Statistical Signal Processing

Lecture 3:
LTI Filtering Random Processes

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Summary Lecture 2:

WSS RP Characteristics: Auto and cross-correlation function:

$$r_x(k) = E[x(n)x^*(n-k)] \quad r_{xy}(k) = E[x(n)y^*(n-k)]$$

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$$r_x(k) = E[x(n)x^*(n-k)] \quad r_{xy}(k) = E[x(n)y^*(n-k)]$$

Properties:

Property 1: If $x(n) \in \mathbb{C}$ then $r_x(k) = r_x^*(-k)$ (conjugate symmetric).
If $x(n) \in \mathbb{R}$ then $r_x(k) = r_x(-k)$ (symmetric).

Property 2: $r_x(0) = E[|x(n)|^2] \geq 0$.

Property 3: $r_x(0) \geq |r_x(k)| \quad \forall k$

Property 4: If $\exists k_0 : r_x(k_0) = r_x(0) \Rightarrow r_x(k)$ is periodic with period k_0 and further

$$E[|x(n) - x(n-k_0)|^2] = 0$$

$x(n)$ is said to be *mean-square periodic*.

Property 1: If $x(n), y(n) \in \mathbb{C}$ then $r_{yx}(k) = r_{xy}^*(-k)$ (change of order index arguments!)
If $x(n), y(n) \in \mathbb{R}$ then $r_{yx}(k) = r_{xy}(-k)$(not symmetric)

Property 2: $|r_{xy}(k)| \leq \sqrt{r_x(0)r_y(0)}$

Property 3: $|Re\Big(r_{xy}(k)\Big)| \leq \frac{1}{2}[r_x(0) + r_y(0)]$

`AutoCor.m`, `CrossCorXY.m`, `CrossCorYX.m`

Summary Lecture 2:

Autocorrelation Function of a (WSS) RPs $\{x(n)\}$:

$$\begin{aligned} r_x(k) &= E[x(n)x^*(n-k)] \\ &= E[x(t+k)x^*(t)] \\ &= E[x(t)x^*(t+k)]^* \\ &= r_x^*(-k) \end{aligned}$$

$$\begin{aligned} P_x(z) &= \sum_{k=-\infty}^{\infty} r_x(k) z^{-k} \\ &= \sum_{k=-\infty}^{\infty} r_x^*(-k) z^{-k} \\ &= \sum_{t=-\infty}^{\infty} r_x^*(t)(z)^t = \left[\sum_{t=-\infty}^{\infty} r_x(t)(z^*)^t \right]^* \\ &= P_x^*(1/z^*) \end{aligned}$$

`Powspec.m`

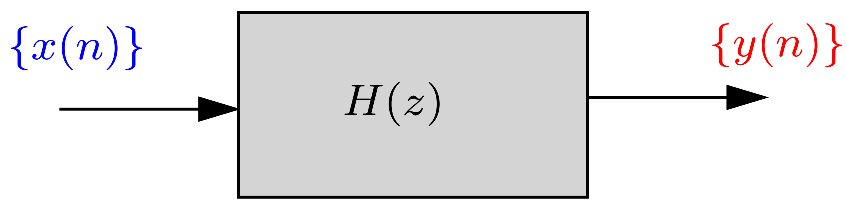
Summary Lecture 2:

Cross-correlation Function between two (WSS) RPs $\{x(n)\}$ and $\{y(n)\}$:

$$
\begin{aligned}
r_{xy}(k) &= E[x(n)y^*(n-k)] \\
&= E[x(t+k)y^*(t)] \\
&= E[y(t)x^*(t+k)]^* \\
&= r_{yx}^*(-k)
\end{aligned}
$$

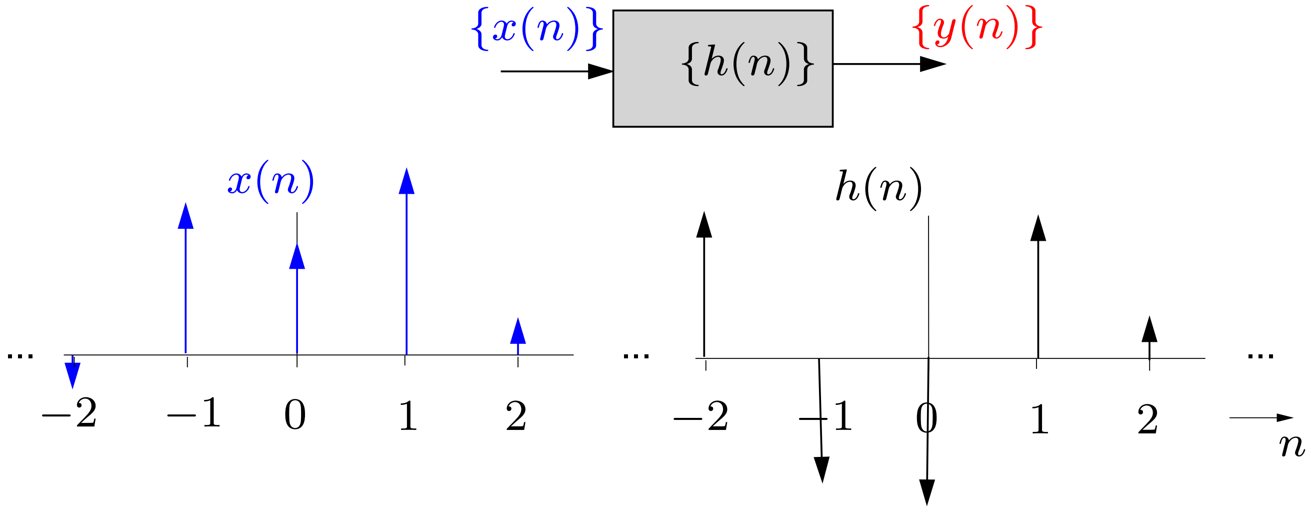
$$
\begin{aligned}
P_{xy}(z) &= \sum_{k=-\infty}^{\infty} r_{xy}(k)z^{-k} \\
&= \sum_{t=-\infty}^{\infty} r_{xy}(-t)z^{t} \\
&= \left[\sum_{t=-\infty}^{\infty} r_{xy}^*(-t)(z^*)^{t}\right]^* \\
&= \left[\sum_{t=-\infty}^{\infty} r_{yx}(t)(z^*)^{t}\right]^* \\
&= [P_{yx}(1/z^*)]^* \\
&= P_{yx}^*(1/z^*)
\end{aligned}
$$

Part I: Filtering Random Processes

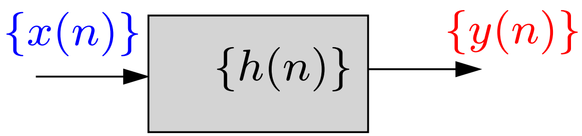


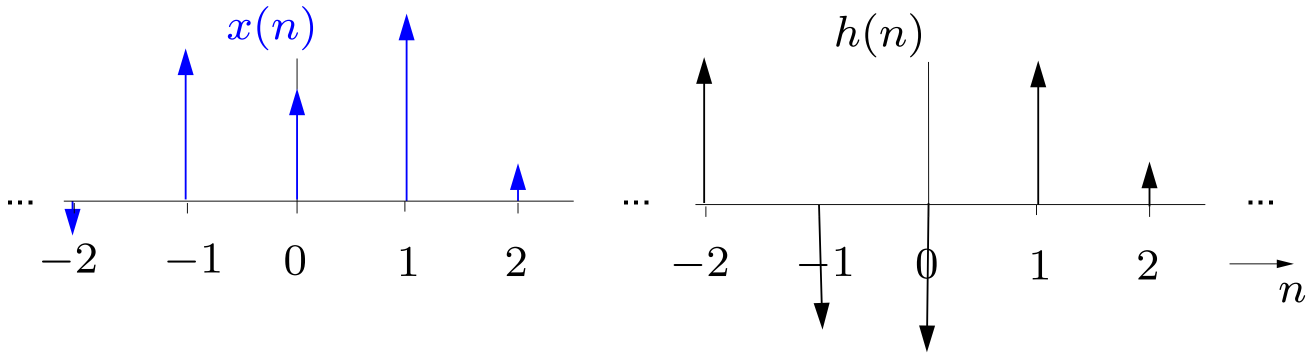
1. **Signals and Systems recap**
2. Preservation of WSS by LTI filtering
3. Cross- and Auto correlation function/spectra due to LTI filtering
4. Applications

Representing Signals and Systems



Representing Signals and Systems





Both $\{x(n)\}$ and $\{h(n)\}$ are functions and can be represented in terms of (orthogonal) basis functions.

