CS 229, Fall 2018

Problem Set #1 Solutions: Supervised Learning

1.

(a)

$$\begin{aligned}
\frac{\partial J(\theta)}{\partial \theta_j} &= -\frac{1}{m}\sum_{i=1}^{m} y^{(i)}\frac{g(\theta^T x^{(i)})[1-g(\theta^T x^{(i)})]}{g(\theta^T x^{(i)})}x_j^{(i)} - (1-y^{(i)})\frac{g(\theta^T x^{(i)})[1-g(\theta^T x^{(i)})]}{1-g(\theta^T x^{(i)})}x_j^{(i)} \\
&= -\frac{1}{m}\sum_{i=1}^{m} y^{(i)}[1-g(\theta^T x^{(i)})]x_j^{(i)} - (1-y^{(i)})g(\theta^T x^{(i)})x_j^{(i)} \\
&= \frac{1}{m}\sum_{i=1}^{m}[g(\theta^T x^{(i)}) - y^{(i)}]x_j^{(i)}
\end{aligned}$$

$$\nabla_\theta J(\theta) = \frac{1}{m}X^T(g(X\theta)-Y)$$

$$H_{jk} = \frac{\partial^2 J(\theta)}{\partial\theta_j\partial\theta_k} = \frac{1}{m}\sum_{i=1}^{m} g(\theta^T x^{(i)})[1-g(\theta^T x^{(i)})]x_j^{(i)}x_k^{(i)}$$

$$H = \frac{1}{m}[X^T\cdot g(X\theta)\cdot(1-g(X\theta))]X$$

$$\begin{aligned}
z^T H z &= \frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} g(\theta^T x^{(i)})[1-g(\theta^T x^{(i)})]x_j^{(i)}x_k^{(i)}z_j z_k \\
&= \frac{1}{m}\sum_{i=1}^{m} g(\theta^T x^{(i)})[1-g(\theta^T x^{(i)})][(x^{(i)})^T z]^2 \geq 0
\end{aligned}$$

(c)

$$\begin{aligned}
p(y=1|x) &= \frac{p(x|y=1)p(y=1)}{p(x|y=1)p(y=1)+p(x|y=0)p(y=0)} \\
&= \frac{\exp\{-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\}\phi}{\exp\{-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\}\phi + \exp\{-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\}(1-\phi)} \\
&= \frac{1}{1+\exp\{\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\}\frac{1-\phi}{\phi}} \\
&= \frac{1}{1+\exp\{-[(\Sigma^{-1}(\mu_1-\mu_0))^T x + \frac{1}{2}(\mu_0+\mu_1)^T\Sigma^{-1}(\mu_0-\mu_1) - \ln(\frac{1-\phi}{\phi})]\}}
\end{aligned}$$

$$\theta = \Sigma^{-1}(\mu_1-\mu_0)$$

$$\theta_0 = \frac{1}{2}(\mu_0+\mu_1)^T\Sigma^{-1}(\mu_0-\mu_1) - \ln(\frac{1-\phi}{\phi})$$

(d)

$$\mu_{y^{(i)}} = 1\{y^{(i)}=0\}\mu_0 + 1\{y^{(i)}=1\}\mu_1$$

$$p(x^{(i)}|y^{(i)};\mu_0,\mu_1,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(x^{(i)}-\mu_{y^{(i)}})^T\Sigma^{-1}(x^{(i)}-\mu_{y^{(i)}})\Big\}$$

$$p(y^{(i)};\phi) = \phi^{1\{y^{(i)}=1\}}(1-\phi)^{1-1\{y^{(i)}=1\}}$$

$$\begin{aligned}
\ell &= \sum_{i=1}^{m}\log p(x^{(i)}|y^{(i)};\mu_0,\mu_1,\Sigma) + \sum_{i=1}^{m}\log p(y^{(i)};\phi) \\
&= \sum_{i=1}^{m}\log\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(x^{(i)}-\mu_{y^{(i)}})^T\Sigma^{-1}(x^{(i)}-\mu_{y^{(i)}})\Big\} + \sum_{i=1}^{m}\log\phi^{1\{y^{(i)}=1\}}(1-\phi)^{1-1\{y^{(i)}=1\}} \\
&= -\frac{mn}{2}\log(2\pi) - \frac{m}{2}\log|\Sigma| - \frac{1}{2}\sum_{i=1}^{m}(x^{(i)}-\mu_{y^{(i)}})^T\Sigma^{-1}(x^{(i)}-\mu_{y^{(i)}}) \\
&\quad + \sum_{i=1}^{m}1\{y^{(i)}=1\}\log\phi + \Big(m-\sum_{i=1}^{m}1\{y^{(i)}=1\}\Big)\log(1-\phi)
\end{aligned}$$

$$\frac{\partial\ell}{\partial\phi} = \frac{1}{\phi}\sum_{i=1}^{m}1\{y^{(i)}=1\} + \frac{1}{\phi-1}(m-\sum_{i=1}^{m}1\{y^{(i)}=1\})$$

$$\frac{\partial\ell}{\partial\mu_{y^{(i)}}} = \Sigma^{-1}\sum_{i=1}^{m}(x^{(i)}-\mu_{y^{(i)}})$$

