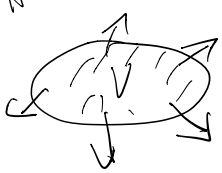
$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathcal{U},\ t>0. \quad u = u(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}. \\ u = g & \text{on } \partial\mathcal{U} \\ u(x,0) = u_0 & \text{in } \mathcal{U} \end{cases}$$

$f = 0$ homogeneous.

$f \neq 0$ non-homogeneous.

<u>Physical interpretation</u>:

Diffusion law: $\underbrace{\frac{d}{dt}\int_V u\,dx}_{\text{rate of change}} = -\int_{\partial V} \underbrace{F \cdot n}_{\text{net flux}}\, ds(x)$



$F = -a\nabla u$. $\Rightarrow$ In many situations, $F \propto \nabla u$.

$$\Rightarrow \int_V (\partial_t u - \Delta u)\, dx = 0$$

- Fundamental Solutions

$u(x,t) \longmapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$

$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t) \Leftarrow$ Look for a solution u to be invariant under this dilation scaling.

Let $\lambda = \frac{1}{t}$, $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$

$$\partial_t u - \Delta u = -\alpha t^{-\alpha-1} v\left(\frac{x}{t^\beta}\right) + (-\beta) t^{-\alpha-\beta-1} x \cdot Dv\left(\frac{x}{t^\beta}\right) - t^{-\alpha-2\beta} \Delta v\left(\frac{x}{t^\beta}\right) = 0$$

$$\Rightarrow \alpha t^{2\beta-1} v(y) + \beta t^{2\beta-1} y \cdot \nabla y + v''(y) = 0 \qquad \left(y = \frac{x}{t^\beta}\right)$$

Take $\beta = \frac{1}{2}$ $\quad \alpha v(y) + \frac{1}{2} y \cdot \nabla y + v''(y) = 0.$

Take v radial, $v(y) = w(|y|)$ $\Longrightarrow$ $\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0.$

Set $\alpha = \frac{n}{2}$ $\Longrightarrow$ $(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0$

$\Longrightarrow r^{n-1} w' + \frac{1}{2} r^n w = \text{const}$

Let $r^{n-1} w' + \frac{1}{2} r^n w = 0$

$\Rightarrow w' = -\frac{1}{2} r w \Rightarrow w = b e^{-\frac{r^2}{4}}$

$$\Rightarrow u(x,t) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

Def $\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & (x\in\mathbb{R}^n,\ t>0) \\ 0 & (x\in\mathbb{R}^n,\ t<0) \end{cases}$ （可约定 $\Phi(x,0)=0$ $x\neq 0$ $\Phi(0,0)$ 无意义.）

is called the fundamental solution of the heat equation.

The choice of $b = \frac{1}{(4\pi)^{\frac{n}{2}}}$ comes from the following consideration.

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1 \quad (\forall t>0) \qquad (\text{Gauss distribution})$$

Remark: $\Phi(x,t)$ is singular at $(0,0)$

· Initial value problem: $\begin{cases} \partial_t u - \Delta u = 0 & x\in\mathbb{R}^n,\ t>0. \\ u = g & x\in\mathbb{R}^n,\ t=0. \end{cases}$

$(\partial_t - \Delta)\Phi(x,t) = 0$, for $t>0$.

$\Rightarrow (\partial_t - \Delta_x)\Phi(x-y,t) = 0$, $\forall y\in\mathbb{R}^n$.

$\Rightarrow (\partial_t - \Delta_x)\int_{\mathbb{R}^n} \Phi(x-y,t)\,g(y)\,dy = 0$.

Therefore, $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\,g(y)\,dy$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy.$$

is a solution of $\partial_t u - \Delta u = 0$.

Indeed, $u\in C^\infty(\mathbb{R}^n\times(0,+\infty))$, $u(x,0) = g(x)$.

We prove these in the following.

Thm: Assume $g\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, u is given as above. Then:

(i) $u\in C^\infty(\mathbb{R}^n\times(0,+\infty))$

(ii) $(\partial_t u - \Delta u)(x,t) = 0$

(iii) $\lim_{\substack{(x,t)\to(x,0)\\ t>0}} u(x,t) = g(x_0)$ for $\forall x_0\in\mathbb{R}^n$.

Pf: (i) $\because$ $\frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \in C^\infty$ with uniformly bounded derivatives of all orders on $\mathbb{R}^n\times[\delta,+\infty)$ for each $\delta>0$.

$\therefore$ $u\in C^\infty(\mathbb{R}^n\times(0,\infty))$

(ii) Obvious.

(iii) For any given $x_0\in\mathbb{R}^n$ $\varepsilon>0$, take $\delta>0$. s.t. $|g(x)-g(x_0)|<\frac{\varepsilon}{2}$. when $|x-x_0|<\delta$.

