Chapter 12

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2020.06.24

Contents

1 Topological Spaces

Definition 1.1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:
(1) $\varnothing$ and X in $\mathcal{T}$.
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Example1 *Example* 1.1. indiscrete topology, trivial(discrete) topology, finite complement topology

Definition 1.2. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are 2 topologies on a given set X.
If $\mathcal{T}' \supset (\supsetneq)\mathcal{T}$, we call $\mathcal{T}'$ is **(strictly)finer** than $\mathcal{T}$; if $\mathcal{T}' \subset (\subsetneq)\mathcal{T}$, we call $\mathcal{T}'$ is **(strictly)coarser** than $\mathcal{T}$.

2 Basis for a Topology

Definition 2.1. If X is a set, a **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X(called **basis elements**) such that:
(1) For $\forall x \in X$, there is at least one basis element B containing x.
(2) If x belongs to the intersection of 2 basis elements B_1 and B_2, then $\exists$ a basis element $B_3 \ni x$ *s.t.* $B_3 \subset B_1 \cap B_2$.

Remark. One can check the collection $\mathcal{T}$ generated by the basis $\mathscr{B}$ is a topology on X by definition easily.

Lemma 2.1. *$\mathcal{T}$ equals the collection of all unions of elements of $\mathscr{B}$, that is, $\mathcal{T} = \{\bigcup_{B \in \mathscr{B}'} B \mid \mathscr{B}' \subset \mathscr{B}\}$*

Lemma 2.2 (the criterion to check whether a collection of open sets is a topological space's basis)**.** *Suppose $\mathscr{C}$ is a collection of open sets of a topological space X s.t. $\forall$ open set $U \subset X$ and $\forall x \in U, \exists C \in \mathscr{C}$ s.t. $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology of X.*

lemma2 **Lemma 2.3** (lemma for check the finer/coarser)**.** *TFAE:*
(1) $\mathcal{T}' \supset \mathcal{T}$;
(2) For $\forall x \in X$ and each basis element $B \in \mathscr{B} \ni x$, $\exists B' \in \mathscr{B}'$ s.t. $x \in B' \subset B$.

Example 2.1. Now one can see that the collection of all cicular regions in the plane generates the same topology as the collection of all rectangles.

stdtop *Example* 2.2. Three topologies on the real line $\mathbb{R}$
(1) **standard topology** on the real line:
generated by the collection of all open intervals in the real line.
(2) **lower limit topology** on $\mathbb{R}$(denoted by $\mathbb{R}_l$):
generated by the collection of all half-open intervals of the form $[a, b)$
(3) **K-topology** on $\mathbb{R}$(denoted by $\mathbb{R}_K$):
Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, then $\mathbb{R}_K$ is generated by all open intervals and all sets of the form $(a, b) - K$.

Remark. $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.(Easy to check)

Definition 2.2. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$, that is, $\mathcal{T} = \{\bigcup_{B \in \mathscr{B}'} B \mid \mathscr{B}' \subset \mathscr{B}\}$, where $\mathscr{B} = \{S_1 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}, i = 1, \cdots, n, n \in \mathbb{Z}_+\}$.

3 The Order Topology

Suppose X is a set having a simple order relation $<$ in this section.

$$\begin{aligned}
(a, b) &= \{x \mid a < x < b\}, \text{open interval in X} \\
[a, b] &= \{x \mid a \leq x \leq b\}, \text{closed interval in X} \\
[a, b) &= \{x \mid a \leq x < b\}, \text{half-open interval in X} \\
(a, b] &= \{x \mid a < x \leq b\}, \text{half-open interval in X}
\end{aligned}$$

Definition 3.1. Let $\mathscr{B}$ be the collection of all sets of the following types:
(1) All open sets in X.
(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.
The collection of $\mathscr{B}$ is a basis, the topology it generated is called the **order topology**.

Example 3.1. (1) $\mathbb{Z}_+$ form an ordered set with a smallest element. The order topology is the discrete topology, for every one-point set is open:

$$\{n\} = \begin{cases} (n-1, n+1) & n > 1 \\ [1, 2) & n = 1 \end{cases}$$

(2) Let $X = \{1, 2\} \times \mathbb{Z}_+$ in the dictionary order, the topology on X is not the discrete topology for any basis element containing 2×1 contains points like $1 \times n$.

Definition 3.2. For $a \in X$, there are 4 subsets called **rays** determined by a: $(a, +\infty), (-\infty, a), [a, +\infty), (-\infty, a]$. The first two types are open rays, and the other two are closed rays.

Remark. The open rays form a subbasis for the order topology on X, for $(a, b) = (a, +\infty) \cap (-\infty, b)$.

Remark. If X has a largest element b_0, $(a, +\infty) = (a, b_0]$; otherwise, $(a, +\infty) = \bigcup_{x > a, x \in X} (a, x)$.

4 The Product Topology on $X \times Y$

Definition 4.1. Let X, Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U, V are open subset of X, Y respectively.

Remark. We can see from a figure that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$.

Theorem 4.1. *$\mathscr{B}, \mathscr{C}$ are bases for the topology of X and Y respectively, then the collection*

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B}, C \in \mathscr{C}\}$$

is a basis for the topology $X \times Y$.

Example 4.1. We have studied a standard topology on $\mathbb{R}$(stdtop 2.2): the order topology. The product of this topology is the standard topology on $\mathbb{R}^2$: the order topology in the dictionary order.

We introduce the following fuctions called projections to express the product topology in terms of a subbasis, which is useful to do so sometimes.

Definition 4.2. Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x, y) = x$, $\pi_2 : X \times Y \to Y$ be defined by $\pi_2(x, y) = y$. The 2 maps are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

Remark. Projections are onto(surjective) unless $X($ or Y or $X \times Y) = \varnothing$.

Theorem 4.2. *The collection $\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid U \text{ open in } Y\}$ is a subbasis for the product topology on $X \times Y$.*

Proof. Let $\mathcal{T}$ be the product topology, $\mathcal{S}$ generates $\mathcal{T}'$. Note that $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, then it's easy to prove $\mathcal{T} \subset (\supset)\mathcal{T}'$ respectively. □

Remark. From the above theorem, one can observe that the product topology is the coarest topology among the topologies on $X \times Y$ which ensures π_1, π_2 are continuous.

5 The Subspace Topology

Definition 5.1. If $Y \subset X$, the collection $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

Definition 5.2 (Another definition for subspace topology)**.** Subspace topology is the coarest topology such that the imbedding(c.f. Definition 7.4) $i : A \to X,\ i(x) = x,\ A \subset X$ is continuous.

Lemma 5.1. *If $\mathscr{B}$ is a basis for X, then the collection $\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$ is a basis for the subspace topology on Y.*

Remark. One needs to be careful when dealing with the case that Y is a subspace of X. We say that a set U is **open in Y(or open relative to Y)** if it belongs to the topology of Y.

Lemma 5.2. *Let Y be a subspace of X, if U is open in Y and Y is open in X, then U is open in X.*

Now we will focus on the subspace topology of an order topology. Let X be an ordered set in the order topology and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. However, the resulting order toplogy on Y need not to be the same as the topology that Y inherits as a subspace of X.(See the examples below.)

