Basic Topology

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Contents

1 Topological Spaces

Definition 1.1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:
(1) $\varnothing$ and X in $\mathcal{T}$.
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Example1 *Example* 1.1. indiscrete topology, trivial(discrete) topology, finite complement topology

Definition 1.2. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are 2 topologies on a given set X.
If $\mathcal{T}' \supset (\supsetneq)\mathcal{T}$, we call $\mathcal{T}'$ is **(strictly)finer** than $\mathcal{T}$; if $\mathcal{T}' \subset (\subsetneq)\mathcal{T}$, we call $\mathcal{T}'$ is **(strictly)coarser** than $\mathcal{T}$.

2 Basis for a Topology

Definition 2.1. If X is a set, a **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X(called **basis elements**) such that:
(1) For $\forall x \in X$, there is at least one basis element B containing x.
(2) If x belongs to the intersection of 2 basis elements B_1 and B_2, then $\exists$ a basis element $B_3 \ni x$ $s.t.$ $B_3 \subset B_1 \cap B_2$.

Remark. One can check the collection $\mathcal{T}$ generated by the basis $\mathscr{B}$ is a topology on X by definition easily.

Lemma 2.1. *$\mathcal{T}$ equals the collection of all unions of elements of $\mathscr{B}$, that is, $\mathcal{T} = \{\bigcup_{B \in \mathscr{B}'} B \mid \mathscr{B}' \subset \mathscr{B}\}$*

Lemma 2.2 (the criterion to check whether a collection of open sets is a topological space's basis)**.** *Suppose $\mathscr{C}$ is a collection of open sets of a topological space X s.t. $\forall$ open set $U \subset X$ and $\forall x \in U, \exists C \in \mathscr{C}$ s.t. $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology of X.*

lemma2

Lemma 2.3 (lemma for check the finer/coarser)**.** *TFAE:*
(1) $\mathcal{T}' \supset \mathcal{T}$;
(2) For $\forall x \in X$ and each basis element $B \in \mathscr{B} \ni x$, $\exists B' \in \mathscr{B}'$ s.t. $x \in B' \subset B$.

Example 2.1. Now one can see that the collection of all cicular regions in the plane generates the same topology as the collection of all rectangles.

stdtop

Example 2.2. Three topologies on the real line $\mathbb{R}$
(1) **standard topology** on the real line:
generated by the collection of all open intervals in the real line.
(2) **lower limit topology** on $\mathbb{R}$(denoted by $\mathbb{R}_l$):
generated by the collection of all half-open intervals of the form $[a, b)$
(3) **K-topology** on $\mathbb{R}$(denoted by $\mathbb{R}_K$):
Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, then $\mathbb{R}_K$ is generated by all open intervals and all sets of the form $(a, b) - K$.

Remark. $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.(Easy to check)

Definition 2.2. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$, that is, $\mathcal{T} = \{\bigcup_{B \in \mathscr{B}'} B \mid \mathscr{B}' \subset \mathscr{B}\}$, where $\mathscr{B} = \{S_1 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}, i = 1, \cdots, n, n \in \mathbb{Z}_+\}$.

3 The Order Topology

Suppose X is a set having a simple order relation $<$ in this section.

$$\begin{aligned}
(a, b) &= \{x \mid a < x < b\}, \text{open interval in X} \\
[a, b] &= \{x \mid a \leq x \leq b\}, \text{closed interval in X} \\
[a, b) &= \{x \mid a \leq x < b\}, \text{half-open interval in X} \\
(a, b] &= \{x \mid a < x \leq b\}, \text{half-open interval in X}
\end{aligned}$$

Definition 3.1. Let $\mathscr{B}$ be the collection of all sets of the following types:
(1) All open sets in X.
(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.
The collection of $\mathscr{B}$ is a basis, the topology it generated is called the **order topology**.

Example 3.1. (1) $\mathbb{Z}_+$ form an ordered set with a smallest element. The order topology is the discrete topology, for every one-point set is open:

$$\{n\} = \begin{cases} (n-1, n+1) & n > 1 \\ [1, 2) & n = 1 \end{cases}$$

(2) Let $X = \{1, 2\} \times \mathbb{Z}_+$ in the dictionary order, the topology on X is not the discrete topology for any basis element containing 2×1 contains points like $1 \times n$.

Definition 3.2. For $a \in X$, there are 4 subsets called **rays** determined by a: $(a, +\infty), (-\infty, a), [a, +\infty), (-\infty, a]$. The first two types are open rays, and the other two are closed rays.

Remark. The open rays form a subbasis for the order topology on X, for $(a, b) = (a, +\infty) \cap (-\infty, b)$.

Remark. If X has a largest element b_0, $(a, +\infty) = (a, b_0]$; otherwise, $(a, +\infty) = \bigcup_{x > a, x \in X} (a, x)$.

4 The Product Topology on $X \times Y$

Definition 4.1. Let X, Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U, V are open subset of X, Y respectively.

Remark. We can see from a figure that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$.

Theorem 4.1. *$\mathscr{B}, \mathscr{C}$ are bases for the topology of X and Y respectively, then the collection*

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B}, C \in \mathscr{C}\}$$

is a basis for the topology $X \times Y$.

Example 4.1. We have studied a standard topology on $\mathbb{R}$(stdtop 2.2): the order topology. The product of this topology is the standard topology on $\mathbb{R}^2$: the order topology in the dictionary order.

We introduce the following fuctions called projections to express the product topology in terms of a subbasis, which is useful to do so sometimes.

Definition 4.2. Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x, y) = x$, $\pi_2 : X \times Y \to Y$ be defined by $\pi_2(x, y) = y$. The 2 maps are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

Remark. Projections are onto(surjective) unless $X($ or Y or $X \times Y) = \varnothing$.

Theorem 4.2. *The collection $\mathcal{S} = \{\pi_1^{-1}(U) \mid U$ open in $X\} \cup \{\pi_2^{-1}(V) \mid U$ open in $Y\}$ is a subbasis for the product topology on $X \times Y$.*

Proof. Let $\mathcal{T}$ be the product topology, $\mathcal{S}$ generates $\mathcal{T}'$. Note that $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, then it's easy to prove $\mathcal{T} \subset (\supset) \mathcal{T}'$ respectively. □

Remark. From the above theorem, one can observe that the product topology is the coarest topology among the topologies on $X \times Y$ which ensures π_1, π_2 are continuous.

5 The Subspace Topology

Definition 5.1. If $Y \subset X$, the collection $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

Definition 5.2 (Another definition for subspace topology)**.** Subspace topology is the coarest topology such that the imbedding(c.f. Definition 7.4) $i : A \to X,\ i(x) = x,\ A \subset X$ is continuous.

Lemma 5.1. *If $\mathscr{B}$ is a basis for X, then the collection $\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$ is a basis for the subspace topology on Y.*

Remark. One needs to be careful when dealing with the case that Y is a subspace of X. We say that a set U is **open in Y(or open relative to Y)** if it belongs to the topology of Y.

Lemma 5.2. *Let Y be a subspace of X, if U is open in Y and Y is open in X, then U is open in X.*

Now we will focus on the subspace topology of an order topology. Let X be an ordered set in the order topology and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. However, the resulting order toplogy on Y need not to be the same as the topology that Y inherits as a subspace of X.(See the examples below.)

