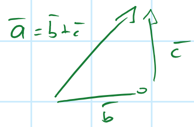
Linear independence

Wednesday, September 18, 2019 4:01 PM

↳ What makes a set of vectors independent?

→ A vector that can be described as a linear combination of the other vectors present in the set is said to be dependent.

Else, independent!



↳ There are 2 methods commonly used to show whether the vectors are dependent or not.

Source: Linear independ...

1) Reduce matrise

Form the expression Tx = 0. Row reduce it. If a single solution can be found for each x_i in x, then it is linearly dependent.
Intuitively speaking, if the vectors are linearly dependent, a transformation using them as unit vectors will collapse the dimensions of the vector x into a smaller number, because one of the unit vectors in T can be written as another.

2) Take determinant

This follows the idea from above, but only works with square matrices. If T causes a collapse of dimensions, then that is equivalent to its determinant being zero. (Because the transformation volume gets nullified.)

Rank & Nullity

Wednesday, September 18, 2019 4:03 PM

Definition:

a) the rank (column space / row space)

↳ The maximum number of linearly independent vectors in a **matrix** is equal to the number of non-zero rows in its row echelon **matrix**. Therefore, to find the **rank of a matrix**, we simply transform the **matrix** to its row echelon form and count the number of non-zero rows.

↳ the number of zero columns is therefore the nullity.

Intuition of rank and null space:

Let's suppose that the matrix A represents a physical system. As an example, let's assume our system is a rocket, and A is a matrix representing the directions we can go based on our thrusters. So what do the null space and the column space represent?

Well let's suppose we have a direction that we're interested in. Is it in our column space? If so, then we can move in that direction. The column space is the set of directions that we can achieve based on our thrusters. Let's suppose that we have three thrusters equally spaced around our rocket. If they're all perfectly functional then we can move in any direction. In this case our column space is the entire range. But what happens when a thruster breaks? Now we've only got two thrusters. Our linear system will have changed (the matrix A will be different), and our column space will be reduced.

What's the null space? The null space are the set of thruster intructions that completely waste fuel. They're the set of instructions where our thrusters will thrust, but the direction will not be changed at all.

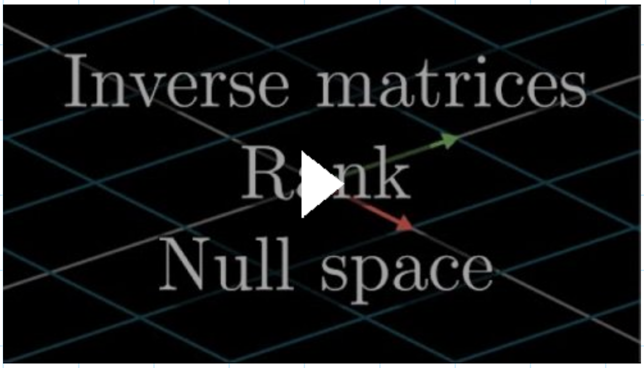
Another example: Perhaps A can represent a rate of return on investments. The range are all the rates of return that are achievable. The null space are all the investments that can be made that wouldn't change the rate of return at all.

Another example: room illumination. The range of A represents the area of the room that can be illuminated. The null space of A represents the power we can apply to lamps that don't change the illumination in the room at all.

https://math.stackexchange.com/questions/21131/physical-meaning-of-the-null-space-of-a-matrix

Also good:

Inverse matrices, column space and null space | Essence of linear algebra, chapter 7



Relation between Rank space and eigenvectors

↳ Rank space is the space a transformation can operate in.

Eigenvectors on the other hand are state where the solution is working.

Remember that a solution to a sys. is $\bar{x} = \bar{\eta} e^{\lambda t}$
and the general solution is $\bar{x} = c\bar{x}_1 + c\bar{x}_2$
$= c\bar{\eta}_1 e^{\lambda_1 t} + c\bar{\eta}_2 e^{\lambda_2 t} \ldots$

Therefore the space the eigenvectors span of eigenvalues $\neq 0$ are contained in the range space of the sys., but the range space may be bigger.

The space the eigenvectors span of eigenvalues $= 0$ however ARE the same as the null space!

Sources:

If you look at a matrix as a linear operator, $T(v) = Av$ then the column space is just the range of that linear operator. Eigenvectors for **non-zero** eigenvalues will be members of the range (if $Av = \lambda v$, then $A(\lambda^{-1} v) = v$).

So the span of the eigenvectors with non-zero eigenvalues, is contained in the column space.

(...and the span of the eignevectors with eigenvalue zero **is** the null space.)

Gram Schmidt Process

Wednesday, September 18, 2019 4:06 PM

What is the Gram-Schmidt process?

→ We form a new set of vectors that are orthogonal using as a (space) basis the previous set of vectors.

Source: Gram–Schmidt proc...

↳ Method:

The Gram–Schmidt process [edit]

We define the projection operator by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the inner product of the vectors **u** and **v**. This operator projects the vector **v** orthogonally onto the line spanned by vector **u**. If **u** = **0**, we define $\mathrm{proj}_{\mathbf{0}}(\mathbf{v}) := 0$. i.e., the projection map $\mathrm{proj}_{\mathbf{0}}$ is the zero map, sending every vector to the zero vector.

The Gram–Schmidt process then works as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\ \ \vdots & &\ \ \vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

Review sine cosine table

Tuesday, November 5, 2019   12:29 PM

angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
sin	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$
cos	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$	$-\frac{\sqrt{1}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{4}}{2}$
tan	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{2}}$	$\sqrt{\frac{3}{1}}$	■	$-\sqrt{\frac{3}{1}}$	$-\sqrt{\frac{2}{2}}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{0}{4}}$
cot	■	$\sqrt{\frac{3}{1}}$	$\sqrt{\frac{2}{2}}$	$\sqrt{\frac{1}{3}}$	0	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{2}}$	$-\sqrt{\frac{3}{1}}$	■
csc	■	$\frac{2}{\sqrt{1}}$	$\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{4}}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{1}}$	■
sec	$\frac{2}{\sqrt{4}}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{1}}$	■	$-\frac{2}{\sqrt{1}}$	$-\frac{2}{\sqrt{2}}$	$-\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{4}}$

How to take the inverse matrix

Wednesday, September 18, 2019 3:53 PM

To find the inverse of a 2x2 matrix: swap the positions of a and d, put negatives in front of b and c, and divide everything by the **determinant** (ad-bc).

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

Basic methods:

https://www.wikihow.com/Find-the-Inverse-of-a-3x3-Matrix

3 Easy Ways to Find the Inverse of a 3x3 Matrix - wikiHow

-> Method 1 (Cramer's rule):

$$S^{-1} = \frac{1}{\det(S)} adj(S)$$

Where $adj(S) = C^T$, where C is the cofactor matrix of S:

Consider a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Its cofactor matrix is

$$\mathbf{C} = \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix},$$

-> Method 2 (Gauss-Jordan elimination) is very straightforward, but lengthy. (Lots of calculations, prone to mistakes, but better option for large matrices.)

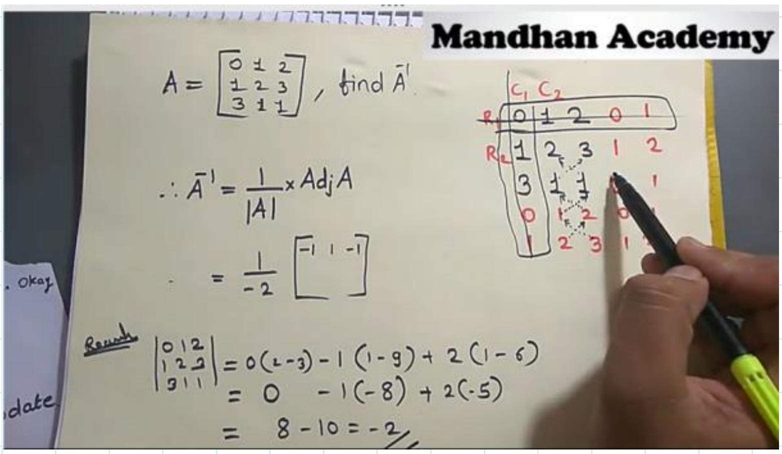
2> $AA^{-1} = I \Rightarrow [A \,\vdots\, I] = A^{-1}$

↳ just like we would solve for a vector!

Shortcut method 3x3 matrix:

Describes Method 1 in a bit more detail. Is fast!

Shortcut Method to Find A inverse of a 3x3 Matrix



↳ Check if it works with 4x4 matrise as well! No! Does not work with 4x4! Use Gaussian elimination method.

Example from: https://semath.info/src/inverse-cofactor-ex4.html

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

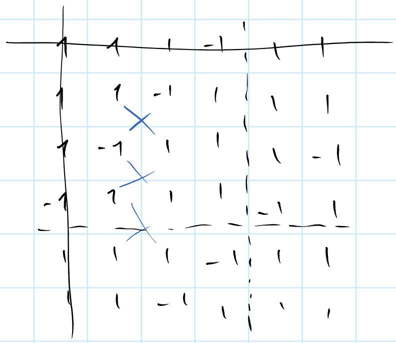
$\Rightarrow$ Cramers rule says that: $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$\therefore$ Step 1 - find determinant

We would find that $\det(A) = -16$

Step 2 - find adjoint

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow$$



$\therefore \operatorname{adj}(A) = \begin{bmatrix} 0 & -2 & 0 & -2 \\ & & & \\ & & & \end{bmatrix}$ ⟵ This is already wrong, should be [-4, -4, -4, 4]!

Method 3: Use Cayley-Hamilton theorem

The Cayley-Hamilton theorem states that for all square matrices, the characteristic equation of such a matrix is also its solution.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Its characteristic polynomial is given by

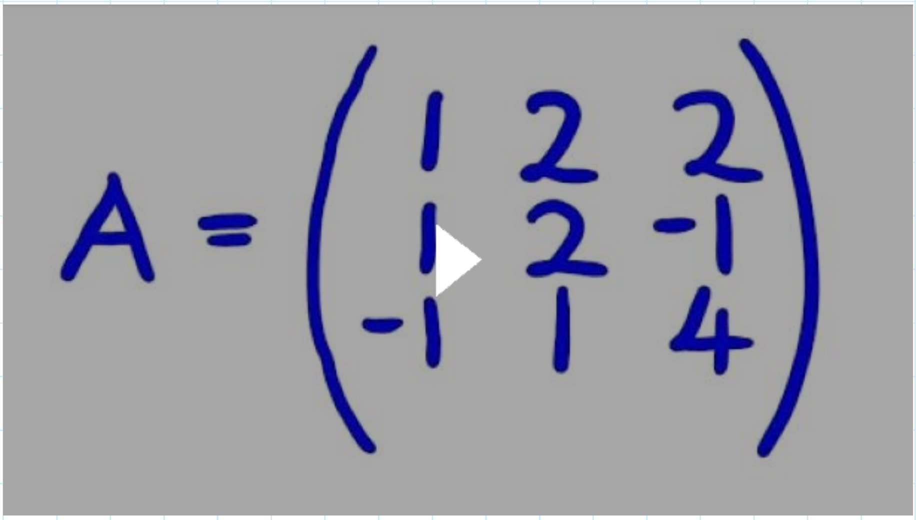
$$p(\lambda) = \det(\lambda I_2 - A) = \det\begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} = (\lambda - 1)(\lambda - 4) - (-2)(-3) = \lambda^2 - 5\lambda - 2.$$

then

$$A^2 - 5A - 2I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And this is true for any power. So $A^3 - 5A^2 - 2A = 0$ is also true.

This video shows how the Cayley-Hamilton theorem can be used to calculate the inverse:
[Cayley-Hamilton Theorem: Inverse of 3x3 Matrix]



Review Solving System of Differential Equations

Thursday, October 17, 2019 1:20 PM

As usual, Pauls online notes to the rescue!

Differential Equations - Solutions to Systems

Paul's Online Notes

Home / Differential Equations / Systems of DE's / Solutions to Systems

Section 5-5 : Solutions To Systems

Now that we've got some of the basics out of the way for systems of differential equations it's time to start thinking about how to solve a system of differential equations. We will start with the homogeneous system written in matrix form,

$$\vec{x}\,' = A\vec{x} \tag{1}$$

where, A is an $n \times n$ matrix and $\vec{x}$ is a vector whose components are the unknown functions in the system.

Now, if we start with $n = 1$then the system reduces to a fairly simple **linear** (or **separable**) first order differential equation.

$$x' = ax$$

and this has the following solution,

$$x(t) = \mathbf{c}e^{a\,t}$$

So, let's use this as a guide and for a general n let's see if

$$\vec{x}(t) = \vec{\eta}\,\mathbf{e}^{r\,t} \tag{2}$$

will be a solution. Note that the only real difference here is that we let the constant in front of the exponential be a vector. All we need to do then is plug this into the differential equation and see what we get. First notice that the derivative is,

$$\vec{x}\,'(t) = r\vec{\eta}\,\mathbf{e}^{r\,t}$$

So, upon plugging the guess into the differential equation we get,

$$\boxed{\begin{aligned} r\vec{\eta}\mathbf{e}^{r\,t} &= A\vec{\eta}\mathbf{e}^{r\,t} \\ (A\vec{\eta} - r\vec{\eta})\,\mathbf{e}^{r\,t} &= \vec{0} \\ (A - rI)\,\vec{\eta}\mathbf{e}^{r\,t} &= \vec{0} \end{aligned}}$$

Now, since we know that exponentials are not zero we can drop that portion and we then see that in order for (2) to be a solution to (1) then we must have

$$(A - rI)\,\vec{\eta} = \vec{0}$$

$$\frac{dx}{dt} = a\,x$$

$$dx = a\,x\,dt$$

$$\frac{1}{x}\,dx = a\,dt$$

$$\int \frac{1}{x}\,dx = a\int dt$$

$$\ln(x) = a\,t + c$$

$$x = e^{at+c}$$

$$x = C\,e^{at}$$

$$x = x_0\,e^{at}$$

Or, in order for (2) to be a solution to (1), r and $\vec{\eta}$ must be an eigenvalue and eigenvector for the matrix A.

