- Hamilton – Jacobi equation

$$\begin{cases} \partial_t u + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0, t \vee 0) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

H : Hamiltonian $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

$$\begin{cases} \dot{x} = H_p(p, x) \\ \dot{p} = -H_x(p, x) \end{cases} \Rightarrow \text{Hamilton's ODE.}$$

From last section, we know that $H(x(s), p(s)) = \text{const}$

Actually, Hamilton's ODE arise in the classical calculus of variations and in mechanics. We first show how it is derived from a variational principle:

Consider the action functional:

$$I[w(\cdot)] = \int_0^t L(\dot{w}(s), w(s))\, ds \qquad \left(\dot{w}(s) = \frac{d}{ds} w\right)$$

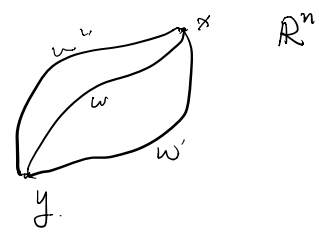
for $w \in \mathcal{A} = \{ w(\cdot) \in C^2([0,t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x \}$

w belong to this admissible class

where $L = L(v, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, smooth, given, is called the Lagrangian.

Basic problem: which $x(\cdot) \in \mathcal{A}$ satisfying

$$I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)] \quad ?$$



Thm: If $x(\cdot) \in \mathcal{A}$ is a solution to the above minimizing problem, then:

$$-\frac{d}{ds}\left(D_v L(\dot{x}(s), x(s))\right) + D_x L(\dot{x}(s), x(s)) = 0 \quad (\star)$$

a calculus of variations problem

Euler–Lagrange equation.

本质上讲有些类似是"导数 = 0" 于是类对应极小值点. $F'(x) = 0$.

second order ODE system (n equation) $\to$ $x(s)$ n variables

the proof similar to Dirichlet principle

Pf: $w(s) = x(s) + \varepsilon y(s)$ $\quad y(0) = 0$, $y(t) = 0$. y smooth the $w \in \mathcal{A}$.

$$0 \le I[w] - I[x] = \int_0^t \left[L(\dot{x}(s) + \varepsilon \dot{y}(s), x(s) + \varepsilon y(s)) - L(\dot{x}(s), x(s)) \right] ds$$

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^t [\quad]\, ds \ge 0$$

$$\Rightarrow \int_0^t \left[D_v L(\dot{x}(s), x(s))\, \dot{y}(s) + D_x L(\dot{x}(s), x(s))\, y(s) \right] ds \ge 0.$$

$$\Rightarrow \int_0^t \left[-\frac{d}{ds} D_v L(\dot{x}(s), x(s)) + D_x L(\dot{x}(s), x(s)) \right] y(s)\, ds \ge 0,$$

for $\forall y$ smooth. $y(0) = y(t) = 0$.

$$\Rightarrow -\frac{d}{ds} D_v L(\dot{x}(s), x(s)) + D_x L(\dot{x}(s), x(s)) = 0.$$

$\square$

Def A solution of the Euler-Lagrange equation is called a critical point of $I[\cdot]$.

Remark: So every minimizer is a critical point, but a critical point need not be a minimizer.

Euler-Lagrange equation : n second-order ODEs

We want to change it into $2n$ first-order ODEs.
A simple method is to let $\alpha(s) = \dot{x}(s)$

$$\begin{cases} \dot{x}(s) = \alpha(s) \\ -\frac{d}{ds} D_v L(\alpha(s), x(s)) + D_x L(\alpha(s), x(s)) = 0. \end{cases}$$

$$\Longleftrightarrow \begin{cases} \dot{x}(s) = \alpha(s) \\ -\underbrace{D_{vv} L(\alpha(s), x(s))}_{n\times n \text{ matrix}} \cdot \dot{\alpha}(s) - D_{vx} L(\alpha(s), x(s))\, \alpha(s) + D_x L(\alpha(s), x(s)) = 0 \end{cases}$$

We don't like this term. This is difficult for us to solve.