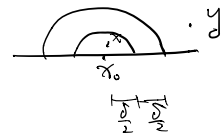
Consider $|x-x_0|<\frac{\delta}{2}$. Then we have

$$|u(x,t)-g(x_0)| = \left|\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}(g(y)-g(y_0))\,dy\right|$$

$$\le \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(\int_{B_\delta(x_0)} + \int_{\mathbb{R}^n\setminus B_\delta(x_0)}\right) e^{-\frac{|x-y|^2}{4t}}|g(y)-g(y_0)|\,dy$$

$$< \frac{\varepsilon}{2} + 2M\int_{\mathbb{R}^n\setminus B_\delta(x_0)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}\, dy.$$

$$\leq \frac{\varepsilon}{2} + 2M\cdot\int_{\mathbb{R}^n\setminus B_\delta(x_0)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|y-x_0|^2}{16t}}\, dy$$



$x_0 - y = z\sqrt{t}$

$$= \frac{\varepsilon}{2} + 2M\underbrace{\int_{\mathbb{R}^n\setminus B_{\frac{\delta}{\sqrt{t}}}(0)} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{16}}\, dz}$$

$|y-x| \geq |y-x_0| - |x-x_0|$

$\geq |y-x_0| - \frac{\delta}{2}$

$\geq \frac{1}{2}|y-x_0|$

$\because \frac{\delta}{\sqrt{t}} \to +\infty$ as $t \to 0$.

$e^{-\frac{|z|^2}{16}} \to 0$ as $|z| \to \infty$

$\therefore$ Choose t sufficient small s.t. $\underbrace{\quad\quad}_{} < \varepsilon$

$\Rightarrow |u(x,t) - g(x_0)| \to 0$ as $|x-x_0| + |t| \to 0$. $\square$.

Remark: ① $\Phi(x,t)$ satisfies $\begin{cases} \partial_t \Phi - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ \boxed{\Phi(\cdot,0) = \delta_0 \quad \text{on } \mathbb{R}^n.} \end{cases}$

δ_0 Dirac 分布.
理解成 $C_c^\infty(\mathbb{R}^n)$ 函数空间的泛函

$\langle \delta_0, \varphi \rangle = \varphi(0) \to$ Dirac measure, $\varphi \in C_0^\infty(\mathbb{R}^n)$

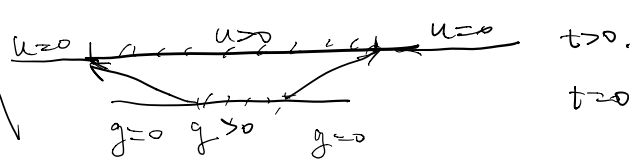
$$\int_{\mathbb{R}^n} \Phi(x,0)\varphi(x)\,dx = \varphi(0)$$

$$\boxed{\lim_{t\to 0}\int_{\mathbb{R}^n} \Phi(x,t)\varphi(x)\,dx = \varphi(0)}$$

② Infinite propagation speed:

If $g \geq 0$, $g \not\equiv 0$, then $u > 0$ for $\forall x \in \mathbb{R}^n$, $t > 0$.

(Finite propagation speed $\Leftarrow$ Wave equation



)

· Non-homogeneous problem : $\begin{cases} \partial_t u - \Delta u = f(x,t) & x \in \mathbb{R}^n,\ t>0. \\ u = 0 & t=0 \end{cases}$ (IH)

Consider the homogeneous problem starting at time $s>0$.

$$\begin{cases} (\partial_t u_s - \Delta u_s)(x,t) = 0 & t>s,\ x\in\mathbb{R}^n. \\ u_s(x,t) = f(x,s) & x\in\mathbb{R}^n. \end{cases}$$

$u_s : \mathbb{R}^n \times [s,+\infty) \longrightarrow \mathbb{R}$

$$u_s(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,\ t-s)\, f(y,s)\, dy.$$

Duhamel's Principle: $\underbrace{u(x,t) = \int_0^t u_s(x,t)\, ds}_{\text{satisfies (IH)}}$ (☆IH)

Formally, 形式上计算,

$$\partial_t u - \Delta u = u_t(x,t) + \int_0^t (\partial_t - \Delta) u_s(x,t)\, ds.$$

$$= f(x,t) + 0 = f(x,t)$$

$t \to 0$; $\left| \int_0^t u_s(x,t)\, ds \right| = \left| \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds \right|$

$$\leq M \cdot t \longrightarrow 0.$$

(If $|f| \leq M$. $\Rightarrow |u_s| \leq M \Rightarrow \int_0^t u_s \leq Mt$)

$\because \int_{\mathbb{R}^n} \Phi\, dy = 1$

<u>Thm</u> : Let u be given by (☆IH). Assume $f \in C_1^2(\mathbb{R}^n \times [0,+\infty))$ has compact support in $\mathbb{R}^n \times [0,+\infty)$

i) $u \in C_1^2(\mathbb{R}^n \times (0,+\infty))$

ii) $\partial_t u - \Delta u = f(x,t)$ $(x\in\mathbb{R}^n,\ t>0)$

iii) $\lim\limits_{\substack{(x,t)\to(x_0,0) \\ t>0}} u(x,t) = 0$ for $\forall x \in \mathbb{R}^n$

Pf: Omitted. 与前面的几个类似定理证明类似.