Therefore, in order to solve (1) we first find the eigenvalues and eigenvectors of the matrix A and then we can form solutions using (2). There are going to be three cases that we'll need to look at. The cases are real, distinct eigenvalues, complex eigenvalues and repeated eigenvalues.

None of this tells us how to completely solve a system of differential equations. We'll need the following couple of facts to do this.

Fact

1. If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions to a homogeneous system, (1), then

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

is also a solution to the system.

2. Suppose that A is an n x n matrix and suppose that $\vec{x}_1(t)$, $\vec{x}_2(t)$, ..., $\vec{x}_n(t)$ are solutions to a homogeneous system, (1). Define,

$$X = (\vec{x}_1 \; \vec{x}_2 \; \cdots \; \vec{x}_n)$$

In other words, X is a matrix whose *ith* column is the i^{th} solution. Now define,

$$W = \det(X)$$

We call W the **Wronskian**. If $W \neq 0$ then the solutions form a **fundamental set of solutions** and the general solution to the system is,

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

Note that if we have a fundamental set of solutions then the solutions are also going to be linearly independent. Likewise, if we have a set of linearly independent solutions then they will also be a fundamental set of solutions since the Wronskian will not be zero.

 Page Last Modified : 6/3/2018

Review Eigenvalues

Monday, January 20, 2020 11:09 AM

Finding Eigenvalues:

For A being a nxn matrix:

- If A is **diagonal**, the eigenvalues are on the diagonal.
- If A is **upper or lower triangular**, the eigenvalues are on the diagonal.
- If A has **rows or columns with zeros everywhere except a value on the diagonal,** that value is an eigenvalue of the system. Proceed by reducing A by that row or column and find the rest of the eigenvalues of the reduced A.

- Using Block matrices:

By definition, an eigenvalue λ of the block matrix A satisfies

$$\det\begin{pmatrix} B-\lambda I & C \\ 0 & D-\lambda I \end{pmatrix} = 0.$$

Using a property of block matrix determinants, we have

$$\det\begin{pmatrix} B-\lambda I & C \\ 0 & D-\lambda I \end{pmatrix} = \det(B-\lambda I)\det(D-\lambda I) = 0$$

Thus the eigenvalues of B, D are also the eigenvalues of A.

Checking if matrix is negative definite:

There may be no need to calculate the eigenvalues!

Example from wikipedia:

Definiteness of matrices

Definition

A minor of A of order k is principal if it is obtained by deleting $n-k$ rows and the $n-k$ columns with the same numbers. The leading principal minor of A of order k is the minor of order k obtained by deleting the last $n-k$ rows and columns.

minus!

Notation

We write D_k for the leading principal minor of order k. There are $\binom{n}{k}$ principal minors of order k, and we write Δ_k for any of the principal minors of order k.

Theorem

Let A be a symmetric $n \times n$ matrix. Then we have:

- *A is positive definite $\Leftrightarrow D_k > 0$ for all leading principal minors*
- *A is negative definite $\Leftrightarrow (-1)^k D_k > 0$ for all leading principal minors*
- *A is positive semidefinite $\Leftrightarrow \Delta_k \geq 0$ for all principal minors*
- *A is negative semidefinite $\Leftrightarrow (-1)^k \Delta_k \geq 0$ for all principal minors*

p.d. if all $D_k > 0$

n.d. if all odd $D_k < 0$

includes leading principal minors!

A is **indefinite** if the computed determinants do not match any of these criteria

If A is a square matrix, then the **minor** of the entry in the *i*-th row and *j*-th column (also called the ***(i,j)* minor**, or a **first minor**[1]) is the determinant of the submatrix formed by deleting the i-th row and j-th column. This number is often denoted $M_{i,j}$. The (i,j) **cofactor** is obtained by multiplying the minor by $(-1)^{i+j}$.

To illustrate these definitions, consider the following 3 by 3 matrix,

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix}$$

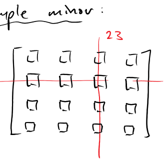
To compute the minor $M_{2,3}$ and the cofactor $C_{2,3}$, we find the determinant of the above matrix with row 2 and column 3 removed.

$$M_{2,3} = \det\begin{bmatrix} 1 & 4 & \square \\ \square & \square & \square \\ -1 & 9 & \square \end{bmatrix} = \det\begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = (9-(-4)) = 13$$

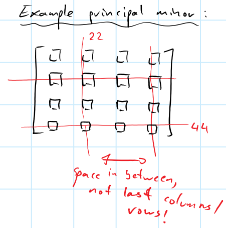
So the cofactor of the (2,3) entry is

$$C_{2,3} = (-1)^{2+3}(M_{2,3}) = -13.$$

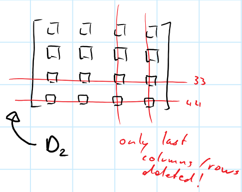
Example minor:



Example principal minor:



Example leading principal minor:



Example

Determine the definiteness of the symmetric 3×3 matrix

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{pmatrix}$$

Solution (Continued)

Let us instead try to use the leading principal minors. They are:

$$D_1 = 1, \quad D_2 = \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = -14, \quad D_3 = \begin{vmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{vmatrix} = -109$$

Let us compare with the criteria in the theorem:

- *Positive definite: $D_1 > 0, D_2 > 0, D_3 > 0$*
- *Negative definite: $D_1 < 0, D_2 > 0, D_3 < 0$*
- *Positive semidefinite: $\Delta_1 \geq 0, \Delta_2 \geq 0, \Delta_3 \geq 0$ for all principal minors*
- *Negative semidefinite: $\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0$ for all principal minors*

The principal leading minors we have computed do not fit with any of these criteria. We can therefore conclude that A is indefinite.

Or – when calculating **eigenvalues**:

- Positive (semi-)definite: all eigenvalues larger (or equal) than/to zero
- Negative (semi-)definite: all eigenvalues smaller (or equal) than/to zero
- Indefinite: when both positive and negative eigenvalues occur

↳ You can use this also for control theory!
If A is negative definite = stable!

Other facts:

- If A is **symmetric**, then it is guaranteed to have n eigenvalues and n eigenvectors.

A useful formula for the eigenvalues of a 2×2 matrix For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ satisfies

$$\boxed{p(\lambda) = \det(A) - \lambda \operatorname{Tr}(A) + \lambda^2,}$$ or $p(\lambda) = \lambda^2 - \operatorname{Tr}(A)\lambda + \det(A)$

where $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ is the determinant of A and $\operatorname{Tr}(A) = a_{11} + a_{22}$ is the trace of A. By setting the characteristic polynomial to zero, we find that the eigenvalues are given by

$$\lambda_{1,2} = \frac{\operatorname{Tr}(A)}{2} \pm \frac{1}{2}\sqrt{\operatorname{Tr}^2(A) - 4\det(A)}.$$

Please feel free to use this formula when solving your problems. Note that it is not necessary to compute the square root in order to classify the equilibrium at zero.

A useful test for stability For 2×2 matrices, the Routh–Hurwitz stability criterion takes a simple form. The eigenvalues of the matrix A are all in the open left half-plane $\mathcal{I}m(s) < 0$ if and only if

$$\boxed{\operatorname{Tr}(A) < 0 \quad \text{and} \quad \det(A) > 0.}$$

You can use this test to determine if the equilibrium point is stable.

Problem 1A Write the assumptions down that were made for the classification of the equilibrium points of two-dimensional linear time invariant systems $\dot{x}(t) = Ax(t)$.

Determinant

10 September 2020 11:42

Finding the determinant:

Method 1: Cofactor expansion

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(So you always go along one row or column.)

The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix, regardless of the sign of a_{ij} itself. The factor $(-1)^{i+j}$ determines the following checkerboard pattern of signs:

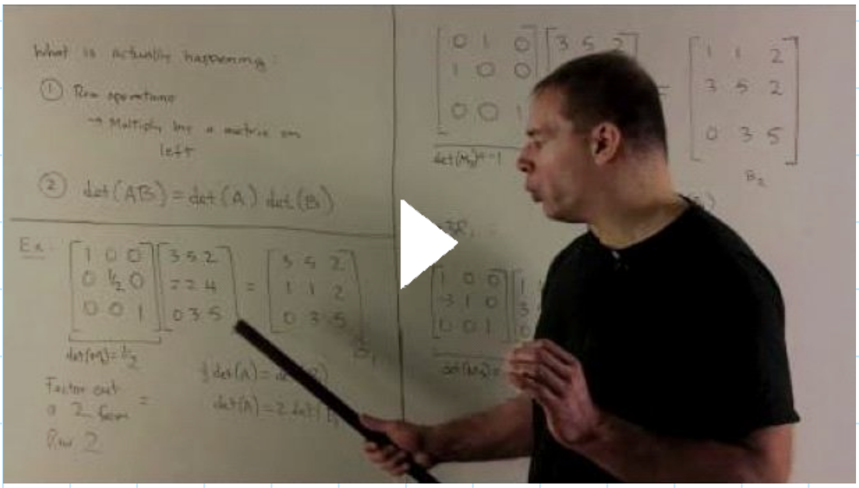
$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Method 2: Rule of Sarrus Shortcut method (only works for 3x3!)

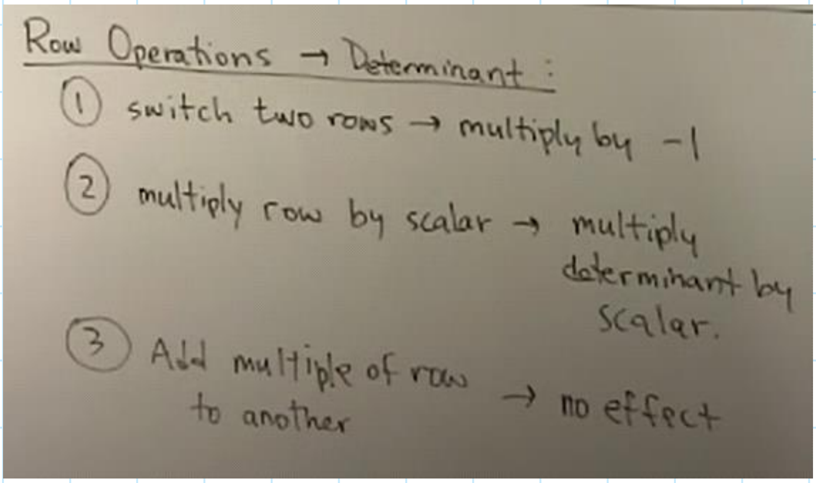
[Finding the determinant of a 3x3 matrix method 1 | Matrices | Precalculus | Khan Academy]

Method 3: Using row operations (Some extra rules, but great especially for larger matrices!)

[Example of Determinant Using Row Echelon Form]



Basically follow these rules:



Check out more methods here:

DeterminantComputa...

Determinant properties:

Some properties of Determinants

- $\det(\mathbf{A}) = \det(\mathbf{A}^T)$

  The value of the determinant of a matrix doesn't change if we transpose this matrix (change rows to columns)

- $\alpha^n \cdot \det(\mathbf{A}) = \det(\alpha \cdot \mathbf{A})$ α is a scalar, A is $n \times n$ matrix

  If we multiply a scalar α to a $n \times n$ matrix **A**, then the value of the determinant will change by a factor α^n !

- If an entire row or an entire column of **A** contains only zero's, then $\det(A) = 0$

  This makes sense, since we are free to choose by which row or column we will expand the determinant. If we choose the one containing only zero's, the result of course will be zero. And - it must be zero for all other possible expansions, too.

- $$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

  If two determinants differ by just one column, we can add them together by just adding up these two columns. For example:

  $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 9 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 9 \\ 1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 3 \\ 4 & 3 & 9 \\ 1 & 0 & 1 \end{vmatrix}$$

How do determinants behave in elementary row operations?

- Interchange of two rows: $\det(\mathbf{A}^*) = -\det(\mathbf{A})$

  The **sign** of the determinant **will change** if you interchange two rows - this has to do with the checkerboard pattern of the coefficients C_{ij} !

- Corresponding entries in two rows are proportional $\Rightarrow \det(\mathbf{A}) = 0$

  If the entries of two rows turn out to be proportional to each other we are able to eliminate one of these row entirely during Gauss elimination: all entries of one row eventually will become zero. Since we can choose this particular row as the one we expand the determinant by the result will become zero!

- Adding a multiple of a row/column to another $\det(\mathbf{A}^*) = \det(\mathbf{A})$

  All other elementary row operations will not affect the value of the determinant!

- We can use Gauss elimination to reduce a determinant to a triangular form!!!

We can use Gauss elimination to reduce a determinant to a triangular form….

Benefit: After this, we only have to multiply the diagonal elements with each other.

This turns out to be an enormous help to us: instead of calculating endlessly one minor after the other, we now have the alternative to simply make our determinant upper triangular using the same row operations we used during Gauss elimination.

However, there is one (but only one) thing we have to pay attention to: **If we exchange two rows with each other, the sign of the determinant will switch**. We **must** keep track on how often we interchange rows!

Let us try out what we just learned. Let us use Gauss elimination in order to obtain the following determinant:

$$D=\begin{vmatrix}2 & 2 & 3\\ 4 & 3 & 9\\ 1 & 0 & 1\end{vmatrix}$$

$$\begin{vmatrix}2 & 2 & 3\\ 4 & 3 & 9\\ 1 & 0 & 1\end{vmatrix}\begin{matrix} \\ \text{row2-2}\cdot\text{row1}\\ \text{row3-0.5}\cdot\text{row1}\end{matrix}$$

gives

$$\begin{vmatrix}2 & 2 & 3\\ 0 & -1 & 3\\ 0 & -1 & -0.5\end{vmatrix}\begin{matrix} \\ \\ \text{row3-row2}\end{matrix}$$

gives

$$\begin{vmatrix}2 & 2 & 3\\ 0 & -1 & 3\\ 0 & 0 & -3.5\end{vmatrix}=2\cdot(-1)\cdot(-3.5)=7$$

Is this the right result?

