2.1 Transport Equation

$$\partial_t u(t,x) + b\cdot\nabla u(t,x) = 0 \qquad (1)$$

$u: [0,+\infty)\times\mathbb{R}^n \qquad x=(x_1,x_2,\cdots,x_n) \qquad b=(b_1,b_2,\cdots,b_n)\in\mathbb{R}^n$ is given.

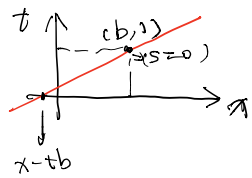
Suppose u is smooth.

Let $Z(s) = u(t+s, x+sb) \quad (s\in\mathbb{R}) \qquad (t,x$ are given$)$.

$$\partial_s Z(s) = (\partial_t u + \nabla_x u\cdot b)(t+s, x+sb) = 0.$$

$\Rightarrow$ $Z(s)$ is independent of s.

$\Rightarrow$ Z is constant on the line $(t+s, x+sb)$ $(s\in\mathbb{R})$



$(t,x)+s(1,b)$

$$Z(0) = u(t,x) = Z(-t) = u(0, x-tb)$$

Initial problem. $\begin{cases}\partial_t u + b\cdot\nabla u = 0 \\ u(0,x) = u_0(x)\end{cases}$ $\longrightarrow$ initial value $\quad t>0,\ x\in\mathbb{R}^n \qquad (2)$

$$u(t,x) = u(0, x-tb) = u_0(x-tb) \qquad (3)$$

If u_0 is C^1 $\Rightarrow$ (3) is a solution of (2)

If u_0 is not C^1 $\Rightarrow$ (2) has no C^1 solution

- Nonhomogeneous problem:

$$\begin{cases}\partial_t u + b\cdot\nabla u = f(t,x) \\ u(0,x) = u_0(x)\end{cases}$$

$$Z(s) = u(t+s, x+sb)$$
$$\partial_s Z = (\partial_t u + b\cdot\nabla x)(t+s, x+sb) = f(t+s, x+sb)$$

$$\Rightarrow Z(0) - Z(-t) = \int_{-t}^{0} f(t+s, x+sb)\,ds$$

$$\Rightarrow u(t,x) = \underbrace{u_0(x-tb)}_{\text{初值}} + \underbrace{\int_0^t f(s, x-tb+sb)\,ds}_{\text{非齐次项}} \quad (t\geq 0)$$

Remark: This method is a special case of the "method of characteristics" 特征线法.

§3.2.

2.2 Poisson equation

$-\Delta u = f$ (1)

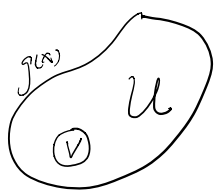
$\Delta u = 0$ (2) $\longrightarrow$ Laplace equation

$u : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$

$u|_{\partial\mathcal{U}} = g(x)$ (Dirichlet B.C.)

u: density of some physical quantity at equilibrium 物理量平衡态的密度

(e.g. chemical concentration, temperature, ...)



$\vec{F}$: flux density. 流密度 (通常假设 $\vec{F} = -a\nabla u$)

$\forall$ region $V \subseteq \mathcal{U}$, $\int_{\partial V} \vec{F}\cdot\vec{n}\, dS(x) = 0$.

$$\int_{\partial V} \vec{F}\cdot\vec{n}\, dS(x) = \int_V \operatorname{div}\vec{F}\, dx = 0$$

$$\Rightarrow -a\int_V \Delta u\, dx = 0 \Rightarrow \Delta u = 0.$$

Harmonic functions.

<u>Def</u> $u \in C^2(\mathcal{U})$ satisfies $\Delta u = 0$. We say u is a harmonic function

Properties of harmonic functions.

- mean-value property
- regularity
- Liouville Thm.
- Harnack's inequality
- maximum principle
- derivative estimate
- analyticity

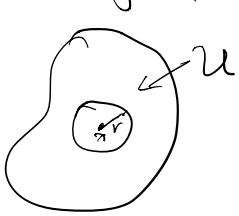
Mean-value property.

<u>Thm</u>: $u \in C^2(\mathcal{U})$ is harmonic, then $u(x) = \fint_{\partial B_r(x)} u(y)\, dS(y)$ 1st MVP

(Notation: $\fint_\Omega = \frac{\int_\Omega (\ )}{|\Omega|}$) $= \fint_{B_r(x)} u(y)\, dy$ 2nd MVP

$\forall B_r(x) \subset \mathcal{U}$

Pf: (1) $\phi(r) = \fint_{\partial B_r(x)} u(y)\, dS(y)$



$$\underset{y = x + rz}{=} \fint_{\partial B_1(0)} u(x+rz)\, dS(z)$$

$$\phi'(r) = \fint_{\partial B_1(0)} \nabla_x u(x+rz)\cdot z\, dS(z)$$

$$= \fint_{\partial B_r(x)} \nabla u(y)\cdot\frac{y-x}{r}\, dS(y)$$

$$= \frac{\int_{\partial B_r(x)} \nabla u(y) \cdot \vec{n} \, dS(y)}{|\partial B_r(x)|}$$

$$= \frac{\int_{B_r(x)} \operatorname{div}(\nabla u(y)) \, dy}{|\partial B_r(x)|} \underset{\because \operatorname{div}(\nabla u(y)) = \Delta u = 0.}{=} 0$$

$\because \lim_{r \to 0} \phi(r) = u(x)$ $\therefore$ ✓ $\square$

(2) $$\int_{B_r(x)} u(y) \, dy = \int_0^r \left[\int_{\partial B_r(x)} u \, dS(y) \right] dr$$

$$= \int_0^r u(x) |\partial B_r| \, dr = u(x) |B_r|$$

$\square$

Actually 1st-MVP $\Leftrightarrow$ 2nd-MVP

<u>Thm</u>: C^2 + MVP $\Rightarrow$ Harmonic

Pf: $\because \phi(r) = u(x)$ $\forall r$

$\Rightarrow \phi'(r) = 0 \Rightarrow \int_{B_r(x)} \Delta u \, dy = 0$ for $\forall B_r(x) \subset \mathcal{U}$

$\Rightarrow \Delta u = 0$ $\square$

Remarks: C^1 + MVP $\Rightarrow$ Harmonic
C^0 + MVP $\Rightarrow$ Harmonic

· Maximum Principle

<u>Thm</u> (strong maximum principle)

$u \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ harmonic

Then (i) $\max_{\overline{\mathcal{U}}} u = \max_{\partial \mathcal{U}} u$

(ii) If $\mathcal{U}$ is connected, and $u(x_0) = \max_{\overline{\mathcal{U}}} u$ ($x_0 \in \mathcal{U}$)

then $u \equiv$ const in $\mathcal{U}$

Pf: It suffices to prove (ii) (If $\exists x_0 \in \mathcal{U}$, $u(x_0) = \max_{\overline{\mathcal{U}}} u$, then (i) ✓)

Let $M = u(x_0) = \max_{\overline{\mathcal{U}}} u$ $A = \{x \in \mathcal{U} \mid u(x) = M\}$

① A is relatively closed in $\mathcal{U}$ ($y_i \to y_0 \in \mathcal{U}$, then $y_i \in A$ ($\forall i$) $\Rightarrow y_0 \in A$)

($\because u$ is continuous $\therefore$ ① ✓)

② A is open

$$\left(\begin{array}{l} \text{For } x\in A \quad u(x)=M \quad \text{Take } B_r(x)\subset \mathcal{U}, \\ M=u(x)= \fint_{B_r(x)} u(y)\,dy \le M \\ \therefore u(y)=M \quad \forall y\in B_r(x) \quad \therefore B_r(x)\subset A \quad \therefore A \text{ is open} \end{array}\right)$$

