SC4025 Mid-Term Exam (2009)

All your answers have to be provided with solid arguments; you have to explicitly refer to results on the slides or the books! Just giving results or answers (even if correct) will not be awarded with credits. Wrong arguments will lead to a reduction of credits.

Make sure to write down your student id-number on each page you hand in!

Problem 1

Let the following system with two inputs u_1, u_2 and two outputs y_1, y_2 be given:

$$\ddot{z}_1 + \ddot{z}_2 + \dot{z}_1 + z_1 = u_1 + u_2, \quad \ddot{z}_2 + \dot{z}_1 + z_2 = u_2, \quad y_1 = z_1, \quad y_2 = \dot{z}_2.$$

a) With $x = (z_1, \dot{z}_1, z_2, \dot{z}_2)$ re-write this system into the form $\dot{x} = Ax + Bu$, $y = Cx$.

b) Verify that the system is controllable.

c) Is the system is stabilizable by using the control input u_2 only (i.e. for $u_1 = 0$)?

d) Find a matrix $F \in \mathbb{R}^{2\times 4}$ such that $u = -Fx$ (using both control inputs) stabilizes the system.

Problem 2

A genetic circuit is modeled as

$$\dot{m} = \frac{2}{1+p^2} - \alpha m - u, \quad \dot{p} = \alpha m - p$$

with state (m, p), control input u and some parameter $\alpha \neq 0$.

a) Show that there exists a unique real equilibrium point for this system if the control input vanishes.

b) Linearize the system around this equilibrium.

c) For which parameter values $\alpha \neq 0$ is the linearization asymptotically stable?

d) Is the linearization controllable?

e) Determine a state-feedback gain matrix F which places the eigenvalues of the linearization into $\{-2, -1\}$.

f) Suppose we want to use this state-feedback gain in order to control the original nonlinear system. Can we do this without affecting the location of the equilibrium and achieving local asymptotical stability?

Problem 3

Suppose you are given the system

$$\dot{x} = Ax + Bu = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x.$$

a) Is it possible to reach the state $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ from $x(0) = 0$ at time $T = 1$?

b) Compute e^{At}.

c) Compute $C(sI - A)^{-1}B$.

d) Use the Hautus test to determine the uncontrollable modes.

e) Is the system stabilizable?

f) Does there exist F such that the derivative of $e^{(A-BF)t}$ converges to zero for $t \to \infty$?

SC4025 Mid-Term Exam (2010)

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Problem 1 (25 points)

Consider the following spring-mass system:

$$m\ddot{z} + kz = 0$$

where the output is the displacement z.

a) (5p) Obtain a state-space representation for the above system in the form of

$$\dot{x} = Ax$$

$$y = Cx.$$

b) (10p) Is A diagonalizable? If yes, find the transformation that diagonalizes A. If not, then compute its Jordan form and a corresponding transformation matrix.

c) (5p) Compute e^{At}.

d) (5p) Derive an expression for the response of the system with initial condition x_0.

Problem 2 (40 points)

Consider the motion of a space vehicle about the principle axes of inertia. The dynamics of the space vehicle is given by the following equations:

$$P\dot{\omega}_x - (Q-R)\omega_y\omega_z = T_x$$

$$Q\dot{\omega}_y - (R-P)\omega_z\omega_x = T_y$$

$$R\dot{\omega}_z - (P-Q)\omega_x\omega_y = T_z$$

where P, Q and R are the moments of inertia about the principle axes; ω_x, ω_y and ω_z denote the angular velocities about the principle axes; T_x, T_y and T_z are the control torques.

a) (5p) Obtain a system description in terms of the first-order differential equation:

$$\dot{x} = f(x,u), \quad x(0) = x_0.$$

b) (15p) Assume that the space vehicle is tumbling in orbit. It is desired to stop the tumbling by applying control torques, which are assumed to be

$$T_x = k_1 P\omega_x, \quad T_y = k_2 Q\omega_y, \quad T_z = k_3 R\omega_z.$$

Determine the closed-loop equilibria of the system.

