Optimization: Multi-objective optimization

Multi-objective optimization

In practice: vector of objectives that must be traded off in some way:

$$\begin{aligned}\min_{x\in\mathbb{R}^n} &\ F(x)\\ \text{s.t.}\ \ g(x) &\leqslant 0\\ h(x) &= 0\end{aligned}$$

with F a vector-valued function

- Pareto optimality
- Algorithms ($\rightarrow$ scalar objective function)
  - weighted-sum strategy, ε-constraint method, goal attainment

Pareto optimality

$$\begin{aligned}\min_{x\in\mathbb{R}^n} \; & F(x) \\ \text{s.t.} \; & g(x) \leqslant 0 \\ & h(x) = 0\end{aligned}$$

F is vector

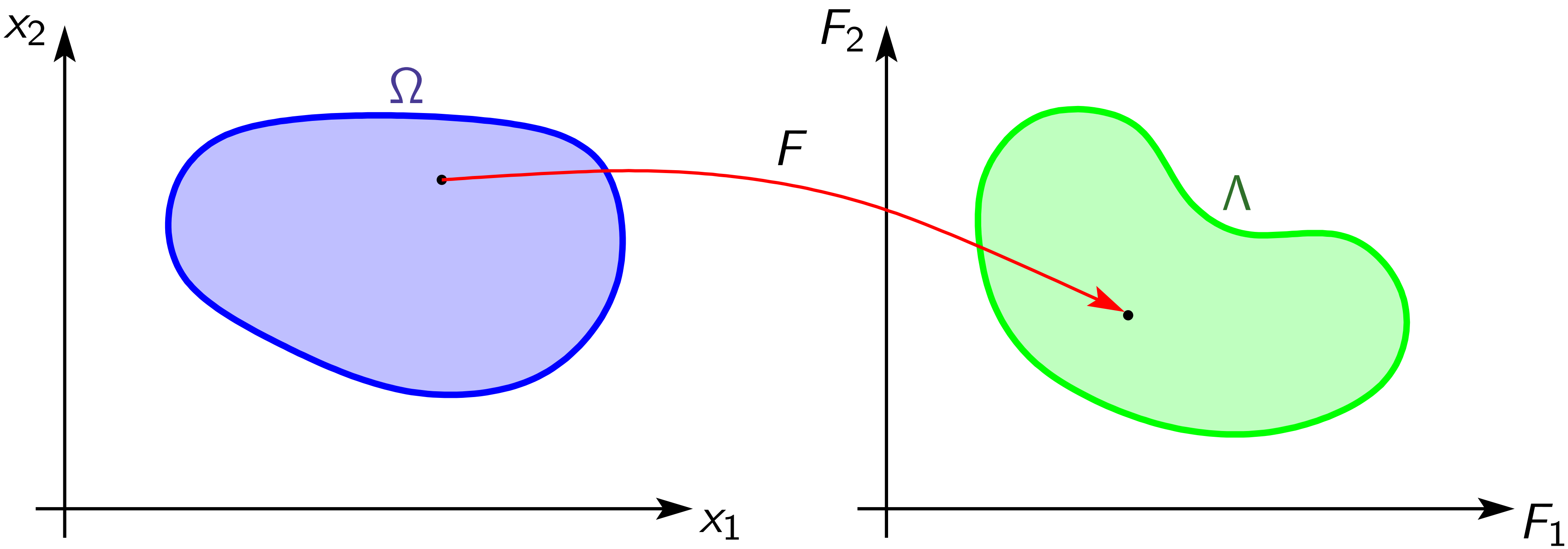
$\rightarrow$ if components of F are competing, no unique solution!

