Optimization: Linear Programming

Linear programming — Introductory example

Brewery problem:

- Selling up to 100 boxes of beer per day
- Operation up to 14 hours a day
- 1 hour for 10 boxes of draft beer
- 2 hours for 10 boxes of dark beer
- 1 box of draft beer yields 20 dollars
- 1 box of dark beer yields 30 dollars

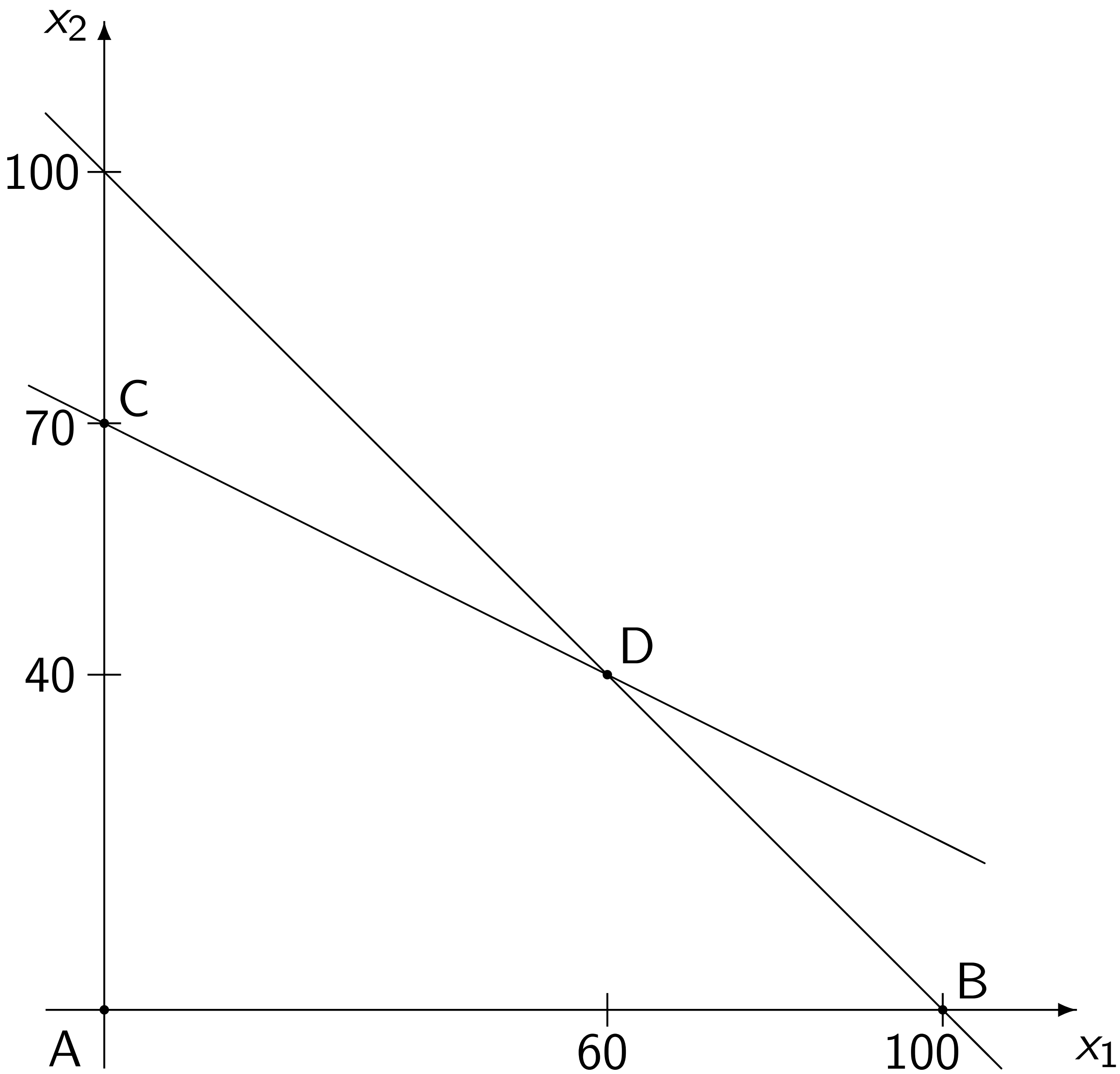
Objective: Maximize profit

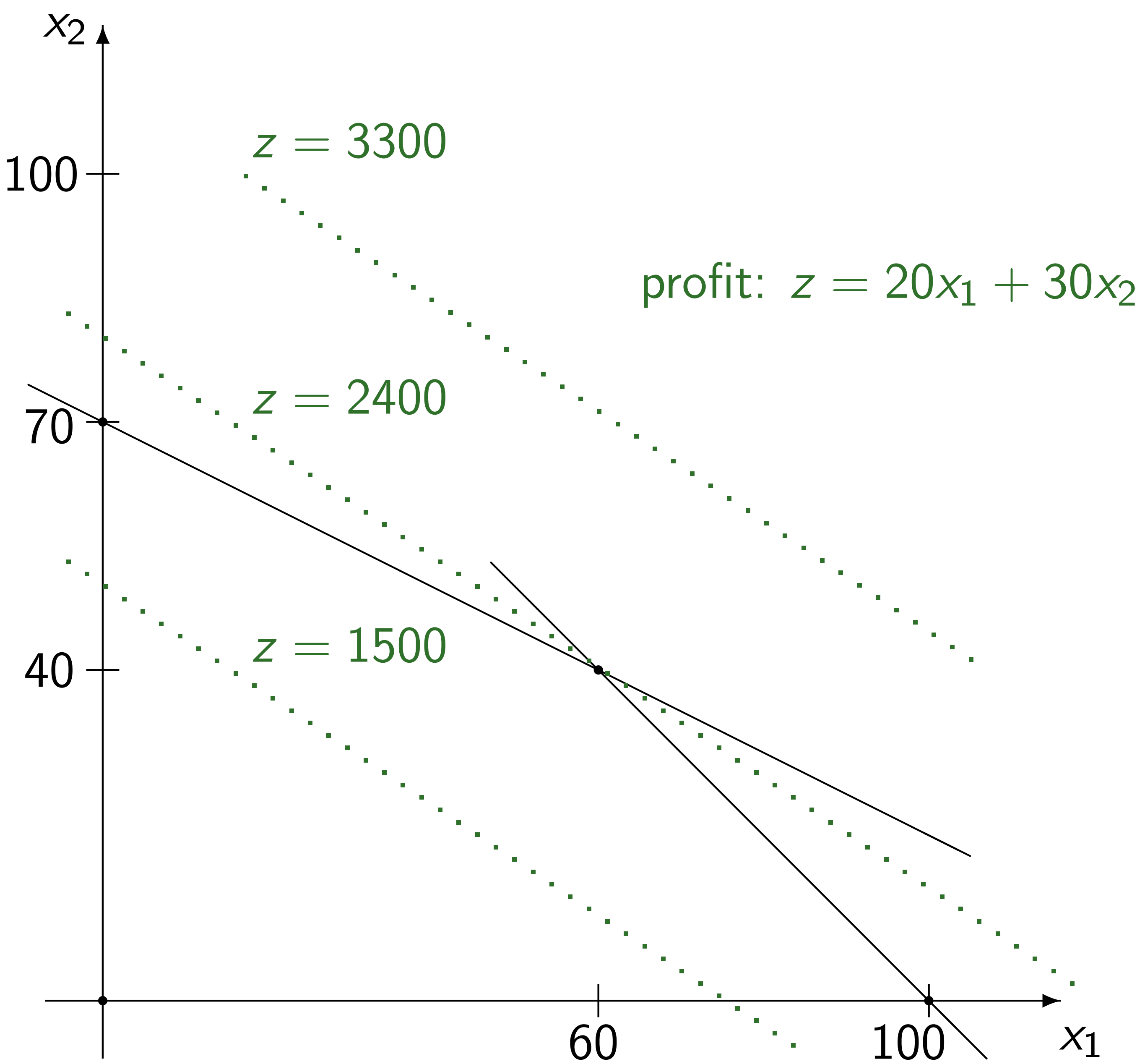
- x_1 = boxes of draft beer
- x_2 = boxes of dark beer