· General problem: $\begin{cases} \partial_t u - \Delta u = f & t>0 \\ u = g & t=0 \end{cases}$ (GP)

$$\begin{cases} \partial_t u_1 - \Delta u_1 = f \\ u_1(0) = 0 \end{cases} \qquad \begin{cases} \partial_t u_2 - \Delta u_2 = 0 \\ u_2(0) = g \end{cases}$$

$$\Rightarrow u(x,t) = u_2 + u_1$$

$$= \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds$$

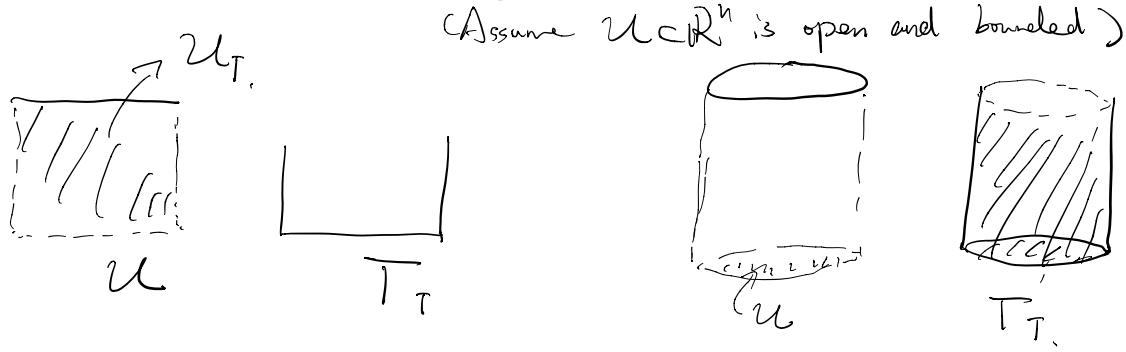
(☆GP)

· Mean-value formula.

<u>Notations</u>: parabolic cylinder: $\mathcal{U}_T = \mathcal{U} \times (0, T]$

parabolic boundary: $\Gamma_T = \partial_p \mathcal{U}_T = \overline{\mathcal{U}_T} - \mathcal{U}_T$

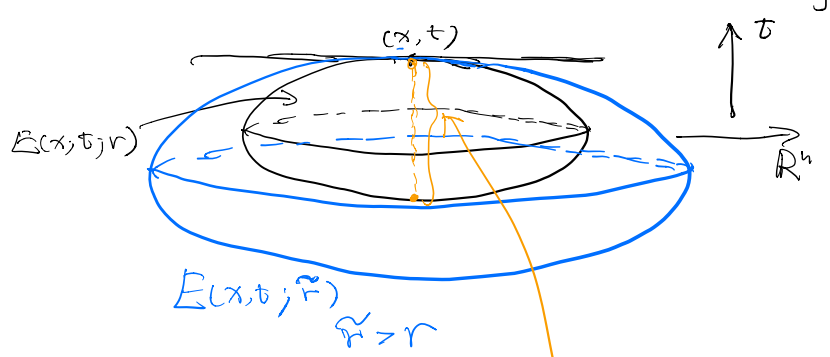
(Assume $\mathcal{U} \subset \mathbb{R}^n$ is open and bounded)



· <u>Def</u>: For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define a "heat ball"

$$E(x,t;r) := \left\{ (y,s) \in \mathbb{R}^{n+1} \;\middle|\; s \le t,\ \Phi(x-y, t-s) \ge \frac{1}{r^n} \right\}.$$

a region in space-time, the boundary of which is a level set of $\Phi(x-y, t-s)$



$$(4\pi(t-s))^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} = \frac{1}{r^n}$$

$$e^{-\frac{|x-y|^2}{4(t-s)}} = \left(\frac{4\pi(t-s)}{r^2}\right)^{\frac{n}{2}}$$

$$\Rightarrow -\frac{|x-y|^2}{4(t-s)} = -\frac{n}{2} \ln \frac{4\pi(t-s)}{r^2}$$

$$\Leftrightarrow \frac{|x-y|^2}{2(t-s)} = n \ln \frac{r^2}{4\pi(t-s)}$$

$$\Rightarrow \frac{r^2}{4\pi(t-s)} > 1 \Leftrightarrow t - s < \frac{r^2}{4\pi}$$

$$t - \frac{r^2}{4\pi} < s \le t$$

$\Rightarrow$ $u(x,t)$ does not depend upon future times

Thm Let $u \in C^2_1(\mathcal{U}_T)$ solve the heat equation. Then:
(space C^2, time C^1)

