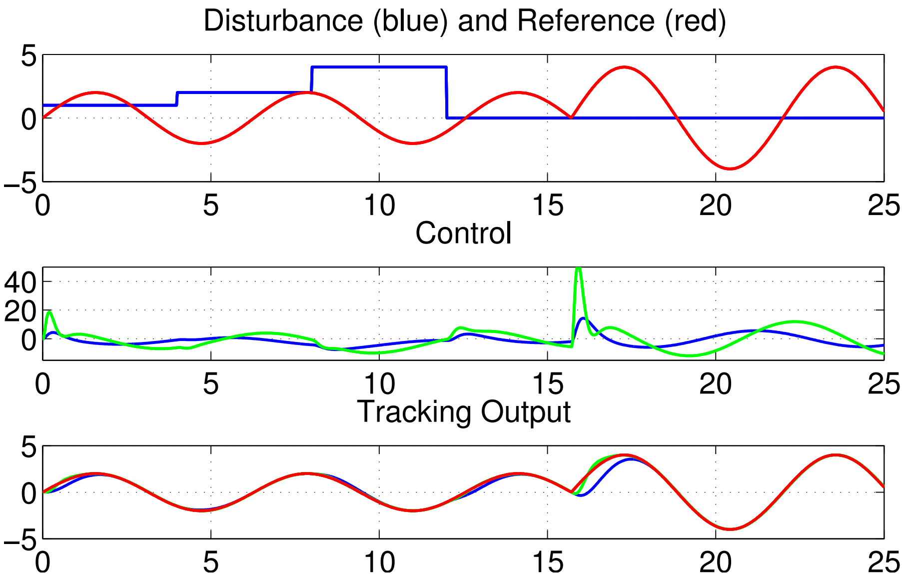
# Motor Example

We arrive at the following nice simulation results:



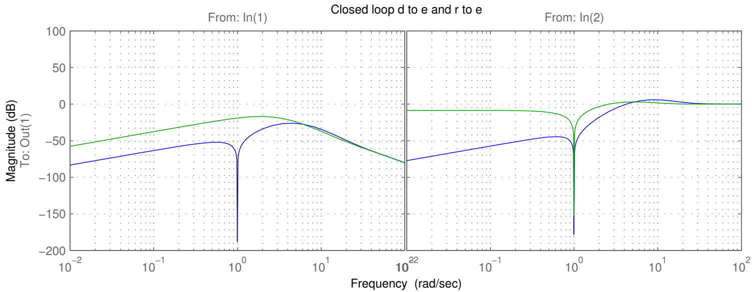
# Motor Example

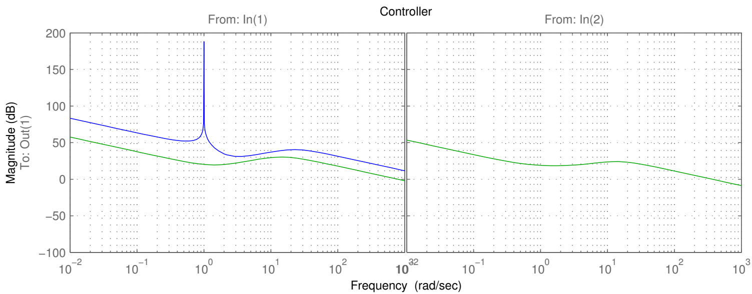
If only the tracking error is measurable $(y = \phi - r)$ we obtain:



# Related Bode Plots

Two-degrees (green) versus error feedback (blue) designs:



## Discussion

The following observations can be made in case that $y = e$:

- The constructed controller incorporates a model of the generalized disturbances. Under specific hypothesis it can be shown that this is a necessary property for a controller to achieve regulation. This fact is often called the **internal-model principle**.

- If the disturbances are directly measured, a regulator typically places **zeros** of the controlled system at the to-be-suppressed frequencies by "rightly balancing the measurements". This can be sensitive to modeling errors (non-robustness).

- If the disturbances are not measured, the controller typically has **poles** at the to-be-suppressed frequencies which creates zeros of the closed-loop system at those frequencies. Usually, designs are more robust. For constant disturbances the controller has **integral action**!

- There is a systematic way to design **robust regulators**.

# Covered in Lecture 6

- Synthesis for performance
  Tracking and disturbance rejection, one- and two-degrees of freedom controller design
- Disturbance models
  constant disturbances, sinusoidal disturbances
- Internal model principle
  state-feedback design, the regulator equation, disturbance estimators, output feedback design