Example 5.1. $Y = [0, 1] \subset \mathbb{R}$, in the subspace topology. Then the open sets in Y are of the following types:

$$(a, b) \cap Y = \begin{cases} (a, b) & a, b \in Y \\ [0, b) & b \in Y, a \notin Y \\ (a, 1] & a \in Y, b \notin Y \\ Y \text{ or } \varnothing & a, b \notin Y \end{cases}$$

Note that these sets form a basis for the order topology on Y. So in this example, the subspace topology and its order topology are the same.

Example 5.2. Let $Y = [0, 1) \cup \{2\} \subset \mathbb{R}$. In the subspace topology on Y, $\{2\}$ is open($\because \{2\} = (\frac{3}{2}, \frac{5}{2}) \cap Y$), but in theorder topology on Y, $\{2\}$ is not open. Any basis element for the order topology on Y that contains 2 is of the form: $\{x \mid x \in Y, a < x \leq 2, a \in Y\}$.

Example 5.3. Let $I = [0, 1]$. The dictionary order topology on $I \times I$ is not the same as the suspace topology on $I \times I$ obtained from the dictionary topology on $\mathbb{R} \times \mathbb{R}$ although the dictionary order on $I \times I$ is just a restriction of dictionary order on the plane on $I \times I$. You can see this because $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in $I \times I$ in the subspace topology, but is not open in the order topology. See Figure 1
The set $I \times I$ in the dictionary order topology will be called the **ordered square**, denoted by I_o^2.

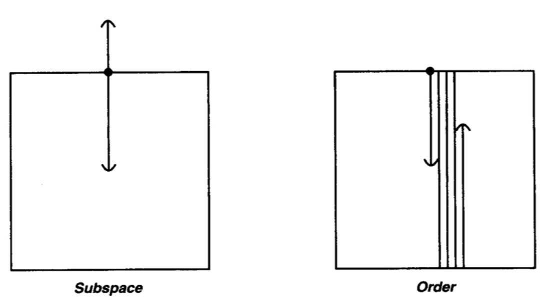

Figure 1: Subspace and Order topology on $I \times I$

Lemma 5.3. *The order topology induced on $Y \subset$ the subspace topology on Y.*

Proof. $\because$ the order topology on Y has a subbasis $\{\{x \in Y \mid x < a\} \mid a \in Y\} \cup \{\{x \in Y \mid x > a\} \mid a \in Y\}$, and $\{x \in Y \mid x < a\} = (-\infty, a) \cap Y$, $\therefore$ the open sets in the order topology is also open in the subspace toplogy. $\square$

Definition 5.3. Given an ordered set X, we say $Y \subset X$ is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Note that intervals and rays in X are convex in X.

Theorem 5.1 (A sufficient condition for 'order topology=subspace topology')**.** *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Similar to the above lemma. One should also consider the subbasis. □

Remark. To avoid ambiguity, whenever X is an order set in the order topology and $Y \subset X$, we shall assume that Y is given the subspace topology unless specified otherwise.

6 Closed sets and Limit Points

Definition 6.1. A subset A of a topological space X is asaid to be **closed** if the set $X - A$ is open.

Lemma 6.1. *Let $Y \subset X$, then A is closed in $Y \iff$ it equals the intersection of a closed set of X with Y.*

Lemma 6.2. *Let $Y \subset X$, if A is closed in Y, Y is closed in X, then A is closed in X.*

Definition 6.2. Given a subset $A \subset X$. The **interior** of A is defined as the union of all open sets contained in A, the **closure** of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by $IntA$ or $\mathring{A}$ and the closure of A is denoted as ClA or $\bar{A}$.

Obviously, $IntA$ is the largest open set contained in A, and $\bar{A}$ is the smalles closed set containing A; furthermore, $IntA \subset A \subset \bar{A}$. If A is open, $A = IntA$; if A is closed, $A = \bar{A}$.

Lemma 6.3. *Let Y be a subspace of X, A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Definition 6.3. We introduce 2 terminology. We shall say that a set A **intersects** a set B if $A \cap B \neq \varnothing$. They say that A is a **neighborhood** of x if A contains an open set containing x.

Theorem 6.1. *Let A be a subset of the topological space X.*

(a) Then $x \in \bar{A} \iff$ every neighborhood of x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A} \iff$ every basis element B containing x intersects A.

Remark. $x \notin \bar{A} \iff \exists U \ni x$ open that does not intersect A.

Definition 6.4. If A is a subset of the topological space X and $x \in X$, we say that x is a **limit point**(or "**cluster point**", or "**point of accumulation**") of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if $x \in \overline{A - \{x\}}$. We denote all the limit point of A as A'.

Lemma 6.4. $\bar{A} = A \cup A'$.

Corollary 6.1. *A is closed $\iff A' \subset A$*

Definition 6.5. In an arbitrary topological space, one says that a sequence $x_1, x_2, \cdots$ of points of the space X **converges** to the point $x \in X$ provided that, corresponding to each neighborhood U of x, $\exists N$ such that $x_n \in U, \forall n \geq N$.

Remark. In an arbitraty topological space, a sequence may converge to more than one point. So we need to impose an additional condition that rule out this situation.

Definition 6.6. (1) A topological space X is called a **Hausdorff Space (T_2 space)** if for $\forall x_1 \neq x_2 \in X, \exists$ neighborhood U_1, U_2 of x_1, x_2, respectively, such that $U_1 \cap U_2 = \varnothing$.

(2) The condition that finite point sets be closed is called the T_1 **axiom**, which is weaker than the Hausdorff condition(c.f. Example 6.1). This means that a Hausdorff space satisfies T_1 axiom.(See the lemma below)

Lemma 6.5. *Every finite point set in a Hausdorff space X is closed.($T_2 \Rightarrow T_1$)*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. $\square$

egT1 *Example* 6.1. The real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed.

Remark. Most of the spaces that are important to mathematicians are Hausdorff spaces.

Theorem 6.2. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Remark. If the sequence $\{x_n\}$ of a Hausdorff space X converges to the point x of X, we often write $x_n \to x$, we say that x is the **limit** of the sequence $\{x_n\}$.

Theorem 6.3. *Let X be a T_1 space; let A be a subset of X. Then x is a limit point of $A \iff$ every neighborhood of x contains infinitely many points of A.*

Proposition 6.1. *We have following properties:*

(1) Every simply ordered set is a Hausdorff space in the order topology.
(2) The product of two T_i space is a T_i space.
(3) A subspace of a T_i space is a T_i space.$(i = 1, 2)$

Proposition 6.2. *This property is about metric topology:*
The Hausdorff axiom is satisfied by every metric topology.

Lemma 6.6 (The sequence lemma(c.f. Lemma 9.2)). *Let $A \subset X$. If $\exists \{x_i\}_{i=1}^{+\infty} \subset A$, s.t. x_i converge to x, then $x \in \bar{A}$; the converse holds if X is metrizable. Furthermore, if X is metrizable, then $x \in A' \iff \exists \{x_i\}_{i=1}^{+\infty} \subset A (x_i \neq x)$ s.t. $x_i \to x$.*

Definition 6.7. If $A \subset X$, we define the **boundary** of A by the equation

$$BdA = \bar{A} \cap \overline{(X - A)}.$$

7 Continuous Functions

Definition 7.1. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, $f^{-1}(V)$ is open in X.