$\rightarrow$ non-inferior solution (=Pareto optimal point)

x^* is *Pareto optimal* if there does not exist another feasible point $\tilde{x}$ such that

$$F(\tilde{x}) \leqslant F(x^*) \quad \text{and} \quad F_i(\tilde{x}) < F_i(x^*) \quad \text{for some } i$$

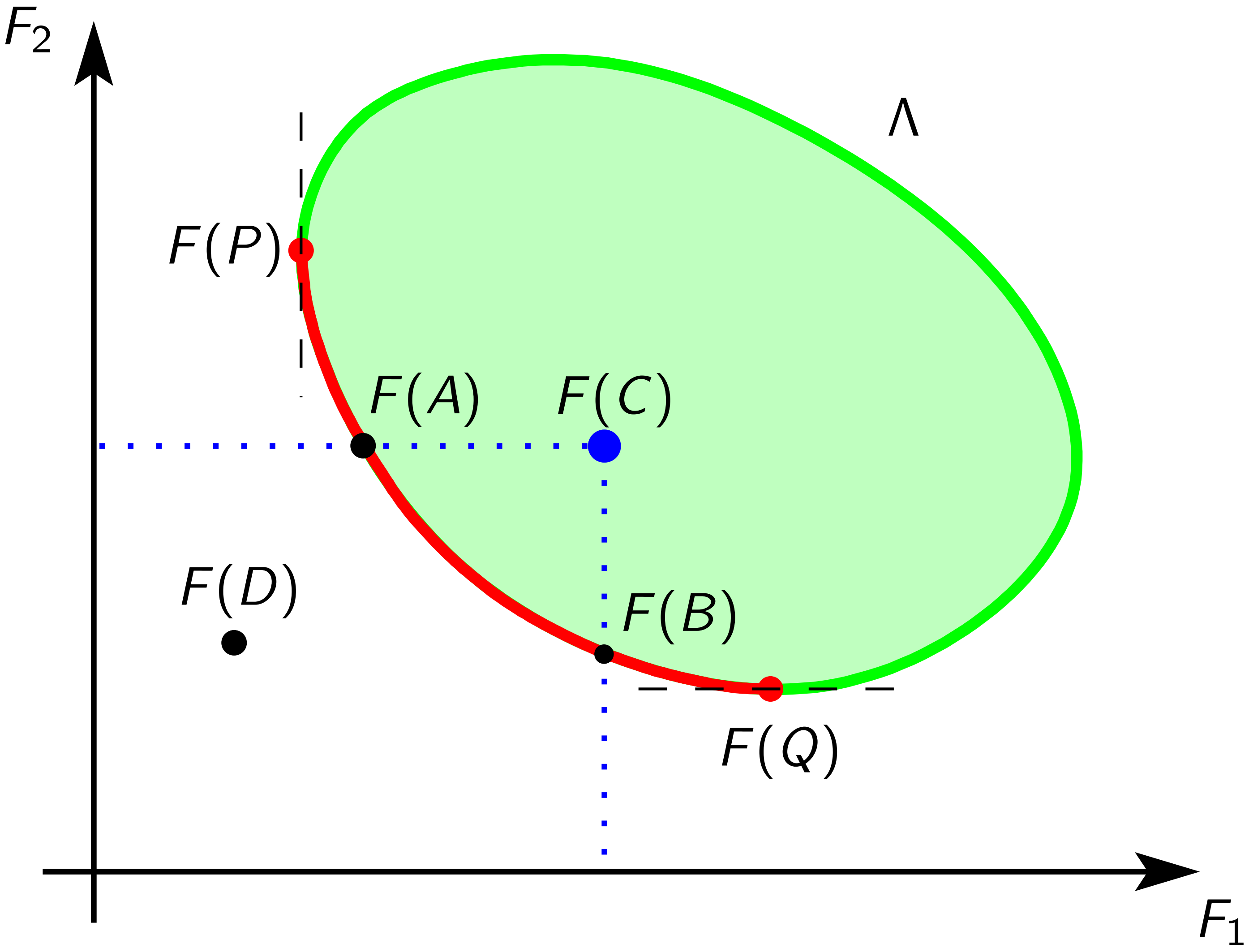
Pareto optimality (continued)



Ω feasible set for x

Λ feasible set for F

Pareto optimality (continued)



A, B, C are feasible; D is infeasible

A, B are Pareto optimal (i.e., no $\tilde{x}$ with $F(\tilde{x}) \leqslant F(x^*)$ and $F_i(\tilde{x}) < F_i(x^*)$)