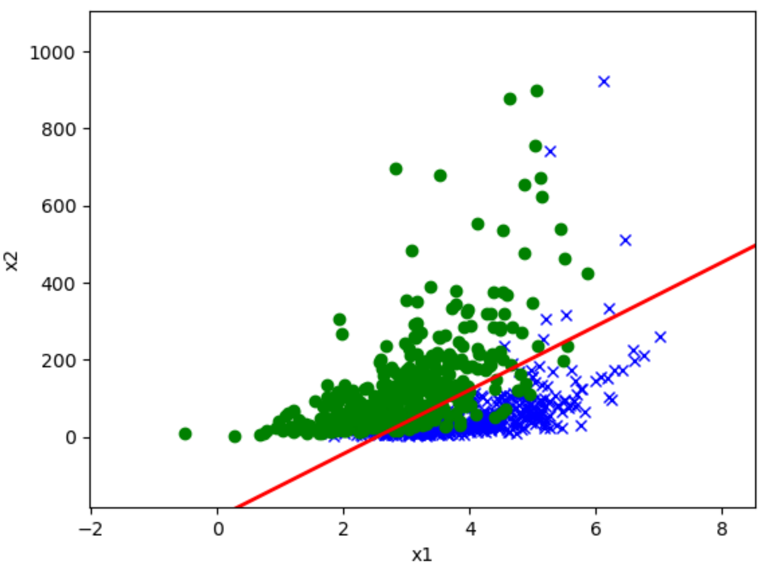
$$\frac{\partial\mu_{y^{(i)}}}{\partial\mu_0} = 1\{y^{(i)}=0\},\ \frac{\partial\mu_{y^{(i)}}}{\partial\mu_1} = 1\{y^{(i)}=1\}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\mu_0} &= \frac{\partial\ell}{\partial\mu_{y^{(i)}}}\frac{\partial\mu_{y^{(i)}}}{\partial\mu_0} = \Sigma^{-1}\sum_{i=1}^{m}\Big(x^{(i)}1\{y^{(i)}=0\} - \mu_0 1\{y^{(i)}=0\}\Big) \\
\frac{\partial\ell}{\partial\mu_1} &= \frac{\partial\ell}{\partial\mu_{y^{(i)}}}\frac{\partial\mu_{y^{(i)}}}{\partial\mu_1} = \Sigma^{-1}\sum_{i=1}^{m}\Big(x^{(i)}1\{y^{(i)}=1\} - \mu_1 1\{y^{(i)}=1\}\Big)
\end{aligned}$$

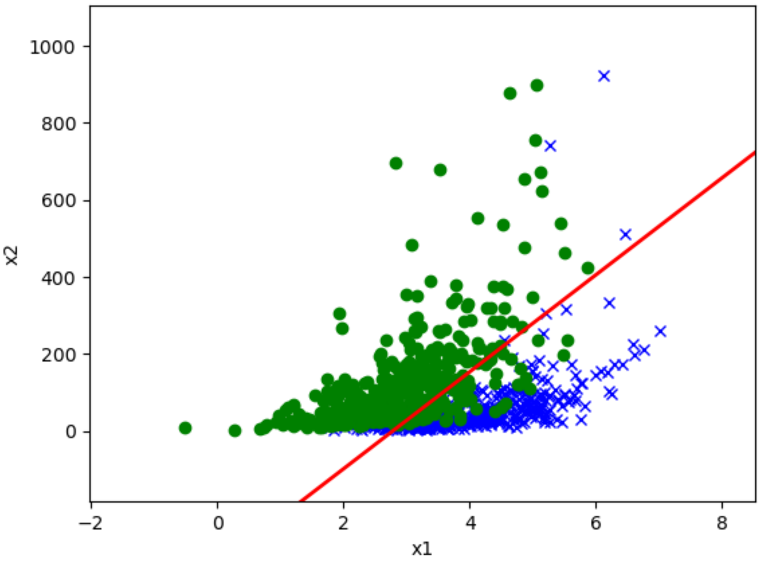
$$\frac{\partial\ell}{\partial\Sigma} = -\frac{m}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\Big(\sum_{i=1}^{m}(x^{(i)}-\mu_{y^{(i)}})(x^{(i)}-\mu_{y^{(i)}})^T\Big)\Sigma^{-1}$$