Let us verify it:

$$D=\begin{vmatrix}2 & 2 & 3\\ 4 & 3 & 9\\ 1 & 0 & 1\end{vmatrix}=2\begin{vmatrix}3 & 9\\ 0 & 1\end{vmatrix}-2\begin{vmatrix}4 & 9\\ 1 & 1\end{vmatrix}+3\begin{vmatrix}4 & 3\\ 1 & 0\end{vmatrix}=$$

$$=2\cdot3-2\cdot(4-9)+3\cdot(-3)=6-8+18-9=7$$

Same result!

In this example it was sort of hard to tell which method is easier. But imagine some 6^{th} order determinant containing almost no zero's…..

Try to obtain

$$D = \begin{vmatrix} 1 & 1 & 5 & 4 & 2 & 1 \\ 2 & -3 & 8 & -2 & 3 & 1 \\ 1 & 2 & -1 & 3 & 2 & 2 \\ -1 & 3 & 1 & 1 & 2 & 5 \\ 4 & 4 & 3 & 5 & -3 & 5 \\ 3 & 2 & 3 & 4 & 2 & 1 \end{vmatrix}$$

using both methods…..

There are more interesting properties of determinants …

Determinant of a Product of Matrices:

A, B are $n \times n$ matrices, then

$$\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{B} \cdot \mathbf{A}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

This is kind of surprising since we know that – in general – we cannot reverse the order of matrix multiplication:

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

Let us demonstrate this new feature:

Demonstration:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$\Rightarrow \det(\mathbf{A}) = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0$ and also $\Rightarrow \det(\mathbf{B}) = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$

Therefore, $\det(\mathbf{A}) \cdot \det(\mathbf{B}) = 0$. Yes, of course: A determinant can be zero even if not a single entry of the determinant itself is zero!

How about the matrix products?

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \det(\mathbf{A} \cdot \mathbf{B}) = \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = (-1) - (-1) = 0$$

$$\mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(\mathbf{B} \cdot \mathbf{A}) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

Therefore, indeed, $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{B} \cdot \mathbf{A}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$

Let's do this with another example – one, where the determinants do not equal to zero (just to make sure…). Let consider

$$\mathbf{A}=\begin{bmatrix}3 & 1\\ 1 & -1\end{bmatrix} \quad \text{and} \quad \mathbf{B}=\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}$$

Then,

$$\det(\mathbf{A})=\begin{vmatrix}3 & 1\\ 1 & -1\end{vmatrix}=-3-1=-4 \quad \text{and} \quad \det(\mathbf{B})=\begin{vmatrix}1 & 1\\ -1 & 2\end{vmatrix}=2-(-1)=3$$

and, for the products we get

$$\mathbf{A}\cdot\mathbf{B}=\begin{bmatrix}3 & 1\\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2 & 5\\ 2 & -1\end{bmatrix}\Rightarrow \det(\mathbf{A}\cdot\mathbf{B})=\begin{vmatrix}2 & 5\\ 2 & -1\end{vmatrix}=-12$$

and

$$\mathbf{B}\cdot\mathbf{A}=\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}3 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}4 & 0\\ -1 & -3\end{bmatrix}\Rightarrow \det(\mathbf{B}\cdot\mathbf{A})=\begin{vmatrix}4 & 0\\ -1 & -3\end{vmatrix}=-12$$

It seems to work!

Rank in terms of determinants

Here is another conclusion that might become handy:

For a $n\times n$ matrix **A**:

$\det(\mathbf{A})\neq 0 \Leftrightarrow \mathbf{A}^{-1}$ exists (or $\operatorname{rank}\mathbf{A}=n$)

If the value of a nth order determinant is not equal zero, then the rank of the associated matrix must be n.

This makes perfect sense: Since we can use Gauss elimination in order to simplify the calculation of our determinant eventually an entire row of the determinant has to be filled with zero's in order to let the determinant become zero. But then the rank of the associated matrix would be smaller than n – it would be n-1 the most.

Later on we will see that this is also an indicator if the inverse of a matrix does exist….

Calculating matrix powers

01 October 2020 10:05

Method 1: Brute force

This is the most basic method. You calculate one power of a matrix, and then keep assembling.

$A^2 = A \cdot A$

$A^3 = A \cdot A \cdot A = A^2 \cdot A$

Etc.

Can use also smart compositions then, like:

$A^9 = A^3 A^3 A^3$

Method 2: Cayley-Hamilton theorem

The Cayley-Hamilton theorem states that for all square matrices, the characteristic equation of such a matrix is also its solution.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Its characteristic polynomial is given by

$$p(\lambda) = \det(\lambda I_2 - A) = \det\begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} = (\lambda - 1)(\lambda - 4) - (-2)(-3) = \lambda^2 - 5\lambda - 2.$$

then

$$A^2 - 5A - 2I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And this is true for any power. So the following expressions are also all true:

$A^3 - 5A^2 - 2A = 0$

$A - 5I - 2A^{-1} = 0$ $\longleftarrow$ Can be used to calculate the inverse! See here. (Although the bigger the matrix, the bigger the characteristic equation, and for small matrices we already have quick methods.

Method 3: Diagonalisation with eigenvectors

If A is diagonalisable (there are n eigenvectors for the $n * n$ matrix A), then we can use the eigenvector decomposition to find the power of a matrix.

Given that $A = SDS^{-1}$

then

$A^3 = SDS^{-1} \cdot SDS^{-1} \cdot SDS^{-1}$ <- The $S \cdot S^{-1}$ cancel!

$\quad = SD^3S^{-1}$

Taking the power of a diagonal matrix is simple, because you simply take the power of the diagonal elements. Therefore once we have found D, it is simple to compute D^n in order to then find $A^n = SD^nS^{-1}$.

Method 4: Binomial formula and the Jordan matrix

The previous method only works if the matrix is diagonalisable. In Control theory though we also often end up with non-diagonalisable matrices, that can then be converted to Jordan matrices. ($A = SJS^{-1}$).

These are almost diagonal, in the sense that if one defines block matrices for every block in the Jordan matrix, it is. Therefore each matrix block can be evaluated on its own. It is then a fact that using the binomial equation, one can obtain any power of the Jordan block like:

$$J_k(\lambda)^n = \begin{bmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots & \cdots & \binom{n}{k-1}\lambda^{n-k+1} \\ & \lambda^n & \binom{n}{1}\lambda^{n-1} & \cdots & \cdots & \binom{n}{k-2}\lambda^{n-k+2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ & & & & & \lambda^n \end{bmatrix}$$

Where $\binom{n}{k}$ are the binomial coefficients that follow from pascals triangle, where n is the row in Pascals triangle.

Derivation:

This all is derived from the binomial equation. For example:

$J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, then the eigenvalue of this Jordan block is 3 with algebraic multiplicity of 2.

This can then be written like:

$J = 3I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \lambda I_2 + N$

Then to take the power of J to J^3, it is the same as:

$J^3 = \left(\lambda I_2 + N\right)^3$

Now note that with matrices, powers in brackets like these generally result in every combination of the elements. For example:

$(A+B)^2 = A^2 + AB + BA + B^2$

$(A+B)^3 = A^3 + AAB + ABA + ABB + BAA + BAB + BBA + B^3$

It is only when A and B commute that we can compress it to the familiar form of the binomial equation. That is:

$(A+B)^2 = A^2 + 2AB + B^2 \quad \Leftrightarrow \quad AB = BA$ (Double arrow means iff.)

That is the case for us! After all $I_2 N = N I_2$.

Therefore:

$$\begin{aligned} J^3 = \left(\lambda I_2 + N\right)^3 &= \binom{3}{0}\lambda^3 I_2^3 + \binom{3}{1}\lambda^2 I_2^2 N + \binom{3}{2}\lambda I_2 N^2 + \binom{3}{3} N^3 \\ &= 1 \cdot \lambda^3 I_2^3 + 3 \cdot \lambda^2 I_2^2 N + 3 \cdot \lambda I_2 N^2 + 1 \cdot N^3 \end{aligned}$$

In addition, the numbers that are off the diagonal of the Jordan matrix are always 1's. An important pattern arises when we take the powers of these.

For example:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$M^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore M^n = 0 \Leftrightarrow n > 2$; or in words, if the Jordan block is dimension $n = 3$, then all powers bigger than $n - 1$ will be zero!

Therefore, going back to calculate J^3, with n of J being equal to 2, all N matrices higher than the power of $n - 1 = 1$ are going to be zero. Thus:

$$\begin{aligned} J^3 &= 1 \cdot \lambda^3 I_2^3 + 3 \cdot \lambda^2 I_2^2 N + 3 \cdot \lambda I_2 N^2 + 1 \cdot N^3 \\ &= 1 \cdot \lambda^3 I_2^3 + 3 \cdot \lambda^2 I_2^2 N \\ &= \begin{bmatrix} 3^3 & 0 \\ 0 & 3^3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 & 3^2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + \begin{bmatrix} 0 & 27 \\ 0 & 0 \end{bmatrix} \\ &= 27 I_2 \end{aligned}$$

Or, in general form:

$$J^3 = \begin{bmatrix} \binom{3}{0}\lambda^3 & \binom{3}{1}\lambda^2 \\ 0 & \binom{3}{0}\lambda^3 \end{bmatrix} = \begin{bmatrix} 1 \cdot \lambda^3 & 3 \cdot \lambda^2 \\ 0 & 1 \cdot \lambda^3 \end{bmatrix}$$

which is the same structure as shown above.

Rank properties

20 October 2020 18:44

- The rank of an $m \times n$ matrix is a nonnegative integer and cannot be greater than either m or n. That is,

$$\operatorname{rank}(A) \leq \min(m, n).$$

A matrix that has rank $\min(m, n)$ is said to have *full rank*; otherwise, the matrix is *rank deficient*.

- If A is a square matrix (i.e., $m = n$), then A is invertible if and only if A has rank n (that is, A has full rank).

And for it to have full rank, the determinant of it cannot be zero, or no eigenvalue can be equal to zero. That also implies that if a matrix is positive or negative definite, it is invertible.

- If B is any $n \times k$ matrix, then

$$\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B)).$$

- If B is an $n \times k$ matrix of rank n, then

$$\operatorname{rank}(AB) = \operatorname{rank}(A).$$

- If C is an $l \times m$ matrix of rank m, then

$$\operatorname{rank}(CA) = \operatorname{rank}(A).$$

- **Sylvester's rank inequality**: if A is an $m \times n$ matrix and B is $n \times k$, then

$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \leq \operatorname{rank}(AB).$$ [i]

This is a special case of the next inequality.

Let us define the matrix

$$W_k = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} \quad \text{for any} \quad k = 1, 2, \ldots.$$

If A has dimension n then W_n is the observability matrix of (A, C).

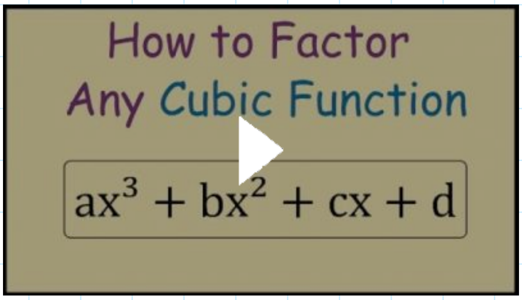
- The null space of W_{k+1} is always contained in the null space of W_k.
- The rank of W_{k+1} is not smaller than the rank of W_k.
- If the rank of W_{k+1} equals the rank of W_k then the null space of W_{k+1} equals the null space of W_k.
- If the rank of W_{k+1} equals the rank of W_k then the null space of $W_{k+\nu}$ is equal to the null space of W_k for all $\nu = 2, 3, \ldots$.
- If the rank of W_{k+1} equals the rank of W_k then the null space of W_k is the unobservable subspace of (A, C).

How to solve 3rd degree polynomials

Tuesday, September 17, 2019 2:30 PM

Source of similar procedure (1st step is different/simplified)

↳ How to factor a cubic function
Cowan Academy



↳ Basically 2 steps:

1) Use rational roots theorem to find possible roots. => Test them to find one.
2) Use the one root to factor out equation.

Example shown below (taught at 3me) simplifies step 1, which is alright if 3rd degree term has 1 in front.

Let's say that we have the following polynomial to solve:

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

Step 1 – try $\lambda = \{0, -1, 1\}$

$\lambda = 0$? => $-2 = 0$ ✗ $\therefore \lambda \neq 0$

$\lambda = -1$? => $(-1) + 2(1) - (-1) - 2 =$
$-1 + 2 + 1 - 2 =$
$-3 + 3 = 0$ ✓ $\therefore \lambda = -1$

$\lambda = 1$? => $1 + 2 - 1 - 2 =$
$3 - 3 = 0$ ✓ $\therefore \lambda = 1$

Already found 2 solutions!

Summary

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

↳ Find the possible roots:

$a_n = \{1\}$ $a_0 = \{1, 2\}$

$\therefore \pm \frac{a_0}{a_n} = \pm \{\frac{1}{1}, \frac{2}{1}\}$

$\Rightarrow \{1, -1, 2, -2\}$

↳ Test out roots:

$\lambda = 1 \Rightarrow 1^3 + 2 - 1 - 2 = 0$ ✓

$\therefore \lambda = 1$ is a root.

Already found 2 solutions!

Step 2 - Factor out

↳ We first look whether we can factor out a λ by itself. (this would mean $\lambda = 0$ is a solution)

If the equation was e.g. $\lambda^3 - \lambda^2 = 0$

then we can rewrite it as
$\lambda(\underbrace{\lambda^2 - \lambda}_{\text{solve this!}}) = 0$

However in our original example we cannot do this.
($\because \lambda = 0$ not a solution)

↳ If we try: $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$
↳ $\lambda(\lambda^2 + 2\lambda - 1 - \frac{2}{\lambda}) = 0$

Bad!

$\therefore$ We need to factor out using one of our solutions.

$\Rightarrow \underline{\lambda = 1}$

$\therefore (\lambda - 1) \cdot (\dots)$ What we want to find out...

