Optimization: Quadratic Programming

Quadratic programming

Quadratic programming (QP):

TYPE 1:

$$\min_x f(x) = \min_x \frac{1}{2} x^T H x + c^T x$$

$$A x \leqslant b$$

$$x \geqslant 0$$

TYPE 2: Standard form

$$\min_x f(x) = \min_x \frac{1}{2} x^T H x + c^T x$$

$$A x = b$$

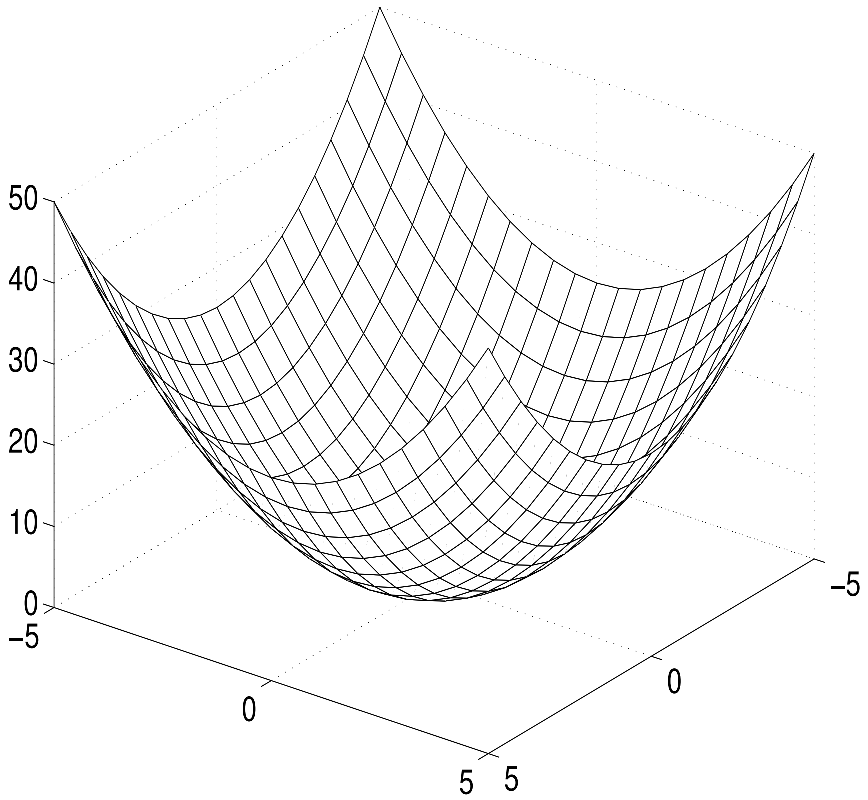
$$x \geqslant 0$$

TYPE 1 $\implies$ TYPE 2 (see later)

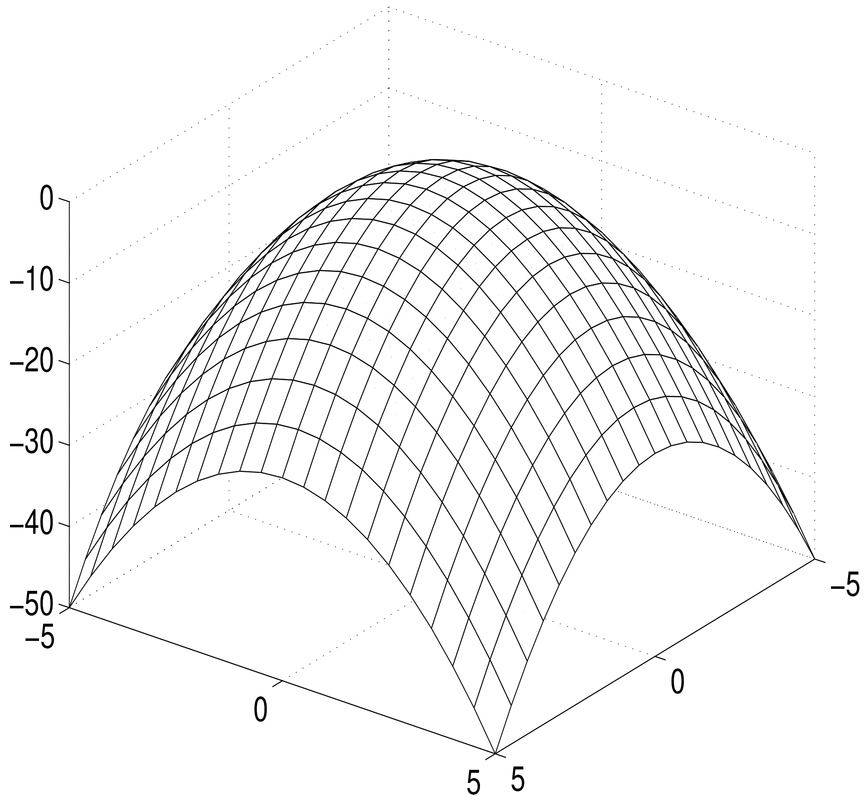
Unconstrained QP: Analytic solution: $x = -H^{-1} c$