So let $p(s) = D_v L(\dot{x}(s), x(s))$

For all $x, p \in \mathbb{R}^n$, we assume that $p = D_v L(v, x)$ has a unique solution v which is a smooth function of (x, p), denoted by $v = v(p, x)$

Def: The Hamilton H associated with the Lagrangian L is

$$H(p, x) = p \cdot v(p, x) - L(v(p, x), x) \qquad (p, x \in \mathbb{R}^n)$$

where $v(p, x)$ is defined (LT)

Laplace transformation.

Thm If $x(s)$ is a solution to the Euler-Lagrange equation, then the corresponding $x(s), p(s)$ satisfy the Hamilton's equations :

$$\begin{cases} \dot{x}(s) = D_p H(p(s), x(s)) \\ \dot{p}(s) = -D_x H(p(s), x(s)) \end{cases} \qquad \text{for } 0 \le s \le t.$$

Moreover, $H(p(s), x(s))$ is constant.

Pf: $D_p H(p(s), x(s)) = v(p(s), x(s)) + p(s) \cdot D_p v(p(s), x(s))$

$$- \underbrace{D_v L(v(p, x), x)}_{= p} D_p v(p, x)$$

$$= v(p(s), x(s)) = \dot{x}(s)$$

$$D_x H(p(s), x(s)) = p \cdot D_x v(p, x) - D_v L(v(p, x), x) \cdot D_x v(p, x) - D_x L(v(p, x), x)$$

$$= -D_x L(v(p, x), x)$$

By defn of p. $= -D_x L(\dot{p}(x), x(s))$

By Euler-Lagrange equ. $= -\frac{d}{ds}(D_v L(\dot{x}(s), x(s))) = -\dot{p}(s)$

Moreover,

$$\frac{d}{ds} H(p(s), x(s)) = D_p H(p(s), x(s)) \cdot \dot{p}(s) + D_x H(p(s), x(s)) \cdot \dot{x}(s) = 0$$

$\therefore$ $H(p(s), x(s))$ is constant. $\square$.

Now let us try to find a connection between the Hamilton-Jacobi PDE and the calculus of variations problem.

$$I[x(\cdot)] = \min_{w \in \mathcal{A}} \underbrace{I[w(\cdot)]}_{\int_0^{\tau} L(\dot{w}(s), w(s))\, ds}$$

To simplify, we also drop the x-dependence in the Hamiltonian, then $H = H(p)$.

- Legendre transform.

For given Lagrangian $L: \mathbb{R}^n \to \mathbb{R}$, which is convex and $\frac{L(v)}{|v|} \to +\infty$ as $|v| \to +\infty$.

we define the Legendre transform of L is $L^*(p) = \sup_{v \in \mathbb{R}^n} \{vp - L(v)\}$ $(p \in \mathbb{R}^n)$

Remark: $p = D_v L(v)$ v satisfies $p = D_v L(v)$

<u>Thm</u>: If $L^*(p) = H(p)$, then $H^*(p) = L(p)$.

Pf: ① First, we prove H is convex

We need to verify that

$H(tp_1 + (1-t)p_2) \leq t H(p_1) + (1-t) H(p_2)$. $\forall t \in (0,1)$, $p_1, p_2 \in \mathbb{R}^n$

$$\begin{aligned}
H(tp_1 + (1-t)p_2) &= \sup_{v \in \mathbb{R}^n} \{v \cdot (tp_1 + (1-t)p_2) - L(v)\} \\
&= \sup_{v \in \mathbb{R}^n} \{t(v \cdot p_1 - L(v)) + (1-t)(v \cdot p_2 - L(v))\} \\
&\leq t \cdot \sup_{v \in \mathbb{R}^n} \{v \cdot p_1 - L(v)\} + (1-t) \sup_{v \in \mathbb{R}^n} \{v \cdot p_2 - L(v)\} \\
&= t H(p_1) + (1-t) H(p_2)
\end{aligned}$$

$\therefore$ H is convex

② Second, we should prove that $\frac{H(v)}{|v|} \to +\infty$ as $|v| \to +\infty$.

