Summary Control Theory plus exam question solving guide

Control Theory (Technische Universiteit Delft)

Lecture 0: Linear Algebra

Linear dependency

The set of vector $x=\{x_1,x_2,...,x_m\}$ is said to be **linearly dependent** if there exists real numbers $\alpha_1,\alpha_2,...,\alpha_m$ not all zero, such that:

$$\alpha_1 x_1+\alpha_2 x_2+...+\alpha_m x_m=0$$

If the only set of α_i for which the above equation holds is $\alpha_1=0,\alpha_2=0,...,\alpha_m=0$, then the set of vectors $x=\{x_1,x_2,...,x_m\}$ is said to be **linearly independent**.

Matrix spaces

The **range space** of A is defined as all possible linear combinations of all columns of A. The rank of A is defined as the dimension of the range space or, equivalently, the number of linearly independent columns in A.

A vector x is called a null vector of A if $Ax=0$. The **null space** of A consists of all its null vectors. The nullity is defined as the maximum number of linearly independent null vectors of A and is related to the rank by:

$$\text{Nullity(A)} \; = \; \text{number of columns of A} \; - \; \text{rank(A)}$$

Orthonormalization

Gram-Schmidt procedure

$$\begin{aligned}
&u_1:=v_1 && q_1:=\frac{u_1}{||u_1||}\\
&u_2:=v_2-(q_1^T v_2)q_1 && q_2:=\frac{u_2}{||u_2||}\\
&\quad\vdots\\
&u_m=e_m-\sum_{k=1}^{m-1}(q_k^T e_m)q_k && q_m:=\frac{u_m}{||u_m||}
\end{aligned}$$

Lecture 1: Dynamical Systems

Interconnection

Linear series interconnection

$$\dot{x}_1 = A_1 x_1 + B_1 u_1, \quad \dot{x}_2 = A_2 x_2 + B_2 u_2$$

$$y_1 = C_1 x_1 + D_1 u_1, \quad y_2 = C_2 x_2 + D_2 u_2, \quad \zeta = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\dot{\zeta} = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \zeta + \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} u_1, \quad y_2 = \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix} \zeta + D_2 D_1 u_1$$

Linear parallel interconnection

$$\dot{\zeta} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \zeta + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u_1, \quad y_2 = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \zeta + (D_2 + D_1) u_1$$

Lecture 2: Solutions of Linear Systems

State-Coordinate Change

An arbitrary non-singular (or invertible) matrix $T \in \mathbb{R}^{nxn}$ defines a **state-coordinate transformation** $z = Tx$.
Suppose that $TAT^{-1} = diag(\lambda_1,...,\lambda_n)$. Then the unique solution of $\dot{x} = Ax, \quad x(0) = x_0$ is given by

$$x(t) = [T^{-1} diag(e^{\lambda_1 t},...,e^{\lambda_n t} T] x_o$$

Some Facts

- If (λ, e) is a pair of eigenvalue/eigenvector then so is $(\bar{\lambda}, \bar{e})$
- Any matrix A does have at most n eigenvalues.
- For each eigenvalue one can compute at most n linearly independent eigenvectors (by computing a basis of the null space of $\lambda I - A$).
- Eigenvalues are not modified by a coordinate change.
- If $g < n$ then A **cannot** be diagonalized by a coordinate change.

Matrix Exponential

For any $A \in \mathbb{R}^{nxn}$ and $t \in \mathbb{R}$ define

$$e^{At} \quad := \quad \lim_{N \to \infty} \sum_{k=0}^{N} \frac{1}{k!} (At)^k \quad = \quad \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k$$

General A

For any $A \in \mathbb{R}^{nxn}$ the unique solution of $\dot{x} = Ax$, $x(0) = x_0$ is

$$x(t) = e^{At} x_0 \quad \text{for any } A\text{, even non-diagonizable}$$

Jordan Form

For any $A \in \mathbb{R}^{nxn}$ there exits a non-singular matrix S such that

$$S^{-1} A S = J = \begin{pmatrix} J_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_g \end{pmatrix} \text{ with } J_l = \begin{pmatrix} \lambda_l & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \lambda_l & 1 \\ 0 & \dots & 0 & \lambda_l \end{pmatrix}$$

About Jordan form and generalized eigenvectors

True / ordinary eigenvectors:

$$(\lambda I - A) e = 0$$

Generalized eigenvectors:

$$(\lambda I - A)^k e = 0$$

Furthermore, the number of generalized eigenvectors = number of ones above the main diagonal in the Jordan form.

