SC4025 Mid-Term Exam Solutions (2009)

Problem 1

a) After subtracting the second from the first equation, the system is equivalent to

$$\ddot{z}_1 + z_1 - z_2 = u_1, \quad \ddot{z}_2 + \dot{z}_1 + z_2 = u_2, \quad y_1 = z_1, \quad y_2 = \dot{z}_2.$$

or to

$$\ddot{z}_1 = -z_1 + z_2 + u_1, \quad \ddot{z}_2 = -\dot{z}_1 - z_2 + u_2, \quad y_1 = z_1, \quad y_2 = \dot{z}_2.$$

Hence

$$\begin{pmatrix} \dot{z}_1 \\ \ddot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ \dot{z}_1 \\ z_2 \\ \dot{z}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ \dot{z}_1 \\ z_2 \\ \dot{z}_2 \end{pmatrix}.$$

b) We have

$$\begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

which has full row rank. Hence the Kalman matrix has full row rank; the system is controllable.

c) Yes, since the Kalman matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix}$$

is non-singular. Controllable systems are stabilizable.

d) $F = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ does the job since

$$A - BF = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right)$$

is Hurwitz since block-diagonal with Hurwitz-blocks.

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Problem 2

a) Equilibria are determined by

$$f(m,p,u) = \begin{pmatrix} \frac{2}{1+p^2} - \alpha m - u \\ \alpha m - p \end{pmatrix} = 0.$$

For $u = 0$ we get $\alpha m = \frac{2}{1+p^2}$ and hence $\frac{2}{1+p^2} - p = 0$. We need to solve $2 - p - p^3 = 0$. Clearly $p_e = 1$ is one solution. Since $f(p) = 2 - p - p^3$ has the derivative $-1 - 3p^2$ which is negative, the graph of this function is strictly decreasing; hence f can have at most one real zero. Therefore there exists only one equilibrium:

$$(m_e, p_e, u_e) = (\frac{1}{\alpha}, 1, 0).$$

b) We have

$$\partial_{m,p} f(m,p,u) = \begin{pmatrix} -\alpha & \frac{-4p}{(1+p^2)^2} \\ \alpha & -1 \end{pmatrix} \quad \text{and} \quad \partial_u f(m,p,u) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The linearization at the equilibrium hence is

$$\dot{x}_\Delta = \begin{pmatrix} -\alpha & -1 \\ \alpha & -1 \end{pmatrix} x_\Delta + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u_\Delta.$$

c) The trace of A is $-(1+\alpha)$ and the determinant is 2α. Hence A is Hurwitz if and only if $\alpha > 0$.

d) The Kalman matrix is $\begin{pmatrix} -1 & \alpha \\ 0 & -\alpha \end{pmatrix}$ and hence non-singular since $\alpha \neq 0$. Therefore the linearization is controllable.

e) Clearly $A - BF$ with $F = \begin{pmatrix} f_1 & f_2 \end{pmatrix}$ reads as

$$\begin{pmatrix} -\alpha + f_1 & -1 + f_2 \\ \alpha & -1 \end{pmatrix}.$$

We can hence choose $f_1 = \alpha - 2$ and $f_2 = 1$.

f) Control the nonlinear system with $u = -f_1(m - \frac{1}{\alpha}) - f_2(p-1)$. This does not change the equilibrium since the control action vanishes for $m = 1/\alpha$ and $p = 1$. The controlled system reads as

$$\dot{m} = \frac{2}{1+p^2} - \alpha m + f_1(m - \frac{1}{\alpha}) - f_2(p-1), \quad \dot{p} = \alpha m - p.$$

Its linearization at the equilibrium must, and actually does, equal

$$A - BF = \begin{pmatrix} -\alpha + f_1 & -1 + f_2 \\ \alpha & -1 \end{pmatrix}.$$

The direct method of Lyapunov implies that the equilibrium is locally asymptotically stable.

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Problem 3

a) The Kalman matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly the given vector cannot be written as a linear combination of these columns (due to the last row, the last element); hence it is not in the range-space. Answer is no.

b) Note that A is block-diagonal, which simplifies the computations a lot (see slides 2-32 and 2-33)! According to slides 2-33 and 2-34 we have

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t\right) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Therefore

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{0t} \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

c) $Ce^{At}B = \cos(t)$ and hence $C(sI - A)^{-1}B = \frac{s}{1+s^2}$, since the transfer matrix is the Laplace transform of the impulse response.

d) Consider

$$\begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -1 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \end{pmatrix}.$$

For $\lambda = 0$ the rank drops. If $\lambda \neq 0$ and if we take any linear combination of the rows with coefficients p_1, p_2, p_3 and set it equal to zero, we get four equations; from the third we infer $p_3 = 0$; from the fourth we get $p_2 = 0$; then either the first or the second imply $p_1 = 0$. Hence the rank does not drop for other λ. The only uncontrollable mode is $\{0\}$.

e) No, since there exists an uncontrollable mode in the closed right-half plane.

f) Let us choose $F = \begin{pmatrix} -1 + f_1 & f_2 & 0 \end{pmatrix}$ to get

$$A - BF = \begin{pmatrix} 0 & 1 & 0 \\ f_1 & f_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can choose f_1 and f_2 to assign the eigenvalues -1 and -2 to the left-upper block. Then $e^{(A-BF)t}$ consists of linear combinations of e^{-t}, e^{-2t} and e^{0t}. Hence the derivative will be a linear combination of e^{-t} and e^{-2t} only, and hence converges to zero for $t \to \infty$.

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SC4025 Mid-Term Exam Solutions (2010)

Problem 1

a) Defining the state vector as $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T = \begin{pmatrix} z & \dot{z} \end{pmatrix}^T$ leads to

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where $\omega = \sqrt{\frac{k}{m}}$.

b) Let us check the eigen values of A.

$$\det(\lambda I - A) = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda_{1,2} = \pm i\omega$$

As the matrix A has distinct eigen values, it is possible to diagonalize A.

In order to construct the transformation matrix we have to get eigen vector corresponding to each eigen value.
For $\lambda = i\omega$, it is easy to get the corresponding eigen vector as follows

$$(\lambda_1 I - A)V_1 = \begin{pmatrix} i\omega & -1 \\ \omega^2 & i\omega \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{12} \end{pmatrix} = 0 \Longrightarrow V_1 = \begin{pmatrix} -i/\omega \\ 1 \end{pmatrix}.$$

For $\lambda = -i\omega$, the eigen vector V_2 is the complex conjugate of the first eigen vector V_1, i.e.

$$V_2 = V_1^* = \begin{pmatrix} i/\omega \\ 1 \end{pmatrix}.$$

Now, the transformation matrix $T = S^{-1}$ where

$$S = (V_1 \quad V_2) = \begin{pmatrix} -i/\omega & i/\omega \\ 1 & 1 \end{pmatrix}$$

and

$$\Lambda = TAT^{-1} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}.$$

c) Using the property given on slide 5 of Lecture 2 we have,

$$\begin{aligned} e^{At} &= T^{-1}e^{\Lambda t}T = Se^{\Lambda t}S^{-1} \\ &= \begin{pmatrix} -i/\omega & i/\omega \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} i\omega/2 & 1/2 \\ -i\omega/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{e^{i\omega t}+e^{-i\omega t}}{2} & \frac{1}{\omega}(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}) \\ -\omega(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}) & \frac{e^{i\omega t}+e^{-i\omega t}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \end{aligned}$$

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d) The response of the system states are given as follows

$$x(t) = e^{At}x(0).$$

The system response is given as follows

$$\begin{aligned} y(t) &= Cx(t) = Ce^{At}x_0 \\ &= \cos(\omega t)x_1(0) + \frac{1}{\omega}\sin(\omega t)x_2(0) \end{aligned}$$

or

$$z(t) = \cos(\omega t)z(0) + \frac{1}{\omega}\sin(\omega t)\dot{z}(0)$$

Problem 2

a) By choosing $x = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}^T$ and $u = \begin{pmatrix} T_x & T_y & T_z \end{pmatrix}^T$ we obtain

$$f(x,u) = \begin{pmatrix} \frac{Q-R}{P}x_2x_3 + \frac{u_1}{P} \\ \frac{R-P}{Q}x_1x_3 + \frac{u_2}{Q} \\ \frac{P-Q}{R}x_1x_2 + \frac{u_3}{R} \end{pmatrix}.$$

b) Using the expressions provided for the control torques, the dynamics of the closed-loop autonomous system will be

$$f_{cl}(x) = \begin{pmatrix} k_1x_1 + \frac{Q-R}{P}x_2x_3 \\ k_2x_2 + \frac{R-P}{Q}x_1x_3 \\ k_3x_3 + \frac{P-Q}{R}x_1x_2 \end{pmatrix}.$$

The equilibria of the closed-loop system can be obtained by solving for all x_e such that $f_{cl}(x_e) = 0$. One trivial equilibrium is $x_{e1} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$. It is also easy to see, that if any of the three states x_1, x_2, x_3 are zero at an equilibrium, then all of them must be zero. We can get the second equilibrium by solving $f_{cl}(x) = 0$ with the assumption of nonzero x_1, x_2, x_3:

$$x_1 = -\frac{Q-R}{Pk_1}x_2x_3, \quad x_2 = -\frac{R-P}{Qk_2}x_1x_3, \quad x_3 = -\frac{P-Q}{Rk_3}x_1x_2,$$

From the first two expressions we get

$$x_2 = \frac{(R-P)(Q-R)}{PQk_1k_2}x_3^2x_2,$$

which (using the assumption of $x_2 \neq 0$) leads to

$$x_{3e} = \pm\sqrt{\frac{PQk_1k_2}{(R-P)(Q-R)}}.$$

Using similar arguments, the other four equilibrium states are obtained as

$$x_{2e} = \pm\sqrt{\frac{PRk_1k_3}{(P-Q)(Q-R)}}, \quad x_{1e} = \pm\sqrt{\frac{RQk_2k_3}{(P-Q)(R-P)}}.$$

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c) It is easy to see that the zero equilibrium (with the corresponding zero control torques) is the one that this question is targeted at. In order to check whether $V(x)$ is a valid Lyapunov function, we have to see if $\frac{\partial V(x)}{\partial x} \cdot f_{cl}(x) \leq 0$ holds for all x.

$$\frac{\partial V(x)}{\partial x} \cdot f_{cl}(x) = \begin{pmatrix} 2P^2 x_1 & 2Q^2 x_2 & 2R^2 x_3 \end{pmatrix} f_{cl}(x) = 2\left(P^2 k_1 x_1^2 + Q^2 k_2 x_2^2 + R^2 k_3 x_3^2\right)$$

The above quadratic expression is non-positive for all x iff $k_1, k_2, k_3 \leq 0$. Thus, the provided function $V(x)$ qualifies as a candidate Lyapunov function as long as $k_1, k_2, k_3 \leq 0$.

d) Asymptotic stability of the closed-loop equilibrium $x_e = \mathbf{0}$ is ensured by the Lyapunov stability theorem if for all x (besides x_e) the conditions $V(x) > V(x_e)$ and $\frac{\partial V(x)}{\partial x} \cdot f_{cl}(x) < 0$ hold. Since $V(x) = P^2 x_1^2 + Q^2 x_2^2 + R^2 x_3^2$ is positive definite and $V(x_e) = 0$, the first condition is clearly satisfied. The second condition is ensured by $k_1, k_2, k_3 < 0$, as can be easily seen from the answer to the previous subquestion.

e) The indirect method of Lyapunov uses the linearized system's stability at an equilibrium point to infer asymptotic stability of the equilibrium for the original nonlinear system. The linearized closed-loop state matrix at the equilibrium $x_e = \mathbf{0}$ is given by the following partial derivative:

$$A_{cl} = \left.\frac{\partial f_{cl}(x)}{\partial x}\right|_{x=0} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$$

Since $k_1, k_2, k_3 < 0$, the above matrix is clearly Hurwitz (all eigenvalues have negative real parts), thus the linearized system is stable.