Definition: *The unit Sample function* $\Delta(n) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$

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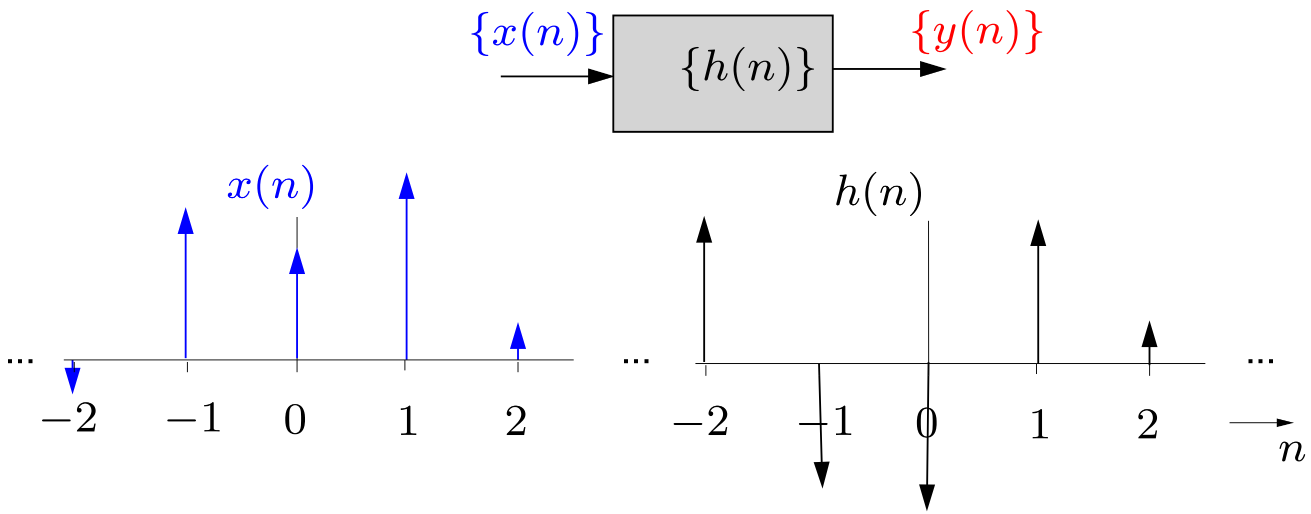

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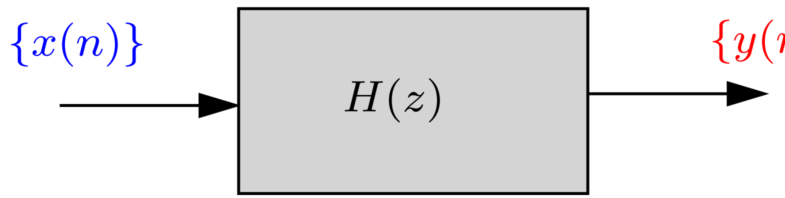
Definition: *The unit Sample function* $\Delta(n) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$

Corollary: Doubly-∞ sequences are represented as:

$$x(n) = \sum_{\ell=-\infty}^{\infty} x(\ell)\Delta(n-\ell) \qquad h(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)\Delta(n-\ell)$$

Linear Time-invariant (LTI) Systems



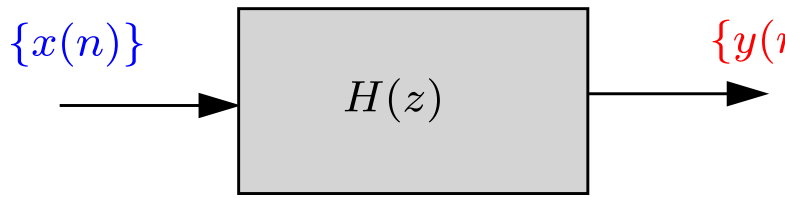
Property 1 - Linearity (L): A system $H[.]$ is *linear* if for any 2 inputs $x_1(n)$, $x_2(n)$ and any 2 constants $a, b \in \mathbb{C}$,

$$H[ax_1 + bx_2](n) = aH[x_1](n) + bH[x_2](n)$$

Property 2 - Shift-invariant (SI) - (Time-invariant (TI)):

$$\text{If } y(n) = H[x](n) \Rightarrow y(n-\ell) = H[x(\cdot - \ell)](n)$$

Input-Output relation for LTI Systems



Let $H[\cdot]$ be a linear, shift-invariant system with impulse response $\{h(n)\}$, then the response for an arbitrary input $\{x(n)\}$ is given by:

$$
\begin{aligned}
y(n) &= H\big[x\big](n) = H\Big[\sum_{\ell=-\infty}^{\infty} x(\ell)\Delta(\cdot-\ell)\Big](n) \\
&= \sum_{\ell=-\infty}^{\infty} x(\ell) H\Big[\Delta(\cdot-\ell)\Big](n) = \sum_{\ell=-\infty}^{\infty} x(\ell) h(n-\ell) \\
y(n) &= x(n) \star h(n) = h(n) \star x(n)
\end{aligned}
$$

z-transform of a series

Rational: The z-transform is a generalization of the DTFT, defined as:

$$
\begin{aligned}
x(n) &= \sum_{\ell=-\infty}^{\infty} x(\ell)\Delta(n-\ell) \\
\mathcal{Z}\big[\Delta(\cdot-\ell)\big] = z^{-\ell} \quad & \Downarrow \\
X(z) &= \mathcal{Z}[x] = \sum_{\ell=-\infty}^{\infty} x(\ell) z^{-\ell}
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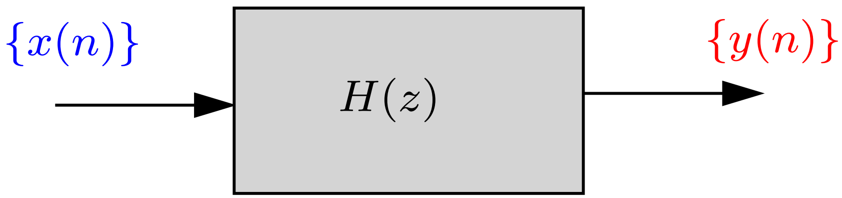
Existence : The z transform is only defined for those values of $z \in \mathbb{C}$ for which the series converges. These values determine the Region of Convergence (ROC).

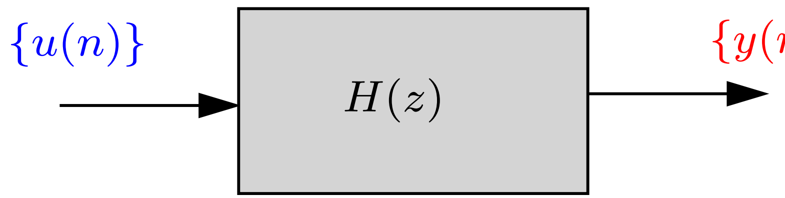
z-transform of an input-output relation



Let $H[\cdot]$ be a linear, time-invariant system with impulse response $\{h(n)\}$, let $\{x(n)\}$ be an arbitrary input then the z-transform of the output (assuming all z-transforms exist) satisfies:

$$Y(z) = H(z)X(z)$$

Stability and Causality of LTI systems



Property 3 - Stability: System $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$ is *stable* if

$$|z| = 1 \subset \mathrm{ROC} \Leftrightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Property 4 - Causality, anti-causality: A stable mixed causal, anti-causal LTI system $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$ can be split into:

$$H(z) = \underbrace{\sum_{n=1}^{\infty} h(-n)z^{n}}_{\text{anti-causal} \quad \left([\mathrm{H(z)}]_{-}\right)} + \underbrace{\sum_{n=0}^{\infty} h(n)z^{-n}}_{\text{causal} \quad \left([\mathrm{H(z)}]_{+}\right)}$$

each having its particular ROC (see example next).