Computation of e^{At}

Using the Jordan form of A:

$$e^{At} = Se^{Jt}S^{-1} = Sdiag(e^{J_1 t},...,e^{J_g t})S^{-1}$$

$$e^{J_l t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & ... & \frac{t^{d-2}}{(d-2)!} & \frac{t^{d-1}}{(d-1)!} \\ 0 & 1 & t & ... & \frac{t^{d-3}}{(d-3)!} & \frac{t^{d-2}}{(d-2)!} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & ... & 1 & t \\ 0 & 0 & 0 & ... & 0 & 1 \end{pmatrix} e^{\lambda_1 t}$$

Asymptotic Stability

The system $\dot{x} = Ax$ is asymptotically stable if and only if all the eigenvalues of A have a negative real part. Matrices A with this property are called **Hurwitz**. A real $2x2$ matrix A is Hurwitz iff $trace(A) < 0$ and $det(A) > 0$. A special variant of the (Hurwitz) A matrix has a specific exponential:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & * \end{pmatrix} \quad \text{where } k,m > 0$$

$$e^{At} = \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

Lyapunov Stability

All solutions of the system $\dot{x} = Ax$ are bounded for $t \to \infty$ iff all eigenvalues of A have a non-positive real part, and all Jordan block of eigenvalues with real part zero have dimension 1.

Lyapunov Functions

A continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ is said to be a **Lyapunov function** for the nonlinear system $\dot{x} = f(x)$ if:

$$[\partial_x V(x)] \cdot f(x) \leq 0 \text{ for all } x \in \mathcal{D}$$

Suppose $V(x)$ is a Lyapunov function for $\dot{x} = f(x)$ with $f(x_e) = 0$.

1. If $V(x) > V(x_e)$ for all $x \in \mathcal{D} \setminus \{x_e\}$ then x_e is stable.
2. If $V(x) > V(x_e)$ and $\partial_x V(x) f(x) < 0$ for all $x \in \mathcal{D} \setminus \{x_e\}$ then x_e is asymptotically stable.

Indirect Method of Lyapunov

Using the linearization of $\dot{x} = f(x)$ at x_e:

$$\dot{x}_\Delta = Ax_\Delta \text{ with } A = \partial_x f(x_e)$$

Suppose that the linearization is asymptotically stable (A Hurwitz). Then x_e is a locally asymptotically stable equilibrium of $\dot{x} = f(x)$.

Superposition Principle / Variation-of-Constants Formula

For a given input function $u(t)$ and the initial condition $x(0)=x_0$, the unique system response is given by:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

The output response hence is a **convolution integral**, which depends **linearly** on both x_0 and $u(\cdot)$ (**superposition principle**)

$$y(t) = Cx(t)+Du(t) = Ce^{At}x_0 + \int_0^t \left[Ce^{A(t-\tau)}B\right]u(\tau)d\tau + Du(t)$$

Step and Impulse Response

The step- and impulse-responses of the system are given by

$$\int_0^t \left[Ce^{A\tau}B\right]d\tau + D \quad \text{and} \quad Ce^{At}B + D\delta(t)$$

and can be obtained by applying m steps/impulses for each input.

Transfer Matrices

$G(s)=C(sI-A)^{-1}B+D$ is called the **transfer matrix** corresponding to the system with state-space description $\dot{x}=Ax+Bu$, $y=Cx+Du$. $G(s)$ is said to be **stable** if all its poles have negative real parts. The matrices (A,B,C,D) of a **realization** of $G(s)$ are **never unique**, as state-coordinate changes **do not change** the transfer matrix.

$$z = Tx$$
$$\dot{z} = \tilde{A}z+\tilde{B}u, \; y = \tilde{C}z+\tilde{D}u$$
$$\text{with}$$
$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{pmatrix}$$

Some notions:

- $(sI-A)^{-1}$ is the Laplace transform of e^{At};
- The transfer matrix is the Laplace transform of the impulse response $h(t)$.

Lecture 3: Controllability and Stabilizability

Controllability

A linear system is **controllable** (and by that also reachable) if the **controllability matrix** or **Kalman matrix** has full row rank. For controllable systems, one can steer any initial state $x_0 \in \mathbb{R}(K)$ at time zero to any final state $x_f \in \mathbb{R}^n$ at time $T > 0$.

$$K = \begin{pmatrix} B & AB & A^2B & ... & A^{n-1}B \end{pmatrix}$$

Also, the **reachable set** $\mathscr{R}_T$ is equal to the range space $R(K)$ of the Kalman matrix K. So if the desired final state is in the range space of K it is **reachable**.

State-Coordinate Change

The Kalman matrices K of (A,B) and $\tilde{K}$ of $(\tilde{A},\tilde{B})$ are related as $\tilde{K} = TK$. Therefore **controllability is invariant** under state-coordinate change.