Proposition 7.1. *Let $\mathcal{S}($ or $\mathscr{B})$ is a subbasis(or basis) of Y, then f is continuous $\iff \forall V \in \mathcal{S}($ or $\mathscr{B}), f^{-1}(V)$ is open in X.*

Example 7.1. Let $\mathbb{R}$ denote the set of real numbers in its usual topology, and let $\mathbb{R}_l$ denote the same set in the lower limit topology. Let $f : \mathbb{R} \to \mathbb{R}_l$ be the identity function, $f(x) = x$. Then f is not a continuous function: $f^{-1}[a, b) = [a, b)$ is not open in $\mathbb{R}$.

Theorem 7.1. *Let $f : X \to Y$, TFAE:*

(1) f is continuous.
(2) For every subset A of X, $f(\bar{A}) \subset \bar{f}(A)$.
(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.
(4) For $\forall x \in X, \forall$ neighborhood V of $f(x), \exists$ a neighborhood U of x s.t. $f(U) \subset V$.

Proof. We only need to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and that (1) $\Rightarrow$ (4) $\Rightarrow$ (1). □

Definition 7.2. Let $f : X \to Y$ be a bijection, if both f and its inverse $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Proposition 7.2. *Let $f : X \to Y$ be a bijection, TFAE:*
(1) f is a homeomorphism.
(2) $f(U)$ is open in $Y \iff U$ is open in X.

Definition 7.3. The above theorem shows that a homeomorphism gives us a bijective correspondence not only between 2 sets(X, Y) but also between the collections of open sets of X and Y. A property that also holds under homeomorphism$(X, f(X)$ both holds, f homeomorphism) is called a **topological property**.

defn7.4 **Definition 7.4.** Now suppose f is injective. Let Z be the image set $f(X)$ as a subspace of Y, then the restriction $\tilde{f} : X \to Z$ is bijective. If it happens to be a homeomorphism, we say that the map $f : X \to Y$ is a **(topological) imbedding** of X in Y.

Example 7.2. A bijective function can be continuous without being a homeomorphism.

One such function is the identity map $g : \mathbb{R}_l \to \mathbb{R}$. Another is the following: Let S^1 be the unit circle as a subspace of the plane $\mathbb{R}^2$, and let $f : [0, 1) \to S^1, f(t) = (cos2\pi t, sin2\pi t)$. $f[0, \frac{1}{4})$ is not open, so f is not a homeomorphism.

As a result, consider $h : [0, 1) \to \mathbb{R}^2$ obtained from f by expanding the range. h is an example of a continuous injective map that is not an imbedding.

Proposition 7.3 (Rules for constructing continuous functions)**.** *Let X, Y, Z be topological spaces, $A \subset X$ as subspace,*

(1)(Constant functions) Given $y_0 \in Y, f(x) = y_0, \forall x \in X$ is continuous.

(2)(Inclusion) The inclusion function $j : A \to X$ is continuous.

(3)(Composites) If $f : X \to Y, g : Y \to Z$ are continuous, so as $g \circ f : X \to Z$.

(4)(Restricting the domain) If $f : X \to Y$ is continuous, then $f|_A : A \to Y$ is continuous.

(5)(Restricting or expanding the range) Let $f : X \to Y$ is continuous, $f(X) \subset Z \subset Y \subset W$ are both subspaces, then $X \to Z$ and $X \to W$ obtained by restricting or expanding the range are continuous.

(6)(Local formulation of continuity) If $f : X \to Y$ is continuous, $X = \bigcup_\alpha U_\alpha, U_\alpha$ open, then $f|_{U_\alpha}$ is continuous for each α.

Theorem 7.2 (The pasting lemma)**.** *Let $X = A \cup B$, where A, B are closed in X. Let $f : A \to Y, g : B \to Y$ be continuous. If $f(x) = g(x), \forall x \in A \cap B$, then the function $h : X \to Y$ defined below*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is a continuous.

Remark. This theorem also holds if A, B are open sets in X, this is just a special case of the "local formulation of continuity" rule in preceding proposition.

Example 7.3. The pasting lemma only holds for finite number of closed sets. Let

$$A_i = \begin{cases} \{\frac{1}{i}\}, & i \in \mathbb{Z}^+ \\ \{0\}, & i = 0 \end{cases}, f_i : A_i \to \mathbb{R}, f_i(\frac{1}{i}) = \begin{cases} 1, & i \in \mathbb{Z}^+ \\ 0, & i = 0 \end{cases}$$

then A_i are all disjoint, if pasting all functions on $\bigcup_{i \in \mathbb{Z}^+} A_i$ to get a function g, g is not continuous.

thm7.3 **Theorem 7.3** (Maps into products). *Let $f : A \to X \times Y, f(a) = (f_1(a), f_2(a))$, then f is continuous $\iff$ f_1, f_2 are continuous.*

The maps f_1, f_2 are called the ***coordinate functions*** *of f.*

Proof. "$\Rightarrow$"$\because f_i(a) = \pi_i(f(a)), i = 1, 2; \forall a \in A$ and π_i is the projection, which is continuous.

"$\Leftarrow$"$\because f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. □

Remark. $f : A \times B \to Y$ may not be continuous if it is continuous in each variable separately. $f : \mathbb{R}^2 \to \mathbb{R}$ be defined below is continuous in each variable separately but f is not continuous(Consider $g(x) = f(x \times x)$).

$$f(x \times y) = \begin{cases} \frac{xy}{x^2+y^2} & x \times y \neq 0 \times 0 \\ 0 & x \times y = 0 \times 0 \end{cases}$$

8 The Product Topology

Definition 8.1. Let us take the collection of all sets of the form $\Pi_{\alpha \in J} U_\alpha$ as a basis for a topology on the product space $\Pi_{\alpha \in J} X_\alpha$, where U_α is open in X_α for $\forall \alpha \in J$. The topology generated by the basis is called the **box topology**.

Definition 8.2. Let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$, and let $\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$. The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this space, $\Pi_{\alpha \in J} X_\alpha$ is called the **product space**.

Proposition 8.1 (Basis of this 2 topologies and their comparison). *The box or product topology on ΠX_α has as basis all sets of the form ΠU_α. For each element of a basis of the product topology, $U_\alpha \neq X_\alpha$ holds for only finitely α.*

From above, we know that product topology $\subset$ box topology.(If J is finite, then these two are equal; otherwise, it is a strictly finer relation.)

Remark. We can get a stronger proposition that product topology $\subset$ uniform topology $\subset$ box topology(c.f. Theorem thm9 9.1).

Remark. Whenever we consider ΠX_α, we shall assume it is given the product topology unless specified otherwise.