$$\begin{cases}
\frac{\partial\ell}{\partial\phi} &= 0 \\
\frac{\partial\ell}{\partial\mu_0} &= 0 \\
\frac{\partial\ell}{\partial\mu_1} &= 0 \\
\frac{\partial\ell}{\partial\Sigma} &= 0
\end{cases}
\Rightarrow
\begin{cases}
\phi &= \frac{1}{m}\sum_{i=1}^{m}1\{y^{(i)}=1\} \\
\mu_0 &= \frac{\sum_{i=1}^{m}1\{y^{(i)}=0\}x^{(i)}}{\sum_{i=1}^{m}1\{y^{(i)}=0\}} \\
\mu_1 &= \frac{\sum_{i=1}^{m}1\{y^{(i)}=1\}x^{(i)}}{\sum_{i=1}^{m}1\{y^{(i)}=1\}} \\
\Sigma &= \frac{1}{m}\sum_{i=1}^{m}(x^{(i)}-\mu_{y^{(i)}})(x^{(i)}-\mu_{y^{(i)}})^T
\end{cases}$$

(f)

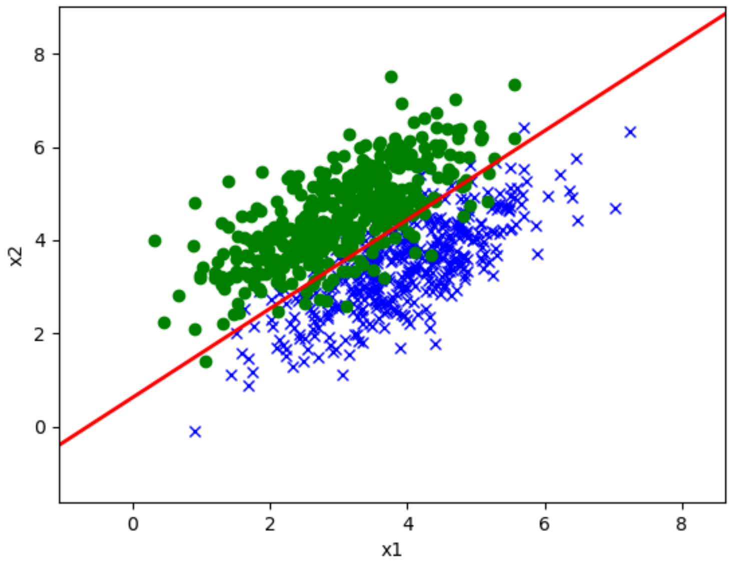


logistic regression

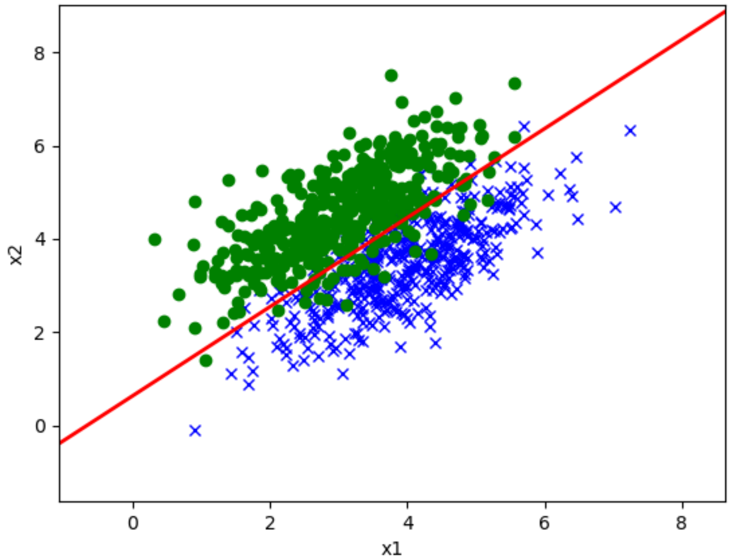


GDA

(g)



logistic regression



GDA

On Dataset 1 GDA perform worse than logistic regression.

Because $p(x|y)$ may be not Gaussian distribution.

(h)

Box-Cox transformation.

2.

