Pendulum Equations

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1. Planar Pendulum

Choosing the sign of θ in Figure 1 to anti-clockwise the equations become

$$\begin{aligned} J_p\ddot\theta - m_p\ell\ddot x\cos\theta - m_pg\ell\sin\theta &= 0 \\ -m_p\ell\ddot\theta\cos\theta + m_t\ddot x + m_p\ell\dot\theta^2\sin\theta &= F \end{aligned} \tag{1}$$

where m_p pendulum mass, m_c cart mass, $m_t = m_p + m_c$, J_p moment of inertia of the pendulum with respect to its pivot axis, ℓ distance from pendulum axis of rotation to its center of mass. The equations of motion can be written as

$$\begin{pmatrix} J_p & -m_p\ell\cos\theta \\ -m_p\ell\cos\theta & m_t \end{pmatrix} \begin{pmatrix} \ddot\theta \\ \ddot x \end{pmatrix} + \begin{pmatrix} -m_p\ell g\sin\theta \\ m_p\ell\dot\theta^2\sin\theta \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Solving for the second derivatives gives

$$\begin{aligned} \begin{pmatrix} \ddot\theta \\ \ddot x \end{pmatrix} &= \frac{1}{den} \begin{pmatrix} m_t & m_p\ell\cos\theta \\ m_p\ell\cos\theta & J_p \end{pmatrix} \begin{pmatrix} m_p\ell g\sin\theta \\ -m_p\ell\dot\theta^2\sin\theta + F \end{pmatrix} \\ &= \frac{1}{den} \begin{pmatrix} m_pm_t\ell g\sin\theta - m_p^2\ell^2\dot\theta^2\sin\theta\cos\theta + m_p\ell F\cos\theta \\ m_p^2\ell^2 g\sin\theta\cos\theta - m_pJ_p\ell\dot\theta^2\sin\theta + J_pF \end{pmatrix} \end{aligned}$$

where

$$den = m_tJ_p - m_p^2\ell^2\cos^2\theta$$

Introducing a transformed control variable v defined by

$$v = \frac{m_p^2\ell^2 g\sin\theta\cos\theta - m_pJ_p\ell\dot\theta^2\sin\theta + J_pF}{m_tJ_p - m_p^2\ell^2\cos^2\theta} \tag{2}$$

which has physical interpretation as the acceleration of the cart the equations become

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= \frac{m_p\ell g}{J_p}\sin\theta + v\frac{m_p\ell}{J_p}\cos\theta \\ \frac{d^2x}{dt^2} &= v \end{aligned}$$

Introducing a new time variable $\tau = \omega_0 t$ where

$$\omega_0 = \sqrt{\frac{m_p \ell g}{J_p}} \tag{3}$$

is the natural frequency of the pendulum. The normalized equations for the pendulum on the cart can then becomes

$$\begin{aligned} \frac{d^2\theta}{d\tau^2} &= \sin\theta + \frac{v}{g}\cos\theta \\ \frac{d^2x}{d\tau^2} &= \frac{v}{\omega_0^2} = \frac{J_p}{m_p \ell g} v \end{aligned}$$

Introduce the scaled control variable

$$u = \frac{v}{g} = \frac{m_p^2 \ell^2 g \sin\theta\cos\theta - m_p J_p \ell \dot{\theta}^2 \sin\theta + J_p F}{g(m_t J_p - m_p^2 \ell^2 \cos^2\theta)} \tag{4}$$

and the equations become

$$\begin{aligned} \frac{d^2\theta}{d\tau^2} &= \sin\theta + u\cos\theta \\ \frac{d^2x}{d\tau^2} &= \frac{g}{\omega_0^2} u = \frac{J_p}{m_p \ell} u \end{aligned}$$

Finally we replace the state variable x by

$$\xi = \frac{m_p \ell}{J_p} x \tag{5}$$

and we obtain the normalized form of the equations

$$\begin{aligned} \frac{d^2\theta}{d\tau^2} &= \sin\theta + u\cos\theta \\ \frac{d^2\xi}{d\tau^2} &= u \end{aligned} \tag{6}$$

This model no explicit parameter but there is an implicit parameter, the maximum value of u which corresponds to the maximum acceleration of the cart, which has a significant influence on the behavior of the system.