Problem 3

a) The controllability matrix of the system is

$$K = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 2 & -3 & 5 \\ 1 & -2 & 4 \\ 0 & 1 & -3 \end{pmatrix},$$

which after elementary column operations can be written as

$$\begin{pmatrix} 2 & -3 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This clearly shows that $\det K = 0$, thus the system is not controllable.

b) Using a simple permutation of the states (interchanging x_2 and x_3), we can write the state matrices as

$$\tilde{A} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

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The eigenvalues of the system can be clearly observed on the diagonal of $\tilde{A}$, and they are determined to be $\{-1,-1,-2\}$. Since all eigenvalues have negative real part, the system is stabilizable, thus there exists a feedback gain that drives all states to zero asymptotically (just pick zero as a state feedback gain).

c) Since $\tilde{A}, \tilde{B}$ are in a nice form, let's try to determine a state-feedback gain $\tilde{F}$ in this coordinate system that places the eigenvalues at $\{-1,-2,-3\}$ directly. Denoting $\tilde{F} = \begin{pmatrix} \tilde{f}_1 & \tilde{f}_2 & \tilde{f}_3 \end{pmatrix}$ leads to

$$\tilde{A} - \tilde{B}\tilde{F} = \begin{pmatrix} -1-2\tilde{f}_1 & -2\tilde{f}_2 & -1-2\tilde{f}_3 \\ 0 & -1 & 1 \\ -\tilde{f}_1 & -\tilde{f}_2 & -2-\tilde{f}_3 \end{pmatrix}.$$

By choosing $\tilde{f}_1 = 0$, we can see that the resulting closed-loop matrix is block-diagonal, with an eigenvalue at $\lambda_1 = -1$, and the remaining two eigenvalues determined by the eigenvalues of the following sub-matrix

$$\begin{pmatrix} -1 & 1 \\ -\tilde{f}_2 & -2-\tilde{f}_3 \end{pmatrix}.$$

Observing the characteristic equation of this matrix,

$$(\lambda_{2,3}+1)(\lambda_{2,3}+2+\tilde{f}_3)+\tilde{f}_2 = 0,$$

shows that we need to solve the above equation (with $\lambda_2 = -2, \lambda_3 = -3$) for $\tilde{f}_2$ and $\tilde{f}_3$. From simple calculations, we obtain $\tilde{f}_2 = \tilde{f}_3 = 2$, thus the state-feedback gain matrix for the permuted state matrices will be $\tilde{F} = \begin{pmatrix} 0 & 2 & 2 \end{pmatrix}$. The state-feedback gain matrix for the original order of the states requires flipping the entries that correspond to the second and the third states, which in this case leads to the same gain, i.e.: $F = \begin{pmatrix} 0 & 2 & 2 \end{pmatrix}$.

Bonus Questions

a) The equilibrium $\theta = \theta_0$ can only be maintained if the cart is accelerating with a value that is equal to θ_0.

b) Notice that the system is in controllability normal form, where (using the notation from the lecture slides) $A_{11} = 0$, $A_{12} = 1$, $A_{22} = 0$, $B_1 = 1$. Thus, the system is not controllable, the eigenvalues of A_{22} cannot be moved by any control law. Alternatively, check the rank of the controllability matrix, which in this case will be $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

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SC4025 Solution Mid-Term Exam (2011)

Problem 1 (5+5=10 Points)

a) State-space model of the system:

$$\dot{x} = \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{pmatrix} x + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} i_s$$

modes of the system:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda + \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda + \frac{3}{2} \end{pmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

$$\lambda_1 = -1, \lambda_2 = -2$$

in order to calculate the mode-shapes, we have:

$$Av_i = \lambda_i v_i$$

$$\text{for } \lambda_1 = -1 \longrightarrow \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = -1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

Hence, $v_1 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\text{for } \lambda_2 = -2 \longrightarrow \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = -2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

Hence, $v_2 = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

b) The solution of the system is as follows:

$$x(t) = e^{At}x(0) + \int_a^b e^{A(t-\tau)} B i_s(\tau) d\tau$$

To Compute the matrix e^{At}, we have several approaches:

(i) Taking the inverse of $(SI - A)$.

(ii) Using Caily-Hamilton theorem and ...

(iii) Using the Jordan form.

(iv) Leverrier algorithm.

We go through the first approach;

$$(SI - A)^{-1} = \begin{pmatrix} s + 3/2 & -1/2 \\ -1/2 & s + 3/2 \end{pmatrix}^{-1}$$

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$$= \frac{1}{(s+1)(s+2)} \begin{pmatrix} s+3/2 & 1/2 \\ 1/2 & s+3/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{0.5}{s+1} + \frac{0.5}{s+2} & \frac{0.5}{s+1} - \frac{0.5}{s+2} \\ \frac{0.5}{s+1} - \frac{0.5}{s+2} & \frac{0.5}{s+1} + \frac{0.5}{s+2} \end{pmatrix}$$

If $x(0) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then we will have:

$$x(t) = \pounds(sI - A)^{-1}\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x(t)_{i_s} = e^{-t}\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x(t)_{i_s}$$

If $x(0) = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ then we will have:

$$x(t) = \pounds(sI - A)^{-1}\beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x(t)_{i_s} = e^{-2t}\beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x(t)_{i_s}$$

Conclusion : For the homogenous part of the answer (caused by intial condition), we see that if we set the initial vector equals to each of the mode-shapes, the response will consist of that mode only, and the other mode (eigen-value) will not appear in the response. There is also a physical representation (circuit-wise!) for this, but since some of you maybe don't have any background in Electrical Engineering, the mentioned answer was enough for us.

Problem 2(7+5+3+5=20 Points)

a)

$$det(\lambda I - A) = det(\begin{pmatrix} \lambda+8 & 8 & 0 \\ -8 & \lambda-7 & -1 \\ -7 & -8 & \lambda+1 \end{pmatrix}) = \lambda^3 + 2\lambda^2 + \lambda = 0 \rightarrow \lambda_1 = \lambda_2 = -1,\ \lambda_3 = 0$$

The nullity of the matrix $(-1)I - A$ is 1, so there is only one independent eigen-vector associated with $\lambda = -1$. But we need two eigen-vectors for $\lambda = -1$. The second one can be calculated using the generalized eigen-vectors procedure.

In order to find the eigen-vector, one can use the following equation:

$$Av_1 = \lambda_1 v_1$$

Then for $\lambda_1 = -1$ and $\lambda_3 = 0$ we can obtain the following vectors:

$$v_1 = \begin{pmatrix} 8 \\ -7 \\ -8 \end{pmatrix},\quad v_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

And for the last one, using $(A - (-1)I)v_2 = v_1$, we obtain:

$$v_2 = \begin{pmatrix} 8 \\ -8 \\ -7 \end{pmatrix}$$

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Then we reach the transformation matrix S which consists of the previously obtained eigen-vectors:

$$S = [v_1 v_2 v_3] = \begin{pmatrix} 8 & 8 & 1 \\ -7 & -8 & -1 \\ -8 & -7 & -1 \end{pmatrix}$$

Hence, the Jordan form can be computed using the following:

$$J = S^{-1}AS = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b)

$$\begin{cases} e^{At} = Se^{Jt}S^{-1} \\ e^{Jt} = \begin{pmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{cases} \longrightarrow e^{At} = \begin{pmatrix} 16e^{-t} + 8te^{-t} - 15 & 8e^{-t} - 8 & 8e^{-t} + 8te^{-t} - 8 \\ 15 - 7te^{-t} - 15e^{-t} & 8 - 7e^{-t} & 8 - 7te^{-t} - 8e^{-t} \\ 15 - 8te^{-t} - 15e^{-t} & 8 - 8e^{-t} & 8 - 8te^{-t} - 7e^{-t} \end{pmatrix}$$

c) in order to find the answer of this question, we should first compute the controllability matrix:

$$[B\ AB\ A^2B] = \begin{pmatrix} 2 & -24 & 0 \\ 1 & 24 & -3 \\ 1 & 21 & 3 \end{pmatrix}$$

The rank of this matrix is 3. It means that the system is fully controllable and the mentioned state can be achieved using a proper control input.

d) One way to check this is to find the input-output transfer function of the two systems and check whether they are equal or not:

$$G(s) = C(sI - A)^{-1}B + D$$

$$G_1(s) = \frac{3s+3}{s^2+2s+1} \quad G_2(s) = \frac{4s^2+6s+2}{s^3+2s^2+s}$$

So obviously they are not equivalent. The other way to find the answer is to compute the eigen-values of the second system:

$$det(\lambda I - A_2) = \lambda(\lambda+1)^2 \longrightarrow \lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 0$$

but you should also consider the nullity of $(-1)I - A$, cause this will determine the number and the size of the Jordan blocks. The nullity is 2, so two independent eigen-vectors can be computed for $\lambda = -1$. Thus, the Jordan form of A_2 could be of the form:

$$J_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, obviously there is no similarity transformation between these two systems.

Problem 3 (7+5+5+8=25 Points)

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a) Write the nonlinear system in a state-space form

$$x_1 = \theta, \quad \dot{x}_1 = \dot{\theta} = x_2,$$
$$\dot{x}_2 = -(\frac{g}{l}\sin(x_1) + x_2)$$
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -(\frac{g}{l}\sin(x_1) + x_2) \end{pmatrix} \Rightarrow \dot{x} = f(x)$$

The equilibria of the nonlinear system can be obtained by solving for all x_e such that $f(x_e) = 0$. This nonlinear system has infinitely many equilibrium points in $\mathbb{R}^2$ given by $(x_{1e}, x_{2e}) = (n\pi, 0),\ n = 0, \pm 1, \pm 2, ...$

b) In order to check whether $V(x)$ is a valid Lyapunov function, we have to see if $\frac{\partial V(x)}{\partial x} \cdot f(x) \leq 0$ holds for all x:

$$\frac{\partial V(x)}{\partial x} = \begin{pmatrix} \frac{2g}{l}\sin x_1 & x_2 \end{pmatrix}$$
$$\frac{\partial V(x)}{\partial x} \cdot f(x) = -x_2^2$$

The above expression is non-positive for all x. Thus, the provided function $V(x)$ qualifies as a candidate Lyapunov function.

c) The asymptotic stability of the equilibrium x_e is ensured by the Lyapunov stability theorem. The following conditions should hold for all x:

$$V(x) > V(x_e) \quad \texttt{and} \quad \frac{\partial V(x)}{\partial x} \cdot f(x) < 0.$$

We will analyze the stability of the two equilibrium points, i.e. $x_e^{(1)} = (0, 0)$ and $x_e^{(2)} = (\pi, 0)$:

(i) For the first equilibrium $x_e^{(1)} = (0, 0)$, it is evident that both conditions for stability are satisfied. Hence, the system is asymptotically stable around this equilibrium point.

(ii) For the second equilibrium $x_e^{(2)} = (\pi, 0)$, the second condition holds but the first condition cannot be satisfied. Hence, this equilibrium point is not stable.

d) The indirect method of Lyapunov uses the linearized description of the system around an equilibrium point to verify the asymptotic stability of the original nonlinear system around the equilibrium point.

The Jacobian matrix $\frac{\partial f}{\partial x}$ is given by

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -1 \end{pmatrix}$$

$$A_1 = \frac{\partial f}{\partial x}\Big|_{x = x_e^{(1)} = (0,0)} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -1 \end{pmatrix} \qquad A_2 = \frac{\partial f}{\partial x}\Big|_{x = x_e^{(2)} = (\pi,0)} = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -1 \end{pmatrix}$$

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Since $\text{trace}(A_1) < 0$ and $\det(A_1) > 0$, A_1 is Hurwitz and therefore the first equilibrium point $x_e^{(1)} = (0, 0)$ is asymptotically stable. In contrast, $\text{trace}(A_2) < 0$ and $\det(A_2) < 0$, implying that A_1 is not Hurwitz. Therefore, the second equilibrium point $x_e^{(2)} = (\pi, 0)$ is unstable.