Proposition 8.2. *Let A_α be a subspace of X_α for $\forall \alpha$, then ΠA_α is a subspace of ΠX_α, either box or product topology is given.*

Proposition 8.3. *If X_α is hausdorff for each α, then X_α is also Hausdorff in both the box and product topologies.*

Proof. We only need to prove for the product topology case.($\because$ product topology $\subset$ box topology, $\therefore$ the proposition that product topology is Hausdorff indicates that the other.) □

Theorem 8.1. *If $A_\alpha \subset X_\alpha, \forall \alpha$; then $\Pi \bar{A}_\alpha = \overline{\Pi A_\alpha}$ in both box and product topologies.*

Theorem 8.2 (A generalization of theorem thm7.3 7.3). *$f : Y \to (\Pi X_\alpha$, product topology$)$ is continuous $\iff$ $\pi_\alpha \circ f$ is continuous for each α.*

The above theorem is false for box topology, see the example below.

Example 8.1. Let $\mathbb{R}^\omega := \Pi_{n=1}^{+\infty} \mathbb{R}$, define $f : \mathbb{R} \to \mathbb{R}^\omega, f(t) = (t, t, \cdots, t)$.

$\because f^{-1}\Big((t-1, t+1) \times (t - \frac{1}{2}, t + \frac{1}{2}) \times \cdots \times (t - \frac{1}{n}, t + \frac{1}{n}) \times \cdots\Big) = \bigcap_{n=1}^{+\infty} (t - \frac{1}{n}, t + \frac{1}{n}) = \{t\}$ is not open when given box topology,

$\therefore$ the above theorem is false for box topology.

9 The Metric Topology

Definition 9.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ satisfies:
(1) $d(x, y) \geq 0,\ \forall x, y \in X$; equality holds $\iff x = y$.
(2) $d(x, y) = d(y, x),\ \forall x, y \in X$.
(3) $d(x, y) + d(y, z) \geq d(x, z),\ \forall x, y, z \in X$.

Definition 9.2. If d is a metric on the set X, then the collection of all $\epsilon-$balls $B_d(x, \epsilon)$, $x \in X, \epsilon > 0$ is a basis for a topology on X, called the **metric topology** induced by d.

Definition 9.3 (Another definition for metric topology)**.** A set U is open in the metric topology induced by d $\iff \forall y \in U,\ \exists \delta > 0$, s.t. $B_d(y, \delta) \subset U$.

Example 9.1. Given a set X, define

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

The topology it induces is the discrete topology(Example 1.1).

Definition 9.4. If X is a topological space, X and (Y, d) is homeomorphic, then X is called **metrizable**. There may exists different metric that gives different topologies on a metrizable space.

Definition 9.5. (X, d) is called a **bounded metric space** if $\exists M > 0$, s.t. $\forall x, y \in X, d(x, y) \leq M$.

prop9 **Proposition 9.1.** *Let (X, d) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ as $\bar{d}(x, y) = min\{d(x, y), 1\}$. Then $\bar{d}$ is a metric induces the same topology as d(i.e. $id : (X, d) \to (X, \bar{d})$ is a homeomorphism). The metric $\bar{d}$ is called the **standard bounded metric** corresponding to d.*

Lemma 9.1 (Metric space case for Lemma 2.3)**.** *Consider (X, d), (X, d'), then $\mathcal{T}'$ is finer than $\mathcal{T}$ $\iff$ $\forall x \in X, \forall \epsilon > 0, \exists \delta > 0$, s.t. $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.*

Corollary 9.1. *(X, d) and $(X, \bar{d})$ is homeomorphic.*

Proposition 9.2. *Let $(X, \rho_1), (Y, \rho_2)$ be metric spaces, then the following three are metric functions on $X_1 \times \cdots \times Y$:*
$d_1(x, y) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2)$;
$d_2(x, y) = \sqrt{\rho_1(x_1, y_1)^2 + \rho_2(x_2, y_2)^2}$;
$d_\infty(x, y) = max\{\rho_1(x_1, y_1), \rho_2(x_2, y_2\}$; and the induced topology of these three metric are all the product topology on $X \times Y$.

Proof. $\because d_\infty(x, y) \leq d_1(x, y) \leq \sqrt{2} d_2(x, y) \leq 2 d_\infty(x, y)$, and then apply the above lemma. □

Proposition 9.3. *For $\mathbb{R}^n$,*
*$d_2(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ is called the **euclidean metric** on $\mathbb{R}^n$;*
*$d_\infty(x, y) = max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$ is called the **square metric** on $\mathbb{R}^n$.*
The induced topology of $d_2(x, y), d_\infty(x, y)$ is the product topology on $\mathbb{R}^n$.

Proof. $\because d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$ □

Remark. It is not natural to generalize the metric $d_2(x, y), d_\infty(x, y)$ to $\mathbb{R}^\omega$, for the series need not converge and the supremum does not always make sense.

defn9 **Definition 9.6.** Given an index set J and a collection of metric space $(X_\alpha, d_\alpha)_{\alpha \in J}$, define a metric $\bar{\rho}$ on $\Pi_{\alpha \in J} X_\alpha$ as $\bar{\rho}(x, y) = sup\{\bar{d}_\alpha(x_\alpha, y_\alpha) \mid \alpha \in J\}$, the induced topology is called the **uniform topology**, where $\bar{d}_\alpha$ is the standard bounded topology(c.f. Proposition 9.1) in X_α.

thm9 **Theorem 9.1.** *On $\Pi_{\alpha\in J}X_\alpha$, product topology $\subset$ uniform topology $\subset$ box topology. If J is an infinite set, these three topologies are distinct with each other.*

Proof. The proof is a simple application of the original definition. The last statement can be shown by given examples as the following: Given $x \in \Pi X_i$, then $\Pi\{y_i | d(x_i, y_i) < \frac{1}{i}\}$ is open in box topology but not in uniform topology; $\Pi\{y_i | d(x_i, y_i) < \frac{1}{2}\}$ is open in uniform topology but not in prouduct topology □

Theorem 9.2 (When $|J|$ is countable, product topology is metrizable)**.** *Given a collection of metric space $(X_i, d_i)_{i\in\mathbb{N}^*}$, define a metric $D(x, y)$ on $\Pi_{i=1}^{+\infty}X_i$ as $D(x, y) = sup\{\frac{\bar{d}_i(x_i,y_i)}{i}\}$, then the topology induced by the metric D is the product topology on $\Pi_{i=1}^{+\infty}X_i$.*

Proof. (1) $\because y \in B_D(x, r) \iff D(x, y) < r \iff sup\{\frac{\bar{d}_i(x_i,y_i)}{i}\} < r, \therefore$ take N s.t. $\frac{1}{N} < r$, then $y \in \Pi_{i=1}^{N-1}B_{d_i}(x_i, r) \times \Pi_{i=N}^{+\infty}X_i \subset B_D(x, r)$. (2) Conversely, for the basis element of product topology $\Pi_{i=1}^{N-1}U_i \times \Pi_{i=N}^{+\infty}X_i \ni y$, take $\epsilon > 0$ s.t. $y_i \in B_{d_i}(x_i, r) \subset U_i, i = 1, 2, \cdots, N-1$, then $y \in B_D(x, \epsilon/N) \subset \Pi_{i=1}^{N-1}U_i \times \Pi_{i=N}^{+\infty}X_i$. □

Definition 9.7. A space X is said to have a **countable basis at the point** x if there exists a countable collection $\{U_n\}_{n\in\mathbb{Z}_+}$ of neighborhoods of x s.t. $\forall$ neighborhood U of x, $\exists n$, s.t. $U_n \subset U$. A space X that has a countable basis at each of its points is said to satisfy the **first countability axiom**.