Example of the causal, anti-causal interpretation

A causal system gives a bounded output (for a bounded input) by the (*forward recursion*):

$$\begin{aligned} y(n) &= 0.9y(n-1) + x(n-1) \xrightarrow{\mathcal{Z}} \\ Y(z) &= \frac{z^{-1}}{1 - 0.9z^{-1}} X(z) \end{aligned}$$

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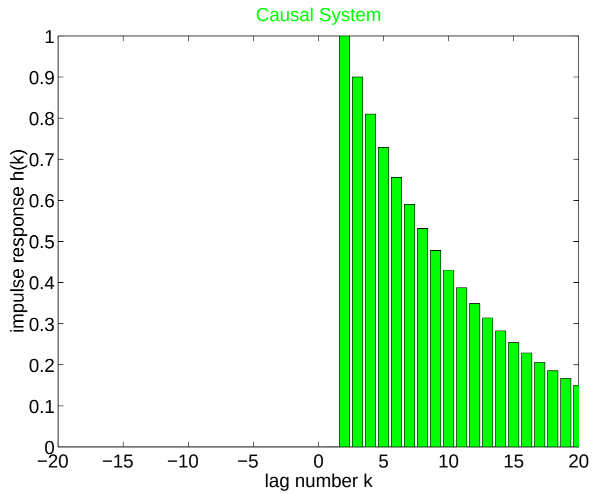
$$= \left(z^{-1} + 0.9z^{-2} + 0.81z^{-3} + \cdots\right) X(z)$$



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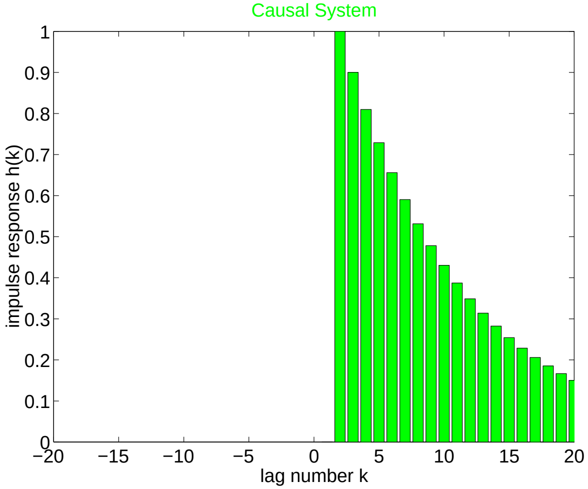

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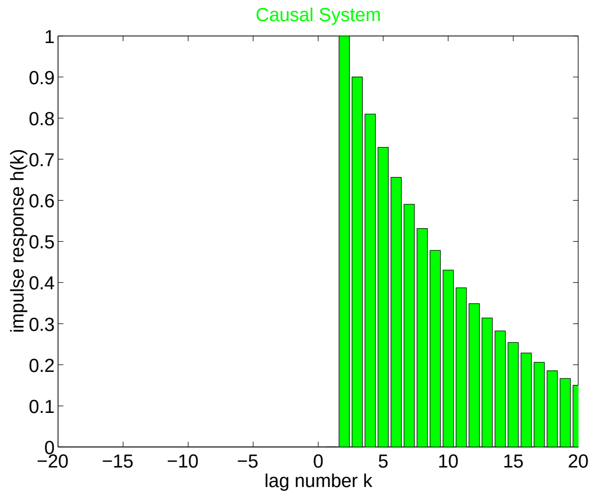

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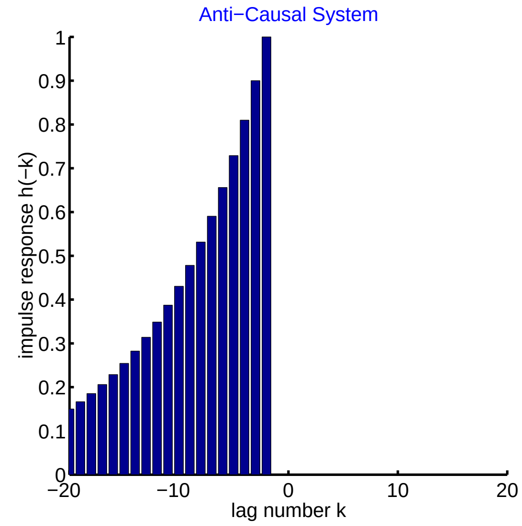

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$$\begin{aligned} \mathrm{ROC}_{\mathrm{causal}} &: \quad |0.9z^{-1}| < 1 \Leftrightarrow |z| > 0.9 \\ \mathrm{ROC}_{\mathrm{acausal}} &: \quad |0.9z| < 1 \Leftrightarrow |z| < \frac{1}{0.9} \end{aligned}$$

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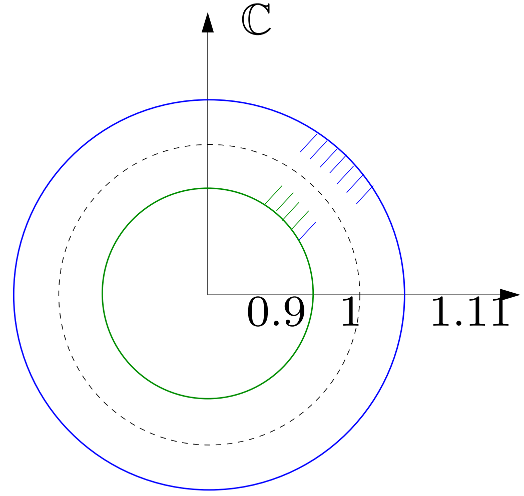

ROC$\left(\left[\frac{z^{-1}}{1-0.9z^{-1}} + \frac{z}{1-0.9z}\right]\right)$

Linear Filtering

1. Signals and Systems recap
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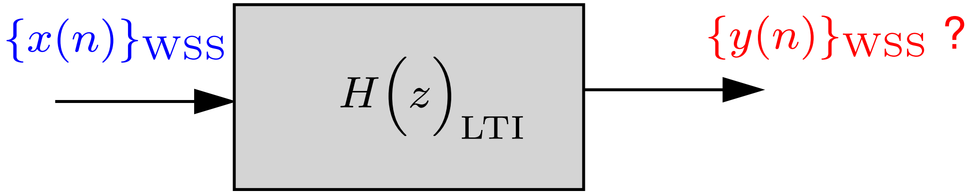
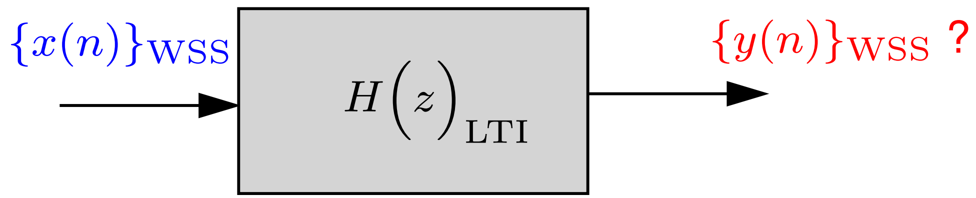
Preservation of WSS for $H(z)$ stable



$$y(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell)$$

- 1 - First requirement:

Preservation of WSS for $H(z)$ stable



$$y(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell)$$

- 1 - First requirement:

$$\begin{aligned} E[y(n)] &= \sum_{\ell=-\infty}^{\infty} h(\ell)E[x(n-\ell)] = \sum_{\ell=-\infty}^{\infty} h(\ell)\mu_x \\ &= \left(H(z)\Big|_{z=1} \right)\mu_x = H(1)\mu_x \end{aligned}$$

Provided $H(z)|_{z=1}$ exists (is bounded), it can be concluded that $E[y(n)]$ is constant (not dependent on time n).

Preservation of WSS for $H(z)$ stable (II)

- 2 - Second requirement:

$$r_y(n, n-k) = E[y(n)y^*(n-k)] \quad \text{not dependent on } n?$$

Preservation of WSS for $H(z)$ stable (II)

- 2 - Second requirement:

$$r_y(n, n-k) = E[y(n)y^*(n-k)] \quad \text{not dependent on } n?$$

$$\begin{aligned}
&= E\Big[\sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell) \sum_{p=-\infty}^{\infty} h^*(p)x^*(n-k-p)\Big] \\
&= \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(\ell)h^*(p)E[x(n-\ell)x^*(n-k-p)] \\
&= \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(\ell)h^*(p)r_x(n-\ell, n-k-p)
\end{aligned}$$

Preservation of WSS for $H(z)$ stable (II)

- 2 - Second requirement:

$$r_y(n, n-k) = E[y(n)y^*(n-k)] \quad \text{not dependent on } n?$$

$$\begin{aligned}
&= E\Big[\sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell) \sum_{p=-\infty}^{\infty} h^*(p)x^*(n-k-p)\Big] \\
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&= \sum_{\ell=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} h(\ell)h^*(p)r_x(n-\ell, n-k-p)
\end{aligned}$$

Therefore, if $x(n)$ WSS, $r_x(n-\ell, n-k-p) = r_x(k+p-\ell)$ we have that $r_y(n, n-k)$ is independent of n.