(a)

$$P(y=1|t=1,x)P(t=1|x)P(x) = P(y=1,t=1,x) = P(t=1|y=1,x)P(y=1|x)P(x)$$

$$P(t=1|x) = P(y=1|x)\frac{P(t=1|y=1,x)}{P(y=1|t=1,x)}$$

$$P(t=1|y=1,x) = 1,\ \ P(y=1|t=1,x) = P(y=1|t=1)$$

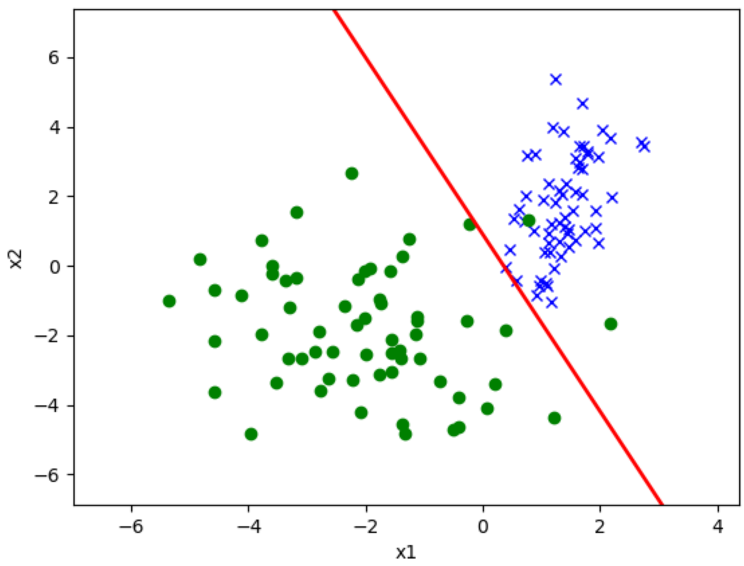
$$P(t=1|x) = \frac{P(y=1|x)}{P(y=1|t=1)}$$

$$P(y=1|t=1) = \alpha$$

(b)

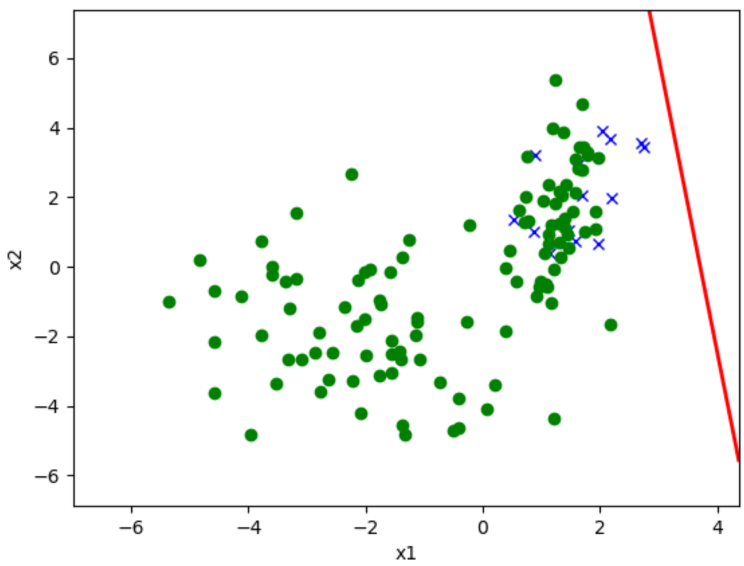
$$h(x) \approx p(y=1|x) = p(t=1|x)\alpha \approx \alpha \quad \text{for all } x \in V_{+}$$

(c)



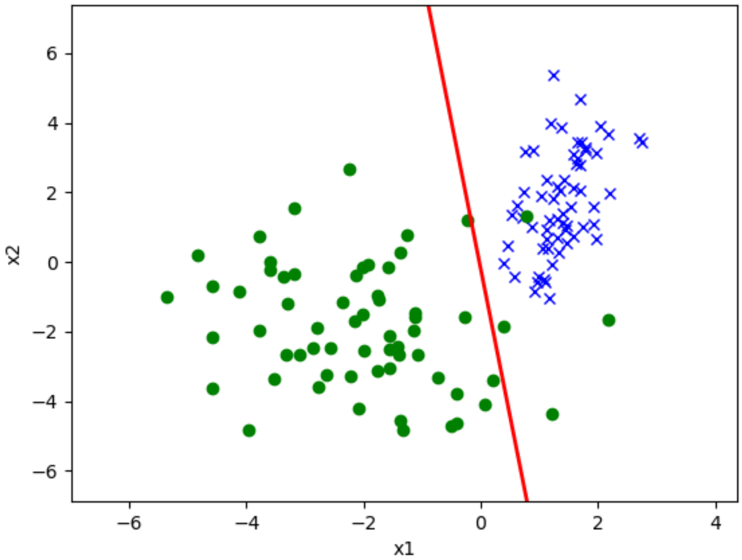
train use t-label

(d)



train use y-label

(e)



train use y-label, rescale by α

3.

