2.3 Heat Equation

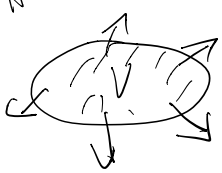
$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathcal{U}, t>0. \quad u = u(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}. \\ u = g & \text{on } \partial\mathcal{U} \\ u(x,0) = u_0 & \text{in } \mathcal{U} \end{cases}$$

$f = 0$ homogeneous.

$f \neq 0$ non-homogeneous.

<u>Physical interpretation</u>:

Diffusion law: $\underbrace{\frac{d}{dt}\int_V u\,dx}_{\text{rate of change}} = -\int_{\partial V} \underbrace{F \cdot n}_{\text{net flux}}\, ds(x)$



$F = -a\nabla u$. $\Rightarrow$ In many situations, $F \propto \nabla u$.

$$\Rightarrow \int_V (\partial_t u - \Delta u)\, dx = 0$$

- Fundamental Solutions

$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$

$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t) \Leftarrow$ Look for a solution u to be invariant under this dilation scaling.

Let $\lambda = \frac{1}{t}$, $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$

$$\partial_t u - \Delta u = -\alpha t^{-\alpha-1} v\left(\frac{x}{t^\beta}\right) + (-\beta) t^{-\alpha-\beta-1} x \cdot Dv\left(\frac{x}{t^\beta}\right) - t^{-\alpha-2\beta} \Delta v\left(\frac{x}{t^\beta}\right) = 0$$

$$\Rightarrow \alpha t^{2\beta-1} v(y) + \beta t^{2\beta-1} y \cdot \nabla y + v''(y) = 0 \qquad \left(y = \frac{x}{t^\beta}\right)$$

Take $\beta = \frac{1}{2}$ $\quad \alpha v(y) + \frac{1}{2} y \cdot \nabla y + v''(y) = 0.$

Take v radial, $v(y) = w(|y|)$ $\Longrightarrow$ $\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0.$

Set $\alpha = \frac{n}{2}$ $\Longrightarrow$ $(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0$

$$\Rightarrow r^{n-1} w' + \frac{1}{2} r^n w = \text{const}$$

Let $r^{n-1} w' + \frac{1}{2} r^n w = 0$

$$\Rightarrow w' = -\frac{1}{2} r w \Rightarrow w = b e^{-\frac{r^2}{4}}$$

$$\Rightarrow u(x,t) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

Def $\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & (x\in\mathbb{R}^n,\ t>0) \\ 0 & (x\in\mathbb{R}^n,\ t<0) \end{cases}$ （可约定 $\Phi(x,0)=0$ $x\neq 0$ $\Phi(0,0)$ 无意义.）

is called the fundamental solution of the heat equation.

The choice of $b = \frac{1}{(4\pi)^{\frac{n}{2}}}$ comes from the following consideration.

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1 \quad (\forall t>0) \qquad (\text{Gauss distribution})$$

Remark: $\Phi(x,t)$ is singular at $(0,0)$

· Initial value problem: $\begin{cases} \partial_t u - \Delta u = 0 & x\in\mathbb{R}^n,\ t>0. \\ u = g & x\in\mathbb{R}^n,\ t=0. \end{cases}$

$(\partial_t - \Delta)\Phi(x,t) = 0$, for $t>0$.

$\Rightarrow (\partial_t - \Delta_x)\Phi(x-y,t) = 0$, $\forall y\in\mathbb{R}^n$.

$\Rightarrow (\partial_t - \Delta_x)\int_{\mathbb{R}^n} \Phi(x-y,t)\,g(y)\,dy = 0$.

Therefore, $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\,g(y)\,dy$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy.$$

is a solution of $\partial_t u - \Delta u = 0$.

Indeed, $u\in C^\infty(\mathbb{R}^n\times(0,+\infty))$, $u(x,0) = g(x)$.

We prove these in the following.

Thm: Assume $g\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, u is given as above. Then:

(i) $u\in C^\infty(\mathbb{R}^n\times(0,+\infty))$

(ii) $(\partial_t u - \Delta u)(x,t) = 0$

(iii) $\lim_{\substack{(x,t)\to(x,0)\\ t>0}} u(x,t) = g(x_0)$ for $\forall x_0\in\mathbb{R}^n$.

Pf: (i) $\because$ $\frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \in C^\infty$ with uniformly bounded derivatives of all orders on $\mathbb{R}^n\times[\delta,+\infty)$ for each $\delta>0$.

$\therefore u\in C^\infty(\mathbb{R}^n\times(0,\infty))$

(ii) Obvious.

(iii) For any given $x_0\in\mathbb{R}^n$ $\varepsilon>0$, take $\delta>0$. s.t. $|g(x)-g(x_0)|<\frac{\varepsilon}{2}$. when $|x-x_0|<\delta$.

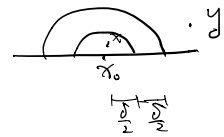
Consider $|x-x_0|<\frac{\delta}{2}$. Then we have

$$|u(x,t)-g(x_0)| = \left|\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}(g(y)-g(y_0))\,dy\right|$$

$$\leq \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(\int_{B_\delta(x_0)} + \int_{\mathbb{R}^n\setminus B_\delta(x_0)}\right) e^{-\frac{|x-y|^2}{4t}}|g(y)-g(y_0)|\,dy$$

$$< \frac{\varepsilon}{2} + 2M\int_{\mathbb{R}^n\setminus B_\delta(x_0)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}\, dy.$$

$$\leq \frac{\varepsilon}{2} + 2M\cdot\int_{\mathbb{R}^n\setminus B_\delta(x_0)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|y-x_0|^2}{16t}}\, dy$$



$x_0 - y = z\sqrt{t}$

$$= \frac{\varepsilon}{2} + 2M\underbrace{\int_{\mathbb{R}^n\setminus B_{\frac{\delta}{\sqrt{t}}}(0)} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{16}}\, dz}$$

$|y-x| \geq |y-x_0| - |x-x_0|$
$\geq |y-x_0| - \frac{\delta}{2}$
$\geq \frac{1}{2}|y-x_0|$

$\because \frac{\delta}{\sqrt{t}} \to +\infty$ as $t \to 0$.

$e^{-\frac{|z|^2}{16}} \to 0$ as $|z| \to \infty$

$\therefore$ Choose t sufficient small s.t. $\sim\sim\sim < \varepsilon$

$\Rightarrow |u(x,t) - g(x_0)| \to 0$ as $|x-x_0| + |t| \to 0$.

$\square$.

Remark: ① $\Phi(x,t)$ satisfies $\begin{cases} \partial_t \Phi - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ \boxed{\Phi(\cdot,0) = \delta_0 \quad \text{on } \mathbb{R}^n.} \end{cases}$

δ_0 Dirac 分布.
理解成 $C_c^\infty(\mathbb{R}^n)$ 函数空间的泛函

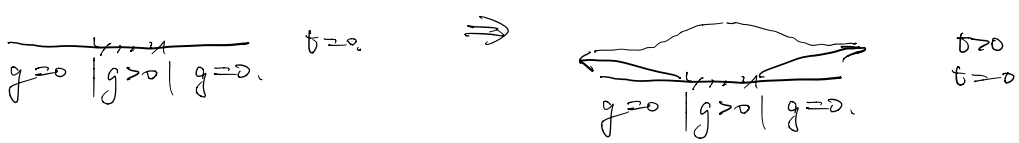
$\langle \delta_0, \varphi \rangle = \varphi(0) \to$ Dirac measure
$\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \Phi(x,0)\varphi(x)\,dx = \varphi(0)$$

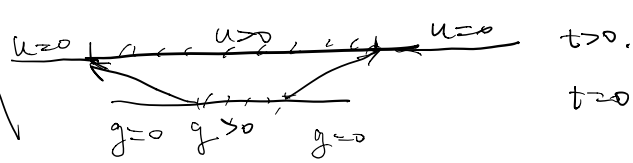
$$\boxed{\lim_{t\to 0}\int_{\mathbb{R}^n} \Phi(x,t)\varphi(x)\,dx = \varphi(0)}$$

② Infinite propagation speed:

If $g \geq 0$, $g \not\equiv 0$, then $u > 0$ for $\forall x \in \mathbb{R}^n$, $t > 0$.



