Wave equation

$\Box u = \partial_t^2 u - \Delta u \longrightarrow$ wave operator

$u = u(x,t) \quad (x \in \mathbb{R}^n,\ t \geq 0)$

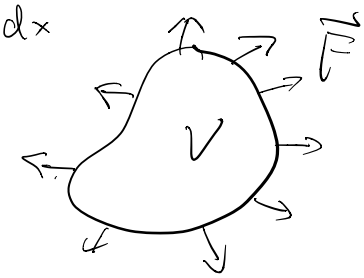
$\Box u = 0$ (homogeneous) or $\Box u = f$ (non-homogeneous)

Describe the movement of "wave"

$u(x,t)$: the displacement in some direction of point x at time $t > 0$.

$V \subset \mathcal{U}$ is a smooth domain.

The acceleration : $\frac{d^2}{dt^2}\int_V u\,dx = \int_V \partial_t^2 u\,dx$



$\sum F = ma$ assume $m = 1$.

$\Rightarrow$ The net contact force :

$$-\int_{\partial V} F \cdot n \, dS(x) = \int_V \partial_t^2 u\,dx$$

$$\overset{(1)}{\int_V (-\operatorname{div}\vec{F})\,dx}$$

$$\Rightarrow \partial_t^2 u = -\operatorname{div} F$$

Assumption $F = F(\nabla u) \approx -k \cdot \nabla u \quad (F \propto \nabla u)$

$$\Rightarrow \partial_t^2 u = k \Delta u. \quad (k > 0)$$

Initial condition: $\begin{cases} u(x,0) = u_0(x) & \longrightarrow \text{displacement} \\ \partial_t u(x,0) = u_1(x) & \longrightarrow \text{velocity} \end{cases}$

· Solution for $n=1$

a) $\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & (x,t) \in \mathbb{R} \times (0, +\infty) \\ u(x,0) = g(x) & x \in \mathbb{R} \\ \partial_t u(x,0) = h(x) \end{cases}$

$\Rightarrow (\partial_t + \partial_x)(\partial_t - \partial_x) u = 0 \quad \rightarrow$ The PDE can be factored.

Step 1 : Let $v(x,t) = (\partial_t - \partial_x) u$

$\Rightarrow \partial_t v + \partial_x v = 0. \longrightarrow$ transport equation.

$\Rightarrow v(x,t) = a(x-t)$ (Apply the solution formula to transport equation)

Then $a(x) = v(x,0) = (\partial_t - \partial_x) u(x,0) = h(x) - g'(x)$

Step 2: $\partial_t u - \partial_x u = v(x,t) \longrightarrow$ nonhomogeneous transport equation

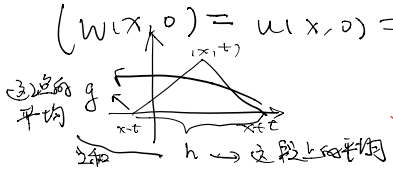
Let $w(x,t) = u(x-t, t)$

$\partial_t w = (\partial_t u - \partial_x u)(x-t, t) = v(x-t, t) = a(x-2t)$

$\Rightarrow w(x,t) = w(x,0) + \int_0^t a(x-2\tau)\,d\tau$

$\Rightarrow u(x,t) = w(x+t, t) = w(x+t, 0) + \int_0^t a(x+t-2\tau)\,d\tau$

$(w(x,0) = u(x,0) = g(x)) \Rightarrow g(x+t) + \int_0^t (h(x+t-2\tau) - g'(x+t-2\tau))\,d\tau$



$$u(x,t) = \frac{1}{2}\big(g(x+t) - g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

d'Alembert's formula.