It is convenient to use the normal form for analysis and simulations. To make real experiments it is however necessary to know the transformations between the real pendulum and the normal form. Summarizing we find that the variables are related as

$$
\begin{aligned}
\xi &= \frac{m_p\ell}{J_p}x \\
x &= \frac{J_p}{m_p\ell}\xi \\
\tau &= \omega_0 t = t\sqrt{\frac{m_p\ell g}{J_p}} \\
t &= \frac{1}{\omega_0}\tau = \tau\sqrt{\frac{J_p}{m_p\ell g}} \\
\frac{d\theta}{d\tau} &= \frac{1}{\omega_0}\frac{d\theta}{dt} \\
\frac{d\theta}{dt} &= \omega_0\frac{d\theta}{d\tau} \\
\frac{d\xi}{d\tau} &= \frac{m_p\ell}{\omega_0 J_p}\frac{dx}{dt} \\
\frac{dx}{dt} &= \frac{\omega_0 J_p}{m_p\ell}\frac{d\xi}{d\tau} \\
u &= \frac{v}{g} = \frac{m_p^2\ell^2 g\sin\theta\cos\theta - m_pJ_p\ell\dot\theta^2\sin\theta + J_pF}{g(m_tJ_p - m_p^2\ell^2\cos^2\theta)} \\
F &= \frac{m_tJ_p - m_p\ell^2\cos^2\theta}{J_p}gu - \frac{m_p^2\ell^2 g}{J_p}\sin\theta\cos\theta + m_p\ell\dot\theta^2\sin\theta
\end{aligned}
\tag{7}
$$

2. Furuta Pendulum

The equations for the Furuta pendulum are

$$
\begin{aligned}
J_p\ddot\theta - m_pr\ell\ddot\varphi\cos\theta - J_p\dot\varphi^2\sin\theta\cos\theta - m_pg\ell\sin\theta &= 0 \\
(J_a + J_p\sin^2\theta)\ddot\varphi - m_pr\ell\ddot\theta\cos\theta + 2J_p\dot\varphi\dot\theta\sin\theta\cos\theta + m_pr\ell\dot\theta^2\sin\theta &= T
\end{aligned}
\tag{8}
$$

where J_p is the moment of inertia of the pendulum with respect to its pivot, J_a is the moment of inertia of the arm with respect to the axis of rotation, ℓ is the distance from the pendulum center of mass to the pivot and r the length of the arm. The equations of motion can be written as

$$
\begin{pmatrix} J_p & -m_p\ell r\cos\theta \\ -m_p\ell r\cos\theta & J_a + J_p\sin^2\theta \end{pmatrix}
\begin{pmatrix} \ddot\theta \\ \ddot\varphi \end{pmatrix}
+
\begin{pmatrix} -\frac{1}{2}J_p\dot\varphi^2\sin 2\theta - m_p\ell g\sin\theta \\ J_p\dot\theta\dot\varphi\sin 2\theta + m_p\ell r\dot\theta^2\sin\theta - T \end{pmatrix} = 0
$$

To solve for the angular accelerations we first compute

$$
\begin{aligned}
&\det\begin{pmatrix} J_p & -m_p\ell r\cos\theta \\ -m_p\ell r\cos\theta & J_a + J_p\sin^2\theta \end{pmatrix} = J_aJ_p + J_p^2\sin^2\theta - m_p^2\ell^2r^2\cos^2\theta \\
&= J_aJ_p - m_p^2\ell^2r^2 + (J_p^2 + m_p^2\ell^2r^2)\sin^2\theta
\end{aligned}
$$

and

$$
a = \begin{pmatrix} J_a + J_p\sin^2\theta & m_p\ell r\cos\theta \\ m_p\ell r\cos\theta & J_p \end{pmatrix}
\begin{pmatrix} \frac{1}{2}J_p\dot\varphi^2\sin 2\theta + m_p\ell g\sin\theta \\ -J_p\dot\theta\dot\varphi\sin 2\theta - m_p\ell r\dot\theta^2\sin\theta + T \end{pmatrix}
$$