Linear Filtering

1. Signals and Systems recap
2. Preservation of WSS by LTI filtering
3. **Cross- and Auto correlation function/spectra due to LTI filtering**
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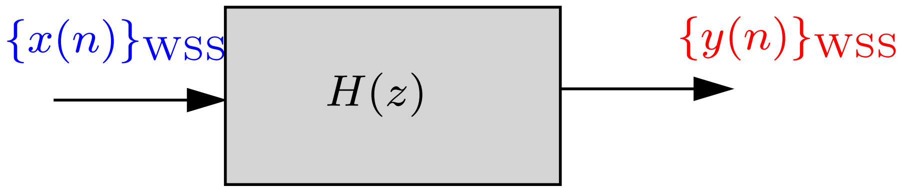
Use of z-transform

Given a WSS RP $\{x(n)\}$ with auto-covariance function $\{r_x(k)\}_{k=-\infty}^{\infty}$, then its z-transform is given as:

$$P_x(z) = \mathcal{Z}\left[r_x\right](z) = \sum_{k=-\infty}^{\infty} r_x(k) z^{-k}$$

Its power spectrum is $P_x(e^{j\omega})$.

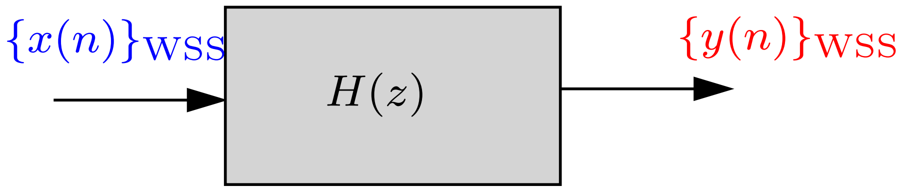
Cross-correlation function



$$y(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell)$$

$$r_{yx}(k) \quad = \quad E[y(n)x^*(n-k)]$$

Cross-correlation function



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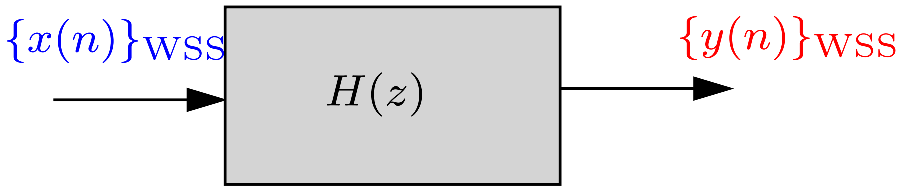
$$\begin{aligned} r_{yx}(k) &= E[y(n)x^*(n-k)] \\ &= \sum_{\ell=-\infty}^{\infty} h(\ell)E[x(n-\ell)x^*(n-k)] \end{aligned}$$

Cross-correlation function



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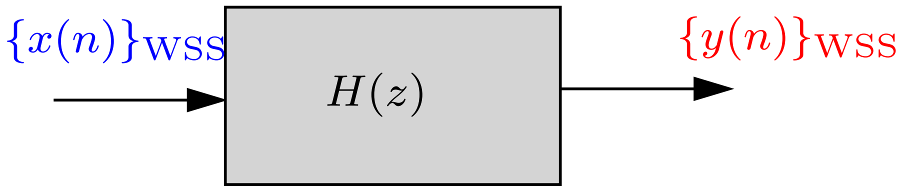

$$y(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell)$$

$$P_{yx}(z) = \sum_{k=-\infty}^{\infty} \underbrace{\sum_{\ell=-\infty}^{\infty} h(\ell)r_x(k-\ell)}_{r_{yx}(k)} z^{-}$$

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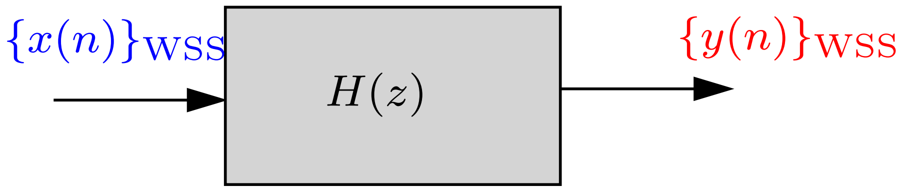

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$$= \sum_{\ell=-\infty}^{\infty} h(\ell)\mathcal{Z}\Big[r_x(\cdot-\ell)\Big](z)$$

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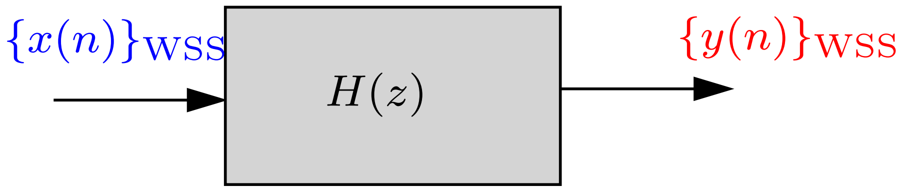

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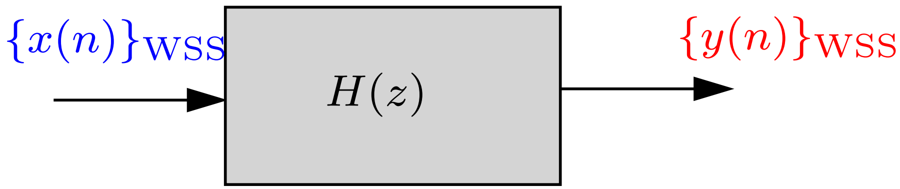
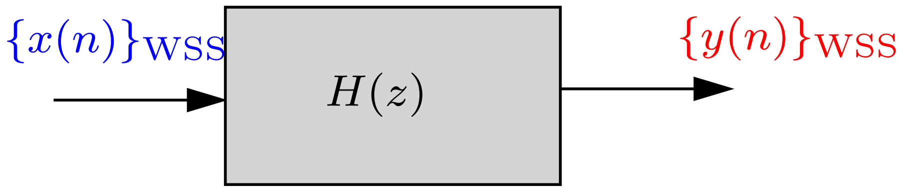
Cross-correlation function



$$y(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell)$$

$$\begin{aligned} r_{xy}(k) &= E[x(n+k)y^*(n)] \\ &= \sum_{\ell=-\infty}^{\infty} h^*(\ell)E[x(n+k)x^*(n-\ell)] \\ &= \sum_{\ell=-\infty}^{\infty} h^*(\ell)r_x(k+\ell) \end{aligned}$$

Cross-correlation function

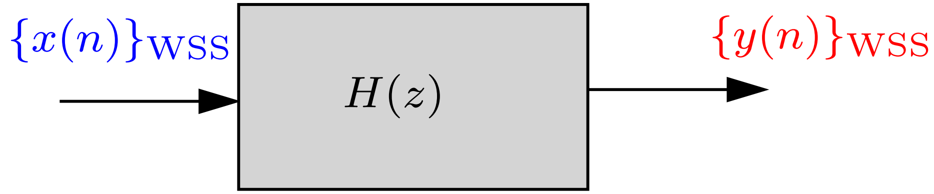


$$y(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell)$$

$$\begin{aligned} r_{xy}(k) &= E[x(n+k)y^*(n)] \\ &= \sum_{\ell=-\infty}^{\infty} h^*(\ell)E[x(n+k)x^*(n-\ell)] \\ &= \sum_{\ell=-\infty}^{\infty} h^*(\ell)r_x(k+\ell) \end{aligned}$$

$$\begin{aligned} P_{xy}(z) &= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} h^*(\ell)r_x(k+\ell)z \\ &= \sum_{\ell=-\infty}^{\infty} h^*(\ell)\mathcal{Z}\Big[r_x(\cdot+\ell)\Big](z) \\ &= \sum_{\ell=-\infty}^{\infty} h^*(\ell)z^{\ell}P_x(z) \\ &= H^*(1/z^*)P_x(z) \end{aligned}$$