Example 5.1. $Y = [0, 1] \subset \mathbb{R}$, in the subspace topology. Then the open sets in Y are of the following types:

$$(a, b) \cap Y = \begin{cases} (a, b) & a, b \in Y \\ [0, b) & b \in Y, a \notin Y \\ (a, 1] & a \in Y, b \notin Y \\ Y \text{ or } \varnothing & a, b \notin Y \end{cases}$$

Note that these sets form a basis for the order topology on Y. So in this example, the subspace topology and its order topology are the same.

Example 5.2. Let $Y = [0, 1) \cup \{2\} \subset \mathbb{R}$. In the subspace topology on Y, $\{2\}$ is open($\because \{2\} = (\frac{3}{2}, \frac{5}{2}) \cap Y$), but in theorder topology on Y, $\{2\}$ is not open. Any basis element for the order topology on Y that contains 2 is of the form: $\{x \mid x \in Y, a < x \leq 2, a \in Y\}$.

example5.3 *Example* 5.3. Let $I = [0, 1]$. The dictionary order topology on $I \times I$ is not the same as the suspace topology on $I \times I$ obtained from the dictionary topology on $\mathbb{R} \times \mathbb{R}$ although the dictionary order on $I \times I$ is just a restriction of dictionary order on the plane on $I \times I$. You can see this because $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in $I \times I$ in the subspace topology, but is not open in the order topology. See Figure 1
The set $I \times I$ in the dictionary order topology will be called the **ordered square**, denoted by I_o^2.

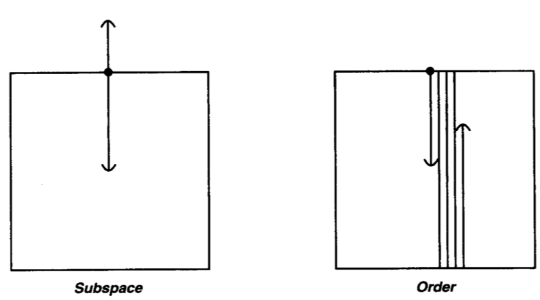

Figure 1: Subspace and Order topology on $I \times I$ fig:5.1

Lemma 5.3. *The order topology induced on $Y \subset$ the subspace topology on Y.*

Proof. $\because$ the order topology on Y has a subbasis $\{\{x \in Y \mid x < a\} \mid a \in Y\} \cup \{\{x \in Y \mid x > a\} \mid a \in Y\}$, and $\{x \in Y \mid x < a\} = (-\infty, a) \cap Y$, $\therefore$ the open sets in the order topology is also open in the subspace toplogy. $\square$

Definition 5.3. Given an ordered set X, we say $Y \subset X$ is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Note that intervals and rays in X are convex in X.

Theorem 5.1 (A sufficient condition for 'order topology=subspace topology')**.** *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Similar to the above lemma. One should also consider the subbasis. $\square$

Remark. To avoid ambiguity, whenever X is an order set in the order topology and $Y \subset X$, we shall assume that Y is given the subspace topology unless specified otherwise.

6 Closed sets and Limit Points

Definition 6.1. A subset A of a topological space X is asaid to be **closed** if the set $X - A$ is open.

Lemma 6.1. *Let $Y \subset X$, then A is closed in $Y \iff$ it equals the intersection of a closed set of X with Y.*

Lemma 6.2. *Let $Y \subset X$, if A is closed in Y, Y is closed in X, then A is closed in X.*

Definition 6.2. Given a subset $A \subset X$. The **interior** of A is defined as the union of all open sets contained in A, the **closure** of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by $IntA$ or $\mathring{A}$ and the closure of A is denoted as ClA or $\bar{A}$.

Obviously, $IntA$ is the largest open set contained in A, and $\bar{A}$ is the smalles closed set containing A; furthermore, $IntA \subset A \subset \bar{A}$. If A is open, $A = IntA$; if A is closed, $A = \bar{A}$.

Lemma 6.3. *Let Y be a subspace of X, A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Definition 6.3. We introduce 2 terminology. We shall say that a set A **intersects** a set B if $A \cap B \neq \varnothing$. They say that A is a **neighborhood** of x if A contains an open set containing x.

Theorem 6.1. *Let A be a subset of the topological space X.*

(a) Then $x \in \bar{A} \iff$ every neighborhood of x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A} \iff$ every basis element B containing x intersects A.

Remark. $x \notin \bar{A} \iff \exists U \ni x$ open that does not intersect A.

Definition 6.4. If A is a subset of the topological space X and $x \in X$, we say that x is a **limit point**(or "**cluster point**", or "**point of accumulation**") of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if $x \in \overline{A - \{x\}}$. We denote all the limit point of A as A'.

Lemma 6.4. $\bar{A} = A \cup A'$.

Corollary 6.1. *A is closed $\iff A' \subset A$*

Definition 6.5. In an arbitrary topological space, one says that a sequence $x_1, x_2, \cdots$ of points of the space X **converges** to the point $x \in X$ provided that, corresponding to each neighborhood U of x, $\exists N$ such that $x_n \in U, \forall n \geq N$.

Remark. In an arbitraty topological space, a sequence may converge to more than one point. So we need to impose an additional condition that rule out this situation.

Definition 6.6. (1) A topological space X is called a **Hausdorff Space (T_2 space)** if for $\forall x_1 \neq x_2 \in X, \exists$ neighborhood U_1, U_2 of x_1, x_2, respectively, such that $U_1 \cap U_2 = \varnothing$.

(2) The condition that finite point sets be closed is called the T_1 **axiom**, which is weaker than the Hausdorff condition(c.f. Example 6.1). This means that a Hausdorff space satisfies T_1 axiom.(See the lemma below)

Lemma 6.5. *Every finite point set in a Hausdorff space X is closed.($T_2 \Rightarrow T_1$)*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. □

egT1 *Example* 6.1. The real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed.

Remark. Most of the spaces that are important to mathematicians are Hausdorff spaces.

Theorem 6.2. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Remark. If the sequence $\{x_n\}$ of a Hausdorff space X converges to the point x of X, we often write $x_n \to x$, we say that x is the **limit** of the sequence $\{x_n\}$.

Theorem 6.3. *Let X be a T_1 space; let A be a subset of X. Then x is a limit point of $A \iff$ every neighborhood of x contains infinitely many points of A.*

Proposition 6.1. *We have following properties:*

(1) Every simply ordered set is a Hausdorff space in the order topology.

(2) The product of two T_i space is a T_i space.

(3) A subspace of a T_i space is a T_i space.($i = 1, 2$)

Proposition 6.2. *This property is about metric topology:*

The Hausdorff axiom is satisfied by every metric topology.

Lemma 6.6 (The sequence lemma(c.f. Lemma 9.2))**.** *Let $A \subset X$. If $\exists \{x_i\}_{i=1}^{+\infty} \subset A$, s.t. x_i converge to x, then $x \in \bar{A}$; the converse holds if X is metrizable. Furthermore, if X is metrizable, then $x \in A' \iff \exists \{x_i\}_{i=1}^{+\infty} \subset A (x_i \neq x)$ s.t. $x_i \to x$.*

Definition 6.7. If $A \subset X$, we define the **boundary** of A by the equation

$$BdA = \bar{A} \cap \overline{(X - A)}.$$

7 Continuous Functions

Definition 7.1. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, $f^{-1}(V)$ is open in X.

