$$\begin{cases} \partial_t u + \partial_x F(u) = 0 & \text{in } \mathbb{R}\times(0,+\infty) \\ u = g & \text{on } \mathbb{R}\times\{t=0\} \end{cases} \qquad (1)$$

$F: \mathbb{R}\to\mathbb{R}$, $g: \mathbb{R}\to\mathbb{R}$.

In §3.2, we find that there does not in general exist a smooth solution of (1), existing for all times $t>0$.
So we look for some sort of weak or generalized solution.

<u>Def</u>. We call $v: \mathbb{R}\times[0,+\infty)\to\mathbb{R}$ is a test function if v is smooth with compact support.

multiply the PDE (1) by smooth v, and then integrate by parts to transfer the derivatives onto v.

$$0 = \int_0^\infty\int_{-\infty}^{+\infty} (\partial_t u + \partial_x F(u))\, v\, dx\, dt$$

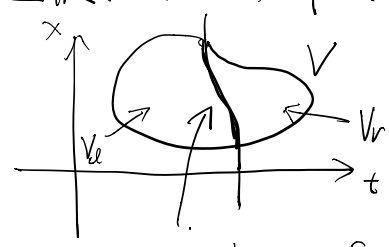
$$= -\int_{-\infty}^{+\infty} uv\, dx\Big|_{t=0} - \int_0^{+\infty}\int_{-\infty}^{+\infty} u\cdot\partial_t v\, dx\, dt - \int_0^\infty\int_{-\infty}^{+\infty} F(u)\,\partial_x v\, dx\, dt$$

$\because$ supp v is compact

$$\Rightarrow \int_0^{+\infty}\int_{-\infty}^{+\infty} (u\,\partial_t v + F(u)\,\partial_x v)\, dx\, dt + \int_{-\infty}^{+\infty} g\, v(x,0)\, dx = 0. \qquad (2)$$

<u>Def</u> We say that $u\in L^\infty(\mathbb{R}\times(0,+\infty))$ is an <u>integral solution</u> of (1) if (2) holds for all test functions v. 积分解.

What can we deduce?
We consider a simple case: $V\subset\mathbb{R}\times(0,+\infty)$, open.



Smooth curve C

Assume
$V = V_l \cup V_r \cup C$, $\partial V_l \cap \partial V_r = C$.
u, $\partial_t u$, $\partial_x u$ are uniformly continuous in V_l, and V_r.
and u is an integral solution

Then in V_l and V_r.
$\because$ $\partial_t u, \partial_x u$ uniformly continuous, (2) holds
$\Rightarrow u\in C^1$
$\Rightarrow 0 = \int_0^\infty\int_{-\infty}^{+\infty} (\partial_t u + \partial_x F(u))\, v\, dx\, dt$ holds for $\forall$ test function v with compact support in V_l (or V_r)
$\Rightarrow \partial_t u + \partial_x F(u) = 0$. pointwisely in V_l and V_r.

Now, select a test function $v\in C_c^\infty(V)$

$$0 = \int_0^{+\infty}\int_{-\infty}^{+\infty} \left(u\partial_t v + F(u)\partial_x v\right) dx\, dt$$

$$= \left(\iint_{V_\ell} + \iint_{V_r}\right) \left(u\partial_t v + F(u)\partial_x v\right) dx\, dt$$

$$= \iint_{V_\ell} \underbrace{\left(-\partial_t u + \partial_x F(u)\right)}_{=0} v\, dx\, dt + \int_{\partial V_\ell} \left(\vec{n}\cdot(F(u), u)\right) v\, dl,$$

unit outer normal of V_ℓ → 时空区域的外法向.

$$\iint_{V_r} \underbrace{\left(-\partial_t u + \partial_x F(u)\right)}_{=0} v\, dx\, dt + \int_{\partial V_r} \left(\vec{n}\cdot(F(u), u)\right) v\, dl,$$

unit outer normal of V_r

$$= \int_C \vec{n}\cdot\left((F(u_\ell), u_\ell) - (F(u_r), u_r)\right) v\, dl$$

$\because$ $v=0$ on $\partial V_\ell \cap \partial V$, $\partial V_r \cap \partial V$.

