Introduction

Notations: $u: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function

$\frac{\partial}{\partial x_i} u = \partial_i u \quad (1 \le i \le n)$

$Du = (\partial_1 u, \partial_2 u, \cdots \partial_n u) \xrightarrow{:=} \text{grad } u := \nabla u$ "gradient" of u. (梯度)

$D^2 u = (\partial_i \partial_j u)_{1 \le i,j \le n} \longrightarrow$ "Hessian" of u

multi-index: $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \quad \alpha_i \ge 0. \quad \alpha_i \in \mathbb{Z}.$ 多重指标.

$\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u$

$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \longrightarrow$ order of α (length) 称α的阶/模长.

$\alpha! := \alpha_1! \, \alpha_2! \cdots \alpha_n!$

$D^k u = \{ D^\alpha u \mid |\alpha| = k \} \in \mathbb{R}^{n^k}$ ($k=1$ gradient, $k=2$ Hessian)

$= \{ \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} u \mid i_1, \cdots, i_k \in \{1, 2, \cdots, n\} \}$

记号上稍微有点混用.

后面我们考虑的基本均为光偏导的次序无关 $\partial_i \partial_j = \partial_j \partial_i$ 的这种函数.

★ Laplacian of u: $\Delta u = \text{tr}(D^2 u) = \sum_{i=1}^n \partial_i^2 u$

Divergence (散度) of $F = (F_1, F_2, \cdots F_n)$: $\text{div} F = \partial_1 F_1 + \partial_2 F_2 + \cdots + \partial_n F_n$

$= \nabla \cdot F \qquad \nabla = (\partial_1, \cdots, \partial_n)$

梯度算子与F作内积

$\Delta u = \text{div}(\text{grad } u)$

$= \nabla \cdot (\nabla u)$

curl (旋度) of $v = (v_1, v_2, v_3) \qquad n = 3$

$\text{curl } v = (\partial_2 v_3 - \partial_3 v_2, \ \partial_3 v_1 - \partial_1 v_3, \ \partial_1 v_2 - \partial_2 v_1)$

$= \nabla \times v$

<u>Def</u>. $F(D^k u, D^{k-1} u, \cdots, Du, u, x) = 0. \quad (x \in \mathcal{U} \subseteq \mathbb{R}^n)$

is called a kth-order PDE.

$F: \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{U} \to \mathbb{R}^\ell$ is a given function

$u: \mathcal{U} \to \mathbb{R}^m$ is the unknown

通常要求 $m = \ell$. 即: 方程个数与未知函数个数相等.

Def i) Linear PDE: $\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x)$, a_α, f are given

e.g. $-\Delta u(x) = f(x)$

It is homogeneous if $f \equiv 0$.

non-linear:

ii) semi-Linear: $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(D^{k-1}u, \cdots, Du, u, x)$

↳ 最高阶导数项是线性的，且前面系数不依赖于u.

iii) quasi-Linear: 拟线性 $\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, Du, u, x) D^\alpha u = f(D^{k-1}u, \cdots, Du, u, x)$

iv) full-nonlinear:

e.g. $\partial_1^2 u \, \partial_2^2 u = f(u, x)$

几何分析中有一类经常遇到的 full-nonlinear PDE:

$$-\mathrm{Det}(\nabla^2 u) = f(x, u, \nabla u)$$

↳ Monge-Ampere equation

Examples: 1) Linear PDE:

Poisson equation: $-\Delta u = f(x)$ ($f \equiv 0$, Laplace eq.)

Heat equation: $\partial_t u - \Delta u = f(x)$

Wave equation: $\partial_t^2 u - \Delta u = f(x)$

Transport equation: $\partial_t u + a \cdot \nabla u = f(x)$

2) Some important PDEs in physics and mechanics

- Maxwell equ.s

$$\begin{cases} \partial_t E = \nabla \times B \\ \partial_t B = -\nabla \times E \\ \nabla \cdot E = \nabla \cdot B = 0 \end{cases} \quad E, B: \mathbb{R}^3 \to \mathbb{R}^3$$

E: electric field
B: magnetic field

- Euler / Navier-Stokes equ.s for incompressible, viscous flow

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \\ \nabla u = 0 \end{cases} \quad u \in \mathbb{R}^3 - \text{velocity}, \quad p \in \mathbb{R} - \text{pressure}.$$

$\nu = 0$: Euler equ.s
$\nu > 0$: Navier-Stokes equ.s

其光滑解存在性 — 七个千禧问题之一.

