Statistical Signal Processing

Lecture 4: Inverse Problem From Power Spectrum/Autocorrelation to Simulation Model

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Overview

INVERSE Problems - Part I: Frequency Domain

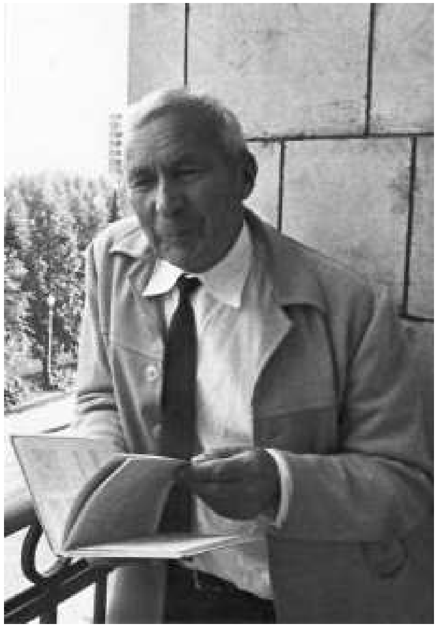
- From Power Spectra (Frequency domain) to generating a stochastic process
- Chapter 7.1-7.3

INVERSE Problems - Part II: Time Domain

- From Autocorrelation (Time domain) to generating a stochastic process
- Chapter 7.4

Example: Kolmogorov Turbulence Model

"A Mathematician doing physics"



"... combine theoretical studies with the analysis of experimental results ..."

Example: Kolmogorov Turbulence Model

"A Mathematician doing physics"

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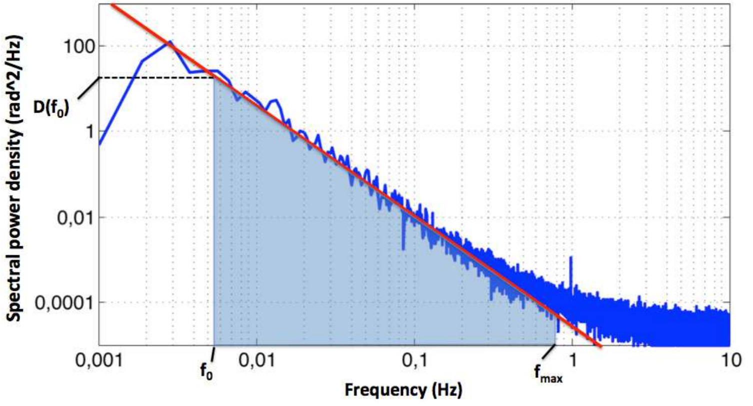
Work on Turbulence modeling based on Experimental results (Power Spectra):



$$P_\phi(f) = 0.023 r_0^{-\frac{5}{3}} f^{-\frac{11}{3}}$$

with r_0 - the Fried parameter
and f - the spatial frequency

Summary Lecture 3

Example Lecture 3: We work with Rational Transfer functions, such as:

$$H(z) = \underbrace{\frac{z}{1-0.9z}}_{\text{anti-causal} \quad \left([\text{H(z)}]_-\right)} + \underbrace{\frac{z^{-1}}{1-0.9z^{-1}}}_{\text{causal} \quad \left([\text{H(z)}]_+\right)} \qquad \text{ROC}\Big(H(z)\Big) \supset \Gamma$$

with the pole(s) $z = \frac{1}{0.9}$ of $[H(z)]_-$ outside the unit circle and the pole(s) $z = 0.9$ of $[H(z)]_+$ inside the unit circle.

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ARMA(p,q) Lecture 3:

$$\{v(n)\}_{\mathrm{ZMWN}(\sigma_v^2)} \longrightarrow \boxed{\frac{B(z)}{A(z)}} \longrightarrow \{x(n)\}$$

Then the Power spectrum $P_x(z)$ equals:

$$P_x(z) = \sigma_v^2 \frac{B_q(z)B_q^*(1/z^*)}{A_p(z)A_p^*(1/z^*)} \quad \text{Property}: \ P_x(z) = P_x^*(1/z^*)$$

For example $P_x(z)$ given as $\frac{1-2.5z^{-1}+z^{-2}}{1-2.05z^{-1}+z^{-2}}$.

A simple formulation of our first inverse problem

Given a “Power” spectrum as a rational function:

$$P_x(z) = \frac{1 - 2.5z^{-1} + z^{-2}}{1 - 2.05z^{-1} + z^{-2}}$$

Can you determine an ARMA model (including σ_v) with transfer function:

$$H(z) = \frac{B_q(z)}{A_p(z)}$$

such that,

$$P_x(z) = \sigma_v^2 \frac{B_q(z) B_q^*(1/z^*)}{A_p(z) A_p^*(1/z^*)}$$

Highly non-unique?