For $\forall \lambda > 0$,

$$\begin{aligned}
H(p) = \sup_{v \in \mathbb{R}^n} \{v \cdot p - L(v)\} &\geq \sup_{|v| = \lambda} \{v \cdot p - L(v)\} \\
&\geq \lambda |p| - L\left(\frac{\lambda p}{|p|}\right) \\
&\geq \lambda |p| - \max_{B_\lambda(0)} L(v)
\end{aligned}$$

$$\Rightarrow \frac{H(p)}{|p|} \geq \lambda - \frac{1}{|p|} \max_{B_r(0)} L(v)$$

$$\geq \frac{\lambda}{2}$$

$$\downarrow$$

$$p \to +\infty.$$

$$\therefore \frac{H(p)}{|p|} \to +\infty \quad \text{as} \quad |p| \to +\infty.$$

② We prove that $H^*(w) \leq L(w)$, $H^*(w) \geq L(w)$.

$$H^*(w) = \sup_{p \in \mathbb{R}^n} \{ w \cdot p - H(p) \}$$

$$\leq \sup_{p \in \mathbb{R}^n} \{ w \cdot p - (p \cdot w - L(w)) \} = \sup_{p \in \mathbb{R}^n} \{ L(w) \} = L(w)$$

$$\downarrow$$

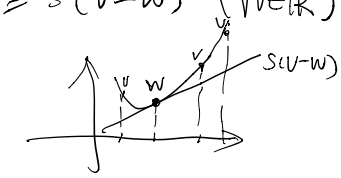
$$\because H(p) \geq pw - L(w)$$

On the other hand,

$\because$ $L(v)$ is convex

$\therefore$ we can find $s \in \mathbb{R}^n$, s.t. $L(v) - L(w) \geq s(v-w)$ $(\forall v \in \mathbb{R}^n)$

Another property (definition) of convexity.



$$\therefore H^*(w) \geq ws - H(s) = w \cdot s - \sup_v \{ s \cdot v - L(v) \}$$

$$= w \cdot s - (s \cdot w - L(w)) = L(w)$$

$\square$

Remark: If $L = H^*$, $H = L^*$

We say L and H are dual convex functions.

For dual convex functions, we have the following property:

$$\begin{cases} p \cdot v = L(v) + H(p) \\ p = D_v L(v) \\ v = D_p H(p) \end{cases} \quad \text{if } D_v L,\ D_p H \text{ exist.}$$

对给定p，使得L^*取sup的v ←

或是对给定v，使H^*p取sup的p. ←

- Hopf-Lax formula

The characteristic equations for the H-J equation are

$$\begin{cases} \dot{p}(s) = 0 \\ \dot{z}(s) = DH(p) \cdot p - H(p) \\ \dot{x}(s) = DH(p) \end{cases}$$

$\longrightarrow$ Hamilton's equation

$\Longleftrightarrow$ A minimizing problem for $L = H^*$

$\because \quad L(\dot{x}(s)) = H^*(\dot{x}(s)) = \sup_{v\in\mathbb{R}^n}\{ v\cdot\dot{x}(s) - H(v)\}$

$$= \sup_{v\in\mathbb{R}^n}\{ v\cdot DH(p) - H(v)\}$$

$\therefore \quad \dot{z}(s) \le L(\dot{x}(s))$

$\Rightarrow \quad u(x(t)) \le \int_0^t L(\dot{x}(s))\,ds + \underbrace{u(x(0))}_{g(x(0))}$

Therefore, we only have to solve the minimization problem

$$u(x,t) = \inf_{w\in C^1}\left\{ \int_0^t L(\dot{w}(s))\,ds + g(w(0)) \;\middle|\; w(t) = x \right\} \quad (*)$$

Suppose L is convex and $\lim_{|x|\to+\infty} \frac{L(x)}{|x|} = +\infty$,

$g:\mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous (i.e. $Lip(g) := \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \left\{\frac{|g(x)-g(y)|}{|x-y|}\right\} < +\infty$)

($g\in C^1 \Rightarrow g$ Lipschitz)

All the theorems below, we admit this assumption.

Thm: If $x\in\mathbb{R}^n$, $t>0$, the solution of the minimization problem $(*)$ is $u(x,t) = \min_{y\in\mathbb{R}^n}\left\{ t\,L\left(\frac{x-y}{t}\right) + g(y)\right\}$ $(\star)$ (Hopf-Lax formula)

($\because \lim_{|y|\to+\infty} t\,L\left(\frac{x-y}{t}\right) + g(y) = +\infty \quad \therefore$ the minimum point can be attained.)