$\Rightarrow (\lambda - 1) / \lambda^3 + 2\lambda^2 - \lambda - 2 \backslash \lambda^2 + 3\lambda + 2$

↳ In order to get λ^3 I could do $(\lambda - 1) \cdot \boxed{\lambda^2}$ 1
but I get:
$\lambda^3 - \lambda^2$
Subtract from original

$3\lambda^2 - \lambda - 2$

In order to get $3\lambda^2$ I could do $(\lambda - 1) \cdot \boxed{3\lambda}$ 2
but I get:

$\therefore \lambda = 1$ is a root.

↳ Factorise

$(\lambda - 1) / \lambda^3 + 2\lambda^2 - \lambda - 2$

Subtract!

$\underline{\underline{\lambda^2}}(\lambda - 1) = \underline{\lambda^3 - \lambda^2}$

$3\lambda^2 - \lambda - 2$

$\underline{\underline{3\lambda}}(\lambda - 1) = \underline{3\lambda^2 - 3\lambda}$

$2\lambda - 2$

$\underline{\underline{2}}(\lambda - 1) = \underline{2\lambda - 2}$

0 ✓

$\therefore \underline{\underline{(\lambda - 1)(\lambda^2 + 3\lambda + 2)}}$

In order to get $3\lambda^2$ I could do $(\lambda-1)\boxed{3\lambda}$

but I get:

$$3\lambda^2 - 3\lambda$$

Subtract from previous:

$$2\lambda - 2$$

$$\rightarrow (\lambda-1)\cdot\boxed{2} = \underline{2\lambda - 2}$$

$$0$$

We get zero if done correctly!

$$\therefore \Rightarrow \underline{\underline{(\lambda-1)}}(\lambda^2 + 3\lambda + 2)$$

↳ This we can solve now!

↳ 1) Try factors out for succesful factorisation.

Note $(\boxed{\ }\lambda^2 + \boxed{SUM}\lambda + \boxed{PRODUCT})$

(2) Just use quadratic formula to find roots:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Multiplicity

Thursday, September 19, 2019 3:27 PM

The **algebraic multiplicity** of an eigenvalue e is the power to which ($\lambda - e$) divides the characteristic polynomial.

The **geometric multiplicity** of an eigenvalue is the number of linearly independent eigenvectors associated with it. That is, it is the dimension of the nullspace of $A - eI$.

From <https://www.google.com/search?q=algebraic+and+geometric+multiplicity&oq=algebraic+vs&aqs=chrome.1.69i57j0l5.6419j0j0&sourceid=chrome&ie=UTF-8>

Jordan form

Friday, September 20, 2019 10:33 AM

Important source:

Differential Equations...

The Jordan form is simply the result of applying diagonilisation to a matrise. $\Rightarrow$ $J = S\,A\,S^{-1}$

Matrix to be diagonilized

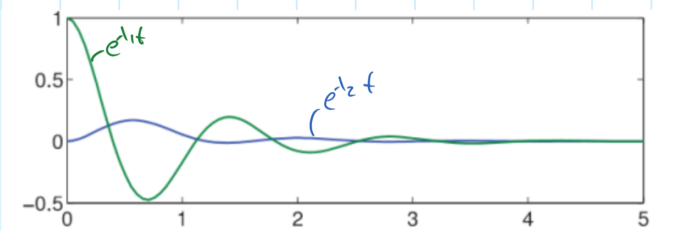
In general we have discovered that in order to diagonilise matrise A, we collect its eigenvectors into matrise S.

$$\Rightarrow S = [e_1, e_2, e_3 \dots e_n] \text{ for } A = [n \times n]$$

We can then compute J. This matrise will now contain the eigenvalues of A in its diagonal and indeed be a diagonal matrise.

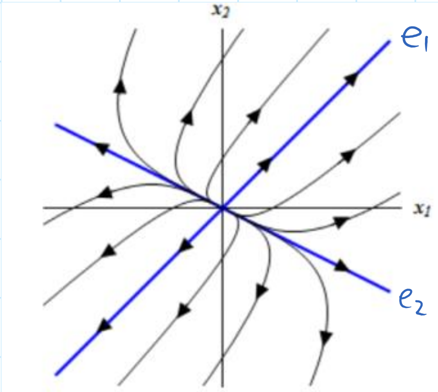
$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

This matrise is great for analysis!
We can study the influence of each eigenvalue!



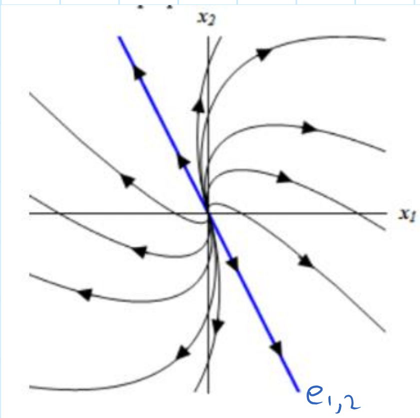
→ But what happens when we have repeated eigenvalues?

When we look at phase plots, when we have 2 distinct eigenvalues for 2x2 matrix, we could get something like this:



→ Clearly the 2 eigenvectors are enough to describe fully the motion of the system.

With repeated eigenvalues though, we get plots like these:

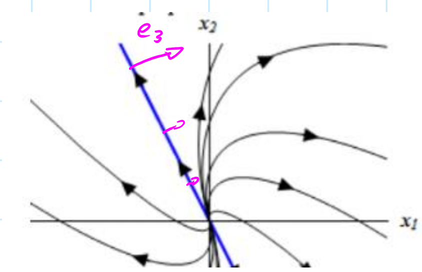


Clearly more stuff is happening than described only by the repeated eigenvector!

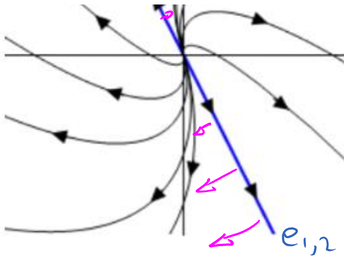
What if we could go along the axis of the repeated eigenvector?

We should then be able to say, from that new reference (sliding) point, a new direction as to how the system behaves.

A higher order eigenvalue if you will.



e3 allows us now to fully describe the motion of the system! ✓



the motion of the system! ✓

We can therefore use this observation to find what e_3 is using math.

We found e_1 and e_2 (where $e_1 = e_2$) using $A\underline{e} = \lambda \underline{e}$ ← key aspect. A only scales vector e by a factor λ.

$$(A - \lambda I)\underline{e} = 0$$

Now we want to go along the eigenvector $e_{1,2}$ and find a 3^{rd} direction.

$$\therefore \; A\underline{e}_3 = \underline{e}_{1,2} + \lambda \underline{e}_3$$

We know this one!

$$\hookrightarrow \underline{(A - \lambda I)\underline{e}_3 = \underline{e}_{1,2}}$$

↳ If we test this solution, then, like shown in Pauls online notes, we see it works! ✓

Eigenvectors that are built up like e_3 are called generalised eigenvectors.

The solution of the system ends up then not being just $e^{\lambda t}$,

but $t e^{\lambda t}$ or higher order ($\frac{t^n}{n!} e^{\lambda t}$ → depends on generalised eigenvectors!)

This represents our sliding motion!

n = How often we have to slide and form a new vector.

Similar matrices

27 October 2020 11:51

A and B are said to be similar matrices if $A = SBS^{-1}$. Among other this implies that A and B share the same eigenvalues.

Intuition behind similarity of matrices

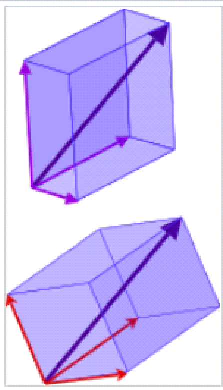
P is a linear transformation which takes things to a new basis. after applying B, you bring your (transformed) basis back to where it used to be (transformed). If this is the same thing as your original transformation (A), then A and B must fundamentally be the same transform, in the different basis. I visualize it with a rotation example: rotate to a new basis (P), do your rotation B, and rotate back (P^{-1}). If this is the same as one rotation A, then they were the same.

$Start \to^{P} Rotated$

$\downarrow A \qquad\qquad \downarrow B$

$End \leftarrow^{P^{-1}} Twice$

https://math.stackexchange.com/questions/149703/intuition-for-similar-matrix

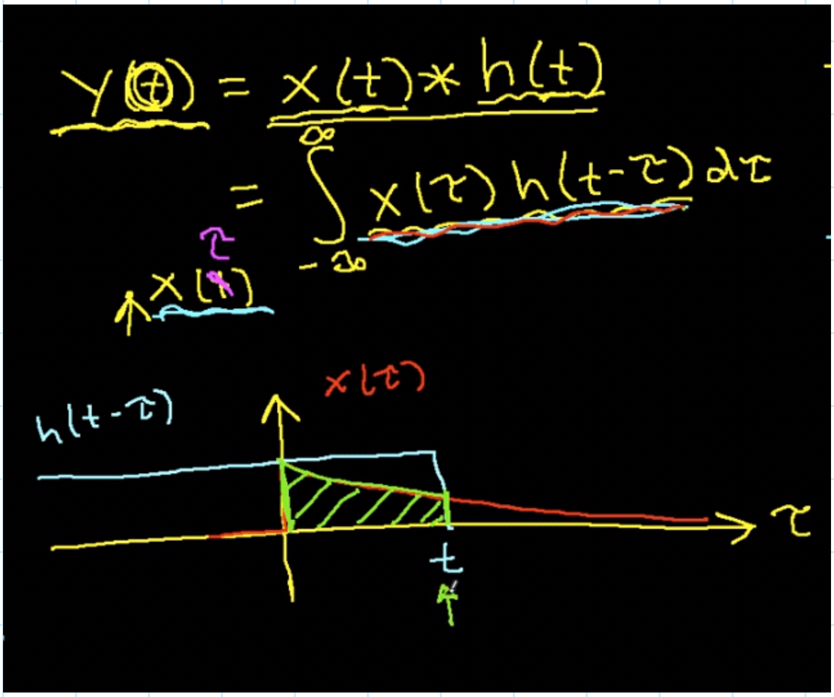


The same vector can be represented in two different bases (purple and red arrows).

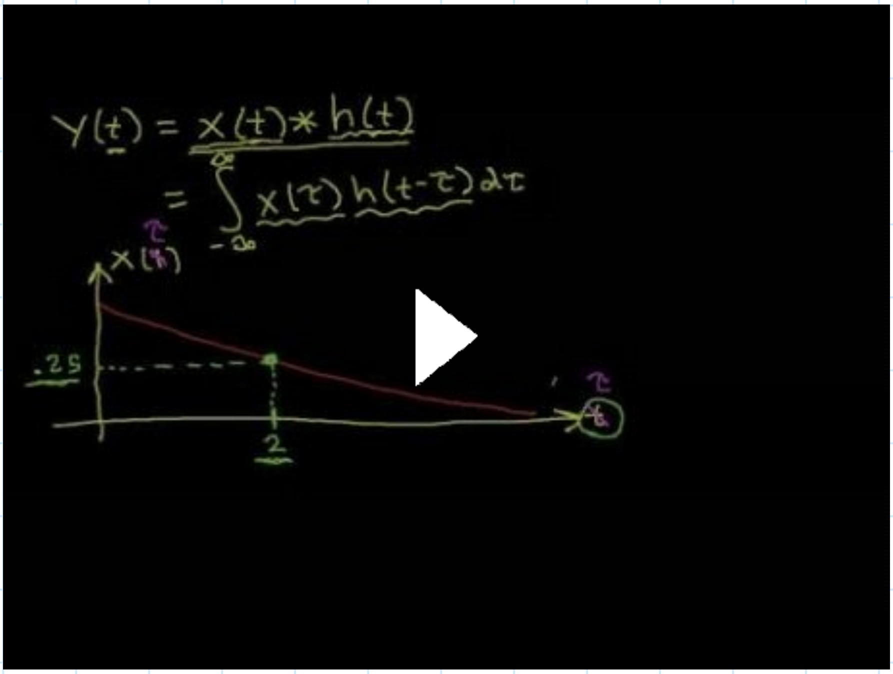
Basis (linear algebra) - ...

Review Convolutions

Wednesday, September 25, 2019 11:26 AM



Video that explains the math:
[Convolution-What's τ got to do with it?]()



A better way to understand what the convolution actually is, here two resources:

I prefer sound to Terry Tao's light. Listen to my voice through a wall. At each moment in time, you hear not just what I am saying now, but also some reverberation from what I said moments ago. So if I make a sound given by $f(t)$ (density of air), you hear a linear combination $h(0)f(t) + h(1)f(t-1) + h(2)f(t-2) + \ldots$, or a continuous version of that, i.e. $h * f$. The function $h(\tau)$ is how much you hear from τ seconds before the current time. If $h(\tau)$ decays slowly, my voice is muffled by reverb.

Fourier theory shows that recovering my voice $f(t)$ is difficult when $\hat{h}(\xi)$ is very small at some frequencies ξ: the wall doesn't vibrate at those frequencies.

If $h(\tau) \neq 0$ for some negative τ, you can hear me before I speak!

share cite improve this answer edited Jun 22 at 8:19 answered Sep 22 '13 at 18:27 Ben McKay 16.1k 2 33 64

This next resource is a bit clearer:

David Greenspan, Meteor dev, Etherpad creator
Answered Jan 3, 2014

Suppose you have a special laser pointer that makes a star shape on the wall. You tape together a bunch of these laser pointers in the shape of a square. The pattern on the wall now is the convolution of a star with a square.

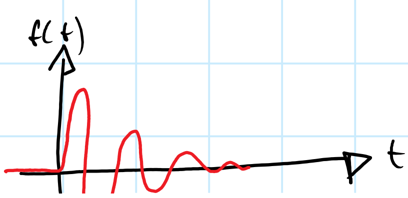
Convolution has a lot of applications in undergrad physics, electrical engineering, and signal processing. Instead of the two dimensions of a wall, take the one dimension of time. You have an object on a spring that, when you give it a kick, starts swinging back and forth a little. Under certain assumptions, the system is "linear," meaning that if you give it a second kick, the resulting behavior of the object can be found by summing the response to the first kick alone and the response to the second kick alone. If you are tracking the x-coordinate of the object as a function of time as it swings back and forth, you can graph the response to a kick (an instantaneous force) as a function x = f(t). In fact, there is a single function f(t) representing the "impulse response" of the system, and you can calculate the response of the system to any series of kicks by adding up scaled copies of it. If you express the "kick force" as a function of time, k(t), then the behavior of the object is the convolution of f(t) with k(t). The kick function k could be series of impulses, or it could itself be a continuous function. The object could be in a magnetic field, for example, or connected to something that pulls on it with varying force.