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds \quad \text{for } \forall E(x,t;r) \subset \mathcal{U}_T$$

Remark:

We have $1 = \frac{1}{4r^n} \iint_{E(x,t;r)} \frac{|x-y|^2}{(t-s)^2}\, dy\, ds$. (Calculated below)

So we call the above Thm as a mean-value property for the heat equ.

Calculation of the integral:

$$\frac{1}{4r^n} \int_{t-\frac{r^2}{4\pi}}^{t} \left(\int_{|x-y| \le \sqrt{2n(t-s)\ln\frac{r^2}{4\pi(t-s)}}} \frac{|x-y|^2}{(t-s)^2}\, dy \right) ds.$$

$$= \frac{1}{4r^n} \cdot \int_{t-\frac{r^2}{4\pi}}^{t} \left(\int_0^{\sqrt{2n(t-s)\ln\frac{r^2}{4\pi(t-s)}}} \frac{R^2}{(t-s)^2} \cdot \omega(n) R^{n-1}\, dR \right) ds.$$

$$= \frac{\omega(n)}{4r^n(n+2)} \int_{t-\frac{r^2}{4\pi}}^{t} \left(2n(t-s)\ln\frac{r^2}{4\pi(t-s)} \right)^{\frac{n+2}{2}} \cdot \frac{1}{(t-s)^2}\, ds$$

Let $v = \ln\frac{r^2}{4\pi(t-s)}$, $\Rightarrow t-s = \frac{r^2}{4\pi e^v}$

$$= \frac{n\alpha(n)}{4r^n(n+2)} \int_0^\infty \left(2n \frac{r^2}{4\pi e^v} \cdot v \right)^{\frac{n+2}{2}} \cdot \frac{4\pi e^v}{r^2}\, dv.$$

$$= \frac{n\alpha(n)}{\cancel{4r^n}(n+2)} \cdot \int_0^\infty \frac{n^{\frac{n+2}{2}} \cdot \cancel{r^{n+2}} \cdot v^{\frac{n+2}{2}}}{2^{\frac{n+2}{2}} \cdot \pi^{\frac{n+2}{2}} e^{\frac{n+2}{2}v}} \cdot \frac{\cancel{4\pi} e^v}{\cancel{r^2}}\, dv$$

$$= \frac{n^{\frac{n+4}{2}}\alpha(n)}{2^{\frac{n+2}{2}}(\sqrt{\pi})^n(n+2)} \underbrace{\int_0^\infty e^{-\frac{n}{2}v} \cdot v^{\frac{n+2}{2}}\, dv}$$

$$= \frac{n^{\frac{n+4}{2}}}{2^{\frac{n+2}{2}} \cdot (\sqrt{\pi})^n \cdot (n+2)} \cdot \underbrace{\boxed{\frac{(\sqrt{\pi})^n}{\Gamma(\frac{n}{2}+1)}}}_{\alpha(n)} \cdot \underbrace{\left(\frac{2}{n}\right)^{\frac{n+4}{2}} \cdot \Gamma\left(\frac{n+4}{2}\right)}$$

$$= \frac{2}{n+2} \cdot \frac{\Gamma(\frac{n}{2}+2)}{\Gamma(\frac{n}{2}+1)} = 1$$

Remark: $\sup_{x \in \mathbb{R}^n} u(x,t) \le \sup_{x \in \mathbb{R}^n} u(x,s) \quad (t > s)$

Pf of Thm:

WLOG, Let $x=0$, $t=0$. and $E(r)=E(0,0;r)$. Consider:

$$\phi(r) = \frac{1}{r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2}\,dy\,ds$$

$$= \iint_{E(1)} u(ry,r^2s)\frac{|y|^2}{s^2}\,dy\,ds$$

$$\phi'(r) = \iint_{E(1)} (y\cdot\nabla u + 2rs\,\partial_s u)(ry,r^2s)\cdot\frac{|y|^2}{s^2}\,dy\,ds$$

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \underbrace{(y\cdot\nabla u}_{A} + \underbrace{2s\,\partial_s u}_{B})(y,s)\cdot\frac{|y|^2}{s^2}\,dy\,ds$$