$\because \mathcal{U}$ is connected. $A\subset\mathcal{U}$ $\therefore A=\mathcal{U}$

i.e. $u=\text{const}$ in $\mathcal{U}$ $\square$

Remark: ① The uniqueness of solution to "Poisson equ. Dirichlet Problem"

$$\begin{cases} \Delta u=f & \text{in } \mathcal{U} \\ u=g & \text{on } \partial\mathcal{U} \end{cases} \qquad u\in C^2(\mathcal{U})\cap C^1(\overline{\mathcal{U}})$$

Take solutions $u_1, u_2 \Rightarrow u_1-u_2\big|_{\partial\mathcal{U}}=0$, $\Delta(u_1-u_2)=0$

$\Rightarrow u_1-u_2\big|_{\mathcal{U}}\equiv 0 \Rightarrow u_1\equiv u_2$ on $\mathcal{U}$

By strong maximum principle. $\therefore$ Unique Solution $\square$

② If $g\ge 0$ $(g\not\equiv 0)$ then $u>0$ in $\mathcal{U}$

Prove by contradiction ($u(x_0)\le 0$, $x_0\in\mathcal{U}$.
Apply strong maximum principle to $-u$)

· Regularity

<u>Thm</u>: $u\in C(\mathcal{U})$ satisfies the mean value property, then $u\in C^\infty(\mathcal{U})$

<u>Cor.</u> $u\in C^2(\mathcal{U})$ + MVP $\Rightarrow$ u harmonic

<u>Cor.</u> u harmonic $\Rightarrow u\in C^\infty$

Pf: Preparation: Given a function $\eta\in C_0^\infty$ (bounded support.) $\Rightarrow \eta_\varepsilon=\frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})\in C^\infty$. 陶哲轩的技巧 PDE中常用

$$u_\varepsilon = u\overset{\text{convolution}}{*}\eta_\varepsilon = \int_{\mathcal{U}} u(y)\,\underbrace{\eta_\varepsilon(x-y)}\,dy$$

then u_ε is C^∞,

this is because $\rightarrow$ bounded & bounded support & cont.

$\Rightarrow \partial u_\varepsilon = \int_{\mathcal{U}} u(y)\,\underbrace{\partial\eta_\varepsilon(x-y)}\,dy$ (By Real Analysis, week 10-11, 1.3)

$\Rightarrow \cdots$

$\Rightarrow \partial^k u_\varepsilon$ exists, $\partial^k u_\varepsilon=\int_{\mathcal{U}} u(y)\,\partial_x^k\eta_\varepsilon(x-y)\,dy$.

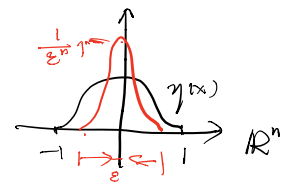
$\Rightarrow u_\varepsilon\in C^\infty(\mathcal{U})$

Pf. Let $\eta(x) = \eta(r)$ $(r = |x|) \in C^\infty(\mathbb{R}^n)$

(1) $\operatorname{supp} \eta \subseteq B_1(0)$

(2) $\eta \geq 0$

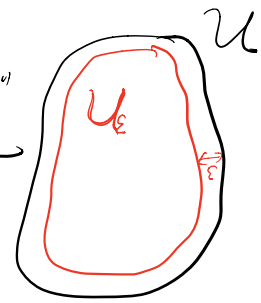
(3) $\int_{\mathbb{R}^n} \eta(x)\,dx = \int_{B_1(0)} \eta(x)\,dx = 1$



$\eta_\varepsilon(x) = \eta(\frac{x}{\varepsilon})\,\varepsilon^{-n}$

$\Rightarrow \int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = 1$

"$u_\varepsilon = u$ in $\mathcal{U}_\varepsilon = \{x \in \mathcal{U} \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$"



$\Rightarrow u \in C^\infty(\mathcal{U}_\varepsilon)$

$\varepsilon \to 0 \Rightarrow u \in C^\infty(\mathcal{U})$ $\square$

Proof of this:

For $x \in \mathcal{U}_\varepsilon$, $u_\varepsilon(x) = \int_{\mathcal{U}} u(y)\,\eta_\varepsilon(x-y)\,dy$

$$= \int_{\mathcal{U}} u(y) \frac{1}{\varepsilon^n} \eta\left(\frac{x-y}{\varepsilon}\right) dy \quad \left(\begin{array}{l} x \in \mathcal{U}_\varepsilon \\ B_\varepsilon(x) \subset \mathcal{U} \end{array}\right)$$

$$= \int_{B_\varepsilon(x)} u(y) \frac{1}{\varepsilon^n} \eta\left(\frac{x-y}{\varepsilon}\right) dy \quad \left(\begin{array}{l} \eta\left(\frac{x-y}{\varepsilon}\right) = 0 \\ \text{for } y \notin B_\varepsilon(x) \end{array}\right)$$

$z = \frac{x-y}{\varepsilon} \in B_1(0)$

$$= \int_{B_1(0)} u(x - \varepsilon z)\,\eta(z)\,dz$$

$$= \int_0^1 \left[\int_{\partial B_r(0)} u(x - \varepsilon r w)\,\eta(r)\,dw \right] dr$$

$$= \int_0^1 \eta(r)\,dr \left[\int_{\partial B_r(0)} u(x - \varepsilon r w)\,dw \right]$$

MVP.

$$= \int_0^1 \eta(r)\,dr\; u(x)\,\omega(n)\,r^{n-1}$$

$$= u(x)\,\omega(n) \underbrace{\int_0^1 r^{n-1} \eta(r)\,dr}_{\int_0^1 \eta(r) \left(\int_{\partial B_r(0)} 1\,dx \right) dr}$$

$= u(x)$

(mollification trick)

$\int_{B_1(0)} \eta(x)\,dx = 1$

$\square$

- Derivative estimates

<u>Thm</u>: u is harmonic in $\mathcal{U}$, then $|D^\alpha u(x)| \leq \dfrac{n^k e^{k-1} k!}{r^k} \max\limits_{y\in B_r(x)} |u(y)|$

$(\forall |\alpha| = k \geq 1)$

(Evans Book has proven that $|D^\alpha u(x)| \leq C \cdot \|u\|_{L^1(B_r(x))}$)

Pf: $k=1$ u harmonic $\Rightarrow$ u is smooth $\Rightarrow$ $\partial_i u$ smooth $\Rightarrow$ $\partial_i u$ harmonic ($\Delta(\partial_i u) = 0$.)

$$\Rightarrow |\partial_i u(x)| \underset{(MVP)}{=} \left| \fint_{B(x,r)} \partial_i u \, dx \right| \underset{(\text{divergence Thm})}{=} \frac{1}{\alpha(n)\cdot r^n} \left| \int_{\partial B(x,r)} u \cdot n_i \, dS \right|$$

n_i: unit outer normal of $\partial B(x,\frac{r}{s})$

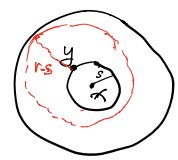
$$\leq \frac{1}{\alpha(n)\cdot r^n} \cdot \max_{B(x,r)} |u(y)| \cdot |\partial B(x,r)| \quad (\because |n_i| \leq |n| = 1)$$

$$= \frac{\omega(n)\cdot r^{n-1}}{\alpha(n)\cdot r^n} \cdot \max_{B(x,r)} |u(y)| = \frac{n}{r} \cdot \max_{B(x,r)} |u(y)|$$

