§ 3.1 Nonlinear first order PDE

$F(Du, u, x) = 0 \quad x \in \mathcal{U} \ (\subseteq \mathbb{R}^n, \text{ open})$

$u : \mathcal{U} \to \mathbb{R}$.

Notations: $F = F(p, z, x) \quad p \in \mathbb{R}^n, \ z \in \mathbb{R}, \ x \in \mathcal{U} \qquad p \sim Du, \ z \sim u$

$$\begin{cases} D_p F = (F_{p_1}, F_{p_2}, \cdots, F_{p_n}) \\ D_z F = F_z \\ D_x F = (F_{x_1}, F_{x_2}, \cdots, F_{x_n}) \end{cases}$$

Boundary condition: $u = g$ on $\Gamma \subset \partial\mathcal{U} \qquad g : \Gamma \to \mathbb{R}$ is given.

Sometimes, you don't need to know $u = g$ on $\partial\mathcal{U}$ to ensure the uniqueness of solution.
You only need $u = g$ on $\Gamma \subsetneq \partial\mathcal{U}$.
This is because the PDE is 1st-order

- Complete integrals.

$F(Du, u, x) = 0$.

Assume $\exists A \in \mathbb{R}^n$, open, s.t. for $\forall a = (a_1, a_2, \cdots, a_n) \in A$, we have a C^2 solution $u = u(x; a)$

a parameter

We write $(D_a u, D^2_{x,a} u) = \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & \vdots & & \vdots \\ u_{a_n} & u_{x_1 a_n} & \cdots & u_{x_n a_n} \end{pmatrix}_{n \times (n+1)}$

<u>Def</u> A C^2-solution $u = u(x; a)$ is called a complete integral in $\mathcal{U} \times A$ if:

(1) $u(x; a)$ is a solution for each $a \in A$;

(2) $\operatorname{rank}(D_a u, D^2_{x,a} u) = n \quad (\forall x \in \mathcal{U}, \ a \in A)$

Remark: The condition (2) ensures $u(x; a)$ "depends on all the n independent parameters. $a_1, \cdots, a_n$"

Pf: We prove this by contradiction.
Suppose $B \subset \mathbb{R}^{n-1}$ open. for each $b \in B$. $v = v(x; b)$ is a solution of PDE. Suppose also $\exists \psi \in C^1 \ \psi : A \to B$.

$\psi = (\psi^1, \psi^2, \cdots, \psi^{n-1})$, s.t. $u(x; a) = v(x; \psi(a)) \quad (x \in \mathcal{U}, a \in A)$

that is, we are supposing the function $u(x; a)$ "really depends only on the $n-1$ parameters $b_1, \cdots, b_{n-1}$".

But then $u_{x_i a_j}(x; a) = \sum_{k=1}^{n-1} v_{x_i b_k}(x; \psi(a)) \psi^k_{a_j}(a) \quad i, j = 1, \cdots, n$

$$\Rightarrow \det(D^2_{xa}u) = \sum_{k_1,\cdots,k_n=1}^{n-1} v_{x_1 b_{k_1}} v_{x_2 b_{k_2}} \cdots v_{x_n b_{k_n}} \cdot \det\begin{pmatrix} \psi^{k_1}_{a_1} & \cdots & \psi^{k_1}_{a_n} \\ \vdots & & \vdots \\ \psi^{k_n}_{a_1} & \cdots & \psi^{k_n}_{a_n} \end{pmatrix}$$

$$= 0.$$

$\because k_1, \cdots, k_n \in \{1, \cdots, n-1\} \therefore \exists\, k_i = k_j$ for each arrangement

Similarly, $\because u_{a_j}(x;a) = \sum_{k=1}^{n-1} v_{b_k}(x;\psi(a))\,\psi^k_{a_j}(a) \quad j=1,\cdots,n.$

$\therefore$ the determinant of each $n\times n$ submatrix of $(D_a u, D^2_{xa} u)$ equals zero

$\therefore \operatorname{rank}(D_a u, D^2_{xa} u) < n \Rightarrow$ a Contradiction!

$\square$.

<u>Example</u>: 1) Clairaut's equation:

$x\cdot Du + f(Du) = u \quad f: \mathbb{R}^n \to \mathbb{R}$ is given

$u(x;a) = a\cdot x + f(a)$ is a complete integral for $a \in \mathbb{R}^n$

2) The eikonal equation: $|Du| = 1$

$u(x;a,b) = a\cdot x + b \quad (a \in S^{n-1},\ b \in \mathbb{R})$ is a complete integral

3) Hamilton–Jacobi equation: $\partial_t u + H(Du) = 0$.

$u(x,t;a,b) = a\cdot x - tH(a) + b \quad (x \in \mathbb{R}^n,\ t>0.\ a \in \mathbb{R}^n,\ b \in \mathbb{R})$ is a complete integral.

· Envelops (包络)

Assume $u = u(x;a)$ solves the equation $F(Du, u, x) = 0$. $\begin{pmatrix} x \in \mathcal{U} \subseteq \mathbb{R}^n \\ a \in A \subseteq \mathbb{R}^n \end{pmatrix}$

Consider $D_a u(x;a) = 0 \quad (x \in \mathcal{U},\ a \in A)$

Suppose the above equation has a C^1 solution $a = \phi(x)$,

i.e. $D_a u(x,a)\big|_{a=\phi(x)} = 0$.

Then $v(x) = u(x;\phi(x))$ is a solution of $F(Du, u, x) = 0$.

<u>Def</u> We call $v(x)$ is the <u>envelope</u> of the function $\{u(\cdot;a)\}_{a\in A}$

$v(x)$ is also called <u>the singular integral</u> of the equation $F(Du,u,x)=0$

Pf: $F(Dv, v, x) = F(Du(x;\phi(x)) + \underbrace{D_a u(x;\phi(x))}_{=0}\cdot D_x\phi(x),\ u(x,\phi(x)),\ x)$

$= F(Du(x;\phi(x)),\ u(x,\phi(x)),\ x) = 0.$

<u>Example</u>: $u^2(1+|Du|^2) = 1$

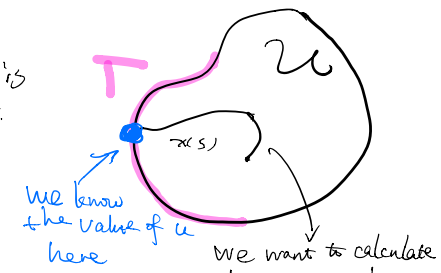
A complete integral: $u(x;a) = \pm(1-|x-a|^2)^{\frac{1}{2}} \quad (|x-a|<1)$

$D_a u = \dfrac{\mp(x-a)}{(1-|x-a|^2)^{\frac{1}{2}}} \Rightarrow a = \phi(x) = x \Rightarrow u = \pm 1$ are singular integrals

§3.2 Characteristics (特征线)

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \mathcal{U} \\ u = g & \text{on } \Gamma \subset \partial\mathcal{U} \end{cases} \qquad F, g \text{ are smooth.}$$

We try to find $x(s)$ such that $u(x(s))$ is a solution of the equation satisfies a ODE:



$$\partial_s u(x(s)) = \sum_{i=1}^{n} \partial_i u(x(s))\, \partial_s x_i(s) \quad ②$$

$$\partial_s(\partial_i u(x(s))) = \sum_{j=1}^{n} \partial_i\partial_j u(x(s))\, \partial_i x_j(s)$$

$$F(Du, u, x) = 0 \xRightarrow{\partial_i} \sum_{j=1}^{n} F_{p_j}\, \partial_i\partial_j u + F_z \cdot \partial_i u + F_{x_i} = 0.$$

In order to get rid of the "dangerous" second derivative terms we take $\partial_s x_j(s) = F_{p_j}(D_u(x(s)), u(x(s)), x(s))$ ① $j = 1, 2, \dots, n$.

then

$$\partial_s(\partial_i(u(x(s))) + F_z(\underbrace{Du(x(s))}_{p(s) = Du(x(s))}, u(x(s)), x(s)) \cdot \partial_i u(x(s)) + F_{x_i}(Du(x(s)), u(x(s)), x(s)) = 0 \quad ③$$

Let $p_i(s) = \partial_i u(x(s))$ $\quad z(s) = u(x(s))$ then:

$$(☆)\begin{cases} \partial_s x(s) = F_p(p(s), z(s), x(s)) & ① \\ \partial_s z(s) = p(s) \cdot \partial_s x(s) = p(s) \cdot F_p(p(s), z(s), x(s)) & ② \\ \partial_s p(s) = -F_z(p(s), z(s), x(s))\, p(s) - F_x(p(s), z(s), x(s)) & ③ \end{cases}$$

2n+1-dim ODE system.