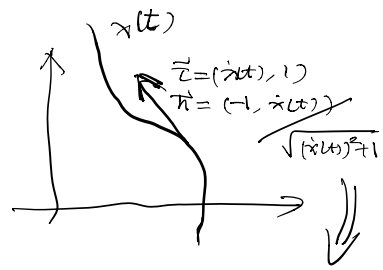
→ u_ℓ denotes the limit from the left.

$$\Rightarrow n^1\cdot\left(F(u_\ell) - F(u_r)\right) + n^2(u_\ell - u_r) = 0$$

$$\Rightarrow \dot{x}(t)(u_\ell - u_r) = F(u_\ell) - F(u_r)$$

$$\Rightarrow \dot{x}(t) = \frac{F(u_\ell) - F(u_r)}{u_\ell - u_r}$$

along the curve $C = \{x(t)\}$.



$$\frac{n^1}{n^2} = -\frac{1}{\dot{x}(t)}$$

展望: 对于分片光滑的积分解, 其在间断面上应满足此式.

Notation: $[[u]] = u_\ell - u_r$ is the jump in u across C

$[[F(u)]] = F(u_\ell) - F(u_r)$. -- $F(u)$. ——

$\sigma = \dot{x}(t)$ is the speed of the curve C.

$$\Rightarrow \sigma = \frac{[[F(u)]]}{[[u]]} \qquad \left([[F(u)]] = \sigma[[u]]\right)$$

Rankine-Hugoniot condition along the shock curve C

激波曲线

A very important condition

$\sigma = 0$ means that $F(u_\ell) = F(u_r)$

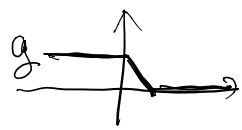
$u_\ell \neq u_r$

a example: $F(u) = \frac{1}{2}u^2$ $\quad u_\ell = +a$, $u_r = -a$

Example: Burger's equation:

$$\begin{cases} \partial_t u + \partial_x\left(\frac{1}{2}u^2\right) = 0. & t>0. \\ u = g & t=0 \end{cases}$$

$$g(x) = \begin{cases} 1 & x<0, \\ 1-x & 0\le x\le 1 \\ 0 & x>1 \end{cases}$$



Apply method of characteristic to solve:

$$F(\overbrace{p, p_{n+1}, z}^{q}, \overbrace{x, x_{n+1}}^{y}) = p_{n+1} + z\cdot p$$

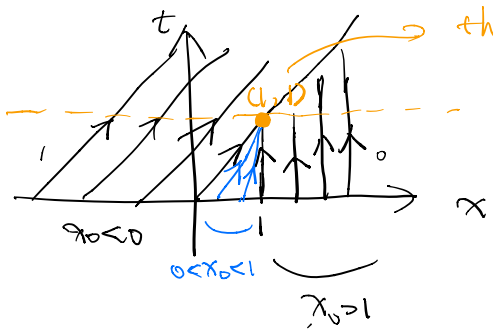
$$D_q F = (z, 1)$$

$$\begin{cases} \dot{x}(s) = z(s) \\ \dot{x}_{n+1}(s) = 1 \\ \dot{z}(s) = 0 \end{cases} \Rightarrow \begin{cases} x(s) = z(0)s + x_0 \\ x_{n+1}(s) = s \\ z(s) = z(0) \end{cases}$$

$$\begin{cases} s = t \\ x = z(0)s + x_0. \\ z(0) = g(x_0) \end{cases}$$

u is constant along the projected characteristics $(g(x_0)s + x_0, s)$

The characteristics are plotted below.

