$\partial_t u(t,x) + b\cdot\nabla u(t,x) = 0$ (1)

$u: [0,+\infty)\times\mathbb{R}^n$ $x=(x_1,x_2,\cdots,x_n)$ $b=(b_1,b_2,\cdots,b_n)\in\mathbb{R}^n$ is given.

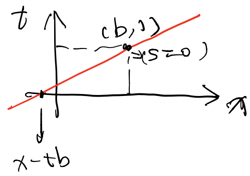
Suppose u is smooth.

Let $Z(s) = u(t+s, x+sb)$ $(s\in\mathbb{R})$ (t,x are given).

$$\partial_s Z(s) = \left(\partial_t u + \nabla_x u\cdot b\right)(t+s, x+sb) = 0.$$

$\Rightarrow$ $Z(s)$ is independent of s.

$\Rightarrow$ Z is constant on the line $(t+s, x+sb)$ $(s\in\mathbb{R})$



$(t,x)+s(1,b)$

$$Z(0) = u(t,x) = Z(-t) = u(0, x-tb)$$

Initial problem. $\begin{cases} \partial_t u + b\cdot\nabla u = 0. & t>0,\ x\in\mathbb{R}^n \\ u(0,x) = u_0(x) & \longrightarrow \text{initial value} \end{cases}$ (2)

$$u(t,x) = u(0, x-tb) = u_0(x-tb) \qquad (3)$$

If u_0 is C^1 $\Rightarrow$ (3) is a solution of (2)

If u_0 is not C^1 $\Rightarrow$ (2) has no C^1 solution

· Nonhomogeneous problem:

$$\begin{cases} \partial_t u + b\cdot\nabla u = f(t,x) \\ u(0,x) = u_0(x) \end{cases}$$

$Z(s) = u(t+s, x+sb)$

$\partial_s Z = (\partial_t u + b\cdot\nabla x)(t+s, x+sb) = f(t+s, x+sb)$

$\Rightarrow$ $Z(0) - Z(-t) = \int_{-t}^{0} f(t+s, x+sb)\,ds$

$\Rightarrow$ $u(t,x) = \underbrace{u_0(x-tb)}_{\text{初值}} + \underbrace{\int_0^t f(s, x-tb+sb)\,ds}_{\text{非齐次项}}$ $(t\geq 0)$

Remark: This method is a special case of the "method of characteristics" 特征线法.

§3.2.