Proposition 9.4. *A metrizable space satisfy the first countability axiom.*

Lemma9 **Lemma 9.2** (The sequence lemma)**.** *Let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X has a countable basis at x.*

Proof. $\Leftarrow$: take $x_n \in U_1 \cap \cdots \cap U_n \cap A$, then $\{x_n\}$ converges to x. □

Theorem 9.3. *Let $f : X \to Y$ and assume that X has a countable basis at x , then f is continuous at $x \iff$ for every sequence x_n converging to x, the sequence $f(x_n)$ converges to $f(x)$.*

Proof. $\Rightarrow$: Let $V \ni f(x)$ open in Y, x_n converges to x, then $x_n \in f^{-1}(V), \forall n > N$, then $f(x_n) \in V, \forall n > N$.

$\Leftarrow$: Let $A \ni f(x)$ closed in Y, if $f^{-1}(A)$ is not closed, take $x \in \overline{f^{-1}(A)} \setminus f^{-1}(A)$, then apply the sequence lemma, there exists a sequence $\{x_n\} \subset f^{-1}(A)$ converges to x, by the assumption, $f(x_n)$ converges to $f(x)$, then $f(x) \in \bar{A} = A$, this is contradict to $x \in \overline{f^{-1}(A)} \setminus f^{-1}(A)$, so $f^{-1}(A)$ is closed. □

lemma9.3 **Lemma 9.3.** *Let $(X, d_1), (Y, d_2)$ metriable, then $f : X \to Y$ is continuous at $x \iff \forall\epsilon > 0, \exists\delta > 0$, s.t. if $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$.*

Proposition 9.5. *Let X be a topological space.If $f, g : X \to \mathbb{R}$ are continuous function, then $f + g, f - g, f\dot{g}$ are continuous. If $g(x) \neq 0, \forall x$, then f/g is continuous.*

Proof. $f + g : X \xrightarrow{(f,g)} \mathbb{R}^2 \xrightarrow{+} \mathbb{R}$, the first map is continuous by applying Theorem 7.3, the second map is continuous by applying Lemma 9.3. Thus $f + g$ is continuous. □

Definition 9.8. Let $f_n : X \to (Y, d)$ be a sequence of functions, then we say f_n **converges uniformly** to $f : X \to Y$ if $\forall\epsilon > 0, \exists N$, s.t. $\forall n > N, \forall x \in X$ $d(f_n(x), f(x)) < \epsilon$.

Theorem 9.4 (Uniform limit theorem)**.** *Let $f_n : X \to (Y, d)$ be a sequence of continuous functions, if f_n converges uniformly to f, then f is continuous.*

Remark. The notion of uniform convergence is related to the Definition 9.6 of the uniform metric. Consider the space $\mathbb{R}^X$ of all functions $f : X \to \mathbb{R}$ in the uniform metric $\bar{\rho}$. One can check that $f_n : X \to \mathbb{R}$ converges uniformly to $f \iff f_n$ converges to f when they are considered as elements of $(\mathbb{R}^X, \bar{\rho})$.

At last, we give some examples of spaces that are not metrizable.

Example 9.2 ($\mathbb{R}^\omega$ in the box topology is not metrizable). We shall show that the sequence lemma does not hold for $\mathbb{R}^\omega$ with box topology.(So it does not satisfy the first countability axiom) Let $A = \{(x_1, x_2, \cdots) \mid x_i > 0, \forall i \in \mathbb{Z}_+\}$, then $0 \in \bar{A}$; however, for any sequence $\{a_n\}, a_n = (x_{1n}, x_{2n}, \cdots, x_{in}, \cdots)$, we can construct a basis element B for the box topology by $B = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots$. Then $0 \in B$, but $\{a_n \mid n \in \mathbb{Z}_+\} \cap B = \varnothing$, thus a_n does not converges to 0.

Example 9.3 ($\mathbb{R}^J$ in the product topology is not metrizable when J is an uncountable index set). We also show that the sequence lemma does not hold for this.(So it does not satisfy the first countability axiom.) Let $A = \{(x_\alpha) \mid$ only finitely many α satisfies that $x_\alpha \neq 1\}$, then $0 \in \bar{A}$; however, for any sequence $\{a_n\} \subset A$, let J_n denote those indices α for which the α-th coordinate of a_n is different from 1. Then $|J_n| < +\infty$, so $I = \bigcup J_n$ is countable, thus $\exists$ an index $\beta \in J \setminus I$. So for each a_n, its β-th coordinate is 1. We can conclude that $\pi_\beta^{-1}((-1,1))$ is a neighborhood of 0 that contains no points of a_n, so the sequence does not converge to 0, which contradicts the sequence lemma.

10 The Quotient Topology

Definition 10.1. Let X be a topological space, $\sim$ is a equivalence relation on X. Define a surjective map $p : X \to X/\sim$ as $p(a) = [a] := \{b \in X \mid b \sim a\}$ Then $\mathcal{F} = \{U \in X/\sim \mid p^{-1}(U) \text{ is open in } X\}$ is a topology on $X/\sim$. We call this topology as the **quotient topology** on $X/\sim$, ($X/\sim$, quotient topology) is called a **quotient space** of X.

Definition 10.2. We say a subset C of X is a **saturated set** with respect to $\sim$, if it contains every set $p^{-1}(y)$ it intersects. Thus C is saturated $\iff C = p^{-1}(B)$ for some $B \subset X/\sim \iff C = p^{-1}(p(C))$.

Remark. The open set V in $X/\sim$ is the image of a saturated open set $p^{-1}(V)$.

Example 10.1. Define $\sim$ on $[0,1]$: $x \sim y \iff x, y \in \{0,1\}$ or $x = y$. Then $[0,1]/\sim \xrightarrow{f} S^1, f(x) = e^{2\pi i x}$ is homeomorphism.

Example 10.2. Define $\sim$ on $[0,1] \times [0,1]$: $(0,y) \sim (1,y), (x,0) \sim (1-x,1)$, then $[0,1] \times [0,1]/\sim$ is called **Klein bottle**.

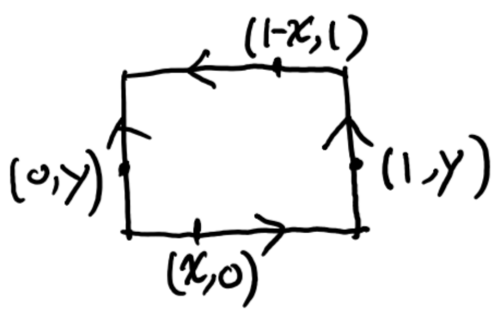

Figure 2: Klein bottle

Definition 10.3. Let X, Y be topological spaces, let $p : X \to Y$ be a surjective map. The map p is called a **quotient map** provided that U is open in $Y \iff p^{-1}(U)$ is open in X.

Remark. (1) $X \to X/\sim$ is a quotient map as the definition above.

(2) For a quotient map $p : X \to Y$, we can define $\sim$ as: $x_1 \sim x_2 \iff p(x_1) = p(x_2)$, then $X/\sim = \{p^{-1}(y) \mid y \in Y\}$ is homeomorphic to Y.