Remark:

$\because$ $\partial_s F(p(s), z(s), x(s)) = F_p(-F_z \cdot p - F_x) + pF_p \cdot F_z + F_x \cdot F_p = 0.$

$\therefore$ If $F(p(0), z(0), x(0)) = 0$.

then $F(p(s), z(s), x(s)) = F(p(0), z(0), x(0)) = 0$, $\forall s$

i.e. If we know $u = g$ on $\Gamma \subset \partial\mathcal{U}$.

and $x(s) \cap \Gamma = x(s_0)$, WLOG. $s_0 = 0$.

$\Rightarrow$ we can calculate u all along the curve $x(s)$

Def We call (☆) be the characteristic equations of $F(Du, u, x) = 0$.

$x(s)$: the projected characteristic.

Example: 1) F linear

$$F(Du, u, x) = b(x) \cdot \nabla u(x) + c(x) u(x) = 0.$$

$$F(p, z, x) = b(x)\, p + c(x) \cdot z \qquad D_p F = b(x)$$

$$\begin{cases} \partial_s x(s) = b(x(s)) & ① \\ \partial_s z(s) = p(s) \cdot b(s) \overset{\because F=0}{=} -c(x(s)) \cdot z(s) & ② \\ \partial_s p(s) = -c(x(s))\, p(s) - D_x b(x(s))\, p(s) - D_x c(x(s)) \cdot z(s) & ③ \end{cases}$$

$\because$ We only want to know $z(s)$

$\therefore$ we only need to solve $\begin{cases} ① \\ ② \end{cases}$

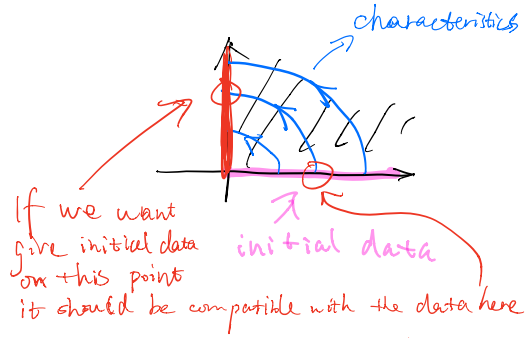
An explicit example:

$$\begin{cases} x_1 \partial_2 u - x_2 \partial_1 u = u & \text{in } \mathcal{U} = \{x_1>0,\ x_2>0\} \\ u = g & \text{on } \Gamma = \{x_1>0,\ x_2=0\} \end{cases}$$

In this example, $\vec{b} = (-x_2, x_1)$ $\quad c = -1$.

$$① \Longleftrightarrow \begin{cases} \dot{x}^1 = -x^2, & \dot{x}^2 = x^1 \\ \dot{z} = z \end{cases}$$

$$② \Longleftrightarrow$$



$$\Longrightarrow \begin{cases} x^1(s) = x^0 \cos s \\ x^2(s) = x^0 \sin s \\ z(s) = z^0 e^s = g(x^0) e^s \end{cases}$$

where $x^0 \geq 0$, $0 \leq s \leq \frac{\pi}{2}$.

Fix a point $(x_1, x_2) \in \mathcal{U}$. We select $s>0$, $x^0>0$ so that $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 \cos s, x^0 \sin s)$, that is, $x^0 = \sqrt{x_1^2 + x_2^2}$, $s = \arctan\left(\frac{x_2}{x_1}\right)$

$$\Longrightarrow u(x_1, x_2) = u(x^1(s), x^2(s)) = z(s) = g(x^0) e^s$$
$$= g\left(\sqrt{x_1^2+x_2^2}\right) \cdot e^{\arctan\left(\frac{x_2}{x_1}\right)}$$

$\square$

2) F quasilinear:

$$F(Du, u, x) = b(x, u(x)) \cdot Du(x) + c(x, u(x)) = 0.$$

$$F(p, z, x) = b(x, z) \cdot p + c(x, z). \qquad F_p = b(x, z)$$

$$\Longrightarrow \begin{cases} \partial_s x(s) = b(x(s), z(s)) \\ \partial_s z(s) = -c(x(s), z(s)) \\ \partial_s p(s) = -D_x b(x(s), z(s)) \cdot p(s) - D_x c(x(s), z(s)) \end{cases}$$

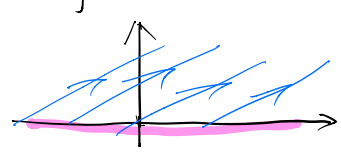
这2个已经构成一组封闭的ODE系统了

An explicit example:

$$\begin{cases} \partial_1 u + \partial_2 u = u^2 & \text{in } \mathcal{U} = \{x_2>0\} \\ u = g & \text{on } \Gamma = \{x_2=0\} = \partial\mathcal{U}. \end{cases}$$

$\vec{b} = (1, 1)$, $\quad c = -x^2$

$$\Longrightarrow \begin{cases} \dot{x}^1 = 1, & \dot{x}^2 = 1 \\ \dot{z} = z^2 \end{cases}$$



$$\Longrightarrow \begin{cases} x^1(s) = x^0 + s, \quad x^2(s) = s \\ z(s) = \dfrac{z^0}{1 - s z^0} = \dfrac{g(x^0)}{1 - s g(x^0)} \end{cases}$$

where $x^0 \in \mathbb{R}$, $s \geq 0$, provided the denominator is not zero.

$$\Longrightarrow u(x_1, x_2) = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}.$$

(make sense only if $1 - x_2 g(x_1 - x_2) \neq 0$.)

$\square$

3) F fully nonlinear

the general form is difficult, we only consider a special one.

$$\begin{cases} \partial_1 u \cdot \partial_2 u = u & \text{in } \mathcal{U} = \{x_1 > 0\} \\ u = x_2^2 & \text{on } \Gamma = \{x_1 = 0\} = \partial\mathcal{U} \end{cases}$$

$F(p, z, x) = p_1 p_2 - z. \quad F_p = (p_2, p_1) \quad F_z = -1$

$$\begin{cases} \partial_s x(s) = (p_2, p_1) \iff \partial_s x_1 = p_2, \ \partial_s x_2 = p_1 \\ \partial_s z = 2p_1 p_2 \\ \partial_s p = p \iff \partial_s p_1 = p_1, \ \partial_s p_2 = p_2 \end{cases}$$

$$\Rightarrow \begin{cases} p_1(s) = p_1(0) e^s \quad p_2(s) = p_2(0) \cdot e^s \\ x_1(s) = p_2(0) \cdot (e^s - 1) + x_1(0) \quad x_2(s) = p_1(0)(e^s - 1) + x_2(0) \\ z(s) = p_1(0) p_2(0)(e^{2s} - 1) + z(0) \end{cases}$$

Let $x_1(0) = 0$, $x_2(0) = x_0$, then

$$\Rightarrow \begin{aligned} p_2(0) &= \partial_2 u(x(0)) \overset{\because u = x_2^2 \text{ on } T}{=} 2x_2(0) = 2x_0 \\ p_1(0) &= \partial_1 u(x(0)) \underset{\text{by the original PDE}}{=} \frac{z(0)}{p_2(0)} \underset{\because u = x_2^2 \text{ on } T}{=} \frac{x_2(0)}{2} = \frac{1}{2} x_0 \end{aligned}$$

$$\Rightarrow \begin{cases} x_1(s) = 2x_0(e^s - 1), \quad x_2(s) = \frac{x_0}{2}(e^s - 1) + x_0 = \frac{x_0}{2}(e^s + 1) \\ z(s) = x_0^2 \cdot e^{2s} \\ p_1(s) = \frac{x_0}{2} e^s, \quad p_2(s) = 2x_0 e^s \end{cases}$$