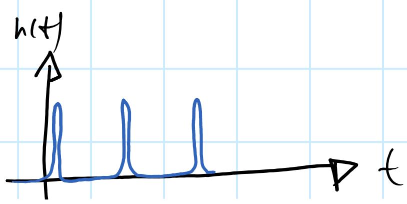
Own Summary:

Let's take the spring-damper example from above, actuated by impulses. Then we can consider two functions:

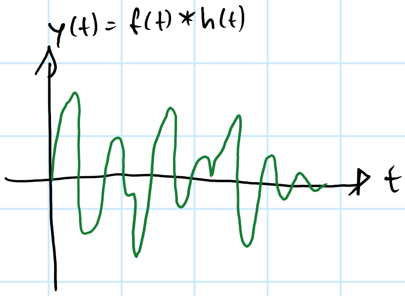
- Function 1, f(t), is the impulse response of a system.
- Function 2, h(t), describes when these impulses happen.

Lets draw out f(t) and h(t) on separate graphs:

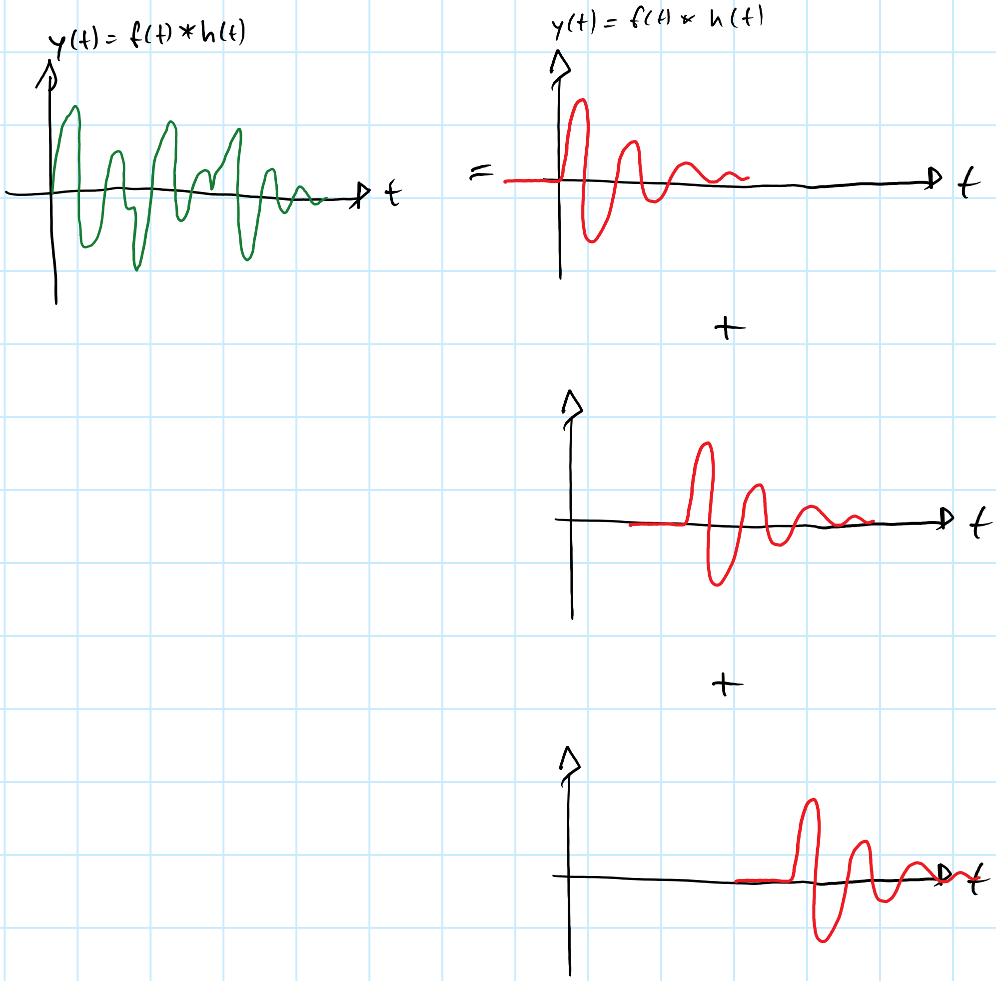




Clearly, if we want to see how the system responds in all of t, we basically actuate f(t) with the first impulse, then we let it decay as defined by f(t), until the second impulse arrives, where f(t) is "reactivated", etc.
The resulting graph, the convolution of f(t) and h(t), we would expect it to look something like this then:



Clearly, this was put together through superposition/linear combination:



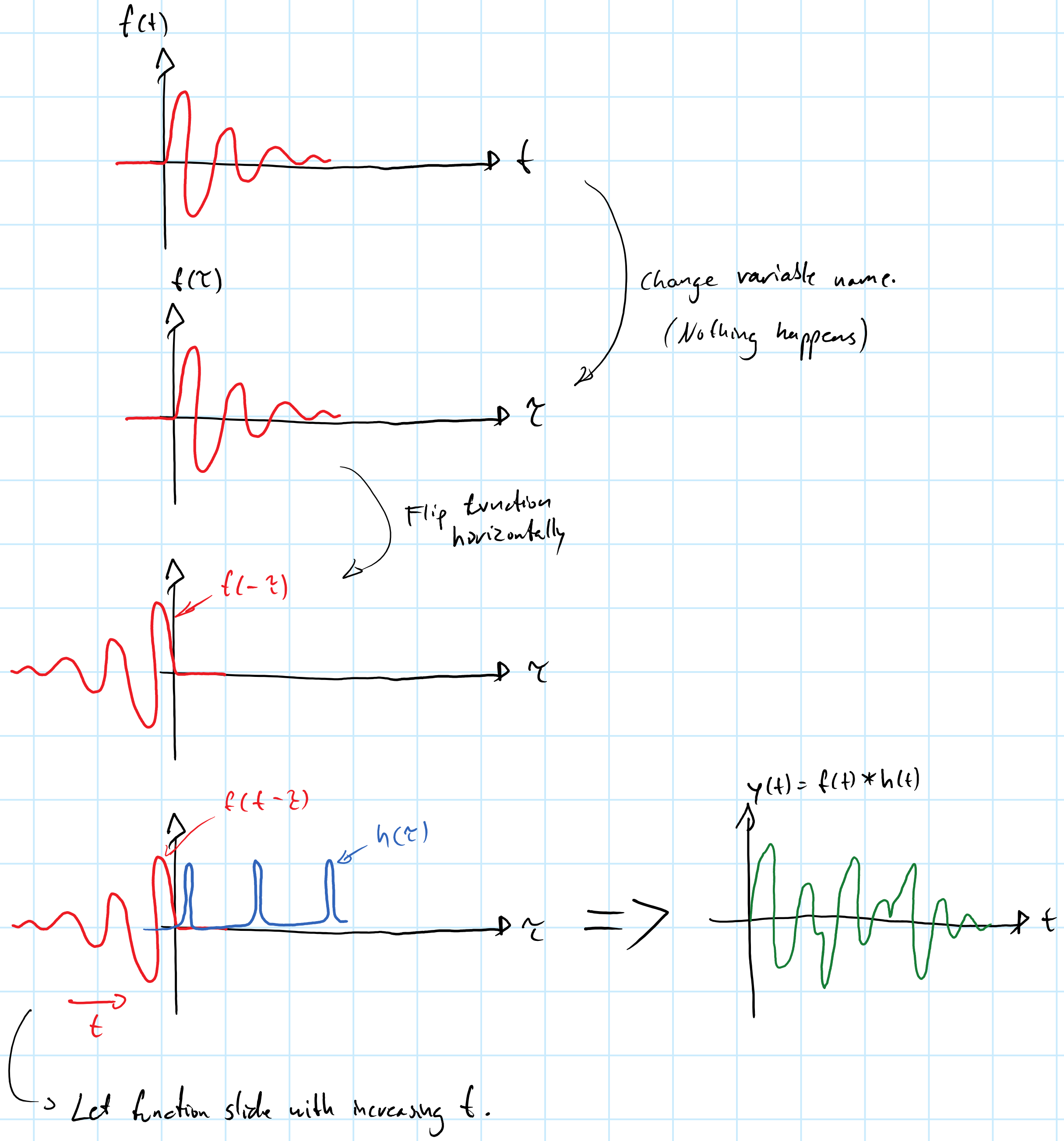
Mathematically we could express this then as:
h(0)f(t) + h(1)f(t-1) + h(2)f(t-2)

From this one can understand how one arrives at the integral expression. One simply generalises the case. Thus:

A summation!

$$y(t) = h(t) * f(t) = \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau$$

The very confusing sliding example: This example is often used to explain convolution. The above explanation is definitely clearer to explain where the math comes from, but this example shows how the math is actually done in a visual way. One starts by realising that we have taken f(t), flipped it horizontally, and then let it slide across h(t) by multiplying the two together. (This is what is explained in the video above.) Clearly one can then obtain the response of the system.



Review Integration

24 September 2020 11:08

MATHS IS FUN

Integration by Substitution

"Integration by Substitution" (also called "u-Substitution" or "The Reverse Chain Rule") is a method to find an integral, but only when it can be set up in a special way.

The first and most vital step is to be able to write our integral in this form:

$$\int f(g(x))\, g'(x)\, dx$$

Note that we have **g(x)** and its derivative **g'(x)**

Like in this example:

$$\int \cos(x^2)\, 2x\, dx$$

Here **f=cos**, and we have **g=x²** and its derivative **2x**
This integral is good to go!

When our integral is set up like that, we can do **this substitution**:

$$\int f(\underbrace{g(x)}_{u})\, \underbrace{g'(x)\, dx}_{du}$$

$$\int f(u)\, du$$

Then we can **integrate f(u)**, and finish by **putting g(x) back as u**.

Like this:

Example: $\int \cos(x^2)\, 2x\, dx$

We know (from above) that it is in the right form to do the substitution:

$$\int \cos(\underbrace{x^2}_{u})\, \underbrace{2x\, dx}_{du}$$

$$\int \cos(u)\, du$$

Now integrate:

MATHS IS FUN

Integration by Parts

Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

You will see plenty of examples soon, but first let us see the rule:

$$\int u\, v\, dx = u \int v\, dx - \int u' \left(\int v\, dx\right) dx$$

- **u** is the function u(x)
- **v** is the function v(x)
- **u'** is the derivative of the function u(x)

As a diagram:

$$\int u\ v\ dx$$

$$u' \quad \int v\, dx$$

$$u \int v\, dx - \int u' \left(\int v\, dx\right) dx$$

Let's get straight into an example, and talk about it after:

Example: What is $\int x \cos(x)\, dx$?

OK, we have **x** *multiplied by* **cos(x)**, so integration by parts is a good choice.

First choose which functions for **u** and **v**:

- u = x
- v = cos(x)

So now it is in the format **∫u v dx** we can proceed:

Differentiate **u**: u' = x' = 1

Integrate **v**: $\int v\, dx = \int \cos(x)\, dx = \sin(x)$ *(see Integration Rules)*

Now we can put it together:

$$\int \cos(u)\, du = \sin(u) + C$$

And finally put $\mathbf{u=x^2}$ back again:

$$\sin(x^2) + C$$

So $\int \mathbf{\cos(x^2)\ 2x\ dx = \sin(x^2) + C}$

That worked out really nicely! (Well, I knew it would.)

But this method only works on *some* integrals of course, and it may need rearranging:

Example: $\int \cos(x^2)\ 6x\ dx$

Oh no! It is **6x**, not **2x** like before. Our perfect setup is gone.

Never fear! Just rearrange the integral like this:

$$\int \cos(x^2)\ 6x\ dx = 3\int \cos(x^2)\ 2x\ dx$$

(We can pull constant multipliers outside the integration, see Rules of Integration.)

Then go ahead as before:

$$3\int \cos(u)\, du = 3\sin(u) + C$$

Now put $\mathbf{u=x^2}$ back again:

$$3\sin(x^2) + C$$

Done!

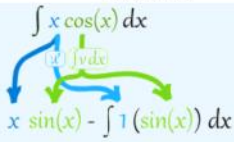
Now let's try a slightly harder example:

Example: $\int x/(x^2+1)\ dx$

Let me see ... the derivative of x^2+1 is $2x$ ... so how about we rearrange it like this:

$$\int x/(x^2+1)\ dx = ½\int 2x/(x^2+1)\ dx$$

Then we have:



Simplify and solve:

$$\rightarrow x\sin(x) - \int \sin(x)\, dx$$

$$\rightarrow x\sin(x) + \cos(x) + C$$

So we followed these steps:

- Choose u and v
- Differentiate u: u'
- Integrate v: $\int v\ dx$
- Put u, u' and $\int v\ dx$ into: $\mathbf{u\int v\ dx - \int u'\ (\int v\ dx)\ dx}$
- Simplify and solve

In English, to help you remember, $\int \mathbf{u\ v\ dx}$ becomes:

(u integral v) minus integral of (derivative u, integral v)

Let's try some more examples:

Example: What is $\int \ln(x)/x^2\ dx$?

First choose u and v:

- $u = \ln(x)$
- $v = 1/x^2$

Differentiate u: $\ln(x)' = 1/x$

Integrate v: $\int 1/x^2\ dx = \int x^{-2}\ dx = -x^{-1} = -1/x$ *(by the power rule)*

$$\tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$\tfrac{1}{2}\int 1/(u)\,du$$

Then integrate:

$$\tfrac{1}{2}\int 1/u\ du = \tfrac{1}{2}\ln(u) + C$$

Now put $\mathbf{u=x^2+1}$ back again:

$$\tfrac{1}{2}\ln(x^2+1) + C$$

And how about this one:

Example: $\int (x+1)^3\,dx$

Let me see ... the derivative of x+1 is ... well it is simply 1.

