LQ Optimal Control

- Stability and the Lyapunov equation
- Linear Quadratic Optimal Control
- Solution with completion of squares
- The algebraic Riccati equation
- Robustness properties
- Cheap control and asymptotic properties

Related Reading

[AM]: Chapters 4.4, 6.4 and [F]: Chapters 9.1-9.5

Lyapunov Functions for Linear Systems

We have analyzed asymptotic stability of the linear system

$$\dot{x} = Ax = f(x)$$

by a direct consideration of e^{At}. It sheds a new light on linear stability analysis and prepares for later if we use Lyapunov theory.

Since the system is linear, let us try to use a (homogenous) **quadratic** Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$. Such functions are described by

$$V(x) = x^T P x \quad \text{with a symmetric matrix} \quad P \in \mathbb{R}^{n \times n}.$$

For applying the Lyapunov theorem (Lecture 2) we need to consider

$$\partial_x V(x) f(x) = 2x^T P A x = x^T [A^T P + PA] x.$$

Remark. Although some formulas for derivatives might not be familiar to you, they can all be verified by the usual rules for scalar functions.

Recap: Some Facts from Linear Algebra

Let $Q \in \mathbb{R}^{n \times n}$ be symmetric ($Q = Q^T$). Then

- Q has n real eigenvalues, counting multiplicities.
- The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
- The eigenspaces are mutually orthogonal.
- All Jordan blocks of Q have dimension 1.
- Q is orthogonally diagonalizable, i.e. $S^{-1}QS = \Lambda$ with $S^{-1} = S^T$.

Recap: Some Facts from Linear Algebra

Let $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ be symmetric ($Q = Q^T$ and $R = R^T$).

1. Q is positive semi-definite iff either one of these conditions hold:
   - $x^T Q x \geq 0$ for all $x \in \mathbb{R}^n$
   - all its eigenvalues are non-negative
   - it can be written as $C^T C$ (with C of full row rank)

2. R is positive definite iff either one of these conditions hold:
   - $u^T R u > 0$ for all $u \in \mathbb{R}^m$ that are not zero
   - all its eigenvalues are positive
   - it can be written as $U^T U$ with a square and invertible U

3. The Euclidean norm $\|x\|$ of a vector $x \in \mathbb{R}^n$ is defined by

   $$\|x\|^2 = x^T x = x_1^2 + \cdots + x_n^2.$$

   Clearly $\|x\| \geq 0$ and equality holds iff $x = 0$.

Examples

- The matrix $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$ is not symmetric.
- The matrix $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ is positive definite. It can be written as

$$\begin{pmatrix} 1.34 & -0.45 \\ -0.45 & 0.89 \end{pmatrix}^2 \quad \text{or} \quad \begin{pmatrix} 1.41 & -0.71 \\ 0 & 0.71 \end{pmatrix}^T \begin{pmatrix} 1.41 & -0.71 \\ 0 & 0.71 \end{pmatrix}.$$

  Hence U as on the previous slide can even be chosen upper-triangular.
- The diagonal elements of positive definite matrices must be positive.
- If a positive semi-definite matrix has a zero on the diagonal, then the corresponding row and column must be zero.
- The matrix $\begin{pmatrix} 2 & -1 \\ -1 & \frac{1}{2} \end{pmatrix}$ is positive semi-definite. It equals

$$\begin{pmatrix} 1.41 \\ -0.71 \end{pmatrix} \begin{pmatrix} 1.41 & -0.71 \end{pmatrix}.$$

Lyapunov Conditions for Asymptotic Stability

The Lyapunov theorem (Lecture 2) requires to make sure that

$$x^T P x > 0 \ \text{ and } \ x^T[A^T P + PA]x < 0 \ \text{ for all } \ x \neq 0.$$

We hence arrive at the following result.

If there exists a positive definite P such that $A^T P + PA$ is negative definite then $\dot{x} = Ax$ is (globally) asymptotically stable.

This result follows from general Lyapunov theory. On the next slide we actually provide a direct proof.

In **practice** we choose and fix any negative definite Q (such as for example $Q = -I$) and solve the linear equation

$$A^T P + PA = Q$$

for P. If P turns out to be positive definite then A is Hurwitz.

Proof

For some small positive α the matrix $A^T P + PA + \alpha P$ is still negative definite. Therefore $x^T[A^T P + PA + \alpha P]x \leq 0$ for all $x \in \mathbb{R}^n$ and hence

$$x^T[A^T P + PA]x \leq -\alpha\, x^T Px. \qquad (\star)$$

For any x_0 we need to show that $x(t) = e^{At}x_0 \to 0$ for $t \to \infty$. Define

$$v(t) = x(t)^T Px(t) \geq 0.$$

We then infer with the help of ($\star$) that

$$\dot{v}(t) = \frac{d}{dt}x(t)^T Px(t) = x(t)^T[A^T P + PA]x(t) \leq -\alpha x(t)^T Px(t) = -\alpha v(t).$$

Hence $r(t) = \dot{v}(t) + \alpha v(t) \leq 0$. By the variation-of-constants formula $0 \leq v(t) = v(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-\tau)} r(\tau)\, d\tau \leq v(0)e^{-\alpha t} \to 0$ for $t \to \infty$.