For given $(x_1, x_2) \in \mathcal{U}$. choose (x_0, s) such that:

$$\begin{cases} x_1(s) = 2x_0(e^s - 1) = x_1 \\ x_2(s) = \frac{x_0}{2}(e^s + 1) = x_2 \end{cases}$$

$$\Rightarrow \begin{cases} x_0 = \frac{4x_2 - x_1}{4} \\ e^s = \frac{x_1 + 4x_2}{4x_2 - x_1} \end{cases}$$

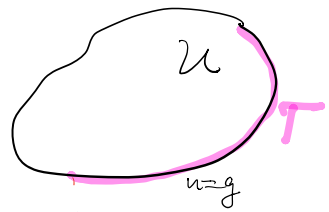
$$\Rightarrow u(x_1, x_2) = z(s) = (x_0)^2 e^{2s} = \frac{(x_1 + 4x_2)^2}{16} \text{ is a solution.}$$

The boundary condition given in above 3 examples are all suitable for us to find "good" solutions.

Now we discuss how to give a good boundary condition in the following.

· Boundary conditions

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \mathcal{U} \\ u = g & \text{on } T \subset \partial\mathcal{U} \end{cases}$$



Let $\{\tau_i\}_{i=1}^{n-1}$ be a local orthogonal 标准正交基 normal bases on T.
ν is the unit outer normal on T at x_0.

$$\text{Then } \begin{cases} \tau_i \cdot Du = \partial_{\tau_i} g \\ F(Du, u, x) = 0 \end{cases} \text{ on } T \longrightarrow F\Big(\sum_{i=1}^{n-1} \tau_i \partial_{\tau_i} g + \nu \cdot \partial_\nu u,\ u,\ x\Big) = 0.$$

$$Du = \underbrace{\Big(\sum_{i=1}^{n-1} \tau_i \otimes \tau_i + \nu \otimes \nu\Big)}_{\text{恒同矩阵}} \cdot Du$$

That is, $\begin{cases} \tau_i \cdot p^0 = \partial_{\tau_i} g & (\star)_1 \\ F(p^0, z^0, x^0) = 0. & (\star)_2 \end{cases} \Big\} (\star)$

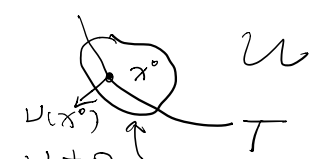
when (p^0, z^0, x^0) is the initial data of characteristic ODE.
$(\star)$ is called compatibility condition on boundary data. (给的ODE边界条件需满足★才可)

$(\star)_2$ is $F\Big(\sum_{i=1}^{n-1} \tau_i \partial_{\tau_i} g + \nu \partial_\nu u,\ g(x_0),\ x_0\Big) = 0$
which gives a condition for $\nu \cdot Du$ → 其第2式给了一个u沿向导数的条件.

<u>Def</u> A triple (p^0, z^0, x^0) verifying $(\star)$ is called <u>admissible</u>.
可容许初值的定义.

<u>Lemma</u>. If $F_p(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$,
then there is a unique solution of $q = q(y)$ such that:

$$\begin{cases} \tau_i(y) \cdot q(y) = \partial_{\tau_i} g(y) \\ F(q(y), g(y), y) = 0. \end{cases} \quad \text{for all } y \in T \text{ close to } x^0.$$



If $F_p \cdot \nu \neq 0$.
then $\exists$ a neighbourhood of x^0,
s.t. in the neighbourhood, there is a collection of "initial value"

Pf: We define the mapping $G: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as below:

$$G(q, y) = (G^1(q, y), G^2(q, y), \cdots, G^n(q, y)),$$

where $\begin{cases} G^i(q, y) = \tau_i(y) \cdot q - \partial_{\tau_i} g(y) & i = 1, \cdots, n-1 \\ G^n(q, y) = F\Big(\sum_{i=1}^{n-1} \tau_i(y)(\tau_i(y) \cdot q) + \nu(y)(\nu(y) \cdot q),\ g(y),\ y\Big) \end{cases}$

$$\Rightarrow D_q G(q,y) = \begin{pmatrix} \tau_1(y) \\ \tau_2(y) \\ \vdots \\ \tau_{n-1}(y) \\ \left[F_p(q, g(y), y)\cdot \nu(y)\right]\cdot \nu(y) + \sum_{i=1}^{n-1}\left[F_p(q,g,y)\cdot \tau_i(y)\right]\tau_i(y) \end{pmatrix}_{n\times n}$$

$$\Rightarrow \det D_q G(q,y)$$

$$= \underbrace{\left[F_p(q, g(y), \nu(y))\cdot \nu(y)\right]}_{\neq 0 \text{ near } x^0} \cdot \underbrace{\det\begin{pmatrix} \tau_1(y) \\ \tau_2(y) \\ \vdots \\ \tau_{n-1}(y) \\ \nu(y) \end{pmatrix}}_{\neq 0} \neq 0 \text{ near } x^0.$$

By the condition of the lemma,

By the definition of basis

$\therefore$ The Implicit Function Thm ensures we can uniquely solve the identity $G(q,y)=0$ for $q=q(y)$ explicitly, provided y is close enough to x^0.

$\square$

Remark: If $F_p(p^0, z^0, x^0)\cdot \nu(x^0) \neq 0$, we call it the nonchaculteristic boundary data.

$\downarrow$

In fact, $\partial_s(x(s))\big|_{s=0} = F_p(p^0, z^0, x^0)$.

so this boundary data mean that the characteristics $x(s)$ cannot be a tangent to the boundary.

- Local solutions

For given $x^0 \in \Gamma$ and (p^0, z^0, x^0) is an admissible triple of boundary data. Then we can find a function $q(\cdot)$ near x^0 such that $p^0 = q(x^0)$ and $(q(y), g(y), y)$ is admissible for $\forall y$ close to x^0.

Consider:
$$\begin{cases} \dot p(s) = -F_x(p(s), z(s), x(s)) - F_z(p(s), z(s), x(s)) \\ \dot z(s) = F_p(p(s), z(s), x(s))\cdot p(s) \\ \dot x(s) = F_p(p(s), z(s), x(s)) \end{cases}$$

with the initial condition: $p(0) = q(y)$, $z(0) = g(y)$, $x(0) = y$.

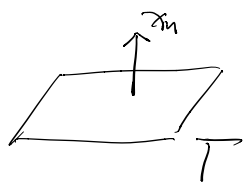
We have a solution: $p = p(y,s)$, $z = z(y,s)$, $x = x(y,s)$

<u>Lemma</u> (local invertibility)

Assume $F_p(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$, then there exists an open interval $I \subseteq \mathbb{R}$ $(0 \in I)$, a neighborhood W of x^0 in T, and a neighborhood V of x^0 in $\mathbb{R}^n$ such that $\forall x \in V$, $\exists$ unique $(s, y) \in I \times W$ satisfies $x = x(y, s)$.

In addition, the mapping $x \mapsto (s, y)$ are C^2.

Pf: In order to simplify the relevant calculations, it is convenient first to change variables, so as to "flatten out" part of the boundary $\partial \mathcal{U}$.



We assume that near x^0 the surface T is flat.

Then the condition "$F_p(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$" change to "$F_{p_n}(p^0, z^0, x^0) \neq 0$."