(Finite propagation speed $\Leftarrow$ Wave equation



)

· Non-homogeneous problem : $\begin{cases} \partial_t u - \Delta u = f(x,t) & x \in \mathbb{R}^n,\ t>0. \\ u = 0 & t=0 \end{cases}$ (IH)

Consider the homogeneous problem starting at time $s>0$.

$$\begin{cases} (\partial_t u_s - \Delta u_s)(x,t) = 0 & t>s,\ x\in\mathbb{R}^n. \\ u_s(x,t) = f(x,s) & x\in\mathbb{R}^n. \end{cases}$$

$u_s : \mathbb{R}^n \times [s,+\infty) \longrightarrow \mathbb{R}$

$$u_s(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,\ t-s)\, f(y,s)\, dy.$$

Duhamel's Principle: $\underbrace{u(x,t) = \int_0^t u_s(x,t)\, ds}_{\text{satisfies (IH)}}$ (☆IH)

Formally, 形式上计算,

$$\partial_t u - \Delta u = u_t(x,t) + \int_0^t (\partial_t - \Delta) u_s(x,t)\, ds.$$

$$= f(x,t) + 0 = f(x,t)$$

$t \to 0$; $\left| \int_0^t u_s(x,t)\, ds \right| = \left| \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds \right|$

$$\leq M \cdot t \longrightarrow 0.$$

(If $|f| \leq M$. $\Rightarrow |u_s| \leq M \Rightarrow \int_0^t u_s \leq Mt$)

$\because \int_{\mathbb{R}^n} \Phi\, dy = 1$

<u>Thm</u> : Let u be given by (☆IH). Assume $f \in C_1^2(\mathbb{R}^n \times [0,+\infty))$ has compact support in $\mathbb{R}^n \times [0,+\infty)$

i) $u \in C_1^2(\mathbb{R}^n \times (0,+\infty))$

ii) $\partial_t u - \Delta u = f(x,t)$ $(x\in\mathbb{R}^n,\ t>0)$

iii) $\lim\limits_{\substack{(x,t)\to(x_0,0) \\ t>0}} u(x,t) = 0$ for $\forall x \in \mathbb{R}^n$

Pf: Omitted. 与前面的证明类似.

· General problem: $\begin{cases} \partial_t u - \Delta u = f & t>0 \\ u = g & t=0 \end{cases}$ (GP)

$$\begin{cases} \partial_t u_1 - \Delta u_1 = f \\ u_1(0) = 0 \end{cases} \qquad \begin{cases} \partial_t u_2 - \Delta u_2 = 0 \\ u_2(0) = g \end{cases}$$

$$\Rightarrow u(x,t) = u_2 + u_1$$

$$= \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds$$

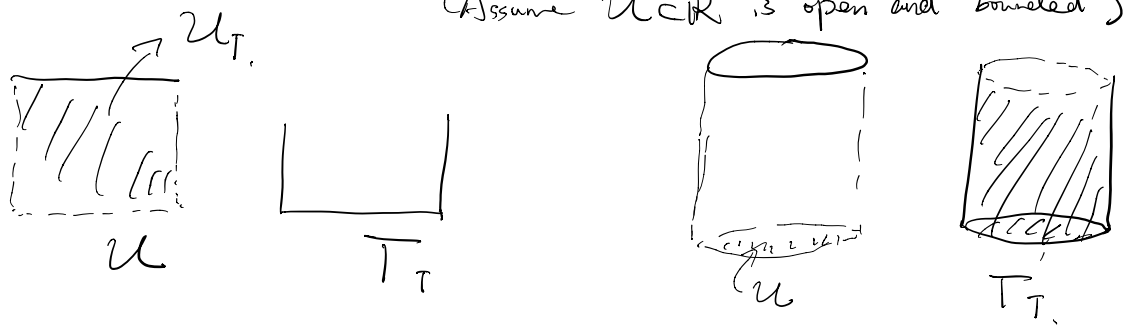
(☆GP)

· Mean-value formula.

<u>Notations</u>: parabolic cylinder: $\mathcal{U}_T = \mathcal{U} \times (0, T]$

parabolic boundary: $\Gamma_T = \partial_p \mathcal{U}_T = \overline{\mathcal{U}_T} - \mathcal{U}_T$

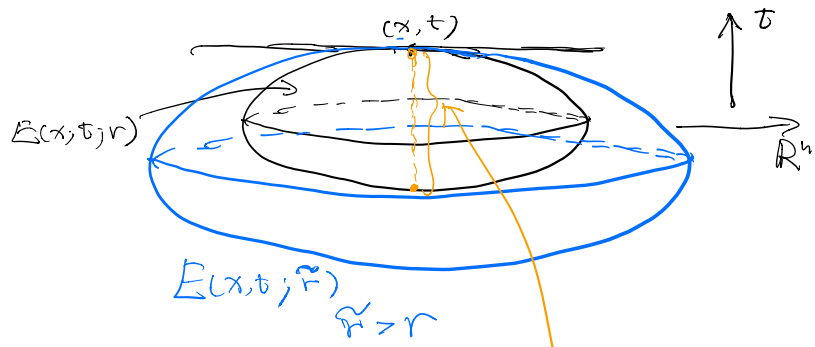
(Assume $\mathcal{U} \subset \mathbb{R}^n$ is open and bounded)



· <u>Def</u>: For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define a "heat ball"

$$E(x,t;r) := \left\{ (y,s) \in \mathbb{R}^{n+1} \;\middle|\; s \le t,\ \Phi(x-y, t-s) \ge \frac{1}{r^n} \right\}.$$

a region in space-time, the boundary of which is a level set of $\Phi(x-y, t-s)$



$$(4\pi(t-s))^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} = \frac{1}{r^n}$$

$$e^{-\frac{|x-y|^2}{4(t-s)}} = \left(\frac{4\pi(t-s)}{r^2}\right)^{\frac{n}{2}}$$

$$\Rightarrow -\frac{|x-y|^2}{4(t-s)} = -\frac{n}{2} \ln \frac{4\pi(t-s)}{r^2}$$

$$\Leftrightarrow \frac{|x-y|^2}{2(t-s)} = n \ln \frac{r^2}{4\pi(t-s)}$$

$$\Rightarrow \frac{r^2}{4\pi(t-s)} > 1 \Leftrightarrow t - s < \frac{r^2}{4\pi}$$

$$t - \frac{r^2}{4\pi} < s \le t$$

$\Rightarrow$ $u(x,t)$ does not depend upon future times

Thm Let $u \in C^2_1(\mathcal{U}_T)$ solve the heat equation. Then:
(space C^2, time C^1)

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds \quad \text{for } \forall E(x,t;r) \subset \mathcal{U}_T$$

Remark:

We have $1 = \frac{1}{4r^n} \iint_{E(x,t;r)} \frac{|x-y|^2}{(t-s)^2}\, dy\, ds$. (Calculated below)

So we call the above Thm as a mean-value property for the heat equ.

Calculation of the integral:

$$\frac{1}{4r^n} \int_{t-\frac{r^2}{4\pi}}^{t} \left(\int_{|x-y| \le \sqrt{2n(t-s)\ln\frac{r^2}{4\pi(t-s)}}} \frac{|x-y|^2}{(t-s)^2}\, dy \right) ds.$$

$$= \frac{1}{4r^n} \cdot \int_{t-\frac{r^2}{4\pi}}^{t} \left(\int_0^{\sqrt{2n(t-s)\ln\frac{r^2}{4\pi(t-s)}}} \frac{R^2}{(t-s)^2} \cdot \omega(n) R^{n-1}\, dR \right) ds.$$

$$= \frac{\omega(n)}{4r^n(n+2)} \int_{t-\frac{r^2}{4\pi}}^{t} \left(2n(t-s)\ln\frac{r^2}{4\pi(t-s)} \right)^{\frac{n+2}{2}} \cdot \frac{1}{(t-s)^2}\, ds$$

Let $v = \ln\frac{r^2}{4\pi(t-s)}$, $\Rightarrow t-s = \frac{r^2}{4\pi e^v}$

$$= \frac{n\alpha(n)}{4r^n(n+2)} \int_0^\infty \left(2n \frac{r^2}{4\pi e^v} \cdot v \right)^{\frac{n+2}{2}} \cdot \frac{4\pi e^v}{r^2}\, dv.$$

$$= \frac{n\alpha(n)}{\cancel{4r^n}(n+2)} \cdot \int_0^\infty \frac{n^{\frac{n+2}{2}} \cdot \cancel{r^{n+2}} \cdot v^{\frac{n+2}{2}}}{2^{\frac{n+2}{2}} \cdot \pi^{\frac{n+2}{2}} e^{\frac{n+2}{2}v}} \cdot \frac{\cancel{4\pi} e^v}{\cancel{r^2}}\, dv$$

$$= \frac{n^{\frac{n+4}{2}}\alpha(n)}{2^{\frac{n+2}{2}}(\sqrt{\pi})^n(n+2)} \underbrace{\int_0^\infty e^{-\frac{n}{2}v} \cdot v^{\frac{n+2}{2}}\, dv}$$

$$= \frac{n^{\frac{n+4}{2}}}{2^{\frac{n+2}{2}} \cdot (\sqrt{\pi})^n \cdot (n+2)} \cdot \underbrace{\boxed{\frac{(\sqrt{\pi})^n}{\Gamma(\frac{n}{2}+1)}}}_{\alpha(n)} \cdot \underbrace{\left(\frac{2}{n}\right)^{\frac{n+4}{2}} \cdot \Gamma\left(\frac{n+4}{2}\right)}$$

$$= \frac{2}{n+2} \cdot \frac{\Gamma(\frac{n}{2}+2)}{\Gamma(\frac{n}{2}+1)} = 1$$