$$\max_{x_1, x_2} \; 20\,x_1 + 30\,x_2$$

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leqslant & 100 \\ 0.1\,x_1 & + & 0.2\,x_2 & \leqslant & 14 \\ & & x_1,\; x_2 & \geqslant & 0 \end{array}$$

$$\begin{array}{lll} x_1 = 0 & x_2 = 0 & \text{profit} = 0 \\ x_1 = 100 & x_2 = 0 & \text{profit} = 2000 \\ x_1 = 0 & x_2 = 70 & \text{profit} = 2100 \\ x_1 = 60 & x_2 = 40 & \text{profit} = 2400 \end{array}$$


x_2
100
C
70
D
40
B
A
60
100
x_1


x_2
$z = 3300$
100
profit: $z = 20x_1 + 30x_2$
$z = 2400$
70
$z = 1500$
40
60
100
x_1

General linear programming problem

Minimize the objective function

$$f(x_1, x_2, \ldots, x_n) = c_1\, x_1 + c_2\, x_2 + \cdots + c_n\, x_n$$

with respect to

$$\begin{array}{cccc} a_{11}\, x_1 & + \cdots + & a_{1n} x_n & = & b_1 \\ a_{21}\, x_1 & + \cdots + & a_{2n} x_n & = & b_2 \\ \vdots & & & & \vdots \\ a_{m1}\, x_1 & + \cdots + & a_{mn} x_n & = & b_m \end{array}$$

$$x_i \geqslant 0 \quad \text{for} \quad i = 1, \ldots, n$$

In matrix notation, minimize the objective function

$$f(x) = c^T\, x$$

with respect to

$$A\, x \;=\; b$$

$$x \geqslant 0$$

Rewriting into standard form

- Standard form for linear programming

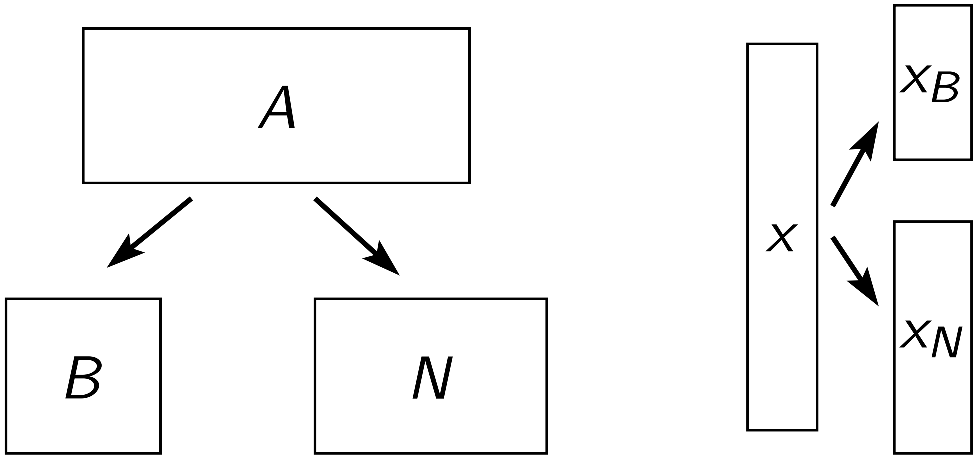
$$\min\ c^T x \qquad \text{s.t. } Ax = b,\ x \geqslant 0$$

- Rewriting other forms into standard form:
  - max $c^T x \to -\min(-c^T)x$
  - $Ax \leqslant b \to$ introduce dummy variables $s \geqslant 0$ with $Ax + Is = b$
  - $x \in \mathbb{R}^n \to$ split x into positive and negative parts: $x = x^+ - x^-$ with $x^+, x^- \geqslant 0$