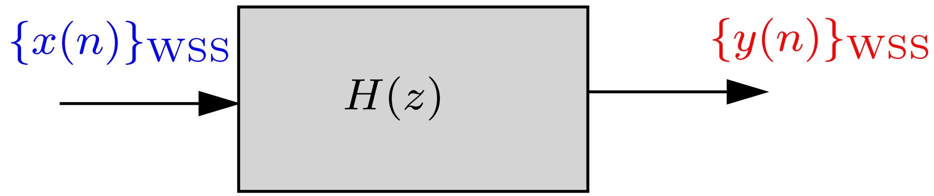
Correlation function and Power Spectra



$$r_y(k) = \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(\ell) h^*(p) r_x(k+p-\ell)$$

Therefore, $P_y(z)$ equals,

Correlation function and Power Spectra

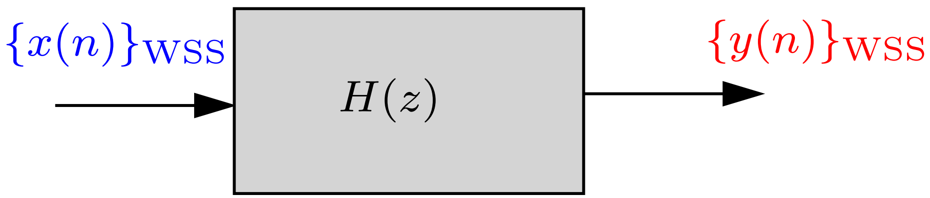


$$r_y(k) = \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(\ell) h^*(p) r_x(k+p-\ell)$$

Therefore, $P_y(z)$ equals,

$$\begin{aligned}
P_y(z) &= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(\ell) h^*(p) r_x(k+p-\ell) z^{-k} \\
&= \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(\ell) h^*(p) \;\; \mathcal{Z}\Big[r_x\big(\cdot - (\ell - p)\big)\Big](z) \\
&= \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(\ell) z^{-\ell} \;\; h^*(p) z^{p} \;\; P_x(z) \\
&= H(z) H^*(1/z^*) P_x(z)
\end{aligned}$$

Summary Cross- and Auto (Power) Spectra

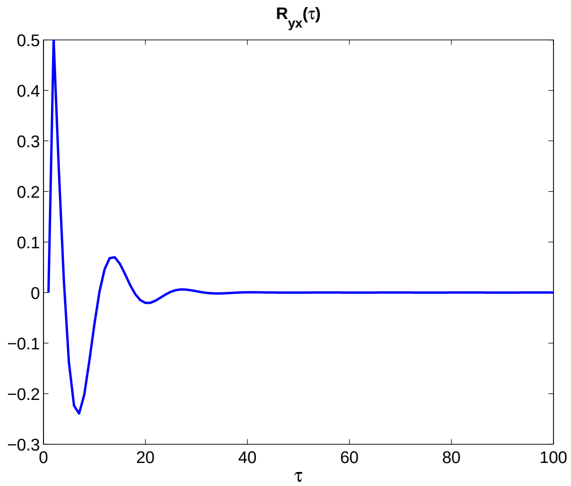


$$
\begin{aligned}
P_{yx}(z) &= H(z)P_x(z) \\
P_{xy}(z) &= H^*(1/z^*)P_x(z) \\
P_y(z) &= H(z)H^*(1/z^*)P_x(z)
\end{aligned}
$$

Linear Filtering

1. Signals and Systems recap
2. Preservation of WSS by LRI filtering
3. Cross- and Auto correlation function/spectra due to LTI filtering
4. **Applications**

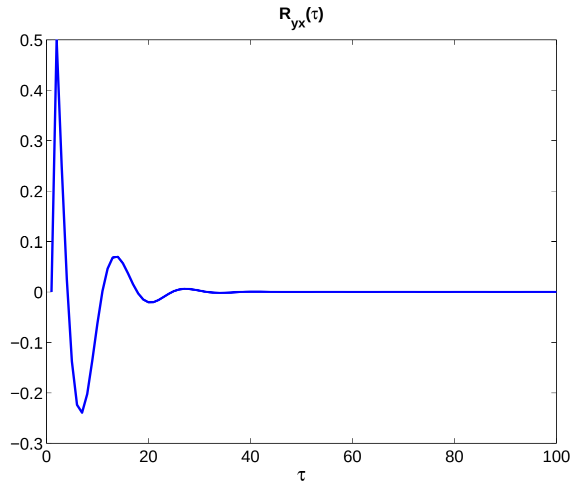
Application 1: Estimating the Impulse Response



$$r_{yx}(k) = g(k) \star r_x(k)$$

Experimental conditions: $x(n)$ ZMWN with $\sigma_x^2 = 1$.

Application 1: Estimating the Impulse Response



$$r_{yx}(k) = g(k) \star r_x(k)$$

Experimental conditions: $x(n)$ ZMWN with $\sigma_x^2 = 1$. Therefore $r_{yx}(k) = g(k)$.

Consequences:

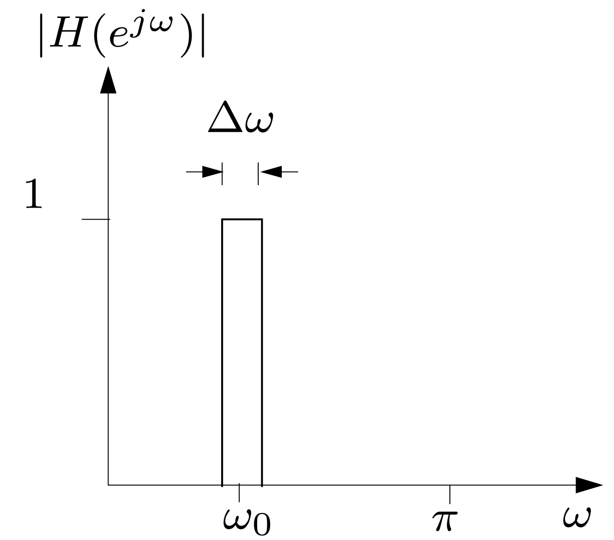
- $P_{yx}(e^{j\omega}) = G(e^{j\omega})$
- $P_y(e^{j\omega}) = |G(e^{j\omega})|^2$

Application 2: Proving Positivity of the Power Spectrum

The Power Spectrum of a WSS RP $x(n)$ satisfies, $\boxed{P_x(e^{j\omega}) \geq 0}$

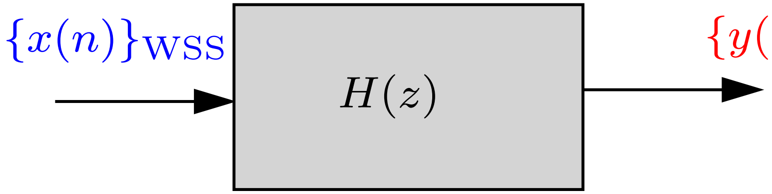
In order to show this, consider $y(n)$ the output of a LTI filter with magnitude response show in the figure, such that $y(n)$ is WSS. Then following property 3 of the Power spectrum,

$$\begin{aligned} E[|y(n)|^2] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_y(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 P_x(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{\omega_0-\frac{\Delta\omega}{2}}^{\omega_0+\frac{\Delta\omega}{2}} P_x(e^{j\omega})d\omega \geq 0 \end{aligned}$$



And this for all ω_o and $\Delta\omega \Rightarrow P_x(e^{j\omega}) \geq 0$.

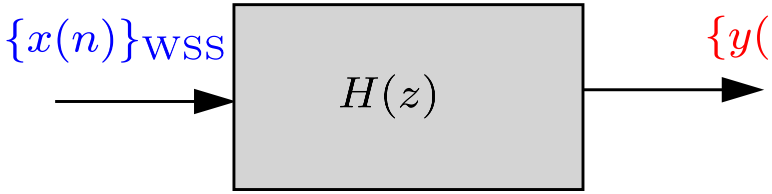
Application 3: Updating the cross-correlation function/spectra



Given: $P_{\boxed{\text{y}}x}(z)\Big(= H(z)P_x(z)\Big)$ and $U(z) = G(z)Y(z)$.