Proposition 10.1. *The composites of two quotient maps is a quotient map.*

Theorem 10.1. *Let $p : X \to Y$ be a quotient map, and the map $g : X \to Z$ satisfies that $\forall y \in Y, g|_{p^{-1}(y)}$ is a constant map, then (1) $\exists! f : Y \to Z$, s.t. $f \circ p = g$. (2) And f is continuous $\iff$ g is continuous;(3) f is a quotient map $\iff$ g is quotient map.*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, dashed, "\exists! f"'] & \end{tikzcd}$$

Figure 3:

Proof. (3)"$\Leftarrow$": $\because$ g is continuous, $\therefore$ f is continuous. Let $f^{-1}(U)$ is open, then $p^{-1}(f^{-1}(U)) = g^{-1}(U)$ is open, $\therefore U$ is open. $\square$

Example 10.3 (Not a quotient map). Let $X = \bigcup_{n=1}^{+\infty}[0,1] \times \{n\}, Z = \{x \times \frac{x}{n} \mid x \in [0,1]\}$ both with the subspace topology of $\mathbb{R}^2$. $g : X \to Z$ be defined as $g(x \times n) = x \times \frac{x}{n}$, thus the quotient space $X^* = \bigcup_{x \times y \in Z \setminus \{0 \times 0\}}\{x \times \frac{x}{y}\} \cup \{0 \times n \mid n \in \mathbb{N}^*\}$. However, g is not a quotient map because $A = \{\frac{1}{n} \times n \mid n \in \mathbb{N}^*\}$ is a saturated closed set in X, but $g(A) = \{\frac{1}{n} \times \frac{1}{n^2} \mid n \in \mathbb{N}^*\}$ is not a closed set in Z. This phenomenon is because the quotient space X^* identify the subset $\{0\} \times \mathbb{N}^*$ to a point.

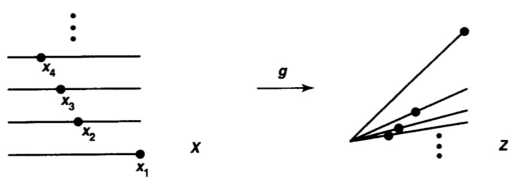

Figure 4: g is not a quotient map

Example 10.4 (The product of two quotient maps need not be a quotient map). Let $X = \mathbb{R}, A = \mathbb{Z}_+$, and $p : X \to X/A$ be the quotient map(X/A means that the space is obatained from X by identifying the subset A to a point b), $id : \mathbb{Q} \to \mathbb{Q}$ be a quotient map, then $p \times id : X \times \mathbb{Q} \to X/A \times \mathbb{Q}$ is not a quotient map.

Let $c_n = \frac{\sqrt{2}}{n}$, consider all the U_n like the shadowed area below for each $n(U_n$ is lying above or beneath both of the straight lines with slopes 1 and -1, respectively, and between the vertical lines $x = n - \frac{1}{4}, x = n + \frac{1}{4}$)(See Figure figure10 5).

Let $U = \bigcup U_n$, then U is a saturated open set in $X \times \mathbb{Q}$.

Assume $(p \times id)(U)$ is also open, then $([1] \times 0) \in X/A \times \mathbb{Q}$, hence $(p \times id)(U)$ contains an open set of the form $W \times \{y \in (-q,q) \mid y \in \mathbb{Q}\}$ where $p^{-1}(W)$ is open in X and $\mathbb{Z}_+ \subset p^{-1}(W)$. Thus $\exists \epsilon_n > 0, q > 0$, s.t. $\bigcup_{n=1}^{+\infty}(n - \epsilon_n, n + \epsilon_n) \times \big((-q,q) \cap \mathbb{Q}\big)$. However, $\exists N, \exists q > c_N$, $(N - \epsilon_N, N + \epsilon_N) \times \big((-q,q) \cap \mathbb{Q}\big) \nsubseteq U$, which leads
eg10.4 to a contradiction!

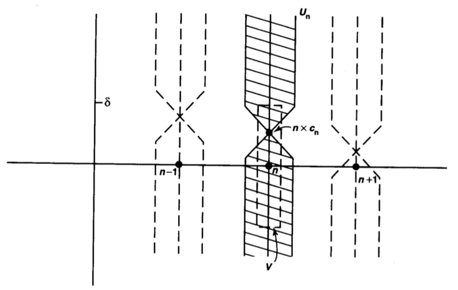

Figure 5: For Example 10.4 figure10

Let $p: X \to Y$ be a quotient map, $A \subset Y$ with the subspace topology, then $p|_A : A \to p(A)$ need not be a quotient map in general. However, one have the following theorem.

Theorem 10.2. *Let $p: X \to Y$ be a quotient map, A be a subspace of X that is saturated with respect to p, then*

(1) If A is either open or closed in X, then $p|_A$ is a quotient map.

(2) If p is either an open or closed map, then $p|_A$ is a quotient map.

Proof. We only need to notice that:

(1) For U is open in A, $(p|_A)^{-1}(U) = p^{-1}(U)$ is open. (This equation is because A is saturated.);

(2) For $p^{-1}(U) = V \cap A$, $U = p(V) \cap p(A)$.

Then other steps are simple. □

11 Connected Spaces

Definition 11.1. Let X be a topological space. A **separation** of X is a pair of sets U, V, where $X = U \cup V$, U, V are disjoint nonempty open sets. The space X is said to be **connected** if there does not exist a separation of X.

Remark. Obviously, the definition of connected space can be formulated as:

The only subsets of X that are both open and closed are $\varnothing$ and X itself.

Lemma 11.1 (Another way to formulate the definition of connectedness of subspace)**.** *If Y is a subspace of X, a **separation** of Y is a pair of disjoint nonempty sets A, B whose union is Y, and satisfying $A' \cap B = A \cap B' = \varnothing$.*

Proof. Hint: $\bar{A} \cap Y = \bar{A} \cap (A \cap B) = A \cap (\bar{A} \cap B)$. □

lemma11 **Lemma 11.2.** *If C, D form a separation of X, and if Y is a connected subspace of X, then $Y \subset C$ or $Y \subset D$.*

prop11.1 **Proposition 11.1.** *If $A_\alpha, \alpha \in J$ is a collection of connected subspace of X, and $\bigcap A_\alpha \neq \varnothing$, then $\bigcup A_\alpha$ is connected.*

Proof. Prove by contradiction. Suppose $Y = C \cup D$ form a separation and $x \in \bigcap A_\alpha$. Then WLOG, $x \in C$, applying Lemma 11.2, $\bigcup A_\alpha \subset C$, contradiction! □

prop11.2 **Proposition 11.2.** *Let A be a connected subspace of X, $A \subset B \subset \bar{A}$, then B is also connected.*

Corollary 11.1. *A is connected $\Rightarrow$ $\bar{A}$ is connected.*

prop11.3 **Proposition 11.3.** *If X is connected and f is continuous, then the image space $f(X)$ is connected.*

thm11 **Theorem 11.1.** *If X, Y are connected, then $X \times Y$ is connected.*

Proof.