For $x = x(y, s)$ $\dfrac{\partial x^j}{\partial y_i} := x^j_{y_i}(x^0, 0) = \begin{cases} \delta_{ij} & j = 1, 2, \cdots, n-1 \\ 0 & j = n \end{cases}$

$$\partial_s x(x^0, 0) = F_p(p^0, z^0, x^0) \neq 0.$$

Thus $\det D_{(y,s)} x(x^0, 0) = \det \begin{pmatrix} 1 & & & F_{p_1}(p^0, z^0, x^0) \\ & 1 & & \vdots \\ & & \ddots & \vdots \\ & & 1 & \\ & & & F_{p_n}(p^0, z^0, x^0) \end{pmatrix}$

$$= F_{p_n}(p^0, z^0, x^0) \neq 0.$$

$\therefore$ The Inverse function thm gives us the result:

For given $x \in V$, we can locally find a unique solution to the equation

$$\begin{cases} x = x(y, s) \\ \text{for } y = y(x),\ s = s(x). \end{cases}$$

$\square$

Then we define $\begin{cases} u(x) := z(y(x), s(x)) \\ p(x) := p(y(x), s(x)) \end{cases}$ $(*)$, $x \in V$, s, y as in above Lemma's result

We will check the function u defined above is C^2 and satisfying the PDE with the boundary condition.

We restate the fact in the following thm, which is our principal assertion.

Thm (Local Existence thm).
The function u defined above in (*) is C^2 and solves the PDE:

$$F(Du(x), u(x), x) = 0 \quad (x \in V)$$

with the boundary condition $u(x) = g(x)$ for $x \in \Gamma \cap V$.

Pf: Omitted (cf. Evans §3.2 Thm 2)

Applications

- F Linear: $\begin{cases} F(Du, u, x) = b(x) \cdot Du(x) + c(x) \cdot u(x) = 0. \ (x \in \mathcal{U}) \\ u = g \quad \text{on } \Gamma \end{cases}$

Noncharacteristic assumption: $b(x_0) \cdot \nu(x_0) \neq 0$.

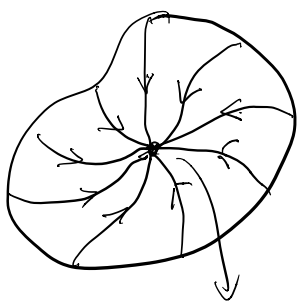
$$\begin{cases} \dot{x}(s) = b(x(s)) \\ \dot{z}(s) = b(x(s)) \cdot p(s) = -c(x(s))\, z(s) \\ \dot{p}(s) = \cdots \end{cases} \quad \Rightarrow \text{don't need it}$$

If $c = 0$, $\dot{z}(s) = 0 \Rightarrow z(s) = \text{const}$ along each trajectory $x(s)$

We consider 3 cases of the trajectories of the ODE:

$$\dot{x}(s) = b(x(s))$$

Case a) flow to an attracting point



WLDG, $x = 0$

$\because$ $z(s) = \text{const}$ along each trajectory $x(s)$

$$\begin{cases} b \cdot Du = 0 \text{ in } \mathcal{U} \\ u = g \text{ on } \Gamma \end{cases}$$

Invoking "Local Existence thm"

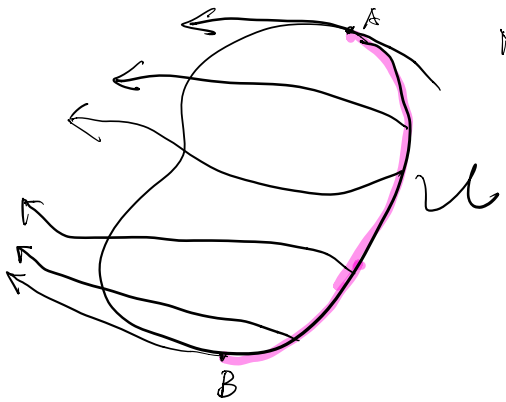
$\Rightarrow \exists$ a unique solution u defined near Γ.

$\Rightarrow u(x(s)) \equiv u(x(0)) = g(x^0)$ for each solution of the ODE $\dot{x}(s) = b(x(s))$ with the initial condition $x(0) = x^0 \in \Gamma$.

However, this solution cannot be smoothly continued to all of $\mathcal{U}$ unless $g \equiv \text{const}$:

$\because$ Each trajectory takes on different values near 0 (the attracting point)

Case b) flow across a domain



Now Suppose trajectories look like this, i.e. we assume that each trajectory of the ODE enters $\mathcal{U}$ precisely once, somewhere through the set

$$T := \{x \in \partial\mathcal{U} \mid b(x)\cdot\nu(x) < 0\}$$

and exits $\mathcal{U}$ precisely once.

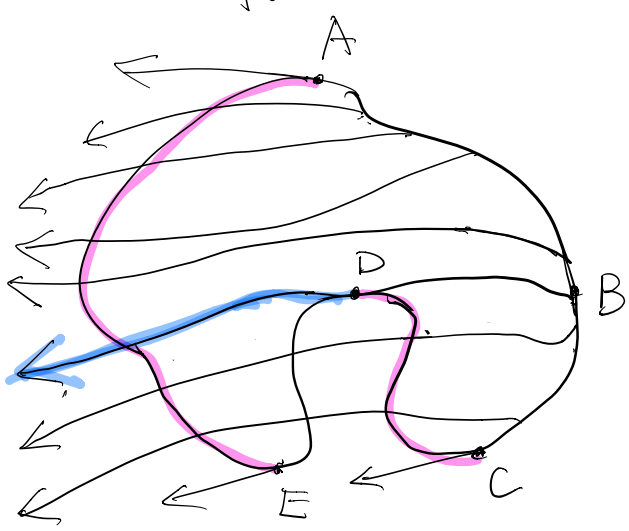
(except those through the characteristic points A, B)

(A, B's trajectories are tangent to $\partial\mathcal{U}$)

Only on the part of $\partial\mathcal{U}$ can we give initial data.

这是叫特征线
指向$\mathcal{U}$内部.
和$b\cdot\nu\neq 0$.

Case c) flow with characteristic points

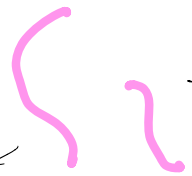


We can now define u to be constant along trajectories, but then u will be discontinuous (unless $g(B)=g(D)$)

D: characteristic point

local existence theory fails near D.

along this trajectory, you cannot know its uniqueness and existence of solution of ODE.

$T \cap$  $= \phi$

You cannot give initial data on this part of $\partial\mathcal{U}$.

· F quasilinear: $\begin{cases} F(Du, u, x) = b(x,u)\cdot Du + c(x,u) = 0. \\ u = g \text{ on } T. \end{cases}$

Noncharacteristic assumption: $b(x^0, z^0)\cdot\nu(x^0) \neq 0$. $(z^0 = g(x^0))$

We consider a concrete example:

<u>Conservation law</u>:

$$\begin{cases} G(Du, \partial_t u, u, x, t) = \partial_t u + \operatorname{div} F(u) = \partial_t u + F'(u)\cdot Du = 0 \quad (x,t)\in\mathbb{R}^n\times(0,+\infty) \\ u = g \text{ on } T = \mathbb{R}^n\times\{t=0\} \end{cases}$$

$F = (F_1(u), F_2(u), \dots, F_n(u)) : \mathbb{R}\to\mathbb{R}^n$

Let $t = x_{n+1}$, then the characteristic ODE:

$$\begin{cases} \partial_s x_{n+1} = 1, \\ \partial_s x_i(s) = F_i'(z(s)) \\ \partial_s z(s) = 0. \end{cases} \Rightarrow \begin{cases} x_{n+1}(s) = s, \\ \dot{x}_i(s) = F_i'(z_0) \\ z(s) = z_0 = g(x^0) \end{cases}$$

$\Rightarrow \quad x(s) = F'(g(x^0))\, s + x^0$

$\Rightarrow$ The projected characteristic $y(s) = (x(s), s)$ is a straight line.
$u = \text{const}$ along the line.

<u>Crossing characteristics</u>

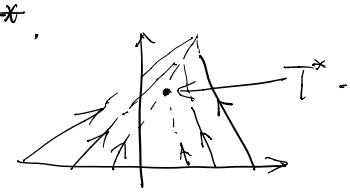
$$u(x(s), s) = z(s) = z_0 = g(x(s) - F'(z_0)\, s)$$
$$= g[x(s) - F'(u(x(s)))\, s]$$

$u(x,t) = g[x - F'(u)t]$ $\to$ Eliminate s, we obtain an implicit formula for u as a function of x and t.