Therefore $\lim_{t\to\infty} v(t) = 0$. Since P is positive definite, it can be written as $U^T U$, U invertible. Then $v(t) = x(t)^T U^T U x(t) = \|Ux(t)\|^2 \to 0$; hence $Ux(t) \to 0$ and thus $U^{-1}Ux(t) = x(t) \to 0$ for $t \to \infty$.

Lyapunov Equation

Let $A \in \mathbb{R}^{n \times n}$ be Hurwitz.

- For every symmetric matrix $Q \in \mathbb{R}^{n \times n}$ the **Lyapunov equation**

$$A^T P + PA = Q$$

  does have a unique symmetric solution $P \in \mathbb{R}^{n \times n}$.

- If Q is negative semi-definite then P is positive semi-definite.
- If Q is negative definite then P is positive definite.

The equation is well-studied also in the case that A is **not** Hurwitz. Then, for any symmetric and negative definite Q:

- either the Lyapunov equation has no solution;
- or there exists a solution but it is not unique;
- or there exists a unique solution but it is not positive definite.

Proof

Since e^{At} decays exponentially to zero for $t \to \infty$ the matrix

$$\boxed{P = -\int_0^\infty e^{A^Tt} Q e^{At}\, dt}$$

is well-defined. Moreover we have

$$A^TP + PA = -\int_0^\infty A^T \left[e^{A^Tt} Q e^{At}\right] + \left[e^{A^Tt} Q e^{At}\right] A\, dt =$$

$$= -\int_0^\infty \frac{d}{dt}\left[e^{A^Tt} Q e^{At}\right]\, dt = -\, e^{A^Tt} Q e^{At}\Big|_{t=0}^{t=\infty} = Q.$$

Hence P solves the Lyapunov equation & "P has opposite sign of Q".

If $\tilde{P}$ is another solution we infer for $\Delta = \tilde{P} - P$ that $A^T\Delta + \Delta A = 0$.
If we define $M(t) = e^{A^Tt}\Delta e^{At}$ we have $M(\infty) = 0$ and

$$\dot{M}(t) = e^{A^Tt} A^T \Delta e^{At} + e^{A^Tt} \Delta A e^{At} = e^{A^Tt}[A^T\Delta + \Delta A]e^{At} = 0.$$

Hence $M(\cdot)$ is constant; thus $\Delta = M(0) = M(\infty) = 0$; hence $P = \tilde{P}$.

Example

The command `lyap(A,R)` solves the equation $AX + XA^T + R = 0$:

```
A=[-2 3;1 1];P=lyap(A',eye(2));eig(P)=[-0.8090;0.3090]
%%
As=[-2 3;1 1]-1.8*eye(2);P=lyap(As',eye(2))
eig(P)=[0.1089;68.2607]
%%
ev=eig(A);
As=A-ev(1)*eye(2);
P=lyap(As',eye(2))

??? Error using ==> lyapslv
Solution does not exist or is not unique.
```

LQ Optimal Control

We have seen that there are many ways to stabilize the linear system

$$\dot{x} = Ax + Bu.$$

The choice of suitable feedback gains by pole-placement is not simple since it is somewhat unclear, in general, how to balance the speed of the state-response and the size of the corresponding control action.

This motivates to **quantify** the average distance of the state-trajectory from 0 and the effort involved in the control action as

$$\int_0^\infty x(t)^T Q x(t)\, dt \quad \text{and} \quad \int_0^\infty u(t)^T R u(t)\, dt$$

respectively, where Q and R are symmetric **weighting matrices** that are positive semi-definite and positive definite respectively.

The weighting matrices allow to put individual emphasis on the different components of the state- and control-trajectories.

LQ Optimal Control

Achieving fast state-convergence to zero with the least possible effort then amounts to minimizing the **cost function**

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

over all trajectories satisfying

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) = 0. \qquad \text{(S)}$$

This is the so-called **linear quadratic** (LQ) optimal control problem.

Let us stress at the outset that other cost criteria might, in practice, better reflect the desired objectives. Actually, general optimal control theory is a very rich field in itself (and developing since the 1960's).

The choice for a quadratic cost and linear systems is motivated by a beautiful mathematical problem solution and fast solution algorithms.