Problem 3 (9+8+4+4=25 Points)

Consider the system $\dot{x} = Ax + Bu$ where

$$A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & -2 \end{pmatrix}, \quad B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

a) Find the state transformation $z = Tx$, $\dot{z} = \tilde{A}z + \tilde{B}u$ such that $\tilde{A}$ is in Jordan form. Determine $\tilde{B}$ as well.

b) Is the state $x_F(T=1) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$ reachable from initial state zero? If no, argue why not. If yes, find the input that steers the system from initial state zero to x_F.
(Hint: In order to avoid unnecessary calculations, consider the problem in the transformed z-coordinates).

c) Is the zero state $x_F(T=1) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$ reachable from the initial state $x_I(0) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$? If no, argue why not. If yes, find the input that steers the system from x_I to x_F.

d) Is the zero state reachable asymptotically? If no, argue why not. If yes, design a state feedback gain that stabilizes the system by placing the closed-loop poles at -1.

SC4025 Mid-Term Exam (2011)

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Make sure to write down your student id-number on each page you hand in!

Problem 1 (5+5=10 Points)

The following algebraic and differential equations describe the depicted circuit:

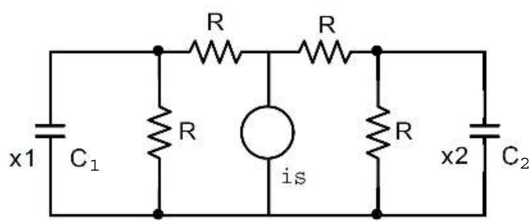
$$i_{c_1} = -\frac{3}{2R}v_{c_1} + \frac{i_s}{2} + \frac{1}{2R}v_{c_2}$$

$$i_{c_2} = \frac{1}{2R}v_{c_1} + \frac{i_s}{2} - \frac{3}{2R}v_{c_2}$$

$$\frac{dv_{c_1}}{dt} = \frac{i_{c_1}}{c_1}$$

$$\frac{dv_{c_2}}{dt} = \frac{i_{c_2}}{c_2}$$

where $R = 1\Omega$ and $C_1 = C_2 = 1F$. The capacitor voltages v_{c_1} and v_{c_2} can be regarded as the states of the system.



a) Find the modes of the system and the corresponding mode-shapes based on a state-space description of the circuit.

b) Find the responses of the system using the mode-shapes as the initial state vectors. You need to calculate a response for each mode-shape. What can be concluded from the obtained responses?

Problem 2 (7+5+3+5=20 Points)

Consider the following state-space system:

$$\dot{x} = \underbrace{\begin{pmatrix} -8 & -8 & 0 \\ 8 & 7 & 1 \\ 7 & 8 & -1 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{B} u \qquad y = \underbrace{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}}_{C} x.$$

a) Compute the Jordan form of A.

b) Compute e^{At}.

c) Is it possible to drive the system from the initial state $x(0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$ to the final state described by $x(t_f) = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$, $t_f = 2.5$?

d) Now consider the following system:

$$\dot{x} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix} x + \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} u \qquad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x.$$

Is this system equivalent to the former one? If yes, prove the equivalence, if not, then explain and show why!

Problem 3 (7+5+5+8=25 Points)

Consider the following nonlinear dynamical system

$$\ddot{\theta}(t) + \dot{\theta}(t) + \frac{g}{l}\sin(\theta(t)) = 0, \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0, \quad t \geq 0$$

where g and l are constants.

a) Calculate all the equilibrium points of the system.

b) Does the function

$$V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1 - \cos\theta)$$

qualify as a Lyapunov function for the given system?

c) Establish whether the nonlinear system is stable around its different equilibrium points.

d) Verify the results of part (c) using the indirect method of Lyapunov.

SC4025 Mid-Term Exam (2012)

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Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (3+6+4+4+3=20 Points)

Suppose you are given the system

$$\dot{x} = Ax + Bu = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x.$$

a) Is it possible to reach the state $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ from $x(0) = 0$ at time $T = 1.2$?

b) Compute e^{At}.

c) Compute $C(sI - A)^{-1}B$. (see Laplace transform table below if needed)

d) Use the Hautus test to determine the uncontrollable modes.

e) Is the system stabilizable?