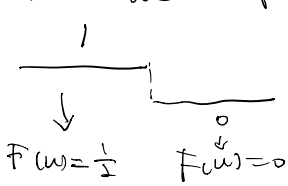


the characteristics crosses at (1,1). thus for $t\ge 1$, characteristics method breaks down.

$$\therefore\ u(x,t) = \begin{cases} 1 & x\le t. \\ \frac{1-x}{t-1} & t\le x\le 1 \\ 0 & x\ge 1 \end{cases} \quad (0\le t<1)$$

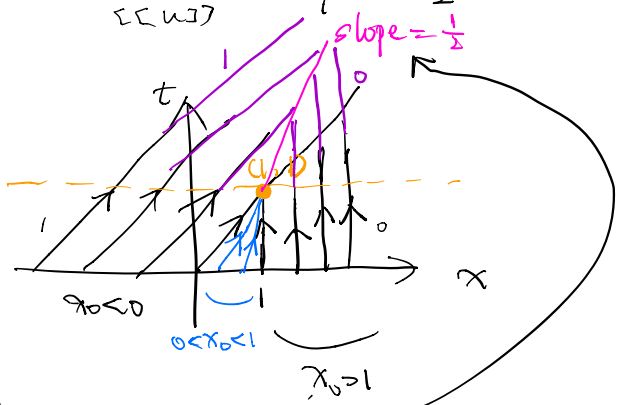
Then we define u for $t\ge 1$

$F(u)=\frac{1}{2}u^2$



$$\Rightarrow \sigma = \frac{[[F(u)]]}{[[u]]} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$ as required by the Rankine-Hugoniot condition.

$\Rightarrow$



If $t \geq 1$

let $s(t) = \frac{1+t}{2}$

$$u(x,t) = \begin{cases} 1 & x < s(t) \\ 0 & x > s(t) \end{cases} \quad (t > 1)$$

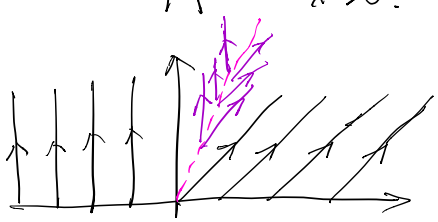
is an integral solution.

#

<u>Example</u>: Still consider Burger's equation:

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0. & t>0. \\ u = g & t=0 \end{cases}$$

$$g(x) = \begin{cases} 0 & x<0 \\ 1 & x>0. \end{cases}$$



$[[u]] = -1$

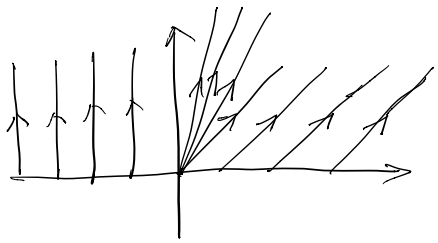
$[[F(u)]] = -\frac{1}{2}$

$\Rightarrow \sigma = \frac{1}{2}$

$$u_1(x,t) = \begin{cases} 0 & x < \frac{1}{2}t \\ 1 & x > \frac{1}{2}t \end{cases}$$ is an integral solution.

However, we can create another solution by writing

$$u_2(x,t) := \begin{cases} 1 & x > t \\ \frac{x}{t} & 0 < x < t \\ 0 & x < 0 \end{cases}$$



(a physical solution)

It is easy to check u_2 is also an integral solution.

Thus, Integral Solutions are not unique!!!

$\downarrow$

Some integral solutions are non-physical.

We introduce Entropy condition to rule out non-physical solutions.

- Lax-Oleinik formula.

F: uniformly convex i.e. $F''(u) \geq \theta > 0$. WLOG, $F(0) = 0$.

Suppose $g \in L^\infty(\mathbb{R})$ and define $h(x) = \int_0^x g(y)\,dy$

then: $\partial_x h = g$.

If $\partial_t u + \partial_x F(u) = 0$. $u = \partial_x w$.

then $\Updownarrow$ $\partial_t \partial_x w + \partial_x F(\partial_x w) = 0$

$\Longleftrightarrow \partial_x \left(\partial_t w + F(\partial_x w)\right) = 0$. $w(x,0) = h(x)$ Hopf-Lax formula.