Let $\psi = \log(\Phi(y,-s)\cdot r^n)$ $\left(\psi=0 \text{ on } \partial E(r)\setminus\{(0,0)\}\right)$

$$= \frac{|y|^2}{4s} - \frac{n}{2}\log(-4\pi s) + n\log r. \qquad (\because s\le t=0)$$

$$B = \frac{1}{r^{n+1}}\iint_{E(r)} 4\partial_s u\cdot y\cdot\nabla\psi\,dy\,ds$$

$$\left(\nabla\psi = \frac{y}{2s},\ y\cdot\nabla\psi = \frac{|y|^2}{2s}\right) \qquad \nabla y = n.$$

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \left(-4\,y\cdot\nabla(\partial_s u)\psi - 4n\,\partial_s u\cdot\psi\right)dy\,ds$$

integration by part with respect to y ($\because \psi=0$ on $\partial E(r)\setminus\{(0,0)\}$)

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \left(-4n\,\partial_s u\,\psi + 4y\cdot\nabla u\cdot\partial_s\psi\right)dy\,ds$$

integration by part with respect to s.

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \left(-4n\psi\,\partial_s u - \frac{2n}{s}\,y\cdot\nabla u - \underbrace{y\cdot\nabla u\,\frac{|y|^2}{s^2}}_{A}\right)dy\,ds.$$

$$\partial_s\psi = -\frac{|y|^2}{4s^2} - \frac{n}{2}\cdot\frac{1}{s}$$

$$\Rightarrow \phi'(r) = \frac{1}{r^{n+1}}\iint_{E(r)} \left(4n\,\partial_s u\cdot\psi + \frac{2n}{s}(y\cdot\nabla u)\right)dy\,ds.$$

$$= -\frac{1}{r^{n+1}}\iint_{E(r)} \left(4n\psi\,\partial_s u + 4n(\nabla\psi\cdot\nabla u)\right)dy\,ds.$$

$$\nabla\psi = \frac{y}{2s}$$

$$= -\frac{1}{r^{n+1}}\iint_{E(r)} 4n\psi\,(\partial_s u - \Delta_y u)\,dy\,ds. = 0$$

integration by part

u is the solution to heatequation

Then $\because \phi(r)\to 4u(0,0)$ as $r\to 0$ $\left(\because \frac{1}{4\pi}\iint_{E(1)}\frac{|y|^2}{s^2}\,dy\,ds = 1 \text{ by the calculation in above remark}\right)$ by the above

$\therefore u(0,0) = \frac{\phi(r)}{4}$ $\square$.

Properties of solutions for the heat equation.

· Strong maximum principle. · Regularity · Local estimates.

· Strong maximum principle.

<u>Thm</u> Assume $u \in C^2(\mathcal{U}_T) \cap C(\overline{\mathcal{U}_T})$ solves the heat equation in $\mathcal{U}_T$. Then

(i) $\max\limits_{\overline{\mathcal{U}_T}} u = \max\limits_{\Gamma_T} u$

(ii) If $\mathcal{U}$ is connected and there exists a point $(x_0, t_0) \in \mathcal{U}_T$, s.t. $u(x_0, t_0) = \max\limits_{\overline{\mathcal{U}_T}} u$, then u is constant in $\overline{\mathcal{U}_{t_0}}$

Only need to prove (ii)

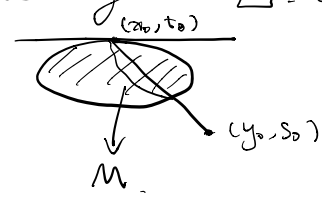
Pf: Assume $M = \max\limits_{\overline{\mathcal{U}_T}} u = u(x_0, t_0)$ $(x_0, t_0) \in \mathcal{U}_T$.

· For $\forall r > 0$, we prove: if $E(x_0, t_0; r) \subset \mathcal{U}_T \Rightarrow u \equiv M$ in $E(x_0, t_0; r)$

Actually, $M = u(x_0, t_0) = \frac{1}{4r^2} \int_{E(x_0, t_0; r)} u(y, s) \frac{|y - x_0|^2}{(s - t_0)^2} \, dy ds \leq M$.

Thus, $u(y, s) \equiv M$. for $(y, s) \in E(x_0, t_0; r)$

· For any line segment L: connecting (x_0, t_0) and another point (y_0, s_0)



$\leftarrow$ We can cover L by a serious of "heat ball"s.

$\Downarrow$

$u(y_0, s_0) = M$.

· For any point $(y, s) \in \mathcal{U}_{t_0}$.

$\because$ $\mathcal{U}_{t_0}$ is path-connected $\therefore$ We have $u \equiv M$ in $\mathcal{U}_{t_0}$.