we have

$$
\begin{aligned}
a_1 &= J_p(J_a + J_p \sin^2\theta)\dot{\varphi}^2 \sin\theta\cos\theta + m_p\ell g(J_a + J_b\sin^2\theta)\sin\theta \\
&\quad - 2m_p\ell r J_p\dot{\theta}\dot{\varphi}\sin\theta\cos^2\theta - m_p^2\ell^2 r^2\dot{\theta}^2\sin\theta\cos\theta + m_p\ell r T\cos\theta \\
a_2 &= m_p\ell r J_p\dot{\varphi}^2\sin\theta\cos^2\theta + m_p^2\ell^2 r g\sin\theta\cos\theta \\
&\quad - 2J_p^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta - m\ell r J_p\dot{\theta}^2\sin\theta + J_p T
\end{aligned}
$$

Replace motor torque T with the a new control variable v defined by

$$
v = \frac{(m_p\ell r J_p\dot{\varphi}^2\cos\theta + m_p^2\ell^2 r g - 2J_p^2\dot{\theta}\dot{\varphi})\sin\theta\cos\theta - m\ell r J_p\dot{\theta}^2\sin\theta + J_p T}{J_a J_p + J_p^2\sin^2\theta - m_p^2\ell^2 r^2\cos^2\theta}
$$

which has physical interpretation as the angular acceleration of the arm. Tedious but straight forward calculations give

$$
\begin{aligned}
\frac{d^2\theta}{dt^2} &= \Big(\frac{d\varphi}{dt}\Big)^2\sin\theta\cos\theta + \frac{m_p\ell g}{J_p}\sin\theta + \frac{m_p\ell r}{J_p}v\cos\theta \\
\frac{d^2\varphi}{dt^2} &= v
\end{aligned}
\tag{9}
$$

Introduce a new time variable $\tau = \omega_0 t$ where

$$
\omega_0 = \sqrt{\frac{m_p\ell g}{J_p}} \tag{10}
$$

is the natural frequency of the pendulum. The normalized equations for the pendulum on the cart can then becomes

$$
\begin{aligned}
\frac{d^2\theta}{d\tau^2} &= \Big(\frac{d\varphi}{d\tau}\Big)^2\sin\theta\cos\theta + \sin\theta + \frac{rv}{g}\cos\theta \\
\frac{d^2\varphi}{d\tau^2} &= \frac{1}{\omega_0^2}v = \frac{J_p}{m_p\ell g}v
\end{aligned}
\tag{11}
$$

Introducing the control variable

$$
u = \frac{rv}{g} \tag{12}
$$

gives

$$
\begin{aligned}
\frac{d^2\theta}{d\tau^2} &= \Big(\frac{d\varphi}{d\tau}\Big)^2\sin\theta\cos\theta + \sin\theta + u\cos\theta \\
\frac{d^2\varphi}{d\tau^2} &= \frac{J_p}{m_p\ell r}u
\end{aligned}
$$

Replacing φ by

$$
\phi = \frac{m_p\ell r}{J_p}\varphi
$$

gives the following form of the equations

$$
\begin{aligned}
\frac{d^2\theta}{d\tau^2} &= a\Big(\frac{d\phi}{d\tau}\Big)^2\sin\theta\cos\theta + \sin\theta + u\cos\theta \\
\frac{d^2\phi}{d\tau^2} &= u
\end{aligned}
\tag{13}
$$

where

$$a = \Big(\frac{J_p}{m_p \ell r}\Big)^2 \tag{14}$$

By introducing a new control variable defined by

$$w = u - a\Big(\frac{d\phi}{d\tau}\Big)^2 \sin\theta \tag{15}$$

equation (13) becomes

$$\begin{aligned}
\frac{d^2\theta}{d\tau^2} &= \sin\theta + w\cos\theta \\
\frac{d^2\phi}{d\tau^2} &= -a\Big(\frac{d\phi}{d\tau}\Big)^2 \sin\theta + w
\end{aligned} \tag{16}$$

The Furuta pendulum can thus be characterized by one explicit parameter a, and one implicit parameter, the maximum acceleration of the arm.