Choice of Weighting Matrices

Often $Q = \mathrm{diag}(q_1, \ldots, q_n)$ and $R = \mathrm{diag}(r_1, \ldots, r_m)$ are taken to be diagonal and the cost then reads as

$$\sum_{k=1}^{n} \int_0^\infty q_k x_k(t)^2 \, dt + \sum_{k=1}^{m} \int_0^\infty r_k u_k(t)^2 \, dt.$$

The scalars $q_k \geq 0$ and $r_k > 0$ allow us to balance the emphasis put on the state- and input-components:

- Large values of q_k or r_k penalize the component $x_k(t)$ or $u_k(t)$ heavier. Therefore these components are expected to be pushed to smaller values by optimal controllers.
- Small values of q_k or r_k allow for larger deviations of $x_k(t)$ from zero or for larger action of $u_k(t)$.
- With $q_k = 0$ no emphasis is put on $x_k(t)$. For technical reasons $r_k = 0$ is not allowed: **All control components have to be penalized.**

Completion of Squares

For any symmetric matrix P and any state-trajectory of (S) we have

$$\frac{d}{dt}x(t)^T P x(t) = \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) =$$

$$= (Ax(t) + Bu(t))^T P x(t) + x(t)^T P (Ax(t) + Bu(t)) =$$

$$= x(t)^T (A^T P + PA) x(t) + x(t)^T P B u(t) + u(t)^T B^T P x(t).$$

Let us analyze the last two terms, by adding the term $u(t)^T R u(t)$, and by exploiting $R = U^T U$. We infer

$$x(t)^T PBu(t) + u(t)^T B^T Px(t) + u(t)^T Ru(t) = -x(t)^T PBR^{-1}B^T Px(t) +$$

$$+ x(t)^T PBR^{-1}B^T Px(t) + x(t)^T PBu(t) + u(t)^T B^T Px(t) + u(t)^T Ru(t) =$$

$$= -x(t)^T PBR^{-1}B^T Px(t) + \|Uu(t) + U^{-T}B^T Px(t)\|^2.$$

This latter step is called **completion of the squares**. Purpose?

Completion of Squares

We now add also $x(t)^T Q x(t)$ and arrive at the following key relation:

$$
\begin{aligned}
\frac{d}{dt} x(t)^T P x(t) + x(t)^T Q x(t) + u(t)^T R u(t) = \\
= x(t)^T [A^T P + PA - PBR^{-1} B^T P + Q] x(t) + \\
+ \| U u(t) + U^{-T} B^T P x(t) \|^2 .
\end{aligned}
$$

This motivates to choose $P = P^T$ as a solution of the following so-called **algebraic Riccati equation (ARE)**

$$
A^T P + PA - PBR^{-1} B^T P + Q = 0.
$$

If that was possible we could infer

$$
\begin{aligned}
\frac{d}{dt} x(t)^T P x(t) + x(t)^T Q x(t) + u(t)^T R u(t) = \\
= \| U u(t) + U^{-T} B^T P x(t) \|^2 .
\end{aligned}
$$

Completion of Squares

If we integrate over $[0,T]$ for $T>0$ we finally arrive at

$$x(T)^T P x(T) + \int_0^T x(t)^T Q x(t) + u(t)^T R u(t)\, dt =$$
$$= x_0^T P x_0 + \underbrace{\int_0^T \|Uu(t) + U^{-T} B^T P x(t)\|^2\, dt}_{\geq 0}.$$

- For any trajectory of (S) we have $x(T) \to 0$ for $T \to \infty$ and thus
$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt \geq x_0^T P x_0.$$
The cost is **not smaller** than $x_0^T P x_0$, no matter which stabilizing control function is chosen.

- **Equality** is achieved exactly when $Uu(t) + U^{-T} B^T P x(t) = 0$ or
$$u(t) = -R^{-1} B^T P x(t) \ \text{ for all } \ t \geq 0.$$

Insights

- Any solution P of the ARE gives us a **lower bound** $x_0^T P x_0$ on the cost function for all admissible control functions.
- The lower bound **is attained** if we can choose the control function to satisfy $u(t) = -R^{-1}B^T P x(t)$. This could be assured as follows:

  1. Solve $\dot{x}(t) = [A - BR^{-1}B^T P]x(t)$ with $x(0) = x_0$.
  2. Then define the control function by $u_*(t) = -R^{-1}B^T P x(t)$.

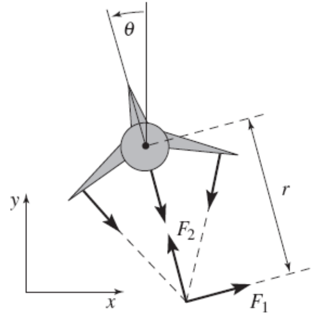
  But we need to make sure that $\lim_{t\to\infty} x(t) = 0$ which requires that

  $$A - BR^{-1}B^T P \quad \text{is Hurwitz.}$$

  If there exists a P as indicated then the constructed input $u_*(\cdot)$ is indeed a **unique optimal open-loop control function**.
- Moreover, the optimal control function can actually be implemented by a **feedback strategy** $u = -Fx$ with gain $F = R^{-1}B^T P$.