Solution methods for multi-objective optimization

Methods for multi-objective optimization

$\rightarrow$ generation and selection of Pareto optimal points

$\rightarrow$ transformation into scalar objective

- Weighted-sum strategy
- ε-constraint method
- Goal attainment method

Weighted-sum strategy

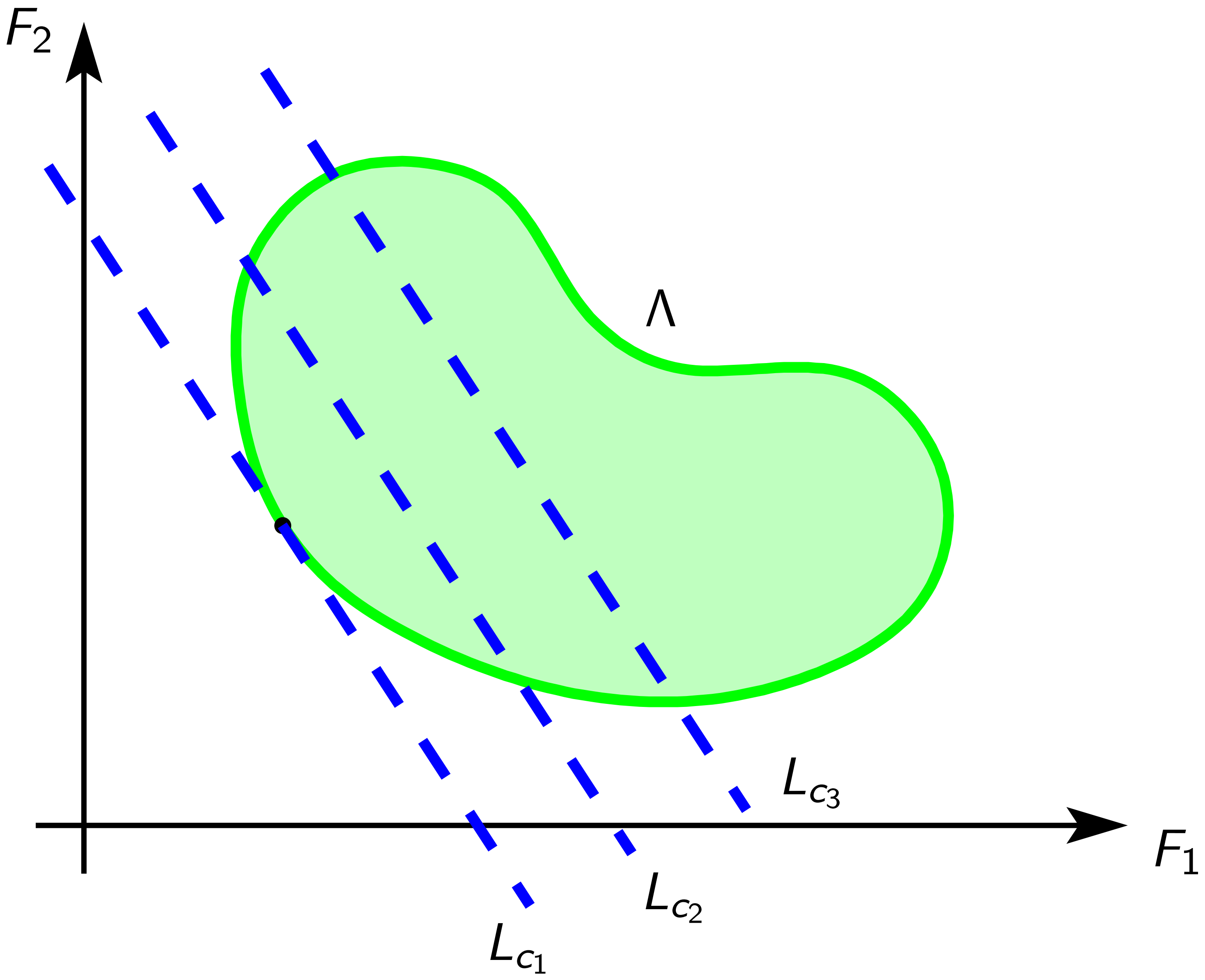
Weighted-sum strategy:

$$\begin{aligned} \min_x \; & \sum_{i=1}^{m} w_i F_i(x) \\ \text{s.t.} \; & g(x) \leqslant 0 \\ & h(x) = 0 \end{aligned}$$