Laplace transform of some functions

$y(t)$	$Y(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Problem 2 (6+4+6=16 Points)

Consider a mechanical system composed of a unit mass attached to a nonlinear spring with a velocity dependent damper:

$$\ddot{q} + B(\dot{q}) + K(q) = 0$$

where

$$B(\dot{q}) = \alpha\dot{q} \quad (\alpha \neq 0)$$
$$K(q) = \beta q^3 \quad (\beta \neq 0)$$

A candidate energy function of the system is:

$$V(q, \dot{q}) = \frac{\dot{q}^2}{2} + \frac{\beta q^4}{4}$$

a) Derive the conditions on α and β under which the system is locally stable at its equilibrium point.

b) Can it be concluded from the Lyapunov stability theory whether the system is locally asymptotically stable at the equilibrium point? Motivate your answer.

c) Verify if the system is locally asymptotically stable at the equilibrium point using the indirect method of Lyapunov.

Problem 3 (6+4+3+3+4=20 Points)

A genetic circuit is modeled as

$$\dot{m} = \frac{2}{1+p^2} - \alpha m - u, \quad \dot{p} = \alpha m - p$$

with state (m, p), control input u and some parameter $\alpha \neq 0$.

a) Show that there exists a unique real equilibrium point for this system if the control input vanishes.

b) Linearize the system around this equilibrium.

c) For which parameter values $\alpha \neq 0$ is the linearization asymptotically stable?

d) Is the linearization controllable?

e) Determine a state-feedback gain matrix F which places the eigenvalues of the linearization into $\{-3, -1\}$.

SC4025 Mid-Term Exam (2013)

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Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (5+5+5+2+5=22 Points)

An industrial plant's waste is fed into a pool of bacteria that transforms the waste into non-polluting forms. Maintaining the bacteria concentrations at effective levels is a critical problem. Under certain circumstances, the bacteria's ability to grow is primarily dependent on the supply of some nourishing organic substrate (e.g., glucose or the waste itself). Let x_1 be the concentration of bacteria in the pool and x_2 be the concentration of the substrate. A simple mathematical model of growth is then given as

$$\begin{aligned} \dot{x}_1 &= \frac{x_2}{x_2 + K} x_1 - D x_1 \\ \dot{x}_2 &= -\frac{x_2}{x_2 + K} x_1 + D(1 - x_2) \end{aligned} \tag{1}$$

where $K > 0$ and $0 < D < 1$ are constants. A stability analysis will reveal whether in the face of slight disturbances the culture will continue to be effective.

a) Determine the equilibrium points of the system (1). How many distinct equilibrium points are there as a function of D and K? What are the conditions on D and K to ensure that *all* equilibrium points are non-negative?

b) Assume that $K = \frac{1-D}{D}$ and determine if the corresponding equilibrium point is asymptotically stable.

c) Consider the equilibrium point $x_e = (0, 1)$. What are the general conditions on D and K such that this equilibrium point is asymptotically stable?

d) Do the conditions on D and K obtained in question c) guarantee that *all* equilibria are physically meaningful for this system?

e) What are general conditions on D and K such that *all other* equilibria (besides $x_e = (0, 1)$) are asymptotically stable?

Problem 2 (4+4+4=12 Points)

Consider the linear time-invariant (LTI) state-space systems given by

$$\dot{x} = A_i x + B_i u, \qquad x(0) = x_0, \qquad \text{for } i = 1, 2, 3$$

where

$$\left[\begin{array}{c|c} A_1 & B_1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 6 & -2 & -4 \end{array}\right], \quad \left[\begin{array}{c|c} A_2 & B_2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 6 & -2 & 4 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_3 & B_3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 6 & -2 & 1 \end{array}\right].$$

a) For which of these systems is it possible to design a stabilizing state feedback of the form $u(t) = -Kx(t)$?

b) For which of these systems is it possible to relocate the eigenvalues of $A - BK$ to any arbitrary location by suitable choice of K?

c) Design a stabilizing feedback controller for the systems obtained in a).