Proposition 7.1. *Let $\mathcal{S}$(or $\mathscr{B}$) is a subbasis(or basis) of Y, then f is continuous $\iff \forall V \in \mathcal{S}$(or $\mathscr{B}$)$, f^{-1}(V)$ is open in X.*

Example 7.1. Let $\mathbb{R}$ denote the set of real numbers in its usual topology, and let $\mathbb{R}_l$ denote the same set in the lower limit topology. Let $f : \mathbb{R} \to \mathbb{R}_l$ be the identity function, $f(x) = x$. Then f is not a continuous function: $f^{-1}[a, b) = [a, b)$ is not open in $\mathbb{R}$.

Theorem 7.1. *Let $f : X \to Y$, TFAE:*

(1) f is continuous.

(2) For every subset A of X, $f(\bar{A}) \subset f(\bar{A})$.

(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For $\forall x \in X, \forall$ neighborhood V of $f(x), \exists$ a neighborhood U of x s.t. $f(U) \subset V$.

Proof. We only need to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and that (1) $\Rightarrow$ (4) $\Rightarrow$ (1). $\square$

Definition 7.2. Let $f : X \to Y$ be a bijection, if both f and its inverse $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Proposition 7.2. *Let $f : X \to Y$ be a bijection, TFAE:*
(1) f is a homeomorphism.
(2) $f(U)$ is open in $Y \iff U$ is open in X.

Definition 7.3. The above theorem shows that a homeomorphism gives us a bijective correspondence not only between 2 sets(X, Y) but also between the collections of open sets of X and Y. A property that also holds under homeomorphism($X, f(X)$ both holds, f homeomorphism) is called a **topological property**.

defn7.4 **Definition 7.4.** Now suppose f is injective. Let Z be the image set $f(X)$ as a subspace of Y, then the restriction $\tilde{f} : X \to Z$ is bijective. If it happens to be a homeomorphism, we say that the map $f : X \to Y$ is a **(topological) imbedding** of X in Y.

Example 7.2. A bijective function can be continuous without being a homeomorphism.

One such function is the identity map $g : \mathbb{R}_l \to \mathbb{R}$. Another is the following: Let S^1 be the unit circle as a subspace of the plane $\mathbb{R}^2$, and let $f : [0, 1) \to S^1, f(t) = (cos2\pi t, sin2\pi t)$. $f[0, \frac{1}{4})$ is not open, so f is not a homeomorphism.

As a result, consider $h : [0, 1) \to \mathbb{R}^2$ obtained from f by expanding the range. h is an example of a continuous injective map that is not an imbedding.

Proposition 7.3 (Rules for constructing continuous functions)**.** *Let X, Y, Z be topological spaces, $A \subset X$ as subspace,*

(1)(Constant functions) Given $y_0 \in Y, f(x) = y_0, \forall x \in X$ is continuous.

(2)(Inclusion) The inclusion function $j : A \to X$ is continuous.

(3)(Composites) If $f : X \to Y, g : Y \to Z$ are continuous, so as $g \circ f : X \to Z$.

(4)(Restricting the domain) If $f : X \to Y$ is continuous, then $f|_A : A \to Y$ is continuous.

(5)(Restricting or expanding the range) Let $f : X \to Y$ is continuous, $f(X) \subset Z \subset Y \subset W$ are both subspaces, then $X \to Z$ and $X \to W$ obtained by restricting or expanding the range are continuous.

(6)(Local formulation of continuity) If $f : X \to Y$ is continuous, $X = \bigcup_\alpha U_\alpha, U_\alpha$ open, then $f|_{U_\alpha}$ is continuous for each α.

Theorem 7.2 (The pasting lemma)**.** *Let $X = A \cup B$, where A, B are closed in X. Let $f : A \to Y, g : B \to Y$ be continuous. If $f(x) = g(x), \forall x \in A \cap B$, then the function $h : X \to Y$ defined below*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is a continuous.

Remark. This theorem also holds if A, B are open sets in X, this is just a special case of the "local formulation of continuity" rule in preceding proposition.

Example 7.3. The pasting lemma only holds for finite number of closed sets. Let

$$A_i = \begin{cases} \{\frac{1}{i}\}, & i \in \mathbb{Z}^+ \\ \{0\}, & i = 0 \end{cases}, f_i : A_i \to \mathbb{R}, f_i(\frac{1}{i}) = \begin{cases} 1, & i \in \mathbb{Z}^+ \\ 0, & i = 0 \end{cases}$$

then A_i are all disjoint, if pasting all functions on $\bigcup_{i \in \mathbb{Z}^+} A_i$ to get a function g, g is not continuous.

thm7.3 **Theorem 7.3** (Maps into products)**.** *Let $f : A \to X \times Y, f(a) = (f_1(a), f_2(a))$, then f is continuous $\iff$ f_1, f_2 are continuous.*

The maps f_1, f_2 are called the ***coordinate functions*** *of f.*

Proof. "$\Rightarrow$"$\because$ $f_i(a) = \pi_i(f(a)), i = 1, 2; \forall a \in A$ and π_i is the projection, which is continuous.
"$\Leftarrow$"$\because$ $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. $\square$

Remark. $f : A \times B \to Y$ may not be continuous if it is continuous in each variable separately. $f : \mathbb{R}^2 \to \mathbb{R}$ be defined below is continuous in each variable separately but f is not continuous(Consider $g(x) = f(x \times x)$).

$$f(x \times y) = \begin{cases} \frac{xy}{x^2+y^2} & x \times y \neq 0 \times 0 \\ 0 & x \times y = 0 \times 0 \end{cases}$$

8 The Product Topology

Definition 8.1. Let us take the collection of all sets of the form $\Pi_{\alpha \in J} U_\alpha$ as a basis for a topology on the product space $\Pi_{\alpha \in J} X_\alpha$, where U_α is open in X_α for $\forall \alpha \in J$. The topology generated by the basis is called the **box topology**.

Definition 8.2. Let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$, and let $\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$. The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this space, $\Pi_{\alpha \in J} X_\alpha$ is called the **product space**.

Proposition 8.1 (Basis of this 2 topologies and their comparison)**.** *The box or product topology on ΠX_α has as basis all sets of the form ΠU_α. For each element of a basis of the product topology, $U_\alpha \neq X_\alpha$ holds for only finitely α.*

From above, we know that product topology $\subset$ box topology.(If J is finite, then these two are equal; otherwise, it is a strictly finer relation.)

Remark. We can get a stronger proposition that product topology $\subset$ uniform topology $\subset$ box topology(c.f. Theorem 9.1).

Remark. Whenever we consider ΠX_α, we shall assume it is given the product topology unless specified otherwise.