So we can have this:

$$\int (x+1)^3\,dx = \int (x+1)^3 \cdot 1\,dx$$

Then we have:

$$\int (x+1)^3 \cdot 1\,dx$$

$$\int (u)^3\,du$$

Then integrate:

$$\int u^3\,du = (u^4)/4 + C$$

Now put $\mathbf{u=x+1}$ back again:

$$(x+1)^4/4 + C$$

We can take that idea further like this:

Example: $\int (5x+2)^7\,dx$

If it was in THIS form we could do it:

$$\int (5x+2)^7\ \mathbf{5}\ dx$$

So let's make it so by doing this:

$$\frac{1}{5}\int (5x+2)^7\ \mathbf{5}\ dx$$

The $\frac{1}{5}$ and 5 cancel out so all is fine.

And now we can have $\mathbf{u=5x+2}$

$$\frac{1}{5}\int (5x+2)^7 \cdot 5\,dx$$

$$\frac{1}{5}\int (u)^7\,du$$

And then integrate:

$$\frac{1}{5}\int u^7\,du = \frac{1}{5}\,\frac{u^8}{8} + C$$

Now put $\mathbf{u=5x+2}$ back again, and simplify:

$$\frac{(5x+2)^8}{40} + C$$

Now get some practice, OK?

In Summary

When we can put an integral in this form:

$$\int f(g(x))\ g'(x)\ dx$$

Then we can make $\mathbf{u=g(x)}$ and integrate $\int f(u)\,du$

Now put it together:

$$\int \ln x\ \frac{1}{x^2}\,dx$$

$u \quad \int v\,dx$

$$\ln x\ \frac{-1}{x} - \int \frac{1}{x}\left(\frac{-1}{x}\right)dx$$

Simplify:

$$-\ln(x)/x - \int -1/x^2\,dx = -\ln(x)/x - 1/x + C$$

$$-(\ln(x) + 1)/x + C$$

Example: What is $\int \ln(x)\,dx$?

But there is only one function! How do we choose u and v ?

Hey! We can just choose v as being "1":

- u = ln(x)
- v = 1

Differentiate u: $\ln(x)' = 1/x$

Integrate v: $\int 1\,dx = x$

Now put it together:

$$\int \ln x \cdot 1\,dx$$

$u \quad \int v\,dx$

$$\ln x \cdot x - \int \frac{1}{x}(x)\,dx$$

Simplify:

$$x\ln(x) - \int 1\,dx = x\ln(x) - x + C$$

Example: What is $\int e^x\,x\,dx$?

Choose u and v:

- $u = e^x$
- $v = x$

Differentiate u: $(e^x)' = e^x$

Integrate v: $\int x\,dx = x^2/2$

Now put it together:

$$\int e^x\,x\,dx$$

$u \quad \int v\,dx$

$$e^x\,\frac{x^2}{2} - \int e^x\left(\frac{x^2}{2}\right)dx$$

Well, that was a spectacular disaster! It just got more complicated.

Maybe we could choose a different u and v?

Example: $\int e^x\,x\,dx$ (continued)

Choose u and v differently:

- $u = x$
- $v = e^x$

Differentiate u: $(x)' = 1$

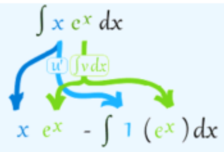
Integrate v: $\int e^x\,dx = e^x$

Now put it together:

Then we can make **u=g(x)** and integrate $\int f(u)\,du$

And finish up by re-inserting **g(x)** where **u** is.

Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10



Simplify:

→ $x\,e^x - e^x + C$

→ $e^x(x-1) + C$

The moral of the story: Choose **u** and **v** carefully!

Choose a **u** that gets simpler when you differentiate it and a **v** that doesn't get any more complicated when you integrate it.

A helpful rule of thumb is I LATE. Choose **u** based on which of these comes first:

- **I**: Inverse trigonometric functions such as $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$
- **L**: Logarithmic functions such as $\ln(x)$, $\log(x)$
- **A**: Algebraic functions such as x^2, x^3
- **T**: Trigonometric functions such as $\sin(x)$, $\cos(x)$, $\tan(x)$
- **E**: Exponential functions such as e^x, 3^x

And here is one last (and tricky) example:

Example: $\int e^x \sin(x)\,dx$

Choose u and v:

- $u = \sin(x)$
- $v = e^x$

Differentiate u: $\sin(x)' = \cos(x)$

Integrate v: $\int e^x\,dx = e^x$

Now put it together:

→ $\int e^x \sin(x)\,dx = \sin(x)\,e^x - \int \cos(x)\,e^x\,dx$

Looks worse, but let us persist! We can use integration by parts **again**:

Choose u and v:

- $u = \cos(x)$
- $v = e^x$

Differentiate u: $\cos(x)' = -\sin(x)$

Integrate v: $\int e^x\,dx = e^x$

Now put it together:

→ $\int e^x \sin(x)\,dx = \sin(x)\,e^x - (\cos(x)\,e^x - \int -\sin(x)\,e^x\,dx)$

Simplify:

→ $\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx$

Now we have the same integral on both sides (except one is subtracted) ...

... so bring the right hand one over to the left and we get:

→ $2\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x)$

Simplify:

→ $\int e^x \sin(x)\,dx = e^x\,(\sin(x) - \cos(x)) / 2 + C$

Footnote: Where Did "Integration by Parts" Come From?

Footnote: Where Did "Integration by Parts" Come From?

It is based on the Product Rule for Derivatives:

$(uv)' = uv' + u'v$

Integrate both sides and rearrange:

$\int (uv)'\,dx = \int uv'\,dx + \int u'v\,dx$

$uv = \int uv'\,dx + \int u'v\,dx$

$\int uv'\,dx = uv - \int u'v\,dx$

Some people prefer that last form, but I like to integrate v' so the left side is simple:

$\int uv\,dx = u\int v\,dx - \int u'(\int v\,dx)\,dx$

Your Turn

Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10

Matrix multiplications

Friday, October 18, 2019 2:08 PM

1) Dot rule method

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} \Rightarrow \begin{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} e \\ g \end{pmatrix}, & \begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} f \\ h \end{pmatrix} \\ \begin{pmatrix} c \\ d \end{pmatrix}\cdot\begin{pmatrix} e \\ g \end{pmatrix}, & \begin{pmatrix} c \\ d \end{pmatrix}\cdot\begin{pmatrix} f \\ h \end{pmatrix} \end{bmatrix}$$

2) Direct multiplication strategy

$$\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

copy

$$= \begin{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cdot 2 + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\cdot 0 , & 2\cdot -1 + -1 \cdot 2 \\ 0 \cdot 2 + 2 \cdot 0 , & 0 \cdot -1 + 2 \cdot 2 \end{bmatrix}$$

(1) Copy matrix 1 by nr. of vectors in matrix 2.

(2) Add component to each vector of result from matrix 2.

$$= \begin{bmatrix} 4 , & -4 \\ 0 & 4 \end{bmatrix}$$

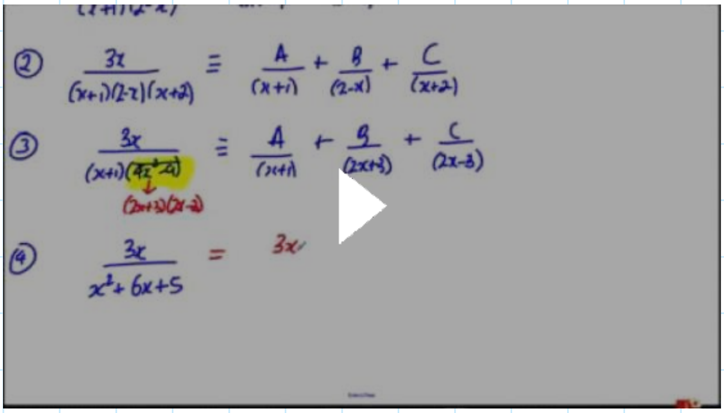
This was also to show that for Jordan matrix J, J^2 is $\begin{bmatrix} J_1^2 & & \\ & J_2^2 & \\ & & J_3^2 \end{bmatrix}$

$\therefore$ we can take the square of individual Jordan blocks!

Review Partial Fraction Decomposition

Monday, October 7, 2019   1:03 PM

[Partial Fractions in 5 minutes]



How to factorise expressions like these.

$\Rightarrow 4x^2 - 4$

1) Use binomial equation (Safe!)

2) Take the square root then add complex conj. (Useful)

$(2x - j2)(2x + j2)$

$4x^2 + 2x2j - 2xj2 + 4$ ✓

3) ② also works without j

$(2x - 2)(2x + 2)$

$4x^2 + 4x - 4x - 4$

④ Guess using sum-product rule

Note! No coefficient! $\rightarrow x^2 + 5x + 6$

$= (x+2)(x+3)$

$= x^2 + 3x + 2x + 6$ ✓

Example:

$\therefore \frac{s^2 a + sb + c}{s(s+1)^2}$ 1) not proper (degree of numerator $\geq$ degree denominator) 2) denominator has repeated factor

$$\therefore \frac{s^2 a + sb + c}{s(s+1)^2} = A + \frac{B}{s} + \frac{C}{(s+1)} + \frac{D}{(s+1)^2}$$

$$s^2 a + sb + c = A s(s+1)^2 + B(s+1)^2 + C s(s+1) + Ds$$

$$s^2 a + sb + c = A s(s^2+2s+1) + B(s^2+2s+1) + Cs^2 + Cs + Ds$$

$$[\ldots] = As^3 + 2As^2 + As + Bs^2 + 2Bs + B + Cs^2 + Cs + Ds$$

$s^3: \quad 0 = A$

$s^2: \quad a = 2A + B + C$

$s: \quad b = A + 2B + C + D$

$1: \quad c = B$

Use now method of undetermined coefficients!

$$\begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & a \\ 1 & 2 & 1 & 1 & b \\ 0 & 1 & 0 & 0 & c \end{array}\right]$$

$R_2 \Leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c \\ 1 & 2 & 1 & 1 & b \\ 2 & 1 & 1 & 0 & a \end{bmatrix}$$

$R_3 - R_1$, $R_4 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c \\ 0 & 2 & 1 & 1 & b \\ 0 & 1 & 1 & 0 & a \end{bmatrix}$$

$R_3 - 2R_2$; $R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c \\ 0 & 0 & 1 & 1 & b-2c \\ 0 & 0 & 1 & 0 & a-c \end{bmatrix}$$

$R_3 \Leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c \\ 0 & 0 & 1 & 0 & a-c \\ 0 & 0 & 1 & 1 & b-2c \end{bmatrix}$$

$R_4 - R_3$

$R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c \\ 0 & 0 & 1 & 0 & a-c \\ 0 & 0 & 0 & 1 & b-c-a \end{bmatrix}$$

$$\therefore \frac{s^2 a + sb + c}{s(s+1)^2} = 0 + \frac{c}{s} + \frac{a-c}{(s+1)} + \frac{b-c-a}{(s+1)^2}$$

Factorisation of polynomials

27 October 2020 17:15

Method 1: Quadratic formula (favorite)

Simply use the quadratic formula. The solutions of it are the roots, which also form the factorisation terms with switched signs.

$$ax^2 + bx + c \to \left(x + \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)^2$$

Only works with 2nd order polynomials, plus with complicated expressions it may be a complicated process to work with due to the square root term.

Method 2: Identities

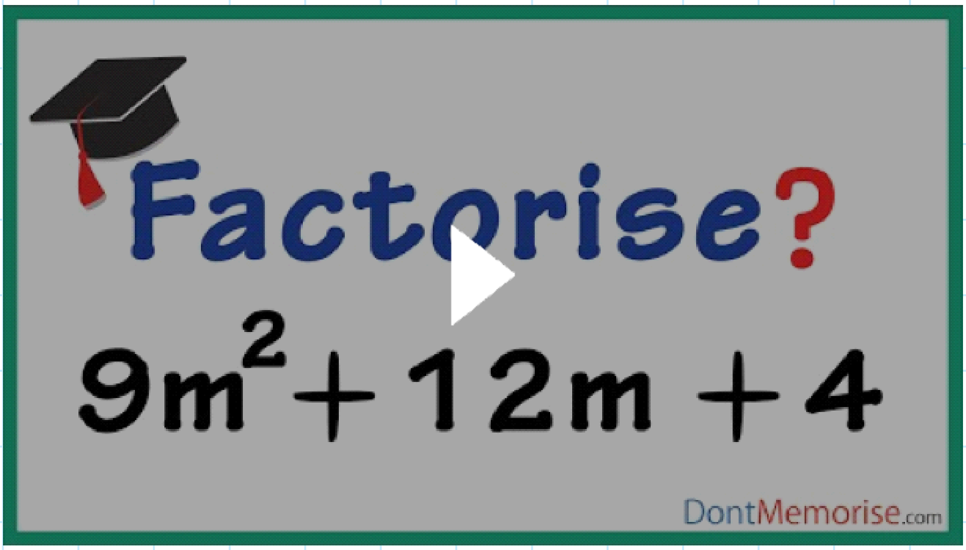
Using identities. We know that:

$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$(a+b)(a-b) = a^2 - b^2$

Therefore, if we see an expression where we can write some terms in squares and the rest are nonlinear terms connecting the two, we try those squares and see if we can resolve the middle.

Example:
$4m^2 + 12m + 9$
We can see that we can write $4m^2 = (2m)^2$ and $9 = 3^2$. Therefore it is tempting to say that this expression equals $(2m+3)^2$. This turns out to be correct given that $2(2m)(3) = 12m$, the term in the middle.

[Factorisation of Polynomials (GMAT/GRE/CAT/Bank PO/SSC CGL) | Don't Memorise]



Pseudo Inverters

Tuesday, October 15, 2019 1:12 PM

Go through this explanation first:

Mathematics and Machine Learning +3

What is the intuition behind pseudo inverse of a matrix?

This question previously had details. They are now in a comment.