Step 1: For $a \in X, b \in Y$, $\{a\} \times Y \cup X \times \{b\}$ is connected. ($\because$ X is homeomorphic to $X \times \{b\}$, Y is homeomorphic to $\{a\} \times Y$ and they share a common point, so applying Proposition 11.1 and 11.3 to get the result)

Step 2: Fix $y_0 \in Y$, then $X \times \{y_0\} \cup \{x\} \times Y, x \in X$ is a collection of connected space with common points $X \times \{y_0\}$, apply Proposition 11.1 again, $\Rightarrow$ $X \times Y$ is connected. □

Remark. The above result is true for finite cartesian product by induction. However, it is not true for infinite cartesian product, see the following examples.

Example 11.1 (($\mathbb{R}^\omega$, box topology) is not connected.).

$A = \{(x_i) \mid x_i$ is bounded $, \forall i\}$ is open and closed. ($\because x \in A \Rightarrow \Pi(x_i - 1, x_i + 1) \subset A$; $x \notin A \Rightarrow \Pi(x_i - 1, x_i + 1) \subset A^c$.)

Example 11.2. Suppose X_i is connected, then ($\Pi_{i=1}^{+\infty} X_i$, product topology) is connected.

Fix $a = (a_i) \in \Pi_{i=1}^{+\infty} X_i$. Let $Y_n = \{x \in \Pi_{i=1}^{+\infty} X_i \mid x_m = a_m, \forall m > n\}$, then Y_n is homeomorphic to $\Pi_{i=1}^{n} X_i$, which is connected; thus Y_n is connected. $\because a \in \bigcap_{i=1}^{+\infty} Y_n, \therefore Y = \bigcup_{i=1}^{+\infty} Y_n$ is connected.(Applying Proposition 11.1) Observe that $\bar{Y} = \Pi_{i=1}^{+\infty} X_i$, so it is connected by Proposition 11.2.

Proposition 11.4 (Intermediate value theorem)**.** *Let $f : X \to Y$ be a continuous map, where X is a connceted space and Y is an ordered set in order topology. Then $\forall a \neq b \in X, \forall r \in Y$ satisfying $f(a) < r < f(b), \exists c \in X$, s.t. $f(c) = r$.*

Proof. Suppose such c does not exist, then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, +\infty)$ form a separation of $f(X)$, which is connected. □

The preceding theorems show us how to construct new connected spaces out of given ones, and tell us how to prove a space is not connected. But we need to find some connected spaces to start with. We shall prove $\mathbb{R}$ is connected. The connectedness of $\mathbb{R}$ only rely on its order properties, not on its algebraic property.

Definition 11.2. A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

(1) L has the least upper bound property.

(2) If $x < y$, then $\exists z$, s.t. $x < z < y$.

Theorem 11.2. *If L is a linear continuum in the order topology, then L is connected and the intervals and rays in L are also connected.*

Proof. We prove that if Y is a convex subspace of L, then Y is connceted. Suppose that Y has a separation $Y = A \cup B$, then take $a \in A, b \in B \Rightarrow [a, b] \subset Y$. Let $A_0 = A \cap [a, b], B_0 = B \cap [a, b], c = supA_0$, then show that $c \notin A_0, c \notin B_0$ respectively, which leads to contradiction! □

Example 11.3. $\mathbb{R}$ is connected.

Example 11.4. The ordered square(c.f. Example 5.3) is a linear continuum.

For $\varnothing \neq A \subset I_o^2$, let $c = sup\pi_1(A)$, then

$$supA = \begin{cases} c \times 0 & c \notin \pi_1(A) \\ c \times sup\big((c \times I) \cap A\big) & c \in \pi_1(A) \end{cases}$$

Definition 11.3. Given points x, y of the space X, a **path** in X from x to y is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$. A space X is **path connected** if every pair of points of X can be joined by a path in X.

Proposition 11.5. *It is easy to see that a path-connected space is connected.*

Proposition 11.6 (Path-connected version for Proposition 11.1)**.** *Let $p \in A_\alpha (\forall \alpha \in J)$, A_α is path-connected, then $\bigcup A_alpha$ is path-connected.*

Proposition 11.7 (Path-connected version for Proposition 11.3)**.** *The continuous image of a path-connected space is path-connected.*

Proposition 11.8 (Path-connected version for Theorem 11.1)**.** *If X, Y are path-connected, then $X \times Y$ is path-connected.*

Example 11.5 (The unit ball $B^n = \{x \mid \|x\| \leq 1\}$ in $\mathbb{R}^n$ is path connected.). Define $\gamma(t) = (1-t)x + ty$ for any given points x, y.

Example 11.6 (The punctured euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.). For x, y, find z s.t. the path between x, z or y, z do not go through the origin.

Example 11.7 (The unit sphere $S^n - 1 = \{x \mid \|x\| = 1\}$). is path connected if $n > 1$. $g(x) = \frac{x}{\|x\|}$ maps the unit ball to the unit sphere continuously.

Example 11.8. The ordered square I_o^2 is connected but not path connected.

Proof. Let $p = 0 \times 0, q = 1 \times 1$, suppose there exist a path γ s.t. $\gamma(0) = p, \gamma(1) = q$, then by the intermediate value theorem, $I_o^2 \subset \gamma([0,1])$, thus $\forall x \in I, U_x = \gamma^{-1}(\{x\} \times (0,1))$ is open in $[0,1]$. These U_x are all disjoint and open, however, this contradicts the fact that there are only countable disjoint open subsets of $[0,1]$. ☐

eg11 *Example* 11.9 (No path-connected version for Proposition 11.2). Let $S = \{x \times sin(\frac{1}{x}) \mid 0 < x \leq 1\}$.

$\because$ S is the image of the connected set $(0,1]$ under a continuous map(by Theorem 7.3), $\therefore$ S is connected. $\therefore$ $\bar{S}$ is connected(by Proposition 11.2).

We call $\bar{S}$ **topologist's sine curve**(illustrated below). It equals the union of S and the vertical interval $\{0\} \times [-1,1]$. We show that $\bar{S}$ is not path connected.

Proof. Suppose $\gamma : [0,1] \to \bar{S}$ is a path joining $(0,0)$ and $(1, sin1)$. Then $\gamma^{-1}(\{0\} \times [-1,1])$ is closed in $[0,1]$, denote its supremum by b, then $b < 1$. Thus $\forall t \in (b,1], \gamma(t) \in S$.

Now we would like to take some t_n, s.t. $t_n \to b, \gamma(t_n) = (*, (-1)^n)$.

Denote $z_n = \frac{1}{2n\pi + (-1)^n \frac{\pi}{2}}$; Now we shall show that $\exists t_n$, s.t. $t_n \to b, x(t_n) = z_n$.