Controllable Canonical Form

If $\dot{x} = Ax + Bu$ has only one input $(m = 1)$ and is controllable, there exists a state-coordinate change such that $\dot{z} = \left[TAT^{-1}\right]z + [TB]u$ is in **controllable canonical form**. To do so, first compute the characteristic polynomial of A:

$$det(A) = 0, \qquad \rightarrow \quad \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

Next using $T^{-1} = S = (S_1 ... S_n)$ such that

$$B = S\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad AS = S\begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & ... & -\alpha_n \\ 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & 1 & 0 \end{pmatrix}$$

With the first n relations we can recursively solve for the columns as

$$\begin{aligned} S_1 &= B \\ S_2 &= (A + \alpha_1 I)B \\ S_3 &= (A^2 + \alpha_1 A + \alpha_2 I)B \\ &... \\ S_n &= (A^{n-1} + \alpha_1 A^{n-2} + ... + \alpha_{n-1} I)B \\ AS_n &= -\alpha_n S_1 \end{aligned}$$

Controllability Normal Form

There exists a state coordinate change which transforms the linear system $\dot{x} = Ax + Bu$ into **controllable normal form**

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix}u$$

such that (A_{11}, B_1) is controllable.

Hautus-Test for Controllability

The pair (A,B) is controllable if and only if every left-eigenvector e of the matrix A satisfies $e*B \neq 0$. Equivalently, the matrix

$$\begin{pmatrix} A-\lambda I & B \end{pmatrix} \quad \text{has full row rank for all } \lambda \in \mathbb{C}$$

Stabilizability

The linear system is **stabilizable** if for each initial state $x_0 \in \mathbb{R}^n$ there exists a control input $u(t)$ for $t \geq 0$ such that the solution of $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$ satisfies

$$\lim_{t\to\infty} x(t) = 0$$

Hautus-Test for Stabilizability

The system $\dot{x} = Ax + Bu$ is stabilizable iff all uncontrollable modes are contained in the open left-half complex plane. Equivalently $\begin{pmatrix} A-\lambda I & B \end{pmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$ with $Re(\lambda) \geq 0$.

State-Feedback Control

A **linear state-feedback** controller with gain F is defined as

$$u = -Fx$$

The controller actually changes the system dynamics from open-loop $\dot{x} = Ax$ to closed-loop $\dot{x} = (A-BF)x$.

Stabilization by State-Feedback

The system $\dot{x} = Ax + Bu$ is stabilizable if and only if there exists some matrix F such that $\dot{x} = (A-BF)x$ is asymptotically stable ($(A-BF)$ is Hurwitz).

Lecture 4: Linear Quadratic Optimal Control

Lyapunov Conditions for Asymptotic Stability

If there exists a positive definite p such that A^TP+PA is negative definite then $\dot{x}=Ax$ is (globally) asymptotically stable. In practice we choose and fix any negative definite Q (such as for example $Q==I$) and solve the linear equation

$$A^TP+PA=Q$$

for P. If P turns out to be positive definite then A is Hurwitz.
Let $A\in\mathbb{R}^{nxn}$ be Hurwitz. For every symmetric matrix $Q\in\mathbb{R}^{nxn}$ the Lyapunov equation

$$\begin{aligned} A^TP+PA &= Q \\ P &= -\int_0^\infty e^{A^Tt}Qe^{At}dt \end{aligned}$$

does have a unique symmetric solution $P\in\mathbb{R}^{nxn}$.

LQ Optimal Control

To balance the speed of the state-response and the size of the corresponding control action we **quantify the average distance** of the state-trajectory from 0 and the effort involved in the control action as

$$\int_0^\infty x(t)^TQx(t)dt \quad \text{and} \quad \int_0^\infty u(t)^TRu(t)dt$$

respectively, where Q and R are **symmetric weighting matrices** that are positive semi-definite and positive definite respectively. Achieving fast state-convergence to zero with the least possible effort then amounts to minimizing the **cost function**

$$\int_0^\infty x(t)^TQx(t)+u(t)^TRu(t)dt$$

over all trajectories satisfying

$$\dot{x}(t)=Ax(t)+Bu(t), \quad x(0)=x_0 \quad \text{and} \quad \lim_{t\to\infty} x(t)=0 \qquad (S)$$

This is the so-called **linear quadratic** (LQ) optimal control problem.

Completion of Squares

Algebraic Riccati equation (ARE)

$$A^TP+PA-PBR^{-1}B^TP+Q=0$$

For any trajectory of (S) we have $x(T)\to 0$ for $T\to\infty$ and thus

$$\int_0^\infty x(t)^TQx(t)+u(t)^TRu(t)dt\geq x_0^TPx_0$$

The cost is **not smaller** than $x_0^TPx_0$, no matter which stabilizing control function is chosen. We can attain the lower bound with $u(t)=-R^{-1}B^TPx(t)$ for all $t\geq 0$. So the optimal control function can actually be implemented by a feedback strategy $u=Fx$ with gain $F=R^{-1}B^TP$.

The Hamiltonian

A key role in solving the ARE is played by the **Hamiltonian** matrix

$$H = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \in \mathbb{R}^{2nx2n}$$

The set of eigenvalues of H on the imaginary axis is equal to the union of the set of uncontrollable modes of (A,B) and of (A^T,Q) on the imaginary axis.

Solution of the LQ-Problem: Main Result

Suppose that (A,B) is stabilizable and (A^T,Q) has no uncontrollable modes on the imaginary axis.