Remark: We can change max $\rightarrow$ min (Apply to $-u$)

Remark: Infinite propagation speed: $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathcal{U}_T \\ u = g & t = 0. \end{cases}$

If $g \geq 0$ ($g \not\equiv 0$) $\Rightarrow u > 0$. for $t > 0$.

Pf: Suppose $\min\limits_{\mathcal{U}_T} u = u(x_0, t_0) \leq 0$.

then $u \equiv 0$ in $\mathcal{U}_{t_0}$

$\Rightarrow g \equiv 0$. Contradiction.

$\therefore \min\limits_{\mathcal{U}_T} u > 0$.

Thm (Uniqueness for bounded domains) $\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathcal{U}_T \\ u = g & \text{on } \Gamma_T. \end{cases}$ has at most one solution

Pf: If u_1, u_2 are 2 solutions.

$w = u_1 - u_2$ $\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \mathcal{U}_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$

Apply strong maximum principle to $\pm w$.

$\Rightarrow w \equiv 0$

<u>Thm</u>: (Maximum principle for the Cauchy problem).

Assume $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$

solves $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

and satisfies $u(x,t) \le A e^{a|x|^2}$ for some $A, a > 0$.

Then $\sup\limits_{\mathbb{R}^n \times [0,T]} u = \sup\limits_{\mathbb{R}^n} g$

Remark: This Thm is not a trivial corollary of the above strong maximum principle, for we do not assume the maximum could not be attained in $\mathbb{R}^n \times (0,T]$

$\downarrow$

It is not a bounded region.

还要加上 $u(x,t) \le Ae^{a|x|^2}$ 这个增长条件 (growth estimate).

没有的话是不对的.

Pf: I) Assume $v(x,t) = u(x,t) - \dfrac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}$ $(x \in \mathbb{R}^n,\ t > 0)$,

$T < \frac{1}{4a}$

where $T + \varepsilon < \frac{1}{4a}$

$\mu > 0$. $y \in \mathbb{R}^n$ $t < T + \varepsilon$.

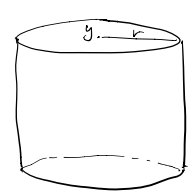
As $\partial_t v - \Delta v = 0$, we have $\max\limits_{\overline{\mathcal{U}}_T} v = \max\limits_{\Gamma_T} v$ $\mathcal{U}_T = B_r(y) \times (0,T]$

① $v(x,0) \le u(x,0) = g(x)$

② if $x \in \partial B_y(r)$,

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon-t)}}$$

$$\le A e^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon)}}$$



As $a < \dfrac{1}{4(T+\varepsilon)}$, we can choose $r \gg 1$ such that $v(x,t) \le \sup\limits_{\mathbb{R}^n} g$

Thus $\max\limits_{\overline{\mathcal{U}}_T} v \le \sup\limits_{\mathbb{R}^n} g$ $\rightsquigarrow$ independent of μ, a, A.

Let $\mu \to 0$. $\Rightarrow \max\limits_{\overline{\mathcal{U}}_T} u \le \sup\limits_{\mathbb{R}^n} g$

II) For $T \geq \frac{1}{4a}$, $[0,T_1] \subset [0,T_1] \cup [T_1, 2T_1] \cup \cdots \cup [kT_1, (k+1)T_1]$

$T \geq \frac{1}{4a}$

Apply result of (I) repeatedly on time intervals,

$$\Rightarrow \max_{[iT_1,(i+1)T_1]} u \leq \max_{\substack{t=iT_1 \\ x\in\mathbb{R}^n}} u \leq \cdots \leq \sup_{\mathbb{R}^n} g.$$

$\left(\text{Take } T_1 = \frac{1}{8a} < \frac{1}{4a}, \quad \varepsilon = \frac{1}{16a}, \quad T_1 + \varepsilon < \frac{1}{4a} \right)$

$\square$.

<u>Thm</u> (Uniqueness for Cauchy problem)

Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of the initial value problem $\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

satisfying the growth estimate: $|u(x,t)| \leq A e^{a|x|^2}$ $(x \in \mathbb{R}^n, 0 \leq t \leq T)$

Remark: We can construct infinitely many solutions of $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$ (c.f. John Chapter 7)

Each of the solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$.
These solutions are "non-physical."