Remark: $\sup_{x \in \mathbb{R}^n} u(x,t) \le \sup_{x \in \mathbb{R}^n} u(x,s) \quad (t > s)$

Pf of Thm:

WLOG, Let $x=0$, $t=0$. and $E(r)=E(0,0;r)$. Consider:

$$\phi(r) = \frac{1}{r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2}\,dy\,ds$$

$$= \iint_{E(1)} u(ry,r^2s)\frac{|y|^2}{s^2}\,dy\,ds$$

$$\phi'(r) = \iint_{E(1)} (y\cdot\nabla u + 2rs\,\partial_s u)(ry,r^2s)\cdot\frac{|y|^2}{s^2}\,dy\,ds$$

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \underbrace{(y\cdot\nabla u}_{A} + \underbrace{2s\,\partial_s u}_{B})(y,s)\cdot\frac{|y|^2}{s^2}\,dy\,ds$$

Let $\psi = \log(\Phi(y,-s)\cdot r^n)$ $\left(\psi=0 \text{ on } \partial E(r)\setminus\{(0,0)\}\right)$

$$= \frac{|y|^2}{4s} - \frac{n}{2}\log(-4\pi s) + n\log r. \qquad (\because s\le t=0)$$

$$B = \frac{1}{r^{n+1}}\iint_{E(r)} 4\partial_s u\cdot y\cdot\nabla\psi\,dy\,ds$$

$$\left(\nabla\psi = \frac{y}{2s},\ y\cdot\nabla\psi = \frac{|y|^2}{2s}\right) \qquad \nabla y = n.$$

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \left(-4\,y\cdot\nabla(\partial_s u)\psi - 4n\,\partial_s u\cdot\psi\right)dy\,ds$$

integration by part with respect to y ($\because \psi=0$ on $\partial E(r)\setminus\{(0,0)\}$)

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \left(-4n\,\partial_s u\,\psi + 4y\cdot\nabla u\cdot\partial_s\psi\right)dy\,ds$$

integration by part with respect to s.

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \left(-4n\psi\,\partial_s u - \frac{2n}{s}\,y\cdot\nabla u - \underbrace{y\cdot\nabla u\,\frac{|y|^2}{s^2}}_{A}\right)dy\,ds.$$

$$\partial_s\psi = -\frac{|y|^2}{4s^2} - \frac{n}{2}\cdot\frac{1}{s}$$

$$\Rightarrow \phi'(r) = \frac{1}{r^{n+1}}\iint_{E(r)} \left(4n\,\partial_s u\cdot\psi + \frac{2n}{s}(y\cdot\nabla u)\right)dy\,ds.$$

$$= -\frac{1}{r^{n+1}}\iint_{E(r)} \left(4n\psi\,\partial_s u + 4n(\nabla\psi\cdot\nabla u)\right)dy\,ds.$$

$$\nabla\psi = \frac{y}{2s}$$

$$= -\frac{1}{r^{n+1}}\iint_{E(r)} 4n\psi\,(\partial_s u - \Delta_y u)\,dy\,ds. = 0$$

integration by part

u is the solution to heatequation

Then $\because \phi(r)\to 4u(0,0)$ as $r\to 0$ $\left(\because \frac{1}{4\pi}\iint_{E(1)}\frac{|y|^2}{s^2}\,dy\,ds = 1 \text{ by the calculation in above remark}\right)$ by the above

$\therefore u(0,0) = \frac{\phi(r)}{4}$ $\square$.

Properties of solutions for the heat equation.

· Strong maximum principle. · Regularity · Local estimates.

· Strong maximum principle.

<u>Thm</u> Assume $u \in C^2(\mathcal{U}_T) \cap C(\overline{\mathcal{U}_T})$ solves the heat equation in $\mathcal{U}_T$. Then

(i) $\max\limits_{\overline{\mathcal{U}_T}} u = \max\limits_{\Gamma_T} u$

(ii) If $\mathcal{U}$ is connected and there exists a point $(x_0, t_0) \in \mathcal{U}_T$, s.t. $u(x_0, t_0) = \max\limits_{\overline{\mathcal{U}_T}} u$, then u is constant in $\overline{\mathcal{U}_{t_0}}$

Only need to prove (ii)

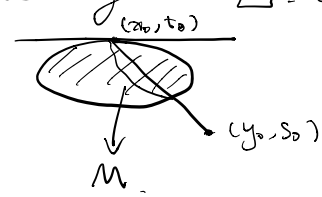
Pf: Assume $M = \max\limits_{\overline{\mathcal{U}_T}} u = u(x_0, t_0)$ $(x_0, t_0) \in \mathcal{U}_T$.

· For $\forall r > 0$, we prove: if $E(x_0, t_0; r) \subset \mathcal{U}_T \Rightarrow u \equiv M$ in $E(x_0, t_0; r)$

Actually, $M = u(x_0, t_0) = \frac{1}{4r^2} \int_{E(x_0, t_0; r)} u(y, s) \frac{|y - x_0|^2}{(s - t_0)^2} \, dy ds \leq M$.

Thus, $u(y, s) \equiv M$. for $(y, s) \in E(x_0, t_0; r)$

· For any line segment L: connecting (x_0, t_0) and another point (y_0, s_0)



$\leftarrow$ We can cover L by a serious of "heat ball"s.

$\Downarrow$

$u(y_0, s_0) = M$.

· For any point $(y, s) \in \mathcal{U}_{t_0}$.

$\because \mathcal{U}_{t_0}$ is path-connected $\therefore$ We have $u \equiv M$ in $\mathcal{U}_{t_0}$.

Remark: We can change max $\rightarrow$ min (Apply to $-u$)

Remark: Infinite propagation speed: $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathcal{U}_T \\ u = g & t = 0. \end{cases}$

If $g \geq 0$ ($g \not\equiv 0$) $\Rightarrow u > 0$. for $t > 0$.

Pf: Suppose $\min\limits_{\mathcal{U}_T} u = u(x_0, t_0) \leq 0$.

then $u \equiv 0$ in $\mathcal{U}_{t_0}$

$\Rightarrow g \equiv 0$. Contradiction.

$\therefore \min\limits_{\mathcal{U}_T} u > 0$.

Thm (Uniqueness for bounded domains) $\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathcal{U}_T \\ u = g & \text{on } \Gamma_T. \end{cases}$ has at most one solution

Pf: If u_1, u_2 are 2 solutions.

$w = u_1 - u_2$ $\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \mathcal{U}_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$

Apply strong maximum principle to $\pm w$.

$\Rightarrow w \equiv 0$

<u>Thm</u>: (Maximum principle for the Cauchy problem).

Assume $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$

solves $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

and satisfies $u(x,t) \le A e^{a|x|^2}$ for some $A, a > 0$.

Then $\sup\limits_{\mathbb{R}^n \times [0,T]} u = \sup\limits_{\mathbb{R}^n} g$

Remark: This Thm is not a trivial corollary of the above strong maximum principle, for we do not assume the maximum could not be attained in $\mathbb{R}^n \times (0,T]$

$\downarrow$

It is not a bounded region.

还要加上 $u(x,t) \le Ae^{a|x|^2}$ 这个增长条件 (growth estimate).

没有的话是不对的.

Pf: I) Assume $v(x,t) = u(x,t) - \dfrac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}$ $(x \in \mathbb{R}^n,\ t > 0)$,

$T < \frac{1}{4a}$

where $T + \varepsilon < \frac{1}{4a}$

$\mu > 0$. $y \in \mathbb{R}^n$ $t < T + \varepsilon$.

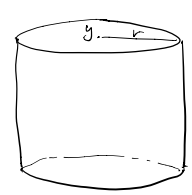
As $\partial_t v - \Delta v = 0$, we have $\max\limits_{\overline{\mathcal{U}}_T} v = \max\limits_{\Gamma_T} v$ $\mathcal{U}_T = B_r(y) \times (0,T]$

① $v(x,0) \le u(x,0) = g(x)$

② if $x \in \partial B_y(r)$,

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon-t)}}$$

$$\le A e^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon)}}$$



As $a < \dfrac{1}{4(T+\varepsilon)}$, we can choose $r \gg 1$ such that $v(x,t) \le \sup\limits_{\mathbb{R}^n} g$

Thus $\max\limits_{\overline{\mathcal{U}}_T} v \le \sup\limits_{\mathbb{R}^n} g$ $\rightsquigarrow$ independent of μ, a, A.

