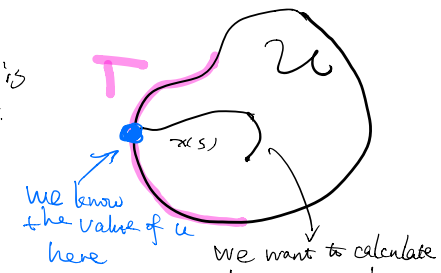
· Characteristics (特征线)

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \mathcal{U} \\ u = g & \text{on } \Gamma \subset \partial\mathcal{U} \end{cases}$$

F, g are smooth.



We try to find $x(s)$ such that $u(x(s))$ is a solution of the equation satisfies a ODE:

$$\partial_s u(x(s)) = \sum_{i=1}^{n} \partial_i u(x(s))\, \partial_s x_i(s) \quad ②$$

$$\partial_s(\partial_i u(x(s))) = \sum_{j=1}^{n} \partial_i\partial_j u(x(s))\, \partial_i x_j(s)$$

$$F(Du, u, x) = 0 \xRightarrow{\partial_i} \sum_{j=1}^{n} F_{p_j}\, \partial_i\partial_j u + F_z \cdot \partial_i u + F_{x_i} = 0.$$

In order to get rid of the "dangerous" second derivative terms we take $\partial_s x_j(s) = F_{p_j}(D_u(x(s)), u(x(s)), x(s))$ ① $j = 1, 2, \dots, n$.

then

$$\partial_s(\partial_i(u(x(s))) + F_z(\underbrace{Du(x(s))}_{p(s) = Du(x(s))}, u(x(s)), x(s)) \cdot \partial_i u(x(s)) + F_{x_i}(Du(x(s)), u(x(s)), x(s)) = 0 \quad ③$$

Let $p_i(s) = \partial_i u(x(s))$ $\quad z(s) = u(x(s))$ then:

$$(☆)\begin{cases} \partial_s x(s) = F_p(p(s), z(s), x(s)) & ① \\ \partial_s z(s) = p(s) \cdot \partial_s x(s) = p(s) \cdot F_p(p(s), z(s), x(s)) & ② \\ \partial_s p(s) = -F_z(p(s), z(s), x(s))\, p(s) - F_x(p(s), z(s), x(s)) & ③ \end{cases}$$

2n+1-dim ODE system.

Remark:

$\because$ $\partial_s F(p(s), z(s), x(s)) = F_p(-F_z \cdot p - F_x) + pF_p \cdot F_z + F_x \cdot F_p = 0.$

$\therefore$ If $F(p(0), z(0), x(0)) = 0$.

then $F(p(s), z(s), x(s)) = F(p(0), z(0), x(0)) = 0$, $\forall s$

i.e. If we know $u = g$ on $\Gamma \subset \partial\mathcal{U}$.

and $x(s) \cap \Gamma = x(s_0)$, WLOG. $s_0 = 0$.

$\Rightarrow$ we can calculate u all along the curve $x(s)$

Def We call (☆) be the characteristic equations of $F(Du, u, x) = 0$.

$x(s)$: the projected characteristic.

Example: 1) F linear

$$F(Du, u, x) = b(x) \cdot \nabla u(x) + c(x) u(x) = 0.$$

$$F(p, z, x) = b(x)\, p + c(x) \cdot z \qquad D_p F = b(x)$$

$$\begin{cases} \partial_s x(s) = b(x(s)) & ① \\ \partial_s z(s) = p(s) \cdot b(s) \overset{\because F=0}{=} -c(x(s)) \cdot z(s) & ② \\ \partial_s p(s) = -c(x(s))\, p(s) - D_x b(x(s))\, p(s) - D_x c(x(s)) \cdot z(s) & ③ \end{cases}$$