Simplex method: Introduction

- $\min_x c^T x \quad \text{subject to} \quad Ax = b$
  $\qquad\qquad\qquad\qquad\qquad\quad x \geqslant 0$

- Split columns of A into 2 groups
  $\rightarrow$ B and N with B square and non-singular



- $Bx_B + Nx_N = b \quad \Rightarrow \quad x_B = B^{-1}(b - Nx_N)$

- $$c^T x = c_B^T x_B + c_N^T x_N = \underbrace{c_B^T B^{-1} b}_{z_0} + \underbrace{(c_N^T - c_B^T B^{-1} N)}_{p^T} x_N$$
  $$= z_0 + p^T x_N$$

Basic solution

- $$\min_x c^T x$$
  $$\begin{aligned} \text{subject to} \quad & Ax = b \\ & x \geqslant 0 \end{aligned}$$
- Split $A \rightarrow B, N$ with B square and non-singular
- $B\,x_B + N\,x_N = b \quad \Rightarrow \quad x_B = B^{-1}(b - Nx_N)$
- $c^T x = z_0 + p^T x_N$
- *Basic solution*: $x_N = 0$, $x_B = B^{-1}b$

  Feasible if $x_B \geqslant 0$

  Corresponding cost: z_0

Basic solutions (continued)

- It can be shown that
  - each vertex of feasible set corresponds to a basic solution
  - optimum of linear programming problem can always be reached in vertex

- By constructing basic solutions we can find solution of linear programming problem

- Number of possible partitionings of A into B and N is finite
  $\rightarrow$ solution of linear programming problem is found in a *finite number of steps*

Basic solutions for the brewery problem

- $$B = \begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  $$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix} \qquad x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  Feasible basic solution: $(x_1, x_2) = (60, 40) \rightarrow D$

- $$B = \begin{bmatrix} 1 & 1 \\ 0.1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix}$$

  $$x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 140 \\ -40 \end{bmatrix} \qquad x_N = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  Not feasible

Basic solutions for the brewery problem

- $B = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix}$ $\quad N = \begin{bmatrix} 1 & 1 \\ 0.2 & 0 \end{bmatrix}$

  $x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 4 \end{bmatrix}$ $\quad x_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  Feasible basic solution: $(x_1, x_2) = (100, 0) \rightarrow B$

- $B = \begin{bmatrix} 1 & 1 \\ 0.2 & 0 \end{bmatrix}$ $\quad N = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix}$

  $x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 70 \\ 30 \end{bmatrix}$ $\quad x_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  Feasible basic solution: $(x_1, x_2) = (0, 70) \rightarrow C$

Basic solutions for the brewery problem

- $B = \begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 1 \\ 0.1 & 0 \end{bmatrix}$

  $x_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 100 \\ -6 \end{bmatrix} x_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  Not feasible

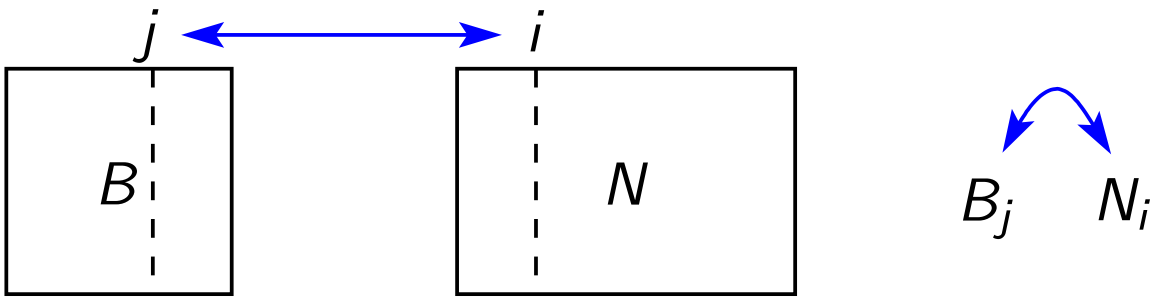
- $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix}$

  $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 14 \end{bmatrix} x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  Feasible basic solution: $(x_1, x_2) = (0, 0) \rightarrow A$

Simplex method

- Switch jth column of B with ith column of N to create new basic solution:



- Column i: largest decrease in objective function

$$z = z_0 + p^T x_N$$
$$\Rightarrow \quad i = \arg\min \{\, p_i \mid p_i < 0 \,\}$$

- Column j: largest feasible step ...