Indeed, if $1 + t\, Dg(x - tF'(u))\cdot F''(u) \neq 0$, the above equation gives an explicit solution u to the original problem.

However, if $n=1$, $F''(u) > 0$, $g' < 0$. this will be false for some time $t > 0$.

$1 + t\, \underbrace{g'(x - tF'(u))}_{<0}\, \underbrace{F''(u)}_{>0} \to 0. \quad t \nearrow T^*.$



· F fully nonlinear.

<u>Example</u> Hamilton-Jacobi equation.

the general Hamilton-Jacobi PDE:

$$G(Du, \partial_t u, u, x, t) = \partial_t u + H(Du, x) = 0.$$

$Du = D_x u$. $\quad q = (p, p_{n+1})$, $\quad y = (x, t)$

$G(q, z, y) = p_{n+1} + H(p, x)$

$D_q G = (D_p H, 1)$ $\quad D_y G = (D_x H, 0)$, $\quad D_z G = 0$.

$$x(s): \begin{cases} \partial_s x^i(s) = H_{p_i}(p(s), x(s)) \\ \partial_s x^{n+1}(s) = 1 \end{cases}$$

$$p(s): \begin{cases} \partial_s p^i(s) = -H_x(p(s), x(s)) \\ \partial_s p^{n+1}(s) = 0 \end{cases}$$

$$z(s): \partial_s z(s) = p^{n+1}(s) + D_p H(p(s), x(s)) \cdot p(s)$$
$$= D_p H(p(s), x(s)) \cdot p(s) - H(p(s), x(s))$$

$$\begin{cases} \partial_s p(s) = -D_x H(p(s), x(s)) & ① \\ \partial_s z(s) = D_p H(p(s), x(s)) \cdot p(s) - H(p(s), x(s)) & ② \\ \partial_s x(s) = D_p H(p(s), x(s)) & ③ \end{cases}$$

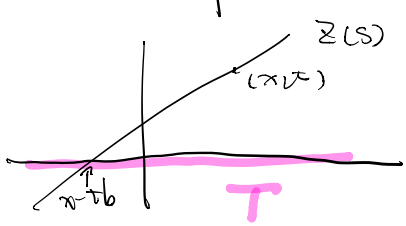
the characteristic equations for the Hamilton-Jacobi equation

①③. $\begin{cases} \partial_s x(s) = D_p H(p, x) \\ \partial_s p(s) = -D_x H(p, x) \end{cases}$ are called the Hamilton's equation.

Observe that the equation for $z(\cdot)$ is trivial, once $x(\cdot)$ and $p(\cdot)$ have been found by solving Hamilton's equations.

$$\partial_s [H(p(s), x(s))] = D_p H \, \partial_s p + D_x H \, \partial_s x$$
$$= -D_p H D_x H + D_x H \cdot D_p H = 0$$
$$\therefore H(p(s), x(s)) = \text{const}.$$

Review: The idea of Characteristics are also used to solve Transport equ.



$z(s) = u(t+s, x+sb)$ ← characteristic

$\partial_s z(s) = 0$.

$\therefore$ for (x, t), we want to know $u(x,t)$

we only need to know $u(x-tb)$ from the initial data.

§23. Hamilton – Jacobi equation

$$\begin{cases} \partial_t u + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0, t\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

H: Hamiltonian $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

$$\begin{cases} \dot{x} = H_p(p, x) \\ \dot{p} = -H_x(p, x) \end{cases} \Rightarrow \text{Hamilton's ODE.}$$

From last section, we know that $H(x(s), p(s)) = \text{const}$

Actually, Hamilton's ODE arise in the classical calculus of variations and in mechanics. We first show how it is derived from a variational principle:

Consider the action functional:

$$I[w(\cdot)] = \int_0^t L(\dot{w}(s), w(s))\, ds \qquad \left(\dot{w}(s) = \frac{d}{ds} w\right)$$

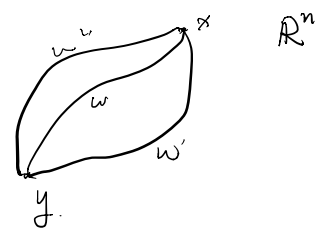
for $w \in \mathcal{A} = \{ w(\cdot) \in C^2([0,t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x \}$

w belong to this admissible class

where $L = L(v, x): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, smooth, given, is called the Lagrangian.

Basic problem: which $x(\cdot) \in \mathcal{A}$ satisfying

$$I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)] \quad ?$$



Thm: If $x(\cdot) \in \mathcal{A}$ is a solution to the above minimizing problem, then:

$$-\frac{d}{ds}\left(D_v L(\dot{x}(s), x(s))\right) + D_x L(\dot{x}(s), x(s)) = 0 \quad (☆)$$

a calculus of variations problem

Euler–Lagrange equation.

本质上讲有些类似是"导数=0"于是类对应极小值点. $F'(x)=0$.

second order ODE system (n equation) $\to$ $x(s)$ n variables

the proof similar to Dirichlet principle

Pf: $w(s) = x(s) + \varepsilon y(s)$ $\quad y(0) = 0$, $y(t) = 0$. y smooth the $w \in \mathcal{A}$.

$$0 \le I[w] - I[x] = \int_0^t \left[L(\dot{x}(s) + \varepsilon \dot{y}(s), x(s) + \varepsilon y(s)) - L(\dot{x}(s), x(s)) \right] ds$$

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^t [\quad]\, ds \ge 0$$

$$\Rightarrow \int_0^t \left[D_v L(\dot{x}(s), x(s))\, \dot{y}(s) + D_x L(\dot{x}(s), x(s))\, y(s) \right] ds \ge 0.$$

$$\Rightarrow \int_0^t \left[-\frac{d}{ds} D_v L(\dot{x}(s), x(s)) + D_x L(\dot{x}(s), x(s)) \right] y(s)\, ds \ge 0,$$

for $\forall y$ smooth. $y(0) = y(t) = 0$.

$$\Rightarrow -\frac{d}{ds} D_v L(\dot{x}(s), x(s)) + D_x L(\dot{x}(s), x(s)) = 0.$$

$\square$

Def A solution of the Euler-Lagrange equation is called a critical point of $I[\cdot]$.

Remark: So every minimizer is a critical point, but a critical point need not be a minimizer.

Euler-Lagrange equation : n second-order ODEs

We want to change it into $2n$ first-order ODEs.
A simple method is to let $\alpha(s) = \dot{x}(s)$

$$\begin{cases} \dot{x}(s) = \alpha(s) \\ -\frac{d}{ds} D_v L(\alpha(s), x(s)) + D_x L(\alpha(s), x(s)) = 0. \end{cases}$$

$$\Longleftrightarrow \begin{cases} \dot{x}(s) = \alpha(s) \\ -\underbrace{D_{vv} L(\alpha(s), x(s))}_{n\times n \text{ matrix}} \cdot \dot{\alpha}(s) - D_{vx} L(\alpha(s), x(s))\, \alpha(s) + D_x L(\alpha(s), x(s)) = 0 \end{cases}$$

We don't like this term. This is difficult for us to solve.

So let $p(s) = D_v L(\dot{x}(s), x(s))$

For all $x, p \in \mathbb{R}^n$, we assume that $p = D_v L(v, x)$ has a unique solution v which is a smooth function of (x, p), denoted by $v = v(p, x)$

Def: The Hamilton H associated with the Lagrangian L is

$$H(p, x) = p \cdot v(p, x) - L(v(p, x), x) \qquad (p, x \in \mathbb{R}^n)$$

where $v(p, x)$ is defined (LT)

Laplace transformation.

Thm If $x(s)$ is a solution to the Euler-Lagrange equation, then the corresponding $x(s), p(s)$ satisfy the Hamilton's equations :

$$\begin{cases} \dot{x}(s) = D_p H(p(s), x(s)) \\ \dot{p}(s) = -D_x H(p(s), x(s)) \end{cases} \qquad \text{for } 0 \le s \le t.$$

Moreover, $H(p(s), x(s))$ is constant.