Thm (Solution of wave equation, $n=1$)

Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and u is given as above, then:

i) $u \in C^2(\mathbb{R}^n \times [0,+\infty))$ (regularity)

ii) $\partial_t^2 u - \partial_x^2 u = 0$ in $\mathbb{R} \times (0,+\infty)$ (satisfying equation)

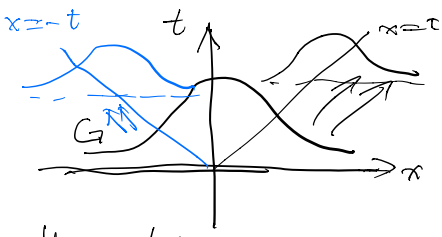
iii) $\lim\limits_{\substack{(x,t)\to(x_0,0)\\ t>0}} (u(x,t), \partial_t u(x,t)) = (g(x_0), h(x_0))$ (initial condition.)

Pf: Omitted. (A direct calculation)

Remark: 1) $u(x,t) = \left(\frac{1}{2} g(x+t) + \frac{1}{2}\int_0^{x+t} h(y)\,dy\right) - \left(\frac{1}{2} g(x-t) + \frac{1}{2}\int_0^{x-t} h(y)\,dy\right)$

$= \underbrace{F(x+t)}_{\text{left travelling solution}} + \underbrace{G(x-t)}_{\text{right travelling solution}}$

travelling solution

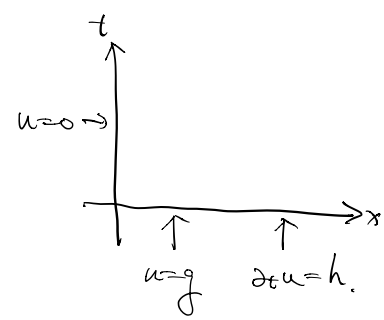


Any solution to $\partial_t^2 u - \partial_x^2 u = 0$ can be decomposed into such 2 kinds of waves.

2) We can see from the d'Alembert's formula that if $g \in C^k$, $h \in C^{k-1}$, then $u \in C^k$

b) $\mathbb{R}_+ \times (0,+\infty) = [0,+\infty) \times (0,+\infty)$

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \text{in } \mathbb{R}_+ \times (0,+\infty) \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = 0 & \text{on } \{x=0\} \times (0,+\infty) \end{cases}$$



Let $\tilde{u}(x,t) = \begin{cases} u(x,t) & x \geq 0,\ t>0 \\ -u(-x,t) & x<0,\ t>0. \end{cases}$

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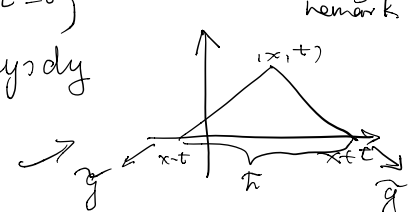
$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x<0 \end{cases}$ $\qquad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x<0 \end{cases}$

We should assume that $\begin{cases} \tilde{g} \in C^2(\mathbb{R}) \\ \tilde{h} \in C^1(\mathbb{R}) \end{cases}$ additionally (Need to add some additional condition to g, h. see the following remark.)

Then: $\begin{cases} \partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } \mathbb{R} \times (0,+\infty) \\ \tilde{u} = \tilde{g} \quad \partial_t \tilde{u} = \tilde{h} & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$

$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{g}(x+t) + \tilde{g}(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy$

$= \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & x \geq t > 0 \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & 0 \leq x \leq t \end{cases}$



Remark: If we want u to be C^2, it is needed that $g(0)=0$, $h(0)=0$, $g''(0)=0$.

$$\left(\begin{array}{l} g(0)=0 \Rightarrow g\in C \\ h(0)=0 \Rightarrow h\in C. \\ \text{We need } u\equiv 0 \text{ on } \{x=0\}\times(0,+\infty) \\ \Rightarrow \text{We need } \partial_t u(0,0)=0. \\ \Rightarrow \text{we need } \partial_x u(0,0)=0 \\ \Rightarrow \text{we need } g''(0)=0. \\ \text{Then } \tilde{g}\in C^2,\ \tilde{h}\in C^1 \\ \therefore \text{By the above thm} \Rightarrow u\in C^2. \end{array}\right)$$

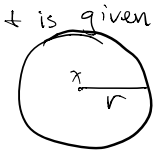
- $n\geq 2$.

$$\begin{cases} \partial_t^2 u-\Delta u=0 & \text{in } \mathbb{R}^n\times(0,+\infty) \\ u=g \quad \partial_t u=h & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases} \qquad \Box=\partial_t^2-\Delta$$

when $n\geq 2$, we cannot do the factorization as $n=1$.

