How to do (minimal) realisations

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There are three methods of increasing complexity:

1) **Standard method:** Convert transfer functions to strictly proper, polynomial expressions. (I.e. standard form). Perform then the realisation like shown in §(L5-35) [1] to the controllable canonical form (ccf) or the observable canonical form (ocf), and stack the realisations using stacking rules shown in §(L5-43).
2) **Gilbert's method:** If transfer functions of single poles are present, then one can use Gilbert's method to complete the realisation. That is: $D + C\left(\frac{1}{s-p}\right)B \rightarrow \begin{bmatrix} pI & B \\ C & D \end{bmatrix}$
   This method is a good "one-shot" method to a minimal realisation.
3) **Common denominator merging and product merging:** Look for common denominators and separate the transfer function $G(s)$ into $G(s) = G_1(s) + G_2(s)$ according to them. This is because the realisation of common denominators can be merged, as explained in §(L5-42). (Note that you will have to perform the realisation into ccf or ocf depending on whether you are merging columns or rows respectively.)
   Try to also minimise the amount of terms that need to be added to $G(s)$, as every addition is essentially a stacking of matrices.
   Finally try to rewrite $G_1(s)$ and $G_2(s)$ in products of smaller matrices, as the realisation of two matrices will also lead to a merging, and thus leads to smaller realisations. (See §(L5-39).)

[1] I use the section symbol § as a symbol for citations and it means "according to.../citing...".

Below the exercises from Set 6 are solved step by step, showing how the methods mentioned above are applied.

5 a) Compute the minimal realisation of $G(s) = \frac{s+1}{s^2-1}$

We could use here brute force and use the standard method (see Method 1), since $G(s)$ is already in strictly proper, polynomial form.

However they are asking here for a minimal realisation, and using the standard method is only a 50-50 gamble. Instead we can directly check if we can rearrange the transfer function to get a realisation that is more likely to be minimal, (see Method 2).

Method 1 – Standard method

$G(s) = \frac{s+1}{s^2-1}$ ← This expression already comes in the standard form: $g(s) = \frac{\beta_1 s^{n-1} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n} + d.$

As shown in §(L5-35), this can be realised into:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccccc|c} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \hline \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n & d \end{array}\right)$$

controllable canonical realization

or

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cccccc|c} -\alpha_1 & 1 & 0 & \cdots & 0 & \beta_1 \\ -\alpha_2 & 0 & 1 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} & 0 & \cdots & 0 & 1 & \beta_{n-1} \\ -\alpha_n & 0 & 0 & \cdots & 0 & \beta_n \\ \hline 1 & 0 & 0 & \cdots & 0 & d \end{array}\right)$$

observable canonical realization

choose.
(I prefer first.)

$\therefore$ A realisation of $G(s) = \frac{s+1}{s^2-1}$ is:

$$G(s) \xrightarrow{\mathcal{R}\,(ccf)} \left[\begin{array}{cc|c} 0 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 0 \end{array}\right]$$

$\hookrightarrow$ Doing the Hautus test though (can use eigenvalue method), the realisation is controllable, but not observable!

NOT MINIMAL!

$\det(A - \lambda I) = -\lambda^2 + 1$

$\lambda^2 - 1 = 0$

$\lambda = \frac{\pm\sqrt{-4(-1)(1)}}{2} = \pm\frac{\sqrt{4}}{2} = \pm 1$

or $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

or $\lambda^2 - 1 = \lambda^2 - 1^2 = (\lambda+1)(\lambda-1)$

$\hookrightarrow$ We could now look for the responsible mode that is unobservable, diagonilise the realisation so that we can then take it out of the matrix expression.

OR, we do from the start a better realisation.

Method 2 - (Gilbert's method)

$G(s) = \frac{s+1}{s^2-1}$. This can be rewritten as follows:

$$\frac{s+1}{s^2-1} = \frac{s+1}{s^2-1^2} = \frac{s+1}{(s+1)(s-1)} = \frac{1}{s-1}$$

Can also use quadratic formula

Can also use partial fractions

This is a single pole expression! We can use Gilbert's method!

Gilbert's method states that if $G(s) = D + C\frac{1}{s-p}B$, then its realisation is $\left[\begin{array}{c|c} pI & B \\ \hline C & D \end{array}\right]$.

In our case:

$$G(s) = \frac{s+1}{s^2-1} = \frac{1}{s-1} = 0 + I\frac{1}{s-1}I \xrightarrow{\mathcal{R}} \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right]$$

This is minimal!

5 b)

$$G(s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}\frac{1}{s+1} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\frac{1}{s-1} + \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}\frac{1}{s-3}.$$

All single poles!

↳ Perfect for Gilbert's method!