Pf: $D_p H(p(s), x(s)) = v(p(s), x(s)) + \underbrace{p(s) \cdot D_p v(p(s), x(s))}_{= D_v L(v(p,x), x)\, D_p v(p,x)}$

$$= v(p(s), x(s)) = \dot{x}(s)$$

$$D_x H(p(s), x(s)) = p \cdot D_x v(p, x) - D_v L(v(p,x), x) \cdot D_x v(p, x) - D_x L(v(p,x), x)$$

$$= -D_x L(v(p, x), x)$$

By defn of p. $= -D_x L(\dot{p}(x), x(s))$

By Euler-Lagrange equ. $= -\frac{d}{ds}(D_v L(\dot{x}(s), x(s))) = -\dot{p}(s)$

Moreover,

$$\frac{d}{ds} H(p(s), x(s)) = D_p H(p(s), x(s)) \cdot \dot{p}(s) + D_x H(p(s), x(s)) \cdot \dot{x}(s) = 0$$

$\therefore$ $H(p(s), x(s))$ is constant. $\square$.

Now let us try to find a connection between the Hamilton-Jacobi PDE and the calculus of variations problem.

$$I[x(\cdot)] = \min_{w \in \mathcal{A}} \underbrace{I[w(\cdot)]}_{\int_0^T L(\dot{w}(s), w(s))\, ds}$$

To simplify, we also drop the x-dependence in the Hamiltonian, then $H = H(p)$.

- Legendre transform.

For given Lagrangian $L: \mathbb{R}^n \to \mathbb{R}$, which is convex and $\frac{L(v)}{|v|} \to +\infty$ as $|v| \to +\infty$.

we define the Legendre transform of L is $L^*(p) = \sup_{v \in \mathbb{R}^n} \{vp - L(v)\}$ $(p \in \mathbb{R}^n)$

Remark: $p = D_v L(v)$ v satisfies $p = D_v L(v)$

<u>Thm</u>: If $L^*(p) = H(p)$, then $H^*(p) = L(p)$.

Pf: ① First, we prove H is convex

We need to verify that

$H(tp_1 + (1-t)p_2) \leq t H(p_1) + (1-t) H(p_2)$. $\forall t \in (0,1)$, $p_1, p_2 \in \mathbb{R}^n$

$$\begin{aligned}
H(tp_1 + (1-t)p_2) &= \sup_{v \in \mathbb{R}^n} \{v \cdot (tp_1 + (1-t)p_2) - L(v)\} \\
&= \sup_{v \in \mathbb{R}^n} \{t(v \cdot p_1 - L(v)) + (1-t)(v \cdot p_2 - L(v))\} \\
&\leq t \cdot \sup_{v \in \mathbb{R}^n} \{v \cdot p_1 - L(v)\} + (1-t) \sup_{v \in \mathbb{R}^n} \{v \cdot p_2 - L(v)\} \\
&= t H(p_1) + (1-t) H(p_2)
\end{aligned}$$

$\therefore$ H is convex

② Second, we should prove that $\frac{H(v)}{|v|} \to +\infty$ as $|v| \to +\infty$.

For $\forall \lambda > 0$,

$$\begin{aligned}
H(p) = \sup_{v \in \mathbb{R}^n} \{v \cdot p - L(v)\} &\geq \sup_{|v| = \lambda} \{v \cdot p - L(v)\} \\
&\geq \lambda |p| - L\left(\frac{\lambda p}{|p|}\right) \\
&\geq \lambda |p| - \max_{B_\lambda(0)} L(v)
\end{aligned}$$

$$\Rightarrow \frac{H(p)}{|p|} \geq \lambda - \frac{1}{|p|} \max_{B_r(0)} L(v)$$

$$\geq \frac{\lambda}{2}$$

$$\downarrow$$

$$p \to +\infty.$$

$$\therefore \frac{H(p)}{|p|} \to +\infty \quad \text{as} \quad |p| \to +\infty.$$

② We prove that $H^*(w) \leq L(w)$, $H^*(w) \geq L(w)$.

$$H^*(w) = \sup_{p \in \mathbb{R}^n} \{ w \cdot p - H(p) \}$$

$$\leq \sup_{p \in \mathbb{R}^n} \{ w \cdot p - (p \cdot w - L(w)) \} = \sup_{p \in \mathbb{R}^n} \{ L(w) \} = L(w)$$

$$\downarrow$$

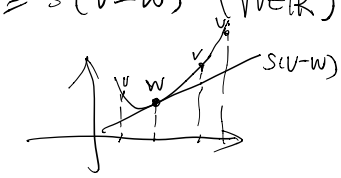
$$\because H(p) \geq pw - L(w)$$

On the other hand,

$\because$ $L(v)$ is convex

$\therefore$ we can find $s \in \mathbb{R}^n$, s.t. $L(v) - L(w) \geq s(v-w)$ $(\forall v \in \mathbb{R}^n)$

Another property (definition) of convexity.



$$\therefore H^*(w) \geq ws - H(s) = w \cdot s - \sup_v \{ s \cdot v - L(v) \}$$

$$= w \cdot s - (s \cdot w - L(w)) = L(w)$$

$\square$

Remark: If $L = H^*$, $H = L^*$

We say L and H are dual convex functions.

For dual convex functions, we have the following property:

$$\begin{cases} p \cdot v = L(v) + H(p) \\ p = D_v L(v) \\ v = D_p H(p) \end{cases} \quad \text{if } D_v L,\ D_p H \text{ exist.}$$

对给定p，使得L^*取sup的v ←

或是对给定v，使H^*p取sup的p. ←

- Hopf-Lax formula

The characteristic equations for the H-J equation are

$$\begin{cases} \dot{p}(s) = 0 \\ \dot{z}(s) = DH(p) \cdot p - H(p) \\ \dot{x}(s) = DH(p) \end{cases}$$

$\longrightarrow$ Hamilton's equation

$\Longleftrightarrow$ A minimizing problem for $L = H^*$

$$\because \quad L(\dot{x}(s)) = H^*(\dot{x}(s)) = \sup_{v\in\mathbb{R}^n}\{v\cdot\dot{x}(s) - H(v)\}$$
$$= \sup_{v\in\mathbb{R}^n}\{v\cdot DH(p) - H(v)\}$$

$$\therefore \quad \dot{z}(s) \le L(\dot{x}(s))$$
$$\Rightarrow \quad u(x(t)) \le \int_0^t L(\dot{x}(s))\,ds + \underbrace{u(x(0))}_{g(x(0))}$$

Therefore, we only have to solve the minimization problem

$$u(x,t) = \inf_{w\in C^1}\left\{\int_0^t L(\dot{w}(s))\,ds + g(w(0)) \;\middle|\; w(t) = x\right\} \quad (*)$$

Suppose L is convex and $\lim_{|x|\to+\infty}\frac{L(x)}{|x|} = +\infty$,
$g:\mathbb{R}^n\to\mathbb{R}$ is Lipschitz continuous (i.e. $Lip(g) := \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}}\left\{\frac{|g(x)-g(y)|}{|x-y|}\right\} < +\infty$)
($g\in C^1 \Rightarrow g$ Lipschitz)

All the theorems below, we admit this assumption.

Thm: If $x\in\mathbb{R}^n$, $t>0$, the solution of the minimization problem $(*)$ is
$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{t\,L\left(\frac{x-y}{t}\right) + g(y)\right\} \quad (\star)$$
(Hopf-Lax formula)

($\because \lim_{|y|\to+\infty} t\,L\left(\frac{x-y}{t}\right) + g(y) = +\infty$ $\therefore$ the minimum point can be attained.)

Pf: Let $w(s) = \frac{(t-s)y + sx}{t}$. then $\dot{w}(s) = \frac{x-y}{t}$ $\quad w(0) = y$.