(a)

$$p(y;\lambda) = \frac{1}{y!}\exp\{\log\lambda \cdot y - \lambda\}$$

$$\begin{cases} b(y) & = \frac{1}{y!} \\ \eta & = \log\lambda \\ T(y) & = y \\ a(\eta) & = e^{\eta} \end{cases}$$

(b)

$$h_\theta(x) = E(y|x;\theta) = \lambda = e^{\eta} = e^{\theta^T x}$$

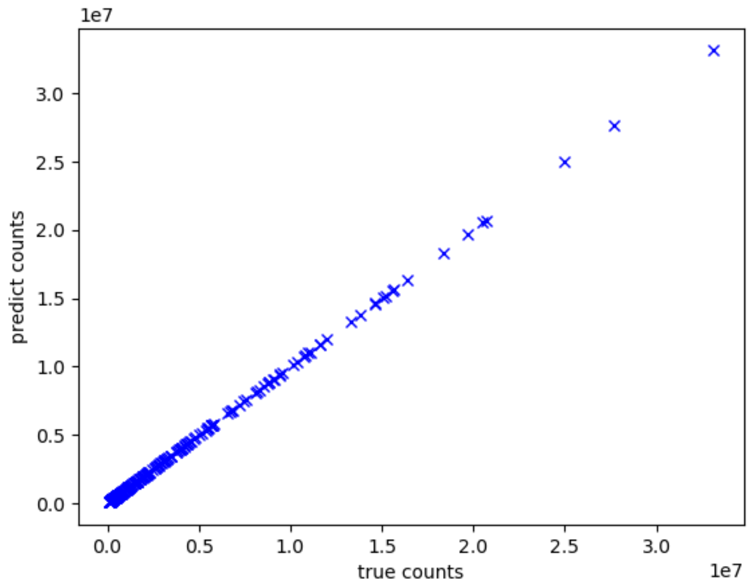
(c)

$$\begin{aligned} \log p(y^{(i)}|x^{(i)};\theta) &= \log \frac{1}{y^{(i)}!}\exp\{\theta^T x^{(i)} y^{(i)} - e^{\theta^T x^{(i)}}\} \\ &= -\log y^{(i)}! + \theta^T x^{(i)} y^{(i)} - e^{\theta^T x^{(i)}} \end{aligned}$$

$$\frac{\partial \log p(y^{(i)}|x^{(i)};\theta)}{\partial \theta_j} = y^{(i)} x_j^{(i)} - e^{\theta^T x^{(i)}} \cdot x_j^{(i)} = (y^{(i)} - e^{\theta^T x^{(i)}})x_j^{(i)}$$

$$\theta_j := \theta_j + \alpha \cdot (y^{(i)} - e^{\theta^T x^{(i)}})x_j^{(i)}$$

(d)



4.

(a)

$$\frac{\partial}{\partial \eta}\int p(y;\eta)dy = 0$$

$$\begin{aligned}
\frac{\partial}{\partial \eta}\int p(y;\eta)dy &= \int \frac{\partial}{\partial \eta} p(y;\eta)dy \\
&= \int b(y)\exp\{\eta y - a(\eta)\}(y - \frac{\partial a(\eta)}{\partial \eta})dy \\
&= \int p(y;\eta)(y - \frac{\partial a(\eta)}{\partial \eta})dy \\
&= \int yp(y;\eta)dy - \frac{\partial a(\eta)}{\partial \eta}\int p(y;\eta)dy \\
&= E[Y;\eta] - \frac{\partial a(\eta)}{\partial \eta}
\end{aligned}$$

$$E[Y;\eta] = E[Y|X;\theta] = \frac{\partial a(\eta)}{\partial \eta}$$

(b)

$$\frac{\partial}{\partial \eta}\int yp(y;\eta)dy = \frac{\partial^2 a(\eta)}{\partial \eta^2}$$

$$\begin{aligned}
\frac{\partial}{\partial \eta}\int yp(y;\eta)dy &= \int y\frac{\partial}{\partial \eta} p(y;\eta)dy \\
&= \int yp(y;\eta)(y - \frac{\partial a(\eta)}{\partial \eta})dy \\
&= \int y^2 p(y;\eta)dy - \frac{\partial a(\eta)}{\partial \eta}\int yp(y;\eta)dy \\
&= E[Y^2;\eta] - E^2[Y;\eta] \\
&= Var[Y;\eta]
\end{aligned}$$

$$Var[Y;\eta] = Var[Y|X;\theta] = \frac{\partial^2 a(\eta)}{\partial \eta^2}$$

(c)