Plot of $x^T H x$

$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, positive definite



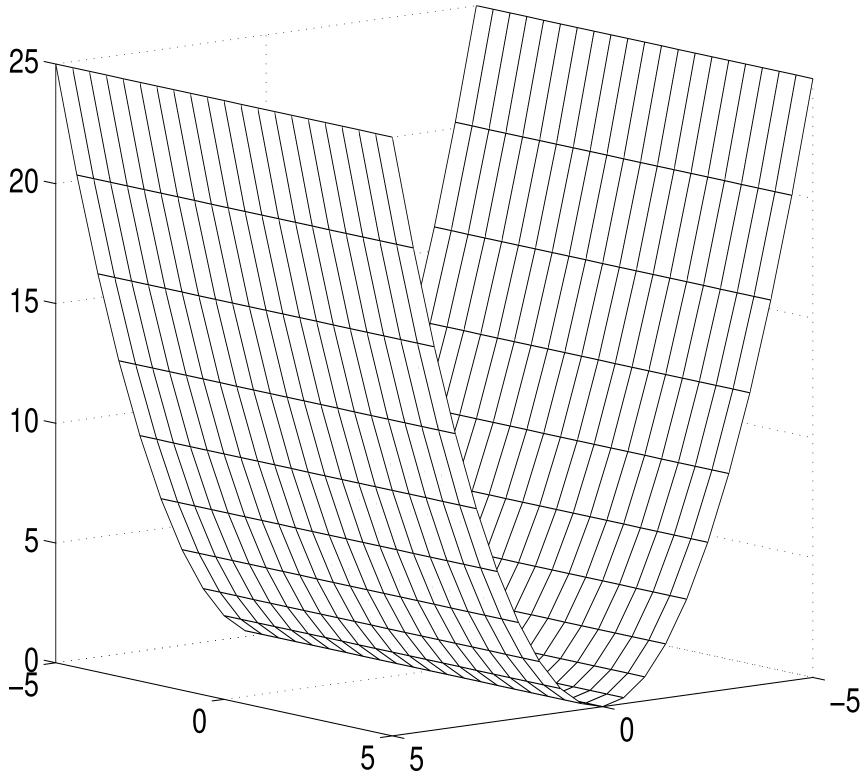
$H = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, negative definite

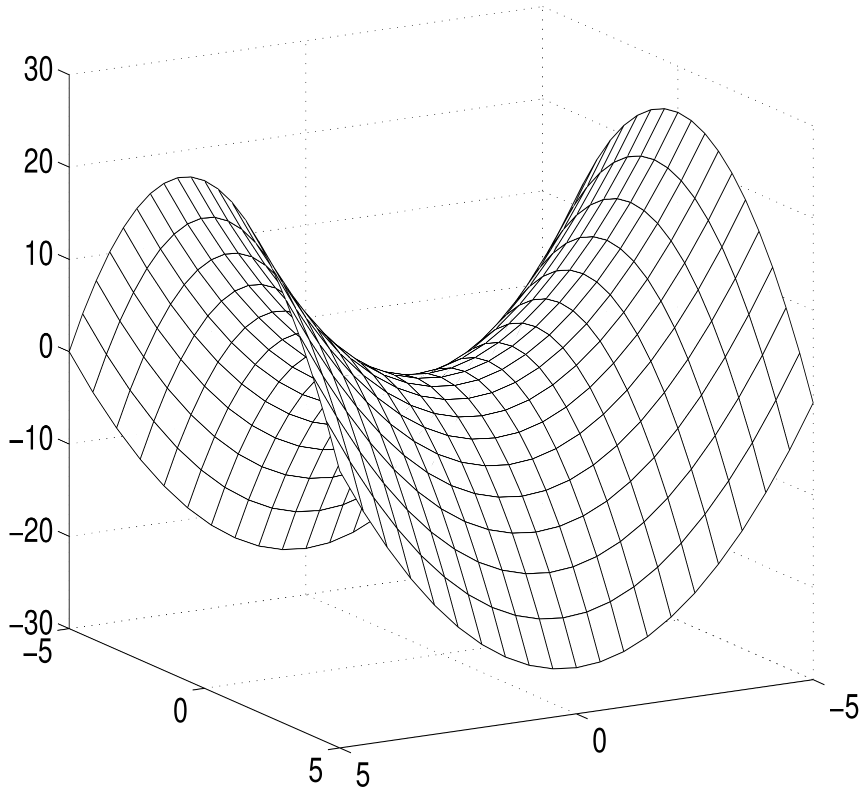


Plot of $x^T H x$

$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, positive semi-definite

$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, indefinite





Type 1 $\to$ Type 2 (= standard form)