Today's Challenge: A simplified Inverse Problem

SPECTRAL FACTORIZATION PROBLEM: Assume that (1) $P_x(z)$ is *rational* **and** (2) $P_x(z)$ is *positive real*, i.e.:

$$P_x(z) = P_x^*(1/z^*) \quad P_x(e^{j\omega}) \in \mathbb{R} \text{ and } P_x(e^{j\omega}) > 0$$

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1. $P_x(z) = \sigma Q(z) Q^*(1/z^*)$
2. $Q(z)$ is causal and stable
3. $Q(z)^{-1}$ is also causal and stable

Remark 1: The property that the inverse of $Q(z)$ is causal and stable is indicated by $Q(z)$ being *minimum phase*.

Remark 2: $Q(z)$ is **unique** if $Q(\infty) = 1$.

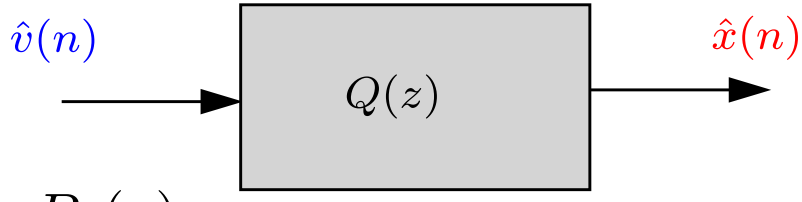
Solving the Spectral Factorization problem

1. **Use of the Spectral Factor**
2. Illustrations of Key Definitions: A rational transfer function, a minimum phase transfer function, $Q(\infty) = 1$.
3. Special form of the power spectrum $P_x(z)$.
   - Constraints on its coefficients
   - Constraints on its poles and zeros
4. A constructive Solution
5. Illustration

Use of the Spectral Factor

Given the unique spectral factorization of the Power spectrum $P_x(z) = P_x^*(1/z^*)$, as $P_x(z) = \sigma Q(z) Q^*(1/z^*)$

1. For a generated ZMWN $\hat{v}(n)$ with variance $\sigma \in \mathbb{R}_+$, we can determine $\hat{x}(n)$ as,



such that, $P_{\hat{x}}(z) = P_x(z)$

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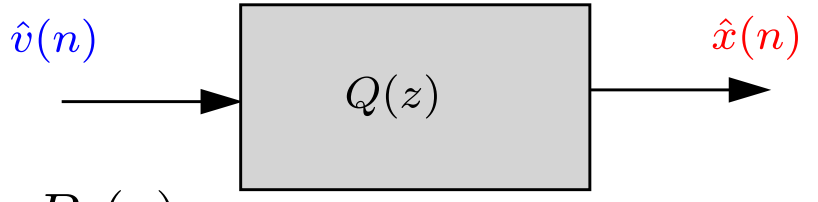

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2. If we have a realization of $x(n)$ we can use this to find the corresponding ZMWN that generated this realization,

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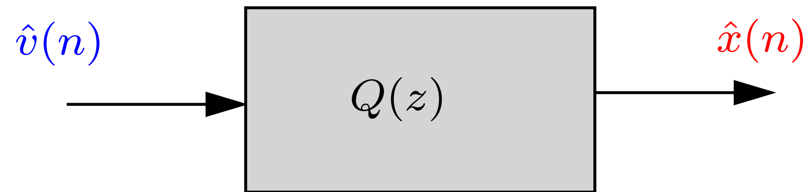

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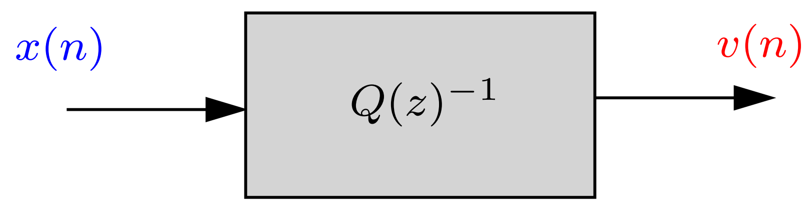

   This filter is called the “whitening” filter.

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3. Special form of the power spectrum $P_x(z)$.
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Illustrations of Key Definitions: minimum phase TF

When $Q(z)$ is a (causal, stable and) minimum-phase rational transfer function given as:

$$Q(z) = \frac{B(z)}{A(z)}$$

then

1. $Q(z)$ is causal and stable,

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Example 2: A causal, rational $Q(z)$ that is stable and minimum-phase is:

$$Q(z) = \frac{1 - 0.5z^{-1}}{1 - 0.8z^{-1}} \quad \text{zero} : z = 0.5 \quad \text{pole} : z = 0.8$$

Remark: Such $Q(z)$ in Example 2 have $Q(\infty) = 1$.