$$\begin{aligned}\ell(\theta) &= -\sum_{i=1}^{m} \log p(y^{(i)}|x^{(i)};\theta) \\ &= \sum_{i=1}^{m} -\log b(y^{(i)}) - \theta^T x^{(i)} y^{(i)} + a(\theta^T x^{(i)})\end{aligned}$$

$$\frac{\partial \ell(\theta)}{\partial \theta_j} = \sum_{i=1}^{m} [a'(\theta^T x^{(i)}) - y^{(i)}] x_j^{(i)}$$

$$H_{jk} = \frac{\partial^2 \ell(\theta)}{\partial \theta_j \theta_k} = \sum_{i=1}^{m} a''(\theta^T x^{(i)}) x_j^{(i)} x_k^{(i)}$$

$$\begin{aligned}z^T H z &= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} a''(\theta^T x^{(i)}) x_j^{(i)} x_k^{(i)} z_j z_k \\ &= \sum_{i=1}^{m} a''(\theta^T x^{(i)}) [(x^{(i)})^T z]^2\end{aligned}$$

$$a''(\theta^T x) = Var[Y|X;\theta] \geq 0 \ \Rightarrow \ z^T H z \geq 0$$

5.

(a)

i.

$$W \in \mathbb{R}^{m \times m}$$

$$W_{ij} = \begin{cases} \frac{1}{2} w^{(i)} & i = j \\ 0 & i \neq j \end{cases}$$

ii.

$$\begin{aligned}\nabla_\theta J(\theta) &= \nabla_\theta (X\theta - y)^T W (X\theta - y) \\ &= \nabla_\theta (\theta^T X^T - y^T) W (X\theta - y) \\ &= \nabla_\theta (\theta^T X^T W X \theta - y^T W X \theta - \theta^T X^T W y + y^T W y) \\ &= \nabla_\theta (\theta^T X^T W X \theta - 2 y^T W X \theta) \\ &= 2 X^T W X \theta - 2 X^T W y\end{aligned}$$

$$\nabla_\theta J(\theta) = 0 \ \Rightarrow \ \theta = (X^T W X)^{-1} X^T W y$$

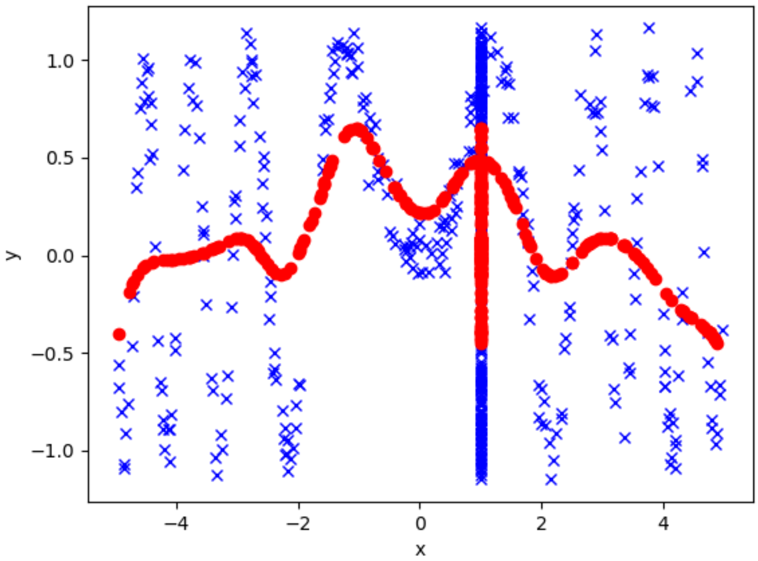
iii.

$$\begin{aligned}\ell(\theta) &= \sum_{i=1}^{m} \log p(y^{(i)}|x^{(i)};\theta) \\ &= \sum_{i=1}^{m} -\log(\sqrt{2\pi}\sigma^{(i)}) - \frac{(y^{(i)} - \theta^T x^{(i)})^2}{2(\sigma^{(i)})^2}\end{aligned}$$

$$w^{(i)} = -\frac{1}{(\sigma^{(i)})^2}$$

$$\frac{\partial \ell(\theta)}{\partial \theta_j} = \sum_{i=1}^{m} \frac{y^{(i)} - \theta^T x^{(i)}}{(\sigma^{(i)})^2} x_j^{(i)}$$