<u>Def</u> $\mathcal{U}(x;r,t):=\fint_{\partial B(x,r)} u(y,t)\,ds(y)$

t is given.



is the average of u on $\partial B(x,r)$

$$G(x,r)=\fint_{\partial B(x,r)} g(y)\,ds(y), \quad H(x,r)=\fint_{\partial B(x,r)} h(y)\,ds(y).$$

<u>Lem</u>: Fix $x\in\mathbb{R}^n$, u is the solution to the initial-value problem the $\mathcal{U}\in C^m(\mathbb{R}_+\times[0,+\infty))$ and

(Euler-Poisson-Darboux Equation) $\begin{cases} \partial_t^2\mathcal{U}-\partial_r^2\mathcal{U}-\frac{n-1}{r}\partial_r\mathcal{U}=0 & \text{in } \mathbb{R}_+\times(0,+\infty) \\ \mathcal{U}=G, \quad \partial_t\mathcal{U}=H & \text{on } \mathbb{R}_+\times\{t=0\} \end{cases}$

Pf: 1) $\mathcal{U}(x;r,t)=\frac{1}{\omega(n)\cdot r^{n-1}}\int_{\partial B(x,r)} u(y,t)\,ds(y)$

$$\underset{(y=x+rz,\ |z|=1)}{=} \frac{1}{\omega(n)}\int_{\partial B(0,1)} u(x+rz,t)\,ds(z)$$

$$\Rightarrow \partial_r\mathcal{U}=\frac{1}{\omega(n)}\int_{\partial B(0,1)}\nabla u(x+rz,t)\cdot z\,ds(z)$$

$$=\frac{1}{\omega(n)\cdot r^{n-1}}\int_{\partial B(x,r)}\nabla u(y,t)\cdot\underbrace{\frac{y-x}{r}}_{=n}\,ds(y)$$

$$=\frac{1}{\omega(n)\cdot r^{n-1}}\int_{B(x,r)}\Delta u(y,t)\cdot dy \qquad (*)$$

$$\Rightarrow \quad r^{n-1}\partial_r \mathcal{U} = \frac{1}{\omega(n)}\int_{B(x,r)} \partial_t^2 u(y,t)\,dy$$

$$\Rightarrow \quad \partial_r\left(r^{n-1}\partial_r \mathcal{U}\right) = \frac{1}{\omega(n)}\int_{\partial B(x,r)} \partial_t^2 u(y,t)\,dS(y) \qquad \partial_r \int_{B(x,r)} = \int_{\partial B(x,r)}$$

$$= \partial_t^2\left(\frac{1}{\omega(n)}\int_{\partial B(x,r)} u(y,t)\,dS(y)\right)$$

$$= r^{n-1}\partial_t^2 \mathcal{U}$$

$$\Rightarrow \quad \partial_t^2 \mathcal{U} - \partial_r^2 \mathcal{U} - \frac{n-1}{r}\partial_r \mathcal{U} = 0. \qquad \square$$

2) $\lim_{r\to 0^+} \mathcal{U}(x;r,t) = u(x,t)$

$\lim_{r\to 0^+} \partial_r \mathcal{U}(x;r,t) = \lim_{r\to 0^+} \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\,dy = 0.$

After some computations. $\lim_{r\to 0^+} \partial_r^2 \mathcal{U}(x;r,t) = \frac{1}{n}\Delta u(x,t)$

$\lim_{r\to 0^+} \partial_r^k \mathcal{U}(x;r,t) = \cdots$

$\Rightarrow \mathcal{U} \in C^m(\mathbb{R}_+ \times [0,+\infty))$ $\square$

Above Euler-Poisson-Darboux equation is similar to 1-dimensional wave equation. Now we need to do some transformation