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SC4025 Mid-Term Exam Solutions (2012)

Problem 1

a) The Kalman matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly the given vector cannot be written as a linear combination of these columns (due to the last row, the last element); hence it is not in the range-space. Answer is no.

b) Note that A is block-diagonal, which simplifies the computations a lot (see slides 2-36 and 2-37)! According to slides 2-37 and 2-38 we have

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t\right) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Therefore

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{0t} \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

c) $Ce^{At}B = \cos(t)$ and hence $C(sI - A)^{-1}B = \frac{s}{1+s^2}$, since the transfer matrix is the Laplace transform of the impulse response.

d) Consider

$$\begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 0 \end{pmatrix}.$$

For $\lambda = 0$ the rank drops. If $\lambda \neq 0$ and if we take any linear combination of the rows with coefficients p_1, p_2, p_3 and set it equal to zero, we get four equations; from the third we infer $p_3 = 0$; from the fourth we get $p_2 = 0$; then either the first or the second imply $p_1 = 0$. Hence the rank does not drop for other λ. The only uncontrollable mode is $\{0\}$.

e) No, since there exists an uncontrollable mode in the closed right-half plane.

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Problem 2

a) We should first write the system in the state space form (2 points). By defining

$$x_1 = q \quad x_2 = \dot{q},$$

the state space representation of the system takes the following form:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\alpha x_2 - \beta x_1^3$$

It follows from the state equations that the only equilibrium of the system is $(0, 0)$ (1 point).

To verify if the system is locally stable at the equilibrium point, we first examine whether the given energy function is a suitable candidate for a Lyapunov function (slide 2-41):

$$\partial_x V(x) f(x) = [\beta x_1^3 \;\; x_2] \begin{bmatrix} x_2 \\ -\alpha x_2 - \beta x_1^3 \end{bmatrix} = \beta x_1^3 x_2 - \alpha x_2^2 - \beta x_1^3 x_2$$

$$= -\alpha x_2^2$$

It turns out that when $\alpha > 0$ the energy function qualifies as a Lyapunov function for the system under study:

$$-\alpha x_2^2 \leq 0 \quad \forall x$$

Moreover, it is evident that $V(x) > V(0) = 0$ for all $x \neq 0$ if $\beta > 0$. Therefore, the system is locally stable at the equilibrium $(0, 0)$ when $\alpha > 0$.

b) From the Lyapunov stability theory we cannot conclude that the system is locally asymptotically stable as for asymptotic stability

$$\partial_x V(x) f(x) < 0 \quad \forall x$$

However, in this case there exist state vectors such as $x = (c, 0)$ $c \neq 0$ for which the above condition is not fulfilled (slide 2-41 and 2-42). To check for asymptotic stability, we can try to make use of the indirect method of Lyapunov (slide 2-44).

c) We first linearize the original system around the equilibrium point:

$$\partial_x f(x_e) = \begin{bmatrix} 0 & 1 \\ -3\beta x_1^2 & -\alpha \end{bmatrix}$$

The linearization allows us to compute the matrix A at the equilibrium point:

$$A = \partial_x f(x_e) = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}$$

It turns out that the eigenvalues of the linearized system are $\{-\alpha, 0\}$. Hence, the matrix A is not Hurwitz since all its eigenvalues do not have a negative real part. Therefore, we cannot conclude local asymptotic stability around the equilibrium.

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Problem 3

a) Equilibria are determined by

$$f(m,p,u) = \begin{pmatrix} \frac{2}{1+p^2} - \alpha m - u \\ \alpha m - p \end{pmatrix} = 0.$$

For $u = 0$ we get $\alpha m = \frac{2}{1+p^2}$ and hence $\frac{2}{1+p^2} - p = 0$. We need to solve $2 - p - p^3 = 0$. Clearly $p_e = 1$ is one solution. Since $f(p) = 2 - p - p^3$ has the derivative $-1 - 3p^2$ which is negative, the graph of this function is strictly decreasing; hence f can have at most one real zero. Therefore there exists only one equilibrium:

$$(m_e, p_e, u_e) = (\frac{1}{\alpha}, 1, 0).$$

b) We have

$$\partial_{m,p} f(m,p,u) = \begin{pmatrix} -\alpha & \frac{-4p}{(1+p^2)^2} \\ \alpha & -1 \end{pmatrix} \quad \text{and} \quad \partial_u f(m,p,u) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The linearization at the equilibrium hence is

$$\dot{x}_\Delta = \begin{pmatrix} -\alpha & -1 \\ \alpha & -1 \end{pmatrix} x_\Delta + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u_\Delta.$$

c) The trace of A is $-(1+\alpha)$ and the determinant is 2α. Hence A is Hurwitz if and only if $\alpha > 0$.

d) The Kalman matrix is $\begin{pmatrix} -1 & \alpha \\ 0 & -\alpha \end{pmatrix}$ and hence non-singular since $\alpha \neq 0$. Therefore the linearization is controllable.

e) Clearly $A - BF$ with $F = \begin{pmatrix} f_1 & f_2 \end{pmatrix}$ reads as

$$\begin{pmatrix} -\alpha + f_1 & -1 + f_2 \\ \alpha & -1 \end{pmatrix}.$$

We can hence choose $f_1 = \alpha - 3$ and $f_2 = 1$.

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SC4025 Mid-Term Exam Solutions (2013)

Problem 1

a) The equilibrium points x_e of the given autonomous system can be computed by setting the state derivatives to zero and solving the corresponding state equations:

$$\begin{aligned}\dot{x}_{e1} = 0 \quad &\rightarrow \quad Dx_{e1} = \frac{x_{e2}}{x_{e2}+K}x_{e1} \\ \dot{x}_{e2} = 0 \quad &\rightarrow \quad D(1-x_{e2}) = \frac{x_{e2}}{x_{e2}+K}x_{e1}\end{aligned}$$

Notice that the pair $x_{e1} = 0, x_{e2} = 1$ will always be a (non-negative) equilibrium regardless of the values of D and K. In addition, the general solution of these two equations yields the following expressions for the remaining equilibrium as a function of D and K:

$$x_{e1} = 1 - \frac{DK}{1-D}, \quad x_{e2} = \frac{DK}{1-D}. \tag{1}$$

Thus, in general there are two distinct equilibria for the system: one is described by (1) for given D and K values, and the other is $x_e = (0, 1)$, which is independent of their values. However, in case when $K = \frac{1-D}{D}$, then there is only one equilibrium of the system, and it is $x_e = (0, 1)$. Given the assumptions that $K > 0$ and $0 < D < 1$, the second equilibrium state will always be positive $x_{e2} > 0$. The first equilibrium state will be non-negative if

$$K \leq \frac{1-D}{D}.$$

b) Based on the expressions for the D, K-dependent equilibrium in (1), $K = \frac{1-D}{D}$ corresponds to the equilibrium $x_e = (0, 1)$. In order to infer asymptotic stability of this equilibrium we can use the indirect method of Lyapunov, which is based on the stability properties of the linearized system at the given equilibrium. The linearized system has the following state matrix:

$$\begin{aligned}A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{x=x_e} = \\ &= \begin{pmatrix} \frac{x_{e2}}{x_{e2}+K} - D & \frac{Kx_{e1}}{(x_{e2}+K)^2} \\ -\frac{x_{e2}}{x_{e2}+K} & -\frac{Kx_{e1}}{(x_{e2}+K)^2} - D \end{pmatrix}\end{aligned} \tag{2}$$

Substituting in $x_e = (0, 1)$ and the assumption $K = \frac{1-D}{D}$ leads to

$$A = \begin{pmatrix} 0 & 0 \\ -D & -D \end{pmatrix},$$

which has an eigenvalue at zero, thus this equilibrium is not asymptotically stable.

c) The linearized system matrix (as obtained in (2)) has the following form for $x_e = (0, 1)$:

$$A = \begin{pmatrix} \frac{1}{1+K} - D & 0 \\ -\frac{1}{1+K} & -D \end{pmatrix}$$

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This is a lower triangular matrix, whose eigenvalues are the diagonal entries. Since D is positive, for asymptotic stability we only need to ensure negativity of the A_{11} entry, i.e., $\frac{1}{1+K} - D < 0$. This leads to the condition

$$K > \frac{1-D}{D}.$$

d) We have established in the answer to c) that the $x_e = (0, 1)$ equilibrium is asymptotically stable iff $K > \frac{1-D}{D}$. However, from question a) we know that the other equilibrium, expressed by (1), will only be non-negative for $K \le \frac{1-D}{D}$. The states in this system correspond to bacteria and substrate *concentrations*, thus only non-negative values make sense physically. As a result, the conditions on D and K obtained in c) for the asymptotic stability of $x_e = (0, 1)$ do not guarantee that *all* equilibria are non-negative (i.e., physically meaningful).

e) Substituting the formulas (1) for the *other* system equilibrium expressed in terms of D and K into the linearized system equations (2), we obtain the following entries for the linearized system matrix:

$$A_{11} = 0, \quad A_{12} = \frac{(1-D)(1-D-DK)}{K},$$
$$A_{21} = -D, \quad A_{22} = -\frac{(1-D)(1-D-DK)}{K} - D.$$

In order to assess the stability of the resulting matrix, we can use the fact that a 2×2 matrix is Hurwitz iff $\text{trace}(A) < 0$ and $\det(A) > 0$. Checking the latter implies $A_{11}A_{22} - A_{12}A_{21} > 0$, which leads to

$$K < \frac{1-D}{D}. \tag{3}$$

It turns out that using this condition and the assumed properties of D and K, the trace is automatically negative, i.e., $A_{11} + A_{22} < 0$. Therefore the condition on D and K such that the corresponding equilibrium is asymptotically stable is given in (3).

Problem 2

Let us first perform the Kalman controllability test for the three systems:

$$\text{rank}\begin{pmatrix} B_1 & A_1B_1 \end{pmatrix} = 2 \quad \rightarrow \quad (A_1, B_1) \text{ is controllable},$$
$$\text{rank}\begin{pmatrix} B_2 & A_2B_2 \end{pmatrix} = 1 \quad \rightarrow \quad (A_2, B_2) \text{ is not controllable},$$
$$\text{rank}\begin{pmatrix} B_3 & A_3B_3 \end{pmatrix} = 1 \quad \rightarrow \quad (A_3, B_3) \text{ is not controllable}.$$

Now for the uncontrollable systems perform the Hautus test for the unstable eigenvalue of $\lambda = 1$:

$$\text{rank}\begin{pmatrix} I - A_2 & B_2 \end{pmatrix} = 2 \quad \rightarrow \quad (A_2, B_2) \text{ is stabilizable},$$
$$\text{rank}\begin{pmatrix} I - A_3 & B_3 \end{pmatrix} = 1 \quad \rightarrow \quad (A_3, B_3) \text{ is not stabilizable}.$$

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a) The correct answer includes the stabilizable systems, i.e., 1 and 2.

b) The correct answer includes the controllable system, i.e., 1.

c) Using a state feedback gain $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$, the closed-loop state matrix can be written for systems 1 and 2 as

$$A_1 - B_1 K = \begin{pmatrix} 1-2K_1 & -2K_2 \\ 6+4K_1 & -2+4K_2 \end{pmatrix}, \quad A_2 - B_2 K = \begin{pmatrix} 1-2K_1 & -2K_2 \\ 6-4K_1 & -2-4K_2 \end{pmatrix}$$

It is immediate to see that choosing for instance $K_2 = 0$ and $K_1 = 1$ (i.e., $K = \begin{pmatrix} 1 & 0 \end{pmatrix}$ leads to a stable closed loop in both cases.