Example

(a) Harrier "jump jet"



(b) Simplified model

Consider Harrier at vertical take-off ([AM] pp.53,141,191) modeled as

$$
\begin{aligned}
m\ddot{x} &= F_1 \cos(\theta) - F_2 \sin(\theta) - c\dot{x}, \\
m\ddot{y} &= F_1 \sin(\theta) + F_2 \cos(\theta) - mg - c\dot{y}, \\
J\ddot{\theta} &= rF_1.
\end{aligned}
$$

Example

With state $z = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$ and input $u = (F_1, F_2)$ put the system into a first-order description and linearize at the equilibrium $u_e = (0, mg)$ and $z_e = (x_e, y_e, 0, 0, 0, 0)$. This leads to

$$\left(\, A \,\middle|\, B \, \right) = \left(\begin{array}{cccccc|cc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -g & -c/m & 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 & 0 & 1/m \\ 0 & 0 & 0 & 0 & 0 & 0 & r/J & 0 \end{array} \right).$$

For a scale model choose the parameters

$$m = 4; \;\; J = 0.0475; \;\; r = 0.25; \;\; g = 9.81; \;\; c = 0.05.$$

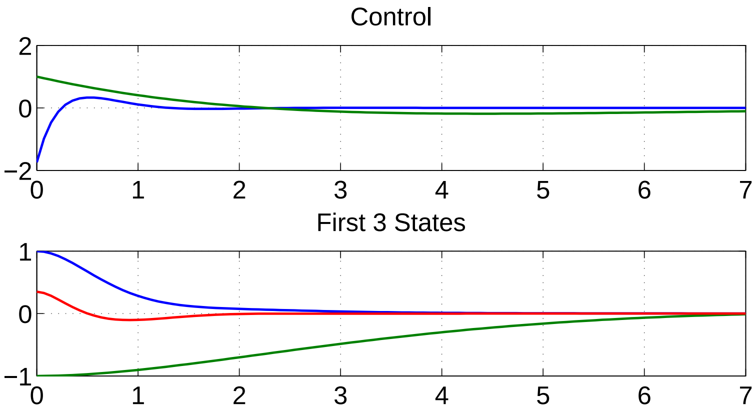
For Q and R we compute with

$$[\mathtt{F}, \mathtt{P}, \mathtt{E}] = \mathtt{lqr}(\mathtt{A}, \mathtt{B}, \mathtt{Q}, \mathtt{R})$$

the LQ-gain F, the stabilizing ARE solution P and the closed-loop eigenvalues $E = \mathrm{eig}(A - BF)$.

Example

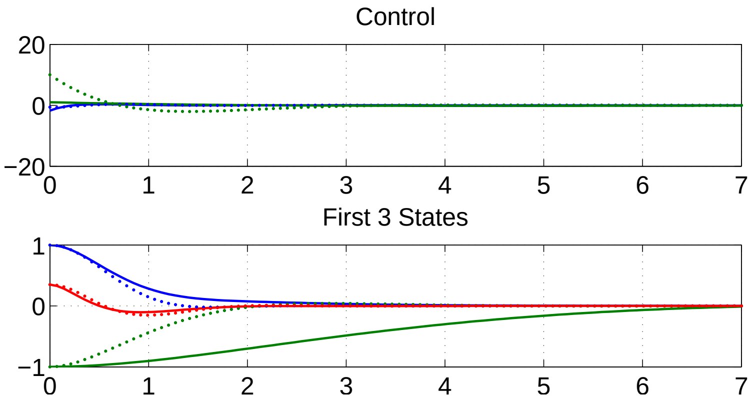
For $Q = I$, $R = I$, $x_0 = (1, -1, 0.35, 0, 0, 0)$ get closed-loop responses



The second state is very slow. Also the first should be somewhat faster. This motivates to increase the penalty (weight) on these states e.g. to $Q = \mathrm{diag}(10, 100, 1, 1, 1, 1)$.

Example

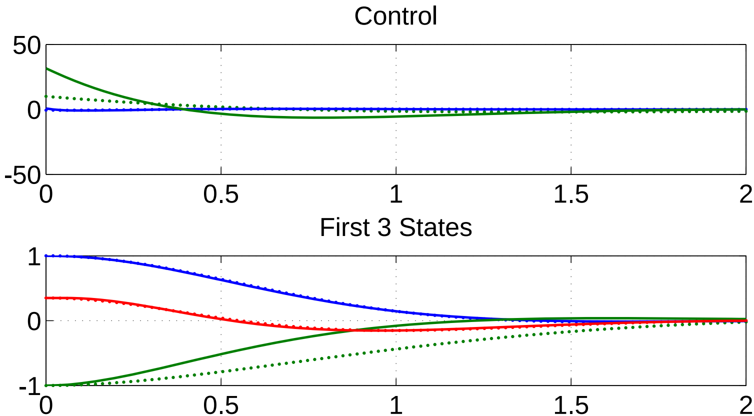
The responses are faster, at the expense of a larger control action:



Let's now allow for an even larger control action by reducing the input weight to $R = 0.1I$.

Example

This speeds up the responses further, but again at the expense of larger control actions:



By reducing $\rho > 0$ in $R = \rho I$ we put less weight on the control input. This typically comes along with high gains in the state-feedback matrix.

Riccati Theory

How does `lqr` work? We need to answer the following question:

Does there exist a solution $P = P^T$ of the algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

such that $A - BR^{-1}B^T P$ is Hurwitz?