- $n=3$. $\quad \partial_t^2 \mathcal{U} - \partial_r^2 \mathcal{U} - \frac{2}{r}\partial_r \mathcal{U} = 0$

$$\partial_r^2(r\mathcal{U}) = \partial_r(\mathcal{U} + r\partial_r \mathcal{U}) = 2\partial_r \mathcal{U} + r\cdot\partial_r^2 \mathcal{U}$$

$$= r\partial_t^2 \mathcal{U} = \partial_t^2(r\mathcal{U})$$

$\therefore$ $r\mathcal{U}$ satisfies 1-dimensional wave equation.

$$\begin{cases} \tilde{\mathcal{U}} = r\mathcal{U} \\ \tilde{G} = rG \\ \tilde{H} = rH \end{cases} \Rightarrow \begin{cases} \partial_t^2 \tilde{\mathcal{U}} - \partial_r^2 \tilde{\mathcal{U}} = 0 & \text{in } \mathbb{R}_+ \times (0,+\infty) \\ \tilde{\mathcal{U}} = \tilde{G}, \quad \partial_t \tilde{\mathcal{U}} = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{\mathcal{U}} = 0 & \text{on } \{r=0\} \times [0,+\infty) \end{cases}$$

By the remark in the previous page, if we want the solution to be C^2. we should need to check $\tilde{G}(0)=0$, $\tilde{H}(0)=0$, $\partial_r^2 \tilde{G}(0) = 0$.

obvious

$$\partial_r^2 \tilde{G}(0) = (r\partial_r^2 G + 2\partial_r G)\big|_{r=0} = \left[\frac{1}{r}\partial_r(r^2 \partial_r G)\right]\Big|_{r=0}.$$

the calculation similar to the equation (*) in previous page

$$= \lim_{r\to 0^+} \frac{1}{\omega_3}\frac{1}{r}\int_{\partial B(x,r)} \Delta g(y)\cdot dS(y) = 0. \quad \checkmark$$

We take $r<t$, then

$$\tilde{\mathcal{U}}(x;r,t)=\frac{1}{2}\left[\tilde{G}(x;r+t)-\tilde{G}(x;t-r)\right]+\frac{1}{2}\int_{t-r}^{t+r}\tilde{H}(y)\,dy.$$

$$u(x,t)=\lim_{r\to 0^+}\mathcal{U}(x;r,t)=\lim_{r\to 0^+}\frac{\tilde{\mathcal{U}}(x;r,t)}{r}$$

$$=\lim_{r\to 0^+}\left\{\frac{1}{2}\left[\frac{\tilde{G}(x;r+t)-\tilde{G}(x;t-r)}{r}\right]+\frac{1}{2r}\int_{t-r}^{t+r}\tilde{H}(y)\,dy\right\}$$

$$=\partial_t\tilde{G}(x;t)+\tilde{H}(x;t)$$

$$=\partial_t\left(tG(x;t)\right)+tH(x;t)$$

$$G(x;t)=\fint_{\partial B(x;t)}g(y)\,dS(y)\qquad H(x;t)=\fint_{\partial B(x;t)}h(y)\,dS(y)$$

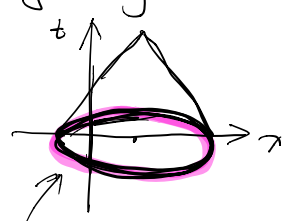
$$\partial_t G(x;t)=\partial_t\fint_{\partial B(0;1)}g(x+tz)\,dS(z)=\fint_{\partial B(0,1)}z\cdot\nabla g(x+tz)\,dS(z)$$

$$=\fint_{\partial B(x;t)}\frac{y-x}{t}\cdot\nabla g(y)\,dS(y)$$

$$\Rightarrow\ u(x;t)=\fint_{\partial B(x;t)}\left[g+th+(y-x)\cdot\nabla g\right](y)\,dS(y)$$

$$=\frac{1}{4\pi t^2}\int_{\partial B(x;t)}\left[g+th+(y-x)\cdot\nabla g\right]dS(y)$$

Kirchhoff's formula.



$n=3$时，仅依赖于这个球面上的初值

而 $n=1$ 时

- $n=2$ $\quad \partial_t^2\mathcal{U}-\partial_r^2\mathcal{U}-\frac{1}{r}\partial_r\mathcal{U}=0.$

No transformation like "$n=3$ case" works.

