State Space Modeling

Topics Covered

- Rotary pendulum modeling.
- Introduction to state-space models.
- Model validation.

Prerequisites

- Integration laboratory experiment.
- First Principles Modeling laboratory experiment.
- Pendulum Moment of Inertia laboratory experiment.
- Rotary pendulum module is attached to the QUBE-Servo 2.

1 Background

1.1 Pendulum Model

The rotary pendulum model is shown in Figure 1.1. The rotary arm pivot is attached to the QUBE-Servo 2 system and is actuated. The arm has a length of L_r, a moment of inertia of J_r, and its angle θ increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive ($V_m > 0$).

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is at $L_p/2$. The moment of inertia about its center of mass is J_p. The rotary pendulum angle α is zero when it is hanging downward and increases positively when rotated CCW.

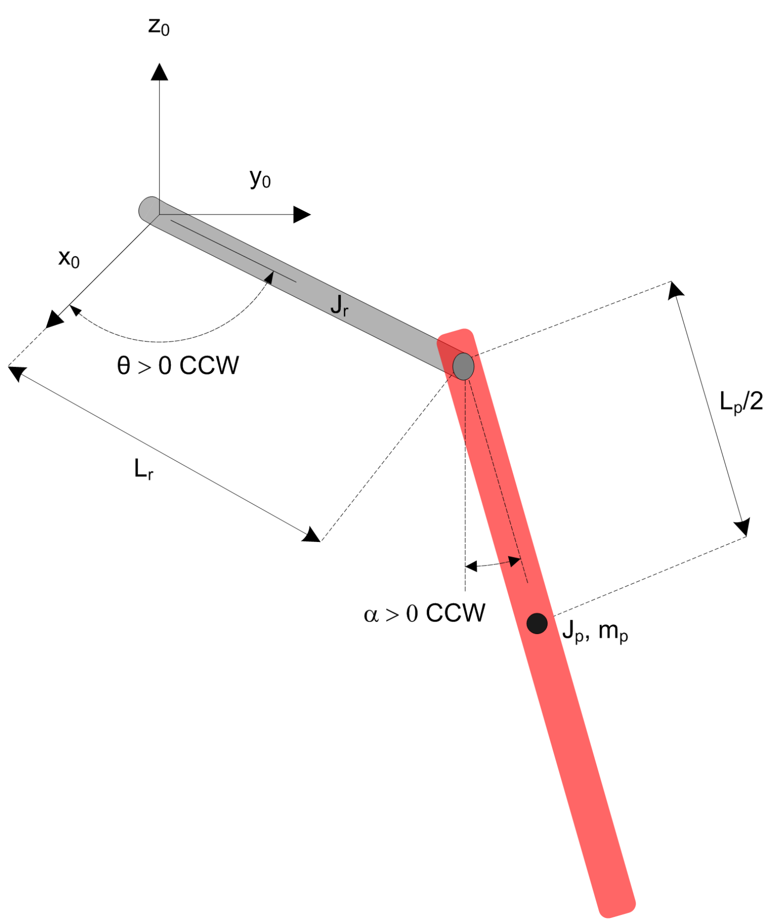


Figure 1.1: Rotary pendulum model

The equations of motion (EOM) for the pendulum system were developed using the Euler-Lagrange method. This systematic method is often used to model complicated systems such as robot manipulators with multiple joints. The total kinetic and potential energy of the system is obtained, then the Lagrangian can be found. A number of derivatives are then computed to yield the EOMs. The complete derivation of the EOM for the pendulum system are presented in the *Rotary Pendulum Modeling Summary* and Maple workbook.

The resultant nonlinear EOM are:

$$\left(m_p L_r^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 \cos(\alpha)^2 + J_r\right) \ddot{\theta} + \left(\frac{1}{2} m_p L_p L_r \cos(\alpha)\right) \ddot{\alpha}$$
$$+ \left(\frac{1}{2} m_p L_p^2 \sin(\alpha)\cos(\alpha)\right) \dot{\theta}\dot{\alpha} - \left(\frac{1}{2} m_p L_p L_r \sin(\alpha)\right) \dot{\alpha}^2 = \tau - D_r \dot{\theta} \tag{1.1}$$
$$\frac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} - \frac{1}{4} m_p L_p^2 \cos(\alpha)\sin(\alpha)\dot{\theta}^2 + \frac{1}{2} m_p L_p g \sin(\alpha) = -D_p \dot{\alpha}. \tag{1.2}$$

with an applied torque at the base of the rotary arm generated by the servo motor as described by the equation:

$$\tau = \frac{k_m \left(V_m - k_m \dot{\theta}\right)}{R_m} \tag{1.3}$$

When the nonlinear EOM are linearized about the operating point, the resultant linear EOM for the rotary pendulum are defined as:

$$\left(m_p L_r^2 + J_r\right) \ddot{\theta} + \frac{1}{2} m_p L_p L_r \ddot{\alpha} = \tau - D_r \dot{\theta}. \tag{1.4}$$

and

$$\frac{1}{2} m_p L_p L_r \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} + \frac{1}{2} m_p L_p g \alpha = -D_p \dot{\alpha}. \tag{1.5}$$