$\because$ We only want to know $z(s)$

$\therefore$ we only need to solve $\begin{cases} ① \\ ② \end{cases}$

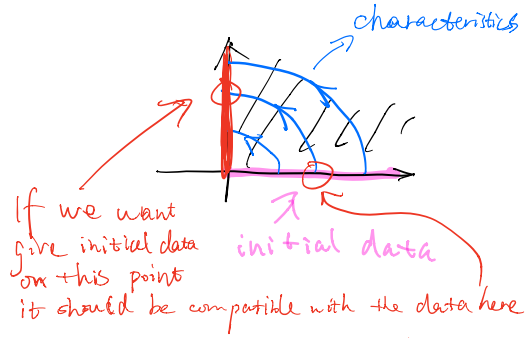
An explicit example:

$$\begin{cases} x_1 \partial_2 u - x_2 \partial_1 u = u & \text{in } \mathcal{U} = \{x_1>0,\ x_2>0\} \\ u = g & \text{on } \Gamma = \{x_1>0,\ x_2=0\} \end{cases}$$

In this example, $\vec{b} = (-x_2, x_1)$ $\quad c = -1$.

$$① \Longleftrightarrow \begin{cases} \dot{x}^1 = -x^2, & \dot{x}^2 = x^1 \\ \dot{z} = z \end{cases}$$

$$② \Longleftrightarrow$$



$$\Longrightarrow \begin{cases} x^1(s) = x^0 \cos s \\ x^2(s) = x^0 \sin s \\ z(s) = z^0 e^s = g(x^0) e^s \end{cases}$$

where $x^0 \geq 0$, $0 \leq s \leq \frac{\pi}{2}$.

Fix a point $(x_1, x_2) \in \mathcal{U}$. We select $s>0$, $x^0>0$ so that $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 \cos s, x^0 \sin s)$, that is, $x^0 = \sqrt{x_1^2 + x_2^2}$, $s = \arctan\left(\frac{x_2}{x_1}\right)$

$$\Longrightarrow u(x_1, x_2) = u(x^1(s), x^2(s)) = z(s) = g(x^0) e^s$$
$$= g\left(\sqrt{x_1^2+x_2^2}\right) \cdot e^{\arctan\left(\frac{x_2}{x_1}\right)}$$

$\square$

2) F quasilinear:

$$F(Du, u, x) = b(x, u(x)) \cdot Du(x) + c(x, u(x)) = 0.$$

$$F(p, z, x) = b(x, z) \cdot p + c(x, z). \qquad F_p = b(x, z)$$

$$\Longrightarrow \begin{cases} \partial_s x(s) = b(x(s), z(s)) \\ \partial_s z(s) = -c(x(s), z(s)) \\ \partial_s p(s) = -D_x b(x(s), z(s)) \cdot p(s) - D_x c(x(s), z(s)) \end{cases}$$

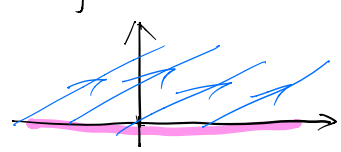
这2个已经构成一组封闭的ODE系统了

An explicit example:

$$\begin{cases} \partial_1 u + \partial_2 u = u^2 & \text{in } \mathcal{U} = \{x_2>0\} \\ u = g & \text{on } \Gamma = \{x_2=0\} = \partial\mathcal{U}. \end{cases}$$

$\vec{b} = (1,1)$, $c = -x^2$

$$\Longrightarrow \begin{cases} \dot{x}^1 = 1, & \dot{x}^2 = 1 \\ \dot{z} = z^2 \end{cases}$$



$$\Longrightarrow \begin{cases} x^1(s) = x^0 + s, \quad x^2(s) = s \\ z(s) = \dfrac{z^0}{1 - s z^0} = \dfrac{g(x^0)}{1 - s g(x^0)} \end{cases}$$

where $x^0 \in \mathbb{R}$, $s \geq 0$, provided the denominator is not zero.

$$\Longrightarrow u(x_1, x_2) = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}.$$

(make sense only if $1 - x_2 g(x_1 - x_2) \neq 0$.)