Problem 3

a) Recall (e.g., slide 2-51) that the state matrices are related to the impulse response as $h(t) = Ce^{At}B + D\delta(t)$. Observing the given functions for $h(t)$ and assuming a diagonal A matrix, it is easy to check that the following state matrices lead to the specified impulse response (see also the similar question #3 in instruction set 3):

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad D = 1.$$

Since

$$e^{At} = \begin{pmatrix} e^{-3t} & te^{-3t} & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we can verify that indeed

$$Ce^{At}B + D\delta(t) = \begin{pmatrix} e^{-3t} & te^{-3t} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \delta(t) = h(t).$$

b) The system is stable in the Lyapunov sense, since $\mathrm{Re}\{\lambda_i\} \leq 0$ for all the eigenvalues, and the zero eigenvalue has multiplicity one. It is because of the zero eigenvalue however, that the system is not asymptotically stable.

c) The transfer function can be computed for instance either by $H(s) = C(sI-A)^{-1}B + D$, or by computing the Laplace transform of the impulse response:

$$H(s) = \mathcal{L}\{h(t)\} = 1 + \frac{1}{(s+3)^2} + \frac{1}{s} = \frac{s^3 + 7s^2 + 16s + 9}{s(s+3)^2}$$

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SC4025 Mid-Term Exam Solutions (2014)

Problem 1

a) The equilibria corresponding to a constant $u_e > 0$ can be computed using the following equations, which are obtained from the state equations by setting the state derivatives to zero:

$$0 = -\frac{1}{L}x_{2e}u_e + \frac{E}{L},$$
$$0 = -\frac{1}{RC}x_{2e} + \frac{1}{C}x_{1e}u_e.$$

From these we can obtain:

$$x_{2e} = \frac{E}{u_e}, \quad x_{1e} = \frac{E}{Ru_e^2}$$

since u_e is assumed to be strictly positive.

b) The linearized system matrices are obtained as:

$$A = \left.\begin{pmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \frac{\partial f_1(x,u)}{\partial x_2} \\ \frac{\partial f_2(x,u)}{\partial x_1} & \frac{\partial f_2(x,u)}{\partial x_2} \end{pmatrix}\right|_{x_e,u_e} = \begin{pmatrix} 0 & -\frac{u_e}{L} \\ \frac{u_e}{C} & -\frac{1}{RC} \end{pmatrix},$$
$$B = \left.\begin{pmatrix} \frac{\partial f_1(x,u)}{\partial u} \\ \frac{\partial f_2(x,u)}{\partial u} \end{pmatrix}\right|_{x_e,u_e} = \begin{pmatrix} -\frac{E}{Lu_e} \\ \frac{E}{RCu_e^2} \end{pmatrix}.$$

c) Based on the expression for A above, we have $\mathrm{trace}(A) = -\frac{1}{RC} < 0$ and $\det(A) = \frac{u_e^2}{LC} > 0$, thus A is a Hurwitz matrix with eigenvalues that have strictly negative real part. Therefore the linearized system is asymptotically stable.

d) Based on the indirect method of Lyapunov (slide 44 of lecture 2), since A is Hurwitz, therefore the given equilibrium of the original nonlinear system is locally asymptotically stable.

e) Using $u(t) = 0$ for all t leads to the following state equations:

$$\dot{x}_1 = \frac{E}{L},$$
$$\dot{x}_2 = -\frac{1}{RC}x_2.$$

Since E and L are positive scalars, these imply that there is no x_{1e} value for which the system will be at an equilibrium. In particular, the state responses will be the following:

$$x_1(t) = x_1(0) + \frac{E}{L}t, \quad \Rightarrow \quad \lim_{t\to\infty} x_1(t) = \infty$$
$$x_2(t) = e^{-\frac{1}{RC}t}x_2(0), \quad \Rightarrow \quad \lim_{t\to\infty} x_2(t) = 0.$$

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f) Using the provided specific value for the input and the parameters, the equilibrium x_e, the function $f(x)$, and the gradient of the function $V(x)$ will become

$$x_e = \begin{pmatrix} 1 & 1 \end{pmatrix}^T,$$
$$f(x) = \begin{pmatrix} 1 - x_2 \\ x_1 - x_2 \end{pmatrix},$$
$$\frac{\partial V(x)}{\partial x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}.$$

Checking whether $V(x)$ qualifies as a Lyapunov function leads to (see slide 41 of Lecture 2):

$$\frac{\partial V(x)}{\partial x} \cdot f(x) = x_1 - x_2^2 \nleq 0$$

over the domain $\mathcal{D}$, since taking e.g., $x = (x_1, x_2) = (1, 0.5) \in \mathcal{D}$ does not satisfy the above inequality. Unfortunately, we cannot conclude anything from this result regarding the stability of the equilibrium, since Lyapunov's Stability Theorem is a sufficient condition for stability, i.e., a negative result simply means that the chosen $V(x)$ did not work out, and thus we could not infer anything about the stability of the equilibrium.

Problem 2

a) Defining the state vector as $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T = \begin{pmatrix} z & \dot{z} \end{pmatrix}^T$ leads to

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where $\omega = \sqrt{\frac{k}{m}}$.

b) Let us check the eigenvalues of A.

$$\det(\lambda I - A) = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda_{1,2} = \pm i\omega$$

Since the matrix A has distinct eigenvalues, it is possible to diagonalize A. In order to construct the transformation matrix we need to compute the eigenvector corresponding to each eigenvalue.

For $\lambda = i\omega$, the corresponding eigenvector can be computed as follows

$$(\lambda_1 I - A)v_1 = \begin{pmatrix} i\omega & -1 \\ \omega^2 & i\omega \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0 \Rightarrow v_1 = \begin{pmatrix} -i/\omega \\ 1 \end{pmatrix}.$$

For $\lambda = -i\omega$, the eigenvector v_2 is the complex conjugate of the first eigenvector v_1, i.e.,

$$v_2 = v_1^* = \begin{pmatrix} i/\omega \\ 1 \end{pmatrix}.$$

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Now, the transformation matrix is $T = S^{-1}$ where

$$S = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -i/\omega & i/\omega \\ 1 & 1 \end{pmatrix},$$

and

$$\Lambda = TAT^{-1} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}.$$

c) Using the property given on slide 5 of Lecture 2 we have,

$$\begin{aligned} e^{At} &= T^{-1}e^{\Lambda t}T = Se^{\Lambda t}S^{-1} \\ &= \begin{pmatrix} -i/\omega & i/\omega \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} i\omega/2 & 1/2 \\ -i\omega/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{e^{i\omega t}+e^{-i\omega t}}{2} & \frac{1}{\omega}\frac{e^{i\omega t}-e^{-i\omega t}}{2i} \\ -\omega\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right) & \frac{e^{i\omega t}+e^{-i\omega t}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}. \end{aligned}$$

d) The response of the system states are given as follows

$$x(t) = e^{At}x(0).$$

The system response is given as follows

$$\begin{aligned} y(t) &= Cx(t) = Ce^{At}x_0 \\ &= \cos(\omega t)x_1(0) + \frac{1}{\omega}\sin(\omega t)x_2(0), \end{aligned}$$

or

$$z(t) = \cos(\omega t)z(0) + \frac{1}{\omega}\sin(\omega t)\dot{z}(0)$$

Problem 3

a) The parallel interconnection of the two systems has the following state matrices (for the formula, see slide 37 of lecture 1):

$$\tilde{A} = \begin{pmatrix} -1 & -2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -15 & 13 \\ 0 & 0 & -7 & 5 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$
$$\tilde{C} = \begin{pmatrix} 0 & 1 & 1 & 0.5 \end{pmatrix}, \quad \tilde{D} = 0.$$

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b) Using the Hautus test, the system is controllable if $\begin{pmatrix} \tilde{A} - \tilde{\lambda} I & \tilde{B} \end{pmatrix}$ remains full row rank for all eigenvalues $\tilde{\lambda}$. Because of the parallel interconnection, this is equivalent to checking if both $\begin{pmatrix} A_1 - \lambda_1 I & B_1 \end{pmatrix}$ remains full row rank for all eigenvalues λ_1 of System 1 and $\begin{pmatrix} A_2 - \lambda_2 I & B_2 \end{pmatrix}$ remains full row rank for all eigenvalues λ_2 of System 2. The eigenvalues of System 1 are: $\tilde{\lambda}_1 = \lambda_{1,1} = 1$ and $\tilde{\lambda}_2 = \lambda_{1,2} = 2$. The eigenvalues of System 2 are: $\tilde{\lambda}_3 = \lambda_{2,1} = -2$ and $\tilde{\lambda}_4 = \lambda_{2,2} = -8$.

This results in four matrices for which we need to check the rank:

$$\begin{pmatrix} A_1 - 1I & B_1 \end{pmatrix} = \begin{pmatrix} -2 & -2 & 1 \\ 3 & 3 & 0 \end{pmatrix},$$
$$\begin{pmatrix} A_1 - 2I & B_1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 1 \\ 3 & 2 & 0 \end{pmatrix},$$
$$\begin{pmatrix} A_2 + 2I & B_2 \end{pmatrix} = \begin{pmatrix} -13 & 13 & 1 \\ -7 & 7 & 1 \end{pmatrix},$$
$$\begin{pmatrix} A_2 + 8I & B_2 \end{pmatrix} = \begin{pmatrix} -7 & 13 & 1 \\ -7 & 13 & 1 \end{pmatrix}.$$

Clearly the first three matrices are full row rank. The fourth, however, has only rank 1 and therefore is not full row rank. This means that $\lambda_{2,2}$ is an uncontrollable mode of the system. Thus eigenvalues $\tilde{\lambda}_1 = 1$, $\tilde{\lambda}_2 = 2$, and $\tilde{\lambda}_3 = -2$ correspond to the controllable modes. Since not all eigenvalues are controllable the system is **not controllable**.

c) Since all eigenvalues with positive real parts ($\tilde{\lambda}_1$ and $\tilde{\lambda}_2$) are controllable, the system is **stabilizable**. In other words, the only uncontrollable mode of the system $\tilde{\lambda}_4 = -8$ is stable.

d) With state feedback the matrix $\tilde{A}$ is changed to $\tilde{A} - \tilde{B}F$ with $F = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \end{pmatrix}$.

This results in:

$$\tilde{A} - \tilde{B}F = \begin{pmatrix} -1 & -2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -15 & 13 \\ 0 & 0 & -7 & 5 \end{pmatrix} - \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \\ 0 & 0 & 0 & 0 \\ F_1 & F_2 & F_3 & F_4 \\ F_1 & F_2 & F_3 & F_4 \end{pmatrix}$$

Since System 2 is already stable (and its poles correspond with two of the desired poles) we will want to keep the poles of that part of the system. This can be ensured by setting $F_3 = F_4 = 0$. The resulting system matrix will then become:

$$\tilde{A} - \tilde{B}F = \begin{pmatrix} -1 - F_1 & -2 - F_2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -F_1 & -F_2 & -15 & 13 \\ -F_1 & -F_2 & -7 & 5 \end{pmatrix}$$

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Clearly in this way the poles of System 2 remain poles of the closed-loop system. The remaining two other poles can be determined by equating the characteristic polynomial of the first subsystem in closed-loop with the desired one:

$$\begin{aligned}\det\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1-F_1 & -2-F_2 \\ 3 & 4 \end{pmatrix}\right) &= 0, \\ (s+1+F_1)(s-4)+6+3F_2 &= 0, \\ s^2+(F_1-3)\,s+2-4F_1+3F_2 &= 0.\end{aligned}$$

The desired pole locations are $s=-2$ and $s=-3$ resulting in the desired characteristic equation:

$$s^2+5s+6=0$$

From this it follows that

$$\begin{aligned} F_1-3 &= 5, \\ 2-4F_1+3F_2 &= 6. \end{aligned}$$

This results in $F_1=8$ and $F_2=12$ and $F=\begin{pmatrix} 8 & 12 & 0 & 0 \end{pmatrix}$.

Thus the system matrix of the closed-loop system becomes

$$\tilde{A}-\tilde{B}F=\begin{pmatrix} -9 & -14 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -8 & -12 & -15 & 13 \\ -8 & -12 & -7 & 5 \end{pmatrix}$$

with the poles at $s_1=-2$, $s_2=-2$, $s_3=-3$, and $s_4=-8$.