Proposition 8.2. *Let A_α be a subspace of X_α for $\forall \alpha$, then ΠA_α is a subspace of ΠX_α, either box or product topology is given.*

Proposition 8.3. *If X_α is hausdorff for each α, then X_α is also Hausdorff in both the box and product topologies.*

Proof. We only need to prove for the product topology case.($\because$ product topology $\subset$ box topology, $\therefore$ the proposition that product topology is Hausdorff indicates that the other.) $\square$

Theorem 8.1. *If $A_\alpha \subset X_\alpha, \forall \alpha$; then $\Pi \bar{A}_\alpha = \overline{\Pi A_\alpha}$ in both box and product topologies.*

Theorem 8.2 (A generalization of theorem 7.3)**.** *$f : Y \to (\Pi X_\alpha, \text{product topology})$ is continuous $\iff$ $\pi_\alpha \circ f$ is continuous for each α.*

The above theorem is false for box topology, see the example below.

Example 8.1. Let $\mathbb{R}^\omega := \Pi_{n=1}^{+\infty} \mathbb{R}$, define $f : \mathbb{R} \to \mathbb{R}^\omega, f(t) = (t, t, \cdots, t)$.
$\because$ $f^{-1}\Big((t-1, t+1) \times (t - \frac{1}{2}, t + \frac{1}{2}) \times \cdots \times (t - \frac{1}{n}, t + \frac{1}{n}) \times \cdots\Big) = \bigcap_{n=1}^{+\infty} (t - \frac{1}{n}, t + \frac{1}{n}) = \{t\}$ is not open when given box topology,
$\therefore$ the above theorem is false for box topology.

9 The Metric Topology

Definition 9.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ satisfies:
(1) $d(x, y) \geq 0,\ \forall x, y \in X$; equality holds $\iff x = y$.
(2) $d(x, y) = d(y, x),\ \forall x, y \in X$.
(3) $d(x, y) + d(y, z) \geq d(x, z),\ \forall x, y, z \in X$.

Definition 9.2. If d is a metric on the set X, then the collection of all $\epsilon-$balls $B_d(x, \epsilon)$, $x \in X, \epsilon > 0$ is a basis for a topology on X, called the **metric topology** induced by d.

Definition 9.3 (Another definition for metric topology)**.** A set U is open in the metric topology induced by d $\iff \forall y \in U,\ \exists \delta > 0$, s.t. $B_d(y, \delta) \subset U$.

Example 9.1. Given a set X, define

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

The topology it induces is the discrete topology(Example 1.1).

Definition 9.4. If X is a topological space, X and (Y, d) is homeomorphic, then X is called **metrizable**. There may exists different metric that gives different topologies on a metrizable space.

Definition 9.5. (X, d) is called a **bounded metric space** if $\exists M > 0$, s.t. $\forall x, y \in X, d(x, y) \leq M$.

Proposition 9.1. *Let (X, d) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ as $\bar{d}(x, y) = min\{d(x, y), 1\}$. Then $\bar{d}$ is a metric induces the same topology as d(i.e. $id : (X, d) \to (X, \bar{d})$ is a homeomorphism). The metric $\bar{d}$ is called the* ***standard bounded metric*** *corresponding to d.*

Lemma 9.1 (Metric space case for Lemma 2.3)**.** *Consider (X, d), (X, d'), then $\mathcal{T}'$ is finer than $\mathcal{T}$ $\iff$ $\forall x \in X, \forall \epsilon > 0, \exists \delta > 0$, s.t. $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.*

Corollary 9.1. *(X, d) and $(X, \bar{d})$ is homeomorphic.*

Proposition 9.2. *Let $(X, \rho_1), (Y, \rho_2)$ be metric spaces, then the following three are metric functions on $X_1 \times \cdots \times Y$:*
$d_1(x, y) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2)$;
$d_2(x, y) = \sqrt{\rho_1(x_1, y_1)^2 + \rho_2(x_2, y_2)^2}$;
$d_\infty(x, y) = max\{\rho_1(x_1, y_1), \rho_2(x_2, y_2\}$; *and the induced topology of these three metric are all the product topology on $X \times Y$.*

Proof. $\because d_\infty(x, y) \leq d_1(x, y) \leq \sqrt{2} d_2(x, y) \leq 2 d_\infty(x, y)$, and then apply the above lemma. □

Proposition 9.3. *For $\mathbb{R}^n$,*
$d_2(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ *is called the* ***euclidean metric*** *on $\mathbb{R}^n$;*
$d_\infty(x, y) = max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$ *is called the* ***square metric*** *on $\mathbb{R}^n$.*
The induced topology of $d_2(x, y), d_\infty(x, y)$ is the product topology on $\mathbb{R}^n$.

Proof. $\because d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$ □

Remark. It is not natural to generalize the metric $d_2(x, y), d_\infty(x, y)$ to $\mathbb{R}^\omega$, for the series need not converge and the supremum does not always make sense.

Definition 9.6. Given an index set J and a collection of metric space $(X_\alpha, d_\alpha)_{\alpha \in J}$, define a metric $\bar{\rho}$ on $\Pi_{\alpha \in J} X_\alpha$ as $\bar{\rho}(x, y) = sup\{\bar{d}_\alpha(x_\alpha, y_\alpha) \mid \alpha \in J\}$, the induced topology is called the **uniform topology**, where $\bar{d}_\alpha$ is the standard bounded topology(c.f. Proposition 9.1) in X_α.

thm9 **Theorem 9.1.** *On $\Pi_{\alpha\in J}X_\alpha$, product topology $\subset$ uniform topology $\subset$ box topology. If J is an infinite set, these three topologies are distinct with each other.*

Proof. The proof is a simple application of the original definition. The last statement can be shown by given examples as the following: Given $x \in \Pi X_i$, then $\Pi\{y_i | d(x_i, y_i) < \frac{1}{i}\}$ is open in box topology but not in uniform topology; $\Pi\{y_i | d(x_i, y_i) < \frac{1}{2}\}$ is open in uniform topology but not in prouduct topology $\square$

Theorem 9.2 (When $|J|$ is countable, product topology is metrizable)**.** *Given a collection of metric space $(X_i, d_i)_{i\in\mathbb{N}^*}$, define a metric $D(x,y)$ on $\Pi_{i=1}^{+\infty}X_i$ as $D(x,y) = sup\{\frac{\bar{d}_i(x_i,y_i)}{i}\}$, then the topology induced by the metric D is the product topology on $\Pi_{i=1}^{+\infty}X_i$.*

Proof. (1) $\because$ $y \in B_D(x,r) \iff D(x,y) < r \iff sup\{\frac{\bar{d}_i(x_i,y_i)}{i}\} < r, \therefore$ take N s.t. $\frac{1}{N} < r$, then $y \in \Pi_{i=1}^{N-1}B_{d_i}(x_i,r) \times \Pi_{i=N}^{+\infty}X_i \subset B_D(x,r)$. (2) Conversely, for the basis element of product topology $\Pi_{i=1}^{N-1}U_i \times \Pi_{i=N}^{+\infty}X_i \ni y$, take $\epsilon > 0$ s.t. $y_i \in B_{d_i}(x_i, r) \subset U_i, i = 1, 2, \cdots, N-1$, then $y \in B_D(x, \epsilon/N) \subset \Pi_{i=1}^{N-1}U_i \times \Pi_{i=N}^{+\infty}X_i$. $\square$

Definition 9.7. A space X is said to have a **countable basis at the point** x if there exists a countable collection $\{U_n\}_{n\in\mathbb{Z}_+}$ of neighborhoods of x s.t. $\forall$ neighborhood U of x, $\exists n$, s.t. $U_n \subset U$. A space X that has a countable basis at each of its points is said to satisfy the **first countability axiom**.