Main problems:

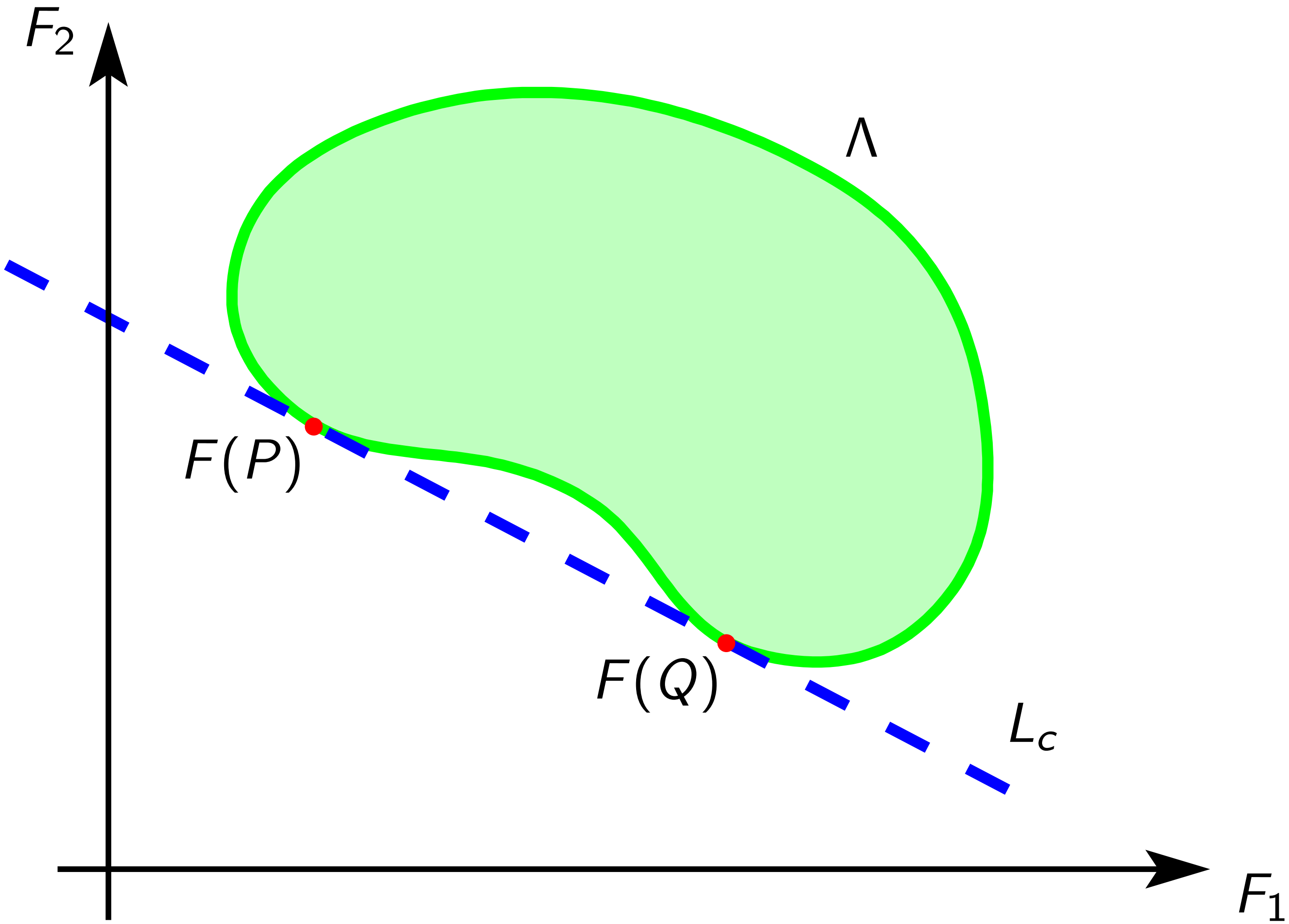
- Assigning appropriate weighting coefficients w_i
- Not all Pareto optimal solutions are accessible by appropriate selection of weights w_i