Problem 3 (5+3+4=12 Points)

Let a linear time-invariant (LTI) system be defined as having the following impulse response:

$$h(t) = \begin{cases} \delta(t) + te^{-3t} + 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

with $\delta(t)$ being a Dirac impulse.

a) Find a state-space model given by the state matrices A, B, C, D for this system. Verify that the obtained state space model indeed has $h(t)$ as impulse response.

b) Is the obtained state-space model Lyapunov stable? Is it asymptotically stable? Explain your answer.

c) Compute the transfer function of the system.

SC4025 Mid-Term Exam (2014)

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Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (3+3+3+3+3+3=18 Points)

Consider the nonlinear model $\dot{x} = f(x, u)$ of an AC/DC converter, given by

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{L}x_2 u + \frac{E}{L}, \\ \dot{x}_2 &= -\frac{1}{RC}x_2 + \frac{1}{C}x_1 u. \end{aligned}$$

where $x_1(t), x_2(t), u(t) \in \mathbb{R}$ and E, R, C and L are positive scalars.

a) Let $u(t) = u_e > 0$ with u_e constant. Compute all corresponding equilibria of the system.

b) Write down the linearized model of the system at the following equilibrium:

$$x_{1e} = \frac{E}{Ru_e^2}, \quad x_{2e} = \frac{E}{u_e}, \quad u_e > 0.$$

c) Study the stability properties of the linearized model determined in part b).

d) Based on your answer in c), what can you conclude about the stability of this equilibrium of the original nonlinear system?

e) Assume that $u(t) = 0$ is chosen as the input signal for all t (in order to boost the energy of the system). Show that in this mode of operation the system does not have any equilibria and that $\lim_{t\to\infty} x_1(t) = \infty$ and $\lim_{t\to\infty} x_2(t) = 0$.

f) Consider again the equilibrium given in part b) and further assume $u(t) = u_e = 1$ and $E = R = C = L = 1$. Check if the function $V(x) \ : \ \mathcal{D} \to \mathbb{R}$ with

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

specified over the domain

$$\mathcal{D} = \left\{x \in \mathbb{R} \mid \|x\|^2 \leq 10\right\}$$

qualifies as a Lyapunov function for the resulting autonomous system. What can we conclude regarding stability of the equilibrium from this result?

Problem 2 (3+6+6+3=18 Points)

Consider the following spring-mass system:

$$m\ddot{z} + kz = 0$$

where the output is the displacement z.

a) Obtain a state-space representation for the above system in the form of

$$\dot{x} = Ax$$

$$y = Cx.$$

b) Is A diagonalizable? If yes, find the transformation that diagonalizes A. If not, then compute its Jordan form and a corresponding transformation matrix.

c) Compute e^{At}.

d) Derive an expression for the response (output) of the system with initial condition x_0 using trigonometric functions.

Problem 3 (3+6+3+6=18 Points)

Consider two systems with the following state space descriptions.

$$\begin{gathered} A_1 = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ C_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D_1 = 0, \end{gathered} \tag{1}$$

$$\begin{gathered} A_2 = \begin{pmatrix} -15 & 13 \\ -7 & 5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ C_2 = \begin{pmatrix} 1 & 0.5 \end{pmatrix}, \quad D_2 = 0. \end{gathered} \tag{2}$$

a) Give the state space description of the system resulting from connecting systems (1) and (2) in parallel.

b) Determine if the resulting system is controllable.

c) Determine if the resulting system is stabilizable.

d) Determine a state feedback gain F such that the closed-loop system is stable. (*Hint:* You may try to choose the poles of the closed-loop system to be at $s_1 = -2$, $s_2 = -2$, $s_3 = -3$, and $s_4 = -8$.)

SC4025 Mid-Term Exam (2015)

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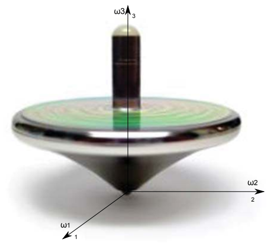
Problem 1 (2+3+3+3=11 Points)

The dynamics of a symmetric spinning-top (see figure) with nonzero principal moments of inertia $I_1 = I_2 \neq I_3$ are described by the following equations:

$$
\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3,\\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1,\\
I_3\dot{\omega}_3 &= 0.
\end{aligned}
$$



where $\omega = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix}^T$ are the angular velocities around the three axes.

a) Determine the equilibria ω_e of the system assuming that $\omega_3 \neq 0$.

b) Linearize the system at the equilibrium points ω_e determined in a).

c) Use the indirect Lyapunov method to study whether ω_e in a) are locally asymptotically stable.

d) Use the direct Lyapunov method to show that $\omega_e = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$ is a locally stable equilibrium point. Is it also locally asymptotically stable? Hint: choose the rotational energy

$$V(\omega) = \frac{1}{2}\left(I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2\right)$$

as candidate Lyapunov function with $I = I_1 = I_2$.