Pf: Let $w(s) = \frac{(t-s)y + sx}{t}$. then $\dot{w}(s) = \frac{x-y}{t}$ $\quad w(0) = y$.

Then the solution of $(*) \le \int_0^t L(\dot{w}(s))\,ds + g(w(0)) = t\,L\left(\frac{x-y}{t}\right) + g(y)$

$\Rightarrow$ the solution of $(*) \le$ the solution of $(\star)$

On the other hand, as L is convex, one has

$$\frac{1}{t}\int_0^t L(\dot{w}(s))\,ds \ge L\left(\frac{1}{t}\int_0^t \dot{w}(s)\,ds\right) = L\left(\frac{w(t)-w(0)}{t}\right)$$

$$\Rightarrow \int_0^t L(\dot{w}(s))\,ds + g(w(0)) \ge t\,L\left(\frac{w(t)-w(0)}{t}\right) + g(w(0))$$

$$\Rightarrow \inf_{w(t)=x}\left(\frac{1}{t}\int_0^t L(\dot{w}(s))\,ds + g(w(0))\right) \ge \inf_{y\in\mathbb{R}^n}\left\{ t\,L\left(\frac{x-y}{t}\right) + g(y)\right\}$$

$\Rightarrow$ the solution of $(*) \ge$ the solution of $(\star)$

the solution of $(*) =$ the solution of $(\star)$

Thm Let $u(x,t)$ be the function given by the Hopf-Lax formula (*) then:

(i) $u(x,t)$ is Lipschitz continuous.

(ii) For $\forall x \in \mathbb{R}^n$, $0 \le s < t$, $u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}$

(iii) $u(x,t)$ is differentiable for a.e. $(x,t) \in \mathbb{R}^n \times (0,+\infty)$, and solves the equation $\begin{cases} \partial_t u(x,t) + H(Du(x,t)) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$ on the differentiable points of u.

Pf: i) For $\forall x, \tilde{x} \in \mathbb{R}^n$, we need to prove $|u(x,t) - u(\tilde{x},t)| \le C|x-\tilde{x}|$

Choose $y \in \mathbb{R}^n$, s.t. $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$

$$u(\tilde{x},t) - u(x,t) = \inf_{z \in \mathbb{R}^n} \left(tL\left(\frac{\tilde{x}-z}{t}\right) + g(z) \right) - \left(tL\left(\frac{x-y}{t}\right) + g(y) \right)$$

$\because \lim_{|y| \to +\infty} tL\left(\frac{x-y}{t}\right) + g(y) = +\infty$

$\therefore$ the minimum point can be attained.

$$\le tL\left(\frac{\tilde{x} - (\tilde{x} - x + y)}{t}\right) + g(\tilde{x} - x + y) - tL\left(\frac{x-y}{t}\right) - g(y)$$

$$= g(\tilde{x} - x + y) - g(y)$$

$$\le Lip(g) \cdot |\tilde{x} - x|$$

$\therefore u$ is Lipschitz continuous. $\square$

(ii) ① Moreover, $u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$

$$\le tL(0) + g(x)$$

Let $x = y$

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$$

$$\ge g(x) + \inf_{y \in \mathbb{R}^n} \left\{ -Lip(g)|x-y| + tL\left(\frac{x-y}{t}\right) \right\}$$

$(\because g(y) - g(x) \ge -Lip(g)|x-y|)$

$$= g(x) + t \inf_{z \in \mathbb{R}^n} \left\{ -Lip(g)|z| + L(z) \right\}$$

$$= g(x) - t \max_{z \in \mathbb{R}^n} \left\{ Lip(g)|z| - L(z) \right\}$$

$$= g(x) - t \max_{w \in B(0, Lip(g))} \max_{z \in \mathbb{R}^n} \left\{ w \cdot z - L(z) \right\}$$

$$= g(x) - t \max_{w \in B(0, Lip(g))} H(w)$$

$$\Rightarrow |u(x,t) - g(x)| \le Ct$$

Let $t \to 0 \Rightarrow u(x,0) = g(x)$

② Then we prove that $u(x,t) = \min_{y\in\mathbb{R}^n} \left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\}$