The parameter a tells how the Furuta pendulum differs from a pendulum on a cart. The system are similar if a is small or if the arm velocity is small, but the difference may be large if a and the arm velocity are large. We have the following upper bound on a

$$a = \left(\frac{J'_p + m_p\ell^2}{m_p \ell r}\right)^2 > \frac{\ell^2}{r^2}.$$

Observing that the differential equation for φ is a Riccati equation the model can also be written as

$$\begin{aligned}
\frac{d^2\theta}{d\tau^2} &= \sin\theta + w\cos\theta \\
\frac{d\phi}{d\tau} &= \frac{x}{y} \\
\frac{dx}{d\tau} &= wy \\
\frac{dy}{d\tau} &= ax\sin\theta
\end{aligned} \tag{17}$$

The following equations give the transformations between the physical equa-

Table 1 Parameters of the original Furuta Pendulum and the Lund copy.

	m_p	J_p	m_a	J_a	ℓ	r	ω_0	a
Furuta	0.098	2.62e-3	0.08	3.65e-3	0.15	0.148	7.38	1.45
Lund	0.035	3.89e-3	0.165	3.53e-3	0.306	0.245	5.23	2.21

tions and the normal form.

$$
\begin{aligned}
\phi &= \frac{m_p \ell r}{J_p}\varphi\\
\varphi &= \frac{J_p}{m_p \ell r}\phi\\
\tau &= \omega_0 t = t\sqrt{\frac{m_p \ell g}{J_p}}\\
t &= \frac{1}{\omega_0}\tau = \tau\sqrt{\frac{J_p}{m_p g \ell}}\\
\frac{d\phi}{d\tau} &= \sqrt{\frac{m_p \ell r^2}{J_p g}}\frac{d\varphi}{dt}\\
\frac{d\varphi}{dt} &= \sqrt{\frac{J_p g}{m_p \ell r^2}}\frac{d\phi}{d\tau}\\
u &= \frac{r\big((m_p \ell r J_p \dot\varphi^2 \cos\theta + m_p^2 \ell^2 r g - 2J_p^2 \dot\theta \dot\varphi)\sin\theta\cos\theta - m\ell r J_p \dot\theta^2 \sin\theta + J_p T\big)}{\big(J_a J_p - m_p^2 \ell^2 r^2 + (J_p^2 + m_p^2 \ell^2 r^2)\sin^2\theta\big)g}\\
A &= J_a J_p - m_p^2 \ell^2 r^2 + (J_p^2 + m_p^2 \ell^2 r^2)\sin^2\theta\\
B &= (m_p \ell r J_p \dot\varphi^2 \cos\theta + m_p^2 \ell^2 r g - 2J_p^2 \dot\theta\dot\varphi)\sin\theta\cos\theta - m\ell r J_p \dot\theta^2 \sin\theta\\
T &= \frac{Agu - Br}{rJ_p}
\end{aligned} \tag{18}
$$

3. Parameters

Table 1 gives the parameters for some implementations The maximum acceleration of the pivot is computed as follows. For small angles we have

$$
\begin{aligned}
J_p \ddot\theta - m_p r \ell \ddot\varphi - m_p g \ell \theta &= 0\\
J_a \ddot\varphi - m_p \ell r \ddot\theta &= T
\end{aligned}
$$

elimination of $\ddot\theta$ gives for $\theta = 0$

$$\ddot\varphi = \frac{J_p}{J_a J_p - m_p^2 r^2 \ell^2} T$$

The maximum acceleration of the pendulum pivot is thus

$$a_{max} = \frac{J_p}{J_a J_p - m_p^2 r^2 \ell^2} r T_{max}$$

Elimination of $\ddot{\varphi}$ gives

$$\left(J_p - \frac{m_p^2 r^2 \ell^2}{J_a}\right)\ddot{\theta} - m_p g \ell \theta = \frac{m_p r \ell}{J_a} T$$

The frequency of oscillation of the pendulum now becomes

$$\omega_{osc} = \sqrt{\frac{m_p g \ell J_a}{J_a J_p - m_p^2 r^2 \ell^2}}$$