We can however also rewrite the 2nd and 3rd term with smaller matrices, which will result in a smaller realisation (thus more likely to be minimal).

reference column

1st column (like reference column

2nd column = 2 · reference column

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\frac{1}{s+1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}\frac{1}{s+1}\begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \frac{1}{s-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s-1} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

has the form: $0 + \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{s-3} I$

$$\therefore\ G(s) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \frac{1}{s+1} \begin{bmatrix} 1 & 2 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s-1} \begin{bmatrix} 1 & 0 \end{bmatrix}}_{G_2(s)} + \underbrace{\begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{s-3}}_{G_3(s)}$$

$R(G_1(s))$, $R(G_2(s))$, $R(G_3(s))$:

$$\left[\begin{array}{c|cc} -1 & 1 & 2 \\ \hline 1 & 1 & 0 \\ -2 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|cc} 3 & & 1 & \\ & 3 & & 1 \\ \hline -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Then, using the fact mentioned in §(L5-39):

If $G_1(s)$, $G_2(s)$ have realizations (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2) then $G_1(s)G_2(s)$ and $G_1(s) + G_2(s)$ have the realizations

$$\left(\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right).$$

we simply stack our realisations of $G(s) = G_1(s) + G_2(s) + G_3(s)$.

This results in:

$$\left[\begin{array}{cccc|cc} -1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ \hline 1 & 0 & -2 & 2 & 1 & 0 \\ -2 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

This realisation is minimal!

The final, badass problem:

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5 c) $G(s) = \begin{pmatrix} \frac{s+2}{s+1} & 1 & \frac{1}{s} \\ \frac{s+2}{(s+1)^2} & \frac{1}{s} & \frac{1}{s^2} \end{pmatrix}$

§(L5-42), if $G(s)$ has common denominators in either a row or column, the realisations can be merged together, instead of using stacking methods. This will result into smaller matrices ≡ more likely to be minimal!

WARNING! This is not mentioned in §(L5-42), but for a row merging of realisations to be possible, the realisations have to be in the observable canonical form!

(And for column merging, use ccf.)

∴ row ⟷ ocf
column ⟷ ccf

When going back to $G(s) = \begin{pmatrix} \frac{s+2}{s+1} & 1 & \frac{1}{s} \\ \frac{s+2}{(s+1)^2} & \frac{1}{s} & \frac{1}{s^2} \end{pmatrix}$

with a bit of trickery one can see that the columns can be put with common denominators. (We will therefore need to make our realisations into ccf.)

That is:

$$G(s) = \begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & \frac{s}{s} & \frac{s}{s^2} \\ \frac{(s+2)}{(s+1)^2} & \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s}{s} & 0 \\ 0 & \frac{1}{s} & 0 \end{bmatrix}}_{G_2(s)} + \underbrace{\begin{bmatrix} 0 & 0 & \frac{s}{s^2} \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}}_{G_3(s)}$$

Now we could try to already do the realisations. However remember that we would prefer to use merging techniques over stacking techniques. (Stacking techniques are shown in § (L5-43).)

And the summation of two transfer functions like $G(s) = G_1(s) + G_2(s)$, mentioned also in § (L5-39), also results in stacking! ☹

And we want to avoid this as much as possible!

Our expression now is of the form of $G(s) = G_1(s) + G_2(s) + G_3(s)$. We must ask ourselves, can we reduce the number of terms?

In fact we can! We can see that the common denominators of $G_2(s)$ and $G_3(s)$ can also be made to have one common denominator for both $G_2(s)$ and $G_3(s)$. Let's combine the two then!

$$G(s) = \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s}{s} & 0 \\ 0 & \frac{1}{s} & 0 \end{bmatrix}}_{G_2(s)} + \underbrace{\begin{bmatrix} 0 & 0 & \frac{s}{s^2} \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}}_{G_3(s)}$$

$$= \begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{s^2}{s^2} & 0 \\ 0 & \frac{s}{s^2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{s}{s^2} \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s^2}{s^2} & \frac{s}{s^2} \\ 0 & \frac{s}{s^2} & \frac{1}{s^2} \end{bmatrix}}_{G_2(s)}$$

$$\underbrace{\begin{bmatrix} \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} \quad \underbrace{\begin{bmatrix} 0 & \frac{s}{s^2} & \frac{1}{s^2} \end{bmatrix}}_{G_2(s)}$$

Beside the merging technique using common denominators as shown in §(L5-42), the product of to transfer functions $G(s) = G_1(s) \cdot G_2(s)$ also results in a merging technique! (See §(L5-39)).

Can we therefore make our $G(s)$ expression better? Using products of smaller matrices (which will result in smaller realisations).

Once again, yes we can!

$$G(s) = \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s^2}{s^2} & \frac{s}{s^2} \\ 0 & \frac{s}{s^2} & \frac{1}{s^2} \end{bmatrix}}_{G_2(s)}$$

$$= \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} \\ \frac{(s+2)}{(s+1)^2} \end{bmatrix}}_{G_{11}(s)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{G_{12}(s)} + \underbrace{\begin{bmatrix} \frac{s}{s^2} \\ \frac{1}{s^2} \end{bmatrix}}_{G_{21}(s)} \underbrace{\begin{bmatrix} 0 & s & 1 \end{bmatrix}}_{G_{22}(s)}$$

Now the terms of G_1 are quite "nice". That is, the left matrix consists of proper functions, and G_{12} is just constants.