Weighted-sum strategy (continued)



Line L_c: points for which $w^T F(x) = c$

Weighted-sum strategy (continued)



$\rightarrow$ Pareto-optimal points between $F(P)$ and $F(Q)$ are not accessible

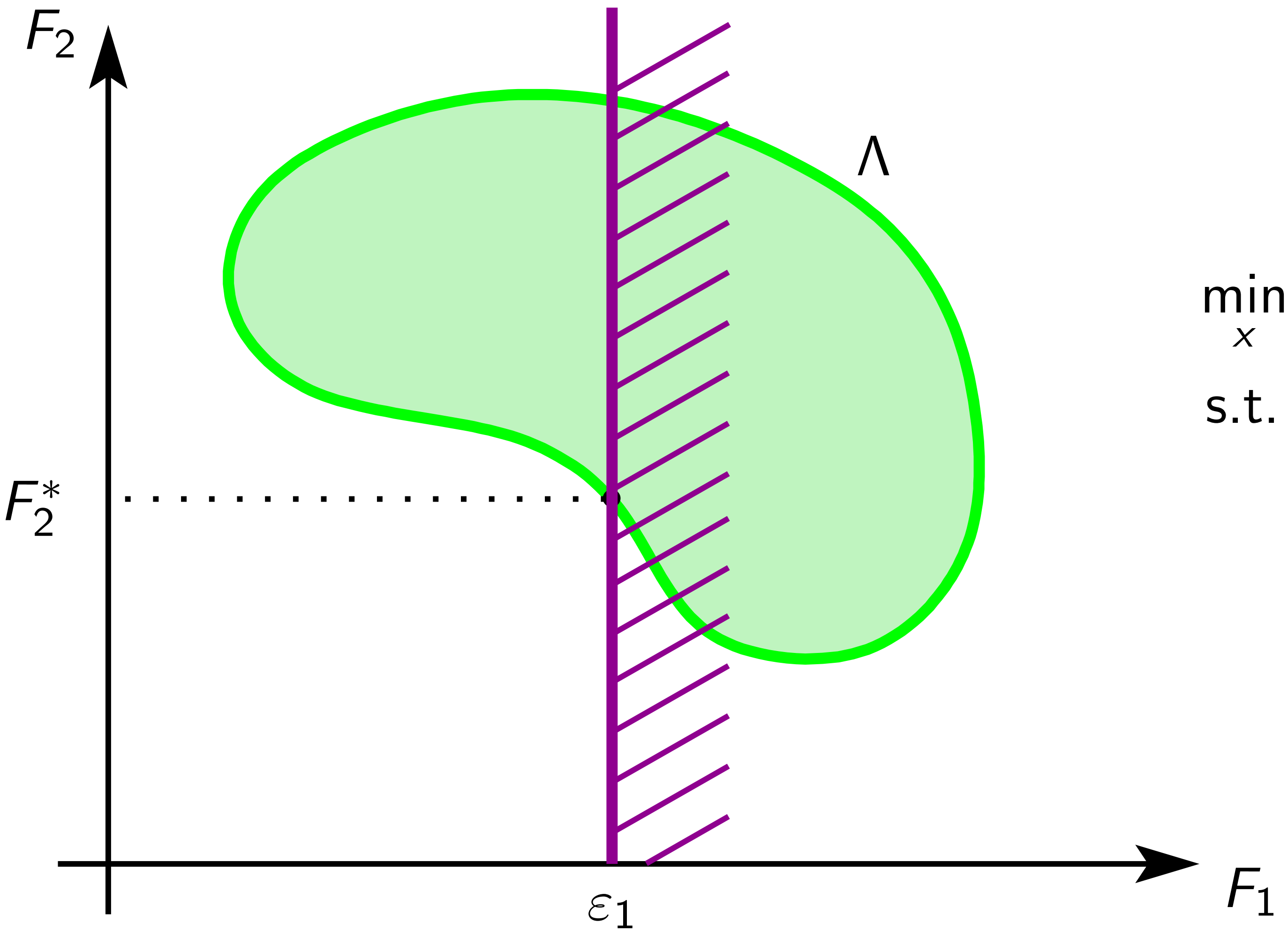
ε-constraint method

ε-constraint method:

$\rightarrow$ minimize primary objective F_{i_p}
and express other objectives in form of inequality constraints

$$
\begin{aligned}
&\min_x \ F_{i_\mathrm{p}}(x) \\
&\text{s.t.} \ \ F_i(x) \leqslant \varepsilon_i \quad \text{for } \ i = 1, \ldots, m \ \text{ with } \ i \neq i_\mathrm{p} \\
&\qquad g(x) \leqslant 0 \\
&\qquad h(x) = 0
\end{aligned}
$$

ε-constraint method (continued)



$$
\begin{aligned}
\min_{x} \quad & F_2(x) \\
\text{s.t.} \quad & F_1(x) \leqslant \varepsilon_1 \\
& g(x) \leqslant 0 \\
& h(x) = 0
\end{aligned}
$$

ε-constraint method (continued)

- ε-constraint method is able to identify Pareto optimal solutions that are not obtainable using weighted-sum technique
- Problems:
  - suitable selection of ε
    e.g., if ε is too small, then maybe no feasible solution
  - hard constraints are used in ε-constraint method
    $\leftrightarrow$ expressing true design objectives

Goal attainment method

Goal attainment method:

Define vector of design goals:

$$F^{\text{goal}} = [\, F_1^{\text{goal}} \;\; F_2^{\text{goal}} \;\; \ldots \;\; F_m^{\text{goal}} \,]^T$$

Allow objectives to be under- or overachieved
$\rightarrow$ designer can be relatively imprecise about initial design goals

Relative degree of under- or over-achievement of goals is controlled by weighting coefficients w

Goal attainment method (continued)

$$
\begin{aligned}
&\min_{\gamma, x} \gamma \\
\text{s.t.}\quad & F_i(x) - w_i\gamma \leqslant F_i^{\text{goal}} \quad \text{for } i = 1, \ldots, m \\
& g(x) \leqslant 0 \\
& h(x) = 0
\end{aligned}
$$

- This problem can be solved using, e.g., SQP
- Term $w_i\gamma$ in inequality constraints introduces an element of slackness: $F_i(x) \leqslant F_i^{\text{goal}} + w_i\gamma$

  To introduce hard constraint i': set $w_{i'} = 0$
- Implemented in Matlab: `fgoalattain`

Summary

- Multi-objective optimization: Standard form

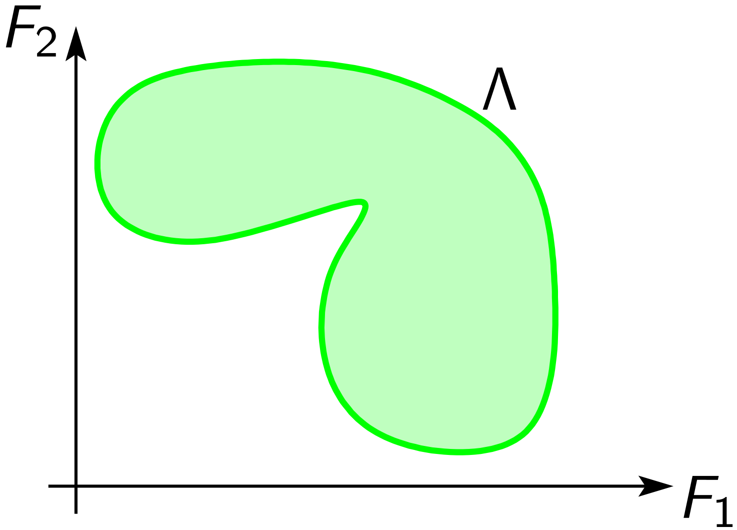
$$\begin{aligned} &\min_x F(x) \\ \text{s.t.} \quad &g(x) \leqslant 0 \\ &h(x) = 0 \end{aligned}$$

- Pareto optimality
- Algorithms for multi-objective objective optimization problems:
  - weighted-sum strategy
  - ε-constraint method
  - goal attainment

$\rightarrow$ scalar optimization problem

Test: Pareto optimal points

Consider vector-valued function F represented by feasible set Λ:



Which of the following red sets is the Pareto optimal set of F?