"$\leq$" For $y\in\mathbb{R}^n$, $0\leq s<t$

Choose $z\in\mathbb{R}^n$, s.t. $u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z)$

$$u(x,t) \leq tL\left(\frac{x-z}{t}\right) + g(z)$$

$$\underset{\downarrow}{\leq} (t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-z}{s}\right) + g(z)$$

($\because$ L is convex)

$$= (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

$$\therefore u(x,t) \leq \min_{y\in\mathbb{R}^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\}$$

"$\geq$" Assume $u(x,t) = tg\left(\frac{x-w}{t}\right) + g(w)$

Let $y = \frac{s}{t}x + \frac{t-s}{t}w$

$$(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

$$\leq (t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-w}{s}\right) + g(w)$$

$\frac{x-y}{t-s} = \frac{y-w}{s}$

$$= (t-s)L\left(\frac{x-w}{t}\right) + sL\left(\frac{x-w}{t}\right) + g(w)$$

$$= tL\left(\frac{x-w}{t}\right) + g(w) = u(x,t)$$

$$\therefore u(x,t) = \min_{y\in\mathbb{R}^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\}$$ □.

(iii) ① $\because$ u is Lipschitz continuous

$\therefore$ u is differentiable for a.e. (x,t)

(This is a property of a Lipschitz continuous function)

② Then we prove that for $\forall (x,t)\in\mathbb{R}^n\times(0,+\infty)$,

u is differentiable at (x,t)

we have $\partial_t u + H(Du) = 0$.

"$\leq$" We only need prove $\partial_t u + v\cdot Du(x,t) \leq L(v)$

(then $\partial_t u + H(Du)$
$= \partial_t u + \max\{v\cdot Du - L(v)\} \leq 0.$)

$$u(x+\varepsilon v, t+\varepsilon) = \min_{y\in\mathbb{R}^n}\left\{\varepsilon L\left(\frac{x+\varepsilon v-y}{\varepsilon}\right) + u(y,t)\right\}$$

$$\leq \varepsilon\cdot L(v) + u(x,t)$$

let $\varepsilon\to 0$, $\Rightarrow \partial_t u + v\cdot Du(x,t) \leq L(v)$

"$\geq$" Let $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$ and $s = t-\varepsilon$, $y = \frac{s}{t}x + \frac{\varepsilon z}{t}$

Then $u(x,t) - u(y,s) \geq tL\left(\frac{x-z}{t}\right) + g(z) - \left(sL\left(\frac{y-z}{s}\right) + g(z)\right)$

$$\left(\begin{array}{l} \because u(y,s) \leq sL\left(\frac{y-z}{s}\right) + g(z) \\ u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z) \end{array}\right)$$

$$= (t-s)L\left(\frac{x-z}{t}\right)$$

$$\Rightarrow u(x,t) - u\left(x + \varepsilon\frac{z-x}{t},\ t-\varepsilon\right) \geq \varepsilon L\left(\frac{x-z}{t}\right)$$

$\varepsilon \to 0$, $\Rightarrow \partial_t u + H(Du) \geq \partial_t u + \frac{z-x}{t}\cdot Du - L\left(\frac{z-x}{t}\right) \geq 0$

$\square$

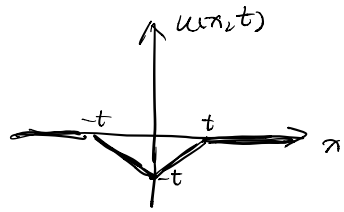
Remark: We have constructed a solution <u>differentiable and satisfying the H-J equation a.e.</u>.

☆: This kind of solution is not unique!!!

Example: $\begin{cases} \partial_t u + |Du|^2 = 0, & t>0 \\ u = 0, & t=0 \end{cases}$ $(n=1)$

Solution $u \equiv 0$

or $u = \begin{cases} 0 & |x|>t \\ x-t & 0 \leq x \leq t \\ -x-t & -t \leq x \leq 0 \end{cases}$



To make sure "uniqueness", we introduce "weak solutions"

· Weak solutions.