5 Answers

Pragyaditya Das, 1 year of extensive ML. Not done with ml-004. Done with PR&ML by Bishop.
Answered May 30, 2016

Originally Answered: What is the intuition behind pseudo inverse of a matrix ?

I have had two three courses on Linear Algebra (2nd Semester), Matrix Theory (3rd Semester) and Pattern Recognition (6th Semester).

Pseudo-inverse is a very common concept in any subject that involves any mathematical acumen.

Let us first look into the Inverse of a Matrix and then intuitively come into the Pseudo-Inverse.

Inverse of a Matrix A is given by,

$$A^{-1} = \frac{1}{|A|} . adj(A)$$

We can see that the term A^{-1} depends upon the $|A|$ value.

Thus, $|A| \neq 0$ because, if $|A| = 0$, then $\frac{1}{|A|} = \infty$.

In other words, A must be non-singular.

Another factor to look into is, *A must be square, because, |A| exists for square matrices only.*

This is the ideal case, now let us take the case where |A| is not square or singular.

In that case, we use the Moore-Penrose Pseudoinverse.

> Wikipedia says,
> In mathematics , and in particular linear algebra , a **pseudoinverse** A+ of a matrix A is a generalization of the inverse matrix [1] . The most widely known type of matrix pseudoinverse is the **Moore–Penrose pseudoinverse**, which was independently described by E. H. Moore [2] in 1920, Arne Bjerhammar [3] in 1951 and Roger Penrose [4] in 1955. Earlier, Fredholm had introduced the concept of a pseudoinverse of integral operators in 1903. When referring to a matrix, the term pseudoinverse, without further specification, is often used to indicate the Moore–Penrose pseudoinverse. The term generalized inverse is sometimes used as a synonym for pseudoinverse. A common use of the pseudoinverse is to compute a 'best fit' (least squares) solution to a system of linear equations that lacks a unique solution (see below under § Applications). Another use is to find the minimum (Euclidean) norm solution to a system of linear equations with multiple solutions. The pseudoinverse facilitates the statement and proof of results in linear algebra.The pseudoinverse is defined and unique for all matrices whose entries are real or complex numbers. It can be computed using the singular value decomposition

Related Questions

- What is the difference between the pseudo-inverse and inverse of a matrix?
- Why should I care about the concept of the pseudo-inverse of a matrix?
- What is the mathematical intuition behind the determinant of a matrix? How was its definition conceived and why is it important? What does it ...
- Linear Algebra: What is the intuition behind computing the inverse of a matrix?
- I have been studying matrices for over a year now, no one can prove to me that the determinant of a matrix gives something concrete other than...
- What is the inverse covariance matrix? What is its statistical meaning?

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Pseudoinverse looks like this:

$A^+ = (A^T A)^{-1} A^T$

Now, the problem of A being non-square is solved by the term $A^T A$, since if A is of order $n \times m$, then $A^T A$is of order $m \times m$.

The Problem of it being singular is also solved, as $A$is not comimg in the denominator anymore, so there is no problem if A^+ being ∞

And, let us also remember the basic rules of Matrices as :

1. $(AB)^{-1} = B^{-1}.A^{-1}$
2. $A^{-1}.A = I$
3. $A.I = A$

Now, let us assume $A$is square and non-singular. Then,

$A^+ = (A^T A)^{-1} A^T$

$\implies A^+ = A^{-1}(A^T)^{-1}A^T (from 1)$

$\implies A^+ = A^{-1}.I(from 2)$

$\implies A^+ = A^{-1}(from 3)$

Hence Proved.

I hope its intuitive enough.

Cheers!

Thanks Shreyas S for pointing out the blunder I made. :)

Dabeer Mirza
This post needs at least 1000 more upvotes! :)

Sean Owen, Data Science @ Databricks
Answered Dec 28, 2013 · Upvoted by Vladimir Novakovski, started Quora machine learning team, 2012-2014

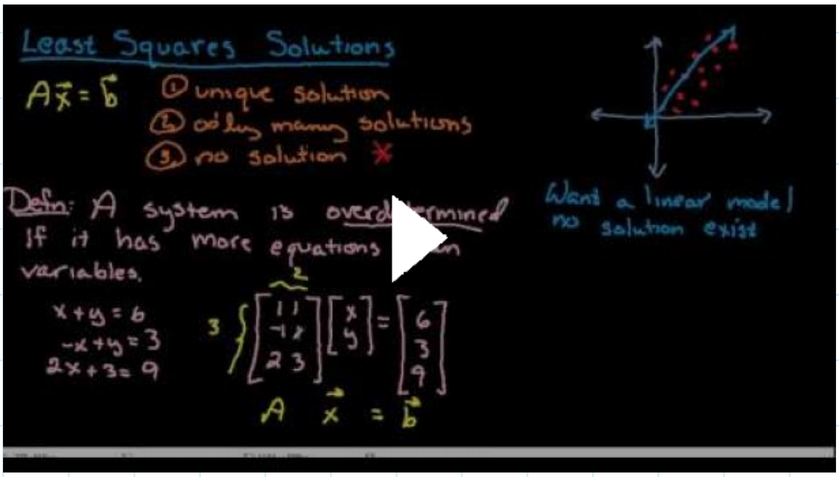
Originally Answered: What is the intuition behind pseudo inverse of a matrix ?

The pinv() (http://www.mathworks.co.uk/help/...) function computes the Moore–Penrose pseudoinverse .

Think of it as a generalization of the inverse. It is defined in for all matrices, but has fewer guaranteed properties as a result. For example, it will be a matrix such that $AA^+A = A$, but not necessarily the stronger usual inverse property $A^{-1}A = AA^{-1} = I$ (the second one implies the first). In the special case where a matrix has an inverse, it will be the same as the pseudo-inverse. So pinv() gives you the inverse where it exists, and still gives something inverse-like everywhere else.

The pseudo-inverse is the ... (more)

It turns out that the pseudo inverse comes from the least square theory. The video below gives a quick introduction into this:
Introduction to the pseudo-inverse



Indeed, if we look back at the formula for finding the solution of an overdetermined system of equations, the least squares formula, we find the following expression:

$\hat{\beta} = \underbrace{(X^T X)^{-1}}_{\text{definition of the pseudo inverse!}} X^T \boldsymbol{y}.$ for $X \hat{b} = y$

Checking observability/controllability

26 October 2020 09:40

Method 1: Observability/Kalman matrix

Compute the Kalman matrix to see whether it has full rank. This method is usually quite lengthy, because you have to calculate the powers of matrices. Sometimes pattern appear, like in the case of a Jordan matrix.

Method 2: Hautus test

The Hautus test is most often the quickest method. The method is summarised below:

Hautus test for controllability

for $A \in \mathbb{R}^{n \times n}$ use

$$[p_1, p_2, \dots p_n]\, A = [\lambda p_1, \lambda p_2, \dots \lambda p_n] \quad , \quad [p_1, p_2, \dots p_n]\, B = 0$$

1) if there is no solution for λ => controllable! :D
2) if there are solutions only for $\lambda < 0$ => stabilisable :)
3) if there are solutions with $\lambda > 0$ => not controllable / stabilisable :(

The solutions for λ are the poles that are uncontrollable and therefore cannot be moved by a feedback matrix F. If $\lambda < 0$ though, the uncontrollable modes are stable!

Hautus test for observability

for $A \in \mathbb{R}^{n \times n}$ use

$$A \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \lambda p_1 \\ \lambda p_2 \\ \vdots \\ \lambda p_n \end{bmatrix} \quad , \quad B \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = 0$$

1) if there is no solution for λ => observable! :D
2) if there are solutions only for $\lambda < 0$ => detectable :)
3) if there are solutions with $\lambda > 0$ => not observable / detectable :(

Method 3: Hautus test version 2

If you can find the eigenvalues of A quickly, then this is also a very nice method.

The actual Hautus test states:

The pair (A, C) is observable if and only if

$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \text{ has full column rank for all } \lambda \in \mathbb{C}.$$

Equivalently, there exists no eigenvector $e \neq 0$ of A with $Ce = 0$

Or, for controllability:

The pair (A, B) is controllable if and only if every left-eigenvector e of the matrix A satisfies $e^*B \neq 0$. Equivalently, the matrix

$$\begin{pmatrix} A - \lambda I & B \end{pmatrix} \text{ has full row rank for all } \lambda \in \mathbb{C}.$$

Note that the weaker forms of these allow for detectability and stabilisability.

Below a very nice example for checking observability from this Homework set.

Control Theory - Homework set #6 Ethel Legrand

Exercise 1

$$A = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -0.1 & -2 & 0 & -0.1 \\ 0 & 1 & 2 & 0 & -0.2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & -0.1 \\ 0 & 0 \end{pmatrix}$$

(a) 3 C-matrices $C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ $C_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$C_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Find the eigenvalues of A (plan to use the Hautus test for observability)

$$\det \begin{pmatrix} \lambda & 1 & 1 & 0 & 0 \\ 0 & \lambda+1 & 0 & 0 & 0 \\ 0 & 1 & \lambda+1 & 0 & 0 \\ 0 & 0.1 & 2 & \lambda & 0.1 \\ 0 & -1 & -2 & 0 & \lambda + 1/5 \end{pmatrix} = \lambda \det \begin{pmatrix} \lambda+1 & 0 & 0 & 0 \\ 1 & \lambda+1 & 0 & 0 \\ 0.1 & 2 & \lambda & 0.1 \\ 0 & \cancel{-0.}\ -1 & -2 & \lambda+0.2 \end{pmatrix}$$

$$= \lambda(\lambda+1) \det \begin{pmatrix} \lambda+1 & 0 & 0 \\ 2 & \lambda & 0.1 \\ -2 & 0 & \lambda+0.2 \end{pmatrix} = \lambda^2(\lambda+1)^2(\lambda+1/5)$$

eigenvalues are $0^{(2)}, -1^{(2)}, -1/5$

For C_1: no eigenvector e of A should have that $Ce = 0$

$\Rightarrow$ Find all e' such that $Ce = 0$ (null space of C_1).

$e' \in \text{null}(C_1)$ if $e' = \begin{pmatrix} 0 \\ a \\ b \\ 0 \\ c \end{pmatrix}$ with a, b, c to be chosen arbitrarily

$\Rightarrow$ does any e' satisfy that $Ae' = \lambda e'$?

$$Ae' = \begin{pmatrix} -a-b \\ -a \\ -a-b \\ -0.1a-2b-0.1c \\ a+2b-0.2c \end{pmatrix}$$

Try $\lambda = 0 \Rightarrow Ae' = 0$

$b = -a$
$-a = 0$
$-c = 0$
$b = 0$

$\Rightarrow$ no, zero vector is the trivial solution

Try $\lambda = -1$

$$Ae' = \begin{pmatrix} 0 \\ -a \\ -b \\ 0 \\ -c \end{pmatrix}$$

$b = -a$
$-a - b = -b$ not possible unless a, b, c 0

(1)

Finally, try $\lambda = -1/5$

$$Ac' = \begin{pmatrix} 0 \\ -a/5 \\ -b/5 \\ 0 \\ -c/5 \end{pmatrix} \rightarrow \text{also not possible}$$

Hence, the pair (A, C_1) is observable.

For C_2:

$$\underline{c}' \in \text{Null}(C_2) \Leftrightarrow \underline{c}' \in \left\{ \begin{pmatrix} a \\ b \\ 0 \\ c \\ 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

Try $\lambda = 0$:

$$Ac' = \begin{pmatrix} -b \\ -b \\ -b \\ -0.1b \\ b \end{pmatrix} \rightarrow \text{okay for } \lambda = 0$$

↗ did not check the other ones

The pair (A, C_2) is not observable ($\lambda = 0$ is an ~~abs~~ unobservable mode.)

For C_3

$$c' \in \text{Null}(C_3) \Leftrightarrow c' \in \left\{ \begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

$$Ac' = \begin{pmatrix} -b \\ -b \\ -b \\ -0.1b \\ b \end{pmatrix}$$

For the same reason as C_2, the pair (A, C_3) is unobservable.

(b) Asymptotically stable error dynamics based on y_3 require (A, C_3) to be detectable

$\Leftrightarrow$ all the unobservable modes are in the left-half open plane.

This is not the case, since $\lambda = 0$ is an unobservable mode.

(c) $A - LC_1$ must be Hurwitz

all the eigenvalues of A are $\{0^{(2)}, -1^{(2)}, -2/5\}$

↳ only these eigenvalues must be replaced

Observe the block-~~diagonal~~ triangular structure of A:

$$\left(\begin{array}{ccc|cc} 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ \hline 0 & -0.1 & -2 & 0 & -1 \\ 0 & 1 & 2 & 0 & -0.2 \end{array}\right) \rightarrow \text{it suffices to put } -1 \text{ at entries } (1,1) \text{ and } (4,5)$$

$$\Rightarrow LC = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(E)

How to do (minimal) realisations

by Marco Delgado Schwartz

Wednesday, October 28, 2020 3:27 PM

There are three methods of increasing complexity:

1) **Standard method:** Convert transfer functions to strictly proper, polynomial expressions. (I.e. standard form). Perform then the realisation like shown in §(L5-35) [1] to the controllable canonical form (ccf) or the observable canonical form (ocf), and stack the realisations using stacking rules shown in §(L5-43).
2) **Gilbert's method:** If transfer functions of single poles are present, then one can use Gilbert's method to complete the realisation. That is: $D + C\left(\frac{1}{s-p}\right)B \rightarrow \begin{bmatrix} pI & B \\ C & D \end{bmatrix}$
   This method is a good "one-shot" method to a minimal realisation.
3) **Common denominator merging and product merging:** Look for common denominators and separate the transfer function $G(s)$ into $G(s) = G_1(s) + G_2(s)$ according to them. This is because the realisation of common denominators can be merged, as explained in §(L5-42). (Note that you will have to perform the realisation into ccf or ocf depending on whether you are merging columns or rows respectively.)
   Try to also minimise the amount of terms that need to be added to $G(s)$, as every addition is essentially a stacking of matrices.
   Finally try to rewrite $G_1(s)$ and $G_2(s)$ in products of smaller matrices, as the realisation of two matrices will also lead to a merging, and thus leads to smaller realisations. (See §(L5-39).)