Illustrations of Key Definitions: Rational TF

When $P_x(z)$ is a rational function, it can be written as:

$$P_x(z) = \frac{N(z)}{D(z)}$$

for $N(z)$ and $D(z)$ finite order polynomials in z.

Example 1: A rational $P_x(z)$ is given as:

$$P_x(z) = \frac{1 - 2.5z^{-1} + z^{-2}}{1 - 2.05z^{-1} + z^{-2}}$$

Remarks:

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Remarks: (1) When $P_x(z)$ is a rational function in z it has *poles* and *zeros* and (2) $P_x(e^{j\omega})$ does not contain discontinuities (due to harmonics): $x(n)$ is a regular RP.

Solving the Spectral Factorization problem

1. Use of the Spectral Factor
2. Illustrations of Key Definitions: A rational transfer function, a minimum phase transfer function, $Q(\infty) = 1$.
3. **Special form of the power spectrum $P_x(z)$.**
   - Constraints on its coefficients
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Special form of the power spectrum $P_x(z)$

The special form of the Power Spectrum $P_x(z) = P_x^*(1/z^*)$ constraints its pole-zero location.

Example 3-b: Let $P_x(z) = az + b + cz^{-1}$, then $P_x(z) = P_x^*(1/z^*)$ yields $a^* = c$, $b^* = b$ and $P_x(z)$ is given as,

$$P_x(z) = az + b + a^* z^{-1} \quad b \in \mathbb{R}, a \in \mathbb{C} \qquad \texttt{rootsPolReal.m}$$

Special form of the power spectrum $P_x(z)$ (II)

LEMMA: If $P_x(z) = \frac{N(z)}{D(z)}$, $N(z), D(z)$ coprime and $P_x(z) = P_x^*(\frac{1}{z^*})$, then,

$$\text{If } p_0 \text{ is a pole of } P_x(z) \Rightarrow \frac{1}{p_0^*} \text{ is a pole of } P_x(z)$$

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Proof: Since $P_x^*(\frac{1}{z^*}) = \frac{N^*(\frac{1}{z^*})}{D^*(\frac{1}{z^*})} \Rightarrow \quad \frac{N(z)}{D(z)} \frac{D^*(\frac{1}{z^*})}{N^*(\frac{1}{z^*})} = 1$

Therefore *perfect* pole-zero cancellation is necessary. Since $N(z)$ and $D(z)$ are coprime, and the order of $N(z)$ is identical to that of $N^*(\frac{1}{z^*})$, $\Rightarrow$

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Hence,

$$\text{If } \exists z_0 \in \mathbb{C} : N(z_0) = 0 \Rightarrow N^*(\frac{1}{z_0^*}) = 0 \Rightarrow N(\frac{1}{z_0^*}) = 0$$

Summary of the Properties of $P_x(z)$

From the properties:

1. $P_x(z) = P_x^*(1/z^*) \Rightarrow$ contains poles p_0 and $1/p_0^*$ and zeros z_0 and $1/z_0^*$.
2. $P_x(e^{j\omega}) > 0$ means that $P_x(z)$ does not contain zeros on the unit circle.
3. Since ROC$\Big(P_x(z)\Big) \supset \Gamma$, $P_x(z)$ does not contain poles on the unit circle.

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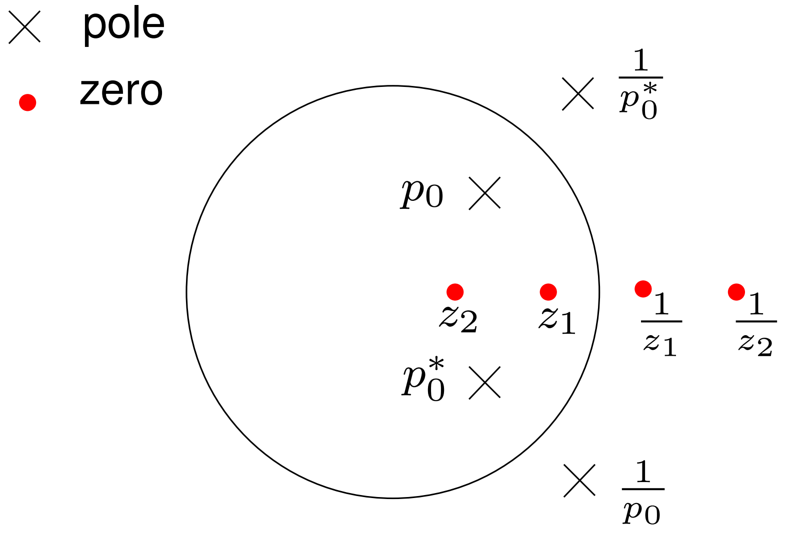
Pole-zero pattern $P_x(z)$