- Then one can compute the unique solution $P = P^T$ of the ARE

  $$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

  for which $A - BR^{-1}B^TP$ is Hurwitz.
- The LQ-optimal control problem has a unique solution.
- The optimal value is $x_0^TPx_0$ and the optimal control strategy can be implemented as a static state-feedback controller:

  $$u = -R^{-1}B^TPx$$

  The closed-loop eigenvalues are equal to those eigenvalues of the Hamiltonian that are contained in the open left half-plane.
- LQ-controller has impressive **generic** stability margins and is robust:
  - The gain can vary in $(\frac{1}{2}, \infty)$ without endangering stability.
  - The phase-margin is at least 60°.
  - The vector margin (distance of NC to -1) is at least 1.

Lecture 5: Observability, Separation Principle, Realization

State Reconstruction

Assuming availability of all states for control is unrealistic. So reconstructing the states is vital in controller design. To assess whether it is possible to uniquely determine the state $x(t)$ for $t \in [0,T]$ based on the knowledge of the input $u(t)$ and output $y(t)$ for $t \in [0,T]$. The linear system $\dot{x} = Ax + Bu, \quad y = Cx + Du$ is **observable** if and only if its observability matrix W **has full column rank**. The null space of W is called **unobservable subspace** of (A,C).

$$W = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Hautus Test for Observability states that the pair (A,C) is observable if and only if

$$\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} \quad \text{has full column rank for all } \lambda \in \mathbb{C}$$

Equivalently, there exists no eigenvector $e \neq 0$ of A with $Ce = 0$. Any $\lambda \in \mathbb{C}$ for which $\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}$ does not have full column rank is called an **unobservable mode** of (A,C).

Observability Normal Form

There exists a state-coordinate change $z = Tx$ (T invertible) that transforms $\dot{x} = Ax + Bu, \quad y = Cx + Du$ into

$$\begin{aligned} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u = \tilde{A}z + \tilde{B}u, \\ y &= \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Du = \tilde{C}z + Du \end{aligned}$$

such that (A_{11},C) is observable.

Observable Canonical Form

If $\dot{x} = Ax + Bu, \quad y = Cx + Du$ has one output (y scalar) and is observable, there exists a coordinate change $z = Tx$ (T invertible) that transforms it into

$$\begin{aligned} \dot{z} &= \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & 0 & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} u = \tilde{A}z + \tilde{B}u \\ y &= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix} z + du = \tilde{C}z + du \end{aligned}$$

Observer gain

An **observer** for the linear system $\dot{x}=Ax+Bu, \quad y=Cx+Du$ is the dynamical system

$$\dot{\hat{x}}=A\hat{x}+Bu+L(y-\hat{y}), \quad \hat{y}=C\hat{x}+Du$$

that is specified by choosing an **observer gain** $L\in\mathbb{R}^{nxk}$. The **error dynamics** is described by $\dot{\tilde{x}}=(A-LC)\tilde{x}$. For observable (A,C) the eigenvalues of $A-LC$ can be placed arbitrarily in $\mathbb{C}$ if they are located symmetrically w.r.t. the real axis.

Detectability

The system $\dot{x}=Ax+Bu, \quad y=Cx+Du$ or the pair (A,C) is said to be **detectable** if there exists some matrix L (of compatible dimensions) such that $A-LC$ is Hurwitz. There is also a **Hautus Test**: (A,C) is detectable iff all its unobservable modes are in the open left half-plane. Equivalently:

$$\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} \quad \text{has full column rank for all } \lambda\in\mathbb{C} \quad \text{with } Re(\lambda)\geq 0.$$

Separation Principle

Combining state feedback and asymptotic reconstruction of the states of the system leads to the **observer-based output-feedback** controller

$$\dot{\hat{x}}=A\hat{x}+Bu+L(y-\hat{y}), \quad \hat{y}=C\hat{x}+Du, \quad u=-F\hat{x}$$

If we interconnect the constructed controller with $\dot{x}=Ax+Bu, \quad y=Cx+Du$ we arrive at the **closed-loop** system description

$$\begin{aligned} \dot{x} &= Ax-BF\hat{x} \\ \dot{\hat{x}} &= (A-LC-BF)\hat{x}+LCx \end{aligned}$$

which is **asymptotically stable**.