Let $\mu \to 0$. $\Rightarrow \max\limits_{\overline{\mathcal{U}}_T} u \le \sup\limits_{\mathbb{R}^n} g$

II) For $T \geq \frac{1}{4a}$, $[0,T_1] \subset [0,T_1] \cup [T_1, 2T_1] \cup \cdots \cup [kT_1, (k+1)T_1]$

$T \geq \frac{1}{4a}$

Apply result of (I) repeatedly on time intervals,

$$\Rightarrow \max_{[iT_1,(i+1)T_1]} u \leq \max_{\substack{t=iT_1 \\ x\in\mathbb{R}^n}} u \leq \cdots \leq \sup_{\mathbb{R}^n} g$$

$\left(\text{Take } T_1 = \frac{1}{8a} < \frac{1}{4a}, \quad \varepsilon = \frac{1}{16a}, \quad T_1 + \varepsilon < \frac{1}{4a} \right)$

$\square$.

<u>Thm</u> (Uniqueness for Cauchy problem)

Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of the initial value problem $\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

satisfying the growth estimate: $|u(x,t)| \leq A e^{a|x|^2} \ (x \in \mathbb{R}^n, 0 \leq t \leq T)$

Remark: We can construct infinitely many solutions of $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$ (c.f. John Chapter 7)

Each of the solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$.
These solutions are "non-physical."

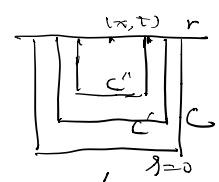
· Regularity

<u>Thm</u>: Suppose $u \in C^2_1(\mathcal{U}_T)$ solves the heat equation in $\mathcal{U}_T$. Then $u \in C^\infty(\mathcal{U}_T)$

Pf: Let $C(x,t;r) = \{(y,s) \mid |x-y| \leq r, \ t - r^2 \leq s \leq t\}$

Fixed $(x_0,t_0) \in \mathcal{U}_T$, we choose $r \ll 1$

s.t. $\begin{cases} C := C(x_0,t_0;r) \\ C' = C(x_0,t_0;\frac{2}{3}r) \\ C'' = C(x_0,t_0;\frac{r}{3}) \end{cases}$ $\mathcal{U}_T$



· Choose $\zeta(x,t)$ smooth supported in C: $\begin{cases} \zeta(x,t) = 1 & \text{in } C' \\ \zeta(x,t) = 0 & \text{in } \mathbb{R}^n \times (0,t_0) \setminus C \end{cases}$

a cut-off function

· $v(x,t) = \zeta(x,t) \cdot u(x,t)$

Then. $\begin{cases} \partial_t v - \Delta v = \zeta \partial_t u + u \cdot \partial_t \zeta - (\zeta \cdot \Delta u + 2D\zeta \cdot Du + u \cdot \Delta \zeta) \\ \qquad = (\partial_t \zeta - \Delta \zeta) u - 2D\zeta \cdot Du \triangleq f(x,t) \\ v = 0 \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$\because \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds$ $\quad \therefore$ By the uniqueness for cauchy problem (above thm)

$$\Rightarrow v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds \qquad (☆☆)$$

$$= \iint_C \Phi(x-y, t-s) [(\partial_s \zeta - \Delta \zeta) u - 2D\zeta \cdot Du]\, dy\, ds.$$

$\because \zeta \equiv 0$ off C

$$= \iint_C \left[\Phi(x-y, t-s)(\partial_s \zeta + \Delta \zeta) + 2 D_y \Phi(x-y, t-s) \cdot D_y \zeta(y,s) \right] u(y,s)\, dy\, ds \quad (*)$$

integration the last term by parts, $\Phi(x-y, t-s) \cdot u \cdot D_y \zeta \Big|_C = 0$.

Above formula holds if we assume $u \in C^\infty$

Only when $f \in C_1^2(\mathbb{R}^n \times [0,+\infty))$ and has compact support; can we apply the thm for non-homogeneous problem's result (☆II) to get (☆☆) above.

If u only satisfies $u \in C_1^2(\mathcal{U}_T)$
we derive $(*)$ with $u_\varepsilon = \eta_\varepsilon * u \in C^\infty$ replacing u.
η_ε being the standard mollifier in the variables x and t.
then u_ε is a C^∞-solution to the heat equation in $(\mathcal{U}_T)_\varepsilon$.
and let $\varepsilon \to 0$. $\Rightarrow u \in C_1^2$ ✓

Let $K(x,t;y,s) = \Phi(x-y, t-s)(\partial_s \zeta + \Delta \zeta) + 2 D_y \Phi(x-y, t-s) \cdot D_y \zeta(y,s)$

$$= \iint_{C \setminus C'} K(x,t;y,s)\, u(y,s)\, dy\, ds \qquad \left(K(x,t;y,s) = 0 \text{ for } (y,s) \in C' \right)$$

If $(x,t) \in C''$

$$u(x,t) = v(x,t) = \iint_{C \setminus C'} K(x,t;y,s)\, u(y,s)\, dy\, ds.$$

For $(y,s) \in C \setminus C'$,
$\because \Phi$ only has singularities when $t-s$ near 0,
and when $(x,t) \in C''$, $t-s \geq C_0 > 0$.
$\therefore K(x,t;y,s)$ is smooth in $(x,t) \in C''$
i.e. $K(x,t;y,s) \in C^\infty$ in $(x,t) \in C''$
$\therefore u(x,t) \in C^\infty$ in $(x,t) \in C''$

- Local derivative estimates

<u>Thm</u> For $\forall\, C(x,t;\frac{r}{2}) \subset C(x,t;r) \subset \mathcal{U}_T$ and solutions for the heat equation,

$$\max_{C(x,t;\frac{r}{2})} \left| D_x^k D_t^\ell u \right| \leq \frac{C_{k,\ell}}{r^{k+2\ell+n+2}} \|u\|_{L^1(C(x,t;r))}$$

Pf: WLOG, Let $(x,t) = (0,0)$,
$C(\frac{1}{2}) = C((0,0),\frac{1}{2})$, $C(\frac{3}{2}) = C((0,0),\frac{3}{2})$, $C(1) = C((0,0),1) \subset \mathcal{U}_T$

Then as in the proof of above thm, we have

$$u(x,t) = \int_{C(1) \setminus C(\frac{3}{2})} K(x,t;y,s)\, u(y,s)\, dy\, ds \quad (\forall (x,t) \in C(\tfrac{1}{2}))$$

Consequently, $|D_x^k D_t^\ell u| \leq \int_{C(1) \setminus C(\frac{3}{2})} \underbrace{|D_x^k D_t^\ell K(x,t;y,s)|}_{\leq C_{k,\ell}} |u(y,s)|\, dy\, ds.$

$\leq C_{k,\ell} \|u\|_{L^1(C(1))}$

This is because $(x,t) \in C(\frac{1}{2})$, $(y,s) \in C(1) \setminus C(\frac{3}{2})$, $\mathrm{dist}(x,y) \geq C_1 > 0$, $\mathrm{dist}(s,t) \geq C_2 > 0$

Then, for generalized r, let $v(x,t) = u(rx, r^2 t)$

$D_x^k D_t^\ell v(x,t) = r^{2\ell+k} D_x^k D_t^\ell u(rx, r^2 t)$ and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}} \|u\|_{L^1(C(r))}$

then we have the desired result $\square$

· Energy methods

Uniqueness & Backwards uniqueness.

