Nonlinear first order PDE

$F(Du, u, x) = 0 \quad x \in \mathcal{U}\ (\subseteq \mathbb{R}^n,\ \text{open})$

$u : \mathcal{U} \to \mathbb{R}$.

Notations: $F = F(p, z, x) \quad p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in \mathcal{U} \quad p \sim Du,\ z \sim u$

$$\begin{cases} D_p F = (F_{p_1}, F_{p_2}, \cdots, F_{p_n}) \\ D_z F = F_z \\ D_x F = (F_{x_1}, F_{x_2}, \cdots, F_{x_n}) \end{cases}$$

Boundary condition: $u = g$ on $\Gamma \subset \partial \mathcal{U} \quad g : \Gamma \to \mathbb{R}$ is given.

Sometimes, you don't need to know $u = g$ on $\partial\mathcal{U}$ ~~to~~ to ensure the uniqueness of solution.
You only need $u = g$ on $\Gamma \subsetneq \partial \mathcal{U}$,
— this is because the PDE is 1st-order.

- Complete integrals.

$F(Du, u, x) = 0$.

Assume $\exists A \subseteq \mathbb{R}^n$, open, s.t. for $\forall a = (a_1, a_2, \cdots, a_n) \in A$,
we have a C^2 solution $u = u(x; a)$

a parameter

We write
$$(D_a u,\ D^2_{x,a} u) = \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & \vdots & & \vdots \\ u_{a_n} & u_{x_1 a_n} & \cdots & u_{x_n a_n} \end{pmatrix}_{n \times (n+1)}$$

<u>Def</u> A C^2-solution $u = u(x; a)$ is called a complete integral in $\mathcal{U} \times A$ if:

(1) $u(x; a)$ is a solution for each $a \in A$;

(2) $\operatorname{rank}(D_a u,\ D^2_{x,a} u) = n \quad (\forall x \in \mathcal{U},\ a \in A)$

Remark: The condition (2) ensures $u(x; a)$ "depends on all the n independent parameters $a_1, \cdots, a_n$"

Pf: We prove this by contradiction.
Suppose $B \subset \mathbb{R}^{n-1}$, open. for each $b \in B$. $v = v(x; b)$ is a solution of PDE. Suppose also $\exists \psi \in C^1 \ \psi : A \to B$.

$\psi = (\psi^1, \psi^2, \cdots, \psi^{n-1})$, s.t. $u(x; a) = v(x; \psi(a))$
$(x \in \mathcal{U}, a \in A)$

that is, we are supposing the function $u(x; a)$ "really depends only on the $n-1$ parameters $b_1, \cdots, b_{n-1}$".
But then
$$u_{x_i a_j}(x; a) = \sum_{k=1}^{n-1} v_{x_i b_k}(x; \psi(a))\, \psi^k_{a_j}(a) \quad i,j = 1, \cdots, n$$

$$\Rightarrow \det(D^2_{xa}u) = \sum_{k_1,\dots,k_n=1}^{n-1} \nu_{x_1 b_{k_1}} \nu_{x_2 b_{k_2}} \cdots \nu_{x_n b_{k_n}} \cdot \det\begin{pmatrix} \psi^{k_1}_{a_1} & \cdots & \psi^{k_1}_{a_n} \\ \vdots & & \vdots \\ \psi^{k_n}_{a_1} & \cdots & \psi^{k_n}_{a_n} \end{pmatrix}$$

$$= 0.$$

$\because$ $k_1,\dots,k_n \in \{1,\dots,n-1\}$ $\therefore \exists\, k_i = k_j$ for each arrangement

Similarly, $\because$ $u_{a_j}(x;a) = \sum_{k=1}^{n-1} \nu_{b_k}(x;\psi(a))\,\psi^{k}_{a_j}(a)$ $j=1,\dots,n.$

$\therefore$ the determinant of each $n\times n$ submatrix of $(D_a u, D^2_{xa}u)$ equals zero

$\therefore$ $\operatorname{rank}(D_a u, D^2_{xa} u) < n \Rightarrow$ a Contradiction!

$\square$

Example: 1) Clairaut's equation:

$x\cdot Du + f(Du) = u$ $\quad f: \mathbb{R}^n \to \mathbb{R}$ is given

$u(x;a) = a\cdot x + f(a)$ is a complete integral for $a\in\mathbb{R}^n$

2) The eikonal equation: $|Du| = 1$

$u(x;a,b) = a\cdot x + b$ $\quad (a\in S^{n-1},\ b\in\mathbb{R})$ is a complete integral

3) Hamilton-Jacobi equation: $\partial_t u + H(Du) = 0$.

$u(x,t;a,b) = a\cdot x - tH(a) + b$ $\quad (x\in\mathbb{R}^n,\ t>0,\ a\in\mathbb{R}^n,\ b\in\mathbb{R})$ is a complete integral.

· Envelops (包络)

Assume $u = u(x;a)$ solves the equation $F(Du,u,x)=0$. $\begin{pmatrix} x\in\mathcal{U}\subseteq\mathbb{R}^n \\ a\in A\subseteq\mathbb{R}^n\end{pmatrix}$

Consider $D_a u(x;a) = 0$ $\quad (x\in\mathcal{U},\ a\in A)$

Suppose the above equation has a C^1 solution $a=\phi(x)$,

i.e. $D_a u(x,a)\big|_{a=\phi(x)} = 0$.

Then $v(x) = u(x;\phi(x))$ is a solution of $F(Du,u,x)=0$.

Def We call $v(x)$ is the envelope of the function $\{u(\cdot;a)\}_{a\in A}$

$v(x)$ is also called the singular integral of the equation $F(Du,u,x)=0$ 奇异积分

Pf: $F(Dv, v, x) = F(Du(x;\phi(x)) + \underbrace{D_a u(x;\phi(x))}_{=0}\cdot D_x\phi(x),\ u(x,\phi(x)),\ x)$

$$= F(Du(x;\phi(x)),\ u(x,\phi(x)),\ x) = 0.$$

Example: $u^2(1+|Du|^2) = 1$

A complete integral: $u(x;a) = \pm(1-|x-a|^2)^{\frac12}$ $\quad (|x-a|<1)$

$D_a u = \dfrac{\mp(x-a)}{(1-|x-a|^2)^{\frac12}} \Rightarrow a=\phi(x)=x \Rightarrow u=\pm1$ are singular integrals.