Problem 2 (6+6=12 Points)

Given a linear, time-invariant, continuous-time system

$$
\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t),
$$

$$
y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},
$$

a) Find a piecewise constant input of the form

$$u(t) = \begin{cases} \alpha & ,0 \leq t < 1, \\ \beta & ,1 \leq t \leq 2, \end{cases}$$

which transfers the system from the initial state $x(0) = \begin{pmatrix} 0 & 2 \end{pmatrix}^T$ at time $t = 0$ to the final state $x(2) = \begin{pmatrix} 2 & 0 \end{pmatrix}^T$ at time $t = 2$. If this is not possible, then argue why not.

b) Let $x(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$. Find a finite input $u(t)$ of the form

$$u(t) = \begin{cases} \alpha & ,0 \leq t < 1, \\ \beta & ,1 \leq t \leq 2, \\ 0 & ,2 < t, \end{cases}$$

such that $y(t) = t$ for all $t \geq 2$. If this is not possible, then argue why not.

Problem 3 (2+5+3+3+2=15 Points)

Suppose you are given the system

$$\dot{x} = Ax + Bu = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x.$$

a) Is it possible to reach the state $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ from $x(0) = 0$ at time $T = 2.015$?

b) Compute e^{At}.

c) Compute $C(sI - A)^{-1}B$. (see Laplace transform table below if needed)

d) Use the Hautus test to determine the uncontrollable modes.

e) Is the system stabilizable?

Laplace transform of some functions

$y(t)$	$Y(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

SC42015 Mid-Term Exam (2016)

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Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (1+2+3+3+1+2+1=13 Points)

Consider the following LTI system

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

with system matrices

$$A = \begin{pmatrix} -4 & -5 & -5 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

a) Verify that the system is unstable.

b) Is it possible to stabilize the system such that the poles of the closed loop become $\{-2, -2, -2\}$?

c) Find a state transformation $z = Tx$, $\dot{z} = \tilde{A}z + \tilde{B}u, y = \tilde{C}z$ such that the resulting $\tilde{A}$ state matrix is in Jordan canonical form. Determine the transformed input and output matrices $\tilde{B}$, $\tilde{C}$ as well.

d) Determine a state-feedback gain in the transformed coordinates $u = \tilde{K}z$ such that all the poles of the resulting closed-loop system are placed at -4 and $\tilde{K}$ has the least possible Frobenius norm[1].

e) Using the computed transformation matrix, determine a state-feedback gain K for the original system such that the poles are all placed at -4.

f) Show that there exists no control input such that the state $x_F = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ can be reached starting from zero initial condition.

g) Show that, on the other hand, starting from initial state $x(0) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, there does exist a control input such that as time tends to infinity the zero state can be asymptotically reached.

[1]The Frobenius, or the "entry-wise" vector 2-norm is $\|M\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |m_{ij}|^2\right)^{\frac{1}{2}}$

Problem 2 (2+3+3+3=11 Points)

Consider the nonlinear autonomous system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2(x_3 - x_1) \\ x_1^2 - 1 \\ -x_1 x_3 \end{pmatrix}.$$

a) Find the equilibrium point(s).

b) Find the linearized system about each equilibrium point.

c) For each case in part b), what does Lyapunov theory tell us about the stability of the nonlinear system near the equilibrium point?

d) Consider using the following Lyapunov function candidate to investigate stability at each equilibrium point:

$$V(x_1, x_2, x_3) = (x_1 - 1)^2 + x_2^2 + x_3^2$$

Can we conclude the same as in part c) ?