Then the solution of $(*) \le \int_0^t L(\dot{w}(s))\,ds + g(w(0)) = t\,L\left(\frac{x-y}{t}\right) + g(y)$

$\Rightarrow$ the solution of $(*) \le$ the solution of $(\star)$

On the other hand, as L is convex, one has
$$\frac{1}{t}\int_0^t L(\dot{w}(s))\,ds \ge L\left(\frac{1}{t}\int_0^t \dot{w}(s)\,ds\right) = L\left(\frac{w(t)-w(0)}{t}\right)$$
$$\Rightarrow \int_0^t L(\dot{w}(s))\,ds + g(w(0)) \ge t\,L\left(\frac{w(t)-w(0)}{t}\right) + g(w(0))$$
$$\Rightarrow \inf_{w(t)=x}\left(\frac{1}{t}\int_0^t L(\dot{w}(s))\,ds + g(w(0))\right) \ge \inf_{y\in\mathbb{R}^n}\left\{t\,L\left(\frac{x-y}{t}\right) + g(y)\right\}$$

$\Rightarrow$ the solution of $(*) \ge$ the solution of $(\star)$

the solution of $(*) =$ the solution of $(\star)$

Thm Let $u(x,t)$ be the function given by the Hopf-Lax formula (*) then:

(i) $u(x,t)$ is Lipschitz continuous.

(ii) For $\forall x \in \mathbb{R}^n$, $0 \le s < t$, $u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}$

(iii) $u(x,t)$ is differentiable for a.e. $(x,t) \in \mathbb{R}^n \times (0,+\infty)$, and solves the equation $\begin{cases} \partial_t u(x,t) + H(Du(x,t)) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$ on the differentiable points of u.

Pf: i) For $\forall x, \tilde{x} \in \mathbb{R}^n$, we need to prove $|u(x,t) - u(\tilde{x},t)| \le C|x-\tilde{x}|$

Choose $y \in \mathbb{R}^n$, s.t. $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$

$$u(\tilde{x},t) - u(x,t) = \inf_{z \in \mathbb{R}^n} \left(tL\left(\frac{\tilde{x}-z}{t}\right) + g(z) \right) - \left(tL\left(\frac{x-y}{t}\right) + g(y) \right)$$

$\because \lim_{|y| \to +\infty} tL\left(\frac{x-y}{t}\right) + g(y) = +\infty$

$\therefore$ the minimum point can be attained.

$$\le tL\left(\frac{\tilde{x} - (\tilde{x} - x + y)}{t}\right) + g(\tilde{x} - x + y) - tL\left(\frac{x-y}{t}\right) - g(y)$$

$$= g(\tilde{x} - x + y) - g(y)$$

$$\le Lip(g) \cdot |\tilde{x} - x|$$

$\therefore u$ is Lipschitz continuous. $\square$

(ii) ① Moreover, $u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$

$$\le tL(0) + g(x)$$

Let $x = y$

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$$

$$\ge g(x) + \inf_{y \in \mathbb{R}^n} \left\{ -Lip(g)|x-y| + tL\left(\frac{x-y}{t}\right) \right\}$$

$(\because g(y) - g(x) \ge -Lip(g)|x-y|)$

$$= g(x) + t \inf_{z \in \mathbb{R}^n} \left\{ -Lip(g)|z| + L(z) \right\}$$

$$= g(x) - t \max_{z \in \mathbb{R}^n} \left\{ Lip(g)|z| - L(z) \right\}$$

$$= g(x) - t \max_{w \in B(0, Lip(g))} \max_{z \in \mathbb{R}^n} \left\{ w \cdot z - L(z) \right\}$$

$$= g(x) - t \max_{w \in B(0, Lip(g))} H(w)$$

$$\Rightarrow |u(x,t) - g(x)| \le Ct$$

Let $t \to 0 \Rightarrow u(x,0) = g(x)$

② Then we prove that $u(x,t) = \min_{y\in\mathbb{R}^n} \left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\}$

"$\leq$" For $y\in\mathbb{R}^n$, $0\leq s<t$

Choose $z\in\mathbb{R}^n$, s.t. $u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z)$

$$u(x,t) \leq tL\left(\frac{x-z}{t}\right) + g(z)$$

$$\underset{\downarrow}{\leq} (t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-z}{s}\right) + g(z)$$

($\because$ L is convex)

$$= (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

$$\therefore u(x,t) \leq \min_{y\in\mathbb{R}^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\}$$

"$\geq$" Assume $u(x,t) = tg\left(\frac{x-w}{t}\right) + g(w)$

Let $y = \frac{s}{t}x + \frac{t-s}{t}w$

$$(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

$$\leq (t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-w}{s}\right) + g(w)$$

$\frac{x-y}{t-s} = \frac{y-w}{s}$

$$= (t-s)L\left(\frac{x-w}{t}\right) + sL\left(\frac{x-w}{t}\right) + g(w)$$

$$= tL\left(\frac{x-w}{t}\right) + g(w) = u(x,t)$$

$$\therefore u(x,t) = \min_{y\in\mathbb{R}^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\}$$ □.

(iii) ① $\because$ u is Lipschitz continuous

$\therefore$ u is differentiable for a.e. (x,t)

(This is a property of a Lipschitz continuous function)

② Then we prove that for $\forall (x,t)\in\mathbb{R}^n\times(0,+\infty)$,

u is differentiable at (x,t)

we have $\partial_t u + H(Du) = 0$.

"$\leq$" We only need prove $\partial_t u + v\cdot Du(x,t) \leq L(v)$

(then $\partial_t u + H(Du)$
$= \partial_t u + \max\{v\cdot Du - L(v)\} \leq 0.$)

$$u(x+\varepsilon v, t+\varepsilon) = \min_{y\in\mathbb{R}^n}\left\{\varepsilon L\left(\frac{x+\varepsilon v-y}{\varepsilon}\right) + u(y,t)\right\}$$

$$\leq \varepsilon\cdot L(v) + u(x,t)$$

let $\varepsilon\to 0$, $\Rightarrow \partial_t u + v\cdot Du(x,t) \leq L(v)$

"$\geq$" Let $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$ and $s = t-\varepsilon$, $y = \frac{s}{t}x + \frac{\varepsilon z}{t}$

Then $u(x,t) - u(y,s) \geq tL\left(\frac{x-z}{t}\right) + g(z) - \left(sL\left(\frac{y-z}{s}\right) + g(z)\right)$

$$\left(\begin{array}{l} \because u(y,s) \leq sL\left(\frac{y-z}{s}\right) + g(z) \\ u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z) \end{array}\right)$$

$$= (t-s)L\left(\frac{x-z}{t}\right)$$

$$\Rightarrow u(x,t) - u\left(x + \varepsilon\frac{z-x}{t},\ t-\varepsilon\right) \geq \varepsilon L\left(\frac{x-z}{t}\right)$$

$\varepsilon \to 0$, $\Rightarrow \partial_t u + H(Du) \geq \partial_t u + \frac{z-x}{t}\cdot Du - L\left(\frac{z-x}{t}\right) \geq 0$

$\square$

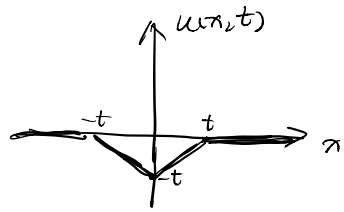
Remark: We have constructed a solution <u>differentiable and satisfying the H-J equation a.e.</u>.

☆: This kind of solution is not unique!!!

Example: $\begin{cases} \partial_t u + |Du|^2 = 0, & t>0 \\ u = 0, & t=0 \end{cases}$ $(n=1)$

Solution $u \equiv 0$

or $u = \begin{cases} 0 & |x|>t \\ x-t & 0 \leq x \leq t \\ -x-t & -t \leq x \leq 0 \end{cases}$



To make sure "uniqueness", we introduce "weak solutions"

· Weak solutions.