Instead, we will regard the original initial-value problem for $n=2$ as a problem for $n=3$ in which the third spatial variable x_3 does not appear.

$$\begin{cases}\tilde{u}(x_1,x_2,x_3,t):=u(x_1,x_2,t) & \tilde{x}=(\overset{(x_1,x_2)}{x},x_3)\\ \tilde{g}(x,x_3)=g(x) & \tilde{h}(x,x_3)=h(x)\end{cases}$$

$$\Rightarrow\begin{cases}\partial_t^2\tilde{u}-\Delta\tilde{u}=0 & \text{in } \mathbb{R}^3\times(0,+\infty)\\ \tilde{u}=\tilde{g},\ \partial_t\tilde{u}=\tilde{h} & \text{on } \mathbb{R}^3\times\{t=0\}\end{cases}$$

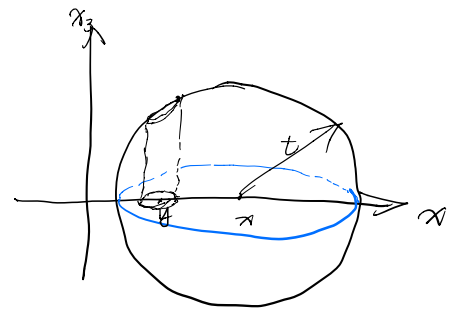
$$\Rightarrow u(x,t) = \tilde{u}(\tilde{x},t)$$

$$= \frac{1}{4\pi t^2}\int_{\underset{S^2}{\partial B(\tilde{x},t)}} \left[\tilde{g} + t\tilde{h} + (\tilde{y}-\tilde{x})\cdot\nabla\tilde{g}\right] ds(\tilde{y})$$

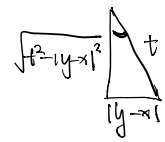
$$= 2\cdot\frac{1}{4\pi t^2}\int_{\underset{S^1}{B(x,t)}} \frac{t}{\sqrt{t^2-|y-x|^2}}\underbrace{(g+th+(y-x)\cdot\nabla g)}_{\text{纵向投影值 } \tilde{g}=g}\, dy$$

上下半球均作投影.

球坐标下面积微元(对比)



$$= \frac{1}{2\pi t}\int_{B(x,t)} \frac{g+th+(y-x)\cdot\nabla g}{\sqrt{t^2-|y-x|^2}}\, dy$$



Poisson's formula.

method of descent

- $n \geq 4$ Omitted
  First consider "n odd" case, then apply method of descent to "n even" case.

- Non-homogeneous problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^n\times(0,+\infty) \\ u = g \quad \partial_t g = h & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases}$$

Denote the solution: $u(x,t) \triangleq \mathcal{M}(f,g,h)(x,t)$

$\mathcal{M}(0,g,h)(x,t) = \cdots$ (We have done)

$\mathcal{M}(f,g,h)(x,t) = \underline{\mathcal{M}(f,0,0)} + \mathcal{M}(0,g,0) + \mathcal{M}(0,0,h)$

Duhamel's principle:

$$\begin{cases} \mathcal{M}(0,g,0)(x,t) = \partial_t \mathcal{M}(0,0,g)(x,t) \\ \mathcal{M}(f,0,0)(x,t) = \int_0^t \mathcal{M}(0,0,f_\tau)(x,t-\tau)\, d\tau \end{cases}$$

Pf: Omitted. (Do some calculation to prove)

Example:

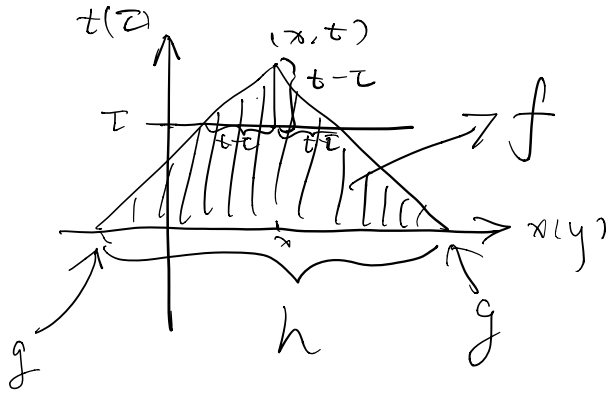
- $n=1$.