Realizations

Realizations of Transfer Functions (SISO)

If $g(s)$ is a proper transfer function, we can represent it as

$$g(s)=\frac{\beta_1 s^{n-1}+\ldots+\beta_{n-1}s+\beta_n}{s^n+\alpha_1 s^{n-1}+\ldots+\alpha_{n-1}s+\alpha_n}+d$$

It is a matter of direct verification that one can then either choose

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)=\left(\begin{array}{cccccc|c} -\alpha_1 & -\alpha_2 & -\alpha_3 & \ldots & -\alpha_n & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ \hline \beta_1 & \beta_2 & \ldots & \beta_{n-1} & \beta_n & d \end{array}\right) \quad \textbf{controllable canonical realization}$$

or

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccccc|c} -\alpha_1 & 1 & 0 & ... & 0 & \beta_1 \\ -\alpha_2 & 0 & 1 & ... & 0 & \beta_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} & 0 & ... & 0 & 1 & \beta_{n-1} \\ -\alpha_n & 0 & 0 & ... & 0 & \beta_n \\ \hline 1 & 0 & 0 & ... & 0 & d \end{array}\right) \quad \textbf{observable canonical realization}$$

Realization consequences of parallel and series interconnection

If $G_1(s)$, $G_2(s)$ have realization (A_1,B_1,C_1,D_1), (A_2,B_2,C_2,D_2) then $G_1(s)G_2(s)$ and $G_1(s)+G_2(s)$ have the realizations

$$\left(\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1+D_2 \end{array}\right)$$

For stacking there are the following realizations

$$\begin{pmatrix} G_1(s) \\ G_2(s) \end{pmatrix} \leftarrow \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{array}\right), \qquad \begin{pmatrix} G_1(s) & G_2(s) \end{pmatrix} \leftarrow \left(\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_2 & D_1 & D_2 \end{array}\right)$$

Minimal Realizations

Starting from an arbitrary realization $G(s)=C(sI-A)^{-1}B+D$ one can systematically construct a new realization

$$G(s)=C_r(sI-A_r)^{-1}B_r+D_r \text{ with } dim(A_r)\leq dim(A)$$

such that (A_r,B_r) is controllable and (A_r,C_r) is observable.
A realization $G(s) = C(sI-A)^{-1}B+D$ is minimal iff (A,B) is controllable and (A,C) is observable. If $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ is another minimal realization of $G(s)$, the realizations are related by a state-coordinate change.

Lecture 6: Tracking and Disturbance Rejection

The Tracking Problem

The goal is to design an asymptotically stabilizing controller for which z asymptotically tracks all **constant** reference signals r:

$$\lim_{t\to\infty}[r-z(t)]=0$$

In order to be able to stabilize the system it is of course required that (A,B) is stabilizable and (A,C) is detectable.

Tracking by Full-Information Feedback

Let us start with the assumption that the signals x and r are available for control. A linear static full-information controller is described by

$$u=-Fx+Gr$$

If applying this controller to the linear system (and neglecting y) we arrive at

$$\dot{x}=(A-BF)x+BGr,\quad z=(\tilde{C}-\tilde{D}F)x+\tilde{D}Gr$$

Since the system should be stabilized, we choose F such that $A-BF$ is Hurwitz. Then the steady-state response of the controlled system is

$$z=\left[\tilde{D}-(\tilde{C}-\tilde{D}F)(A-BF)^{-1}B\right]Gr$$

For asymptotic tracking we would like to guarantee that $z=r$ for all possible reference inputs r. This requires to take G with

$$\left[\tilde{D}-(\tilde{C}-\tilde{D}F)(A-BF)^{-1}B\right]G=I$$

Tracking by Output-Feedback

If x is not measurable, we can choose L such that $A-LC$ is Hurwitz and design the observer

$$\dot{\hat{x}}=A\hat{x}+Bu+L(y-\hat{y}),\quad \hat{y}=C\hat{x}+Du$$

in order to asymptotically reconstruct the state. Then the separation principle motivates to control the system - with the gains F and G designed for full-information feedback - as

$$u=-F\hat{x}+Gr$$

The constructed output-feedback controller (based on the measured signals r and y) stabilizes the system and achieves tracking.

Solvability Condition

To condition required to prove existence of a tracking and disturbance rejecting controller is to find a solution to the equation

$$\begin{pmatrix}A & B\\ \tilde{C} & \tilde{D}\end{pmatrix}\begin{pmatrix}\Pi\\ \Gamma\end{pmatrix}+\begin{pmatrix}B_d\\ \tilde{D}_d\end{pmatrix}=0$$

Output Feedback Control

Steps to design a regulator:

- (A,B) is stabilizable and $\left(\begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C & D_d \end{pmatrix}\right)$ is detectable;
- the regulator equation $\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} + \begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix} = 0$ is solvable.

If the answers are yes then choose

- F, L such that $A - BF$, $\begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix} - L\begin{pmatrix} C & D_d \end{pmatrix}$ are Hurwitz;
- $G = \Gamma + F\Pi$ where Pi, $Gamma$ satisfy the regulator equation.