Proposition 9.4. *A metrizable space satisfy the first countability axiom.*

Lemma9 **Lemma 9.2** (The sequence lemma)**.** *Let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X has a countable basis at x.*

Proof. $\Leftarrow$: take $x_n \in U_1 \cap \cdots \cap U_n \cap A$, then $\{x_n\}$ converges to x. $\square$

Theorem 9.3. *Let $f : X \to Y$ and assume that X has a countable basis at x , then f is continuous at $x \iff$ for every sequence x_n converging to x, the sequence $f(x_n)$ converges to $f(x)$.*

Proof. $\Rightarrow$: Let $V \ni f(x)$ open in Y, x_n converges to x, then $x_n \in f^{-1}(V), \forall n > N$, then $f(x_n) \in V, \forall n > N$.

$\Leftarrow$: Let $A \ni f(x)$ closed in Y, if $f^{-1}(A)$ is not closed, take $x \in \overline{f^{-1}(A)} \setminus f^{-1}(A)$, then apply the sequence lemma, there exists a sequence $\{x_n\} \subset f^{-1}(A)$ converges to x, by the assumption, $f(x_n)$ converges to $f(x)$, then $f(x) \in \bar{A} = A$, this is contradict to $x \in \overline{f^{-1}(A)} \setminus f^{-1}(A)$, so $f^{-1}(A)$ is closed. $\square$

lemma9.3 **Lemma 9.3.** *Let $(X, d_1), (Y, d_2)$ metriable, then $f : X \to Y$ is continuous at $x \iff \forall\epsilon > 0, \exists\delta > 0$, s.t. if $d_1(x,y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$.*

Proposition 9.5. *Let X be a topological space.If $f, g : X \to \mathbb{R}$ are continuous function, then $f + g, f - g, f\dot{g}$ are continuous. If $g(x) \neq 0, \forall x$, then f/g is continuous.*

Proof. $f + g : X \xrightarrow{(f,g)} \mathbb{R}^2 \xrightarrow{+} \mathbb{R}$, the first map is continuous by applying Theorem 7.3, the second map is continuous by applying Lemma 9.3. Thus $f + g$ is continuous. $\square$

Definition 9.8. Let $f_n : X \to (Y, d)$ be a sequence of functions, then we say f_n **converges uniformly** to $f : X \to Y$ if $\forall\epsilon > 0, \exists N$, s.t. $\forall n > N, \forall x \in X$ $d(f_n(x), f(x)) < \epsilon$.

Theorem 9.4 (Uniform limit theorem)**.** *Let $f_n : X \to (Y, d)$ be a sequence of continuous functions, if f_n converges uniformly to f, then f is continuous.*

Remark. The notion of uniform convergence is related to the Definition 9.6 of the uniform metric. Consider the space $\mathbb{R}^X$ of all functions $f : X \to \mathbb{R}$ in the uniform metric $\bar{\rho}$. One can check that $f_n : X \to \mathbb{R}$ converges uniformly to $f \iff f_n$ converges to f when they are considered as elements of $(\mathbb{R}^X, \bar{\rho})$.

At last, we give some examples of spaces that are not metrizable.

Example 9.2 ($\mathbb{R}^\omega$ in the box topology is not metrizable). We shall show that the sequence lemma does not hold for $\mathbb{R}^\omega$ with box topology.(So it does not satisfy the first countability axiom) Let $A = \{(x_1, x_2, \cdots) \mid x_i > 0, \forall i \in \mathbb{Z}_+\}$, then $0 \in \bar{A}$; however, for any sequence $\{a_n\}, a_n = (x_{1n}, x_{2n}, \cdots, x_{in}, \cdots)$, we can construct a basis element B for the box topology by $B = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots$. Then $0 \in B$, but $\{a_n \mid n \in \mathbb{Z}_+\} \cap B = \varnothing$, thus a_n does not converges to 0.

Example 9.3 ($\mathbb{R}^J$ in the product topology is not metrizable when J is an uncountable index set). We also show that the sequence lemma does not hold for this.(So it does not satisfy the first countability axiom.) Let $A = \{(x_\alpha) \mid$ only finitely many α satisfies that $x_\alpha \neq 1\}$, then $0 \in \bar{A}$; however, for any sequence $\{a_n\} \subset A$, let J_n denote those indices α for which the α-th coordinate of a_n is different from 1. Then $|J_n| < +\infty$, so $I = \bigcup J_n$ is countable, thus $\exists$ an index $\beta \in J \setminus I$. So for each a_n, its β-th coordinate is 1. We can conclude that $\pi_\beta^{-1}((-1,1))$ is a neighborhood of 0 that contains no points of a_n, so the sequence does not converge to 0, which contradicts the sequence lemma.

10 The Quotient Topology

Definition 10.1. Let X be a topological space, $\sim$ is a equivalence relation on X. Define a surjective map $p : X \to X/\sim$ as $p(a) = [a] := \{b \in X \mid b \sim a\}$ Then $\mathcal{F} = \{U \in X/\sim \mid p^{-1}(U)$ is open in $X\}$ is a topology on $X/\sim$. We call this topology as the **quotient topology** on $X/\sim$, ($X/\sim$, quotient topology) is called a **quotient space** of X.

Definition 10.2. We say a subset C of X is a **saturated set** with respect to $\sim$, if it contains every set $p^{-1}(y)$ it intersects. Thus C is saturated $\iff C = p^{-1}(B)$ for some $B \subset X/\sim \iff C = p^{-1}(p(C))$.

Remark. The open set V in $X/\sim$ is the image of a saturated open set $p^{-1}(V)$.

Example 10.1. Define $\sim$ on $[0,1]$: $x \sim y \iff x, y \in \{0,1\}$ or $x = y$. Then $[0,1]/\sim \xrightarrow{f} S^1, f(x) = e^{2\pi i x}$ is homeomorphism.

Example 10.2. Define $\sim$ on $[0,1] \times [0,1] : (0,y) \sim (1,y), (x,0) \sim (1-x,1)$, then $[0,1] \times [0,1]/\sim$ is called **Klein bottle**.

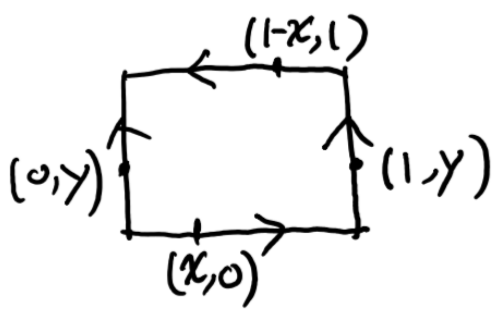

Figure 2: Klein bottle

Definition 10.3. Let X, Y be topological spaces, let $p : X \to Y$ be a surjective map. The map p is called a **quotient map** provided that U is open in $Y \iff p^{-1}(U)$ is open in X.

Remark. (1) $X \to X/\sim$ is a quotient map as the definition above.

(2) For a quotient map $p : X \to Y$, we can define $\sim$ as: $x_1 \sim x_2 \iff p(x_1) = p(x_2)$, then $X/\sim = \{p^{-1}(y) \mid y \in Y\}$ is homeomorphic to Y.