$\square$

3) F fully nonlinear

the general form is difficult, we only consider a special one.

$$\begin{cases} \partial_1 u \cdot \partial_2 u = u & \text{in } \mathcal{U} = \{x_1 > 0\} \\ u = x_2^2 & \text{on } \Gamma = \{x_1 = 0\} = \partial\mathcal{U} \end{cases}$$

$F(p, z, x) = p_1 p_2 - z. \quad F_p = (p_2, p_1) \quad F_z = -1$

$$\begin{cases} \partial_s x(s) = (p_2, p_1) \iff \partial_s x_1 = p_2, \ \partial_s x_2 = p_1 \\ \partial_s z = 2p_1 p_2 \\ \partial_s p = p \iff \partial_s p_1 = p_1, \ \partial_s p_2 = p_2 \end{cases}$$

$$\Rightarrow \begin{cases} p_1(s) = p_1(0) e^s \quad p_2(s) = p_2(0) \cdot e^s \\ x_1(s) = p_2(0) \cdot (e^s - 1) + x_1(0) \quad x_2(s) = p_1(0)(e^s - 1) + x_2(0) \\ z(s) = p_1(0) p_2(0)(e^{2s} - 1) + z(0) \end{cases}$$

Let $x_1(0) = 0$, $x_2(0) = x_0$, then

$$\Rightarrow \begin{aligned} p_2(0) &= \partial_2 u(x(0)) \overset{\because u = x_2^2 \text{ on } T}{=} 2x_2(0) = 2x_0 \\ p_1(0) &= \partial_1 u(x(0)) \underset{\text{by the original PDE}}{=} \frac{z(0)}{p_2(0)} \underset{\because u = x_2^2 \text{ on } T}{=} \frac{x_2(0)}{2} = \frac{1}{2} x_0 \end{aligned}$$

$$\Rightarrow \begin{cases} x_1(s) = 2x_0(e^s - 1), \quad x_2(s) = \frac{x_0}{2}(e^s - 1) + x_0 = \frac{x_0}{2}(e^s + 1) \\ z(s) = x_0^2 \cdot e^{2s} \\ p_1(s) = \frac{x_0}{2} e^s, \quad p_2(s) = 2x_0 e^s \end{cases}$$

For given $(x_1, x_2) \in \mathcal{U}$. choose (x_0, s) such that:

$$\begin{cases} x_1(s) = 2x_0(e^s - 1) = x_1 \\ x_2(s) = \frac{x_0}{2}(e^s + 1) = x_2 \end{cases}$$

$$\Rightarrow \begin{cases} x_0 = \frac{4x_2 - x_1}{4} \\ e^s = \frac{x_1 + 4x_2}{4x_2 - x_1} \end{cases}$$

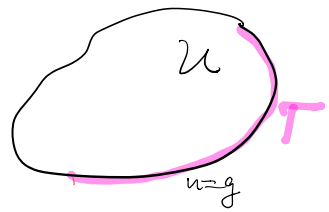
$$\Rightarrow u(x_1, x_2) = z(s) = (x_0)^2 e^{2s} = \frac{(x_1 + 4x_2)^2}{16} \text{ is a solution.}$$

The boundary condition given in above 3 examples are all suitable for us to find "good" solutions.

Now we discuss how to give a good boundary condition in the following.

· Boundary conditions

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \mathcal{U} \\ u = g & \text{on } T \subset \partial\mathcal{U} \end{cases}$$



Let $\{\tau_i\}_{i=1}^{n-1}$ be a local orthogonal 标准正交基 normal bases on T.
ν is the unit outer normal on T at x_0.

$$\text{Then } \begin{cases} \tau_i \cdot Du = \partial_{\tau_i} g \\ F(Du, u, x) = 0 \end{cases} \text{ on } T \longrightarrow F\Big(\sum_{i=1}^{n-1} \tau_i \partial_{\tau_i} g + \nu \cdot \partial_\nu u,\ u,\ x\Big) = 0.$$

$$Du = \underbrace{\Big(\sum_{i=1}^{n-1} \tau_i \otimes \tau_i + \nu \otimes \nu\Big)}_{\text{恒同矩阵}} \cdot Du$$

That is, $\begin{cases} \tau_i \cdot p^0 = \partial_{\tau_i} g & (\star)_1 \\ F(p^0, z^0, x^0) = 0. & (\star)_2 \end{cases} \Big\} (\star)$

when (p^0, z^0, x^0) is the initial data of characteristic ODE.
$(\star)$ is called compatibility condition on boundary data. (给的ODE边界条件需满足$\star$才可)

$(\star)_2$ is $F\Big(\sum_{i=1}^{n-1} \tau_i \partial_{\tau_i} g + \nu \partial_\nu u,\ g(x_0),\ x_0\Big) = 0$

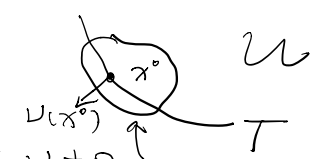
which gives a condition for $\nu \cdot Du$ → 其第2式给了一个u沿向导数的条件.