So if we can find a solution w to the Hamilton-Jacobi equation $\partial_t w + F(\partial_x w) = 0$, then $u = \partial_x w$ is a solution to the conservation Laws equation $\partial_t u + \partial_x F(u) = 0$.

$$w(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + h(y) \right\}$$

w is differentiable a.e. $u(x,t) = \partial_x w$ is a candidate of "solution"

Thm: (Lax–Oleinik formula)

$F: \mathbb{R} \to \mathbb{R}$ is smooth and uniformly convex, $g \in L^\infty(\mathbb{R})$.

(i) $\forall t>0$, for all but at most countably many value of $x \in \mathbb{R}$ a unique point $y(x,t)$ s.t. $\min_{y\in\mathbb{R}} \{ t L(\frac{x-y}{t}) + h(y) \}$

$$= t L\left(\frac{x-y(x,t)}{t}\right) + h(y(x,t)).$$

(ii) $y(x,t)$ is nondecreasing in x.

(iii) $\forall t>0$, $u(x,t) := \partial_x \left(\min_{y\in\mathbb{R}} \{ t L(\frac{x-y}{t}) + h(y) \} \right)$

Lax–Oleinik formula. $L = F^*$

$$= G\left(\frac{x-y(x,t)}{t}\right) \quad \text{for a.e. } x \in \mathbb{R}.$$

where $G = (F')^{-1}$

($\because$ F is uniformly convex $\therefore$ F' strictly increasing $\therefore$ we can define $(F')^{-1}$)

In particular, it holds for a.e. $(x,t) \in \mathbb{R}\times(0,+\infty)$

(iv) $u(x,t)$ is an integral solution of the initial value problem (1)

Remark: This thm gives an integral solution of (1)

Pf: (i) & (ii) $L(v) = \max_{p\in\mathbb{R}^n} \{ v\cdot p - F(p) \}$

$$\underset{\text{Choose } p^*\in\mathbb{R}^n}{=} v\cdot p^* - F(p^*) \geq v\cdot 0 - F(0) = 0.$$

At maximum point p^*, $v = F'(p^*)$

$$\Updownarrow$$

$$G(v) = p^*.$$

$\Rightarrow L(v) = v G(v) - F(G(v))$

$L'(v) = G(v) + v G'(v) - F'(G(v)) G'(v) = G(v)$

$\Rightarrow L(v)$ is C^2 ($F' \in C^1 \Rightarrow (F')^{-1} \in C^1$)

and $L''(v) = G'(v) > 0$.

$\Rightarrow$ L is nonnegative and strictly convex.

Fix $t>0$, $\tilde{L}_x(y) = t L(\frac{x-y}{t}) + h(y)$

$\exists y_1$, s.t. $\tilde{L}_{x_1}(y_1) = \min_y \tilde{L}_{x_1}(y)$

We next claim that

For $x_2 > x_1$, $\tilde{L}_{x_2}(y_1) < \tilde{L}_{x_2}(y)$ (for $\forall y < y_1$)

$x_2 - y_1 = \tau(x_1 - y_1) + (1-\tau)(x_2 - y)$ and $x_1 - y = (1-\tau)(x_1 - y_1) + \tau(x_2 - y)$

for $0 < \tau := \frac{y_1 - y}{x_2 - x_1 + y_1 - y} < 1$. if $y < y_1$

$\because L'' > 0$

$\therefore L\left(\frac{x_2 - y_1}{t}\right) < \tau L\left(\frac{x_1 - y_1}{t}\right) + (1-\tau) L\left(\frac{x_2 - y}{t}\right)$

$L\left(\frac{x_1 - y}{t}\right) < (1-\tau) L\left(\frac{x_1 - y_1}{t}\right) + \tau L\left(\frac{x_2 - y}{t}\right)$