[1] I use the section symbol § as a symbol for citations and it means "according to.../citing...".

Below the exercises from Set 6 are solved step by step, showing how the methods mentioned above are applied.

5 a) Compute the minimal realisation of $G(s) = \frac{s+1}{s^2-1}$

We could use here brute force and use the standard method (see Method 1), since $G(s)$ is already in strictly proper, polynomial form.

However they are asking here for a minimal realisation, and using the standard method is only a 50-50 gamble. Instead we can directly check if we can rearrange the transfer function to get a realisation that is more likely to be minimal, (see Method 2).

Method 1 – Standard method

$G(s) = \frac{s+1}{s^2-1}$ ← This expression already comes in the standard form: $g(s) = \dfrac{\beta_1 s^{n-1} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n} + d.$

As shown in §(L5-35), this can be realised into:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccccc|c} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \hline \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n & d \end{array}\right)$$

controllable canonical realization

or

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cccccc|c} -\alpha_1 & 1 & 0 & \cdots & 0 & \beta_1 \\ -\alpha_2 & 0 & 1 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} & 0 & \cdots & 0 & 1 & \beta_{n-1} \\ -\alpha_n & 0 & 0 & \cdots & 0 & \beta_n \\ \hline 1 & 0 & 0 & \cdots & 0 & d \end{array}\right)$$

observable canonical realization

choose.
(I prefer first.)

$\therefore$ A realisation of $G(s) = \frac{s+1}{s^2-1}$ is:

$$G(s) \xrightarrow{\mathcal{R}\,(ccf)} \left[\begin{array}{cc|c} 0 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 0 \end{array}\right]$$

$\det(A-\lambda I) = -\lambda^2+1$

$\lambda^2 - 1 = 0$

$\lambda = \frac{\pm\sqrt{-4(-1)(1)}}{2} = \pm\frac{\sqrt{4}}{2} = \pm 1$

or $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

or $\lambda^2 - 1 = \lambda^2 - 1^2 = (\lambda+1)(\lambda-1)$

Doing the Hautus test though (can use eigenvalue method), the realisation is controllable, but not observable!

NOT MINIMAL!

We could now look for the responsible mode that is unobservable, diagonalise the realisation so that we can then take it out of the matrix expression.

OR, we do from the start a better realisation.

Method 2 – (Gilbert's method)

$G(s) = \frac{s+1}{s^2-1}$. This can be rewritten as follows:

$$\frac{s+1}{s^2-1} = \frac{s+1}{s^2-1^2} = \frac{s+1}{(s+1)(s-1)} = \frac{1}{s-1}$$

Can also use quadratic formula

Can also use partial fractions

This is a single pole expression! We can use Gilbert's method!

Gilbert's method states that if $G(s) = D + C\frac{1}{s-p}B$, then its realisation is $\left[\begin{array}{c|c} pI & B \\ \hline C & D \end{array}\right]$.

In our case:

$$G(s) = \frac{s+1}{s^2-1} = \frac{1}{s-1} = 0 + I\frac{1}{s-1}I \xrightarrow{\mathcal{R}} \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right]$$

This is minimal!

(5) b)

$$G(s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}\frac{1}{s+1} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\frac{1}{s-1} + \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}\frac{1}{s-3}.$$

All single poles!

↳ Perfect for Gilbert's method!

We can however also rewrite the 2nd and 3rd term with smaller matrices, which will result in a smaller realisation (thus more likely to be minimal).

reference column

1st column (like reference column

2nd column = 2·reference column

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\frac{1}{s+1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}\frac{1}{s+1}\begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \frac{1}{s-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s-1} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

has the form: $0 + \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{s-3} I$

$$\therefore G(s) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \frac{1}{s+1} \begin{bmatrix} 1 & 2 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s-1} \begin{bmatrix} 1 & 0 \end{bmatrix}}_{G_2(s)} + \underbrace{\begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{s-3}}_{G_3(s)}$$

$R(G_1(s))$:
$$\left[\begin{array}{c|cc} -1 & 1 & 2 \\ \hline 1 & 1 & 0 \\ -2 & 0 & 0 \end{array}\right]$$

$R(G_2(s))$:
$$\left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$R(G_3(s))$:
$$\left[\begin{array}{cc|cc} 3 & & 1 & \\ & 3 & & 1 \\ \hline -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Then, using the fact mentioned in §(L5-39):

If $G_1(s)$, $G_2(s)$ have realizations (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2) then $G_1(s)G_2(s)$ and $G_1(s) + G_2(s)$ have the realizations

$$\left(\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right).$$

we simply stack our realisations of $G(s) = G_1(s) + G_2(s) + G_3(s)$.

This results in:

$$\left[\begin{array}{c|c|cc|cc} -1 & 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ \hline 1 & 0 & -2 & 2 & 1 & 0 \\ -2 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

This realisation is minimal!

The final, badass problem:

The final, badass problem:

5 c) $G(s) = \begin{pmatrix} \frac{s+2}{s+1} & 1 & \frac{1}{s} \\ \frac{s+2}{(s+1)^2} & \frac{1}{s} & \frac{1}{s^2} \end{pmatrix}$

§(L5-42), if G(s) has common denominators in either a row or column, the realisations can be merged together, instead of using stacking methods. This will result into smaller matrices ≡ more likely to be minimal!

WARNING! This is not mentioned in §(L5-42), but for a row merging of realisations to be possible, the realisations have to be in the observable canonical form!

(And for column merging, use ccf.)

∴ row ⟷ ocf
column ⟷ ccf

When going back to $G(s) = \begin{pmatrix} \frac{s+2}{s+1} & 1 & \frac{1}{s} \\ \frac{s+2}{(s+1)^2} & \frac{1}{s} & \frac{1}{s^2} \end{pmatrix}$

with a bit of trickery one can see that the columns can be put with common denominators. (We will therefore need to make our realisations into ccf.)

That is:

$$G(s) = \begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & \frac{s}{s} & \frac{s}{s^2} \\ \frac{(s+2)}{(s+1)^2} & \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s}{s} & 0 \\ 0 & \frac{1}{s} & 0 \end{bmatrix}}_{G_2(s)} + \underbrace{\begin{bmatrix} 0 & 0 & \frac{s}{s^2} \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}}_{G_3(s)}$$

Now we could try to already do the realisations. However remember that we would prefer to use merging techniques over stacking techniques. (Stacking techniques are shown in §(L5-43).)

And the summation of two transfer functions like $G(s) = G_1(s) + G_2(s)$, mentioned also in §(L5-39), also results in stacking! ☹

And we want to avoid this as much as possible!

Our expression now is of the form of $G(s) = G_1(s) + G_2(s) + G_3(s)$. We must ask ourselves, can we reduce the number of terms?

In fact we can! We can see that the common denominators of $G_2(s)$ and $G_3(s)$ can also be made to have one common denominator for both $G_2(s)$ and $G_3(s)$. Let's combine the two then!

$$G(s) = \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s}{s} & 0 \\ 0 & \frac{1}{s} & 0 \end{bmatrix}}_{G_2(s)} + \underbrace{\begin{bmatrix} 0 & 0 & \frac{s}{s^2} \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}}_{G_3(s)}$$

$$= \begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{s^2}{s^2} & 0 \\ 0 & \frac{s}{s^2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{s}{s^2} \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s^2}{s^2} & \frac{s}{s^2} \\ 0 & \frac{s}{s^2} & \frac{1}{s^2} \end{bmatrix}}_{G_2(s)}$$

$$\underbrace{\begin{bmatrix} \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} \quad \underbrace{\begin{bmatrix} 0 & \frac{s}{s^2} & \frac{1}{s^2} \end{bmatrix}}_{G_2(s)}$$

Beside the merging technique using common denominators as shown in §(L5-42), the product of to transfer functions $G(s) = G_1(s) \cdot G_2(s)$ also results in a merging technique! (See §(L5-39)).

Can we therefore make our $G(s)$ expression better? Using products of smaller matrices (which will result in smaller realisations).

Once again, yes we can!

$$G(s) = \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} & 0 & 0 \\ \frac{(s+2)}{(s+1)^2} & 0 & 0 \end{bmatrix}}_{G_1(s)} + \underbrace{\begin{bmatrix} 0 & \frac{s^2}{s^2} & \frac{s}{s^2} \\ 0 & \frac{s}{s^2} & \frac{1}{s^2} \end{bmatrix}}_{G_2(s)}$$

$$= \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} \\ \frac{(s+2)}{(s+1)^2} \end{bmatrix}}_{G_{11}(s)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{G_{12}(s)} + \underbrace{\begin{bmatrix} \frac{s}{s^2} \\ \frac{1}{s^2} \end{bmatrix}}_{G_{21}(s)} \underbrace{\begin{bmatrix} 0 & s & 1 \end{bmatrix}}_{G_{22}(s)}$$

Now the terms of G_1 are quite "nice". That is, the left matrix consists of proper functions, and G_{12} is just constants.

G_2 though has a transfer function that is not proper! Can we maybe change that?

$$G_2(s) = \begin{bmatrix} \frac{s}{s^2} \\ \frac{1}{s^2} \end{bmatrix} \begin{bmatrix} 0 & s & 1 \end{bmatrix}$$

pull out $\frac{1}{s}$

$$= \frac{1}{s} \begin{bmatrix} \frac{s}{s} \\ \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 & s & 1 \end{bmatrix}$$

insert $\frac{1}{s}$

$$\begin{bmatrix} \vdots \\ \frac{1}{s} \end{bmatrix} \quad \text{insert } \tfrac{1}{s}$$

$$= \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 & 1 & \frac{1}{s} \end{bmatrix}$$

This is already much better! We have in fact gained two advantages:

1) G_2 now only has proper transfer functions.
2) The transfer functions are of a single pole! We can use Gilbert's method then!

Our final expression for $G(s)$ is:

$$G(s) = \underbrace{\begin{bmatrix} \frac{(s+2)(s+1)}{(s+1)^2} \\ \frac{(s+2)}{(s+1)^2} \end{bmatrix}}_{G_{11}(s)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{G_{12}(s)} + \underbrace{\begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix}}_{G_{21}(s)} \underbrace{\begin{bmatrix} 0 & 1 & \frac{1}{s} \end{bmatrix}}_{G_{22}(s)}$$

This is pretty much as far as we can go with our rearranging.

NOW let's do some realisations!

$\boxed{G_{11}:}$

Not strictly proper!   strictly proper! ✓

$$g_{11,11}(s) = \frac{(s+2)(s+1)}{(s+1)^2} = \frac{s^2+3s+2}{s^2+2s+1} = \frac{(s^2+2s+1)+s+1}{s^2+2s+1} = 1 + \frac{(s+1)}{s^2+2s+1}$$

$$\therefore g_{11,11}(s) \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 1 \end{array}\right]$$

$$g_{11,21}(s) = \frac{(s+2)}{(s+1)^2} = \frac{s+2}{s^2+2s+1}$$

$$\therefore\ g_{11,21}(s) \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 2 & 0 \end{array}\right]$$

$$\leadsto \begin{bmatrix} g_{11,11} \\ g_{11,21} \end{bmatrix} \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$$

$\boxed{G_{12}:}$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \xrightarrow{\mathcal{R}} \left[\begin{array}{|ccc} 1 & 0 & 0 \end{array}\right]$$

This can directly be treated as a D-matrix, which remains unchanged when realised.

Proof:

I just prefer to do my realisations into ccf.

$$g_{12,11}(s) = 1 \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \end{array}\right]$$

$$g_{12,12}(s) = 0 \xrightarrow{\mathcal{R}(ccf)} \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right]$$

$$g_{12,13}(s) = 0 \xrightarrow{\mathcal{R}(ccf)}$$

(see (L5-43)).

Stacking them results in:

$$\left[\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Good news, A is diagonal, $\text{eig}(A) = \{0\}$.

However when checking unobservability, we can see that $\lambda = 0$ is an unobservable mode. (You can do the math or just see how since $C = [0\ 0\ 0]$, whatever the information the system has, we are litterally not getting anything.)

Therefore, to get a minimal realisation of $G_{12}(s)$, take out the parts describing the unobservable mode $\lambda = 0$.

That leaves us only with the D-matrix.

$\boxed{G_1(s)}$

Therefore, the realisation of $G_1(s) = G_{11}(s) \cdot G_{12}(s)$ is, §(L5-38):

I'm showing these lines to make clear this is a D-matrix.

$$G_{11}(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right] ; \quad G_{12}(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$$

$$G_1(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{c|c} A_1 & B_1 D_2 \\ \hline C_1 & D_1 D_2 \end{array}\right] = \left[\begin{array}{cc|ccc} -2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{array}\right]$$

From §(L5-35), middle column and row dissapear! We don't have those!

$\boxed{G_{21}:}$

$$G_{21}(s) = \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix}$$ ← We can use Gilbert's method!

$$G_{21}(s) = \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \begin{bmatrix} 1 \end{bmatrix}$$

$$\therefore G_{21}(s) \xrightarrow{\mathcal{R}(G)} \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \\ 1 & 0 \end{array}\right]$$

$\boxed{G_{22}:}$

$$G_{22}(s) = \begin{bmatrix} 0 & 1 & \frac{1}{s} \end{bmatrix}$$ ← We can use Gilbert's method!

$$G_{22}(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \frac{1}{s} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\therefore G_{22}(s) \xrightarrow{\mathcal{R}(G)} \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{G_2(s):}$

$$G_2(s) = G_{21}(s)\cdot G_{22}(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

FINALLY, let's put together $G(s) = G_1(s) + G_2(s)$, as explained in §(L5-39).

$$G_1(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|ccc} -2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{array}\right] ; \quad G_2(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$G(s) = G_1(s) + G_2(s) \xrightarrow{\mathcal{R}} \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc|ccc} -2 & -1 & \underline{0} & & 1 & 0 & 0 \\ 1 & 0 & & & 0 & 0 & 0 \\ \hline \underline{0} & & 0 & 1 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

And indeed, this realisation can be shown to be

↪ And indeed, this realisation can be shown to be minimal.