- Schrodinger's equ: $i\partial_t u + \Delta u = f(x, u, \nabla u)$

- Einstein equ.s

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$R_{\mu\nu}$ → Ricci curvature tensor of space-time

R – scalar curvature

$g_{\mu\nu}$ – space-time metric

Λ – constant

c – Light speed

$T_{\mu\nu}$ → Energy-momentum tensor

描述时空变化

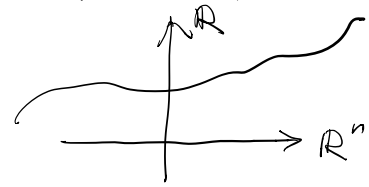
时空变化依赖于物质分布

描述时空物质分布

3) PDE in geometry.

minimal surface: $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$ $:= H$: mean curvature.

$u: \mathbb{R}^n \to \mathbb{R}$



Monge-Ampere equ: $-\det(\nabla^2 u) = f(x, u, \nabla u)$

Gauss curvature

Harmonic map equ.s: $\Delta u + |\nabla u|^2 u = 0$.

调和映照

$u: \Omega \subseteq \mathbb{R}^n \to S^2$

$$u = \arg\min_u \int_\Omega |\nabla u|^2 dx$$

- Study a PDE.
  - Find all explicit solutions?
  - Study the property of solutions.
    - Existence
    - Uniqueness
    - Stability

    } Well-posed (适定)

- Solution:
  - If $D^k u \in C(U)$, we call it classic solution. (Evans Part I)
  - generalized solutions, weak solutions, --- (Evans Part II)
  - Sobolev spaces.

- Boundary Condition:

$u|_{\partial\mathcal{U}} = g$ — Dirichlet B.C. 1st kind B.C.

$\frac{\partial u}{\partial n}\Big|_{\partial\mathcal{U}} = g$ — Neumann B.C. 2nd kind B.C.

$u + \alpha(x)\frac{\partial u}{\partial n}\Big|_{\partial\mathcal{U}} = g$ — 3rd kind B.C.

- Initial condition:

heat equ: $\partial_t u - \Delta u = f$
$u(0,x) = u_0(x)$

Wave equ: $\partial_t^2 u - \Delta u = f$
$u(0,x) = u_0(x)$, $\partial_t u(0,x) = u_1(x)$

- Well-posedness

if solution is of existence uniqueness stability.

- Gauss-Green Formula (divergence theorem) 散度定理. (Appendix C.2)

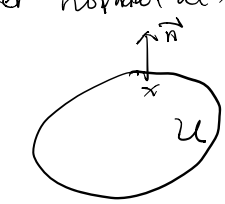
Thm $\mathcal{U} \subset \mathbb{R}^n$ open. $\partial\mathcal{U} \in C^1$ $\vec{F}: \mathcal{U} \to \mathbb{R}^n \in C^1(\mathcal{U}) \cap C(\bar{\mathcal{U}})$

这门课中最基本的工具

then i) $\int_{\mathcal{U}} \partial_i F \, dx = \int_{\partial\mathcal{U}} F \cdot n_i \, dS$ $(i = 1, 2, \cdots, n.)$

ii) $\int_{\mathcal{U}} \operatorname{div}\vec{F} \, dx = \int_{\partial\mathcal{U}} \vec{F} \cdot \vec{n} \, ds(x)$

$\vec{n}$: the unit outer normal at x on $\partial\mathcal{U}$



$n=1$: Newton-Leibniz
$n=2$: Green formula
$n=3$: Gauss formula.