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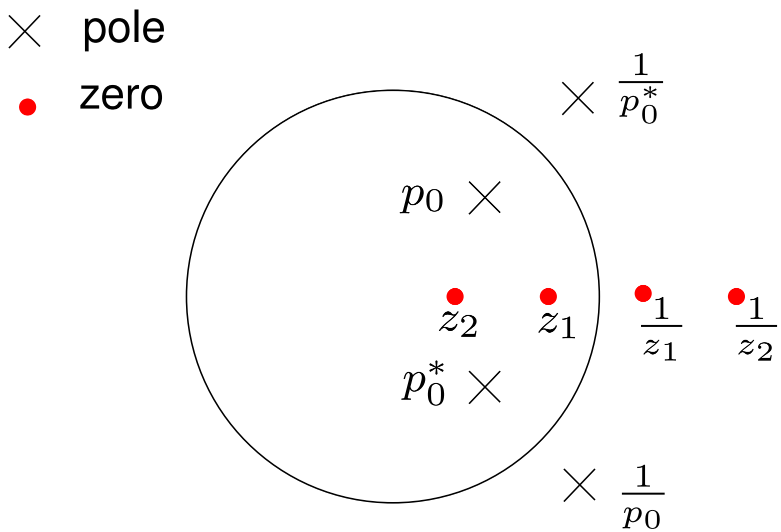
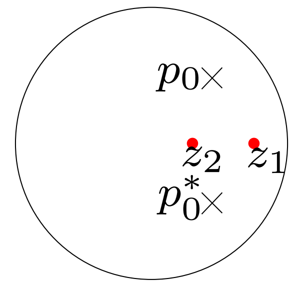


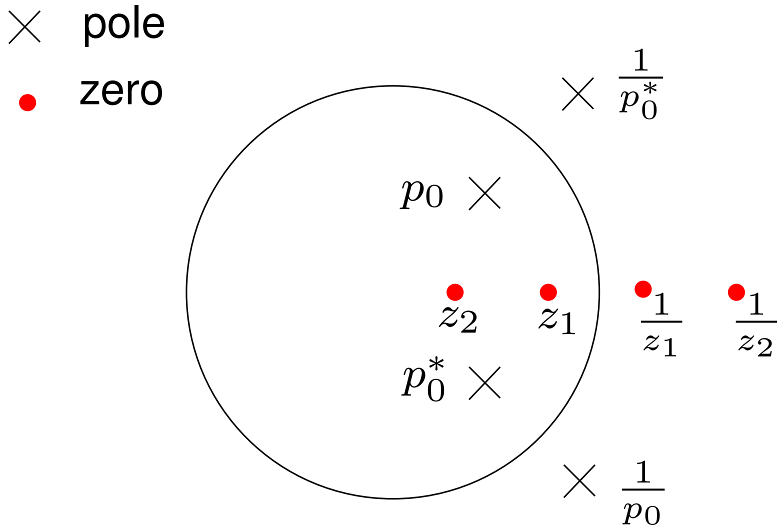
Stable pole-zero part of $P_x(z)$:



$$Q(z) = \frac{(1-z_1 z^{-1})(1-z_2 z^{-1})}{(1-p_0 z^{-1})(1-p_0^* z^{-1})}$$

A constructive Solution to the Spectral Factorization Problem

Pole-zero pattern $P_x(z)$



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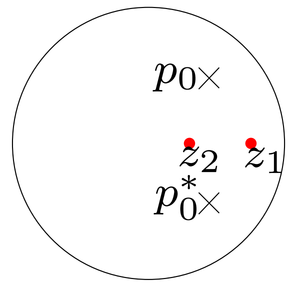

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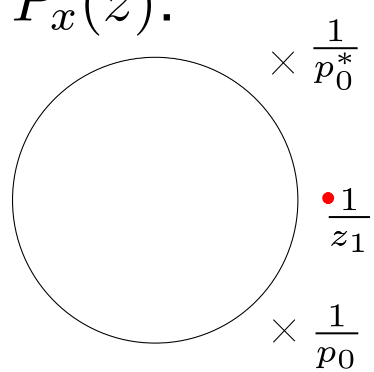
Unstable pole-zero part of $P_x(z)$:



$$Q^*\left(\frac{1}{z^*}\right) = \frac{(1-z_1 z)(1-z_2 z)}{(1-p_0^* z)(1-p_0 z)}$$

A constructive Spectral Factorization Method (Ct'd)