$$\mathcal{M}(0,0,h) = \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

$$\mathcal{M}(0,g,0) = \partial_t \mathcal{M}(0,0,g) = \frac{1}{2}\partial_t \int_{x-t}^{x+t} g(y)\,dy = \frac{1}{2}\left(g(x+t)+g(x-t)\right)$$

$$\mathcal{M}(f,0,0) = \frac{1}{2}\int_0^t \left(\int_{x-(t-\tau)}^{x+t-\tau} f(y,\tau)\,dy\right)d\tau$$

$$= \frac{1}{2}\int_{\substack{|y-x|\le t-\tau \\ 0\le\tau\le t}} f(y,\tau)\,dy\,d\tau$$



- $n=3$

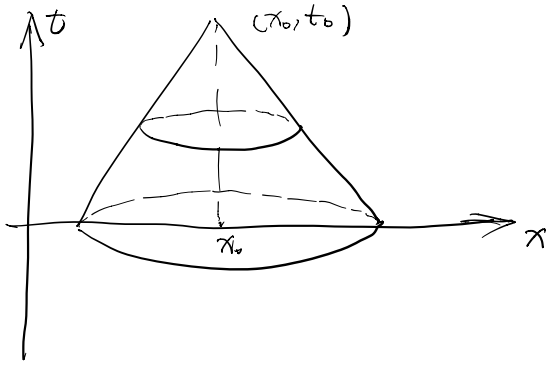
$$\mathcal{M}(f,0,0)(x,t) = \int_0^t \left(\frac{t-\tau}{4\pi(t-\tau)^2}\int f(y,\tau)\,dS(y)\right)d\tau$$

$$= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-\tau)} \frac{f(y,\tau)}{t-\tau}\,dS(y)\,d\tau$$

$$= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dS(y)\,dr$$

$$= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy$$

It is called retarded potential

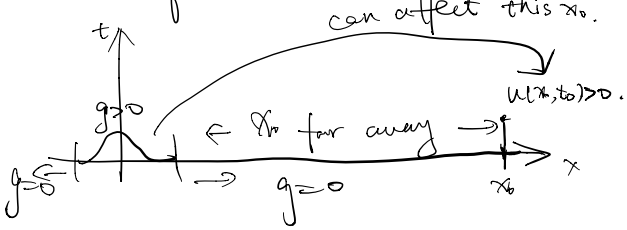


$K(x_0,t_0) = \{(x,t) \mid 0 \le t \le t_0,\ |x-x_0| \le t_0 - t\}$

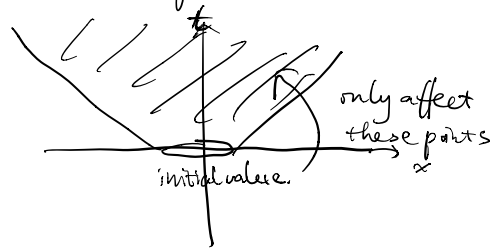
Backward wave cone

$u(x_0,t_0)$ only depends on the value given in $K(x_0,t_0)$.

heat equation:



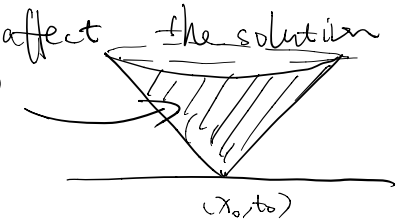
wave equation:



- The given value at (x_0,t_0) only affect the solution in $\{(x,t) \mid |x-x_0| \le t-t_0,\ t \ge t_0\}$

$\Downarrow$

$K^+(x_0,t_0)$



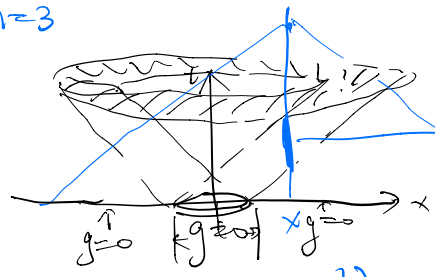
- The initial value g only affect $u\Big|_{\bigcup_{x_0 \in \operatorname{supp} g} K^+(x_0,0)}$.