Problem 3 (5+3+3=11 Points)

Let a linear time-invariant (LTI) system be defined as having the following impulse response:

$$h(t) = \begin{cases} \delta(t) + te^{-3t} + 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

with $\delta(t)$ being a Dirac impulse.

a) Find a state-space model given by the state matrices A, B, C, D for this system. Verify that the obtained state space model indeed has $h(t)$ as impulse response.

b) Is the obtained state-space model Lyapunov stable? Is it asymptotically stable? Explain your answer.

c) Compute the transfer function of the system.

Laplace transform of some functions

$y(t)$	$Y(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

SC42015 Mid-Term Exam (2017)

All your answers have to be provided with solid arguments; you have to explicitly refer to results in the slides or the books! Show clearly how you have arrived at your results! Just giving results or answers (even if correct) will not be awarded with credits.

Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (3+2+3=8 Points)

Consider the following linear time-invariant (LTI) state-space system:

$$\dot{x} = \underbrace{\begin{pmatrix} -8 & -8 & 0 \\ 8 & 7 & 1 \\ 7 & 8 & -1 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{B} u, \quad y = \underbrace{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}}_{C} x$$

a) Compute the Jordan form of A, and the corresponding transformation matrix.

b) Is it possible to drive the system from the initial state $x(0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$ to the final state described by $x(t_f) = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$, with $t_f = 20.17$?

c) Now consider the following system:

$$\dot{x} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix} x + \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

Is this system equivalent to the former one? If yes, prove the equivalence, if not, then explain and show why!

Problem 2 (3+3+3=9 Points)

Consider the LTI state-space systems given by

$$\dot{x} = A_i x + B_i u, \qquad x(0) = x_0, \qquad \text{for } i = 1, 2, 3$$

where

$$\left[\begin{array}{c|c} A_1 & B_1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 6 & -2 & -4 \end{array}\right], \quad \left[\begin{array}{c|c} A_2 & B_2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 6 & -2 & 4 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_3 & B_3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 6 & -2 & 1 \end{array}\right].$$

a) For which of these systems is it possible to design a stabilizing state feedback of the form $u(t) = -Kx(t)$?

b) For which of these systems is it possible to relocate the eigenvalues of $A - BK$ to any arbitrary location by suitable choice of K?

c) Design a stabilizing feedback controller for the systems obtained in a).

Problem 3 (3+3+3+1+3=13 Points)

An industrial plant's waste is fed into a pool of bacteria that transforms the waste into non-polluting forms. Maintaining the bacteria concentrations at effective levels is a critical problem. Under certain circumstances, the bacteria's ability to grow is primarily dependent on the supply of some nourishing organic substrate (e.g., glucose or the waste itself). Let x_1 be the concentration of bacteria in the pool and x_2 be the concentration of the substrate. A simple mathematical model of growth is then given as

$$\begin{aligned} \dot{x}_1 &= \frac{x_2}{x_2 + K} x_1 - D x_1 \\ \dot{x}_2 &= -\frac{x_2}{x_2 + K} x_1 + D(1 - x_2) \end{aligned} \tag{1}$$

where $K > 0$ and $0 < D < 1$ are constants. A stability analysis will reveal whether in the face of slight disturbances the culture will continue to be effective.

a) Determine the equilibrium points of the system (1). How many distinct equilibrium points are there as a function of D and K? What are the conditions on D and K to ensure that *all* equilibrium points are non-negative?

b) Assume that $K = \frac{1-D}{D}$ and determine if the corresponding equilibrium point is asymptotically stable using the indirect method of Lyapunov.

c) Consider the equilibrium point $x_e = (0, 1)$. What are the general conditions on D and K such that this equilibrium point is asymptotically stable?

d) Do the conditions on D and K obtained in question c) guarantee that *all* equilibria are physically meaningful for this system?

e) What are general conditions on D and K such that *all other* equilibria (besides $x_e = (0, 1)$) are asymptotically stable?

SC42015 Mid-Term Exam (2018)

All your answers have to be provided with solid arguments; you have to explicitly refer to results in the slides or the books! Show clearly how you have arrived at your results! Just giving results or answers (even if correct) will not be awarded with credits.

Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (2+4+4+2=12 Points)

Consider the following spring-mass system

$$m\ddot{z} + kz = 0,$$

where the output is the displacement z. For notational convenience, we can define the quantity $\omega = \sqrt{\frac{k}{m}}$.

a) Obtain a state-space representation for the above system in the form of

$$\dot{x} = Ax,$$

$$y = Cx.$$

b) Is A diagonalizable? If yes, find the transformation that diagonalizes A. If not, then compute its Jordan form and a corresponding transformation matrix.

c) Compute e^{At}.

d) Derive an expression for the response (output) of the system with initial condition x_0 using trigonometric functions.

Problem 2 (6+6=12 Points)

Consider the nonlinear system

$$\begin{aligned}
\dot{x}_1 &= x_2 - x_3^3 \\
\dot{x}_2 &= -x_1^3 \\
\dot{x}_3 &= x_1^3 + \alpha(x_3^3 - x_3).
\end{aligned}$$

a) Use the direct Lyapunov method with $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^4$ as Lyapunov function candidate to analyze the stability of the origin, i.e., $x_e = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$, for different values of α.

b) Use the indirect Lyapunov method to analyze the stability of the origin for different values of α.

Problem 3 (9+8+4+4=25 Points)

Consider the system $\dot{x} = Ax + Bu$ where

$$A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & -2 \end{pmatrix}, \quad B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

a) Find the state transformation $z = Tx$, $\dot{z} = \tilde{A}z + \tilde{B}u$ such that $\tilde{A}$ is in Jordan form. Determine $\tilde{B}$ as well.

b) Is the state $x_F(T = 1) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$ reachable from initial state zero? If no, argue why not. If yes, find the input that steers the system from initial state zero to x_F.
(Hint: In order to avoid unnecessary calculations, consider the problem in the transformed z-coordinates).

c) Is the zero state $x_F(T = 1) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$ reachable from the initial state $x_I(0) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$? If no, argue why not. If yes, find the input that steers the system from x_I to x_F.

d) Is the zero state reachable asymptotically? If no, argue why not. If yes, design a state feedback gain that stabilizes the system by placing the closed-loop poles at -1.

SC42015 Mid-Term Exam (2019)

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Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (1+2+3+3+1+1+1=12 Points)

Consider the following LTI system

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

with system matrices

$$A = \begin{pmatrix} -4 & -5 & -5 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

a) Verify that the system is unstable.

b) Is it possible to stabilize the system such that the poles of the closed loop become $\{-2, -2, -2\}$?

c) Find a state transformation $z = Tx$, $\dot{z} = \tilde{A}z + \tilde{B}u, y = \tilde{C}z$ such that the resulting $\tilde{A}$ state matrix is in Jordan canonical form. Determine the transformed input and output matrices $\tilde{B}$, $\tilde{C}$ as well.

d) Determine a state-feedback gain in the transformed coordinates $u = \tilde{F}z$ such that all the poles of the resulting closed-loop system are placed at -4 and $\tilde{F}$ has the least possible Frobenius norm[1].

e) Using the computed transformation matrix, determine a state-feedback gain F for the original system such that the poles are all placed at -4.

f) Show that there exists no control input such that the state $x_f = (1, 1, 1)^\top$ can be reached starting from zero initial condition.

g) Show that, on the other hand, starting from initial state $x(0) = (1, 1, 1)^\top$, there does exist a control input such that as time tends to infinity the zero state can be asymptotically reached.

[1]The Frobenius, or the "entry-wise" vector 2-norm is $\|M\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|m_{ij}|^2\right)^{\frac{1}{2}}$

Problem 2 (1+4+3=8 Points)

Consider a nonlinear model of a pendulum with friction, described by the following state-space system

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -a \sin x_1 - b x_2 \end{pmatrix} = f(x)$$

with $a > 0$, $b \geq 0$ and the state vector defined over the state space domain as $(x_1, x_2)^\top \in (-\pi, \pi] \times \mathbb{R}$.

a) Show that the system has **only** two equilibria in the above domain: $(x_{e1}, x_{e2})^\top = (0, 0)^\top$ and $(x_{e1}, x_{e2})^\top = (\pi, 0)^\top$.

b) Using the indirect method of Lyapunov, which values of a, b allow us to conclude that the equilibrium $(x_{e1}, x_{e2})^\top = (0, 0)^\top$ is asymptotically stable, unstable, or leave us without any conclusion? Answer the same question for the second equilibrium $(x_{e1}, x_{e2})^\top = (\pi, 0)^\top$ as well.

c) Consider only those cases in the above question, for which you could not conclude stability or instability. Then use the following candidate Lyapunov function

$$V(x) = a(1 - \cos x_1) + \frac{1}{2} x_2^2$$

to determine stability using Lyapunov's direct method. Explain how the resulting outcome (stability or instability of the certain case) agrees or disagrees with your physical intuition.