<u>Thm</u>: There exists at most 1 solution $u \in C^2_1(\overline{\mathcal{U}_T})$ of the initial / boundary value problem:
$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathcal{U}_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Pf: $w = u_1 - u_2$
$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \mathcal{U}_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

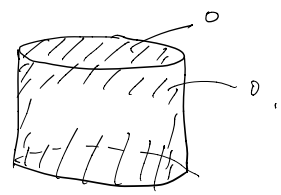
Let $e(t) = \int_{\mathcal{U}} w^2(x,t)\,dx \quad (0 \le t \le T)$.

$$\partial_t e(t) = 2\int_{\mathcal{U}} w \cdot \partial_t w\,dx = 2\int_{\mathcal{U}} w \cdot \Delta w\,dx = -2\int_{\mathcal{U}} |\nabla w|^2 dx \le 0.$$

$\Rightarrow e(t) = e(0) = 0,\ \forall t > 0 \Rightarrow w \equiv 0.$ $\square$

<u>Thm</u>: If u satisfies
$$\begin{cases} \partial_t u - \Delta u = 0. & \text{in } \mathcal{U}_T \\ u = 0 & \text{on } \partial\mathcal{U} \times [0,T] \\ u = 0 & \text{on } \mathcal{U} \times \{t = T\} \end{cases}$$
then $u = 0$ in $\mathcal{U}_T$

Remark:
$$\begin{cases} v(x,t) = u(x, T-t) & (0 \le t \le T) \\ v(x,0) = 0. \quad v|_{\partial\mathcal{U}} = 0. \\ \partial_t v + \Delta v = 0. \end{cases}$$
$\Rightarrow v \equiv 0.$



Pf of thm: Let $e(t) = \int_{\mathcal{U}} |u(x,t)|^2 dx$ Then:

同上一定理证明 $\Rightarrow e'(t) = -2\int_{\mathcal{U}} |\nabla u|^2 dx$

$$e''(t) = -4\int_{\mathcal{U}} \nabla u \cdot \nabla \partial_t u\,dx \overset{\partial_t u = \Delta u.}{=} 4\int_{\mathcal{U}} |\Delta u|^2 dx$$

Since $\int_{\mathcal{U}} |\nabla u|^2 dx = \int_{\mathcal{U}} -u\Delta u\,dx \le \left(\int_{\mathcal{U}} u^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\Delta u|^2 dx\right)^{\frac{1}{2}}$

Cauchy-Schwarz.

$$\Leftrightarrow \left(\frac{1}{2} e'(t)\right)^2 \le e(t) \cdot \frac{e''(t)}{4}$$

$$\Leftrightarrow (e'(t))^2 \le e(t) \cdot e''(t)$$

In fact, this implies $e \equiv 0$. We prove this by contradiction.

If $e(t) \not\equiv 0$ on $[0,T]$. then $\exists\, [t_1, t_2] \subset [0,T]$ s.t. $e(t) > 0$. in $t \in [t_1, t_2)$, $e(t_2) = 0$.

$$(\log e(t))'' = \left(\frac{e'}{e}\right)' = \frac{e''e-(e')^2}{e^2} \geq 0$$

$\Rightarrow$ $\log e(t)$ is convex on $[t_1, t_2)$

$$\Rightarrow \log e\left(\frac{t_1+t_2-\varepsilon}{2}\right) \leq \frac{1}{2}\left(\log e(t_1) + \log e(t_2-\varepsilon)\right)$$

$$\Rightarrow e\left(\frac{t_1+t_2-\varepsilon}{2}\right)^2 \leq e(t_1)\cdot e(t_2-\varepsilon)$$

Let $\varepsilon \to 0$ $\Rightarrow$ $e\left(\frac{t_1+t_2}{2}\right)^2 \leq e(t_1)\, e(t_2) = 0.$

$\Rightarrow$ $e\left(\frac{t_1+t_2}{2}\right) = 0.$

a Contradiction!

$\square$

2.4 Wave equation

$\Box u = \partial_t^2 u - \Delta u \longrightarrow$ wave operator

$u = u(x,t) \quad (x \in \mathbb{R}^n,\ t \geq 0)$

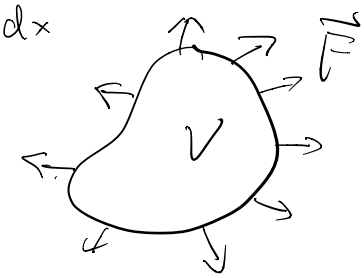
$\Box u = 0$ (homogeneous) or $\Box u = f$ (non-homogeneous)

Describe the movement of "wave"

$u(x,t)$: the displacement in some direction of point x at time $t > 0$.

$V \subset \mathcal{U}$ is a smooth domain.

The acceleration : $\frac{d^2}{dt^2}\int_V u\,dx = \int_V \partial_t^2 u\,dx$



$\sum F = ma$ assume $m = 1$.

$\Rightarrow$ The net contact force :

$$-\int_{\partial V} F \cdot n \, dS(x) = \int_V \partial_t^2 u\,dx$$

$$\overset{(1)}{\int_V (-\operatorname{div}\vec{F})\,dx}$$

$$\Rightarrow \partial_t^2 u = -\operatorname{div} F$$

Assumption $F = F(\nabla u) \approx -k \cdot \nabla u \quad (F \propto \nabla u)$

$$\Rightarrow \partial_t^2 u = k \Delta u. \quad (k > 0)$$

Initial condition: $\begin{cases} u(x,0) = u_0(x) & \longrightarrow \text{displacement} \\ \partial_t u(x,0) = u_1(x) & \longrightarrow \text{velocity} \end{cases}$

· Solution for $n=1$

a) $\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & (x,t) \in \mathbb{R} \times (0, +\infty) \\ u(x,0) = g(x) & x \in \mathbb{R} \\ \partial_t u(x,0) = h(x) \end{cases}$

$\Rightarrow (\partial_t + \partial_x)(\partial_t - \partial_x) u = 0 \rightarrow$ The PDE can be factored.

Step 1 : Let $v(x,t) = (\partial_t - \partial_x) u$

$\Rightarrow \partial_t v + \partial_x v = 0. \longrightarrow$ transport equation.

$\Rightarrow v(x,t) = a(x-t)$ (Apply the solution formula to transport equation)

Then $a(x) = v(x,0) = (\partial_t - \partial_x) u(x,0) = h(x) - g'(x)$

Step 2: $\partial_t u - \partial_x u = v(x,t) \longrightarrow$ nonhomogeneous transport equation

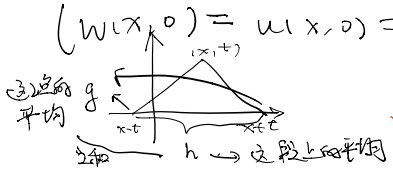
Let $w(x,t) = u(x-t, t)$

$\partial_t w = (\partial_t u - \partial_x u)(x-t, t) = v(x-t, t) = a(x-2t)$

$\Rightarrow w(x,t) = w(x,0) + \int_0^t a(x-2\tau)\,d\tau$

$\Rightarrow u(x,t) = w(x+t, t) = w(x+t, 0) + \int_0^t a(x+t-2\tau)\,d\tau$

$(w(x,0) = u(x,0) = g(x)) \Rightarrow g(x+t) + \int_0^t (h(x+t-2\tau) - g'(x+t-2\tau))\,d\tau$



$$u(x,t) = \frac{1}{2}\big(g(x+t) - g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

d'Alembert's formula.

Thm (Solution of wave equation, $n=1$)

Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and u is given as above, then:

i) $u \in C^2(\mathbb{R}^n \times [0,+\infty))$ (regularity)

ii) $\partial_t^2 u - \partial_x^2 u = 0$ in $\mathbb{R} \times (0,+\infty)$ (satisfying equation)

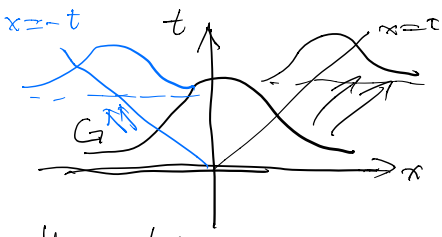
iii) $\lim\limits_{\substack{(x,t)\to(x_0,0)\\ t>0}} (u(x,t), \partial_t u(x,t)) = (g(x_0), h(x_0))$ (initial condition.)

Pf: Omitted. (A direct calculation)

Remark: 1) $u(x,t) = \left(\frac{1}{2} g(x+t) + \frac{1}{2}\int_0^{x+t} h(y)\,dy\right) - \left(\frac{1}{2} g(x-t) + \frac{1}{2}\int_0^{x-t} h(y)\,dy\right)$

$= \underbrace{F(x+t)}_{\text{left travelling solution}} + \underbrace{G(x-t)}_{\text{right travelling solution}}$

travelling solution

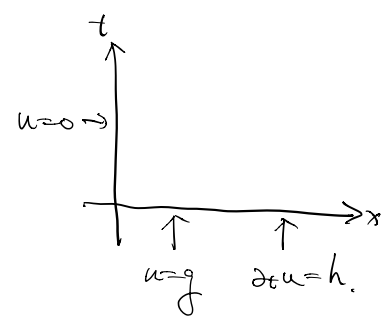


Any solution to $\partial_t^2 u - \partial_x^2 u = 0$ can be decomposed into such 2 kinds of waves.