- If $f=0$ $u(x_0,t_0)$ only depends on the initial data in $|y-x_0| \le t_0$.

$n=3$: $u(x_0,t_0)$ only depends on the initial data on $|y-x_0| = t_0$.

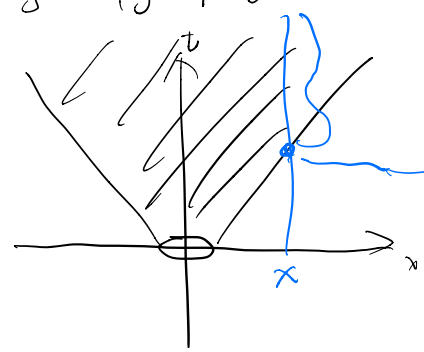
In physics, this reflects the different property of 2-d and 3-d wave.

$n=3$



at x, it is affected only in this time interval. → 清晰的刻画

$n=2$

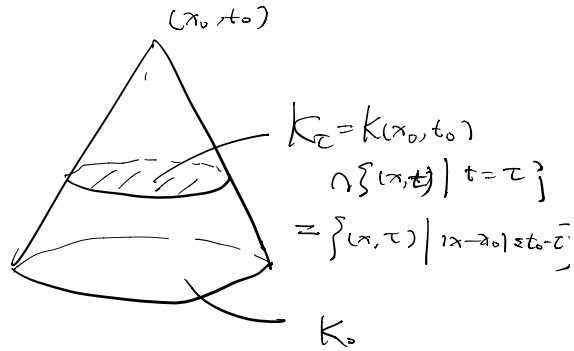


at x, it is affected before this time → 水面上的波一直向外传

Huygens' principle.

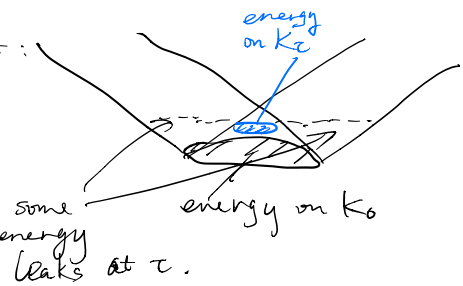
· Energy estimates

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g, \quad \partial_t u = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$



<u>Thm</u> If $u \in C^1(\mathbb{R}^n \times [0,+\infty)) \cap C^2(\mathbb{R}^n \times (0,+\infty))$.
Fix (x_0, t_0). Then $\int_{K_\tau} |\partial_t u|^2 + |\nabla u|^2 dx \le \int_{K_0} |h|^2 + |\nabla g|^2 dx$

Physical viewpoint:



Pf: $E(t) = \int_{K_\tau} (|\partial_t u|^2 + |\nabla u|^2)\, dx$

<u>Lemma</u> $\partial_t \int_{K_t} v(x,t)\, dx = \int_{K_t} \partial_t v(x,t)\, dx - \int_{\partial K_t} v(x,t)\, ds(x)$

Pf: $\partial_t \int_{K_t} v(x,t)\,dx$

$$\underset{x=x_0+(t_0-t)z}{=} \partial_t \int_{|z|\le 1} v(x_0+(t_0-t)z,\ t)\cdot(t_0-t)^n \cdot dz.$$

$$= \int_{|z|\le 1} \Big[\partial_t v(x_0+(t_0-t)z,\ t)\cdot(t_0-t)^n - z\cdot\nabla v(\ ,\ )\cdot(t_0-t)^n - v(\ ,\ )\cdot n(t_0-t)^{n-1}\Big]dz$$

$$= \int_{|z|\le 1} \Big(\partial_t v - z\cdot\nabla_x v - \frac{nv}{t_0-t}\Big)\cdot(t_0-t)^n\,dz$$

$$= \int_{K_t} \Big(\partial_t v \underbrace{- \frac{x-x_0}{t_0-t}\cdot\nabla_x v}_{①} \underbrace{- \frac{nv}{t_0-t}}_{②}\Big)\,dx$$