Proposition 10.1. *The composites of two quotient maps is a quotient map.*

Theorem 10.1. *Let $p : X \to Y$ be a quotient map, and the map $g : X \to Z$ satisfies that $\forall y \in Y, g|_{p^{-1}(y)}$ is a constant map, then (1) $\exists! f : Y \to Z$, s.t. $f \circ p = g$. (2) And f is continuous $\iff$ g is continuous;(3) f is a quotient map $\iff$ g is quotient map.*

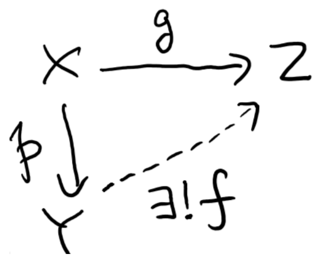

Figure 3:

Proof. (3)"$\Leftarrow$": $\because g$ is continuous, $\therefore f$ is continuous. Let $f^{-1}(U)$ is open, then $p^{-1}(f^{-1}(U)) = g^{-1}(U)$ is open, $\therefore U$ is open. $\square$

Example 10.3 (Not a quotient map). Let $X = \bigcup_{n=1}^{+\infty}[0,1] \times \{n\}, Z = \{x \times \frac{x}{n} \mid x \in [0,1]\}$ both with the subspace topology of $\mathbb{R}^2$. $g : X \to Z$ be defined as $g(x \times n) = x \times \frac{x}{n}$, thus the quotient space $X^* = \bigcup_{x \times y \in Z \setminus \{0 \times 0\}}\{x \times \frac{x}{y}\} \cup \{0 \times n \mid n \in \mathbb{N}^*\}$. However, g is not a quotient map because $A = \{\frac{1}{n} \times n \mid n \in \mathbb{N}^*\}$ is a saturated closed set in X, but $g(A) = \{\frac{1}{n} \times \frac{1}{n^2} \mid n \in \mathbb{N}^*\}$ is not a closed set in Z. This phenomenon is because the quotient space X^* identify the subset $\{0\} \times \mathbb{N}^*$ to a point.

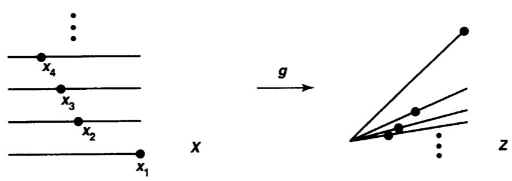

Figure 4: g is not a quotient map

Example 10.4 (The product of two quotient maps need not be a quotient map). Let $X = \mathbb{R}, A = \mathbb{Z}_+$, and $p : X \to X/A$ be the quotient map(X/A means that the space is obatained from X by identifying the subset A to a point b), $id : \mathbb{Q} \to \mathbb{Q}$ be a quotient map, then $p \times id : X \times \mathbb{Q} \to X/A \times \mathbb{Q}$ is not a quotient map.

Let $c_n = \frac{\sqrt{2}}{n}$, consider all the U_n like the shadowed area below for each n(U_n is lying above or beneath both of the straight lines with slopes 1 and -1, respectively, and between the vertical lines $x = n - \frac{1}{4}, x = n + \frac{1}{4}$)(See Figure figure10 5).

Let $U = \bigcup U_n$, then U is a saturated open set in $X \times \mathbb{Q}$.

Assume $(p \times id)(U)$ is also open, then $([1] \times 0) \in X/A \times \mathbb{Q}$, hence $(p \times id)(U)$ contains an open set of the form $W \times \{y \in (-q,q) \mid y \in \mathbb{Q}\}$ where $p^{-1}(W)$ is open in X and $\mathbb{Z}_+ \subset p^{-1}(W)$. Thus $\exists \epsilon_n > 0, q > 0$, s.t. $\bigcup_{n=1}^{+\infty}(n - \epsilon_n, n + \epsilon_n) \times \big((-q,q) \cap \mathbb{Q}\big)$. However, $\exists N, \exists q > c_N$, $(N - \epsilon_N, N + \epsilon_N) \times \big((-q,q) \cap \mathbb{Q}\big) \nsubseteq U$, which leads
eg10.4 to a contradiction!

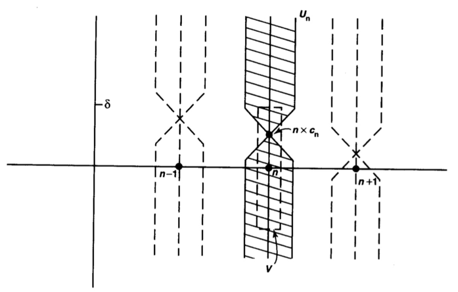

Figure 5: For Example 10.4 figure10

Let $p : X \to Y$ be a quotient map, $A \subset Y$ with the subspace topology, then $p|_A : A \to p(A)$ need not be a quotient map in general. However, one have the following theorem.

Theorem 10.2. *Let $p : X \to Y$ be a quotient map, A be a subspace of X that is saturated with respect to p, then*
(1) If A is either open or closed in X, then $p|_A$ is a quotient map.
(2) If p is either an open or closed map, then $p|_A$ is a quotient map.

Proof. We only need to notice that:
(1) For U is open in A, $(p|_A)^{-1}(U) = p^{-1}(U)$ is open. (This equation is because A is saturated.);
(2) For $p^{-1}(U) = V \cap A$, $U = p(V) \cap p(A)$.
Then other steps are simple. □

11 Connected Spaces

Definition 11.1. Let X be a topological space. A **separation** of X is a pair of sets U, V, where $X = U \cup V$, U, V are disjoint nonempty open sets. The space X is said to be **connected** if there does not exist a separation of X.

Remark. Obviously, the definition of connected space can be formulated as:
The only subsets of X that are both open and closed are $\varnothing$ and X itself.

Lemma 11.1 (Another way to formulate the definition of connectedness of subspace)**.** *If Y is a subspace of X, a **separation** of Y is a pair of disjoint nonempty sets A, B whose union is Y, and satisfying $A' \cap B = A \cap B' = \varnothing$.*

Proof. Hint: $\bar{A} \cap Y = \bar{A} \cap (A \cap B) = A \cap (\bar{A} \cap B)$. □

lemma11 **Lemma 11.2.** *If C, D form a separation of X, and if Y is a connected subspace of X, then $Y \subset C$ or $Y \subset D$.*

prop11.1 **Proposition 11.1.** *If $A_\alpha, \alpha \in J$ is a collection of connected subspace of X, and $\bigcap A_\alpha \neq \varnothing$, then $\bigcup A_\alpha$ is connected.*

Proof. Prove by contradiction. Suppose $Y = C \cup D$ form a separation and $x \in \bigcap A_\alpha$. Then WLOG, $x \in C$, applying Lemma 11.2, $\bigcup A_\alpha \subset C$, contradiction! □

prop11.2 **Proposition 11.2.** *Let A be a connected subspace of X, $A \subset B \subset \bar{A}$, then B is also connected.*

Corollary 11.1. *A is connected $\Rightarrow \bar{A}$ is connected.*

prop11.3 **Proposition 11.3.** *If X is connected and f is continuous, then the image space $f(X)$ is connected.*

thm11 **Theorem 11.1.** *If X, Y are connected, then $X \times Y$ is connected.*

Proof.
Step 1: For $a \in X, b \in Y$, $\{a\} \times Y \cup X \times \{b\}$ is connected. ($\because X$ is homeomorphic to $X \times \{b\}$, Y is homeomorphic to $\{a\} \times Y$ and they share a common point, so applying Proposition 11.1 and 11.3 to get the result)
Step 2: Fix $y_0 \in Y$, then $X \times \{y_0\} \cup \{x\} \times Y, x \in X$ is a collection of connected space with common points $X \times \{y_0\}$, apply Proposition 11.1 again, $\Rightarrow X \times Y$ is connected. □

Remark. The above result is true for finite cartesian product by induction. However, it is not true for infinite cartesian product, see the following examples.