2) We can see from the d'Alembert's formula that if $g \in C^k$, $h \in C^{k-1}$, then $u \in C^k$

b) $\mathbb{R}_+ \times (0,+\infty) = [0,+\infty) \times (0,+\infty)$

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \text{in } \mathbb{R}_+ \times (0,+\infty) \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = 0 & \text{on } \{x=0\} \times (0,+\infty) \end{cases}$$



Let $\tilde{u}(x,t) = \begin{cases} u(x,t) & x \geq 0,\ t>0 \\ -u(-x,t) & x<0,\ t>0. \end{cases}$

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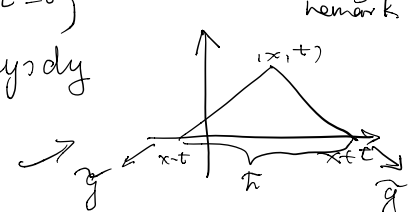
$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x<0 \end{cases}$ $\qquad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x<0 \end{cases}$

We should assume that $\begin{cases} \tilde{g} \in C^2(\mathbb{R}) \\ \tilde{h} \in C^1(\mathbb{R}) \end{cases}$ additionally (Need to add some additional condition to g, h. see the following remark.)

Then: $\begin{cases} \partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } \mathbb{R} \times (0,+\infty) \\ \tilde{u} = \tilde{g},\ \partial_t \tilde{u} = \tilde{h} & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$

$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{g}(x+t) + \tilde{g}(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy$

$= \begin{cases} \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & x \geq t > 0 \\ \frac{1}{2}[g(x+t)-g(t-x)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & 0 \leq x \leq t \end{cases}$



Remark: If we want u to be C^2, it is needed that $g(0)=0$, $h(0)=0$, $g''(0)=0$.

$$\left(\begin{array}{l} g(0)=0 \Rightarrow g\in C \\ h(0)=0 \Rightarrow h\in C. \\ \text{We need } u\equiv 0 \text{ on } \{x=0\}\times(0,+\infty) \\ \Rightarrow \text{We need } \partial_t u(0,0)=0. \\ \Rightarrow \text{we need } \partial_x u(0,0)=0 \\ \Rightarrow \text{we need } g''(0)=0. \\ \text{Then } \tilde{g}\in C^2,\ \tilde{h}\in C^1 \\ \therefore \text{By the above thm} \Rightarrow u\in C^2. \end{array}\right)$$

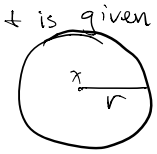
- $n\geq 2$.

$$\begin{cases} \partial_t^2 u-\Delta u=0 & \text{in } \mathbb{R}^n\times(0,+\infty) \\ u=g \quad \partial_t u=h & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases} \qquad \Box=\partial_t^2-\Delta$$

when $n\geq 2$, we cannot do the factorization as $n=1$.

<u>Def</u> $\mathcal{U}(x;r,t):=\fint_{\partial B(x,r)} u(y,t)\,ds(y)$

t is given.



is the average of u on $\partial B(x,r)$

$$G(x,r)=\fint_{\partial B(x,r)} g(y)\,ds(y), \quad H(x,r)=\fint_{\partial B(x,r)} h(y)\,ds(y).$$

<u>Lem</u>: Fix $x\in\mathbb{R}^n$, u is the solution to the initial-value problem the $\mathcal{U}\in C^m(\mathbb{R}_+\times[0,+\infty))$ and

(Euler-Poisson-Darboux Equation) $\begin{cases} \partial_t^2\mathcal{U}-\partial_r^2\mathcal{U}-\frac{n-1}{r}\partial_r\mathcal{U}=0 & \text{in } \mathbb{R}_+\times(0,+\infty) \\ \mathcal{U}=G, \quad \partial_t\mathcal{U}=H & \text{on } \mathbb{R}_+\times\{t=0\} \end{cases}$

Pf: 1) $\mathcal{U}(x;r,t)=\frac{1}{\omega(n)\cdot r^{n-1}}\int_{\partial B(x,r)} u(y,t)\,ds(y)$

$$\underset{(y=x+rz,\ |z|=1)}{=} \frac{1}{\omega(n)}\int_{\partial B(0,1)} u(x+rz,t)\,ds(z)$$

$$\Rightarrow \partial_r\mathcal{U}=\frac{1}{\omega(n)}\int_{\partial B(0,1)}\nabla u(x+rz,t)\cdot z\,ds(z)$$

$$=\frac{1}{\omega(n)\cdot r^{n-1}}\int_{\partial B(x,r)}\nabla u(y,t)\cdot\underbrace{\frac{y-x}{r}}_{=n}\,ds(y)$$

$$=\frac{1}{\omega(n)\cdot r^{n-1}}\int_{B(x,r)}\Delta u(y,t)\cdot dy \qquad (*)$$

$$\Rightarrow \quad r^{n-1}\partial_r \mathcal{U} = \frac{1}{\omega(n)}\int_{B(x,r)} \partial_t^2 u(y,t)\,dy$$

$$\Rightarrow \quad \partial_r\left(r^{n-1}\partial_r \mathcal{U}\right) = \frac{1}{\omega(n)}\int_{\partial B(x,r)} \partial_t^2 u(y,t)\,dS(y) \qquad \partial_r \int_{B(x,r)} = \int_{\partial B(x,r)}$$

$$= \partial_t^2\left(\frac{1}{\omega(n)}\int_{\partial B(x,r)} u(y,t)\,dS(y)\right)$$

$$= r^{n-1}\partial_t^2 \mathcal{U}$$

$$\Rightarrow \quad \partial_t^2 \mathcal{U} - \partial_r^2 \mathcal{U} - \frac{n-1}{r}\partial_r \mathcal{U} = 0. \qquad \square$$

2) $\lim_{r\to 0^+} \mathcal{U}(x;r,t) = u(x,t)$

$\lim_{r\to 0^+} \partial_r \mathcal{U}(x;r,t) = \lim_{r\to 0^+} \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\,dy = 0.$

After some computations. $\lim_{r\to 0^+} \partial_r^2 \mathcal{U}(x;r,t) = \frac{1}{n}\Delta u(x,t)$

$\lim_{r\to 0^+} \partial_r^k \mathcal{U}(x;r,t) = \cdots$

$\Rightarrow \mathcal{U} \in C^m(\mathbb{R}_+ \times [0,+\infty))$ $\square$

Above Euler-Poisson-Darboux equation is similar to 1-dimensional wave equation. Now we need to do some transformation

- $n=3$. $\quad \partial_t^2 \mathcal{U} - \partial_r^2 \mathcal{U} - \frac{2}{r}\partial_r \mathcal{U} = 0$

$$\partial_r^2(r\mathcal{U}) = \partial_r(\mathcal{U} + r\partial_r \mathcal{U}) = 2\partial_r \mathcal{U} + r\cdot\partial_r^2 \mathcal{U}$$

$$= r\partial_t^2 \mathcal{U} = \partial_t^2(r\mathcal{U})$$

$\therefore$ $r\mathcal{U}$ satisfies 1-dimensional wave equation.

$$\begin{cases} \tilde{\mathcal{U}} = r\mathcal{U} \\ \tilde{G} = rG \\ \tilde{H} = rH \end{cases} \Rightarrow \begin{cases} \partial_t^2 \tilde{\mathcal{U}} - \partial_r^2 \tilde{\mathcal{U}} = 0 & \text{in } \mathbb{R}_+ \times (0,+\infty) \\ \tilde{\mathcal{U}} = \tilde{G}, \quad \partial_t \tilde{\mathcal{U}} = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{\mathcal{U}} = 0 & \text{on } \{r=0\} \times [0,+\infty) \end{cases}$$

By the remark in the previous page, if we want the solution to be C^2. we should need to check $\tilde{G}(0)=0$, $\tilde{H}(0)=0$, $\partial_r^2 \tilde{G}(0) = 0$.

obvious

$$\partial_r^2 \tilde{G}(0) = (r\partial_r^2 G + 2\partial_r G)\big|_{r=0} = \left[\frac{1}{r}\partial_r(r^2 \partial_r G)\right]\Big|_{r=0}.$$

the calculation similar to the equation (*) in previous page

$$= \lim_{r\to 0^+} \frac{1}{\omega_3}\frac{1}{r}\int_{\partial B(x,r)} \Delta g(y)\cdot dS(y) = 0. \quad \checkmark$$

We take $r<t$, then

$$\tilde{\mathcal{U}}(x;r,t)=\frac{1}{2}\left[\tilde{G}(x;r+t)-\tilde{G}(x,t-r)\right]+\frac{1}{2}\int_{t-r}^{t+r}\tilde{H}(y)\,dy.$$

$$u(x,t)=\lim_{r\to 0^+}\mathcal{U}(x;r,t)=\lim_{r\to 0^+}\frac{\tilde{\mathcal{U}}(x;r,t)}{r}$$

$$=\lim_{r\to 0^+}\left\{\frac{1}{2}\left[\frac{\tilde{G}(x;r+t)-\tilde{G}(x;t-r)}{r}\right]+\frac{1}{2r}\int_{t-r}^{t+r}\tilde{H}(y)\,dy\right\}$$

$$=\partial_t\tilde{G}(x;t)+\tilde{H}(x;t)$$

$$=\partial_t\left(tG(x;t)\right)+tH(x;t)$$

$$G(x;t)=\fint_{\partial B(x;t)}g(y)\,dS(y)\qquad H(x;t)=\fint_{\partial B(x;t)}h(y)\,dS(y)$$

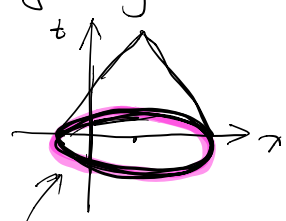
$$\partial_t G(x;t)=\partial_t\fint_{\partial B(0;1)}g(x+tz)\,dS(z)=\fint_{\partial B(0,1)}z\cdot\nabla g(x+tz)\,dS(z)$$

$$=\fint_{\partial B(x;t)}\frac{y-x}{t}\cdot\nabla g(y)\,dS(y)$$

$$\Rightarrow\ u(x;t)=\fint_{\partial B(x;t)}\left[g+th+(y-x)\cdot\nabla g\right](y)\,dS(y)$$

$$=\frac{1}{4\pi t^2}\int_{\partial B(x;t)}\left[g+th+(y-x)\cdot\nabla g\right]dS(y)$$

Kirchhoff's formula.