The finalize the calculations we need to fix the gain such that:

$$\lim_{z\to\alpha} P_x(z) = \sigma \lim_{z\to\alpha} Q(z)Q^*(\frac{1}{z^*})$$

Example 2: Consider the case displayed in the last figure and take $\alpha = 1$:

$$\begin{aligned}
\lim_{z\to 1} Q(z) &= \lim_{z\to 1} \frac{(1-z_1z^{-1})(1-z_2z^{-1})}{(1-p_0z^{-1})(1-p_0^*z^{-1})} \\
&= \frac{(1-z_1)(1-z_2)}{(1-p_0)(1-p_0^*)} \\
\lim_{z\to 1} Q^*(\frac{1}{z^*}) &= \lim_{z\to 1} \frac{(1-z_1z)(1-z_2z)}{(1-p_0^*z)(1-p_0z)} \\
&= \frac{(1-z_1)(1-z_2)}{(1-p_0^*)(1-p_0)}
\end{aligned}$$

From this follows $\sigma\left(\frac{(1-z_1)(1-z_2)}{(1-p_0)(1-p_0^*)}\right)^2 = \lim_{z\to 1} P_x(z)$.

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Illustration

$$P_x(z) = \frac{4}{(1-0.5z^{-1})(1-0.5z)}$$

This (rational) function has the poles 0.5 and 2. Therefore,

$$Q(z) = \frac{1}{1-0.5z^{-1}} \Rightarrow Q^*(\frac{1}{z^*}) = \frac{1}{1-0.5z}$$

With the spectral factorization given as:

$$P_x(z) = \sigma Q(z) Q^*(1/z^*)$$

we can find σ e.g. by considering the above equation for $z=1$:

$$16 = \sigma 4 \Rightarrow \sigma = 4$$

Example 3

Problem: Given $P_x(\omega) = \frac{25-24cos\omega}{26-10cos\omega}$ for $\omega \in [0, \pi]$. Determine a realization of $x(n)$?

Solution:

1. Determine the z-transform $P_x(z)$.
2. Determine a spectral factorization of $P_x(z) = \sigma Q(z)Q^*(\frac{1}{z^*})$.
3. Then filtering a ZMWN(1) sequence $v(n)$ with the filter $\sqrt{\sigma}Q(z)$ delivers a signal that has the given spectrum.

`Ex3_26.m`

Generalization to non-rational Spectra (optional)

Generalization to non-rational Spectra

Recall the Kolmogorov turbulence spectra:

$$P_\phi(f) = 0.023 r_0^{-\frac{5}{3}} f^{-\frac{11}{3}}$$

In order to apply the constructive method for rational Power spectra, we approximate the given Power spectra by a rational function first.

Rational Approximation (Example)

Let the rational approximation be denoted as $\hat{P}_x(z) = \frac{N(z)}{D(z)}$, with e.g.

$$N(z) = az + b + a^* z^{-1} \quad D(z) = dz + e + d^* z^{-1}$$

Then the coefficients $a, d \in \mathbb{C}, b, e \in \mathbb{R}$ are found by solving the following optimization problem,

$$\min_{a,b,d,e} \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_x(\omega) - \frac{N(e^{j\omega})}{D(e^{j\omega})}|^2 d\omega$$

Remark: This is a non-linear optimization problem.

[Intermezzo] Subspace Identification

The problem of approximating multivariable Power spectra where $P_x(e^{j\omega})$ is a matrix by a rational matrix function has been succesfully addressed in our team.

`test.m`

More info provided in the course `sc42025`

The Wold Decomposition

Regular RPs

DEFINITION: A regular WSS random process has a continuous Power Spectrum $P(e^{j\omega})$ in ω and this power spectrum has a spectral factorization:

$$P_x(z) = \sigma_0^2 Q(z) Q^*(\frac{1}{z^*})$$

The Wold Decomposition

WOLD DECOMPOSITION THEOREM: A general WSS random process can be written as,

$$x(n) = x_p(n) + x_r(n)$$

where $x_r(n)$ is a regular RP and $x_p(n)$ is a predictable process, with

$$E[x_r(m)x_p^*(n)] = 0 \quad \forall m, n \text{ i.e. } x_r(m), x_p(n) \text{ are orthogonal}$$

COROLLARY: For a general WSS RP $x(n)$:

$$P_x(e^{j\omega}) = \sigma_0^2 |Q(e^{j\omega})|^2 + \sum_{k=1}^{N} \alpha_k \delta(\omega - \omega_k)$$

Key message from Part I

1. If $P_x(z)$ is polynomial in z with the propety:

$$P_x(z) = P_x^*(\frac{1}{z^*})$$

then $P_x(z)$ is fully known from its causal part only.

Example 0: Let $P_x(z) = az + b + cz^{-1}$ then

$$a = c^* \quad b = b^*$$

Outline of Part II

1. **Problem formulation for ARMA(p,q)-model**
2. From $r_x(k)$ to the parameters $a(i), |b(0)|$ of an AR(p)-model
3. Spectral factorization to determine a set of $b(j)$ parameters of an MA(q)-model.
4. From $r_x(k)$ to the parameters $a(i), b(j)$ and the impulse response parameters $h(m)$ of an ARMA(p,q)-model:
   - The set of equations to determine its $a(i)$-parameters.
   - Retrieving its $b(i)$ and impulse response parameters $h(m)$ using spectral factorization.