Any such P is called a **stabilizing** solution of the ARE.

- The ARE is a **quadratic** matrix equation in the unknown symmetric matrix P. Just to get some feeling think about the case $n = m = 1$.
- Recall that Q is positive semi-definite and R is positive definite. In the sequel we will make use of $Q = C^T C$ and $R = U^T U$, U invertible.
- Clearly, a stabilizing solution can only exist if (A, B) is stabilizable. It is less obvious that (A^T, Q) cannot have uncontrollable modes on the imaginary axis. These two properties also imply existence of P.

The Hamiltonian

A key role in solving the ARE is played by the **Hamiltonian** matrix

$$H = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.$$

Indeed if P solves the ARE we can rearrange it as

$$-Q - A^T P = P[A - BR^{-1}B^T P]$$

in order to infer the following relation:

$$H \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^T P & -BR^{-1}B^T \\ -Q - A^T P & -A^T \end{pmatrix} =$$

$$= \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} A - BR^{-1}B^T P & -BR^{-1}B^T \\ 0 & -[A - BR^{-1}B^T P]^T \end{pmatrix}.$$

A solution P of the ARE allows, hence, to transform H by similarity into a block-triangular form. Many insights can be extracted from here.

The Hamiltonian

Suppose that the ARE has the stabilizing solution P. Then

$$\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}^{-1} H \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^TP & -BR^{-1}B^T \\ 0 & -[A - BR^{-1}B^TP]^T \end{pmatrix}.$$

Since H is similar to the matrix on the right they have the same eigenvalues. Since $A - BR^{-1}B^TP$ is Hurwitz, $-[A - BR^{-1}B^TP]^T$ has all its eigenvalues in the open right half-plane. Therefore

$$H \quad \text{has no eigenvalues on the imaginary axis.}$$

By the lemma below (and proved on the next slide) we conclude that (A^T, Q) has no uncontrollable modes on the imaginary axis.

Lemma. The set of eigenvalues of H on the imaginary axis is equal to the union of the set of uncontrollable modes of (A, B) and of (A^T, Q) on the imaginary axis.

Proof (Somewhat Technical)

If H has the eigenvalue λ on the imaginary axis we infer

$$\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \bar{\lambda}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \text{for some} \quad \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \neq 0.$$

With $R = U^T U$ and $Q = C^T C$ we get

$$Ae_1 - BU^{-1}[BU^{-1}]^T e_2 = \bar{\lambda} e_1 \;\; \text{and} \;\; -C^T C e_1 - A^T e_2 = \bar{\lambda} e_2. \quad (\star)$$

By left-multiplying e_2^* and e_1^* we infer

$$e_2^* A e_1 - \|e_2^* B U^{-1}\|^2 = \bar{\lambda} e_2^* e_1 \;\; \text{and} \;\; -\|Ce_1\|^2 - e_1^* A^T e_2 = \bar{\lambda} e_1^* e_2.$$

The conjugate of the latter is $-\|Ce_1\|^2 - e_2^* A e_1 = \lambda e_2^* e_1$. Adding to the first and exploiting $\bar{\lambda} + \lambda = 0$ (λ is on imaginary axis) implies $\|e_2^* B U^{-1}\|^2 + \|Ce_1\|^2 = 0$ and thus $e_2^* B = 0$ and $Ce_1 = 0$; therefore $e_1^* Q = 0$. By ($\star$) hence $e_1^*(A^T - \lambda I) = 0$ and $e_2^*(A - \lambda I) = 0$. Since either $e_1 \neq 0$ or $e_2 \neq 0$, λ is either an uncontrollable mode of (A^T, Q) or one of (A, B). The **converse** is shown by reversing arguments.

Riccati Theory: Main Result

The algebraic Riccati equation $A^TP + PA - PBR^{-1}B^TP + Q = 0$ has a stabilizing solution if and only if (A, B) is stabilizable and (A^T, Q) has no uncontrollable modes on the imaginary axis.

- Although the Riccati equation might have infinitely many solutions, the stabilizing solution is **unique**. It is also **positive semi-definite**.
- Note that the uncontrollable modes of (A^T, Q) and of (A^T, C^T) are identical in case that $Q = C^TC$. In practice one often just verifies whether any of these pairs is stabilizable.
- We already proved "only if". We provide a proof of "if" that is constructive and forms the basis for the algorithm that is used in Matlab and which is accessible by `are`.
- There is a large body of literature on the algebraic Riccati equation, in particular related to the case that Q is not positive semi-definite.

Proof

By hypothesis (A, B) and (A^T, Q) have no uncontrollable modes on the imaginary axis. Hence H has no eigenvalue on the imaginary axis.

Due to its structure the eigenvalues of H are located symmetrically with respect to the real (obvious) and imaginary (unusual) axis.