Assume: $|\alpha| \leq k-1$ ✓

For $|\alpha| = k$, $\partial^\alpha u = \partial_i \partial^\beta u$, $|\beta| = k-1$

$$|D^\alpha u(x)| \leq \frac{n}{s} \cdot \max_{B(x,s)} |D^\beta u(y)|$$



$$\leq \frac{n}{s} \cdot \max_{B(x,s)} \left(\frac{n^{k-1} e^{k-2} (k-1)!}{(r-s)^{k-1}} \cdot \max_{B(y,r-s)} |u(z)| \right)$$

$$= \frac{n^k e^{k-2} (k-1)!}{s(r-s)^{k-1}} \left(\max_{B(x,s)} \cdot \max_{B(y,r-s)} |u(z)| \right)$$

$$\leq \frac{n^k \cdot e^{k-2} \cdot (k-1)!}{s(r-s)^{k-1}} \cdot \max_{B(x,r)} |u(y)|$$

$$\underset{\text{Let } s = \frac{r}{k}}{=} \frac{n^k \cdot e^{k-2} \cdot (k-1)!}{r^k \cdot \frac{1}{k} \cdot \left(\frac{k-1}{k}\right)^{k-1}} \cdot \max_{B(x,r)} |u(y)|$$

$$= \frac{n^k \cdot e^{k-2} \cdot k!}{r^k} \cdot \underbrace{\left(1 + \frac{1}{k-1}\right)^{k-1}}_{\leq e} \cdot \max_{B(x,r)} |u(y)|$$

$$\leq \frac{n^k \cdot e^{k-1} \cdot k!}{r^k} \cdot \max_{B(x,r)} |u(y)|$$

$\square$

- Liouville Theorem.

<u>Thm</u> Suppose u is a bounded harmonic function on $\mathbb{R}^n$, then $u \equiv$ const

Pf: $|Du(x)| \leq \dfrac{n}{r} \cdot \max\limits_{B(x,r)} |u(y)| \leq \dfrac{n}{r} \cdot M \longrightarrow 0 \quad (r \to \infty)$

$\Rightarrow u \equiv$ const. $\square$.

Remark: If u is upperly/lowerly bounded and harmonic, then $u \equiv \text{const}$

Pf: Hint: Using Harnack's inequality.

· Analyticity

<u>Thm</u>: u is harmonic in $\mathcal{U} \Longrightarrow u$ is analytic in $\mathcal{U}$

Pf: We need to prove:

For any $x_0 \in \mathcal{U}$, $\exists r > 0$, $\sum_{|\alpha|=0}^{N-1} \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha \overset{N\to+\infty}{\rightrightarrows} u(x)$ in $B(x_0, r)$

$$\Longleftrightarrow R_N(x) = u(x) - \sum_{|\alpha|=0}^{N-1} \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha \rightrightarrows 0 \quad (N \to +\infty)$$

$$R_N(x) = \left| \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x-x_0))}{\alpha!}(x-x_0)^\alpha \right| \qquad 0 \le t \le 1$$

$\downarrow$ Mean Value Thm

$$\le \sum_{|\alpha|=N} \frac{|h|^{|\alpha|}}{\alpha!} \cdot \frac{n^N e^{N-1} N!}{r_0^N} \cdot \underbrace{\max_{B(x_0+th, r_0)} |u(y)|}_{\le \max_{B(x_0, 2r_0)} |u(y)|}$$

Let $r_0 := \frac{1}{3} \cdot \mathrm{dist}(x_0, \partial\mathcal{U})$

$x_0 + t(x-x_0) \in B(x_0, r_0)$

$B(x_0 + th, r_0) \subset B(x_0, 2r_0)$

$\downarrow$ Let $h = x - x_0$, then by "derivative estimates"

$$\le \sum_{|\alpha|=N} \frac{r^N \cdot n^N e^{N-1} N!}{\alpha! \, r_0^N} \cdot \max_{B(x_0, 2r_0)} |u(y)|$$

$\downarrow$ Take $r < r_0$ and let $|h| \le r$

$$\left(\text{Observe the fact } \sum_{|\alpha|=N} \frac{N!}{\alpha!} = n^N = (\overbrace{1+1+\cdots+1}^{n})^N \right.$$

$$\left. \text{Calculate } \sum_{|\alpha|=N} \frac{N!}{\alpha!} x^\alpha = \cdots \text{ to prove} \right)$$

$$= \frac{r^N n^{2N} e^{N-1}}{r_0^N} \cdot \max_{B(x_0, 2r_0)} |u(y)|$$

$$= \left(\frac{r n^2 e}{r_0}\right)^N \cdot \frac{1}{e} \cdot \max_{B(x_0, 2r_0)} |u(y)|$$

$$< \frac{1}{2^N} \cdot \frac{1}{e} \cdot \max_{B(x_0, 2r_0)} |u(y)| \rightrightarrows 0 \quad (N \to +\infty)$$

$\downarrow$ (Take $r < \frac{1}{2} \frac{r_0}{n^2 e}$)

$\downarrow$ independent of the choice of $x \in B(x_0, r)$

$\square$.

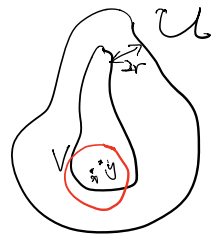
Harnack's inequality

<u>Thm</u> $u \geq 0$ harmonic in $\mathcal{U}$. Then for $\forall$ open and connected $V \subset\subset \mathcal{U}$ $\exists\, c = c(V) > 0$, such that $\sup_V u \leq c \cdot \inf_V u$

Remark: If $\mathcal{U}$ is connected, then V does not need to satisfy connectivity. Just take an open and connected V' s.t. $V \subset V' \subset\subset \mathcal{U}$, then consider the inequality on V'.

Pf: $r = \frac{1}{2}\operatorname{dist}(V, \partial\mathcal{U}) > 0$ $(V \subset\subset \mathcal{U})$

$\forall x, y \in V$, $|y - x| < r$. then:



$$u(x) = \fint_{B(x,2r)} u(z)\,dz$$

$$\geq \frac{\int_{B(y,r)} u(z)\,dz}{|B(x,2r)|} \geq \frac{u(y)\cdot|B(y,r)|}{|B(x,2r)|}$$

$(\because B(y,r) \subset B(x,2r))$

$$= \frac{1}{2^n} u(y)$$

For $\forall x, y \in V$, $\exists\, x_0 = x, x_1, x_2, \cdots, x_m = y$, $\operatorname{dist}(x_i, x_{i+1}) < r$.

$\Rightarrow u(x) \geq \frac{1}{2^{nm}} u(y)$ $\quad m = m(V, r) \to \because V \subset\subset \mathcal{U}$ $\therefore$ Such m has an upper bound, we take the upperbound to be this m, then

$\Rightarrow \inf_V u \geq \frac{1}{2^{nm}} \sup_V u$ $\quad$ m is only dependent of V, r

Cor: $u \leq M$ (or $u \geq -M$) then u is harmonic in $\mathbb{R}^n \Rightarrow u \equiv C$.

Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{U} \\ u = g & \text{on } \partial\mathcal{U} \end{cases}$$

Look for radial harmonic function in $\mathbb{R}^n$, i.e. $u(x) = v(r)$ $(r = |x|)$

$$\partial_i v = v'(r)\cdot\partial_i r = v'(r)\cdot\frac{x_i}{r} \qquad \partial_i r = \frac{x_i}{r}$$

$$\partial_i^2 v = \partial_i\left(v'(r)\partial_i r\right) = v''(r)(\partial_i r)^2 + v'(r)\cdot\partial_i^2 r$$

$$\partial_i^2 r = \partial_i\left(\frac{x_i}{r}\right) = \frac{r - x_i\frac{x_i}{r}}{r^2} = \frac{1}{r} - \frac{x_i^2}{r^3}$$

$$\Rightarrow \Delta u = \sum_{i=1}^n \left[v''(r)\cdot\left(\frac{x_i}{r}\right)^2 + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right]$$

$$= v''(r) + v'(r)\cdot\frac{n-1}{r}$$

$$\Delta u = 0 \iff v''(r) + v'(r)\cdot\frac{n-1}{r} = 0.$$

$$(r^{n-1}v')' = (n-1)r^{n-2}v' + r^{n-1}v'' = r^{n-1}\left(r'' + v'\cdot\frac{n-1}{r}\right) = 0.$$

$$\Rightarrow r^{n-1}v' = a \qquad v' = \frac{a}{r^{n-1}} \quad (a \text{ is const})$$

$$v(r) = \begin{cases} b\log r + c & (n=2) \\ \frac{b}{r^{n-2}} + c & (n\geq 3) \end{cases} \qquad b, c \text{ are constants.}$$

<u>Def</u> The function $\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & n\geq 3 \end{cases} \qquad x\in\mathbb{R}^n\setminus\{0\}$

is called the fundamental solution of Laplace's equation.