Solving for the acceleration terms yields:

$$\ddot{\theta} = \frac{1}{J_T}\left(-\left(J_p + \frac{1}{4} m_p L_p^2\right) D_r \dot{\theta} + \frac{1}{2} m_p L_p L_r D_p \dot{\alpha} + \frac{1}{4} m_p^2 L_p^2 L_r g \alpha + \left(J_p + \frac{1}{4} m_p L_p^2\right) \tau\right). \tag{1.6}$$

and

$$\ddot{\alpha} = \frac{1}{J_T}\left(\frac{1}{2} m_p L_p L_r D_r \dot{\theta} - \left(J_r + m_p L_r^2\right) D_p \dot{\alpha} - \frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2\right) \alpha - \frac{1}{2} m_p L_p L_r \tau\right). \tag{1.7}$$

where

$$J_T = J_p m_p L_r^2 + J_r J_p + \frac{1}{4} J_r m_p L_p^2. \tag{1.8}$$

1.2 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \tag{1.9}$$

and

$$y = Cx + Du \tag{1.10}$$

where x is the state, u is the control input, A, B, C and D are state-space matrices. For the rotary pendulum system, the state and output are defined

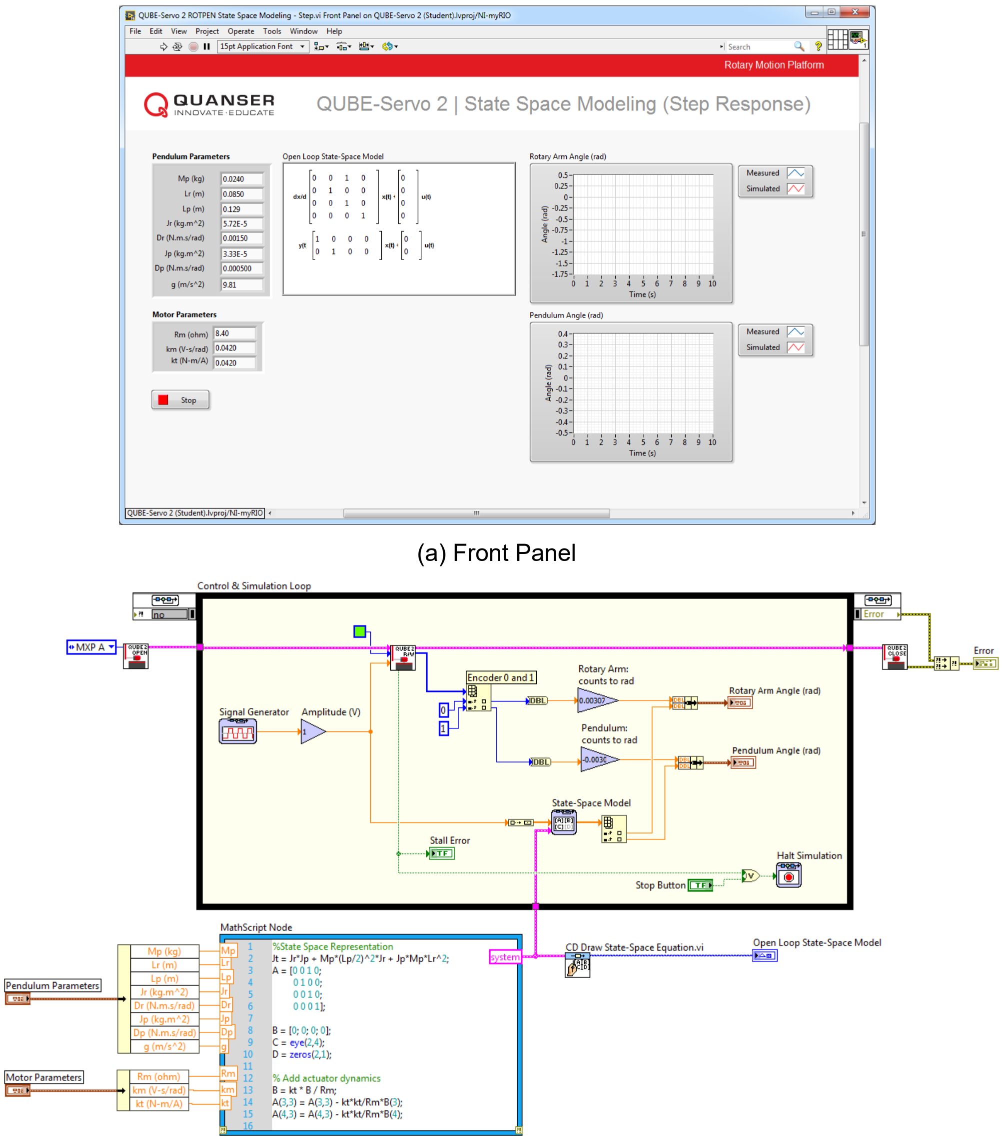
$$x = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}^T \tag{1.11}$$