<u>Def</u>: If a Lipschitz function $u(x,t): \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ satisfies:

(i) $u(x,0) = g(x)$

(ii) $\partial_t u(x,t) + H(Du) = 0$. a.e. $(x,t) \in \mathbb{R}^n \times (0,+\infty)$

(iii) $\exists C>0$, $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C\left(1+\frac{1}{t}\right)|z|^2$ for $\forall x, z \in \mathbb{R}^n$, $t>0$.

Semiconcavity

then u is called a <u>weak solution</u> of the H-J equation.

<u>Thm</u>: If $H \in C^2$ is convex, $\frac{H(p)}{|p|} \to +\infty$ as $|p| \to +\infty$.
and g is Lipschitz, then there is at most one weak solution of the H-J equation.

Pf: Omitted. (Evans Page 132).

Thm If g is semiconcave or H is uniformly convex, then the solution given by the Hopf-Lax formula is a weak solution to the H-J equation.

For this theorem, we introduce 2 definitions and then 2 lemmas, as a result, this theorem is a direct corollary.

Def If $g(x+z) - 2g(x) + g(x-z) \leq C|z|^2$ for some constant C, $\forall x, z \in \mathbb{R}^n$, then g is called semiconcave.

Def A C^2 convex function $H: \mathbb{R}^n \to \mathbb{R}$ is called uniformly convex (with constant $\theta > 0$) if

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\, \xi_i \xi_j \geq \theta |\xi|^2 \text{ for } \forall p, \xi \in \mathbb{R}^n.$$

Lemma: If g is semiconcave, then u given by Hopf-Lax formula satisfies $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C|z|^2$

Pf: Choose $y \in \mathbb{R}^n$ s.t. $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$,

thus: $u(x+z,t) + u(x-z,t) - 2u(x,t)$

$$\leq tL\left(\frac{x+z-(y+z)}{t}\right) + g(y+z) + tL\left(\frac{x-z-(y-z)}{t}\right) + g(y-z) - 2\left(tL\left(\frac{x-y}{t}\right) + g(y)\right)$$

$$= g(y+z) + g(y-z) - 2g(y) \leq C|z|^2$$

$\square$

Lemma If H is uniformly convex with constant θ, then u (given by Hopf-Lax formula) satisfies:

$$u(x+z,t) + u(x-z,t) - 2u(x) \leq \frac{1}{\theta t}|z|^2 \text{ for all } x, z \in \mathbb{R}^n, t > 0$$

Pf: $u(x+z,t) + u(x-z,t) - 2u(x,t)$

$$\leq tL\left(\frac{x+z-y}{t}\right) + g(y) + tL\left(\frac{x-z-y}{t}\right) + g(y) - 2\left(tL\left(\frac{x-y}{t}\right) + g(y)\right)$$

$$= t\left(L\left(\frac{x+z-y}{t}\right) + L\left(\frac{x-z-y}{t}\right) - 2L\left(\frac{x-y}{t}\right)\right)$$

$$\leq \frac{1}{\theta t}|z|^2$$

$$L(v) = \inf_{p \in \mathbb{R}^n} \{v \cdot p - H(p)\}$$

Aim to prove $L(x+z) + L(x-z) - 2L(x) \leq \frac{1}{\theta}|z|^2$.

Take $p_0 \in \mathbb{R}^n$, s.t. $L(x) = x \cdot p_0 - H(p_0)$

then: $L(x+z)+L(x-z)-2L(x)$

$$= \inf_{p\in\mathbb{R}^n}\{(x+z)\cdot p - H(p)\} + \inf_{p\in\mathbb{R}^n}\{(x-z)\cdot p - H(p)\} - 2\left(x\cdot p_0 - H(p_0)\right)$$

$$\leq (x+z)\cdot\left(p_0+\frac{z}{\theta}\right) - H\left(p_0+\frac{z}{\theta}\right) + (x-z)\cdot\left(p_0-\frac{z}{\theta}\right) - H\left(p_0-\frac{z}{\theta}\right) - 2x\cdot p_0 + 2H(p_0)$$

$$\leq \frac{2}{\theta}|z|^2 - H\left(p_0+\frac{z}{\theta}\right) - H\left(p_0-\frac{z}{\theta}\right) + 2H(p_0)$$

$$\leq \frac{2}{\theta}|z|^2 - \frac{1}{\theta}|z|^2 = \frac{1}{\theta}|z|^2$$

$\square$,

$\because \sum_{i,j=1}^n H_{p_ip_j}(p)\xi_i\xi_j \geq \theta|\xi|^2$ for $\forall p,\xi\in\mathbb{R}^n$.