Determine: $P_{\boxed{\text{u}}x}(z)$

Application 3: Updating the cross-correlation function/spectra



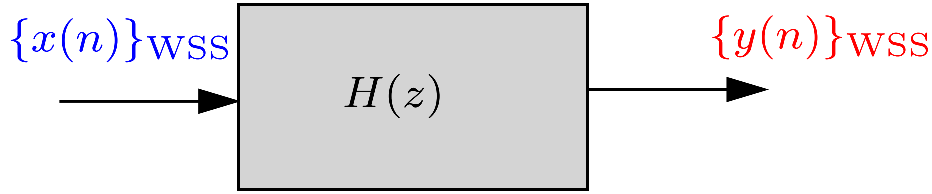
Given: $P_{\boxed{\text{y}}x}(z)\Big(=H(z)P_x(z)\Big)$ and $U(z)=G(z)Y(z)$.

Determine: $P_{\boxed{\text{u}}x}(z)$

Looking for a relationship $U(z) =? \quad X(z)$.

$$\begin{aligned} U(z) &= G(z)Y(z) \\ &= G(z)H(z)X(z) \\ \Rightarrow P_{ux}(z) &= G(z)H(z)P_x(z) \\ &= G(z)P_{yx}(z) \end{aligned}$$

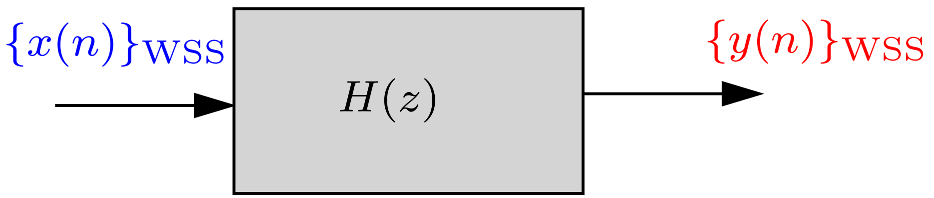
Application 3: Updating the cross-correlation function/spectra



Given: $P_{y\boxed{x}}(z)\Big(=H(z)P_x(z)\Big)$ and $V(z)=G(z)X(z)$.

Determine: $P_{y\boxed{v}}(z)$.

Application 3: Updating the cross-correlation function/spectra



Given: $P_{y\boxed{x}}(z)\left(= H(z)P_x(z)\right)$ and $V(z) = G(z)X(z)$.

Determine: $P_{y\boxed{v}}(z)$. From the previous result we known,

$$P_{vy}(z) = G(z)P_{xy}(z)$$

In order to make use of the given information $P_{yx}(z)$

Application 3: Updating the cross-correlation function/spectra

$$\{x(n)\}_{\text{WSS}} \longrightarrow \boxed{H(z)} \longrightarrow \{y(n)\}_{\text{WSS}}$$

Given: $P_{y\boxed{x}}(z)\Big(=H(z)P_x(z)\Big)$ and $V(z)=G(z)X(z)$.

Determine: $P_{y\boxed{v}}(z)$. From the previous result we known,

$$P_{vy}(z)=G(z)P_{xy}(z)$$

In order to make use of the given information $P_{yx}(z)$ we take the complex conjugate of both sides and change the argument z by $1/z^*$. This yields,

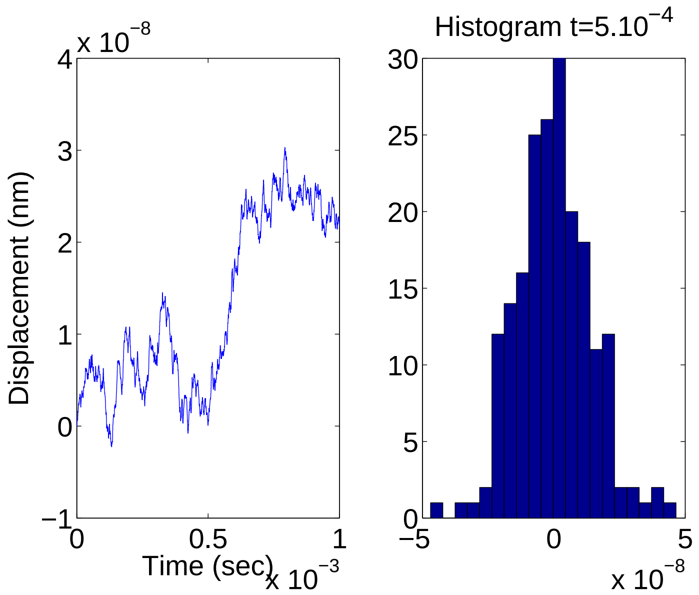
$$P^*_{vy}(1/z^*)=G^*(1/z^*)P^*_{xy}(1/z^*)$$

Using the property of the Cross-correlation spectrum:

$$P_{yv}(z)=G^*(1/z^*)P_{yx}(z)$$

Example 1: Brownian motion

A single realization of a free Brownian particle and the Histogram at time $0.5\mu s$.



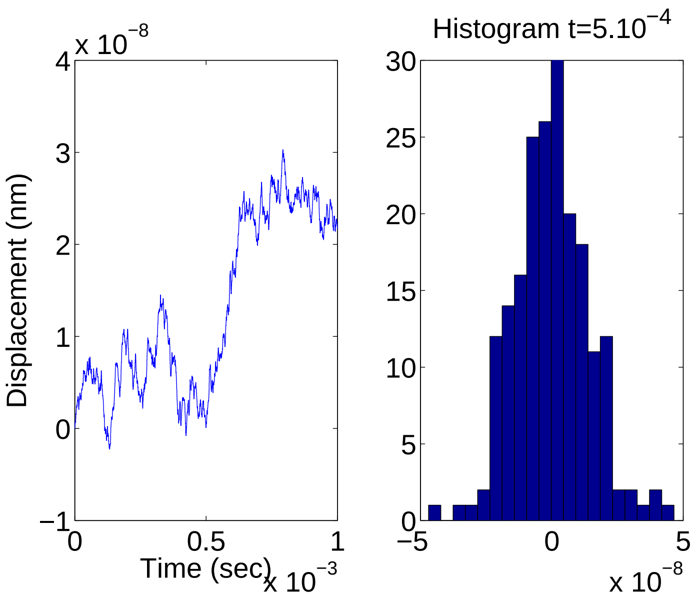
The simulation equation to generate one such realization of the displacement reads:

$$x(n) = a(1)x(n-1) + a(2)x(n-2) + b(0)v($$

with $v(n)$ ZMWN.

Example 1: Brownian motion

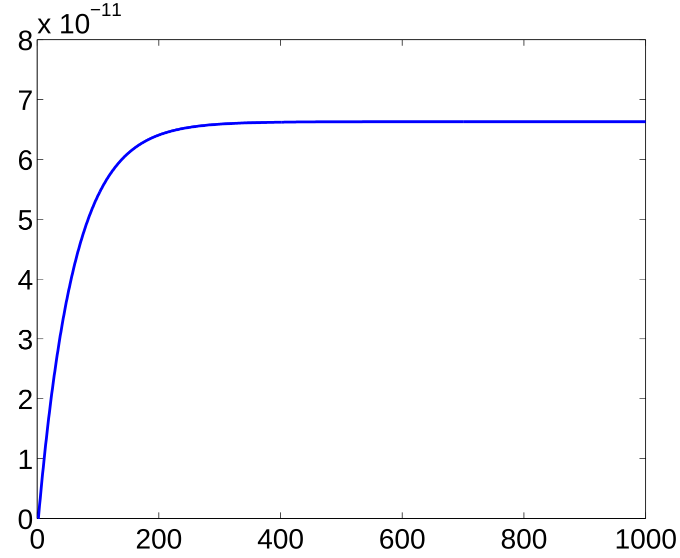
A single realization of a free Brownian particle and the Histogram at time $0.5\mu s$.