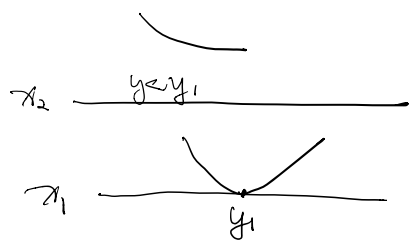
$\Rightarrow L\left(\frac{x_2 - y_1}{t}\right) + L\left(\frac{x_1 - y}{t}\right) < L\left(\frac{x_1 - y_1}{t}\right) + L\left(\frac{x_2 - y}{t}\right)$ ①

$\because \tilde{L}_{x_1}(y_1) = \min_y \tilde{L}_{x_1}(y)$

i.e. $t L\left(\frac{x_1 - y_1}{t}\right) + h(y_1) < t L\left(\frac{x_1 - y}{t}\right) + h(y)$ ②

$t \cdot ① - ②$: $\Rightarrow t L\left(\frac{x_2 - y_1}{t}\right) + h(y_1) < t L\left(\frac{x_2 - y}{t}\right) + h(y)$

i.e. $\tilde{L}_{x_2}(y_1) < \tilde{L}_{x_2}(y)$



Let $y(x,t)$ be the smallest y s.t. $\tilde{L}_x(y(x,t)) = \min_{z \in \mathbb{R}^n} \tilde{L}_x(z)$

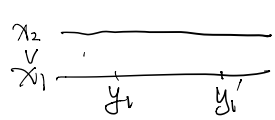
By the claim above,

$\Rightarrow$ $y(x,t)$ is non-decreasing.

Thus y is continuous for all but countably many x.

If $y(x,t)$ is continuous at x, then $y(x,t)$ is the unique y s.t. $\tilde{L}_x(y) = \min_{z \in \mathbb{R}^n} \tilde{L}_x(z)$.

若不是，会有一个跳跃



若对 $y(x,t)$ 在 x_1 处连续，
x_1 对应2个极小值点 y_1, y_1'
则对 $\forall x_2 > x_1$，$L_{x_2}(y)$ 的极小值点 比 y_1' 大，于是 $y(x,t)$ 在 x_1 处有跳跃，矛盾！

连续点处的极小值点是唯一的

(iii) $y(x,t)$ is differentiable a.e. $x \in \mathbb{R}$.

$\Rightarrow L\left(\frac{x-y(x,t)}{t}\right)$ and $h(y(x,t))$ is differentiable a.e.

$$u(x,t) = \frac{\partial}{\partial x}\left[t\, L\left(\frac{x-y(x,t)}{t}\right) + h(y(x,t))\right]$$

$$= t\, L'\left(\frac{x-y(x,t)}{t}\right)\frac{1}{t}\left(1-y'(x,t)\right) + h'(y(x,t))\cdot y'(x,t) \quad ③$$

$\because$ $y(x,t)$ is the one such that $\tilde{L}_x(y(x,t)) = \min_{\xi\in\mathbb{R}^n} \tilde{L}_x(\xi)$,

$$\therefore \min_{z\in\mathbb{R}^n}\left(t\, L\left(\frac{x-y(z,t)}{t}\right) + h(y(z,t))\right) = t\, L\left(\frac{x-y(x,t)}{t}\right) + h(y(x,t))$$

$$\therefore \frac{\partial}{\partial z}\left(t\, L\left(\frac{x-y(z,t)}{t}\right) + h(y(z,t))\right)\Big|_{z=x} = 0.$$

$$\Rightarrow t\, L'\left(\frac{x-y(x,t)}{t}\right)\frac{1}{t}y'(x,t) + h'(y(x,t))\, y'(x,t) = 0$$

By ③ $\Rightarrow u(x,t) = L'\left(\frac{x-y(x,t)}{t}\right) = G\left(\frac{x-y(x,t)}{t}\right)$

(iv) $u(x,t) = \min_{y\in\mathbb{R}}\left\{t\, L\left(\frac{x-y}{t}\right) + h(y)\right\}$ is Lip continuous and differentiable a.e.