$\therefore 2H(p) \leq H(p+q) + H(p-q) - \theta|q|^2$

$$2H(p) - H(p+q) - H(p-q)$$

$$\overset{\text{Taylor's formula}}{=} 2H(p) - H(p) - DH(p)\cdot q - \frac{1}{2}\sum_{i,j=1}^n H_{p_ip_j}(p+\xi)\cdot q_iq_j - H(p) + DH(p)\cdot q - \frac{1}{2}\sum_{i,j=1}^n H_{p_ip_j}(p+\xi)\, q_iq_j$$

$$\leq -\theta|q|^2$$

<u>Example 1</u>: $\begin{cases} \partial_t u + \frac{1}{2}|Du|^2 = 0. & \text{in } \mathbb{R}^n\times(0,+\infty) \\ u = |x| & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases}$ $\longrightarrow$ $g=|x|$ is not semiconcave

(for $\forall C>0$, take $x=0$, $|z|$ sufficiently small, then $|x+z|-2|x|+|x-z|>C|z|^2$)

$H(p)=\frac{1}{2}|p|^2$ $\quad L(v)=\frac{1}{2}|v|^2$ $\longrightarrow$ H is uniformly convex.

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}$$

$$= \min_{y\in\mathbb{R}^n}\left\{t\cdot\frac{1}{2}\left|\frac{x-y}{t}\right|^2+|y|\right\} = \min_{y\in\mathbb{R}^n}\left\{\underbrace{\frac{1}{2t}|x-y|^2+|y|}\right\}$$

$$= \frac{|x|^2}{2t} + \frac{1}{2t}|y|^2 + |y| - \frac{1}{t}x\cdot y$$

$$\geq \frac{|x|^2}{2t} + \underbrace{\frac{1}{2t}|y|^2 + \left(1-\frac{|x|}{t}\right)|y|}$$

"$=$": $y=-kx$, $k>0$. $\qquad$ minimum point: $|y|=|x|-t$

$$\geq \begin{cases} \frac{|x|^2}{2t} & |x|\leq t \\ \frac{|x|^2}{2t} + \frac{1}{2t}(|x|-t)^2 + \left(1-\frac{|x|}{t}\right)(|x|-t) & |x|>t \end{cases}$$

$$= |x| - \frac{t}{2}$$

Another way to obtain the expression for $u(x,t)$

$$0 = D_y\left(\frac{|x-y|^2}{2t}+|y|\right) = \frac{y-x}{t} + \frac{y}{|y|} \quad (y\neq 0)$$

$\Longleftrightarrow$ $y = (|x|-t)\frac{x}{|x|} \neq 0$.

$$\Rightarrow u(x,t) = \begin{cases} |x|-\frac{t}{2} & |x|>t \\ \frac{|x|^2}{2t} & |x|\leq t \end{cases}$$

<u>Example 2</u>: $\begin{cases} \partial_t u + \frac{1}{2}|Du|^2 = 0. & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = -|x| & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$g = -|x|$ satisfies semi-convexity

$\because \; -|x+z| + 2|x| - |x-z| \leq 0 < \frac{1}{2}|z|^2$

$\frac{1}{2}(|x+z| + |x-z|) \geq |x|$

$$\Rightarrow u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{|x-y|^2}{2t} - |y| \right\}$$

$$D_y\left(\frac{|x-y|^2}{2t} - |y| \right) = \frac{y-x}{t} - \frac{y}{|y|} \quad (y \neq 0)$$

and this equals zero if $x = y - \frac{y}{|y|} t$

$$\Rightarrow \quad u(x,t) = -|x| - \frac{t}{2} \quad (x \in \mathbb{R}^n, t > 0)$$