Then the following controller solves the regulation problem:

$$\begin{aligned} \begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{pmatrix} &= \begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + L(y - \hat{y}), \\ u &= \begin{pmatrix} -F & G \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} C & D_d \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + Du \end{aligned}$$

Extension of Design Procedure

In our development we can replace $\dot{d} = 0$ all throughout by $\dot{d} = Sd$ with S having all its **modes in the closed right half-plane**. This allows to design stabilizing controllers that achieve regulation for all signals

$$e^{St}s_0 \quad \text{with arbitrary } s_0 \in \mathbb{R}^{dim(S)}$$

Just like previously we can design a regulator as follows

- (A,B) is stabilizable and $\left(\begin{pmatrix} A & B_d \\ 0 & S \end{pmatrix}, \begin{pmatrix} C & D_d \end{pmatrix}\right)$ is detectable;
- the regulator equation $\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} - \begin{pmatrix} \Pi \\ 0 \end{pmatrix} S + \begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix} = 0$ is solvable.

If the answers are yes then choose

- F, L such that $A - BF$, $\begin{pmatrix} A & B_d \\ 0 & S \end{pmatrix} - L\begin{pmatrix} C & D_d \end{pmatrix}$ are Hurwitz;
- $G = \Gamma + F\Pi$ where Pi, $Gamma$ satisfy the regulator equation.

Then the following controller is a regulator for all d with $\dot{d} = Sd$:

$$\begin{aligned} \begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{pmatrix} &= \begin{pmatrix} A & B_d \\ 0 & S \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + L(y - \hat{y}), \\ u &= \begin{pmatrix} -F & G \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} C & D_d \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + Du \end{aligned}$$

Exam Question Solving

Controllability, observability

Verify that (A,B) is controllable

$$K = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

Is the given system detectable?

Check the observability, if it is not observable check the detectability using the Hautus-Test.

Is the given system stabilizable?

No uncontrollable modes in the closed right-half plane, or just verify that (A,B) is controllable.

Hautus-Test for controllability

$$eA = \lambda e, \quad eB = 0$$

Solve the above equation for e, if e is all zeros the pair (A,B) is controllable. If some λ satisfies the equation with $e \neq 0$ that λ is an uncontrollable mode. If $\lambda < 0$ it is still stabilizable.

Lyapunov Stability

Show that $V(x)$ is indeed a Lyapunov function

- $V(x) > 0 \quad \forall x \neq 0$
- $\dot{V}(x) < 0 \quad \forall x \neq 0$

Determine if the energy function qualifies as a Lyapunov function of the system

Check if the following equation holds

$$\frac{\partial V(x)}{\partial x} \cdot f(x) \leq 0 \quad \forall x$$

Establish where the nonlinear system is stable around its different equilibrium points.

First criterion:

- $V(x) > V(x_e) \quad \forall x \rightarrow$ stability requirement.

Second criterion:

- $\frac{\partial V(x)}{\partial x} \cdot f(x) < 0 \quad \rightarrow$ asymptotic stability;
- $\frac{\partial V(x)}{\partial x} \cdot f(x) = 0 \quad \forall x \neq 0 \rightarrow$ only stability can be concluded;
- $\frac{\partial V(x)}{\partial x} \cdot f(x) \leq 0 \quad \forall x \rightarrow$ only stability can be concluded;
- $\frac{\partial V(x)}{\partial x} \cdot f(x) \nleq 0 \quad \rightarrow$ unstable.

Indirect method of Lyapunov

- Obtain $f = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$;
- Calculate $A = \frac{\partial f}{\partial x}$, fill in the equilibrium point;
- Check if A is Hurwitz, if so the equilibrium is (locally) asymptotically stable

LQ-control

Show that there exists a stabilizing solution to this LQ-problem

- (A,B) stabilizable (or show controllability)
- (A^T,Q) has no uncontrollable modes on the imaginary axis (or show controllability)

Check that the Ricatti equation that needs to be solved has a stabilizing solution.

- (A,B) is stabilizable (or controllable);
- (A,Q) is detectable (no unobservable modes ≥ 0) can also check for observability. Furthermore Q can be extracted from the cost function: $x(t)^T Q x(t)$.

Find the stabilizing solution of the Ricatti equation and the optimal state-feedback gain F that solves the LQ-problem, by using the Hamiltonian and its eigenvectors.

- Calculate the eigenvalue of the Hamiltonian, the negative ones are the eigenvalues λ_n needed;
- Get an eigenvector e_n for each eigenvalue;
- Write down $H\begin{pmatrix} e_1 & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$;
- $H\begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} M_{11}$;
- $P = T_{21} T_{11}^{-1}$;
- $F = R^{-1} B^T P$.

Determine the eigenvalues of the optimal closed-loop system without solving the Ricatti equation.

Just compute the eigenvalues of the Hamiltonian matrix.

Compute the closed-loop eigenvalues of the LQ-optimal system (eigenvectors of the Hamiltonian are known/given)

- $H = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}$;
- Solve $He_1 = e_1 \lambda$ for λ;

Compute the closed-loop eigenvalues of the LQ-optimal system (P known/given)

$$det(sI - A - (-BR^{-1}B^T P)) = 0$$

Determine the optimal state-feedback law that minimizes the given cost function

- Solve the ARE for P
- Implement it: $u = -Fx \rightarrow F = R^{-1}B^T P$

If R is replaced by ρR and $\rho \to \infty$, compute the asymptotic values of the closed-loop eigenvalues.