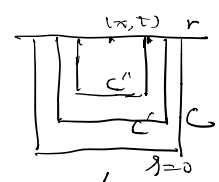
· Regularity

<u>Thm</u>: Suppose $u \in C^2_1(\mathcal{U}_T)$ solves the heat equation in $\mathcal{U}_T$. Then $u \in C^\infty(\mathcal{U}_T)$

Pf: Let $C(x,t;r) = \{(y,s) \mid |x-y| \leq r,\ t-r^2 \leq s \leq t\}$

Fixed $(x_0,t_0) \in \mathcal{U}_T$, we choose $r \ll 1$

s.t. $\begin{cases} C := C(x_0,t_0;r) \\ C' = C(x_0,t_0;\frac{2}{3}r) \\ C'' = C(x_0,t_0;\frac{r}{3}) \end{cases}$ $\mathcal{U}_T$



· Choose $\zeta(x,t)$ smooth supported in C: $\begin{cases} \zeta(x,t) = 1 & \text{in } C' \\ \zeta(x,t) = 0 & \text{in } \mathbb{R}^n \times (0,t_0) \setminus C \end{cases}$

a cut-off function

· $v(x,t) = \zeta(x,t) \cdot u(x,t)$

Then. $\begin{cases} \partial_t v - \Delta v = \zeta \partial_t u + u \cdot \partial_t \zeta - (\zeta \cdot \Delta u + 2D\zeta \cdot Du + u \cdot \Delta \zeta) \\ \qquad = (\partial_t \zeta - \Delta \zeta) u - 2D\zeta \cdot Du \triangleq f(x,t) \\ v = 0 \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$\because \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds$ $\quad \therefore$ By the uniqueness for cauchy problem (above thm)

$$\Rightarrow v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds \qquad (☆☆)$$

$$= \iint_C \Phi(x-y, t-s) [(\partial_s \zeta - \Delta \zeta) u - 2D\zeta \cdot Du]\, dy\, ds.$$

$\because \zeta \equiv 0$ off C

$$= \iint_C \left[\Phi(x-y,t-s)(\partial_s \zeta + \Delta \zeta) + 2 D_y \Phi(x-y,t-s)\cdot D_y \zeta(y,s)\right] u(y,s)\,dy\,ds \quad (*)$$

integration the last term by parts, $\Phi(x-y,t-s)\cdot u \cdot D_y \zeta\big|_{C} = 0$

Above formula holds if we assume $u \in C^\infty$

Only when $f \in C_1^2(\mathbb{R}^n \times [0,+\infty))$ and has compact support; can we apply the thm for non-homogeneous problem's result (☆II) to get (☆☆) above.

If u only satisfies $u \in C_1^2(\mathcal{U}_T)$
we derive $(*)$ with $u_\varepsilon = \eta_\varepsilon * u \in C^\infty$ replacing u.
η_ε being the standard mollifier in the variables x and t.
then u_ε is a C^∞-solution to the heat equation in $(\mathcal{U}_T)_\varepsilon$.
and let $\varepsilon \to 0$. $\Rightarrow u \in C_1^2$ ✓

Let $K(x,t;y,s) = \Phi(x-y,t-s)(\partial_s \zeta + \Delta \zeta) + 2 D_y \Phi(x-y,t-s)\cdot D_y \zeta(y,s)$

$$= \iint_{C\setminus C'} K(x,t;y,s)\, u(y,s)\,dy\,ds \qquad \left(K(x,t;y,s) = 0 \text{ for } (y,s) \in C'\right)$$

If $(x,t) \in C''$
$$u(x,t) = v(x,t) = \iint_{C\setminus C'} K(x,t;y,s)\, u(y,s)\,dy\,ds.$$

For $(y,s) \in C\setminus C'$,
$\because \Phi$ only has singularities when $t-s$ near 0,
and when $(x,t) \in C''$, $t-s \geq C_0 > 0$.
$\therefore K(x,t;y,s)$ is smooth in $(x,t) \in C''$
i.e. $K(x,t;y,s) \in C^\infty$ in $(x,t) \in C''$
$\therefore u(x,t) \in C^\infty$ in $(x,t) \in C''$

- Local derivative estimates

<u>Thm</u> For $\forall\, C(x,t;\frac{r}{2}) \subset C(x,t;r) \subset \mathcal{U}_T$ and solutions for the heat equation,
$$\max_{C(x,t;\frac{r}{2})} \left| D_x^k D_t^\ell u \right| \leq \frac{C_{k,\ell}}{r^{k+2\ell+n+2}} \|u\|_{L^1(C(x,t;r))}$$

Pf: WLOG, Let $(x,t) = (0,0)$,
$C(\frac{1}{2}) = C((0,0),\frac{1}{2})$, $C(\frac{3}{8}) = C((0,0),\frac{3}{8})$, $C(1) = C((0,0),1) \subset \mathcal{U}_T$
Then as in the proof of above thm, we have
$$u(x,t) = \int_{C(1)\setminus C(\frac{3}{8})} K(x,t;y,s)\, u(y,s)\,dy\,ds \quad (\forall (x,t) \in C(\tfrac{1}{2}))$$
Consequently, $$|D_x^k D_t^\ell u| \leq \int_{C(1)\setminus C(\frac{3}{8})} \underbrace{|D_x^k D_t^\ell K(x,t;y,s)|}_{\leq C_{k,\ell}} |u(y,s)|\,dy\,ds.$$
$$\leq C_{k,\ell} \|u\|_{L^1(C(1))}$$