<u>Def</u>: If a Lipschitz function $u(x,t): \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ satisfies:

(i) $u(x,0) = g(x)$

(ii) $\partial_t u(x,t) + H(Du) = 0$. a.e. $(x,t) \in \mathbb{R}^n \times (0,+\infty)$

(iii) $\exists C>0$, $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C\left(1+\frac{1}{t}\right)|z|^2$ for $\forall x, z \in \mathbb{R}^n$, $t>0$.

Semiconcavity

then u is called a <u>weak solution</u> of the H-J equation.

<u>Thm</u>: If $H \in C^2$ is convex, $\frac{H(p)}{|p|} \to +\infty$ as $|p| \to +\infty$.
and g is Lipschitz, then there is at most one weak solution of the H-J equation.

Pf: Omitted. (Evans Page 132).

Thm If g is semiconcave or H is uniformly convex, then the solution given by the Hopf-Lax formula is a weak solution to the H-J equation.

For this theorem, we introduce 2 definitions and then 2 lemmas, as a result, this theorem is a direct corollary.

Def If $g(x+z) - 2g(x) + g(x-z) \leq C|z|^2$ for some constant C, $\forall x, z \in \mathbb{R}^n$, then g is called semiconcave.

Def A C^2 convex function $H: \mathbb{R}^n \to \mathbb{R}$ is called uniformly convex (with constant $\theta > 0$) if

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\, \xi_i \xi_j \geq \theta |\xi|^2 \text{ for } \forall p, \xi \in \mathbb{R}^n.$$

Lemma: If g is semiconcave, then u given by Hopf-Lax formula satisfies $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C|z|^2$

Pf: Choose $y \in \mathbb{R}^n$ s.t. $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$,

thus: $u(x+z,t) + u(x-z,t) - 2u(x,t)$

$$\leq tL\left(\frac{x+z-(y+z)}{t}\right) + g(y+z) + tL\left(\frac{x-z-(y-z)}{t}\right) + g(y-z) - 2\left(tL\left(\frac{x-y}{t}\right) + g(y)\right)$$

$$= g(y+z) + g(y-z) - 2g(y) \leq C|z|^2$$

$\square$

Lemma If H is uniformly convex with constant θ, then u (given by Hopf-Lax formula) satisfies:

$$u(x+z,t) + u(x-z,t) - 2u(x) \leq \frac{1}{\theta t}|z|^2 \text{ for all } x, z \in \mathbb{R}^n, t>0$$

Pf: $u(x+z,t) + u(x-z,t) - 2u(x,t)$

$$\leq tL\left(\frac{x+z-y}{t}\right) + g(y) + tL\left(\frac{x-z-y}{t}\right) + g(y) - 2\left(tL\left(\frac{x-y}{t}\right) + g(y)\right)$$

$$= t\left(L\left(\frac{x+z-y}{t}\right) + L\left(\frac{x-z-y}{t}\right) - 2L\left(\frac{x-y}{t}\right)\right)$$

$$\leq \frac{1}{\theta t}|z|^2$$

$$L(v) = \inf_{p \in \mathbb{R}^n} \{v \cdot p - H(p)\}$$

Aim to prove $L(x+z) + L(x-z) - 2L(x) \leq \frac{1}{\theta}|z|^2$.

Take $p_0 \in \mathbb{R}^n$, s.t. $L(x) = x \cdot p_0 - H(p_0)$

then: $L(x+z)+L(x-z)-2L(x)$

$$= \inf_{p\in\mathbb{R}^n}\{(x+z)\cdot p - H(p)\} + \inf_{p\in\mathbb{R}^n}\{(x-z)\cdot p - H(p)\} - 2\left(x\cdot p_0 - H(p_0)\right)$$

$$\leq (x+z)\cdot\left(p_0+\frac{z}{\theta}\right) - H\left(p_0+\frac{z}{\theta}\right) + (x-z)\cdot\left(p_0-\frac{z}{\theta}\right) - H\left(p_0-\frac{z}{\theta}\right) - 2x\cdot p_0 + 2H(p_0)$$

$$\leq \frac{2}{\theta}|z|^2 - H\left(p_0+\frac{z}{\theta}\right) - H\left(p_0-\frac{z}{\theta}\right) + 2H(p_0)$$

$$\leq \frac{2}{\theta}|z|^2 - \frac{1}{\theta}|z|^2 = \frac{1}{\theta}|z|^2$$

$\square$,

$\because \sum_{i,j=1}^n H_{p_ip_j}(p)\xi_i\xi_j \geq \theta|\xi|^2$ for $\forall p,\xi\in\mathbb{R}^n$.

$\therefore 2H(p) \leq H(p+q)+H(p-q)-\theta|q|^2$

$$2H(p) - H(p+q) - H(p-q)$$

$$\overset{\text{Taylor's formula}}{=} 2H(p) - H(p) - DH(p)\cdot q - \frac{1}{2}\sum_{i,j=1}^n H_{p_ip_j}(p+\xi)\cdot q_iq_j - H(p) + DH(p)\cdot q - \frac{1}{2}\sum_{i,j=1}^n H_{p_ip_j}(p+\xi)\, q_iq_j$$

$$\leq -\theta|q|^2$$

<u>Example 1</u>: $\begin{cases} \partial_t u + \frac{1}{2}|Du|^2 = 0. & \text{in } \mathbb{R}^n\times(0,+\infty) \\ u = |x| & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases}$ $\longrightarrow$ $g=|x|$ is not semiconcave

(for $\forall C>0$, take $x=0$, $|z|$ sufficiently small, then $|x+z|-2|x|+|x-z|>C|z|^2$)

$H(p)=\frac{1}{2}|p|^2$ $\quad L(v)=\frac{1}{2}|v|^2$ $\longrightarrow$ H is uniformly convex.

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}$$

$$= \min_{y\in\mathbb{R}^n}\left\{t\cdot\frac{1}{2}\left|\frac{x-y}{t}\right|^2+|y|\right\} = \min_{y\in\mathbb{R}^n}\left\{\underbrace{\frac{1}{2t}|x-y|^2+|y|}\right\}$$

$$\| $$

$$\frac{|x|^2}{2t}+\frac{1}{2t}|y|^2+|y|-\frac{1}{t}x\cdot y$$

$$\geq \frac{|x|^2}{2t}+\underbrace{\frac{1}{2t}|y|^2+\left(1-\frac{|x|}{t}\right)|y|}$$

"$=$": $y=-kx$, $k>0$. $\qquad$ minimum point: $|y|=|x|-t$

$$\geq \begin{cases} \frac{|x|^2}{2t} & |x|\leq t \\ \frac{|x|^2}{2t}+\frac{1}{2t}(|x|-t)^2+\left(1-\frac{|x|}{t}\right)(|x|-t) & |x|>t \end{cases}$$

$$\| $$

$$|x|-\frac{t}{2}$$

Another way to obtain the expression for $u(x,t)$

$$0 = D_y\left(\frac{|x-y|^2}{2t}+|y|\right) = \frac{y-x}{t}+\frac{y}{|y|} \quad (y\neq 0)$$

$$\Longleftrightarrow y = (|x|-t)\frac{x}{|x|} \neq 0.$$

$$\Longrightarrow u(x,t) = \begin{cases} |x|-\frac{t}{2} & |x|>t \\ \frac{|x|^2}{2t} & |x|\leq t \end{cases}$$

<u>Example 2</u>: $\begin{cases} \partial_t u + \frac{1}{2}|Du|^2 = 0. & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = -|x| & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$g = -|x|$ satisfies semi-convexity

$\because \; -|x+z| + 2|x| - |x-z| \leq 0 < |z|^2$

$\frac{1}{2}(|x+z| + |x-z|) \geq |x|$

$$\Rightarrow u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{|x-y|^2}{2t} - |y| \right\}$$

$$D_y\left(\frac{|x-y|^2}{2t} - |y| \right) = \frac{y-x}{t} - \frac{y}{|y|} \quad (y \neq 0)$$

and this equals zero if $x = y - \frac{y}{|y|} t$

$$\Rightarrow \quad u(x,t) = -|x| - \frac{t}{2} \quad (x \in \mathbb{R}^n, t > 0)$$