<u>Def</u> A triple (p^0, z^0, x^0) verifying $(\star)$ is called <u>admissible</u>.

可容许初值的定义.

<u>Lemma</u>. If $F_p(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$,
then there is a unique solution of $q = q(y)$ such that:

$$\begin{cases} \tau_i(y) \cdot q(y) = \partial_{\tau_i} g(y) \\ F(q(y), g(y), y) = 0. \end{cases} \quad \text{for all } y \in T \text{ close to } x^0.$$



If $F_p \cdot \nu \neq 0$.
then $\exists$ a neighbourhood of x^0,
s.t. in the neighbourhood, there is a collection of "initial value"

Pf: We define the mapping $G: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as below:

$$G(q, y) = (G^1(q, y), G^2(q, y), \cdots, G^n(q, y)),$$

where $\begin{cases} G^i(q, y) = \tau_i(y) \cdot q - \partial_{\tau_i} g(y) & i = 1, \cdots, n-1 \\ G^n(q, y) = F\Big(\sum_{i=1}^{n-1} \tau_i(y)(\tau_i(y) \cdot q) + \nu(y)(\nu(y) \cdot q),\ g(y),\ y\Big) \end{cases}$

$$\Rightarrow D_q G(q,y) = \begin{pmatrix} \tau_1(y) \\ \tau_2(y) \\ \vdots \\ \tau_{n-1}(y) \\ [F_p(q, g(y), y)\cdot \nu(y)]\cdot \nu(y) + \sum_{i=1}^{n-1} [F_p(q, g, y)\cdot \tau_i(y)]\, \tau_i(y) \end{pmatrix}_{n\times n}$$

$$\Rightarrow \det D_q G(q,y)$$

$$= \underbrace{[F_p(q, g(y), \nu(y))\cdot \nu(y)]}_{\neq 0 \text{ near } x^0} \cdot \underbrace{\det \begin{pmatrix} \tau_1(y) \\ \tau_2(y) \\ \vdots \\ \tau_{n-1}(y) \\ \nu(y) \end{pmatrix}}_{\neq 0} \neq 0 \text{ near } x^0.$$

By the condition of the lemma,

By the definition of basis

$\therefore$ The Implicit Function Thm ensures we can uniquely solve the identity $G(q,y)=0$ for $q=q(y)$ explicitly, provided y is close enough to x^0.

$\square$

Remark: If $F_p(p^0, z^0, x^0)\cdot \nu(x^0) \neq 0$, we call it the nonchaculteristic boundary data.

$\downarrow$

In fact, $\partial_s(x(s))\big|_{s=0} = F_p(p^0, z^0, x^0)$.

so this boundary data mean that the characteristics $x(s)$ cannot be a tangent to the boundary.

- Local solutions

For given $x^0 \in \Gamma$ and (p^0, z^0, x^0) is an admissible triple of boundary data. Then we can find a function $q(\cdot)$ near x^0 such that $p^0 = q(x^0)$ and $(q(y), g(y), y)$ is admissible for $\forall y$ close to x^0.

Consider:
$$\begin{cases} \dot p(s) = -F_x(p(s), z(s), x(s)) - F_z(p(s), z(s), x(s)) \\ \dot z(s) = F_p(p(s), z(s), x(s))\cdot p(s) \\ \dot x(s) = F_p(p(s), z(s), x(s)) \end{cases}$$

with the initial condition: $p(0) = q(y)$, $z(0) = g(y)$, $x(0) = y$.

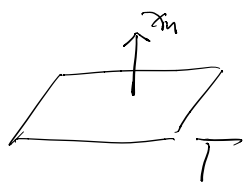
We have a solution: $p = p(y,s)$, $z = z(y,s)$, $x = x(y,s)$

<u>Lemma</u> (local invertibility)

Assume $F_p(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$, then there exists an open interval $I \subseteq \mathbb{R}$ $(0 \in I)$, a neighborhood W of x^0 in T, and a neighborhood V of x^0 in $\mathbb{R}^n$ such that $\forall x \in V$, $\exists$ unique $(s, y) \in I \times W$ satisfies $x = x(y, s)$.