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SC4025 Mid-Term Exam Solutions (2015)

Problem 1

a) To compute the equilibria of the system, we set $\dot{\omega} = 0$ and we compute the vectors ω_e that satisfy the following system of equations:

$$\begin{aligned} 0 &= (I_2 - I_3)\omega_2\omega_3, \\ 0 &= (I_3 - I_1)\omega_3\omega_1, \\ 0 &= 0. \end{aligned}$$

Since $\omega_3 \neq 0$ was given, the solution is $\omega_e = \begin{pmatrix} 0 & 0 & k \end{pmatrix}$, where k is any nonzero real number. Hence, the system has ∞ number of equilibrium points.

b) The linearized system at the equilibrium points ω_e above is given by:

$$\dot{\tilde{\omega}} = \left(\partial \begin{pmatrix} (I_2 - I_3)\omega_2\omega_3 \\ (I_3 - I_1)\omega_3\omega_1 \\ 0 \end{pmatrix} / \partial\omega \right)\Bigg|_{\omega_e} \tilde{\omega} \quad \Rightarrow \quad \dot{\tilde{\omega}} = \underbrace{\begin{pmatrix} 0 & (I_2 - I_3)k & 0 \\ (I_3 - I_1)k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_e} \tilde{\omega},$$

where $\tilde{\omega} := \omega - \omega_e$.

c) According to the indirect Lyapunov method, we study the eigenvalues of A_e. The equilibrium is locally asymptotically stable if A_e is Hurwitz. If we compute the eigenvalues of A_e, we get

$$\lambda(A_e) = \{0, \pm k\sqrt{(I_3 - I_1)(I_2 - I_3)}\} = \{0, \pm k(I_3 - I)j\},$$

where we used the fact that $I_1 = I_2 = I$. Hence, we do not have enough information to conclude that ω_e is a locally asymptotically stable equilibrium point.

d) If we use the suggested Lyapunov function, we immediately note that the rotational energy is

$$V(\omega) > V(\omega_e) = 0 \quad \text{for} \quad \omega \neq 0.$$

If we compute

$$\frac{\partial V(\omega)}{\partial \omega} f(\omega) = \begin{pmatrix} I\omega_1 & I\omega_2 & I_3\omega_3 \end{pmatrix} \begin{pmatrix} (I - I_3)\omega_2\omega_3 \\ (I_3 - I)\omega_1\omega_3 \\ 0 \end{pmatrix} = I(I - I_3)(1-1)\omega_1\omega_2\omega_3 = 0.$$

Hence, according to the direct Lyapunov method, we can conclude that ω_e is locally stable, but we cannot conclude local asymptotic stability, since for that we would need the derivative to be strictly less than zero.

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Problem 2

The system matrices are

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

with state dimension $n = 2$. The controllability matrix of the system is

$$\begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\operatorname{rank}\begin{pmatrix} B & AB \end{pmatrix} = 2$, the system is controllable.

a) For the initial state $x(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and input $u(t)$, the state trajectory of the system is

$$\begin{aligned} x_1(t) &= 0 + \int_0^t u(\tau)d\tau, \\ x_2(t) &= 2 + \int_0^t x_1(\tau)d\tau. \end{aligned}$$

We need to find α, β such that with $u(t)$ of the form

$$u(t) = \begin{cases} \alpha & , 0 \leq t < 1, \\ \beta & , 1 \leq t \leq 2, \end{cases}$$

we have $x_1(2) = 2$ and $x_2(2) = 0$. It can be seen that

$$\begin{aligned} x_1(t) &= \begin{cases} \int_0^t \alpha d\tau & , 0 \leq t \leq 1, \\ \int_0^1 \alpha d\tau + \int_1^t \beta d\tau & , 1 \leq t \leq 2. \end{cases} \\ &= \begin{cases} \alpha t & , 0 \leq t \leq 1, \\ \alpha + \beta(t-1) & , 1 \leq t \leq 2. \end{cases} \\ x_1(2) &= \alpha + \beta. \end{aligned}$$

We have

$$\begin{aligned} x_2(2) &= 2 + \int_0^2 x_1(\tau)d\tau \\ &= 2 + \int_0^1 x_1(\tau)d\tau + \int_1^2 x_1(\tau)d\tau \\ &= 2 + \int_0^1 \alpha\tau d\tau + \int_1^2 [\alpha + \beta(\tau - 1)]d\tau \\ &= 2 + \frac{3}{2}\alpha + \frac{\beta}{2} \end{aligned}$$

To find α and β we solve the system

$$\begin{aligned} \alpha + \beta &= x_1(2) = 2, \\ 2 + \frac{3}{2}\alpha + \frac{\beta}{2} &= x_2(2) = 0. \end{aligned}$$

This system yields $\alpha = -3, \beta = 5$.

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b) For the initial state $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and input $u(t)$, the state trajectory of the system is

$$x_1(t) = \int_0^t u(\tau)d\tau,$$
$$x_2(t) = \int_0^t x_1(\tau)d\tau.$$

If we take $u(t)$ of the form

$$u(t) = \begin{cases} \alpha & ,0 \leq t < 1, \\ \beta & ,1 \leq t \leq 2, \\ 0 & ,2 < t. \end{cases}$$

then

$$\begin{aligned} x_1(t) = \int_0^t u(\tau)d\tau &= \begin{cases} \int_0^t \alpha d\tau & ,0 \leq t \leq 1, \\ \int_0^1 \alpha d\tau + \int_1^t \beta d\tau & ,1 < t \leq 2, \\ \int_0^1 \alpha d\tau + \int_1^2 \beta d\tau + \int_2^t 0 d\tau & ,t > 2. \end{cases} \\ &= \begin{cases} \alpha t & ,0 \leq t \leq 1, \\ \alpha + \beta(t-1) & ,1 < t \leq 2, \\ \alpha + \beta & ,t > 2. \end{cases} \end{aligned}$$

Thus, whenever $t \geq 2$, one has

$$\begin{aligned} x_2(t) = \int_0^t x_1(\tau)d\tau &= \int_0^1 x_1(\tau)d\tau + \int_1^2 x_1(\tau)d\tau + \int_2^t x_1(\tau)d\tau \\ &= \int_0^1 \alpha\tau d\tau + \int_1^2 [\alpha + \beta(\tau - 1)]d\tau + \int_2^t (\alpha+\beta)d\tau \\ &= \frac{\alpha}{2} + \alpha + \frac{\beta}{2} + (\alpha+\beta)(t-2). \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) = t, \forall t \geq 2 &\Longleftrightarrow x_1(t) + x_2(t) = t, \forall t \geq 2 \\ &\Longleftrightarrow \alpha + \beta + \frac{\alpha}{2} + \alpha + \frac{\beta}{2} + (\alpha+\beta)(t-2) = t, \forall t \geq 2 \\ &\Longleftrightarrow \begin{cases} \alpha + \beta & = 1, \\ -\alpha - \beta + \frac{\alpha}{2} + \alpha + \frac{\beta}{2} & = 0. \end{cases} \\ &\Longleftrightarrow \alpha = \beta = \frac{1}{2} \end{aligned}$$

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Problem 3

a) The Kalman matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly the given vector cannot be written as a linear combination of these columns (due to the last row, the last element); hence it is not in the range-space. Answer is no. (Note that it is not enough to answer that the system is not controllable!)

b) Note that A is block-diagonal, which simplifies the computations a lot (see slides 2-36 and 2-37)! According to slides 2-37 and 2-38 we have

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t\right) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Therefore

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{0t} \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

c) $Ce^{At}B = \cos(t)$ and hence $C(sI - A)^{-1}B = \frac{s}{1+s^2}$, since the transfer matrix is the Laplace transform of the impulse response.

d) Consider

$$\begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 0 \end{pmatrix}.$$

For $\lambda = 0$ the rank drops. If $\lambda \neq 0$ and if we take any linear combination of the rows with coefficients p_1, p_2, p_3 and set it equal to zero, we get four equations; from the third we infer $p_3 = 0$; from the fourth we get $p_2 = 0$; then either the first or the second imply $p_1 = 0$. Hence the rank does not drop for other λ. The only uncontrollable mode is $\{0\}$.

e) No, since there exists an uncontrollable mode in the closed right-half plane.

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SC42015 Mid-Term Exam Solutions (2016)

Problem 1

a) The eigenvalues are at $\{1, 1, -4\}$. Since there are eigenvalues with positive real part, the system is unstable.

b) No. Using the Hautus test, it can be shown that -4 is an uncontrollable mode.

c) We can compute two true eigenvectors and one generalized eigenvector, which leads to $T^{-1} = S = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, such that $S^{-1}AS = J = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Using the transformation formulas for the input and output matrices lead to $\tilde{B} = S^{-1}B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and $\tilde{C} = CS = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

d) By equating the closed-loop characteristic equation $\det\left(\lambda I - (J + \tilde{B}\tilde{K})\right)$ with the desired characteristic polynomial $(\lambda + 4)^3$ we obtain $\tilde{K} = \begin{pmatrix} \tilde{k}_1 & -25 & -10 \end{pmatrix}$, with $\tilde{k}_1$ arbitrary. For minimal vector 2-norm we need to choose $\tilde{k}_1 = 0$.

e) $K = \tilde{K}S^{-1} = \begin{pmatrix} 0 & -10 & -35 \end{pmatrix}$.

f) Calculate the range space of $W = \begin{pmatrix} B & AB & A^2B \end{pmatrix}$ and show that x_F is not contained within (or equivalently, show that $\mathbf{rank}\begin{pmatrix} W & x_F \end{pmatrix} = 3 > \mathbf{rank}(W) = 2$).

g) Since the uncontrollable mode -4 is on the left half plane, the system is stabilizable.

Problem 2

a) Setting $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ and solving the nonlinear equations, we obtain $x_{e1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ and $x_{e2} = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}^T$ as the equilibrium points.

b) The Jacobian matrix of the nonlinear function $f(x)$ describing the dynamics is the following:

$$\frac{\partial f(x)}{\partial x} = \begin{pmatrix} -x_2 & -x_1 & x_2 \\ 2x_1 & 0 & 0 \\ -x_3 & 0 & -x_1 \end{pmatrix}.$$

This leads to the following two linearized system matrices:

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x_{e1}} = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \left.\frac{\partial f(x)}{\partial x}\right|_{x_{e2}} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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c) We can use the linearizations from the previous part to determine the stability of the system near the two equilibria: (i) For the first case, the characteristic equation is $(s^2+2)(s+1)=0$. The linear system is marginally stable with two eigenvalues on the imaginary axis. So Lyapunov theory does not tell us whether this system is stable at the equilibrium point x_{e1}. (ii) For the second case, the characteristic equation is $(s^2+2)(s-1)=0$. Thus the system at the equilibrium point x_{e2} is unstable.

d) We can establish that

$$V(x) > V(x_{e1}) = 0 \quad \forall x \neq x_{e1}, \quad \text{and} \quad V(x) \not> V(x_{e2}) = 4 \quad \forall x \neq x_{e2},$$

furthermore

$$\frac{\partial V(x)}{\partial x} f(x) \not\leq 0,$$

by taking for instance $x_1 = x_2 = 2$ and $x_3 = 0$. This implies that we cannot use the provided $V(x)$ function as a Lyapunov function candidate and make any conclusions about the stability properties of the two equilibria.

Problem 3

a) Recall (e.g., slide 2-51) that the state matrices are related to the impulse response as $h(t) = Ce^{At}B + D\delta(t)$. Observing the given functions for $h(t)$ and assuming a block-diagonal A matrix, it is easy to check that the following state matrices lead to the specified impulse response (see also the similar question #3 in instruction set 3):

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad D = 1.$$

Since

$$e^{At} = \begin{pmatrix} e^{-3t} & te^{-3t} & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we can verify that indeed

$$Ce^{At}B + D\delta(t) = \begin{pmatrix} e^{-3t} & te^{-3t} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \delta(t) = h(t).$$

b) The system is stable in the Lyapunov sense, since $\mathrm{Re}\{\lambda_i\} \leq 0$ for all the eigenvalues, and the zero eigenvalue has multiplicity one. It is because of the zero eigenvalue however, that the system is not asymptotically stable.

c) The transfer function can be computed for instance either by $H(s) = C(sI-A)^{-1}B + D$, or by computing the Laplace transform of the impulse response:

$$H(s) = \mathcal{L}\{h(t)\} = 1 + \frac{1}{(s+3)^2} + \frac{1}{s} = \frac{s^3+7s^2+16s+9}{s(s+3)^2}$$

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SC42015 Mid-Term Exam Solutions (2017)

Problem 1

a)

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda+8 & 8 & 0 \\ -8 & \lambda-7 & -1 \\ -7 & -8 & \lambda+1 \end{pmatrix} = \lambda^3 + 2\lambda^2 + \lambda = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 = -1, \lambda_3 = 0.$$