Simplex method (continued)

- Switch jth column of B with ith column of N
- Column i: largest decrease in f: $i = \arg\min\{p_i | p_i < 0\}$
- Column j: largest feasible step

  Current basic solution: $(x_N)_i = 0$, $(x_B)_j \geqslant 0$

  New basic solution: $(x_N)_i > 0$, $(x_B)_j = 0$

$$\left.\begin{aligned} B\,x_B &= b \\ B\,y &= N_i \end{aligned}\right\} \Rightarrow B \underbrace{(x_B - \varepsilon y)}_{\text{New } (x_B)_j} + \underbrace{\varepsilon}_{\text{New } (x_N)_i} N_i = b$$

  New basic solution: $(x_N)_i = \varepsilon > 0$

  $(x_B - \varepsilon y)_j = 0$

$$\Rightarrow \quad j = \arg\min\left\{ \frac{(x_B)_j}{y_j} \,\middle|\, y_j > 0 \right\}$$

- Stop if $p \geqslant 0$

Simplex method applied to brewery problem

1. $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad N = \begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix}$

$$x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 14 \end{bmatrix} \quad x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Feasible basic solution: $(x_1, x_2) = (0, 0)$

$$p^T = \begin{bmatrix} -20 & -30 \end{bmatrix}$$

$$By = N_2 = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$$

$$\frac{(x_B)_1}{y_1} = \frac{100}{1} = 100, \quad \frac{(x_B)_2}{y_2} = \frac{14}{0.2} = 70$$

$\Rightarrow$ interchange B_2 and N_2

Simplex method applied to brewery problem

2. $B = \begin{bmatrix} 1 & 1 \\ 0 & 0.2 \end{bmatrix}$ $\qquad N = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix}$

$$x_B = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 70 \end{bmatrix} \qquad x_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Feasible basic solution: $(x_1, x_2) = (0, 70)$

$$p^T = \begin{bmatrix} -5 & 150 \end{bmatrix}$$

$$By = N_1 = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$\frac{(x_B)_1}{y_1} = \frac{30}{0.5} = 60, \qquad \frac{(x_B)_2}{y_2} = \frac{70}{0.5} = 140$$

$\Rightarrow$ interchange B_1 and N_1

Simplex method applied to brewery problem

3. $B = \begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix} \quad x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Feasible basic solution: $(x_1, x_2) = (60, 40)$

$$p^T = \begin{bmatrix} 10 & 100 \end{bmatrix}$$

$$p \geqslant 0 \;\Rightarrow\; \color{red}{STOP}$$

Optimal solution: $(x_1, x_2) = (60, 40)$

Summary

- Linear programming: Standard form

$$\begin{aligned} \min_x \; & c^T x \\ \text{s.t.} \; & Ax = b \\ & x \geqslant 0 \end{aligned}$$

- Graphical solution
- Basic solutions
- Simplex method
  $\rightarrow$ finds exact solution in finite number of steps

Test: Classification of optimization problems

- Is the next problem a linear programming (LP) problem or can it be recast as an LP problem?

$$\begin{aligned} \min_{x \in \mathbb{R}^4} \ & x_1 - x_2 + 5x_4 \\ \text{s.t.} \ & |x_1 + x_2 - 2x_3 + x_4 + 9| \leqslant 2 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{aligned}$$

- Is the next problem an LP problem or can it be recast as an LP problem?

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \ & |x_1| + |x_2| + |x_3| \\ \text{s.t.} \ & x_1 + x_2 + x_3 \geqslant 3 \\ & x_1 - 2x_2 + 4x_3 \geqslant 1 \end{aligned}$$