This is because $(x,t) \in C(\frac{1}{2})$, $(y,s) \in C(1)\setminus C(\frac{3}{8})$, $\mathrm{dist}(x,y) \geq C_1 > 0$, $\mathrm{dist}(s,t) \geq C_2 > 0$

Then, for generalized r, let $v(x,t) = u(rx, r^2 t)$
$D_x^k D_t^\ell v(x,t) = r^{2\ell+k} D_x^k D_t^\ell u(rx, r^2 t)$ and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}} \|u\|_{L^1(C(r))}$
then we have the desired result $\square$

· Energy methods

Uniqueness & Backwards uniqueness.

<u>Thm</u>: There exists at most 1 solution $u \in C^2_1(\overline{\mathcal{U}_T})$ of the initial / boundary value problem:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathcal{U}_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Pf: $w = u_1 - u_2$

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \mathcal{U}_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

Let $e(t) = \int_{\mathcal{U}} w^2(x,t)\,dx \quad (0 \le t \le T)$.

$$\partial_t e(t) = 2\int_{\mathcal{U}} w \cdot \partial_t w\,dx = 2\int_{\mathcal{U}} w \cdot \Delta w\,dx = -2\int_{\mathcal{U}} |\nabla w|^2 dx \le 0.$$

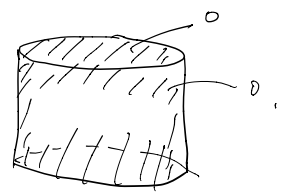
$\Rightarrow e(t) = e(0) = 0,\ \forall t > 0 \Rightarrow w \equiv 0.$ $\square$

<u>Thm</u>: If u satisfies $\begin{cases} \partial_t u - \Delta u = 0. & \text{in } \mathcal{U}_T \\ u = 0 & \text{on } \partial\mathcal{U} \times [0,T] \\ u = 0 & \text{on } \mathcal{U} \times \{t = T\} \end{cases}$

then $u \equiv 0$ in $\mathcal{U}_T$

Remark: $\begin{cases} v(x,t) = u(x, T-t) & (0 \le t \le T) \\ v(x,0) = 0. \quad v|_{\partial\mathcal{U}} = 0. \\ \partial_t v + \Delta v = 0. \end{cases}$

$\Rightarrow v \equiv 0.$



Pf of thm: Let $e(t) = \int_{\mathcal{U}} |u(x,t)|^2 dx$ Then:

同上一定理证明 $\Rightarrow e'(t) = -2\int_{\mathcal{U}} |\nabla u|^2 dx$

$$e''(t) = -4\int_{\mathcal{U}} \nabla u \cdot \nabla \partial_t u\,dx \overset{\partial_t u = \Delta u.}{=} 4\int_{\mathcal{U}} |\Delta u|^2 dx$$

Since $\int_{\mathcal{U}} |\nabla u|^2 dx = \int_{\mathcal{U}} -u\Delta u\,dx \le \left(\int_{\mathcal{U}} u^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\Delta u|^2 dx\right)^{\frac{1}{2}}$

Cauchy-Schwarz.

$\Leftrightarrow \left(\frac{1}{2} e'(t)\right)^2 \le e(t) \cdot \frac{e''(t)}{4}$

$\Leftrightarrow (e'(t))^2 \le e(t) \cdot e''(t)$

In fact, this implies $e \equiv 0$. We prove this by contradiction.

If $e(t) \not\equiv 0$ on $[0,T]$. then $\exists\, [t_1, t_2] \subset [0,T]$ s.t. $e(t) > 0$. in $t \in [t_1, t_2)$, $e(t_2) = 0$.

$$(\log e(t))'' = \left(\frac{e'}{e}\right)' = \frac{e''e-(e')^2}{e^2} \geq 0$$

$\Rightarrow$ $\log e(t)$ is convex on $[t_1, t_2)$

$$\Rightarrow \log e\left(\frac{t_1+t_2-\varepsilon}{2}\right) \leq \frac{1}{2}\left(\log e(t_1) + \log e(t_2-\varepsilon)\right)$$

$$\Rightarrow e\left(\frac{t_1+t_2-\varepsilon}{2}\right)^2 \leq e(t_1)\cdot e(t_2-\varepsilon)$$

Let $\varepsilon \to 0$ $\Rightarrow$ $e\left(\frac{t_1+t_2}{2}\right)^2 \leq e(t_1)\, e(t_2) = 0.$

$\Rightarrow$ $e\left(\frac{t_1+t_2}{2}\right) = 0.$

a Contradiction!

$\square$