Problem Formulation for ARMA(p,q)

Consider the Yule-Walker Equations (Lecture 3) of an ARMA(p,q) model with impulse response parameters $h(m)$ (for $m = 0 : q$):

$$r_x(k) + \sum_{\ell=1}^{p} a(\ell) r_x(k-\ell) = \begin{cases} \sum_{\ell=k}^{q} b(\ell) h^*(\ell - k) & : \quad 0 \leq k \leq q \\ 0 & : \quad k > q \end{cases}$$

then given the Autocorrelation function $r_x(k)$

1. Determine the number N of samples of the Autocorrelation function $r_x(k)$ for $k = 0 : N$ necessary to determine the parameters $a(i), b(j)$.
2. Determine the parameters $a(i), b(j)$ for $i = 1 : p$, $j = 0 : q$ assuming $\sigma_v = 1$ such that the Autocorrelation from the derived ARMA(p,q) model equals $r_x(k)$.

The Yule-Walker equations for AR(p) models

Recall from Lecture 3 (slide 41) that for the AR(p) the Yule-Walker equations reads:

$$r_x(k) + \sum_{\ell=1}^{p} a(\ell) r_x(k-\ell) = |b(0)|^2 \Delta(k) \quad k \geq 0$$

If we take $k = 1:p$ we have p-equations:

$$\begin{bmatrix} r_x(1) & r_x(0) & r_x^*(1) & \cdots & r_x^*(p-1) \\ r_x(2) & r_x(1) & r_x(0) & \cdots & r_x^*(p-2) \\ \vdots & & & \ddots & \\ r_x(p) & r_x(p-1) & r_x(p-2) & \cdots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \\ a(2) \\ \vdots \\ a(p) \end{bmatrix} = 0$$

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From which the parameters $a(i)$ for $i = 1 : p$ can be estimated provided the “system matrix” is invertible.

The Yule-Walker equations for AR(p) models

Recall from Lecture 3 (slide 43) that for the AR(p) the Yule-Walker equations reads:

$$r_x(k) + \sum_{\ell=1}^{p} a(\ell) r_x(k-\ell) = |b(0)|^2 \Delta(k) \quad k \geq 0$$

Knowing the $a(i)$'s, we take $k = 0$ to yield the parameter $|b(0)|$ as,

$$|b(0)|^2 = r_x(0) + \sum_{\ell=1}^{p} a(\ell) r_x^*(\ell)$$

Example 1: AR(1) model

Consider the (real) Autocorrelation function given by:

$$r_x(k) = \frac{4}{3}\left(\frac{1}{2}\right)^{|k|}$$

then determine the parameter a and b in the AR(1) model:

$$x(n) + ax(n-1) = bv(n) \quad v(n) \ \text{ is } \ \mathrm{ZMWN}(1)$$

Example 1: AR(1) model (Ct'd)

Consider the (real) Autocorrelation function given by:

$$r_x(k) = \frac{4}{3}\left(\frac{1}{2}\right)^{|k|}$$

For $k = 1$ in the Yule-Walker equation, we have the single equation:

$$r_x(1) + a r_x(0) = 0 \Rightarrow a = -\frac{r_x(1)}{r_x(0)}$$

For $k = 0$ in the Yule-Walker equation, we have:

$$r_x(0) + a r_x(1) = |b|^2 \Rightarrow |b|^2 = r_x(0) - \frac{r_x(1)^2}{r_x(0)} \Rightarrow |b| = \sqrt{\frac{r_x(0)^2 - r_x(1)^2}{r_x(0)}}$$

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Outline of Part II

1. Problem formulation for ARMA(p,q)-model
2. From $r_x(k)$ to the parameters $a(i), |b(0)|$ of an AR(p)-model
3. **Spectral factorization to determine a set of $b(j)$ parameters of an MA(q)-model.**
4. From $r_x(k)$ to the parameters $a(i), b(j)$ and the impulse response parameters $h(m)$ of an ARMA(p,q)-model:
   - The set of equations to determine its $a(i)$-parameters.
   - Retrieving its $b(i)$ and impulse response parameters $h(m)$ using spectral factorization.

The Yule-Walker equations for MA(q) models

Recall from Lecture 3 (slide 39) that for the MA(q) the Yule-Walker equations reads:

$$r_x(k) = \sum_{\ell=k}^{q} b(\ell)b^*(\ell-k) \quad 0 \leq k \leq q$$

(otherwise the $r_x(k)$'s are zero.). How to find from $r_x(k)$ a set of parameters $b(j)$ for $j = 1 : q$ such that the above equation holds?