With some fixed positive definite matrix R_0 suppose that we choose $R = \rho R_0$ for some scalar $\rho \in (0,\infty)$ to get

$$H = \begin{pmatrix} A & -\frac{1}{\rho} B R_0^{-1} B^T \\ -Q & -A^T \end{pmatrix}$$

For large ρ we try to keep the control effort small. Since $-\frac{1}{\rho} B R_0^{-1} B^T$ approaches 0 for $\rho \to \infty$, the limiting closed-loop eigenvalues are equal to the stable eigenvalues of

$$H = \begin{pmatrix} A & 0 \\ -Q & -A^T \end{pmatrix}$$

Hence they equal the stable eigenvalues of A (open-loop eigenvalues) and of $-A^T$ (open-loop eigenvalues **mirrored on imaginary axis**).

For which initial conditions is the optimal cost equal to c.

Solve $\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}^T P \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = c$.

Feedback Control

Verify whether the system can be stabilized through an (given) feedback, when only output (given) is available from measurements.

Check the eigenvalues of A, if A is Hurwitz it's already stabilizable and detectable without feedback control. If A is not Hurwitz use the Hautus-test for stabilizability and detectability.

Does there exist a gain F such that $(A-BF,C)$ is not observable any more?

- Check if $CB=0$, if that is the case then $W=\begin{pmatrix} C \\ C(A-BF) \end{pmatrix}=\begin{pmatrix} C \\ CA \end{pmatrix}$;
- If $CB \neq 0$ compute the observability matrix W.

Determine a state-feedback gain F such that the eigenvalues of $(A-BF)$ are located at $\{a,b\}$

- Get $(A-BF)$ with $F=[f_1 \; f_2]$;
- Get characteristic polynomial of $(A-BF)$;
- Get characteristic polynomial of desired eigenvalues $(\lambda-a)(\lambda-b)$;
- Equate them.

Is it possible to reject d from e by a full information stabilizing controller? If yes design such a controller

- First make sure $(A-BF)$ is Hurwitz (if A is Hurwitz, you can just choose $F=0$);
- Check if the regulator equation $\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix}+\begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix}=0$ is solvable

If yes to both, then $G=\Gamma+F\Pi$ and we can use the controller below.

Is it possible to design an output feedback controller that asymptotically rejects the input *constant* disturbance? ($\dot{d}=0d$)

- (A,B) is stabilizable and $\left(\begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix},\begin{pmatrix} C & D_d \end{pmatrix}\right)$ is detectable;
- The regular equation $\begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix}+\begin{pmatrix} B_d \\ \tilde{D}_d \end{pmatrix}=0$ is solvable.

If the answers are yes then choose

- F, L, such that $(A-BF)$, $\begin{pmatrix} A & B_d \\ 0 & 0 \end{pmatrix}-L\begin{pmatrix} C & D_d \end{pmatrix}$ are Hurwitz;
- $G=\Gamma+F\Pi$ where Π, Γ satisfy the regulator equation.

Then the following controller is a regulator for d with $\dot{d}=0d$:

$$\begin{aligned}\begin{pmatrix}\dot{\hat{x}}\\ \dot{\hat{d}}\end{pmatrix} &= \begin{pmatrix}A & B_d\\ 0 & 0\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{d}\end{pmatrix}+\begin{pmatrix}B\\ 0\end{pmatrix}u+L(y-\hat{y}),\\ u &= \begin{pmatrix}-F & G\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{d}\end{pmatrix},\quad \hat{y}=\begin{pmatrix}C & D_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{d}\end{pmatrix}+Du.\end{aligned}$$

Is it possible to design an output feedback controller that asymptotically rejects the input *sinusoidal* disturbance? ($\dot{d}=Sd$)

- (A,B) is stabilizable and $\left(\begin{pmatrix}A & B_d\\ 0 & S\end{pmatrix},\begin{pmatrix}C & D_d\end{pmatrix}\right)$ is detectable;
- The regular equation $\begin{pmatrix}A & B\\ \tilde{C} & \tilde{D}\end{pmatrix}\begin{pmatrix}\Pi\\ \Gamma\end{pmatrix}-\begin{pmatrix}\Pi\\ 0\end{pmatrix}S+\begin{pmatrix}B_d\\ \tilde{D}_d\end{pmatrix}=0$ is solvable.

If the answers are yes then choose

- F, L, such that $(A-BF)$, $\begin{pmatrix}A & B_d\\ 0 & S\end{pmatrix}-L\begin{pmatrix}C & D_d\end{pmatrix}$ are Hurwitz;
- $G=\Gamma+F\Pi$ where Π, Γ satisfy the regulator equation.