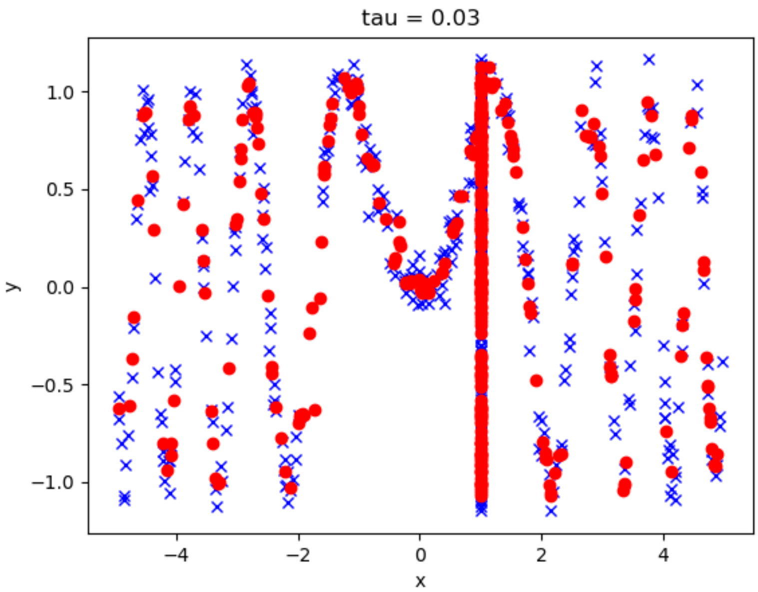
(b)

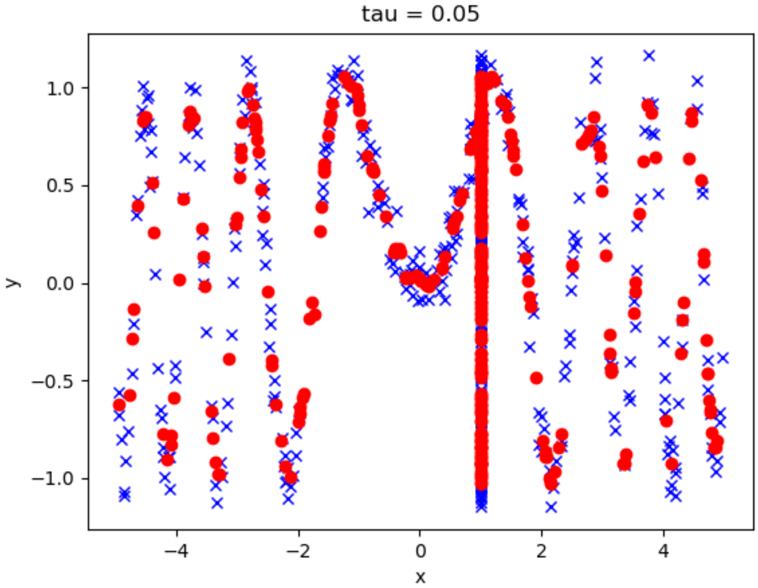


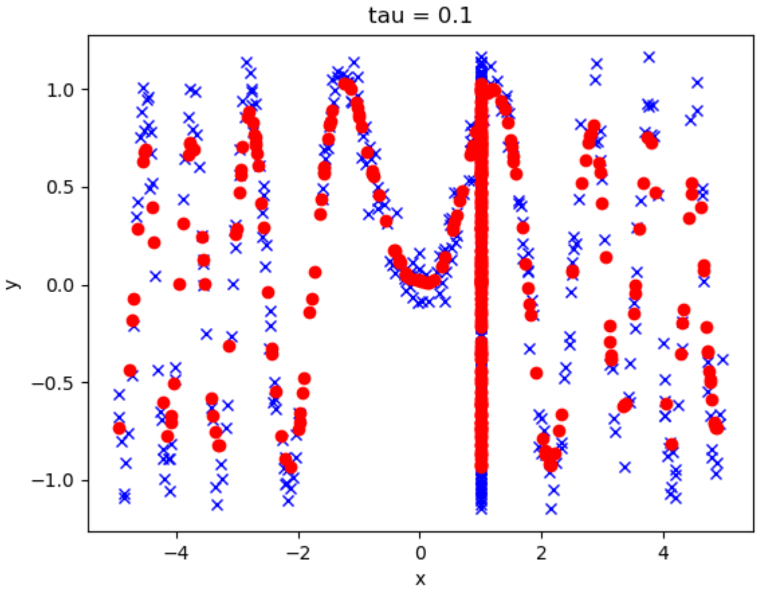
MSE=0.331.