Example 11.1 (($\mathbb{R}^\omega$, box topology) is not connected.).

$A = \{(x_i) \mid x_i \text{ is bounded }, \forall i\}$ is open and closed. ($\because x \in A \Rightarrow \Pi(x_i - 1, x_i + 1) \subset A$; $x \notin A \Rightarrow \Pi(x_i - 1, x_i + 1) \subset A^c$.)

Example 11.2. Suppose X_i is connected, then ($\Pi_{i=1}^{+\infty} X_i$, product topology) is connected.

Fix $a = (a_i) \in \Pi_{i=1}^{+\infty} X_i$. Let $Y_n = \{x \in \Pi_{i=1}^{+\infty} X_i \mid x_m = a_m, \forall m > n\}$, then Y_n is homeomorphic to $\Pi_{i=1}^{n} X_i$, which is connected; thus Y_n is connected. $\because a \in \bigcap_{i=1}^{+\infty} Y_n, \therefore Y = \bigcup_{i=1}^{+\infty} Y_n$ is connected.(Applying Proposition 11.1) Observe that $\bar{Y} = \Pi_{i=1}^{+\infty} X_i$, so it is connected by Proposition 11.2.

Proposition 11.4 (Intermediate value theorem)**.** *Let $f : X \to Y$ be a continuous map, where X is a connceted space and Y is an ordered set in order topology. Then $\forall a \neq b \in X, \forall r \in Y$ satisfying $f(a) < r < f(b), \exists c \in X$, s.t. $f(c) = r$.*

Proof. Suppose such c does not exist, then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, +\infty)$ form a separation of $f(X)$, which is connected. □

The preceding theorems show us how to construct new connected spaces out of given ones, and tell us how to prove a space is not connected. But we need to find some connected spaces to start with. We shall prove $\mathbb{R}$ is connected. The connectedness of $\mathbb{R}$ only rely on its order properties, not on its algebraic property.

Definition 11.2. A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

(1) L has the least upper bound property.

(2) If $x < y$, then $\exists z$, s.t. $x < z < y$.

Theorem 11.2. *If L is a linear continuum in the order topology, then L is connected and the intervals and rays in L are also connected.*

Proof. We prove that if Y is a convex subspace of L, then Y is connceted. Suppose that Y has a separation $Y = A \cup B$, then take $a \in A, b \in B \Rightarrow [a, b] \subset Y$. Let $A_0 = A \cap [a, b], B_0 = B \cap [a, b], c = supA_0$, then show that $c \notin A_0, c \notin B_0$ respectively, which leads to contradiction! □

Example 11.3. $\mathbb{R}$ is connected.

Example 11.4. The ordered square(c.f. Example 5.3) is a linear continuum.

For $\varnothing \neq A \subset I_o^2$, let $c = sup\pi_1(A)$, then

$$supA = \begin{cases} c \times 0 & c \notin \pi_1(A) \\ c \times sup\big((c \times I) \cap A\big) & c \in \pi_1(A) \end{cases}$$

Definition 11.3. Given points x, y of the space X, a **path** in X from x to y is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$. A space X is **path connected** if every pair of points of X can be joined by a path in X.

Proposition 11.5. *It is easy to see that a path-connected space is connected.*

Proposition 11.6 (Path-connected version for Proposition 11.1)**.** *Let $p \in A_\alpha (\forall \alpha \in J)$, A_α is path-connected, then $\bigcup A_alpha$ is path-connected.*

Proposition 11.7 (Path-connected version for Proposition 11.3)**.** *The continuous image of a path-connected space is path-connected.*

Proposition 11.8 (Path-connected version for Theorem 11.1)**.** *If X, Y are path-connected, then $X \times Y$ is path-connected.*

Example 11.5 (The unit ball $B^n = \{x \mid \|x\| \le 1\}$ in $\mathbb{R}^n$ is path connected.). Define $\gamma(t) = (1-t)x + ty$ for any given points x, y.

Example 11.6 (The punctured euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.). For x, y, find z s.t. the path between x, z or y, z do not go through the origin.

Example 11.7 (The unit sphere $S^n - 1 = \{x \mid \|x\| = 1\}$). is path connected if $n > 1$. $g(x) = \frac{x}{\|x\|}$ maps the unit ball to the unit sphere continuously.

Example 11.8. The ordered square I_o^2 is connected but not path connected.

Proof. Let $p = 0 \times 0, q = 1 \times 1$, suppose there exist a path γ s.t. $\gamma(0) = p, \gamma(1) = q$, then by the intermediate value theorem, $I_o^2 \subset \gamma([0,1])$, thus $\forall x \in I, U_x = \gamma^{-1}(\{x\} \times (0,1))$ is open in $[0,1]$. These U_x are all disjoint and open, however, this contradicts the fact that there are only countable disjoint open subsets of $[0,1]$. ☐

Example 11.9 (No path-connected version for Proposition 11.2). Let $S = \{x \times sin(\frac{1}{x}) \mid 0 < x \le 1\}$.

$\because S$ is the image of the connected set $(0,1]$ under a continuous map(by Theorem 7.3), $\therefore S$ is connected. $\therefore \bar{S}$ is connected(by Proposition 11.2).

We call $\bar{S}$ **topologist's sine curve**(illustrated below). It equals the union of S and the vertical interval $\{0\} \times [-1,1]$. We show that $\bar{S}$ is not path connected.

Proof. Suppose $\gamma : [0,1] \to \bar{S}$ is a path joining $(0,0)$ and $(1, sin1)$. Then $\gamma^{-1}(\{0\} \times [-1,1])$ is closed in $[0,1]$, denote its supremum by b, then $b < 1$. Thus $\forall t \in (b,1], \gamma(t) \in S$.

Now we would like to take some t_n, s.t. $t_n \to b, \gamma(t_n) = (*, (-1)^n)$.

Denote $z_n = \frac{1}{2n\pi + (-1)^n \frac{\pi}{2}}$; Now we shall show that $\exists t_n$, s.t. $t_n \to b, x(t_n) = z_n$.