Remark: • $\Delta\Phi = 0$ for $x\in\mathbb{R}^n\setminus\{0\}$ (harmonic except 0)

• $D\Phi(x) = -\frac{1}{n\alpha(n)}\left(\frac{x_1}{|x|^n}, \cdots, \frac{x_n}{|x|^n}\right) = -\frac{1}{n\alpha(n)}\frac{\vec{n}}{|x|^{n-1}}$
$= -\frac{1}{\omega(n)}\cdot\frac{1}{|x|^{n-1}}\cdot\vec{n}$

• $|D\Phi(x)| \leq \frac{c}{|x|^{n-1}}$, $|D^2\Phi(x)| \leq \frac{c}{|x|^n}$ $(x\neq 0)$

• $\int_{\partial\Omega} \partial_n\Phi \, dS(x) = \begin{cases} -1 & 0\in\Omega \\ 0 & 0\notin\Omega. \end{cases}$

基本解的法向导数.

Pf: $0\notin\Omega$: $0 = \int_\Omega \Delta\Phi \, dx = \int_{\partial\Omega} \partial_n\Phi \, dS(x)$

$0\in\Omega$: $0 = \int_{\Omega\setminus B_\varepsilon(0)} \Delta\Phi \, dx = \int_{\partial\Omega} \partial_n\Phi \, dS(x) - \int_{\partial B_\varepsilon(0)} \partial_n\Phi \, dS(x)$

$$= \int_{\partial\Omega} \partial_n\Phi \, dS(x) + 1$$

$$\int_{\partial B_\varepsilon(0)} \partial_n\Phi \, dS(x) = \int_{\partial B_\varepsilon(0)} D\Phi\cdot\vec{n} \, dS(x)$$
$$= -\frac{1}{n\alpha(n)}\frac{1}{\varepsilon^{n-1}}\int_{\partial B_\varepsilon(0)} 1 \, dS(x).$$
$$= -\frac{\omega(n)\cdot\varepsilon^{n-1}}{n\alpha(n)\cdot\varepsilon^{n-1}}$$
$$= -1$$

$-f*\Delta\Phi = f*\delta_0(x) \Leftrightarrow -\Delta u(x) = f(x).$

(In fact, $-\Delta\Phi = \delta_0(x)$ is the Dirac' measure on $\mathbb{R}^n$ giving unit mass to the point 0.
i.e. $\int_{\mathbb{R}^n} \Phi(x)(-\Delta g(x))\,dx = g(0)$. $\forall g\in C^\infty$)

· Let $g(y) = f(x-y)$ (x is given)

$$\int_{\mathbb{R}^n} \Phi(y)(-\Delta g(y))\,dy = f(x)$$

$$\begin{aligned}
-\Delta_x\left(\int_{\mathbb{R}^n} \Phi(y) f(x-y)\,dy\right) &= \int_{\mathbb{R}^n} \Phi(y)(-\Delta_x f(x-y))\,dy \\
&= \int_{\mathbb{R}^n} \Phi(y)(-\Delta_y f(x-y))\,dy \\
&= \int_{\mathbb{R}^n} \Phi(y)(-\Delta g(y))\,dy \\
&= f(x).
\end{aligned}$$

求2阶导时2个负号抵消，因此 Δ_x 与 Δ_y 相同

i.e. $u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,dy$ is a solution to Poisson's equation.

<u>Thm</u>: $f \in C_c^2(\mathbb{R}^n)$. Let $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy$, then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$.

Pf: ① First, we prove that $u \in C^2(\mathbb{R}^n)$

$$u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,dy$$

$$\frac{u(x+\varepsilon e_i) - u(x)}{\varepsilon} = \int_{\mathbb{R}^n} \Phi(y) \underbrace{\frac{f(x+\varepsilon e_i - y) - f(x-y)}{\varepsilon}}_{\varepsilon \to 0 \;\Downarrow\; \partial_i f(x-y) \text{ uniformly in } \mathbb{R}^n.}\,dy$$

这是因为 $f \in C_c^2(\mathbb{R}^n)$，对 f Taylor 展开，结合 f 是紧支集的，以及其2阶导的连续性，
$\Rightarrow$ 二阶导有界，于是收敛是一致的.

$$\Rightarrow \partial_i u := \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon e_i) - u(x)}{\varepsilon}$$

$$= \int_{\mathbb{R}^n} \Phi(y) \cdot \partial_i f(x-y)\,dy \quad \text{is continuous.}$$

Similarly $\Rightarrow$ $\partial_i \partial_j u = \int_{\mathbb{R}^n} \Phi(y)\, \partial_i \partial_j f(x-y)\,dy$ is continuous

i.e. $u \in C^2(\mathbb{R}^n)$.

② Fix $\varepsilon > 0$, $\Delta_x u(x) = \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y)\,dy$.

$$= \underbrace{\int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\,dy}_{\leq I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\,dy}_{\leq J_\varepsilon}.$$

$$|I_\varepsilon| \le M \int_{B(0,\varepsilon)} |\Phi(y)|\,dy \le \begin{cases} c\cdot\varepsilon^2|\ln\varepsilon| & n=2 \\ c\cdot\varepsilon^2 & n\ge 3 \end{cases} \to 0, \text{ as } \varepsilon\to 0$$

calculate directly $= \int_0^\varepsilon |\partial B(0,r)|\cdot|\Phi(r)|\,dr$

$\because f\in C^2$

$\therefore \Delta_x f(x-y)$ uniformly bounded by M in $B(0,\varepsilon)$

Take $R>0$. s.t. $\operatorname{supp} f \subset\subset B(0,R)$

Then $J_\varepsilon = \int_{B(0,R)\setminus B(0,\varepsilon)} \Phi(y)\,\Delta_y f(x-y)\,dy.$

(**Lemma** ☆ Let $u,v\in C^2(\Omega)\cap C^1(\overline{\Omega})$, then

$$\int_\Omega \underbrace{(u\Delta v - v\Delta u)}_{\substack{\| \\ \nabla\cdot(u\nabla v - v\nabla u)}}\,dx = \int_{\partial\Omega} \underbrace{\left(u\cdot\frac{\partial v}{\partial n} - v\cdot\frac{\partial u}{\partial n}\right)}_{\substack{\| \\ (u\nabla v - v\nabla u)\cdot\vec{n}}}\,ds(x)$$

n: the unit outer normal

Pf:)

By the lemma $\Rightarrow$ $J_\varepsilon = \int_{B(0,R)\setminus B(0,\varepsilon)} f(x-y)\,\underbrace{\Delta_y\Phi(y)}_{=0}\,dy.$

$$- \int_{\partial B(0,\varepsilon)} \Big[\underbrace{\Phi(y)\frac{\partial f}{\partial n}(x-y)}_{\int \Phi(y)\frac{\partial f}{\partial n}(x-y)\triangleq K_\varepsilon} - \underbrace{f(x-y)\frac{\partial}{\partial n}\Phi(y)}_{\int f(x-y)\frac{\partial}{\partial n}\Phi(y)\triangleq L_\varepsilon}\Big]\,ds(y)$$

$\because f\equiv 0$ on $\partial B(0,R)$

$$|K_\varepsilon| \le M\int_{\partial B(0,\varepsilon)} |\Phi(y)|\,ds(y) = \begin{cases} c\,\varepsilon\log\varepsilon & n=2 \\ c\,\varepsilon & n\ge 3 \end{cases} \to 0$$

as $\varepsilon\to 0$.

$$L_\varepsilon = \int_{\partial B(0,\varepsilon)} -\frac{1}{n\alpha(n)\,\varepsilon^{n-1}}\cdot f(x-y)\,ds(y)$$

$$\underset{y=\varepsilon z}{=} -\frac{1}{n\cdot\alpha(n)}\int_{\partial B(0,1)} f(x-\varepsilon z)\,ds(z)$$

$$\to -\frac{1}{n\cdot\alpha(n)}\int_{\partial B(0,1)} f(x)\,ds(z) = -f(x)$$

as. $\varepsilon\to 0$.