$n=3$时，仅依赖于这个球面上的初值

而 $n=1$ 时

- $n=2$ $\quad \partial_t^2\mathcal{U}-\partial_r^2\mathcal{U}-\frac{1}{r}\partial_r\mathcal{U}=0.$

No transformation like "$n=3$ case" works.

Instead, we will regard the original initial-value problem for $n=2$ as a problem for $n=3$ in which the third spatial variable x_3 does not appear.

$$\begin{cases}\tilde{u}(x_1,x_2,x_3,t):=u(x_1,x_2,t) & \tilde{x}=(\overset{(x_1,x_2)}{x},x_3)\\ \tilde{g}(x,x_3)=g(x) & \tilde{h}(x,x_3)=h(x)\end{cases}$$

$$\Rightarrow\begin{cases}\partial_t^2\tilde{u}-\Delta\tilde{u}=0 & \text{in } \mathbb{R}^3\times(0,+\infty)\\ \tilde{u}=\tilde{g},\ \partial_t\tilde{u}=\tilde{h} & \text{on } \mathbb{R}^3\times\{t=0\}\end{cases}$$

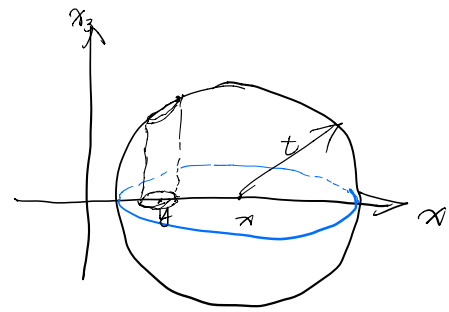
$$\Rightarrow u(x,t) = \tilde{u}(\tilde{x},t)$$

$$= \frac{1}{4\pi t^2}\int_{\underset{S^2}{\partial B(\tilde{x},t)}} \left[\tilde{g} + t\tilde{h} + (\tilde{y}-\tilde{x})\cdot\nabla\tilde{g}\right] ds(\tilde{y})$$

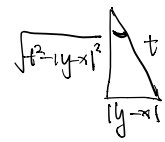
$$= 2\cdot\frac{1}{4\pi t^2}\int_{\underset{S^1}{B(x,t)}} \frac{t}{\sqrt{t^2-|y-x|^2}}\underbrace{(g+th+(y-x)\cdot\nabla g)}_{\text{纵向投影值 } \tilde{g}=g}\, dy$$

上下半球均作投影.

球坐标下面积微元(对比)



$$= \frac{1}{2\pi t}\int_{B(x,t)} \frac{g+th+(y-x)\cdot\nabla g}{\sqrt{t^2-|y-x|^2}}\, dy$$



Poisson's formula.

method of descent

- $n \geq 4$ Omitted
  First consider "n odd" case, then apply method of descent to "n even" case.

- Non-homogeneous problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^n\times(0,+\infty) \\ u = g \quad \partial_t g = h & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases}$$

Denote the solution: $u(x,t) \triangleq \mathcal{M}(f,g,h)(x,t)$

$\mathcal{M}(0,g,h)(x,t) = \cdots$ (We have done)

$\mathcal{M}(f,g,h)(x,t) = \underline{\mathcal{M}(f,0,0)} + \mathcal{M}(0,g,0) + \mathcal{M}(0,0,h)$

Duhamel's principle:

$$\begin{cases} \mathcal{M}(0,g,0)(x,t) = \partial_t \mathcal{M}(0,0,g)(x,t) \\ \mathcal{M}(f,0,0)(x,t) = \int_0^t \mathcal{M}(0,0,f_\tau)(x,t-\tau)\, d\tau \end{cases}$$

Pf: Omitted. (Do some calculation to prove)

Example:

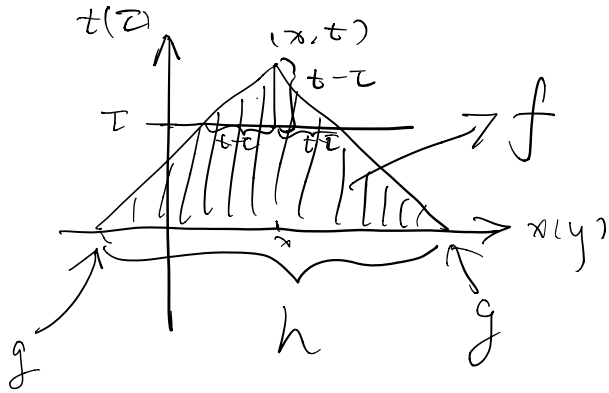
- $n=1$.

$$\mathcal{M}(0,0,h) = \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

$$\mathcal{M}(0,g,0) = \partial_t \mathcal{M}(0,0,g) = \frac{1}{2}\partial_t \int_{x-t}^{x+t} g(y)\,dy = \frac{1}{2}\left(g(x+t)+g(x-t)\right)$$

$$\mathcal{M}(f,0,0) = \frac{1}{2}\int_0^t \left(\int_{x-(t-\tau)}^{x+t-\tau} f(y,\tau)\,dy\right)d\tau$$

$$= \frac{1}{2}\int_{\substack{|y-x|\le t-\tau \\ 0\le\tau\le t}} f(y,\tau)\,dy\,d\tau$$



- $n=3$

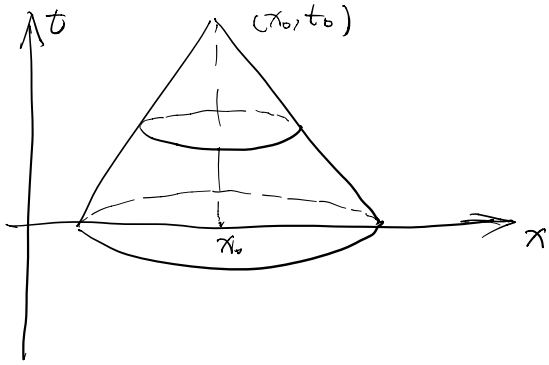
$$\mathcal{M}(f,0,0)(x,t) = \int_0^t \left(\frac{t-\tau}{4\pi(t-\tau)^2}\int f(y,\tau)\,dS(y)\right)d\tau$$

$$= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-\tau)} \frac{f(y,\tau)}{t-\tau}\,dS(y)\,d\tau$$

$$= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dS(y)\,dr$$

$$= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy$$

It is called retarded potential

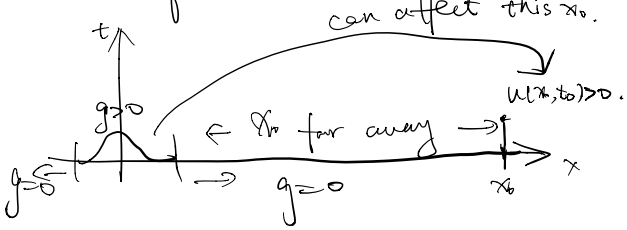


$K(x_0,t_0) = \{(x,t) \mid 0 \le t \le t_0,\ |x-x_0| \le t_0 - t\}$

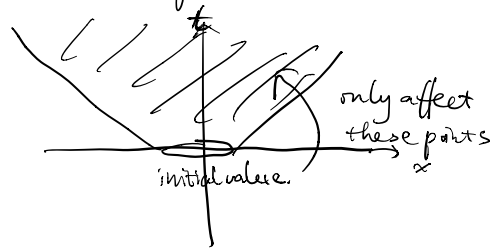
Backward wave cone

$u(x_0,t_0)$ only depends on the value given in $K(x_0,t_0)$.

heat equation:



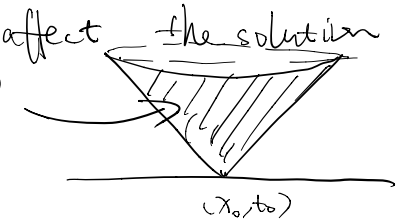
wave equation:



- The given value at (x_0,t_0) only affect the solution in $\{(x,t) \mid |x-x_0| \le t-t_0,\ t \ge t_0\}$

$\Downarrow$

$K^+(x_0,t_0)$



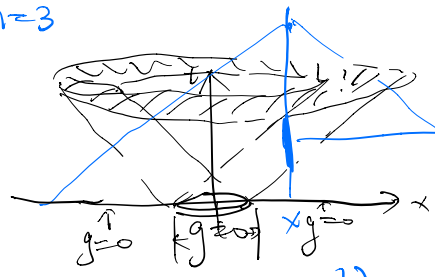
- The initial value g only affect $u\Big|_{\bigcup_{x_0 \in \operatorname{supp} g} K^+(x_0,0)}$.