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$$r_x(k) = \sum_{\ell=k}^{q} b(\ell) b^*(\ell - k) \quad 0 \leq k \leq q$$

Define $r_\infty(k)$ and $b_\infty(j)$ as follows,

$$r_\infty(k) = \begin{cases} r_x(k) & k = -q : q \\ 0 & \text{otherwise} \end{cases} \qquad b_\infty(j) = \begin{cases} b(j) & j = 0 : q \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$P_\infty(z) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} b_\infty(\ell) b_\infty^*(\ell - k) z^{-k} = \sum_{\ell=-\infty}^{\infty} b_\infty(\ell) z^{-\ell} \sum_{m=-\infty}^{\infty} b_\infty^*(m) z^{m}$$

for $m = \ell - k$. Or $P_\infty(z) = B_\infty(z) B_\infty^*(1/z^*)$.

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Therefore,

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Knowing $r_x(k)$ we know $P_x(z)$ (a finite order polynomial in z) and the determination of $B(z)$ is done via a

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Knowing $r_x(k)$ we know $P_x(z)$ (a finite order polynomial in z) and the determination of $B(z)$ is done via a Spectral Factorization.

Example 2: MA(1) model

Consider the (real) Autocorrelation function given by:

$$r_x(-1) = -\frac{1}{2} \quad r_x(0) = \frac{5}{4} \quad r_x(1) = -\frac{1}{2}$$

Then $P_x(z) = -\frac{1}{2}z + \frac{5}{4} - \frac{1}{2}z^{-1}$ The roots of this polynomial are 2 and $\frac{1}{2}$. Therfore,

$$P_x(z) = \alpha(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)$$

Considering $z = 1$ yields $\alpha = 1$.

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$$P_x(z) = \alpha(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)$$

Considering $z = 1$ yields $\alpha = 1$. Concluding: The parameter $b(0) = 1$ and $b(1) = -\frac{1}{2}$.

Outline Part II

1. Problem formulation for ARMA(p,q)-model
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Estimating the $a(i)$'s of an ARMA(p,q) model

Recall the Yule-Walker equations

$$r_x(k) + \sum_{\ell=1}^{p} a(\ell) r_x(k-\ell) = \begin{cases} \sum_{\ell=k}^{q} b(\ell) h^*(\ell-k) & : \quad 0 \leq k \leq q \\ 0 & : \quad k > q \end{cases}$$

If we take $k = q+1, \cdots q+p$ we obtain the p equations:

$$\begin{bmatrix} r_x(q+1) & r_x(q) & \cdots & r_x(q-p+1) \\ \vdots & & \ddots & \\ r_x(q+p) & r_x(q+p-1) & \cdots & r_x(q) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \\ \vdots \\ a(p) \end{bmatrix} = 0$$

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These are called the modified Yule-Walker equations.

Knowing the $a(i)$'s how to find $b(j)$'s?

Recall the Yule-Walker equations

$$r_x(k) + \sum_{\ell=1}^{p} a(\ell) r_x(k-\ell) = \sum_{\ell=k}^{q} b(\ell) h^*(\ell - k)$$

If we know the $a(i)$'s and take $k = 0 : q$,

Knowing the $a(i)$'s how to find $b(j)$'s?

Recall the Yule-Walker equations

$$c(k) = r_x(k) + \sum_{\ell=1}^{p} a(\ell) r_x(k-\ell) = \sum_{\ell=k}^{q} b(\ell) h^*(\ell - k)$$

If we know the $a(i)$'s and take $k = 0 : q$,

Knowing the $a(i)$'s how to find $b(j)$'s?

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$$r_x(k) + \sum_{\ell=1}^{p} a(\ell) r_x(k-\ell) = \sum_{\ell=k}^{q} b(\ell) h^*(\ell-k) = c(k)$$

If we know the $a(i)$'s and take $k = 0 : q$, we can find the coefficients $c(k)$ for $k = 0 : q$ from,

$$\begin{bmatrix} r_x(0) & r_x^*(1) & \cdots & r_x^*(p) \\ r_x(1) & r_x(0) & \cdots & r_x^*(p-1) \\ \vdots & & \ddots & \\ r_x(q) & r_x(q+1) & \cdots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \\ \vdots \\ a(p) \end{bmatrix} = \begin{bmatrix} c(0) \\ c(1) \\ \vdots \\ c(q) \end{bmatrix}$$

Finding the $b(j)$'s

Status: We know the coefficients $c(k)$ for $k = 0 : q$ and their expression:

$$c(k) = \sum_{\ell=k}^{q} b(\ell) h^*(\ell - k) \qquad (1)$$

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From this expression for $c(k)$ we find that for $k > q$ (i.e. $k = q + 1, q + 2, \cdots, \infty$), $c(k) = 0$.