If we combine the last two facts, we conclude that H has n eigenvalues in the open left- and in the open right half-plane. This makes it possible to construct an invertible matrix T such that

$$T^{-1}HT = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}, \quad M_{11} \text{ has size } n \times n \text{ and is Hurwitz.}$$

There are many ways to do this (see remarks below). For example one can choose T such that $T^{-1}HT$ is in Jordan canonical form. Since the ordering of the blocks is free, one can actually achieve the structure by placing the Jordan blocks for all eigenvalues in the open left half-plane first. However, this procedure is numerically not reliable.

Proof (Continued)

This triangularization is motivated by the relation on slide 25. This also leads to the idea of partitioning T into four $n \times n$-blocks as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad \text{which implies} \quad H \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} M_{11}.$$

One can show that T_{11} is invertible and that $T_{21}T_{11}^{-1}$ is real symmetric (no matter how T was computed and even if T is complex).

Again motivated by slide 25 let us hence right-multiply by T_{11}^{-1} to get

$$H \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} T_{11}^{-1} = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} M_{11} T_{11}^{-1}$$

and hence

$$H \begin{pmatrix} I \\ T_{21}T_{11}^{-1} \end{pmatrix} = \begin{pmatrix} I \\ T_{21}T_{11}^{-1} \end{pmatrix} (T_{11}M_{11}T_{11}^{-1}).$$

Let's now hope that the symmetric $P = T_{21}T_{11}^{-1}$ is the desired solution.

Proof (Continued)

Yes it is! Since M_{11} is Hurwitz, the same holds for $M = T_{11} M_{11} T_{11}^{-1}$. The above equation reads as

$$\begin{pmatrix} A - BR^{-1}B^T P \\ -Q - A^T P \end{pmatrix} = H \begin{pmatrix} I \\ P \end{pmatrix} = \begin{pmatrix} I \\ P \end{pmatrix} M = \begin{pmatrix} M \\ PM \end{pmatrix}.$$

- By the first equation $A - BR^{-1}B^T P$ equals M and is hence Hurwitz.
- The second relation can hence be written as

  $$-Q - A^T P = P(A - BR^{-1}B^T P)$$

  which can clearly be rearranged into

  $$0 = A^T P + PA - PBR^{-1}B^T P + Q.$$

  This says that P satisfies the ARE.

How to Block-Triangularize the Hamiltonian?

Let us mention three possibilities to block-triangularize the Hamiltonian:

- Choose T which block-diagonalizes H.

  We have mentioned that one can transform H into the (suitably ordered) Jordan canonical form and extract the first n columns of T.

  In practice H is often diagonalizable. Then these first n columns of T can be taken equal to n linearly independent eigenvectors of H that correspond to eigenvalues of H in the open left half-plane.

- A numerically much more favorable way is to use the **ordered Schur decomposition**: Can always compute a **unitary** matrix T (property $T^{-1} = T^*$) which achieves the required block-triangular form of H.
- Modern algorithms (for large matrices) construct T with symplectic transformations on H that preserve the Hamiltonian structure.

Example

Here is some Matlab code that computes the stabilizing ARE solution:

```
% Check controllability of (A,B) and (A',Q)
[l,u]=lu(ctrb(A,B));u,[l,u]=lu(ctrb(A',Q));u

% Compute transformation based on eigen-decomposition of H
H=[A -B*inv(R)*B';-Q -A'];
[n,n]=size(A);[T,D]=eig(H);Z=[];
for j=1:2*n;
    if real(D(j,j))<0;Z=[Z T(:,j)];end;
end;
T11=Z(1:n,:);T21=Z(n+1:2*n,:);
myP=T21*T11^(-1);
```

Solution of the LQ-Problem: Main Result

Suppose that (A, B) is stabilizable and (A^T, Q) has no uncontrollable modes on the imaginary axis.

- Then one can compute the unique solution $P = P^T$ of the ARE

  $$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

  for which $A - BR^{-1}B^T P$ is Hurwitz.
- The LQ-optimal control problem has a unique solution.
- The optimal value is $x_0^T P x_0$ and the optimal control strategy can be implemented as a static state-feedback controller:

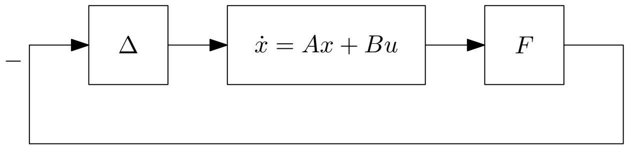
  $$u = -R^{-1}B^T Px.$$

  The closed-loop eigenvalues are equal to those eigenvalues of the Hamiltonian that are contained in the open left half-plane.

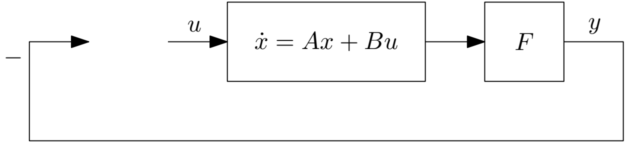
This fundamental result follows directly from the discussion on slide 17.
In Matlab the solution is made available with the command `lqr`.

Robustness Properties

A perfect implementation of a state-feedback controller leads to



with a static gain $\Delta = I$. Classical gain- and phase-margins are obtained by disconnecting Δ and analyzing the transfer matrix $u \to y$ in



This is the so-called loop-gain and equals $L(s) = F(sI - A)^{-1}B$.