The simulation equation to generate one such realization of the displacement reads:

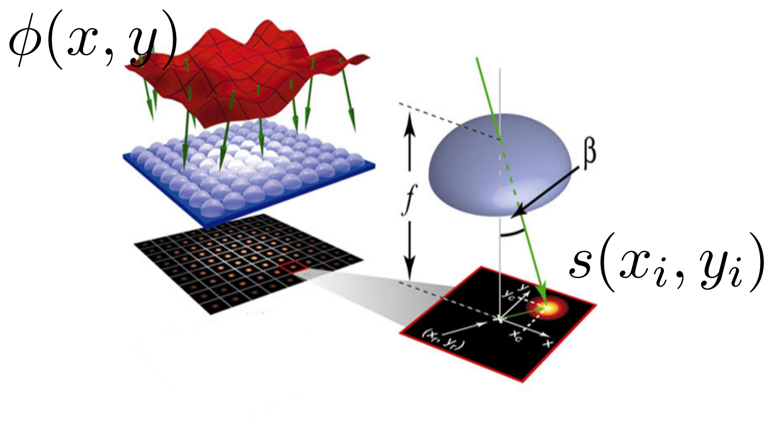
$$x(n) = a(1)x(n-1) + a(2)x(n-2) + b(0)v(n$$

with $v(n)$ ZMWN. The impulse response of this system is.



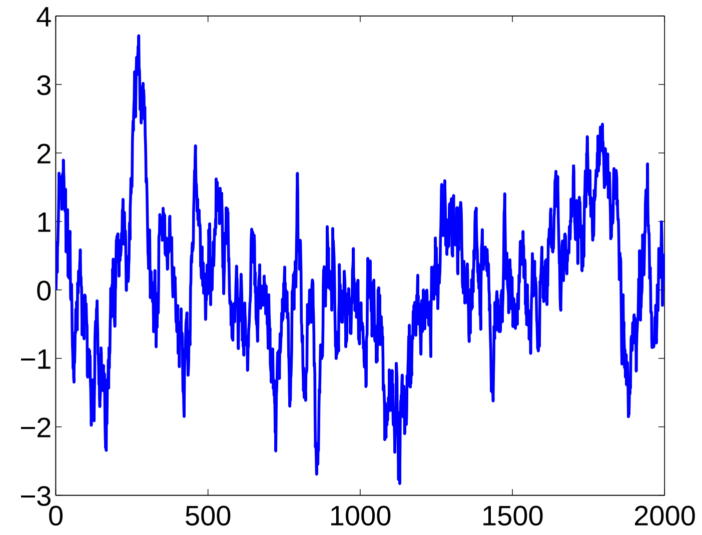
Example 2: Verhaegen's CSI lab

Schematic Shack-Hartmann Sensor



[From M. Konnik, 2010]

A single realization of the spot x-displacement



A signal (based) model (see in 2 lectures):

$$\begin{aligned} x(n) &= a(1)x(n-1) + a(2)x(n-2) + \\ &\quad b(1)v(n-1) + b(1)v(n-2) \end{aligned}$$

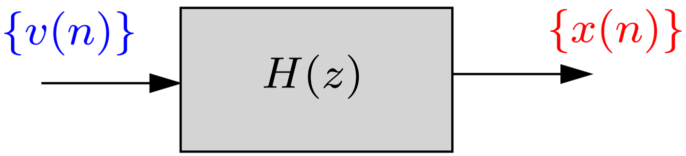
with $v(n)$ ZMWN. Stability? `testARMA.m`

Part II: ARMA Filtering Random Processes

1. **Definition ARMA - AR - MA models**
2. Calculation of Power Spectra
3. Calculation of Auto- and Cross Correlation functions
4. Harmonic Processes
5. Illustrative Examples

The AutoRegressive Moving Average (ARMA) model

Consider filtering zwmn $v(n)$ by the LSI filter $H(z)$ with $E[v(n)^2] = \sigma_v^2$.



When $H(z)$ is of type ARMA(p,q) then the transfer function is given as:

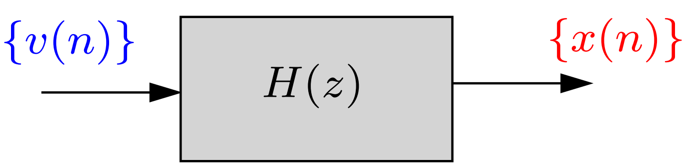
$$H(z) = \frac{\sum_{k=0}^{q} b(k) z^{-k}}{1 + \sum_{k=1}^{p} a(k) z^{-k}} \quad p \geq 1, q \geq 0$$

This corresponds to the following difference equation:

$$\begin{aligned} x(n) + a(1)x(n-1) + \cdots + a(p)x(n-p) &= \\ b(0)v(n) + b(1)v(n-1) + \cdots + b(q)v(n-q) & \end{aligned}$$

The AutoRegressive (AR) model

Consider filtering zwmn $v(n)$ by the LSI filter $H(z)$ with $E[v(n)^2] = \sigma_v^2$.



An AR(p) model is an ARMA(p,0) and has transfer function is given as:

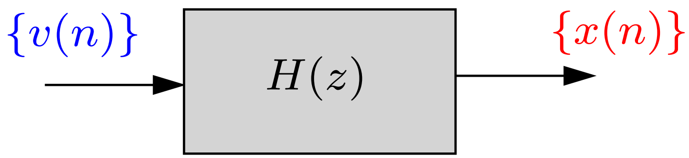
$$H(z) = \frac{b(0)}{1 + \sum_{k=1}^{p} a(k) z^{-k}} \quad p \geq 1$$

This corresponds to the following difference equation:

$$x(n) + a(1)x(n-1) + \cdots + a(p)x(n-p) = b(0)v(n)$$

The Moving Average (MA) model

Consider filtering zwmn $v(n)$ by the LSI filter $H(z)$ with $E[v(n)^2] = \sigma_v^2$.



An MA(q) model is an ARMA(0,q) and has transfer function is given as:

$$H(z) = \sum_{k=0}^{q} b(k) z^{-k} \quad q \geq 0$$

This corresponds to the following difference equation:

$$x(n) = b(0)v(n) + b(1)v(n-1) + \cdots + v(q)v(n-q)$$

Part II:

1. Definition ARMA - AR - MA models
2. **Calculation of Power Spectra**
3. Calculation of Auto- and Cross Correlation functions
4. Harmonic Processes
5. Illustrative Examples

The AutoRegressive Moving Average (ARMA) model

Consider filtering zwmn $v(n)$ by the LSI filter $H(z)$ with $E[v(n)^2] = \sigma_v^2$.

$$\{v(n)\} \longrightarrow \boxed{H(z)} \longrightarrow \{x(n)\}$$

General Rule: With $H(z)$ assumed to be stable, the Power spectrum of x is given as:

$$P_x(z) = H(z)H^*(1/z^*)\sigma_v^2$$

and for $z = e^{j\omega}$ this is,

$$P_x(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_v^2$$

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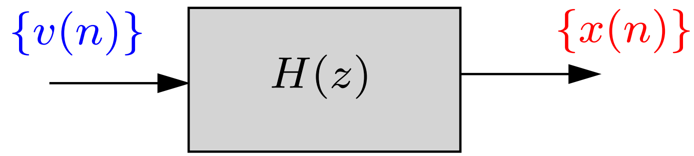

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ARMA(p,q): $H(z) = \frac{\sum_{k=0}^{q} b(k)z^{-k}}{1+\sum_{k=1}^{p} a(k)z^{-k}} = \frac{B_q}{A_p}$

$$P_x(z) = \frac{B_q(z)B_q^*(1/z^*)}{A_p(z)A_p^*(1/z^*)}\sigma_v^2$$

Conclusion about poles/zeros of $P_x(z)$?

The AutoRegressive Moving Average (ARMA) model

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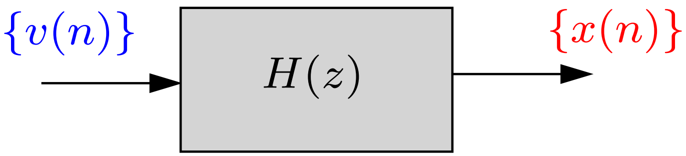

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ARMA(p,q): $H(z) = \frac{\sum_{k=0}^{q} b(k)z^{-k}}{1+\sum_{k=1}^{p} a(k)z^{-k}} = \frac{B_q}{A_p}$

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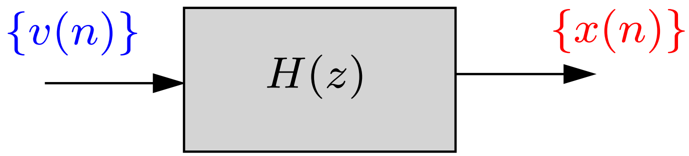
Conclusion about poles/zeros of $P_x(z)$?

$$P_x(e^{j\omega}) = \frac{|B_q(e^{j\omega})|^2}{|A_p(e^{j\omega})|^2}\sigma_v^2$$

ARMAPow.m

The AutoRegressive (AR) model

Consider filtering zwmn $v(n)$ by the LSI filter $H(z)$ with $E[v(n)^2] = \sigma_v^2$.