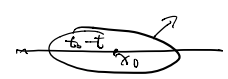
$$\underbrace{\int_{K_t} \frac{x-x_0}{t_0-t}\cdot\overset{\text{gradient}}{\nabla_x v}\,dx}_{①}$$

$$= -\int_{K_t} \underset{\text{divergence}}{\nabla\cdot}\Big(\frac{x-x_0}{t_0-t}\Big)\cdot v\,dx + \int_{\partial K_t} \underset{\text{divergence}}{\nabla\cdot}\Big(\frac{x-x_0}{t_0-t}v\Big)\,ds(x)$$

integrate by parts

$$\underset{\text{divergence thm}}{=} -\int_{K_t} \nabla\cdot\Big(\frac{x-x_0}{t_0-t}\Big)\cdot v\,dx + \int_{\partial K_t} \vec{n}\cdot\frac{x-x_0}{t_0-t}\cdot v\,ds(x).$$

$$\underset{\frac{x-x_0}{t_0-t}=\vec{n}}{=} \underbrace{\int_{K_t} -\frac{nv}{t_0-t}\,dx}_{-②} + \int_{\partial K_t} v\,ds(x)$$



$$\Rightarrow \partial_t \int_{K_t} v(x,t)\,dx = \int_{K_t} \partial_t v(x,t)\,dx - \int_{\partial K_t} v(x,t)\,ds(x)$$

$\square$.

By the lemma, we continue the proof of thm.

$$\partial_t E(t) = \int_{K_t} \underbrace{\partial_t\big((\partial_t u)^2 + |\nabla u|^2\big)}_{\|}\,dx - \int_{\partial K_t} \big((\partial_t u)^2 + |\nabla u|^2\big)\,ds(x)$$

$$= 2\int_{K_t} \partial_t u\,(\underbrace{\partial_t^2 u - \overset{=0}{\Delta u}}) \,dx + 2\int_{\partial K_t} \vec{n}\cdot\nabla u\cdot\partial_t u\,ds(x)$$

$$\le \int_{\partial K_t} \big[(\vec{n}\cdot\nabla u)^2 + (\partial_t u)^2\big]\,ds(x)$$

$$\Rightarrow \partial_t E(t) \le \int_{\partial K_t} \left[(\vec{n}\cdot\nabla u)^2 - |\nabla u|^2 \right] ds(x) \le 0.$$

$$\Rightarrow E(t) \le E(0) = \int_{K_0} \left(h^2 + |\nabla g|^2 \right) dx$$

$\square$.

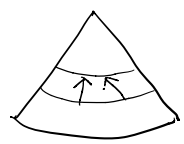
In fact,

$$\partial_t E(t) = -\int_{\partial K_t} \left[(\partial_t u)^2 + |\nabla u|^2 - 2\partial_t u\, \partial_n u \right] ds(x)$$

$$= -\int_{\partial K_t} (\partial_t u - \partial_n u)^2 + \left[|\nabla u|^2 - (\partial_n u)^2 \right] ds(x)$$

$$\partial_t E(t) = 0 \iff \partial_t u = \partial_n u,\ |\nabla u| = \partial_n u \text{ for } u \in \partial K_t$$

i.e. u 仅沿径向传播



Remark:

- If $\operatorname{supp} u$ is compact in $\mathbb{R}^n$ $\longrightarrow$ this 2 requirement is for the step of "integrate by parts"

$$\int_{\mathbb{R}^n} (\partial_t u)^2 + |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} (h^2 + |\nabla g|^2)\, dx$$

- If $u|_{\partial\Omega} = 0$, then $\int_\Omega \left((\partial_t u)^2 + |\nabla u|^2 \right) dx \le \int_\Omega (h^2 + |\nabla g|^2)\, dx$
- Uniqueness: $f, g, h = 0 \Rightarrow u = 0$.