Robustness Properties

Now suppose that $F = R^{-1}B^TP$ is an LQ-optimal gain. Choose any frequency ω and abbreviate $A_\omega = (i\omega I - A)^{-1}$. With the ARE we get:

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

$$(i\omega I - A)^*P + P(i\omega I - A) + PBR^{-1}B^TP = Q$$

$$PA_\omega + A_\omega^*P + A_\omega^*PBR^{-1}B^TPA_\omega = A_\omega^*QA_\omega$$

$$B^TPA_\omega B + B^TA_\omega^*PB + (B^TA_\omega^*PB)R^{-1}(B^TPA_\omega B) = B^TA_\omega^*QA_\omega B$$

$$[I + R^{-1}B^TPA_\omega B]^*R[I + R^{-1}B^TPA_\omega B] - R = B^TA_\omega^*QA_\omega B$$

$$[I + FA_\omega B]^*R[I + FA_\omega B] - R = B^TA_\omega^*QA_\omega B.$$

We hence infer for the loop-gain $L(i\omega) = FA_\omega B$ that

$[I+L(i\omega)]^*R[I+L(i\omega)]-R$ is positive semi-definite for all $\omega \in \mathbb{R}$.

Robustness Properties

This can be interpreted in terms of MIMO robustness. Instead let us consider the case that the system has 1 input only. Then $R > 0$ and $L(i\omega)$ are scalars and we infer

$$|-1 - L(i\omega)| \geq 1 \quad \text{for all} \quad \omega \in \mathbb{R}.$$

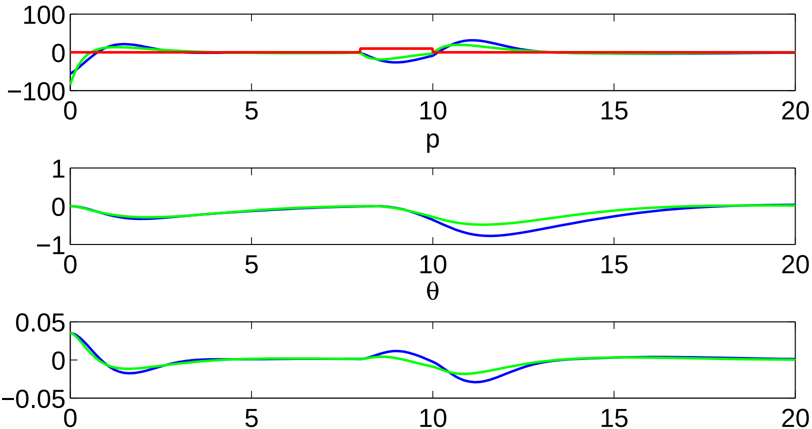
This implies that the Nyquist-curve of L is guaranteed to stay outside a circle of radius 1 around -1.

This implies impressive **generic** stability margins for LQ-controllers:

- The gain can vary in $(\frac{1}{2}, \infty)$ without endangering stability.
- The phase-margin is at least 60^o.
- The vector margin (distance of NC to -1) is at least 1.

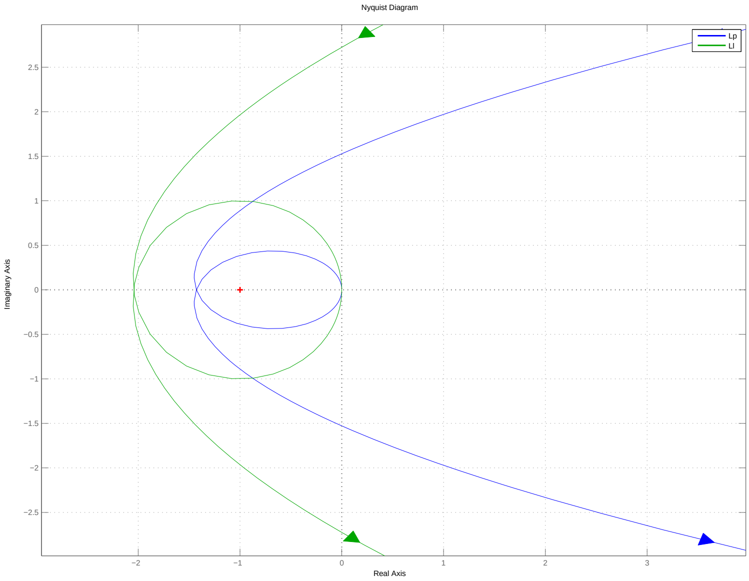
Example: Segway

With the data of [AM] p. 189 and the linearization in the upright position (zero input), we designed a static state-feedback controller by pole-placement in Lecture 3 (blue). With $R = 0.1$, $Q = \mathrm{diag}(100, 1, 1, 1)$ the LQ-responses (green) are improved:



Example: Segway

Robustness is substantially improved, as seen from the Nyquist curves:



Closed-Loop Poles

By slide 24, the closed-loop eigenvalues for the LQ-optimal gain are equal to the eigenvalues of the Hamiltonian in the open left half-plane.