$$\begin{cases} \partial_t w + F(\partial_x w) = 0 & \text{a.e.} \\ w = h & t = 0. \end{cases}$$

Choose $v : \mathbb{R}\times[0,+\infty) \to \mathbb{R}$ test function

then $\partial_x v$ is also a test function (C^∞, supp $\partial_x v$ is compact)

$$\Rightarrow 0 = \int_0^\infty \int_{-\infty}^{+\infty} \left(\partial_t w + F(\partial_x w)\right)\partial_x v \, dx\, dt$$

$$\because \int_0^{+\infty}\int_{-\infty}^{+\infty} \partial_t w\, \partial_x v \, dx\, dt$$

$$= \int_{-\infty}^{+\infty} \partial_x v \cdot w \, dx\Big|_{t=0}^{+\infty} - \int_0^{+\infty}\int_{-\infty}^{+\infty} w\cdot \partial_{xt}^2 v \, dx\, dt$$

$$= -\int_{-\infty}^{+\infty} \partial_x v \cdot w \, dx\Big|_{t=0} - \int_0^{+\infty} w\cdot \partial_t v \, dt\Big|_{x=-\infty}^{x=+\infty} + \int_0^{+\infty}\int_{-\infty}^{+\infty} \partial_t v\, \partial_x w \, dx\, dt$$

$$= \underbrace{-v\cdot w\Big|_{x=-\infty}^{x=+\infty}}_{0} + \int_{-\infty}^{+\infty} v\, \partial_x w \, dx\Big|_{t=0} + \int_0^{+\infty}\int_{-\infty}^{+\infty} \partial_t v\, \partial_x w \, dx\, dt$$

$$= \int_{-\infty}^{+\infty} g v(x,0)\, dx + \int_0^{+\infty}\int_{-\infty}^{+\infty} \partial_t v\, \partial_x w \, dx\, dt$$

$$\therefore 0 = \int_{-\infty}^{+\infty} g v(x,0)\, dx + \int_0^\infty\int_{-\infty}^{+\infty} \left(\partial_x w\, \partial_t v + F(\partial_x w)\partial_x v\right) dx\, dt$$

$\Rightarrow \partial_x w$ is an integral solution. $\square$

· Entropy condition and entropy solution.

<u>Def</u> Entropy condition: $\exists C>0$, s.t. $u(x+z,t)-u(x,t)\le \frac{Cz}{t}$

for all $t>0$, $z>0$ $x\in\mathbb{R}$

<u>Def</u> Entropy solution: An integral solution satisfying entropy condition

<u>Thm</u>: (i) Entropy solution is unique.

(ii) The solution u given by Lax-Oleinik formula is an entropy solution.

Pf: (i) Omitted → 教材

(ii) $u(x,t) = G\left(\frac{x-y(x,t)}{t}\right) \ge G\left(\frac{x-y(x+z,t)}{t}\right)$ $\because G, y$ increasing.

$\ge G\left(\frac{x+z-y(x+z,t)}{t}\right) - (\mathrm{Lip}\,G)\frac{z}{t}$

$= u(x+z,t) - (\mathrm{Lip}\,G)\frac{z}{t}$.

□.

· Riemann's problem —— Piecewise-constant initial data.

$g = \begin{cases} u_l & x<0 \\ u_r & x>0 \end{cases}$ u_l, u_r left / right initial state

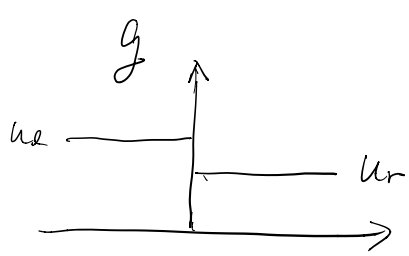
Assume F uniformly convex C^2 $G=(F')^{-1}$.