- Type 1 : $Ax \leqslant b, \quad x \geqslant 0$

$$\Rightarrow \quad Ax + Is = b\,, \quad x, s \geqslant 0$$

$$\Rightarrow \quad [\,A \quad I\,] \begin{bmatrix} x \\ s \end{bmatrix} = b\,, \quad \begin{bmatrix} x \\ s \end{bmatrix} \geqslant 0 \qquad \text{Type 2!}$$

- Type 2 : $Ax = b, \quad x \geqslant 0$

- Type 3 : $Ax \leqslant b, \quad x \in \mathbb{R}^n$

Define $x = x^+ - x^-$ with $x^+, x^- \geqslant 0$

$$\Rightarrow \quad Ax^+ - Ax^- + Is = b\,, \quad x^+, x^-, s \geqslant 0$$

$$\Rightarrow \quad [\,A \quad -A \quad I\,] \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b\,, \quad \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \geqslant 0 \qquad \text{Type 2!}$$

<u>In general:</u> for constrained problem:

$$\min_x f(x) \quad , \quad h(x) = 0 \quad , \quad g(x) \leqslant 0$$

Karush-Kuhn-Tucker conditions:

$$\nabla f(x) + \nabla g(x)\,\mu + \nabla h(x)\,\lambda = 0$$

$$h(x) = 0 \quad , \quad g(x) \leqslant 0$$

$$\mu^T g(x) = 0 \quad , \quad \mu \geqslant 0$$

QP-problem:

$$f(x) = \frac{1}{2} x^T H x + c^T x$$

$$h(x) = A\,x \, - \, b \quad , \quad g(x) = -x$$

Karush-Kuhn-Tucker conditions:

$$A\,x = b \quad , \quad x \geqslant 0$$

$$H\,x + A^T \lambda - \mu = -c$$

$$\mu^T x = 0 \quad , \quad \mu \geqslant 0$$

Modified simplex method

$$
\begin{aligned}
A\,x + u_1 &= b \\
H\,x + A^T\,\lambda - \mu + u_2 &= -c \\
x\,,\ \mu &\geqslant 0 \\
\mu^T\,x &= 0
\end{aligned}
$$

Extended linear programming problem:

$$\min_{u_1,u_2} \sum_i (u_1)_i + \sum_j (u_2)_j$$

$$\begin{bmatrix} A & 0 & 0 & I & 0 \\ H & A^T & -I & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ \mu \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b \\ -c \end{bmatrix}, \quad x\,,\ \mu\,,\ u_1\,,\ u_2 \geqslant 0$$

and so:

$$\min_{x_0} c_0^T\,x_0 \quad , \quad A_0\,x_0 = b_0 \quad , \quad x_0 \geqslant 0$$

but with additional nonlinear constraint $x^T\,\mu = 0$

Modified simplex method

$$
\begin{aligned}
& x^T \mu = 0 && x, \mu \geqslant 0 \\
\Rightarrow \quad & \sum_i x_i \, \mu_i = 0 && x_i, \mu_i \geqslant 0 \\
\Rightarrow \quad & x_i \, \mu_i = 0 && \text{for all } i \\
\Rightarrow \quad & x_i = 0 \quad \text{or} \quad \mu_i = 0 && \text{for all } i
\end{aligned}
$$

$\rightarrow$ extra feasibility constraint for basic solutions
$\rightarrow$ modified simplex method (Wolfe, Lemke)

Finite number of steps!!!

Example

ARX (Auto Regressive eXogenous input) model:

$$y(n+1) + ay(n) = bu(n) + e(n)$$

e: zero-mean white noise

Given: $y(n),\ u(n),\ n = 1, 2, \ldots, N$

$\Rightarrow\ a,\ b$?

$$y(n+1) - \begin{bmatrix} -y(n) & +u(n) \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \hat{e}(n)$$

$$Y \quad - \quad \Phi\ x \quad = \quad E$$

Best estimate for a and b $\rightarrow$ minimize energy of $\hat{e}$

Example (continued)

$$Y - \Phi x = E$$

$$\begin{aligned}
\min_x \sum_{n=1}^{N} \hat{e}^2(n) &= \min_x \; E^T E \\
&= \min_x \,(Y - \Phi x)^T (Y - \Phi x) \\
&= \min_x \; Y^T Y + x^T \Phi^T \Phi x - x^T \Phi^T Y - Y^T \Phi x \\
&= \min_x \; d + \frac{1}{2} x^T H x + c^T x
\end{aligned}$$

$\rightarrow$ quadratic objective function

Stable model $\rightarrow -0.99 \leqslant \hat{a} \leqslant 0.99$
$\rightarrow$ linear constraint

$\Rightarrow$ Quadratic Programming problem

Summary

- Quadratic programming: Standard form

$$\begin{aligned} \min_x \ & \frac{1}{2}x^T Hx + c^T x \\ \text{s.t.} \ & Ax = b \\ & x \geqslant 0 \end{aligned}$$

- Alternative standard forms & conversion
- Modified simplex method
  $\rightarrow$ finds exact solution in finite number of steps