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However from $C(z)$ only the causal part is known. The anti-causal part is unknown and (likely) non-zero. For example,

$$c(-1) = b(-1)h^*(0) + b(0)h^*(1) + \cdots + b(q)h^*(q+1)$$

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$$c(k) = \sum_{\ell=k}^{q} b(\ell) h^*(\ell - k) \qquad (1)$$

From (1) and the following "embedding" (like in the MA(q)-case):

$$\{b(n)\}_{n=-\infty}^{\infty} = \begin{cases} 0 & n < 0 \\ b(n) & 0 \leq n \leq q \\ 0 & n > q \end{cases} \qquad \{h(n)\}_{n=-\infty}^{\infty} = \begin{cases} 0 & n < 0 \\ h(n) & 0 \leq n \end{cases}$$

we find that $\boxed{C(z) = B(z) H^*(1/z^*)}$ (2)

But from $C(z)$ only the causal part is known.

Finding the $b(j)$'s

Status: Let $C(z)$ be denoted as,

$$C(z) = \sum_{k=-\infty}^{\infty} c(k)z^{-k} = \underbrace{\sum_{k=1}^{\infty} c(-k)z^{k}}_{[C(z)]_-} + \underbrace{\sum_{k=0}^{\infty} c(k)z^{-k}}_{[C(z)]_+}$$

with $[C(z)]_+$ known. Further a combination of the expression $C(z) = B(z)H^*(1/z^*)$

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$$C(z) = B(z)\frac{B^*(1/z^*)}{A^*(1/z^*)} \Rightarrow C(z)A^*(1/z^*) = B(z)B^*(1/z^*) = P_y(z)$$

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with $[C(z)]_+$ known. Further a combination of the expression $C(z) = B(z)H^*(1/z^*)$ with the definition $H(z) = \frac{B(z)}{A(z)}$, yields

$$C(z) = B(z)\frac{B^*(1/z^*)}{A^*(1/z^*)} \Rightarrow C(z)A^*(1/z^*) = B(z)B^*(1/z^*) = P_y(z)$$

Since $P_y(z)$ satisfies $P_y(z) = P_y^*(1/z^*)$ it is fully defined by its causal part. Can we find that causal part and subsequently determine $B(z)$ by Spectral Factorization?

The causal part of $P_y(z)$

With $P_y(z)$ equal to $C(z)A^*(1/z^*)$ we have,

$$\begin{aligned} P_y(z) &= C(z)A^*(1/z^*) \quad \text{with} \quad A^*(1/z^*) = 1 + a^*(1)z + \cdots + a^*(p)z^p \\ &= \Big([C(z)]_+ + [C(z)]_-\Big)A^*(1/z^*) \\ &= [C(z)]_+ A^*(1/z^*) + [C(z)]_- A^*(1/z^*) \end{aligned}$$

Therefore,

$$[P_y(z)]_+ = \Big[[C(z)]_+ A^*(1/z^*)\Big]_+$$

Let us further illustrate this calculation and the solution for $B(z)$ by an example.

Example 3 to find the Power spectrum $P_y(z)$ and its min-phase factor

Given: $r_x(0) = 26 \quad r_x(1) = 7 \quad r_x(2) = 3.5$, From the Yule-Walker equations we find that for $p = 1$, $\boxed{a(1) = -\frac{1}{2}}$, and as such,

$$A(z) = 1 - \frac{1}{2} z^{-1}$$

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Again from the Yule-Walker equations we find,

$$[C(z)]_+ = 22.5 - 6z^{-1} \Rightarrow$$

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$$A(z) = 1 - \frac{1}{2}z^{-1}$$

Again from the Yule-Walker equations we find,

$$[C(z)]_+ = 22.5 - 6z^{-1} \Rightarrow$$

$$\begin{aligned} {\color{red}[P_y(z)]_+} &= \Big[{\color{red}[C(z)]_+}A^*(1/z^*)\Big]_+ = \Big[(\frac{45}{2} - 6z^{-1})(1 - \frac{1}{2}z)\Big]_+ \\ &= \Big[-\frac{45}{4}z + (\frac{45}{2} + \frac{6}{2}) - 6z^{-1}\Big]_+ = \frac{51}{2} - 6z^{-1} \end{aligned}$$

and $P_y(z) = -6z + \frac{51}{2} - 6z^{-1}$. The spectral factorization of $P_y(z) = B(z)B^*(1/z^*)$ is,

$$P_y(z) = 24(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{4}z) \Rightarrow B(z) = 2\sqrt{6}(1 - \frac{1}{4}z^{-1})\texttt{inv\_ARMA.m}$$