and

$$y = \begin{bmatrix} \theta & \alpha \end{bmatrix}^T. \tag{1.12}$$

2 In-Lab Exercises

There are several VIs available in the LabVIEW™ *Control Design and Simulation Module* to enable the creation of linear state-space models. Based on the Virtual Instrument (VI) already designed in the Rotary Pendulum Modeling lab, design a VI similar to the one shown in Figure 2.1 that applies a 1 Hz square wave with an amplitude of 1 V to the pendulum system and state-space model.



(a) Front Panel

(b) Block Diagram

Figure 2.1: Applies a step voltage and displays measured and simulated pendulum response.

There are several methods that can be used to create the state-space model in LabVIEW™. The method shown in Figure 2.1 utilizes the *MathScript RT Module* to create the model, which is then passed to the *State-Space* VI for simulation.

Note: Be sure to include the actuator dynamics in your model, as shown in Figure 2.1.

1. Based on the sensors available on the pendulum system, find the C and D matricies in Equation 1.10.

2. Using Equation 1.6 and Equation 1.7 and the defined state in Equation 1.11, derive the linear state-space model of the pendulum system.

3. Input the linear state-space model into the MathScript Node. Using the CD Draw State-Space Equation VI shown in Figure 2.1, verify that you have entered the correct matrices.

4. Set the rotary arm viscous damping coefficient $D_r = 0.0015$ N − m − s/rad, and the pendulum damping coefficient $D_p = 0.0005$ N − m − s/rad. These parameters were found experimentally to reasonably accurately reflect the viscous damping of the system due to effects such as friction, when subject to a step response.

5. Run the VI. The scope response should be similar to Figure 2.2. Attach a screen capture of your scopes. Does your model represent the actual pendulum well? If not, explain why there might be discrepancies.

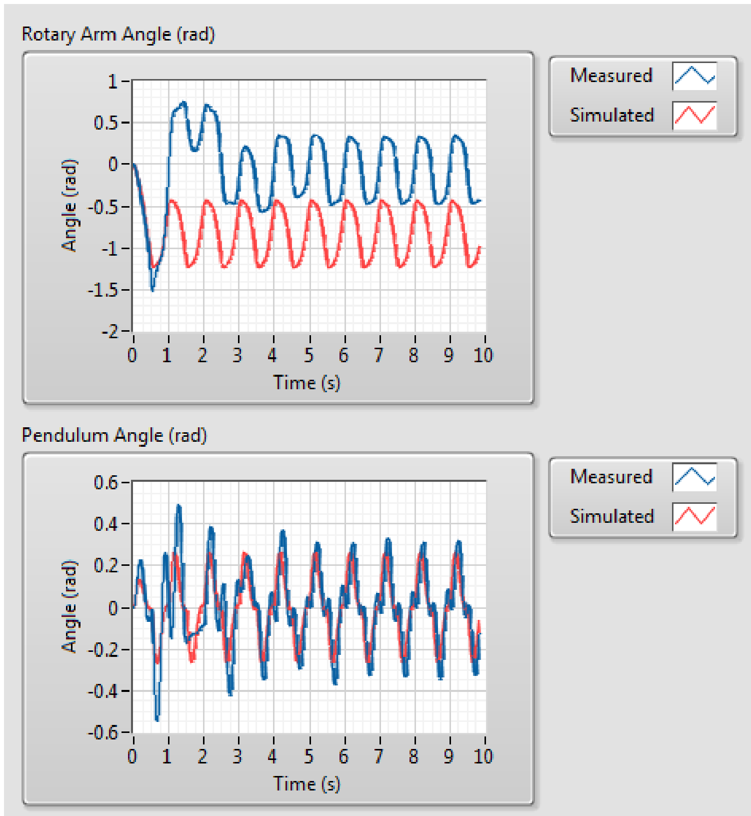


Figure 2.2: Step response of the pendulum system

6. The viscous damping of each pendulum can vary slightly from system to system. If your model does not accurately represent your specific pendulum system, try modifying the damping coefficients D_r and D_p to obtain a more accurate model.

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