In addition, the mapping $x \mapsto (s, y)$ are C^2.

Pf: In order to simplify the relevant calculations, it is convenient first to change variables, so as to "flatten out" part of the boundary $\partial \mathcal{U}$.



We assume that near x^0 the surface T is flat.

Then the condition "$F_p(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$" change to "$F_{p_n}(p^0, z^0, x^0) \neq 0$."

For $x = x(y, s)$ $\dfrac{\partial x^j}{\partial y_i} := x^j_{y_i}(x^0, 0) = \begin{cases} \delta_{ij} & j = 1, 2, \cdots, n-1 \\ 0 & j = n \end{cases}$

$$\partial_s x(x^0, 0) = F_p(p^0, z^0, x^0) \neq 0.$$

Thus $\det D_{(y,s)} x(x^0, 0) = \det \begin{pmatrix} 1 & & & F_{p_1}(p^0, z^0, x^0) \\ & 1 & & \vdots \\ & & \ddots & \vdots \\ & & 1 & \\ & & & F_{p_n}(p^0, z^0, x^0) \end{pmatrix}$

$$= F_{p_n}(p^0, z^0, x^0) \neq 0.$$

$\therefore$ The Inverse function thm gives us the result:

For given $x \in V$, we can locally find a unique solution to the equation

$$\begin{cases} x = x(y, s) \\ \text{for } y = y(x),\ s = s(x). \end{cases}$$

$\square$

Then we define $\begin{cases} u(x) := z(y(x), s(x)) \\ p(x) := p(y(x), s(x)) \end{cases}$ $(*)$, $x \in V$, s, y as in above Lemma's result

We will check the function u defined above is C^2 and satisfying the PDE with the boundary condition.

We restate the fact in the following thm, which is our principal assertion.

Thm (Local Existence thm).
The function u defined above in (*) is C^2 and solves the PDE:

$$F(Du(x), u(x), x) = 0 \quad (x \in V)$$

with the boundary condition $u(x) = g(x)$ for $x \in \Gamma \cap V$.

Pf: Omitted (cf. Evans §3.2 Thm 2)

Applications

- F Linear: $\begin{cases} F(Du, u, x) = b(x) \cdot Du(x) + c(x) \cdot u(x) = 0. \ (x \in \mathcal{U}) \\ u = g \quad \text{on } \Gamma \end{cases}$

Noncharacteristic assumption: $b(x_0) \cdot \nu(x_0) \neq 0$.

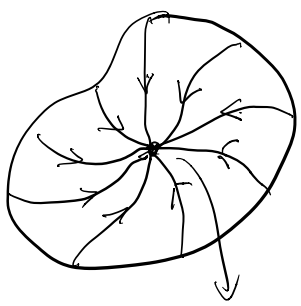
$$\begin{cases} \dot{x}(s) = b(x(s)) \\ \dot{z}(s) = b(x(s)) \cdot p(s) = -c(x(s))\, z(s) \\ \dot{p}(s) = \cdots \end{cases} \quad \Rightarrow \text{don't need it}$$

If $c = 0$, $\dot{z}(s) = 0 \Rightarrow z(s) = \text{const}$ along each trajectory $x(s)$

We consider 3 cases of the trajectories of the ODE:

$$\dot{x}(s) = b(x(s))$$

Case a) flow to an attracting point



WLDG, $x = 0$

$\because$ $z(s) = \text{const}$ along each trajectory $x(s)$

$$\begin{cases} b \cdot Du = 0 \text{ in } \mathcal{U} \\ u = g \text{ on } \Gamma \end{cases}$$

Invoking "Local Existence thm"

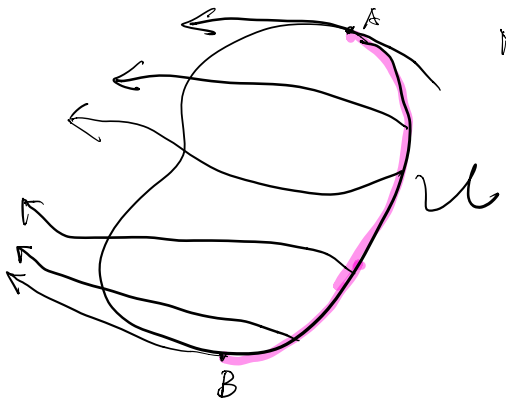
$\Rightarrow \exists$ a unique solution u defined near Γ.