$\therefore -\Delta_x u(x) = f(x).$ $\square$

Remark: The requirements for f can be weakened.

Above thm is valid under far less stringent smoothness requirements for f (see Gilbarg–Trudinger)

Above we discuss the solution on $\mathbb{R}^n$.
Now we are going to find the solution to Dirichlet problem on bounded region.

Green's function

Consider the Dirichlet problem: $\begin{cases} -\Delta u = f & \text{in } \mathcal{U} \\ u = g & \text{on } \partial\mathcal{U} \end{cases}$ (D)

$v = \Phi(y-x)$ in Lemma ☆, $\Omega = \mathcal{U} \setminus B(0,\varepsilon)$, Then:

$$\int_{\mathcal{U}\setminus B(0,\varepsilon)} \Big[u(y) \underbrace{\Delta \Phi(y-x)}_{=0} - \Phi(y-x) \Delta u(y) \Big] dy$$

$$= \int_{\partial\mathcal{U}} \Big[u(y) \cdot \frac{\partial}{\partial n} \Phi(y-x) - \Phi(y-x) \cdot \frac{\partial}{\partial n} u(y) \Big] ds(y).$$

$$- \int_{\partial B(0,\varepsilon)} \Big[\underbrace{u(y) \cdot \frac{\partial}{\partial n} \Phi(y-x)}_{\to -u(x)} - \underbrace{\Phi(y-x) \cdot \frac{\partial}{\partial n} u(y)}_{\to 0} \Big] ds(y)$$

(if u is a solution to (D) then $u \in C^2$.)

Follow the steps as the manipulation of $K_\varepsilon, L_\varepsilon$

$$\Rightarrow u(x) = \int_{\partial\mathcal{U}} \Big[\Phi(y-x) \cdot \frac{\partial}{\partial n} u(y) - \underbrace{u(y)}_{=g(y)} \frac{\partial}{\partial n} \Phi(y-x) \Big] ds(y).$$

$$- \int_{\mathcal{U}} \Phi(y-x) \underbrace{\Delta u(y)}_{=-f(y)} dy. \qquad (A)$$

However, $\frac{\partial u}{\partial n}\Big|_{\partial\mathcal{U}}$ is unknown. We add a harmonic function:

$$\begin{cases} -\Delta \phi^x(y) = 0 & \text{in } \mathcal{U} \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial\mathcal{U}. \end{cases}$$

Then $\int_{\mathcal{U}} \Big[u(y) \underbrace{\Delta \phi^x(y)}_{=0} - \phi^x(y) \underbrace{\Delta u}_{=-f} \Big] dy = \int_{\partial\mathcal{U}} \Big(\underbrace{u(y)}_{=g(y)} \frac{\partial}{\partial n} \phi^x(y) - \phi^x(y) \frac{\partial}{\partial n} u(y) \Big) ds(y)$ (B)

Let $G(x,y) = \Phi(y-x) - \phi^x(y)$, then

(A)+(B) $\Rightarrow$ $u(x) = \underbrace{-\int_{\partial\mathcal{U}} g(y) \frac{\partial G}{\partial n}(x-y)\, ds(y) + \int_{\mathcal{U}} G(x-y) f(y)\, dy}_{(*)}$

(To cancel out the two $=$ part)

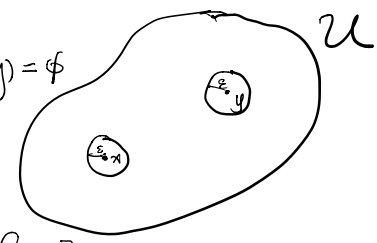
Def: $G(x,y) = \Phi(y-x) - \phi^x(y) \quad (x, y \in \mathcal{U},\ x \neq y)$
is called the Green's function for the region $\mathcal{U}$.

Thm: If $u \in C^2(\overline{\mathcal{U}})$ solves the problem (D),
then u is given by $(*)$ (由上段讨论可).

Remark: $\begin{cases} -\Delta G(x,y) = \delta_x(y) & y \in \mathcal{U}. \\ G(x,y) = 0 & y \in \partial\mathcal{U}. \end{cases}$

Thm: $G(x,y) = G(y,x) \quad (\forall x, y \in \mathcal{U},\ x \neq y)$

Pf: $G_1(z) = G(x,z)$, $G_2(z) = G(y,z)$
We prove: $G_1(y) = G_2(x)$



Take $\varepsilon > 0$, $B_\varepsilon(x), B_\varepsilon(y) \subset \Omega$, $B_\varepsilon(x) \cap B_\varepsilon(y) = \phi$

$$\because \int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds(x)$$

Take $\Omega = \mathcal{U} \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))$ $\quad u = G_1(z)$, $\quad v = G_2(z)$.

$$\int_{\mathcal{U}\setminus(B_\varepsilon(x)\cup B_\varepsilon(y))} (G_1 \cdot \underbrace{\cancel{\Delta G_2}}_{=0} - G_2 \cdot \underbrace{\cancel{\Delta G_1}}_{=0})\,dx = \int_{\partial\mathcal{U}} \left(\underbrace{\cancel{G_1}}_{=0} \frac{\partial G_2}{\partial n} - \underbrace{\cancel{G_2}}_{=0} \frac{\partial G_1}{\partial n}\right) ds(z)$$

$$- \int_{\partial B_\varepsilon(x)} \left(\underbrace{G_1 \frac{\partial G_2}{\partial n}}_{①} - \underbrace{G_2 \frac{\partial G_1}{\partial n}}_{②}\right) ds(z)$$

$$- \int_{\partial B_\varepsilon(y)} \left(\underbrace{G_1 \frac{\partial G_2}{\partial n}}_{③} - \underbrace{G_2 \frac{\partial G_1}{\partial n}}_{④}\right) ds(z)$$

①: $$\left| \int_{\partial B_\varepsilon(x)} G_1 \cdot \frac{\partial G_2}{\partial n}\,ds(z) \right| \leq \int_{\partial B_\varepsilon(x)} |G_1| \cdot \underbrace{\left|\frac{\partial G_2}{\partial n}\right|}_{\leq C} ds(z).$$

$\Rightarrow$ G_2 反常性在 y 附近, $\therefore$ 在 $\partial B_\varepsilon(x)$ 上是调和的, 可微性空间.

$$\leq C \cdot \int_{\partial B_\varepsilon(x)} \left| \underbrace{\Phi(z-x)}_{\leq \frac{C_1}{|z-x|^{n-2}}} - \underbrace{\phi^x(z)}_{\leq C_2} \right| ds(z)$$

$(n \geq 3)$ $\quad$ [For $n=2$, $C(C_1|\ln\varepsilon| + C_2)\varepsilon \to 0$.]