- If $f=0$ $u(x_0,t_0)$ only depends on the initial data in $|y-x_0| \le t_0$.

$n=3$: $u(x_0,t_0)$ only depends on the initial data on $|y-x_0| = t_0$.

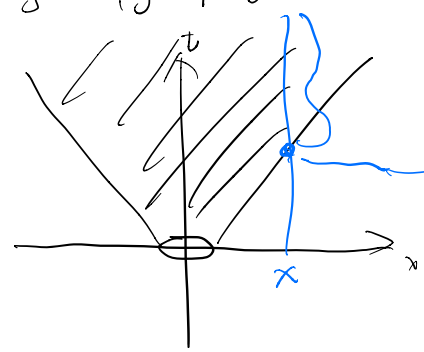
In physics, this reflects the different property of 2-d and 3-d wave.

$n=3$



at x, it is affected only in this time interval. → 清一会吗不到了

$n=2$

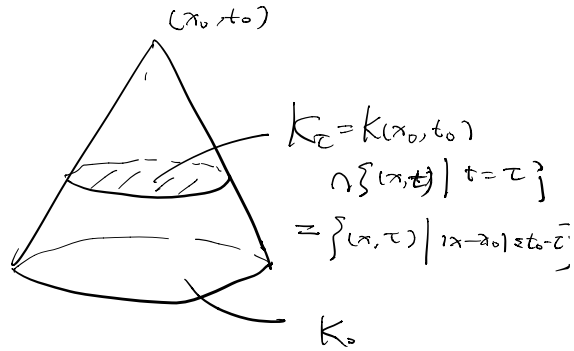


at x, it is affected before this time → 水面上的波一直向外传

Huygens' principle.

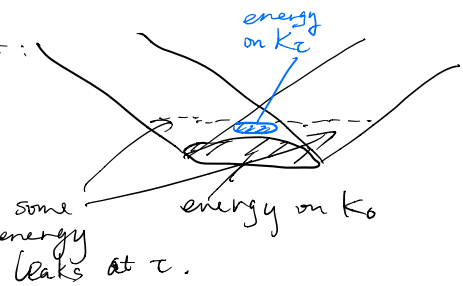
· Energy estimates

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g, \quad \partial_t u = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$



<u>Thm</u> If $u \in C^1(\mathbb{R}^n \times [0,+\infty)) \cap C^2(\mathbb{R}^n \times (0,+\infty))$.
Fix (x_0, t_0). Then $\int_{K_\tau} |\partial_t u|^2 + |\nabla u|^2 dx \le \int_{K_0} |h|^2 + |\nabla g|^2 dx$

Physical viewpoint:



Pf: $E(t) = \int_{K_\tau} (|\partial_t u|^2 + |\nabla u|^2)\, dx$

<u>Lemma</u> $\partial_t \int_{K_t} v(x,t)\, dx = \int_{K_t} \partial_t v(x,t)\, dx - \int_{\partial K_t} v(x,t)\, ds(x)$

Pf: $\partial_t \int_{K_t} v(x,t)\,dx$

$$\underset{x=x_0+(t_0-t)z}{=} \partial_t \int_{|z|\le 1} v(x_0+(t_0-t)z,\ t)\cdot(t_0-t)^n \cdot dz.$$

$$= \int_{|z|\le 1} \Big[\partial_t v(x_0+(t_0-t)z,\ t)\cdot(t_0-t)^n - z\cdot\nabla v(\ ,\ )\cdot(t_0-t)^n - v(\ ,\ )\cdot n(t_0-t)^{n-1}\Big]dz$$

$$= \int_{|z|\le 1} \Big(\partial_t v - z\cdot\nabla_x v - \frac{nv}{t_0-t}\Big)\cdot(t_0-t)^n\,dz$$

$$= \int_{K_t} \Big(\partial_t v \underbrace{- \frac{x-x_0}{t_0-t}\cdot\nabla_x v}_{①} \underbrace{- \frac{nv}{t_0-t}}_{②}\Big)\,dx$$

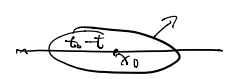
$$\underbrace{\int_{K_t} \frac{x-x_0}{t_0-t}\cdot\overset{\text{gradient}}{\nabla_x v}\,dx}_{①}$$

$$= -\int_{K_t} \underset{\text{divergence}}{\nabla\cdot}\Big(\frac{x-x_0}{t_0-t}\Big)\cdot v\,dx + \int_{\partial K_t} \underset{\text{divergence}}{\nabla\cdot}\Big(\frac{x-x_0}{t_0-t}v\Big)\,ds(x)$$

integrate by parts

$$\underset{\text{divergence thm}}{=} -\int_{K_t} \nabla\cdot\Big(\frac{x-x_0}{t_0-t}\Big)\cdot v\,dx + \int_{\partial K_t} \vec{n}\cdot\frac{x-x_0}{t_0-t}\cdot v\,ds(x).$$

$$\underset{\frac{x-x_0}{t_0-t}=\vec{n}}{=} \underbrace{\int_{K_t} -\frac{nv}{t_0-t}\,dx}_{-②} + \int_{\partial K_t} v\,ds(x)$$



$$\Rightarrow \partial_t \int_{K_t} v(x,t)\,dx = \int_{K_t} \partial_t v(x,t)\,dx - \int_{\partial K_t} v(x,t)\,ds(x)$$

$\square$.

By the lemma, we continue the proof of thm.

$$\partial_t E(t) = \int_{K_t} \underbrace{\partial_t\big((\partial_t u)^2 + |\nabla u|^2\big)}_{\|}\,dx - \int_{\partial K_t} \big((\partial_t u)^2 + |\nabla u|^2\big)\,ds(x)$$

$$= 2\int_{K_t} \partial_t u\,(\underbrace{\partial_t^2 u - \overset{=0}{\Delta u}}) \,dx + 2\int_{\partial K_t} \vec{n}\cdot\nabla u\cdot\partial_t u\,ds(x)$$

$$\le \int_{\partial K_t} \big[(\vec{n}\cdot\nabla u)^2 + (\partial_t u)^2\big]\,ds(x)$$

$$\Rightarrow \partial_t E(t) \le \int_{\partial K_t} \left[(\vec{n}\cdot\nabla u)^2 - |\nabla u|^2 \right] ds(x) \le 0.$$

$$\Rightarrow E(t) \le E(0) = \int_{K_0} \left(h^2 + |\nabla g|^2 \right) dx$$

$\square$.

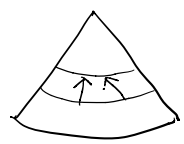
In fact,

$$\partial_t E(t) = -\int_{\partial K_t} \left[(\partial_t u)^2 + |\nabla u|^2 - 2\partial_t u\, \partial_n u \right] ds(x)$$

$$= -\int_{\partial K_t} (\partial_t u - \partial_n u)^2 + \left[|\nabla u|^2 - (\partial_n u)^2 \right] ds(x)$$

$$\partial_t E(t) = 0 \iff \partial_t u = \partial_n u,\ |\nabla u| = \partial_n u \text{ for } u \in \partial K_t$$

i.e. u 仅沿径向传播



Remark:

- If $\operatorname{supp} u$ is compact in $\mathbb{R}^n$ $\longrightarrow$ this 2 requirement is for the step of "integrate by parts"

$$\int_{\mathbb{R}^n} (\partial_t u)^2 + |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} (h^2 + |\nabla g|^2)\, dx$$

- If $u|_{\partial\Omega} = 0$, then $\int_\Omega \left((\partial_t u)^2 + |\nabla u|^2 \right) dx \le \int_\Omega (h^2 + |\nabla g|^2)\, dx$
- Uniqueness: $f, g, h = 0 \Rightarrow u = 0$.