$\Rightarrow u(x(s)) \equiv u(x(0)) = g(x^0)$ for each solution of the ODE $\dot{x}(s) = b(x(s))$ with the initial condition $x(0) = x^0 \in \Gamma$.

However, this solution cannot be smoothly continued to all of $\mathcal{U}$ unless $g \equiv \text{const}$:

$\because$ Each trajectory takes on different values near 0 (the attracting point)

Case b) flow across a domain



Now Suppose trajectories look like this, i.e. we assume that each trajectory of the ODE enters $\mathcal{U}$ precisely once, somewhere through the set

$$T := \{x \in \partial\mathcal{U} \mid b(x)\cdot\nu(x) < 0\}$$

and exits $\mathcal{U}$ precisely once.

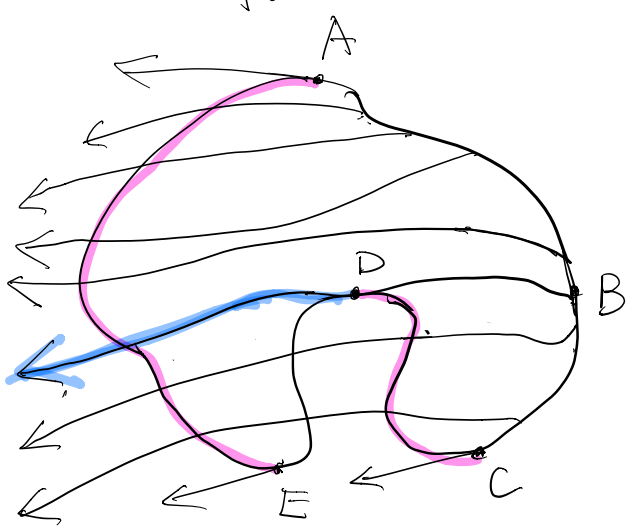
(except those through the characteristic points A, B)

(A, B's trajectories are tangent to $\partial\mathcal{U}$)

Only on the part of $\partial\mathcal{U}$ can we give initial data.

这是叫特征线
指向$\mathcal{U}$内部.
和$b\cdot\nu\neq 0$.

Case c) flow with characteristic points

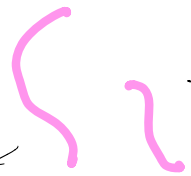


We can now define u to be constant along trajectories, but then u will be discontinuous (unless $g(B)=g(D)$)

D: characteristic point

local existence theory fails near D.

along this trajectory, you cannot know its uniqueness and existence of solution of ODE.

$T \cap$  $= \phi$

You cannot give initial data on this part of $\partial\mathcal{U}$.

· F quasilinear: $\begin{cases} F(Du, u, x) = b(x,u)\cdot Du + c(x,u) = 0. \\ u = g \text{ on } T. \end{cases}$

Noncharacteristic assumption: $b(x^0, z^0)\cdot\nu(x^0) \neq 0$. $(z^0 = g(x^0))$

We consider a concrete example:

<u>Conservation law</u>:

$$\begin{cases} G(Du, \partial_t u, u, x, t) = \partial_t u + \operatorname{div} F(u) = \partial_t u + F'(u)\cdot Du = 0 & (x,t)\in\mathbb{R}^n\times[0,+\infty) \\ u = g \text{ on } T = \mathbb{R}^n\times\{t=0\} \end{cases}$$

$F = (F_1(u), F_2(u), \dots, F_n(u)) : \mathbb{R}\to\mathbb{R}^n$

Let $t = x_{n+1}$, then the characteristic ODE:

$$\begin{cases} \partial_s x_{n+1} = 1, \\ \partial_s x_i(s) = F_i'(z(s)) \\ \partial_s z(s) = 0. \end{cases} \Rightarrow \begin{cases} x_{n+1}(s) = s, \\ \dot{x}_i(s) = F_i'(z_0) \\ z(s) = z_0 = g(x^0) \end{cases}$$

$\Rightarrow \quad x(s) = F'(g(x^0))\, s + x^0$

$\Rightarrow$ The projected characteristic $y(s) = (x(s), s)$ is a straight line.
$u = \text{const}$ along the line.