The nullity of the matrix $((-1)I - A)$ is 1, so there is only one independent eigenvector associated with $\lambda = -1$. (The nullity is the dimension of the null-space, which can be computed by $\dim(A) - \text{rank}(A)$. Since rank is 2 due to the first and last rows being linearly dependent (opposite sign), the nullity is 1.) This implies that the Jordan-block associated with $\lambda = -1$ is of size two. Hence, the Jordan form of A is the following:

$$J = S^{-1}AS = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This means that we have only one true eigenvector for $\lambda = -1$, and we need to compute a generalized eigenvector for the transformation matrix S. In order to find the true eigenvector, one can use the following equation:

$$Av_i = \lambda_i v_i.$$

Then for $\lambda_1 = -1$ and $\lambda_3 = 0$ we can obtain the following vectors:

$$v_1 = \begin{pmatrix} 8 \\ -7 \\ -8 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

For the generalized eigenvector, we can use $(A - (-1)I)\, v_2 = v_1$ to obtain:

$$v_2 = \begin{pmatrix} 8 \\ -8 \\ -7 \end{pmatrix}.$$

The transformation matrix S consists of the previously obtained eigenvectors:

$$S = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 8 & 8 & 1 \\ -7 & -8 & -1 \\ -8 & -7 & -1 \end{pmatrix}.$$

b) In order to find the answer for this question, we should first compute the controllability matrix:

$$\begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 2 & -24 & 0 \\ 1 & 24 & -3 \\ 1 & 21 & 3 \end{pmatrix}.$$

The rank of this matrix is 3 (needs to be shown!). This means that the system is fully controllable and the mentioned final state can be achieved using a proper control input (independent of the final time value).

c) One way to check this is to find the input-output transfer function of the two systems and check whether they are equal or not:

$$G(s) = C(sI - A)^{-1}B + D,$$

$$G_1(s) = \frac{3s+3}{s^2+2s+1}, \quad G_2(s) = \frac{4s^2+6s+2}{s^3+2s^2+s}.$$

Which implies that they are not equivalent. The other (possibly quicker and simpler) way to find the answer is to compute the eigenvalues of the second system:

$$\det(\lambda I - A_2) = \lambda(\lambda+1)^2 = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 = -1, \lambda_3 = 0,$$

which match with those of the first system, however you should also consider the nullity of $((-1)I - A)$, because this will determine the number and the size of the Jordan blocks. The nullity is 2, so two independent eigenvectors can be computed for $\lambda = -1$. Thus, the Jordan form of A_2 will have the form:

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, there is no similarity transformation between these two systems.

Problem 2

Let us first perform the Kalman controllability test for the three systems:

$$\begin{aligned}
\text{rank}\begin{pmatrix} B_1 & A_1B_1 \end{pmatrix} &= 2 \quad \rightarrow \quad (A_1, B_1) \text{ is controllable,} \\
\text{rank}\begin{pmatrix} B_2 & A_2B_2 \end{pmatrix} &= 1 \quad \rightarrow \quad (A_2, B_2) \text{ is not controllable,} \\
\text{rank}\begin{pmatrix} B_3 & A_3B_3 \end{pmatrix} &= 1 \quad \rightarrow \quad (A_3, B_3) \text{ is not controllable.}
\end{aligned}$$

Now for the uncontrollable systems perform the Hautus test for the unstable eigenvalue of $\lambda = 1$:

$$\begin{aligned}
\text{rank}\begin{pmatrix} I - A_2 & B_2 \end{pmatrix} &= 2 \quad \rightarrow \quad (A_2, B_2) \text{ is stabilizable,} \\
\text{rank}\begin{pmatrix} I - A_3 & B_3 \end{pmatrix} &= 1 \quad \rightarrow \quad (A_3, B_3) \text{ is not stabilizable.}
\end{aligned}$$

a) The correct answer includes the stabilizable systems, i.e., 1 and 2.

b) The correct answer includes the controllable system, i.e., 1.

c) Using a state feedback gain $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$, the closed-loop state matrix can be written for systems 1 and 2 as

$$A_1 - B_1 K = \begin{pmatrix} 1-2K_1 & -2K_2 \\ 6+4K_1 & -2+4K_2 \end{pmatrix}, \quad A_2 - B_2 K = \begin{pmatrix} 1-2K_1 & -2K_2 \\ 6-4K_1 & -2-4K_2 \end{pmatrix}$$

It is immediate to see that choosing for instance $K_2 = 0$ and $K_1 = 1$ (i.e., $K = \begin{pmatrix} 1 & 0 \end{pmatrix}$) leads to a stable closed loop in both cases.

Problem 3

a) The equilibrium points x_e of the given autonomous system can be computed by setting the state derivatives to zero and solving the corresponding state equations:

$$\begin{aligned} \dot{x}_{e1} = 0 \quad &\rightarrow \quad D x_{e1} = \frac{x_{e2}}{x_{e2}+K} x_{e1} \\ \dot{x}_{e2} = 0 \quad &\rightarrow \quad D(1-x_{e2}) = \frac{x_{e2}}{x_{e2}+K} x_{e1} \end{aligned}$$

Notice that the pair $x_{e1} = 0, x_{e2} = 1$ will always be a (non-negative) equilibrium regardless of the values of D and K. In addition, the general solution of these two equations yields the following expressions for the remaining equilibrium as a function of D and K:

$$x_{e1} = 1 - \frac{DK}{1-D}, \quad x_{e2} = \frac{DK}{1-D}. \tag{1}$$

Thus, in general there are two distinct equilibria for the system: one is described by (1) for given D and K values, and the other is $x_e = (0, 1)$, which is independent of their values. However, in case when $K = \frac{1-D}{D}$, then there is only one equilibrium of the system, and it is $x_e = (0, 1)$. Given the assumptions that $K > 0$ and $0 < D < 1$, the second equilibrium state will always be positive $x_{e2} > 0$. The first equilibrium state will be non-negative if

$$K \leq \frac{1-D}{D}.$$

b) Based on the expressions for the D, K-dependent equilibrium in (1), $K = \frac{1-D}{D}$ corresponds to the equilibrium $x_e = (0, 1)$. In order to infer asymptotic stability of this equilibrium we can use the indirect method of Lyapunov, which is based on the stability properties of the linearized system at the given equilibrium. The linearized system has the following state matrix:

$$\begin{aligned} A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x=x_e} = \\ = \begin{pmatrix} \frac{x_{e2}}{x_{e2}+K} - D & \frac{K x_{e1}}{(x_{e2}+K)^2} \\ -\frac{x_{e2}}{x_{e2}+K} & -\frac{K x_{e1}}{(x_{e2}+K)^2} - D \end{pmatrix} \end{aligned} \tag{2}$$

Substituting in $x_e = (0, 1)$ and the assumption $K = \frac{1-D}{D}$ leads to

$$A = \begin{pmatrix} 0 & 0 \\ -D & -D \end{pmatrix},$$

which has an eigenvalue at zero, thus the linearization fails to establish stability (we cannot conclude anything).

c) The linearized system matrix (as obtained in (2)) has the following form for $x_e = (0, 1)$:

$$A = \begin{pmatrix} \frac{1}{1+K} - D & 0 \\ -\frac{1}{1+K} & -D \end{pmatrix}$$

This is a lower triangular matrix, whose eigenvalues are the diagonal entries. Since D is positive, for asymptotic stability we only need to ensure negativity of the A_{11} entry, i.e., $\frac{1}{1+K} - D < 0$. This leads to the condition

$$K > \frac{1-D}{D}.$$

d) We have established in the answer to c) that the $x_e = (0, 1)$ equilibrium is asymptotically stable iff $K > \frac{1-D}{D}$. However, from question a) we know that the other equilibrium, expressed by (1), will only be non-negative for $K \leq \frac{1-D}{D}$. The states in this system correspond to bacteria and substrate *concentrations*, thus only non-negative values make sense physically. As a result, the conditions on D and K obtained in c) for the asymptotic stability of $x_e = (0, 1)$ do not guarantee that *all* equilibria are non-negative (i.e., physically meaningful).

e) Substituting the formulas (1) for the *other* system equilibrium expressed in terms of D and K into the linearized system equations (2), we obtain the following entries for the linearized system matrix:

$$A_{11} = 0, \quad A_{12} = \frac{(1-D)(1-D-DK)}{K},$$
$$A_{21} = -D, \quad A_{22} = -\frac{(1-D)(1-D-DK)}{K} - D.$$

In order to assess the stability of the resulting matrix, we can use the fact that a 2×2 matrix is Hurwitz iff $\text{trace}(A) < 0$ and $\det(A) > 0$. Checking the latter implies $A_{11}A_{22} - A_{12}A_{21} > 0$, which leads to

$$K < \frac{1-D}{D}. \tag{3}$$

It turns out that using this condition and the assumed properties of D and K, the trace is automatically negative, i.e., $A_{11} + A_{22} < 0$. Therefore the condition on D and K such that the corresponding equilibrium is asymptotically stable is given in (3).

SC42015 Mid-Term Exam Solutions (2018)

Problem 1

a) Defining the state vector as $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T = \begin{pmatrix} z & \dot{z} \end{pmatrix}^T$ leads to

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where $\omega = \sqrt{\frac{k}{m}}$.

b) Let us check the eigenvalues of A.

$$\det(\lambda I - A) = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda_{1,2} = \pm i\omega$$

Since the matrix A has distinct eigenvalues, it is possible to diagonalize A. In order to construct the transformation matrix we need to compute the eigenvector corresponding to each eigenvalue.

For $\lambda = i\omega$, the corresponding eigenvector can be computed as follows

$$(\lambda_1 I - A)v_1 = \begin{pmatrix} i\omega & -1 \\ \omega^2 & i\omega \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0 \Rightarrow v_1 = \begin{pmatrix} -i/\omega \\ 1 \end{pmatrix}.$$

For $\lambda = -i\omega$, the eigenvector v_2 is the complex conjugate of the first eigenvector v_1, i.e.,

$$v_2 = v_1^* = \begin{pmatrix} i/\omega \\ 1 \end{pmatrix}.$$

Now, the transformation matrix is $T = S^{-1}$ where

$$S = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -i/\omega & i/\omega \\ 1 & 1 \end{pmatrix},$$

and

$$\Lambda = TAT^{-1} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}.$$

c) Using the property given on slide 5 of Lecture 2 we have,

$$\begin{aligned} e^{At} &= T^{-1}e^{\Lambda t}T = Se^{\Lambda t}S^{-1} \\ &= \begin{pmatrix} -i/\omega & i/\omega \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} i\omega/2 & 1/2 \\ -i\omega/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{e^{i\omega t}+e^{-i\omega t}}{2} & \frac{1}{\omega}\frac{e^{i\omega t}-e^{-i\omega t}}{2i} \\ -\omega\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right) & \frac{e^{i\omega t}+e^{-i\omega t}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}. \end{aligned}$$

d) The response of the system states is given as follows

$$x(t) = e^{At}x(0).$$

The system response is given as follows

$$\begin{aligned} y(t) &= Cx(t) = Ce^{At}x_0 \\ &= \cos(\omega t)x_1(0) + \frac{1}{\omega}\sin(\omega t)x_2(0), \end{aligned}$$

or

$$z(t) = \cos(\omega t)z(0) + \frac{1}{\omega}\sin(\omega t)\dot{z}(0)$$

Problem 2

a) For the provided Lyapunov function we have $V(x) > 0$, $V(x) = 0 \Leftrightarrow x = 0$ since it is a sum of squares of the states, and

$$\begin{aligned} \dot{V}(x) &= x_1^3\dot{x}_1 + x_2\dot{x}_2 + x_3^3\dot{x}_3 \\ &= x_1^3(x_2 - x_3^3) + x_2(-x_1^3) + x_3^3(x_1^3 + \alpha(x_3^3 - x_3)) \\ &= x_1^3x_2 - x_1^3x_3^3 - x_1^3x_2 + x_1^3x_3^3 + \alpha x_3^4(x_3^2 - 1) \\ &= \alpha x_3^4(x_3^2 - 1) \end{aligned}$$

Since the term $(x_3^2 - 1)$ is negative for points close to the origin ($x_3 = 0$), for $\dot{V}(x)$ to be nonpositive, we need $\alpha \geq 0$. Therefore, for $\alpha \geq 0$, we have $\dot{V}(x) \leq 0$ for $|x_3| \leq 1$ (i.e., near the equilibrium), and the equilibrium is Lyapunov stable. (For $\alpha > 0$, this means $\dot{V}(x) < 0$ near the equilibrium, and therefore asymptotic stability.) For $\alpha < 0$, we cannot conclude anything about the stability of the origin. Exploring the case of $|x_3| > 1$ does not make sense, since this region is not near the zero equilibrium.

b) For the indirect method we need to linearize the system around origin:

$$\partial_x f(x)|_{x_e} = \left.\begin{pmatrix} 0 & 1 & -3x_3^2 \\ -3x_1^2 & 0 & 0 \\ 3x_1^2 & 0 & \alpha(3x_3^2 - 1) \end{pmatrix}\right|_{x_e} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha \end{pmatrix}$$

This matrix is in Jordan form and has two eigenvalues $\lambda_1 = 0$ (double) and $\lambda_2 = -\alpha$. Since $\lambda_1 = 0$ has multiplicity of two, the linearized system is unstable irrespective of value of α. However, due to the zero eigenvalues, based on the material covered in the course, we can make only the following conclusions. For $\alpha < 0$, there is an unstable eigenvalue (with positive real part), thus the origin is *unstable* for the nonlinear system. For $\alpha \geq 0$, the linearized system does not have any eigenvalue with positive real part, and not all eigenvalues have negative real part, thus we *cannot conclude anything* about the stability of the nonlinear system equilibrium at the origin using the indirect method of Lyapunov.