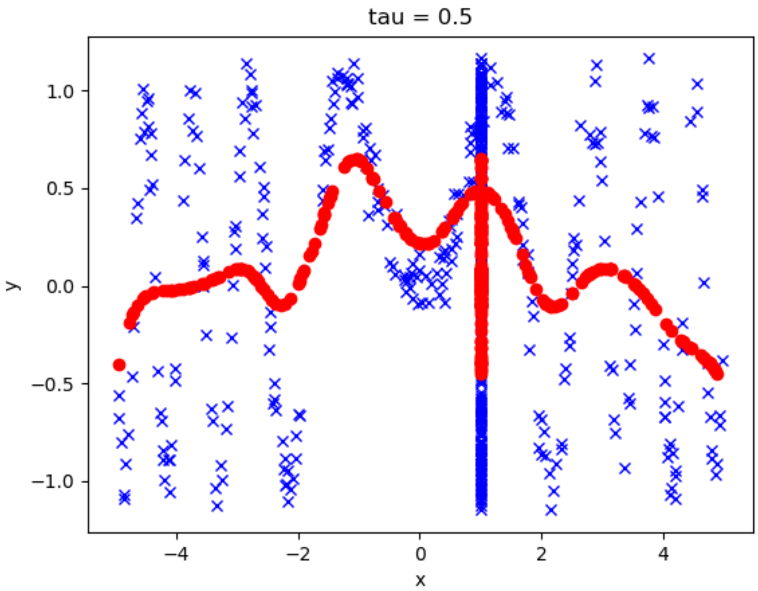
The model seems to be underfitting.

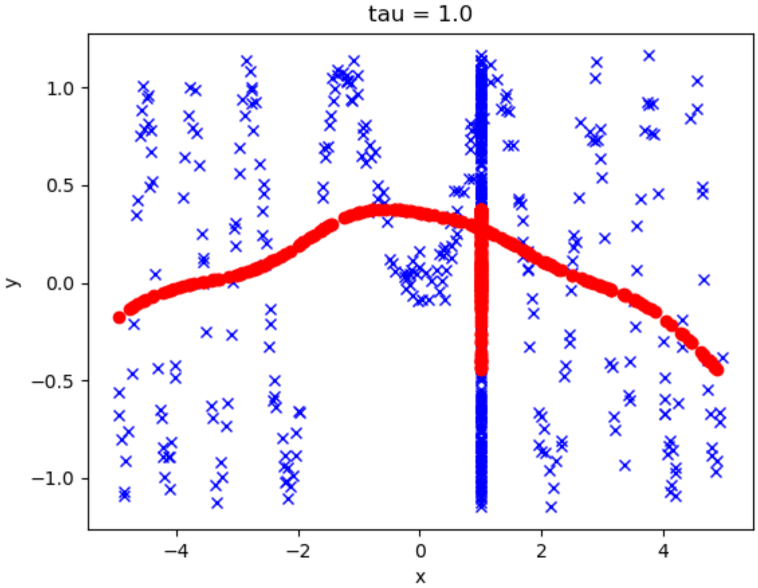
(c)

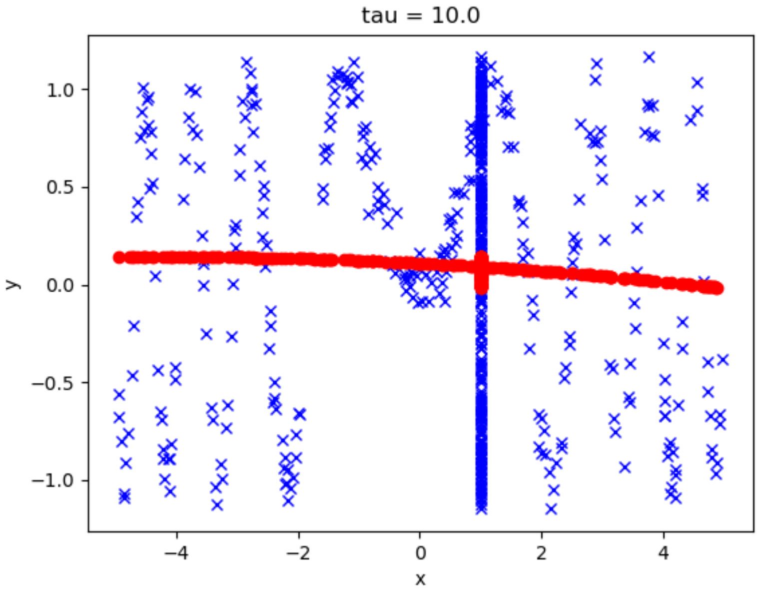



tau = 0.05
1.0
0.5
0.0
−0.5
−1.0
y
−4
−2
0
2
4
x


tau = 0.1
1.0
0.5
0.0
−0.5
−1.0
y
−4
−2
0
2
4
x


tau = 0.5
1.0
0.5
0.0
−0.5
−1.0
y
−4
−2
0
2
4
x





$\tau = 0.05$ achieves the lowest MSE on the valid set.

MSE=0.012 on the valid set, MSE=0.017 on the test set.