$\because x(b) = 0, x(1) = 1$, by the intermediate value theorem, $\exists t_1 \in (b,1]$, s.t. $x(t_1) = z_1$; then by the intermediate value theorem, $\exists t_2 < t_1$, s.t. $x(t_2) = z_2$; we do by induction, then we obatin a decreasing sequence t_n, so it is convergence, however, $\gamma(t_n)$ is not convergent, so this contradicts to γ is continuous. ☐

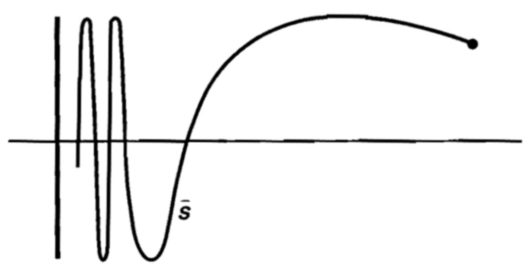

Figure 6: topologist's sine curve

12 Components and Local Connectedness

Definition 12.1. Let X be a topological space, $x \in X$, then:

(1) The biggest connected subspace containing x is called the components (or the connected components) containing x.

(2) The biggest path connected subspace containing x is called the path components containing x.

Proposition 12.1. *(1)The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*

(2)Each component of X is closed.

(3)If X only has finitely many components, then each componentis also open in X.

Proof. (2) Note that the closure of a connected subspace of X is connected.

(3) Note that its complement is a finite union of closed sets. □

Proposition 12.2. *(1)The path components of X are connected disjoint subspaces of X whose union is X, such that each nonempty path connected subspace of X intersects only one of them.*

(2)Each components is the union of some path connected components.

Example 12.1. Each component of $\mathbb{Q}$ consists of a single point.

Example 12.2. Let S be defined as in Example 11.9. Then $\bar{S}$ has 2 path components, one is S and the other is the vertical interval $\{0\} \times [-1, 1]$.

Proof. First, we know from Example 11.9 that $\bar{S}$ is not path connected. So, it has at least 2 path components. Now, we construct a partition that $\bar{S} = S \cup \{0\} \times [-1, 1]$, where S and $\{0\} \times [-1, 1]$ are both path connected. So they are the only 2 path components of $\bar{S}$. □

Example 12.3. $X = \bar{S} \setminus \{0 \times q \mid q \in \mathbb{Q}\}$ is connected, for $S \subset X \subset \bar{S}$.

The path components of $\bar{S}$ are $S, \{(0 \times r)\}_{r \in \mathbb{Q}}$, so $\bar{S}$ has only one component but uncountably many path components.

Definition 12.2. A space X is said to be **locally connected at** x if for every neighborhood U of x, $\exists$ a connected neighborhood V of x contained in U. If X is locally connected at each points of X, X is said to be **locally connected**. **Locally path connected** can be defined as above similarly.

Remark. For some purposes, it is more important that the space satisfy a connectedness condition locally. Roughly speaking, local connectedness means that each point has "arbitrary small" neighborhoods that are connected.

eg12(3) *Example* 12.4. (1) $[-1, 0) \cap (0, 1]$ is not connected but it is locally connected.

(2) $\mathbb{Q}$ are neither connected nor locally connected.

(3) $\bar{S}$ is connected but not locally connected. (Consider 0×0, for every neighborhood U of 0×0, and for any V open s.t. $x \in V \subset U$, $V \cap S$ must have at least 2 disjoint open segement, so there exists a partition. (c.f.Figure 7))

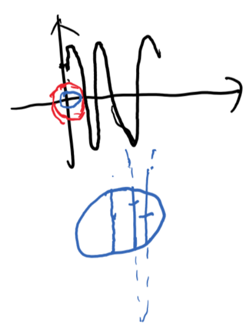

Figure 7: For Example 12.4(3) figure12

Proposition 12.3. *A space X is locally connected $\iff$ for $\forall$ oepn set U, each component of U is open in X.*

prop12 **Proposition 12.4.** *A space X is locally path connected $\iff$ for $\forall$ oepn set U, each path component of U is open in X.*

Theorem 12.1. *If X is locally path connected, then the collection of all components is the same as that of all the path components.*

Proof. Let C be a component, one only need to prove C is path connected.

Let $C = \bigcup_{\alpha \in J} P_\alpha$, by Proposition 12.4, P_α is open($\forall \alpha \in J$), suppose $\#J \neq 1$, then P_{α_0} and $\bigcup_{\alpha \in J \setminus \{\alpha_0\}} P_\alpha$ form a partition, which leads to contradiction. □

13 Compact Spaces

Definition 13.1. A collection $\mathscr{A}$ of subsets of a space X is said to **cover** X or to be a **covering** of X, if $\bigcup_{A \in \mathscr{A}} A = X$. It is called an **open covering** of X if its elements are open subsets of X.

Definition 13.2. A space X is said to be **compact** if every open covering of X contains a finite subcollection that also covers X.

Example 13.1. The real line $\mathbb{R}$ is not compact, for $\mathscr{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$ contains no finite subcollection that covers $\mathbb{R}$.

Example 13.2. $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is compact.

Given an open covering, there is a set U covers 0, then there left finitely many points to be covered.

Lemma 13.1. *Let Y ne a subspace of X. Then Y is compact $\iff$ every covering of Y by sets open in X contains a finite subcollection covering Y.*

thm13.1 **Theorem 13.1.** *Every closed subspace Y of a compact space X is compact.*

Proof. Let $\mathscr{A}$ be a covering of Y by sets open in X, then consdier $\mathscr{B} = \mathscr{A} \cup \{X \setminus Y\}$ □

thm13 **Theorem 13.2.** *Every compact subspace Y of a Hausdorff space X is closed.*

Proof. Let $x_0 \in X \setminus Y$, then for each $y \in Y$, we can choose disjoint open neighborhoods U_y, V_y of x_0, y respectively. Then V_y forms a open covering of Y, therefore, we can choose $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$ cover Y. The open set $V = \bigcup_{i=1}^n V_{y_i}$ is disjoint from the open set $U = \bigcap_{i=1}^n U_{y_i}$, U is a neighborhood of x_0 disjoint from Y. □

Definition 13.3. If a space X satisfies:

(1) One-point sets are closed in X,

(2) For $\forall x \in X$, $\forall$ closed set $C \ni x$, $\exists$ open sets U, V satisfying that $x \in U, C \subset V, U \cap V = \varnothing$,

then X is said to be **regular**.

Definition 13.4. If a space X satisfies:

(1) One-point sets are closed in X,

(2) For $\forall C_1, C_2$ disjoint closed sets in X, $\exists$ open sets U, V satisfying that $C_1 \subset U, C_2 \subset V, U \cap V = \varnothing$,

then X is said to be **normal**.

Corollary 13.1. *If X is a compact Hausdorff space, then X is normal.*

Proof. The statement can be proved by applying the result obtained in the course of the preceding proof of Theorem 13.2. □

Theorem 13.3. *The image of a compact space under a continuous map is compact.*

Theorem 13.4. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We shall prove the images of closed sets of X under f are closed in Y by applying Theorem 13.2, 13.1 □

Theorem 13.5. *Let X_1, X_2 be compact spaces, then $X_1 \times X_2$ is compact.*

Lemma 13.2 (The tube lemma)**.** *Consider the product space $X_1 \times X_2$, where X_2 is compact. If $N \subset X_1 \times X_2$ is an open set containing $\{x\} \times X_2$, $(x \in X_1)$, then $\exists$ open neighborhood $W \ni x$, s.t. $W \times X_2 \subset N$.*