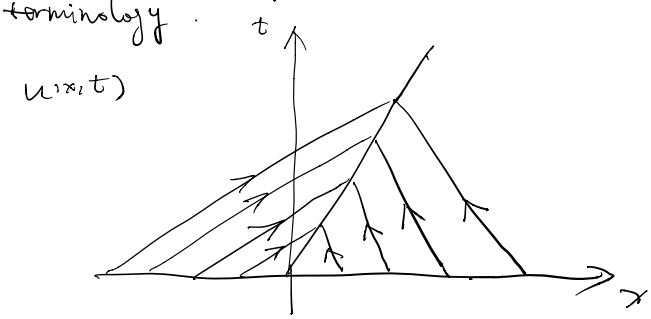
<u>Thm</u>: (i) If $u_l > u_r$, the unique entropy solution is

$u(x,t) = \begin{cases} u_l & \frac{x}{t}<\sigma \\ u_r & \frac{x}{t}>\sigma \end{cases}$. $\left(\sigma = \frac{[\![F(u)]\!]}{[\![u]\!]} = \frac{F(u_l)-F(u_r)}{u_l-u_r}\right)$

(shock solutions) (激波解)

↓ a physics terminology.





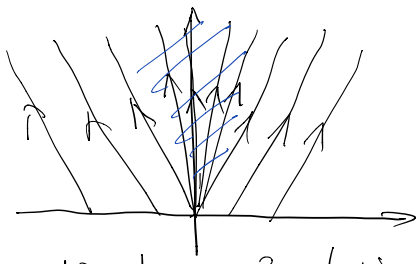
后面的波跑的比前面的快，二者会挤在一起

(ii) If $u_l < u_r$, the unique entropy solution is

$$u(x,t) = \begin{cases} u_l & \text{if } \frac{x}{t} < F'(u_l) \\ G(\frac{x}{t}) & \text{if } F'(u_l) < \frac{x}{t} < F'(u_r) \\ u_r & \text{if } \frac{x}{t} > F'(u_r) \end{cases}$$

(rarefactions solutions) (稀疏波解)

前面的速比
后面的波跑的慢,
二者相差越来越远

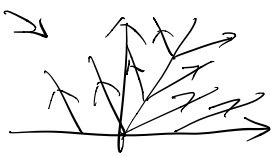


Apply Lax-Oleinik formula to compute these 2 solutions

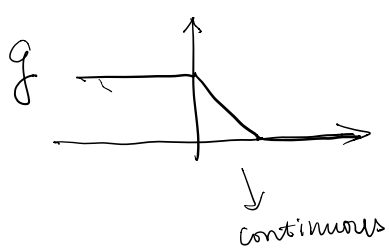
or Check these 2 satisfies entropy condition and are integral solutions directly.

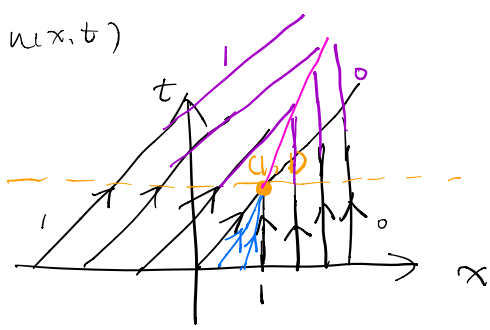
如同 Burger's equation,
$u_l < u_r$ 时, 可以构造出一个
低的激波解

可以验证,
其满足熵条件.



Reconsider Burger's equation with initial data g.





$$u(x,t) \begin{cases} t < 1 & \text{continuous} \\ t \geq 1 & \text{discontinuous} \end{cases}$$

This phenomenon is called blow up → 解的爆破

Solution of PDE can be generally classified as $\begin{cases} \text{blow up at a finite time for smooth initial data} \\ \text{global exist for smooth solutions} \end{cases}$

For Burger's equation, if the initial data g $\exists x_0$, $g'(x_0) < 0$

then $\exists T$.

$u(x,t)$ blow up at finite time T.

(The proof isn't difficult.)