$$\leq C\left(\frac{C_1}{\varepsilon^{n-2}} + C_2\right) \cdot \varepsilon^{n-1} \cdot C_3 \longrightarrow 0 \quad (\text{as } \varepsilon \to 0).$$

②: $$\int_{\partial B_\varepsilon(x)} G_2 \frac{\partial G_1}{\partial n}\,ds(z) = \int_{\partial B_\varepsilon(x)} G_2(z) \left(\frac{1}{n\alpha_n \varepsilon^{n-1}} - \phi^x(z)\right) ds(z)$$

$$\left(G_1 = \Phi(z-x) - \phi^x(z),\ \frac{\partial \Phi}{\partial n}(z-x) = n \cdot \nabla\Phi(z-x) = \frac{1}{n\alpha_n |z-x|^{n-1}} \right)$$

$$= \int_{\partial B_\varepsilon(x)} G_2(z) \frac{1}{n\alpha_n \varepsilon^{n-1}} \, ds(z) + O(\varepsilon).$$

$$\downarrow$$

$$\left(\because \int_{\partial B_\varepsilon(x)} G_2(z) \phi^x(z) \, ds(z) \to 0 \text{ as } \varepsilon \to 0. \right)$$

$$\underset{\text{MVP}}{=} G_2(x) + O(\varepsilon) \longrightarrow G_2(x) \quad \text{as } \varepsilon \to 0.$$

Similarly, we have ③ $\to G_1(y)$ ④ $\to 0$

$\Rightarrow G_2(x) = G_1(y)$

$\square$

Other properties of Green's function.

- $\Delta_y G(x,y) = 0 \quad y \neq x \quad y \in \Omega.$

  $y \to x \quad G(x,y) \to +\infty \xrightarrow[\text{order}]{} \begin{cases} \sim \frac{1}{|x-y|^{n-2}} & n \geq 3 \\ \sim \left| \ln |x-y| \right| & n = 2 \end{cases}$

- For any region $V \subset \mathcal{U}$

$$\int_{\partial V} \frac{\partial G(x,y)}{\partial n} \, ds(y) \begin{cases} = -1 & x \in V \\ = 0 & x \notin V \end{cases}$$

$$G(x,y) = \Phi(x,y) - \phi^x(y)$$

$$\begin{cases} \Delta \phi^x(y) = 0. & y \in \mathcal{U} \\ \phi^x(y) = \Phi(x-y) & y \in \partial \mathcal{U} \end{cases}$$

Next, we discuss how to build Green's function explicitly on some special region $\mathcal{U}$.

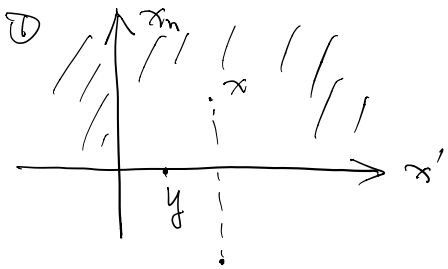
① $\mathcal{U} = \mathbb{R}^n_+ = \{ (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0 \}$

Denote $x' := (x_1, x_2, \cdots, x_{n-1})$

$x := (x_1, x_2, \cdots, x_{n-1}, x_n)$

②. $\mathcal{U} = B_R(0).$

①



$\tilde{x} = (x_1, \cdots, x_{n-1}, -x_n)$

Reflection

$\Delta \Phi(\tilde{x} - y) = 0$. $\quad y \neq \tilde{x}$

$|x-y| = |\tilde{x}-y| \quad y \in \partial \mathbb{R}^n_+$.

$\Rightarrow \Phi(x-y) = \Phi(\tilde{x}-y)$

$$\begin{cases} -\Delta \Phi(\tilde{x}-y) = 0 & y \in \mathbb{R}^n_+ \\ \Phi(\tilde{x}-y) = \Phi(x-y) & y \in \partial \mathbb{R}^n_+ \end{cases}$$

$\Rightarrow \Phi(\tilde{x}-y) = \phi^x(y)$

$\Rightarrow G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$

$(x, y \in \mathbb{R}^n_+)$ is the Green's function for $\mathbb{R}^n_+$

n=2 Φ的特殊不一样

$$\frac{\partial G}{\partial n}(x,y)\Big|_{y \in \partial \mathbb{R}^n_+} = -\partial_{y_n} G(x,y) = -\left[-\frac{1}{n\alpha(n)}\left(\frac{y_n - x_n}{|y-x|^n} - \frac{y_n - \tilde{x}_n}{|y-\tilde{x}|^n}\right)\right].$$

$$= -\frac{1}{n\alpha(n)} \cdot \frac{2x_n}{|x-y|^n}$$

$x_n = -\tilde{x}_n$

$y \in \partial \mathbb{R}^n_+ \Rightarrow |y-x| = |\tilde{y}-x|$.

$\Rightarrow$ The solution of the boundary-value problem $\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial \mathbb{R}^n_+ \end{cases}$

can be written as

$$u(x) = \frac{2x_n}{n\alpha(n)} \cdot \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|y-x|^n}\, dy \quad (\forall x \in \mathbb{R}^n_+). \qquad (*)$$

↳ Poisson formula.

$K(x,y) = \frac{2x_n}{n\alpha(n)} \cdot \frac{1}{|x-y|^n}$ is called Poisson's Kernel.

Remark: $\int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy \to +\infty$ as $x_n \to 0$.

$\sim \int_{\mathbb{R}^{n-1}} \frac{1}{|y|^n}\, dy \to +\infty$.

So $\lim_{x_n \to 0^+} u(x) = g(x', 0)$ is not obvious, We need a following thm.

刚刚证了符合成u这个解了，

还没证这样的u就是解.

We know solution can be written as $(*)$.

Now we should check $(*)$ provide us with a solution of the boundary-value problem.

Thm: Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, u is given as above. Then:

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ (u is a classical solution)

(ii) $\Delta u = 0$ in $\mathbb{R}^n_+$ (satisfy the problem)

(iii) $\lim_{x_n \to 0^+} u(x_1, x_2, \cdots, x_n) = g(x_1, x_2, \cdots, x_{n-1}, 0)$ $\left(\lim_{\substack{x \to x_0 \\ x_0 \in \partial \mathbb{R}^n_+}} u(x) = g(x_0)\right)$

(satisfy boundary condition)

Pf: (i) A direct calculation verifies: $\forall x \in \mathbb{R}^n_+$, $\int_{\partial \mathbb{R}^n_+} K(x,y)\,dy = 1$.

$$\int_{\mathbb{R}^{n-1}} K(x,y)\,dy = \int_0^\infty \omega(n-1)\cdot r^{n-2}\,\frac{2x_n}{n\alpha(n)}\cdot\frac{1}{(x_n^2+r^2)^{\frac{n}{2}}}\,dr$$

$$= 2\cdot\frac{(n-1)\alpha(n-1)}{n\cdot\alpha(n)}\cdot x_n\int_0^\infty \frac{1}{(x_n^2+r^2)^{\frac{n}{2}}}\,r^{n-2}\,dr$$

$$= 2\cdot\frac{(n-1)\alpha(n-1)}{n\,\alpha(n)}\int_0^\infty \frac{1}{\left(\left(\frac{r}{x_n}\right)^2+1\right)^{\frac{n}{2}}}\left(\frac{r}{x_n}\right)^{n-2}\,d\frac{r}{x_n}$$

$$= 2\cdot\frac{(n-1)\alpha(n-1)}{n\alpha(n)}\int_0^\infty \frac{1}{(t^2+1)^{\frac{n}{2}}}\,t^{n-2}\,dt$$

$$= \frac{(n-1)\alpha(n-1)}{n\alpha(n)}\int_0^\infty \frac{1}{(y+1)^{\frac{n}{2}}}\cdot y^{\frac{n-3}{2}}\,dy$$