(In fact, from the answer to a) we could observe that the original nonlinear system equilibrium is Lyapunov stable for $\alpha \geq 0$, even if its linearization is unstable.)

Problem 3

a) The columns of $S = T^{-1}$ are the eigenvectors of A. So, we start by finding the eigenvalues of A:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda - 1 & -1 \\ -1 & 2 & \lambda + 2 \end{pmatrix} = \lambda^2(\lambda + 1) = 0$$

Eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = -1$, with algebraic multiplicity of two and one, respectively. Now, we have to compute the corresponding eigenvectors. For $\lambda_1 = 0$ we have:

$$(\lambda_1 I - A)v = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 2 \end{pmatrix} v = 0$$

Using row operations, we see that $\lambda_1 = 0$ has only one ordinary eigenvector:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \alpha, \ \alpha \in \mathbb{R} \setminus \{0\} .$$

For the generalized eigenvector, we can solve

$$(A - \lambda_1 I)v = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & -2 \end{pmatrix} v = v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

which gives

$$v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \alpha, \ \alpha \in \mathbb{R} .$$

Similarly, for $\lambda_2 = -1$ we have:

$$(\lambda_2 I - A)v = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix} v = 0$$

Row operations give:

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \alpha, \ \alpha \in \mathbb{R} \setminus \{0\} .$$

Therefore, we have

$$S = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad S^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$\tilde{A} = S^{-1}AS = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tilde{B} = S^{-1}B = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

b) To solve the problem in the z-coordinates, we first transform the final state: $z_F = S^{-1}x_F = \begin{pmatrix} -1 & -1 & 0 \end{pmatrix}^T$. The corresponding Kalman matrix is:

$$\tilde{K} = (\tilde{B} \;\; \tilde{A}\tilde{B} \;\; \tilde{A}^2\tilde{B}) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is of rank 2. Now, since $\begin{pmatrix} \tilde{K} & z_F \end{pmatrix}$ is also of rank 2, that is, $z_F \in R(\tilde{K})$, the given final state is reachable. To find the corresponding input, we need to compute the Grammian:

$$\tilde{W}_{T=1} = \int_0^{T=1} e^{\tilde{A}t}\tilde{B}\tilde{B}^T e^{\tilde{A}^T t} dt$$

$$\tilde{W}_{T=1} = \int_0^{T=1} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} dt$$

$$\tilde{W}_{T=1} = \int_0^{T=1} \begin{pmatrix} t^2 & t & 0 \\ t & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} dt = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we have to solve the equation $\tilde{W}_{T=1}\alpha = z_F$ which gives $\alpha = \begin{pmatrix} -6 & 2 & \alpha_3 \end{pmatrix}^T$. Finally, we have the input as:

$$u(\tau) = \tilde{B}^T e^{\tilde{A}^T(1-\tau)}\alpha = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 1-\tau & 1 & 0 \\ 0 & 0 & e^{\tau-1} \end{pmatrix} \begin{pmatrix} -6 \\ 2 \\ \alpha_3 \end{pmatrix} = -6\tau + 4.$$

c) Once again, we start with transforming the initial state into z-coordinates : $z_I = S^{-1}x_I = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. Note that the Jordan form is already in controllability normal form and it shows that $\lambda_2 = -1$ is an uncontrollable mode:

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tilde{B} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Therefore, since $z_{I_3}(0) \neq 0$, it is *not* possible to reach the zero state in finite time. Alternatively, one can show

$$\Delta = z_F - e^{\tilde{A}^T T} z_I = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^{-1} \end{pmatrix}$$

does *not* belong to $R(\tilde{K})$, since

$$rank\begin{pmatrix} \tilde{K} & \Delta \end{pmatrix} = rank\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-1} \end{pmatrix} = 3 \neq rank(\tilde{K}) = 2.$$

d) Once again, we can directly use the Jordan form, $\tilde{A}$, which is also in controllability normal form. Since the uncontrollable mode ($\lambda_2 = -1$) is stable, the system is stabilizable. Alternatively, using Hautus-test for stabilizability, one can show for $\lambda_1 = 0$, i.e., the eigenvalue with non-negative real part, the matrix $\begin{pmatrix} \tilde{A} - \lambda_1 I & \tilde{B} \end{pmatrix}$ has full row rank:

$$rank\begin{pmatrix} \tilde{A} - \lambda_1 I & \tilde{B} \end{pmatrix} = rank\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = 3$$

Now, considering a state feedback in the transformed coordinates $u = -\tilde{F}z = -\begin{pmatrix} \tilde{f}_1 & \tilde{f}_2 & \tilde{f}_3 \end{pmatrix} z$, our first task is to find $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ such that the characteristic polynomial of the closed-loop system,

$$\det(\lambda I - \tilde{A} + \tilde{B}\tilde{F}) = \det\begin{pmatrix} \lambda & -1 & 0 \\ -\tilde{f}_1 & \lambda - \tilde{f}_2 & -\tilde{f}_3 \\ 0 & 0 & \lambda + 1 \end{pmatrix} = (\lambda + 1)(\lambda^2 - \tilde{f}_2\lambda - \tilde{f}_1)$$

has its roots at $\lambda = -1$; that is,

$$(\lambda + 1)(\lambda^2 - \tilde{f}_2\lambda - \tilde{f}_1) = (\lambda + 1)^3 = (\lambda + 1)(\lambda^2 + 2\lambda + 1).$$

Thus, we have $\tilde{f}_1 = -1$, $\tilde{f}_2 = -2$ and $\tilde{f}_3 = *$, i.e., arbitrary. The feedback gain F in the original coordinates is then computed by

$$F = \tilde{F}S^{-1} = \begin{pmatrix} -\tilde{f}_2 & \tilde{f}_1 + \tilde{f}_2 + \tilde{f}_3 & \tilde{f}_2 + \tilde{f}_3 \end{pmatrix} = \begin{pmatrix} 2 & -3 + * & -2 + * \end{pmatrix},$$

or in other words $f_1 = 2$, and $f_3 - f_2 = 1$.

SC42015 Mid-Term Exam Solutions (2019)

Problem 1

a) The eigenvalues are at $\{1, 1, -4\}$. Since there are eigenvalues with positive real part, the system is unstable.

b) No. Using the Hautus test, it can be shown that -4 is an uncontrollable mode.

c) We can compute two true eigenvectors and one generalized eigenvector, which leads to $T^{-1} = S = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, such that $S^{-1}AS = J = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Using the transformation formulas for the input and output matrices lead to $\tilde{B} = S^{-1}B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and $\tilde{C} = CS = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

d) By equating the closed-loop characteristic equation $\det\left(\lambda I - (J + \tilde{B}\tilde{F})\right)$ with the desired characteristic polynomial $(\lambda + 4)^3$ we obtain $\tilde{F} = \begin{pmatrix} \tilde{f}_1 & -25 & -10 \end{pmatrix}$, with $\tilde{f}_1$ arbitrary. For minimal vector 2-norm we need to choose $\tilde{f}_1 = 0$.

e) $F = \tilde{F}S^{-1} = \begin{pmatrix} 0 & -10 & -35 \end{pmatrix}$.

f) Calculate the range space of $K = \begin{pmatrix} B & AB & A^2B \end{pmatrix}$ and show that x_f is not contained within (or equivalently, show that $\textbf{rank}\begin{pmatrix} K & x_f \end{pmatrix} = 3 > \textbf{rank}(K) = 2$).

g) Since the uncontrollable mode -4 is on the left half plane, the system is stabilizable.

Problem 2

a) To determine the equilibria of the system we solve $f(x) = 0$, which implies $x_2 = 0$ and $-a \sin x_1 - bx_2 = 0$. This leads to $a \sin x_1 = 0 \implies x_1 = 0$ or $x_1 = \pi$. Consequently, the only two equilibria of the system are $(x_{e1}, x_{e2}) = (0, 0)$ and $(x_{e1}, x_{e2}) = (\pi, 0)$.

b) The Jacobian matrix of f is given as:

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -a\cos x_1 & -b \end{pmatrix}$$

Let us examine the stability of the origin. Evaluating the Jacobian at the origin leads to

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}$$

Therefore, the eigenvalues of A are as follows

$$\lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 - 4a}$$

For all $a, b > 0$, both eigenvalues of the linearized system have negative real part. Thus, using Lyapunov's indirect method, for $a, b > 0$ the origin of the nonlinear system is asymptotically stable. However, when $b = 0$ then both eigenvalues are on the imaginary axis, implying that we cannot conclude anything about the stability of the equilibrium using the linearized system matrix.

Let us now examine stability of the other equilibrium $(x_{e1}, x_{e2}) = (\pi, 0)$.

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_1,x_2)=(\pi,0)} = \begin{pmatrix} 0 & 1 \\ a & -b \end{pmatrix}$$

The eigenvalues now become

$$\lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 + 4a}$$

For all $a > 0$ and $b \geq 0$, both eigenvalues are real, and one will always belong to the open right half plane (i.e., positive). Thus, the equilibrium $(x_{e1}, x_{e2}) = (\pi, 0)$ is unstable for any $a > 0$, $b \geq 0$.

c) The only case where we could not conclude stability/instability via the linearization is for the equilibrium $(x_{e1}, x_{e2}) = (0, 0)$ when $b = 0$. Let us employ Lyapunov's theorem for stability. For the given Lyapunov function candidate, $V(0) = 0$ and in the given domain $(x_1, x_2) \in (-\pi, \pi] \times \mathbb{R}$ we have that $V(x) > 0$ due to the value limits of the cosine function and the square. Finally, let us examine the derivative of $V(x)$ along the trajectories of the system for this case (i.e., $b = 0$):

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0$$

This shows that all conditions of Lyapunov's theorem are satisfied and we can conclude that the origin is stable in the case of $b = 0$. Our physical intuition suggests that since $b = 0$ corresponds to no friction, the pendulum should show a bounded oscillatory motion around the origin without asymptotic convergence. This agrees with our finding above that shows Lyapunov stability, but does not enable to conclude asymptotic stability.

Problem 3

a) We begin with computing the eigenvalues of the matrix A by solving the characteristic equation

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{pmatrix} = \lambda(\lambda + 3) + 2 = (\lambda + 1)(\lambda + 2) = 0.$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$. For the eigenvector corresponding to $\lambda_1 = -1$, we have

$$(\lambda_1 I - A)v_1 = \begin{pmatrix} -1 & -1 \\ * & * \end{pmatrix} v_1 = 0 \Longrightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Similarly, for $\lambda_2 = -2$, we have

$$(\lambda_2 I - A)v_2 = \begin{pmatrix} -2 & -1 \\ * & * \end{pmatrix} v_2 = 0 \Longrightarrow v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Hence, $A = SJS^{-1}$, where $S = (v_1, v_2)$ and $J = \mathrm{diag}(\lambda_1, \lambda_2)$.

b) First, note that for $t > \ln 2$, the state $x = x_f$ and the input $u(t) = u_3$ are both constant. Thus, the system must be in equilibrium, that is, $Ax_f + Bu_3 = 0$. Solving this equation for u_3 yields $u_3 = 2$.