G_2 though has a transfer function that is not proper! Can we maybe change that?

$$G_2(s) = \begin{bmatrix} \frac{s}{s^2} \\ \frac{1}{s^2} \end{bmatrix} \begin{bmatrix} 0 & s & 1 \end{bmatrix}$$

pull out $\frac{1}{s}$

$$= \frac{1}{s} \begin{bmatrix} \frac{s}{s} \\ \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 & s & 1 \end{bmatrix}$$

insert $\frac{1}{s}$

$$= \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 & 1 & \frac{1}{s} \end{bmatrix}$$

This is already much better! We have in fact gained two advantages:

1) G_2 now only has proper transfer functions.
2) The transfer functions are of a single pole! We can use Gilbert's method then!

Our final expression for $G(s)$ is:

$$G(s) = \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} \\ \frac{(s+2)}{(s+1)^2} \end{bmatrix}}_{G_{11}(s)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{G_{12}(s)} + \underbrace{\begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix}}_{G_{21}(s)} \underbrace{\begin{bmatrix} 0 & 1 & \frac{1}{s} \end{bmatrix}}_{G_{22}(s)}$$

This is pretty much as far as we can go with our rearranging.

NOW let's do some realisations!

$\boxed{G_{11}:}$

Not strictly proper! strictly proper!

$$g_{11,11}(s) = \frac{(s+2)(s+1)}{(s+1)^2} = \frac{s^2+3s+2}{s^2+2s+1} = \frac{(s^2+2s+1)+s+1}{s^2+2s+1} = 1 + \frac{(s+1)}{s^2+2s+1}$$

$$\therefore g_{11,11}(s) \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 1 \end{array}\right]$$

$$g_{11,21}(s) = \frac{(s+2)}{(s+1)^2} = \frac{s+2}{s^2+2s+1}$$

$$\therefore\ g_{11,21}(s) \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 2 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{bmatrix} g_{11,11} \\ g_{11,21} \end{bmatrix} \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$$

$\boxed{G_{12}:}$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \xrightarrow{\mathcal{R}} \left[\begin{array}{|ccc} 1 & 0 & 0 \end{array}\right]$$

This can directly be treated as a D-matrix, which remains unchanged when realised.

Proof:

I just prefer to do my realisations into ccf.

$$g_{12,11}(s) = 1 \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \end{array}\right]$$

$$g_{12,12}(s) = 0 \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right]$$

$$g_{12,13}(s) = 0 \xrightarrow{\mathcal{R}(ccf)}$$

(see (L5-43)).

Stacking them results in:

$$\left[\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Good news, A is diagonal, $\text{eig}(A) = \{0\}$.

However when checking unobservability, we can see that $\lambda = 0$ is an unobservable mode. (You can do the math or just see how since $C = [0\ 0\ 0]$, whatever the information the system has, we are litterally not getting anything.)

Therefore, to get a minimal realisation of $G_{12}(s)$, take out the parts describing the unobservable mode $\lambda = 0$.

That leaves us only with the D-matrix.

$\boxed{G_1(s)}$

Therefore, the realisation of $G_1(s) = G_{11}(s) \cdot G_{12}(s)$ is, $\{$(L5-38):

I'm showing these lines to make clear this is a D-matrix.

$$G_{11}(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right] \quad ; \quad G_{12}(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{|ccc} 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$$

$$G_1(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{c|c} A_1 & B_1 D_2 \\ \hline C_1 & D_1 D_2 \end{array}\right] = \left[\begin{array}{cc|ccc} -2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{array}\right]$$

From §(L5-35), middle column and row dissapear! We don't have those!

$\boxed{G_{21}:}$

$$G_{21}(s) = \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix}$$ ← We can use Gilberts' method!

$$G_{21}(s) = \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \begin{bmatrix} 1 \end{bmatrix}$$

$$\therefore G_{21}(s) \xrightarrow{\mathcal{R}(G)} \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \\ 1 & 0 \end{array}\right]$$

$\boxed{G_{22}:}$

$$G_{22}(s) = \begin{bmatrix} 0 & 1 & \frac{1}{s} \end{bmatrix}$$ ← We can use Gilberts' method!

$$G_{22}(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \frac{1}{s} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\therefore G_{22}(s) \xrightarrow{\mathcal{R}(G)} \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{\boxed{G_2(s):}}$

$$G_2(s) = G_{21}(s) \cdot G_{22}(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

FINALLY, let's put together $G(s) = G_1(s) + G_2(s)$, as explained in §(L5-39).

$$G_1(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|ccc} -2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{array}\right] ; \quad G_2(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$G(s) = G_1(s) + G_2(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc|ccc} -2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

And indeed, this realisation can be shown to be [illegible]

↪ And indeed, this realisation can be shown to be <u>minimal</u>.