$$= \frac{(n-1)\alpha(n-1)}{n\alpha(n)}\cdot B\left(\frac{n-1}{2},\frac{1}{2}\right)$$

$$\because\ \alpha(n) = 2\alpha(n-1)\cdot\int_0^1 (1-t^2)^{\frac{n-1}{2}}\,dt$$

$$= 2\alpha(n-1)\cdot\int_0^1 \xi^{-\frac{1}{2}}\cdot(1-\xi)^{\frac{n-1}{2}}\,d\xi$$

$$= \alpha(n-1)\cdot B\left(\frac{1}{2},\frac{n+1}{2}\right)$$

$$\Rightarrow \int_{\mathbb{R}^{n-1}} K(x,y)\,dy = \frac{n-1}{n}\cdot\frac{\Gamma\left(\frac{n+2}{2}\right)}{\cancel{\Gamma\left(\frac{1}{2}\right)}\cdot\Gamma\left(\frac{n+1}{2}\right)}\cdot\frac{\Gamma\left(\frac{n-1}{2}\right)\cdot\cancel{\Gamma\left(\frac{1}{2}\right)}}{\Gamma\left(\frac{n}{2}\right)}$$

$$= 1.$$

As $g \in L^\infty(\mathbb{R}^{n-1})$, assume $|g| \le M$, for some $M > 0$.

$$\Rightarrow |u(x)| = \int_{\mathbb{R}^{n-1}} K(x,y)\cdot|g(y)|\,dy \le M\cdot\int_{\mathbb{R}^{n-1}} K(x,y)\,dy = M$$

Moreover, $K(x,y)$ is smooth in x $\Rightarrow$ $u \in C^\infty(\mathbb{R}^n_+)$

$$\Rightarrow |\partial_x^k u(x)| = \left|\int \partial_x^k K(x,y)\,g(y)\,dy\right| \le M\cdot\int |\partial_x^k K(x,y)|\,dy < +\infty.$$

g is bounded.

(By Real Analysis Notes Week 10-11 Part 1.3

$\because$ $|K(x,y)\,g(y)| \le M\,K(x,y) \in L^1(\mathbb{R}^{n-1})$.

$|\partial_x K(x,y)\,g(y)| \le M\,\partial_x K(x,y) \in L^1$

$\therefore$ $\partial_x u(x) = \int \partial_x K(x,y)\,g(y)\,dy$.

Induction on k. $\Rightarrow$ $\partial_x^k u(x) < \infty$.)

(ii) $K(x,y) = -\frac{\partial}{\partial y_n} G(x,y)$

$\Delta_x K(x,y) = -\frac{\partial}{\partial y_n} \Delta_x G(x,y) = 0.$

$\Delta_x K(x,y) = 0.$

$\Rightarrow \Delta_x u = \int_{\mathbb{R}^{n-1}} \Delta_x K(x,y) \cdot g(y)\, dy = 0.$

(iii) We need to prove that: Fix $x_0 \in \partial \mathbb{R}^n_+$

For $\forall \varepsilon > 0$, $\exists \delta > 0$, such that: If $|x - x_0| < \delta$, $x_n > 0$,

then $|u(x) - g(x_0)| < \varepsilon$

- We choose $\delta_1 > 0$, s.t. $|y - x_0| < \delta_1 \Rightarrow |g(y) - g(x_0)| < \frac{\varepsilon}{2}$
- When $|x - x_0| < \frac{\delta_1}{2}$

$$|u(x) - g(x_0)| = \left| \int_{\mathbb{R}^{n-1}} \left(K(x,y) g(y) - K(x,y) g(x_0) \right) dy \right|$$

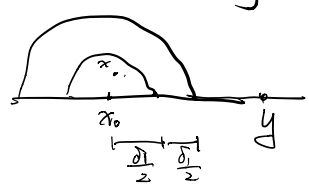
$$\leq \left(\int_{|y-x_0| \leq \delta_1} + \int_{|y-x_0| > \delta_1} \right) |K(x,y)| \cdot |g(y) - g(x_0)|\, dy$$

$$< \frac{\varepsilon}{2} \cdot 1 + \underbrace{2M \int_{|y-x_0| > \delta_1} |K(x,y)|\, dy}_{=: J}$$

when $|y - x_0| > \delta_1$,

$$|K(x,y)| = \frac{2x_n}{n\alpha_n |x-y|^n}$$

$$\leq \frac{2x_n}{n\alpha_n \left(\frac{1}{2}|y-x_0|\right)^n}$$



$|y - x| > \frac{1}{2}|y - x_0|$

$\left(\because |y-x| \geq |y-x_0| - |x-x_0| \geq |y-x_0| - \frac{\delta_1}{2} \geq \frac{|y-x_0|}{2} \right)$

$$\Rightarrow \int_{|y-x_0| \geq \delta_1} |K(x,y)|\, dy$$

$$\leq \int_{|y-x_0| \geq \delta_1} \frac{2^{n+1} x_n}{n\alpha_n |y-x_0|^n}\, dy$$

$$= \frac{2^{n+1} x_n}{n\alpha_n} \underbrace{\int_{|y-x_0| \geq \delta_1} \frac{1}{|y-x_0|^n}\, dy}_{= c(\delta_1)} = \frac{c(\delta_1) \cdot 2^{n+1} x_n}{n\alpha_n}$$

So taking $\delta < \frac{n\alpha_n}{2^{n+3} M c(\delta_1)} \varepsilon$, we have $J < \frac{\varepsilon}{2}$.

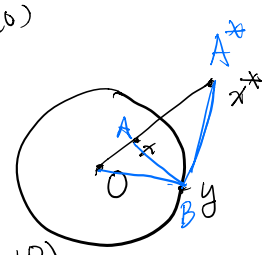
$\delta = \min\left\{ \frac{\delta_1}{2}, \frac{n\alpha_n \varepsilon}{2^{n+3} M \cdot c(\delta_1)} \right\}$, thus $|u(x) - g(x_0)| < \varepsilon$ when $|x - x_0| < \delta$

□

Remark: "Green函数定义的解满足边界条件", 后面会遇到很多此类型结论, 处理手段不同以, 拆为离 x_0 近的 (利用连续性估计) + 离 x_0 远的 (利用其积出来是有界的 x_n, Let $x_n \to 0$)

② $\mathcal{U} = B_R(0)$

$R=1$. We consider $\begin{cases} -\Delta \phi^x(y) = 0 & y \in B_1(0). \\ \phi^x(y) = \Phi(y-x) & y \in \partial B_1(0) \end{cases}$



$x^* = \dfrac{x}{|x|^2}$ $\longrightarrow$ inversion point of x through the unit sphere.

$\Phi(x^* - y)$ is harmonic (of y) in $B_1(0)$.

Thus, $\Phi(|x|(x^*-y))$ is also harmonic in $B_1(0)$

When $y \in \partial B_1(0)$, $|x| \cdot |x^* - y| = |OA| \cdot |A^*B| = |OB| \cdot |AB| = |y| \cdot |x-y| = |x-y|$

$\left(\triangle OAB \sim \triangle OBA^* \Rightarrow \dfrac{|OA|}{|OB|} = \dfrac{|AB|}{|A^*B|} \right)$

$\Rightarrow \Phi(|x|(x^*-y)) = \Phi(x-y) \quad (y \in \partial B_1(0))$

$\Rightarrow G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)) \qquad (x, y \in B_1(0),\ x \neq y)$

$$\frac{\partial G}{\partial n}(x,y)\Big|_{|y|=1} = y \cdot \nabla_y G(x,y)\Big|_{|y|=1} = \frac{y \cdot (x-y)}{n\alpha_n |x-y|^n} - \frac{|x|^2 y \cdot (x^*-y)}{n\alpha_n |x|\,|(y-x^*)|^n}$$

$$= \frac{|x|^2 - 1}{n\alpha_n |x-y|^n}$$

Therefore, the solution of $\begin{cases} -\Delta u = 0 & \text{in } B_1(0) \\ u = g & \text{on } \partial B_1(0) \end{cases}$

can be written as $u(x) = \displaystyle\int_{\partial B_1(0)} \frac{(1-|x|^2)\, g(y)}{n\alpha_n |x-y|^n}\, dy.$

$\begin{cases} \Delta u = 0 & \text{in } B_R(0) \\ u = g & \text{on } \partial B_R(0) \end{cases}$

Take $\tilde{u}(x) = u(Rx)$ solves above problem, $\tilde{g}(x) = g(Rx)$ replacing g.