Problem 3 (3+7=10 Points)

Consider the system

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \quad y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} x,$$

with zero initial state $x(t = 0) = (0, 0)^\top$.

a) Diagonalize the matrix A and find the corresponding coordinate transformation matrix.

b) Consider the control problem of starting from the zero initial state, steering the system to the final state $x_f = (1, 0)^\top$ at $t = \ln 2$, and staying there for all $t \geq \ln 2$. To achieve this, we want to make use of a piecewise linear input $u(t)$ shown in Figure 1. Determine the values of the parameters u_1, u_2, u_3 of the input given in Figure 1 that solves this control problem.

Hint: You might find the following integrals useful:

$$\int e^t t \;\mathrm{d}t = e^t(t - 1) + C,$$

$$\int e^{2t} t \;\mathrm{d}t = \frac{1}{4} e^{2t}(2t - 1) + C.$$

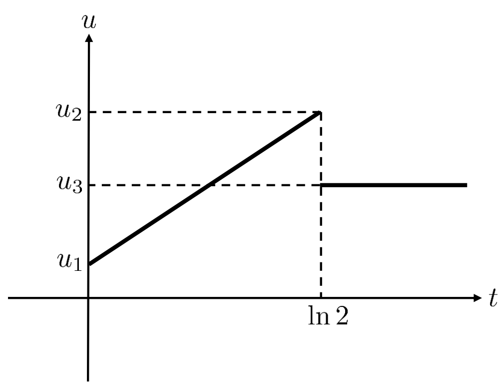

Figure 1: Piece-wise linear input $u(t)$.

SC42015 Mid-Term Exam (2020)

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Work out each of the three problems on a different sheet of paper and make sure to write down your student id-number on each page you hand in!

Problem 1 (4+2+1=7 Points)

Let a linear time-invariant (LTI) system be defined as having the following impulse response:

$$h(t) = \begin{cases} te^{-5t} + e^{-5t} + 2t, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

a) Find a state-space model given by the state matrices A, B, C, D for this system. Verify that the obtained state space model indeed has $h(t)$ as impulse response.

b) Is the obtained state-space model Lyapunov stable? Is it asymptotically stable? Explain your answer.

c) Compute the transfer function of the system, and specify the coefficients of the numerator polynomial.

Laplace transform of some functions

$y(t)$	$Y(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Problem 2 (1+3+2=6 Points)

Consider the following linear system:

$$\dot{x} = Ax + Bu,$$

with the matrices

$$A = \begin{pmatrix} 3 & 8 \\ -1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -4 \\ 1 \end{pmatrix}.$$

a) Is the system stabilizable?

b) Compute e^{At}.

c) Is it possible to design a state feedback gain K such that the closed-loop system $A - BK$ has both modes at -1? If no, then argue why not. If yes, then find a valid gain matrix K that achieves this with minimum Frobenius norm and non-negative entries.

   Note: The Frobenius, or the "entry-wise" vector 2-norm of an $n \times m$ matrix M is the square root of the squared sum of all its terms, i.e., $\|M\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} |M_{ij}|^2 \right)^{\frac{1}{2}}$.

Problem 3 (2+3=5 Points)

Let us consider the following continuous-time nonlinear system with variable $\beta > 0$:

$$\begin{aligned} \dot{x}_1 &= \beta(2 - x_1) + x_1^2 x_2 \\ \dot{x}_2 &= x_1 - x_1^2 x_2, \end{aligned}$$

a) Compute the equilibrium points of the system.

b) Investigate the stability of the equilibrium points using the indirect method of Lyapunov.