$\because$ $x(b) = 0, x(1) = 1$, by the intermediate value theorem, $\exists t_1 \in (b,1]$, s.t. $x(t_1) = z_1$; then by the intermediate value theorem, $\exists t_2 < t_1$, s.t. $x(t_2) = z_2$; we do by induction, then we obatin a decreasing sequence t_n, so it is convergence, however, $\gamma(t_n)$ is not convergent, so this contradicts to γ is continuous. ☐

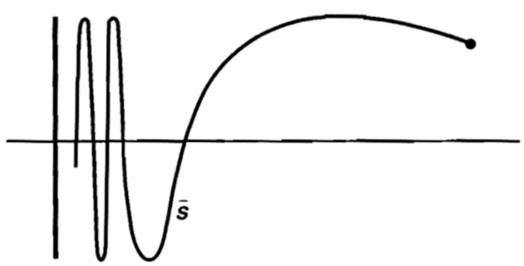

Figure 6: topologist's sine curve

12 Components and Local Connectedness

Definition 12.1. Let X be a topological space, $x \in X$, then:

(1) The biggest connected subspace containing x is called the components (or the connected components) containing x.

(2) The biggest path connected subspace containing x is called the path components containing x.

Proposition 12.1. *(1)The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*

(2)Each component of X is closed.

(3)If X only has finitely many components, then each componentis also open in X.

Proof. (2) Note that the closure of a connected subspace of X is connected.

(3) Note that its complement is a finite union of closed sets. □

Proposition 12.2. *(1)The path components of X are connected disjoint subspaces of X whose union is X, such that each nonempty path connected subspace of X intersects only one of them.*

(2)Each components is the union of some path connected components.

Example 12.1. Each component of $\mathbb{Q}$ consists of a single point.

Example 12.2. Let S be defined as in Example 11.9. Then $\bar{S}$ has 2 path components, one is S and the other is the vertical interval $\{0\} \times [-1, 1]$.

Proof. First, we know from Example 11.9 that $\bar{S}$ is not path connected. So, it has at least 2 path components. Now, we construct a partition that $\bar{S} = S \cup \{0\} \times [-1, 1]$, where S and $\{0\} \times [-1, 1]$ are both path connected. So they are the only 2 path components of $\bar{S}$. □

Example 12.3. $X = \bar{S} \setminus \{0 \times q \mid q \in \mathbb{Q}\}$ is connected, for $S \subset X \subset \bar{S}$.

The path components of $\bar{S}$ are $S, \{(0 \times r)\}_{r \in \mathbb{Q}}$, so $\bar{S}$ has only one component but uncountably many path components.

Definition 12.2. A space X is said to be **locally connected at** x if for every neighborhood U of x, $\exists$ a connected neighborhood V of x contained in U. If X is locally connected at each points of X, X is said to be **locally connected**. **Locally path connected** can be defined as above similarly.

Remark. For some purposes, it is more important that the space satisfy a connectedness condition locally. Roughly speaking, local connectedness means that each point has "arbitrary small" neighborhoods that are connected.

eg12(3) *Example* 12.4. (1) $[-1, 0) \cap (0, 1]$ is not connected but it is locally connected.

(2) $\mathbb{Q}$ are neither connected nor locally connected.

(3) $\bar{S}$ is connected but not locally connected. (Consider 0×0, for every neighborhood U of 0×0, and for any V open s.t. $x \in V \subset U$, $V \cap S$ must have at least 2 disjoint open segement, so there exists a partition. (c.f.Figure 7))

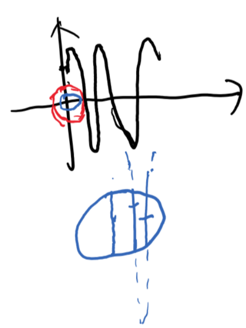

Figure 7: For Example 12.4(3) figure12

Proposition 12.3. *A space X is locally connected $\iff$ for $\forall$ oepn set U, each component of U is open in X.*

prop12 **Proposition 12.4.** *A space X is locally path connected $\iff$ for $\forall$ oepn set U, each path component of U is open in X.*

Theorem 12.1. *If X is locally path connected, then the collection of all components is the same as that of all the path components.*

Proof. Let C be a component, one only need to prove C is path connected.

Let $C = \bigcup_{\alpha \in J} P_\alpha$, by Proposition 12.4, P_α is open($\forall \alpha \in J$), suppose $\#J \neq 1$, then P_{α_0} and $\bigcup_{\alpha \in J \setminus \{\alpha_0\}} P_\alpha$ form a partition, which leads to contradiction. $\square$

13 Compact Spaces

Definition 13.1. A collection $\mathscr{A}$ of subsets of a space X is said to **cover** X or to be a **covering** of X, if $\bigcup_{A \in \mathscr{A}} A = X$. It is called an **open covering** of X if its elements are open subsets of X.

Definition 13.2. A space X is said to be **compact** if every open covering of X contains a finite subcollection that also covers X.

Example 13.1. The real line $\mathbb{R}$ is not compact, for $\mathscr{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$ contains no finite subcollection that covers $\mathbb{R}$.

Example 13.2. $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is compact.

Given an open covering, there is a set U covers 0, then there left finitely many points to be covered.

Lemma 13.1. *Let Y ne a subspace of X. Then Y is compact $\iff$ every covering of Y by sets open in X contains a finite subcollection covering Y.*

thm13.1 **Theorem 13.1.** *Every closed subspace Y of a compact space X is compact.*

Proof. Let $\mathscr{A}$ be a covering of Y by sets open in X, then consdier $\mathscr{B} = \mathscr{A} \cup \{X \setminus Y\}$ $\square$

thm13 **Theorem 13.2.** *Every compact subspace Y of a Hausdorff space X is closed.*

Proof. Let $x_0 \in X \setminus Y$, then for each $y \in Y$, we can choose disjoint open neighborhoods U_y, V_y of x_0, y respectively. Then V_y forms a open covering of Y, therefore, we can choose $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$ cover Y. The open set $V = \bigcup_{i=1}^n V_{y_i}$ is disjoint from the open set $U = \bigcap_{i=1}^n U_{y_i}$, U is a neighborhood of x_0 disjoint from Y. $\square$

Definition 13.3. If a space X satisfies:

(1) One-point sets are closed in X,

(2) For $\forall x \in X$, $\forall$ closed set $C \ni x$, $\exists$ open sets U, V satisfying that $x \in U, C \subset V, U \cap V = \varnothing$,

then X is said to be **regular**.

Definition 13.4. If a space X satisfies:

(1) One-point sets are closed in X,

(2) For $\forall C_1, C_2$ disjoint closed sets in X, $\exists$ open sets U, V satisfying that $C_1 \subset U, C_2 \subset V, U \cap V = \varnothing$,

then X is said to be **normal**.

Corollary 13.1. *If X is a compact Hausdorff space, then X is normal.*

Proof. The statement can be proved by applying the result obtained in the course of the preceding proof of Theorem 13.2. $\square$

Theorem 13.3. *The image of a compact space under a continuous map is compact.*

Theorem 13.4. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We shall prove the images of closed sets of X under f are closed in Y by applying Theorem 13.2, 13.1 $\square$

Theorem 13.5. *Let X_1, X_2 be compact spaces, then $X_1 \times X_2$ is compact.*

Lemma 13.2 (The tube lemma)**.** *Consider the product space $X_1 \times X_2$, where X_2 is compact. If $N \subset X_1 \times X_2$ is an open set containing $\{x\} \times X_2$, $(x \in X_1)$, then $\exists$ open neighborhood $W \ni x$, s.t. $W \times X_2 \subset N$.*