By changing variables to obtain following:

For the ball $B_R(0)$, we have:

$$u(x) = \frac{(R^2 - |x|^2)}{n\alpha_n R} \int_{\partial B_R(0)} \frac{g(y)}{|x-y|^n}\, dy \rightarrow \text{Poisson's formula.}$$

$K(x,y) = \dfrac{R^2 - |x|^2}{n\alpha_n R} \cdot \dfrac{1}{|x-y|^n}$ is called Poisson's kernel for the ball $B_R(0)$

<u>Thm</u>: Assume $g \in C(\partial B_R(0))$ and u is given as above. Then:

(i) $u \in C^\infty(B_R(0))$

(ii) $\Delta u = 0$ in $B_R(0)$

(iii) $\displaystyle\lim_{\substack{x \to x_0 \\ x \in B_R(0)}} u(x) = g(x_0)$ for $\forall x_0 \in \partial B_R(0)$

Pf: Omitted. (Similar to that for the above thm)

• Energy methods → 用于2个特殊区域，均利用了对称、反演这种有特殊性

Powerful 可用于对一般的椭圆型方程组

乘上些测试函数，考虑 L^2 - estimates

Basic idea: multiply test functions and consider L^2 estimates

（有些也会考虑 L^p - estimates

这里是能量估计，而像素级估计，

逐点估计需本得到能量估计后

用其它方法再去做。）

• Uniqueness $\begin{cases} -\Delta u = f & \text{in } \mathcal{U} \\ u = g & \text{on } \partial\mathcal{U}. \end{cases}$ $\mathcal{U}$ is open and bounded. $\partial\mathcal{U} \in C^1$.

(Note that we have already employed the maximum principle to show uniqueness (in "Maximum Principle's Remark.))

Pf: Assume u_1, u_2 are 2 solutions, $w = u_1 - u_2$

$$\Rightarrow \begin{cases} -\Delta w = 0 & \text{in } \mathcal{U} \\ w = 0 & \text{on } \partial\mathcal{U} \end{cases}$$

u_1, u_2 are classic solutions $\Rightarrow w \in C^2$

$$0 = \int_{\mathcal{U}} (-\Delta w)\, w \, dx = \int_{\mathcal{U}} \Big(-\underbrace{\nabla\cdot}_{\text{div}} \big(w \underbrace{\nabla w}_{\text{gradient}} \big) + \underbrace{|\nabla w|^2}_{\text{gradient}} \Big) dx$$

integration by part

Gauss-Green Formula

$$= -\int_{\partial\mathcal{U}} w \frac{\partial w}{\partial n} \, ds(x) + \int_{\mathcal{U}} |\nabla w|^2 \, dx$$

$$\overset{w=0 \text{ on } \partial\mathcal{U}}{=} \int_{\mathcal{U}} |\nabla w|^2 dx$$

$$\Rightarrow \nabla w \equiv 0 \text{ in } \mathcal{U} \Rightarrow w \equiv 0 \text{ in } \mathcal{U}.$$

• Dirichlet's Principle

Consider $I[w] = \int_{\mathcal{U}} \left(\frac{1}{2} |\nabla w|^2 - wf \right) dx$ $\quad f \in C(\overline{\mathcal{U}})$

where $w \in \mathcal{A} = \{ w \in C^2(\overline{\mathcal{U}}) \mid w = g \text{ on } \partial\mathcal{U} \}$.

Energy functional

admissible set

Thm: $u \in \mathcal{A}$, $I[u] = \min_{w \in \mathcal{A}} I[w] \iff u$ solves the boundary-value problem $\begin{cases} -\Delta u = f & \text{in } \mathcal{U} \\ u = g & \text{on } \partial\mathcal{U} \end{cases}$

Remark: The PDE $\Longleftrightarrow$ The minimizing problem.

① $I[w] \geq -c$ (have a lower bound).

② Take $\{w_k\}$ s.t. $I[w_k] \longrightarrow \inf_{w \in \mathcal{A}} I[w]$

③ Prove $\{w_k\}$ has a limit (up to a subsequence)

Assume $w_{i_k} \to w_0$.

④ Prove $w_0 \in \mathcal{A}$ (compactness of $\mathcal{A}$).

⑤ Prove: $I[w_0] \leq \varliminf_{i \to +\infty} [w_i]$ lower semi-continuous of I

$$\Longrightarrow I[w_0] = \inf_{w \in \mathcal{A}} I[w]$$

A general method to find a solution for a PDE with a <u>variational structure</u>.

变分结构.

称可以找到这样的能量泛函的PDE为"具有变分结构的". 对于这种PDE的解他质有比较成熟的体系. —— Evans §8!

Pf: "$\Longrightarrow$" For any $v \in C_c^\infty(\mathcal{U})$

Let $p(t) = I[u+tv] - I[u]$ $\left(\begin{array}{l} \because \forall t \quad p(t) \geq p(0) = 0. \\ \therefore p'(0) = 0 \end{array} \right.$ $\Big)$

$p(0) = 0 \quad p'(0) = 0.$

$$p(t) = \int_{\mathcal{U}} \frac{1}{2} |\nabla(u+tv)|^2 - (u+tv)f - \left(\frac{1}{2}|\nabla u|^2 - uf\right) dx$$

$$= t\int_{\mathcal{U}} (\nabla u \cdot \nabla v - vf)\, dx + \frac{t^2}{2}\int_{\mathcal{U}} |\nabla v|^2 dx$$

$$p'(0) = \int_{\mathcal{U}} (\nabla u \cdot \nabla v - vf)\, dx = 0.$$

$\Vert \to$ integration by part

$$\int_{\mathcal{U}} (-\Delta u - f) v\, dx + \underbrace{\int_{\mathcal{U}} \nabla \cdot (v \nabla u)\, dx}_{\int_{\partial \mathcal{U}} \partial_n u \cdot v\, ds(x) = 0}$$

$\because v \in C_c^\infty(\mathcal{U})$

$\therefore v|_{\partial \mathcal{U}} = 0.$

$$\Longrightarrow \int_{\mathcal{U}} (-\Delta u - f) v\, dx = 0$$

By the arbitrariness of $v \in C_c^\infty(\mathcal{U})$ $\Longrightarrow -\Delta u - f = 0.$

i.e. $-\Delta u = f.$

$\square$.

"$\Leftarrow$" $I[w] - I[u]$ (Let $w = u + v$, $v|_{\partial U} = 0$, $v \in C^2(\overline{U})$)

$$= \int_U \frac{1}{2}|\nabla(u+v)|^2 - (u+v)f - \left(\frac{1}{2}|\nabla u|^2 - uf\right) dx$$

$$= \int_U (\nabla u \cdot \nabla v - vf) + \frac{1}{2}|\nabla v|^2 \, dx$$

$$= \int_U (-\Delta u - f) v + \int_U \frac{1}{2}|\nabla v|^2 dx$$

$$= \int_U \frac{1}{2}|\nabla v|^2 dx \geq 0.$$

$\square$