<u>Crossing characteristics</u>

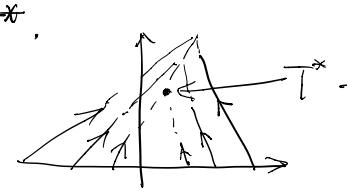
$$u(x(s), s) = z(s) = z_0 = g(x(s) - F'(z_0)\, s)$$
$$= g[x(s) - F'(u(x(s)))\, s]$$

$u(x,t) = g[x - F'(u)t]$ $\to$ Eliminate s, we obtain an implicit formula for u as a function of x and t.

Indeed, if $1 + t\, Dg(x - tF'(u))\cdot F''(u) \neq 0$, the above equation gives an explicit solution u to the original problem.

However, if $n=1$, $F''(u) > 0$, $g' < 0$. this will be false for some time $t > 0$.

$1 + t\, \underbrace{g'(x - tF'(u))}_{<0}\, \underbrace{F''(u)}_{>0} \to 0. \quad t \nearrow T^*.$



· F fully nonlinear.

<u>Example</u> Hamilton-Jacobi equation.

the general Hamilton-Jacobi PDE:

$$G(Du, \partial_t u, u, x, t) = \partial_t u + H(Du, x) = 0.$$

$Du = D_x u$. $\quad q = (p, p_{n+1})$, $\quad y = (x, t)$

$G(q, z, y) = p_{n+1} + H(p, x)$

$D_q G = (D_p H, 1)$ $\quad D_y G = (D_x H, 0)$, $\quad D_z G = 0$.

$$x(s): \begin{cases} \partial_s x^i(s) = H_{p_i}(p(s), x(s)) \\ \partial_s x^{n+1}(s) = 1 \end{cases}$$

$$p(s): \begin{cases} \partial_s p^i(s) = -H_x(p(s), x(s)) \\ \partial_s p^{n+1}(s) = 0 \end{cases}$$

$$z(s): \partial_s z(s) = p^{n+1}(s) + D_p H(p(s), x(s)) \cdot p(s)$$
$$= D_p H(p(s), x(s)) \cdot p(s) - H(p(s), x(s))$$

$$\begin{cases} \partial_s p(s) = -D_x H(p(s), x(s)) & ① \\ \partial_s z(s) = D_p H(p(s), x(s)) \cdot p(s) - H(p(s), x(s)) & ② \\ \partial_s x(s) = D_p H(p(s), x(s)) & ③ \end{cases}$$

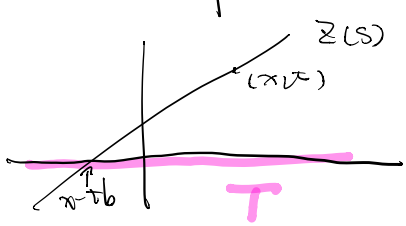
the characteristic equations for the Hamilton-Jacobi equation

①③. $\begin{cases} \partial_s x(s) = D_p H(p, x) \\ \partial_s p(s) = -D_x H(p, x) \end{cases}$ are called the Hamilton's equation.

Observe that the equation for $z(\cdot)$ is trivial, once $x(\cdot)$ and $p(\cdot)$ have been found by solving Hamilton's equations.

$$\partial_s [H(p(s), x(s))] = D_p H \, \partial_s p + D_x H \, \partial_s x$$
$$= -D_p H D_x H + D_x H \cdot D_p H = 0$$
$$\therefore H(p(s), x(s)) = \text{const}.$$

Review: The idea of Characteristics are also used to solve Transport equ.



$z(s) = u(t+s, x+sb)$ ← characteristic

$\partial_s z(s) = 0$.

$\therefore$ for (x, t), we want to know $u(x,t)$

we only need to know $u(x-tb)$ from the initial data.