With some fixed positive definite matrix R_0 suppose that we choose $R = \rho R_0$ for some scalar $\rho \in (0, \infty)$ to get

$$H = \begin{pmatrix} A & -\frac{1}{\rho} B R_0^{-1} B^T \\ -Q & -A^T \end{pmatrix}.$$

For large ρ we try to keep the control effort small. Since $-\frac{1}{\rho} B R_0^{-1} B^T$ approaches 0 for $\rho \to \infty$, the limiting closed-loop eigenvalues are equal to the stable eigenvalues of

$$H = \begin{pmatrix} A & 0 \\ -Q & -A^T \end{pmatrix}.$$

Hence they equal the stable eigenvalues of A (open-loop eigenvalues) and of $-A^T$ (open-loop eigenvalues **mirrored on imaginary axis**).

Cheap Control

For small ρ we allow for a large control effort (i.e. control is "cheap"). Let us use

$$Q = C^T C, \quad R_0^{-1} = U_0 U_0^T \text{ (U_0 invertible)}, \quad G(s) = C(sI - A)^{-1} B U_0.$$

With the Schur-determinant formula (applied twice) we get

$$\det(sI - H) = \det(sI - A)\det(sI + A^T - Q(sI - A)^{-1} B R_0^{-1} B^T/\rho)$$

$$= \det(sI - A)\det(sI + A^T)\det(I - (sI + A^T)^{-1} C^T G(s) U_0^T B^T/\rho) =$$

$$= \det(sI - A)\det(sI + A^T)\det(I - U_0^T B^T (sI + A^T)^{-1} C^T G(s)/\rho) =$$

$$= \det(sI - A)\det(sI + A^T)\det(I - \frac{1}{\rho} G(-s)^T G(s)).$$

In general the zeros of this polynomial are not easy to analyze for $\rho \to 0$. One can show that some zeros move off to ∞, and others move to the zeros of $\det(G(-s)^T G(s))$ if this polynomial does not vanish identically.

Cheap Control - Butterworth Pattern

If $G(s)$ is SISO define $d(s) = \det(sI - A)$ with zeros $p_1, ..., p_n$ and $n(s) = d(s)G(s)$ with zeros $z_1, ..., z_m$. We need to analyze the zeros of

$$d(-s)d(s) + \frac{1}{\rho}n(-s)n(s) = 0. \qquad (\star)$$

For $\rho \to 0$ the following holds (Kwakernaak, Sivan, 1972):

- $2m$ zeros of $(\star)$ approach $\pm z_1, \ldots, \pm z_m$.
- $2(n-m)$ move to ∞ asymptotically along straight lines through the origin with the following angles to the positive real axis:

$$\frac{k\pi}{n-m}, \quad k = 0, 1, \ldots, 2n-2m-1, \quad n-m \text{ odd}$$

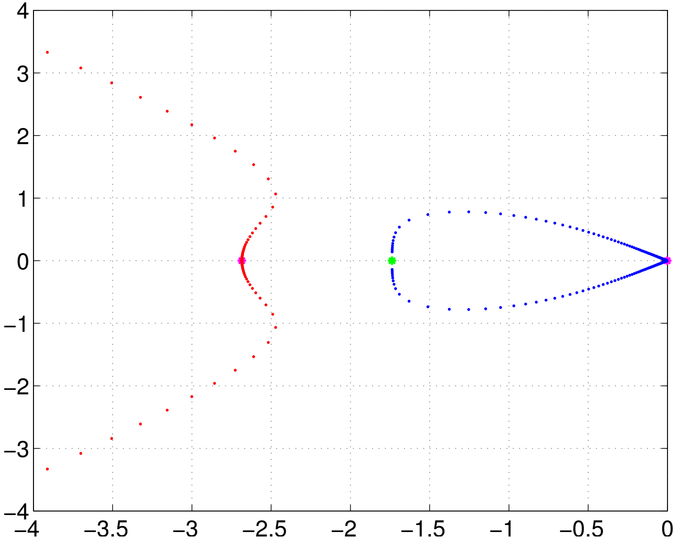
$$\frac{(k+\frac{1}{2})\pi}{n-m}, \quad k = 0, 1, \ldots, 2n-2m-1, \quad n-m \text{ even.}$$

Those in the open left half-plane are the closed-loop eigenvalues.

Example

Segway with $Q = C^T C$ and $C = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}$ as well as $R_0 = 1$.

Magenta: Zeros $d(s)$. Green: Zeros $n(s)$. Eigenvalues for $\rho \in (10^{-6}, 100)$:



Covered in Lecture 4

- Stability revisited

  Quadratic Lyapunov functions, Lyapunov equation
- LQ control

  optimal control, LQ structure, completion of squares

  algebraic Riccati equation
- Riccati theory

  Hamiltonians, stabilizing solutions
- Properties of LQ regulator

  Robustness, cheap control, Butterworth