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and for $z = e^{j\omega}$ this is,

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AR(p): $H(z) = \frac{1}{1+\sum_{k=1}^{p} a(k)z^{-k}} = \frac{1}{A_p(z)}$,

$$P_x(z) = \frac{1}{A_p(z)A_p^*(1/z^*)}\sigma_v^2$$

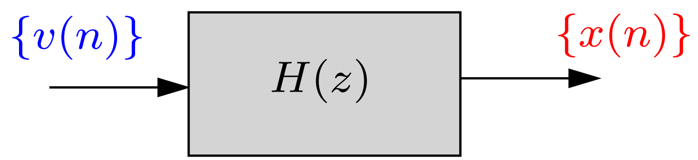
Conclusion about poles/zeros of $P_x(z)$?

$$P_x(e^{j\omega}) = \frac{1}{|A_p(e^{j\omega})|^2}\sigma_v^2$$

`AR1Pow.m` `AR2Pow.m`

The Moving Average (MA) model

Consider filtering zwmn $v(n)$ by the LSI filter $H(z)$ with $E[v(n)^2] = \sigma_v^2$.



General Rule: With $H(z)$ assumed to be stable, the Power spectrum of x is given as:

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and for $z = e^{j\omega}$ this is,

$$P_x(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_v^2$$

MA(q): $H(z) = \sum_{k=0}^{q} b(k)z^{-k}$

$$P_x(z) = B_q(z)B_q^*(1/z^*)\sigma_v^2$$

Conclusion about poles/zeros of $P_x(z)$?

$$P_x(e^{j\omega}) = |B_q(e^{j\omega})|^2\sigma_v^2$$

`MAPow.m`

Part II:

1. Definition ARMA - AR - MA models
2. Calculation of Power Spectra
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The time-domain model:

$$x(n) + \sum_{\ell=1}^{p} a(\ell)x(n-\ell) = \sum_{\ell=0}^{q} b(\ell)v(n-\ell) \quad v(n) \frown \text{ZMWN}(\sigma_v^2)$$

Procedure:

Step 1: Check WSS?

The AutoRegressive Moving Average (ARMA) model

The time-domain model:

$$x(n) + \sum_{\ell=1}^{p} a(\ell)x(n-\ell) = \sum_{\ell=0}^{q} b(\ell)v(n-\ell) \quad v(n) \frown \text{ZMWN}(\sigma_v^2)$$

Procedure:

Step 1: Check WSS? ($H(z)$ is stable).

Step 2: The Yule-Walker Equations:

$$r_x(n) + \sum_{\ell=1}^{p} a(\ell)r_x(n-\ell) = \sigma_v^2 \sum_{\ell=n}^{q} b(\ell)h^*(\ell-n)$$

with $H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$.

Example 3: Illustration for ARMA(1,1) - real case

Given: $H(z) = \frac{1-0.5z^{-1}}{1-0.8z^{-1}}$ and $H(z) = \sum_{m=0}^{\infty} h(m) z^{-m}$ Then it holds,

$$\frac{1-0.5z^{-1}}{1-0.8z^{-1}} = 1 + 0.3z^{-1} + 0.24z^{-2} + \cdots$$

Therefore the Yule-Walker Eq. yields for $\boxed{n = 0}$ ($\sigma_v = 1$):

$$r_x(0) - 0.8 r_x(-1) = h(0) - 0.5h(1) \Rightarrow r_x(0) - 0.8 r_x(1) = h(0) - 0.5h(1)$$

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For $\boxed{n = 1}$: $r_x(1) - 0.8 r_x(0) = 0 - 0.5h(0)$.

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For $\boxed{n=1}$: $r_x(1) - 0.8r_x(0) = 0 - 0.5h(0)$.

$$\begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix} \begin{bmatrix} r_x(0) \\ r_x(1) \end{bmatrix} = \begin{bmatrix} 1 - .15 \\ -0.5 \end{bmatrix} \quad \Rightarrow \quad r_x(0) = \frac{5}{4} \text{ and } r_x(1) = \frac{1}{2}$$

Example 3: Illustration for ARMA(1,1) - real case

Given: $H(z) = \frac{1-0.5z^{-1}}{1-0.8z^{-1}}$ and $H(z) = \sum_{m=0}^{\infty} h(m)z^{-m}$ Then it holds,

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$$r_x(0) - 0.8r_x(-1) = h(0) - 0.5h(1) \Rightarrow r_x(0) - 0.8r_x(1) = h(0) - 0.5h(1)$$

For $\boxed{n=1}$: $r_x(1) - 0.8r_x(0) = 0 - 0.5h(0)$.

$$\begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix} \begin{bmatrix} r_x(0) \\ r_x(1) \end{bmatrix} = \begin{bmatrix} 1 - .15 \\ -0.5 \end{bmatrix} \quad \Rightarrow \quad r_x(0) = \frac{5}{4} \text{ and } r_x(1) = \frac{1}{2}$$

For $\boxed{n>1}$: $r_x(n) - 0.8r_x(n-1) = 0$ `testARMA.m`.

The time-domain model:

$$x(n) + \sum_{\ell=1}^{p} a(\ell)x(n-\ell) = b(0)v(n) \quad v(n) \frown \text{ZMWN}(\sigma_v^2)$$

Follows as a special case from the ARMA(p,0)

$$q = 0 \quad \text{and} \quad h(0) = b(0)$$

the Yule Walker equation for $r_x(k)$ become:

$$r_x(n) + \sum_{\ell=1}^{p} a(\ell)r_x(n-\ell) = \sigma_v^2|b(0)|^2\Delta(n) \quad n \geq 0$$

The Moving Average (MA) model (real-case only)

The time-domain model:

$$x(n) = \sum_{\ell=0}^{q} b(\ell) v(n-\ell) \quad v(n) \frown \text{ZMWN}(\sigma_v^2)$$

Follows as a special case from the ARMA(0,q) calculations with the special form that,

$$h(\ell) = b(\ell) \quad 0 \leq \ell \leq q$$

and therefore the Yule-Walker equations become:

$$r_x(n) = \sigma_v^2 \sum_{\ell=n}^{q} b(\ell) b^*(\ell - n) \quad 0 \leq n \leq q$$

the conjugate symmetric part follows from $r_x(-n) = r_x^*(n)$.

Example 4: Illustration for ARMA(0,3) - real case

Given: $H(z) = b(0) + b(1)z^{-1} + b(2)z^{-2} + b(3)z^{-3}$

Then ($\sigma_v = 1$):

$$
\begin{aligned}
r_x(0) &= |b(0)|^2 + |b(1)|^2 + |b(2)|^2 \\
r_x(1) &= b^*(0)b(1) + b^*(1)b(2) \\
r_x(2) &= b^*(0)b(2)
\end{aligned}
$$

Important Remark: The inverse problem to derive the filter coefficients from the Auto-correlation samples is non-linear!

Part II:

1. Definition ARMA - AR - MA models
2. Calculation of Power Spectra
3. Calculation of Auto- and Cross Correlation functions
4. **Harmonic Processes**
5. Illustrative Examples

Example 4: Harmonic Process

When $x(n)$ is a WSS harmonic process:

$$x(n) = Asin(n\omega_0 + \phi)$$

with A, ϕ uncorrelated random variables with ϕ uniformly distributed, then (see Lecture 4)

$$r_x(k) = \frac{E[A^2]}{2} cos(k\omega_0) \Rightarrow P_x(e^{j\omega}) = \frac{\pi E[A^2]}{2} \left[\Delta(\omega - \omega_0) + \Delta(\omega + \omega_0) \right]$$

`PeriodBias.m`

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`PeriodBias.m`

Remark: By the linearity of the $E[.]$-operator this can easily be extended to the finite sum of harmonic processes.

Next steps forward to improve your chances to succeed ...

Instruction session for explanation of the abstract notions and getting hands-on-exerpience!

Preparation:

Study Chapter 6 (6.1-6.4) (and Chapter 2)

Next Instruction/lecture see Course Overview