Then the following controller is a regulator for d with $\dot{d}=Sd$:

$$\begin{aligned}\begin{pmatrix}\dot{\hat{x}}\\ \dot{\hat{d}}\end{pmatrix} &= \begin{pmatrix}A & B_d\\ 0 & S\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{d}\end{pmatrix}+\begin{pmatrix}B\\ 0\end{pmatrix}u+L(y-\hat{y}),\\ u &= \begin{pmatrix}-F & G\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{d}\end{pmatrix},\quad \hat{y}=\begin{pmatrix}C & D_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{d}\end{pmatrix}+Du.\end{aligned}$$

Is asymptotic (i.e., rejection of disturbance) robustly achieved for the full information controller even if the parameters describing the system change slightly (e.g., A_{21} with an ϵ perturbation)?

- Add ϵ to the system description $\dot{x}=\underbrace{\begin{pmatrix}A_{11} & A_{12}\\ A_{21}+\epsilon & A_{22}\end{pmatrix}}_{A}x+\begin{pmatrix}B\end{pmatrix}u+\begin{pmatrix}B_d\end{pmatrix},\quad y=\begin{pmatrix}C\end{pmatrix}x,\quad e=\begin{pmatrix}\tilde{C}\end{pmatrix}x;$

- Steady-state regulation error value:

$$\begin{aligned}\dot{x} &= (A-BF)x+BGd+B_d d,\quad \text{steady-state so } \dot{x}=0\\ x &= (A-BF)^{-1}(-BG-B_d)d,\\ \\ e &= (\tilde{C}-\tilde{D}F)x+\tilde{D}Gd+\tilde{D}_d d\\ e &= (\tilde{C}-\tilde{D}F)(A-BF)^{-1}(-BG-B_d)d+\tilde{D}Gd+\tilde{D}_d d\end{aligned}$$

What is the model for the considered class of disturbances?

The model they want is $\dot{d}$. If d is a scalar, then of course $\dot{d}=0d$. If d is a sinusoidal signal of frequency ω

$$\dot{d} = Sd \text{ with } S=\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \text{ since } e^{St} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

If $d(t)=e^t(\alpha\cos(2t)+\beta\sin(2t)$ then choose $d=\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ then $\dot{d}=Sd=\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.
If the disturbance is a ramp then $S=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Realizations

Is this realization minimal?

- Check the controllability of (A,B);
- Check the observability of (A,C);

If the realization is controllable and observable it is minimal.

Minimal realizations

$M\frac{1}{s-p} \rightarrow \left(\begin{array}{c|c} pI & I \\ \hline M & 0 \end{array}\right)$

- Try to collect terms;
- Split off $\lim_{s\rightarrow\infty}$ into a separate D matrix;
- Try to collect terms again and split the transfer function in (B·transfer function·C) elements;
- Using the rules for parallel and series interconnection and stacking compute the realization;
- Check if it's minimal.

Can one find a state-coordinate change which transforms the given system into *system of the question*

- Check if both systems are minimal;
- Check if they have the same transfer function

If both checks are positive the systems are related by a state-coordinate change.

Point-to-point Control

$m\ddot{x}(t)+kx(t)=u(t), \quad k,m>0$ **where u(t) is the input force.**

Is it possible to find an input which will drive both the deflection $x(t)$ and the velocity $\dot{x}(t)$ to zero in finite time from arbitrary initial conditions?

- Write into state-space form;
- Verify that the Kalman matrix is non-singular.

Consider $m=k=1$, the initial condition $x(0)=(x_{01},x_{02}=\left(-\frac{\sqrt{2}(\pi^2-8)}{16},0\right)$, and the finite time $T=\frac{\pi}{4}$. Determine a control input signal $u(t)$ defined over $t\in[0,T]$ that drives the specified initial state $x(0)$ to zero in the specified finite time T.

- $u(t) = B^T e^{A^T(T-t)}\alpha$;
- $x(T)=e^{AT}x(0)+\int_0^T e^{A(T-t)}B\underbrace{B^T e^{A^T(T-t)}\alpha}_{u(t)}dt = e^{AT}x(0)+W_T\alpha$;

  Where $W_T = \int_0^T e^{A(T-t)}BB^T e^{A^T(T-t)}dt = \int_0^T e^{At}BB^T e^{A^T t}dt$ is the controllability Gramian of (A,B) at time T. Since we want to drive the states from the given $x(0)$ to zero in time T, we are aiming for $x(T)=0$. From the above equation, this implies

  $$W_T\alpha = -e^{AT}x(0)$$

- The matrix exponential of A has the form

  $$e^{At}=\begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}, \quad \text{with } \omega=\sqrt{\frac{k}{m}}$$

  Fill in the final time and parameters in the matrix exponential

- Compute the Gramian and it's inverse (exists since the system is controllable)
- Use the Gramian to solve $\alpha = -W_T^{-1}e^{AT}x(0)$;
- Construct the input $u(t) = B^T e^{A^T(T-t)}\alpha$.