For $0 \le t \le \ln 2$, we have $u(t) = u_1 + a_1 t$, where $a_1 = (u_2 - u_1)/\ln 2$. The state response of the system can be computed as follows

$$\begin{aligned}
x(t) &= e^{At}\, x(0) + \int_0^t e^{A(t-\tau)}\, B\, u(\tau)\, d\tau \\
&= (0,0)^\top + \int_0^t S e^{J(t-\tau)} S^{-1}\, B\, u(\tau)\, d\tau \\
&= \int_0^t \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{\tau-t} & 0 \\ 0 & e^{2(\tau-t)} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(\tau)\, d\tau \\
&= \int_0^t \begin{pmatrix} e^{\tau-t} - e^{2(\tau-t)} \\ -e^{\tau-t} + 2e^{2(\tau-t)} \end{pmatrix} (u_1 + a_1\tau)\, d\tau.
\end{aligned}$$

The required integrals are

$$\begin{aligned}
\int_0^t e^{\tau-t}\, d\tau &= 1 - e^{-t}, \\
\int_0^t e^{2(\tau-t)}\, d\tau &= \frac{1}{2} - \frac{1}{2}e^{-2t}, \\
\int_0^t e^{\tau-t}\, \tau\, d\tau &= -1 + t + e^{-t}, \\
\int_0^t e^{2(\tau-t)}\, \tau\, d\tau &= -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}.
\end{aligned}$$

Hence,

$$x(t) = u_1 \begin{pmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{pmatrix} + a_1 \begin{pmatrix} -\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t} \\ \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \end{pmatrix}. \tag{1}$$

Recall that the control input must steer the system to the final state x_f at $t = \ln 2$, that is,

$$x(t = \ln 2) = u_1 \begin{pmatrix} \frac{1}{8} \\ \frac{1}{4} \end{pmatrix} + a_1 \begin{pmatrix} \frac{1}{2}\ln 2 - \frac{5}{16} \\ \frac{1}{8} \end{pmatrix} = x_f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence,

$$\begin{cases} 2u_1 + a_1(8\ln 2 - 5) = 16 \\ 2u_1 + a_1 = 0 \end{cases} \Longrightarrow u_1 = \frac{-4}{4\ln 2 - 3},\ a_1 = \frac{8}{4\ln 2 - 3}.$$

Finally,

$$u_2 = u_1 + a_1 \ln 2 = u_1(1 - 2\ln 2) = \frac{8\ln 2 - 4}{4\ln 2 - 3}.$$

Remark: It is possible to exploit the structure of the problem to drastically reduce the required computations for deriving the state response $x(t)$ in (1) to the input $u(t) = u_1 + a_1 t$. Since the system is linear and also initially at rest, it suffices to compute the state response $x^{(1)}(t)$ to the step input $u^{(1)} = 1$, and the state response $x^{(2)}(t)$ to the ramp input $u^{(2)}(t) = t$. Then, we simply have $x(t) = u_1 x^{(1)}(t) + a_1 x^{(2)}(t)$. Also, notice that since the ramp input $u^{(2)}(t)$ is the integral of the step input $u^{(1)}$, the state response $x^{(2)}(t)$ is the integral of the state response $x^{(1)}(t)$. This is exactly like the relationship between the impulse and step responses you have seen in your lecture notes. Finally, note that the second state is the derivative (with respect to time) of the first state ($x_2 = \dot{x}_1$), while $y = x_1$. Hence, one only needs to compute the output response to derive x_1 and then take the derivative of the output response to derive x_2. Putting all these observations together, we have

$$x(t) = u_1 \begin{pmatrix} \int_0^t h(\tau)\, d\tau \\ h(t) \end{pmatrix} + a_1 \begin{pmatrix} \int_0^t \int_0^{\tau_2} h(\tau_1)\, d\tau_1\, d\tau_2 \\ \int_0^t h(\tau)\, d\tau \end{pmatrix},$$

where $h(t)$ is the impulse response of the system, given by

$$\begin{aligned} h(t) &= Ce^{At}B = CSe^{Jt}S^{-1}B \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-t} - e^{-2t}. \end{aligned}$$

Grading.

- b: (1) Computing the eigenvalues
- b: (1) Computing the eigenvectors
- b: (1) Forming the the matrices S and J
- c: (2) Recognizing that the system is in equilibrium and computing u_3
- c: (3) Computing the state response to the first part of the input
- c: (2) Using the final state constraint to compute u_1 and u_2

SC42015 Mid-Term Exam Solutions (2020)

Problem 1

a) Recall (e.g., slide 2-51) that the state matrices are related to the impulse response as $h(t) = Ce^{At}B + D\delta(t)$. Observing the given functions for $h(t)$ and assuming a block-diagonal A matrix, it is easy to check that the following state matrices lead to the specified impulse response (see also the similar question #3 in instruction set 3):

$$A = \begin{pmatrix} -5 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}, \quad D = 0.$$

Since

$$e^{At} = \begin{pmatrix} e^{-5t} & te^{-5t} & 0 & 0 \\ 0 & e^{-5t} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we can verify that indeed

$$Ce^{At}B + D\delta(t) = \begin{pmatrix} e^{-5t} & te^{-5t} & 1 & t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = h(t).$$

b) Although $\mathrm{Re}\{\lambda_i\} \leq 0$ for all the eigenvalues λ_i, the zero eigenvalue has algebraic multiplicity two and geometric multiplicity one, therefore the system is neither Lyapunov nor asymptotically stable.

c) The transfer function can be computed for instance either by $H(s) = C(sI-A)^{-1}B + D$, or by computing the Laplace transform of the impulse response:

$$H(s) = \mathcal{L}\{h(t)\} = \frac{1}{(s+5)^2} + \frac{1}{s+5} + \frac{2}{s^2} = \frac{s^3 + 8s^2 + 20s + 50}{s^2(s+5)^2}$$

Grading.

- a: (3) Correct/valid state matrices are provided (partial points can be given for correct A matrix).
- a: (1) The impulse response of the proposed state matrices is verified.
- b: (1) Correct arguments about Lyapunov stability.

- b: (1) Correct arguments about asymptotic stability.
- c: (1) The transfer function is computed correctly with the numerator polynomial expanded.

Problem 2

a) The matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$. It is not Hurwitz, so it is not stable. Nevertheless, using the Hautus test:

$$\begin{pmatrix} A - \lambda_1 I & B \end{pmatrix} = \begin{pmatrix} 2 & 8 & -4 \\ -1 & -4 & 1 \end{pmatrix},$$

we can determine that the test matrix is full row rank, therefore the only unstable mode $\lambda_1 = 1$ is controllable. Hence the system is stabilizable. (Note: Using the controllability matrix K shows that the system is uncontrollable, however it is still stabilizable).

b) Since matrix A has rank 2 and has two distinct eigenvalues, it has two (regular) eigenvectors and is therefore diagonalizable. For computing e^{At} we first compute a diagonalizing transformation. The eigenvectors u, v are:

$$u: \qquad Au = \lambda_1 u \Rightarrow u = \begin{pmatrix} 2 \\ -\frac{1}{2} \end{pmatrix}$$

$$v: \qquad Av = \lambda_2 v \Rightarrow v = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$$

Therefore, by defining $S := \begin{pmatrix} u & v \end{pmatrix}$ and computing the inverse (either with the direct computation for a 2×2 matrix or using the Gauss-Jordan method), we can verify that

$$S^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \Rightarrow S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, the exponential matrix e^{At} can be computed as

$$\begin{aligned} e^{At} = S\,\mathrm{diag}(e^t, e^{-t})S^{-1} &= \begin{pmatrix} 2 & -1 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2e^t - e^{-t} & 4e^t - 4e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{2}e^{-t} & -e^t + 2e^{-t} \end{pmatrix}. \end{aligned}$$

c) Yes, it is possible. Since the only uncontrollable mode is already at the location where we want it ($\lambda_2 = -1$), we can choose a feedback gain K that places the controllable mode λ_1 at -1 also, and achieves $\lambda(A - BK) = \{-1, -1\}$. Let $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$. Then,

$$A - BK = \begin{pmatrix} 3 & 8 \\ -1 & -3 \end{pmatrix} - \begin{pmatrix} -4k_1 & -4k_2 \\ k_1 & k_2 \end{pmatrix} = \begin{pmatrix} 3 + 4k_1 & 8 + 4k_2 \\ -1 - k_1 & -3 - k_2 \end{pmatrix} \qquad (1)$$

We want the characteristic polynomial of (1) to be equal to $(\lambda+1)^2 = \lambda^2 + 2\lambda + 1$. Then,

$$\begin{aligned}
\det(\lambda I - (A - BK)) &= \det \begin{pmatrix} \lambda - 3 - 4k_1 & -8 - 4k_2 \\ 1 + k_1 & \lambda + 3 + k_2 \end{pmatrix} \\
&= (\lambda - 3 - 4k_1)(\lambda + 3 + k_2) + (8 + 4k_2)(1 + k_1) \\
&= \lambda^2 + (k_2 - 4k_1)\lambda - 4k_1 - 1 + k_2 \\
&= \lambda^2 + 2\lambda + 1 \iff k_1 = 2 + 4k_1.
\end{aligned}$$

Therefore, the gain matrix with positive entries and minimum Frobenius norm that places both eigenvalues at -1 is $K = \begin{pmatrix} 0 & 2 \end{pmatrix}$.

Grading.

- a: (1) Correct conclusion about stabilizability with correct arguments.
- b: (3) The matrix exponential is computed correctly.
- c: (2) The state feedback gain is computed correctly.

Problem 3

a) The equilibrium points can be determined solving the following system

$$\begin{cases} 0 = \beta(2 - x_1) + x_1^2 x_2, \\ 0 = x_1 - x_1^2 x_2 = x_1(1 - x_1 x_2). \end{cases}$$

The second equation is solved by

$$x_1 = 0 \qquad \text{and for} \qquad x_1 x_2 = 1.$$

The solution $x_1 = 0$ does not satisfy the first equation. Substituting $x_1 x_2 = 1$ in the first equation one obtains

$$2\beta - \beta x_1 + x_1 = 0 \qquad \rightarrow \qquad x_1 = \frac{2\beta}{\beta - 1}.$$

Therefore, the only equilibrium point $x_e = \begin{pmatrix} x_{1e} & x_{2e} \end{pmatrix}^T$ of the system for a positive $\beta \neq 1$ is

$$x_{1e} = \frac{2\beta}{\beta - 1}, \qquad x_{2e} = \frac{\beta - 1}{2\beta}.$$

b) Linearizing in the vicinity of the equilibrium point one obtains

$$\dot{x} = \begin{pmatrix} -\beta + 2x_1 x_2 & x_1^2 \\ 1 - 2x_1 x_2 & -x_1^2 \end{pmatrix}\Bigg|_{(x_{1e}, x_{2e})} = \begin{pmatrix} 2 - \beta & \frac{4\beta^2}{(\beta-1)^2} \\ -1 & \frac{-4\beta^2}{(\beta-1)^2} \end{pmatrix}.$$

The characteristic polynomial of the system is

$$s^2 + \left(\beta - 2 + \frac{4\beta^2}{(\beta-1)^2}\right)s + \frac{4\beta^2}{\beta - 1} = 0,$$

from which one obtains

$$s^2 + \frac{\beta^3 + 5\beta - 2}{(\beta-1)^2}s + \frac{4\beta^2}{\beta - 1} = 0.$$

From this characteristic polynomial we can draw the following conclusions

- The eigenvalues are both nonzero (their product is nonzero since $\beta > 0$ and $\beta \neq 1$).
- For $\beta > 1$, the coefficients of this polynomial are both positive (show!), and therefore the real part of both eigenvalues is negative. This implies that the equilibrium point is locally asymptotically stable.
- For $0 < \beta < 1$, the linearized system has at lease one eigenvalue with positive real part and hence the equilibrium point is unstable.

Grading.

- a: (2) The equilibrium point is correctly determined.
- b: (1) Correct linearization around the equilibrium.
- b: (2) Correct conclusion about the stability of the equilibrium as a function of β.