

# ASE o MLqE Story Latest

Runze

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## 1 Problem Description

### 1.1 Uncontaminated Model

Let  $F$  be a distribution on  $\mathcal{X} \in \mathbb{R}^d$ , satisfying  $x^T y \geq 0$  for all  $x, y \in \mathcal{X}$ . We now generate  $m$  i.i.d. graphs under the RDPG( $F$ ) model. First sample  $X_1, \dots, X_n$  independently from distribution  $F$ , and define  $X = [X_1, \dots, X_n]^T \in \mathbb{R}^{n \times d}$ ,  $P = XX^T \in [0, R]^{n \times n}$ , where  $R$  is a constant. Then we can sample  $m$  conditionally i.i.d. symmetric and hollow graphs  $G^{(1)}, \dots, G^{(m)}$ , such that conditioned on  $X$ ,  $G_{ij}^{(t)} \stackrel{\text{ind}}{\sim} \text{Exp}(P_{ij})$  for each  $1 \leq t \leq m$ ,  $1 \leq i < j \leq n$ .

Note: We are now considering the SBM model as a RDPG.

### 1.2 Contaminated Observations

Now we assume the observed edges are contaminated with probability  $\epsilon$ .

Let  $G$  be a distribution on  $\mathcal{Y} \in \mathbb{R}^{d'}$ , satisfying  $x^T y \geq 0$  for all  $x, y \in \mathcal{Y}$ . First sample  $X$  from  $F$  and  $Y$  from  $G$ . Then we sample  $m$  conditionally i.i.d. symmetric and hollow graphs  $A^{(1)}, \dots, A^{(m)}$  such that conditioning on  $X$  and  $Y$ ,  $A_{ij}^{(t)} \stackrel{\text{ind}}{\sim} (1 - \epsilon)\text{Exp}(P_{ij}) + \epsilon\text{Exp}(C_{ij})$  for each  $1 \leq t \leq m$ ,  $1 \leq i < j \leq n$ , where the contamination is a rank- $d'$  matrix  $C = YY^T \in [0, R]^{n \times n}$ ,  $Y \in \mathbb{R}^{n \times d'}$ .

### 1.3 Goal

Given the contaminated observation of adjacency matrices of  $m$  graphs, i.e.  $A^{(1)}, \dots, A^{(m)}$ , we want to estimate the mean of the collection of uncontaminated graphs  $P$ .

## 2 Candidate Estimators

After observing contaminated adjacency matrices of  $m$  graphs  $A^{(1)}, \dots, A^{(m)}$ , we want to propose a good estimator for the mean of the collection of graphs  $P$ .

### 2.1 $\hat{P}^{(1)}$ based on entry-wise MLE

Under the independent edge setting, we can simplify the problem to finding an entry-wise estimate of  $P$ . And MLE is always our first choice, which exists and

happen to be  $\bar{A}$ , the entry-wise mean in this case. For consistency, we define  $\hat{P}^{(1)} = \bar{A}$ .

## 2.2 $\hat{P}^{(q)}$ based on entry-wise ML $q$ E

Since the observations are contaminated, robust estimators are preferred. A modified MLE estimator, the maximum likelihood L- $q$  estimator [2, 6], is considered in this case. Note that there might be multiple solution to the ML $q$  equation, we define the ML $q$ E to be the largest solution (which is still less than MLE when the model is exponential distribution). Denote  $\hat{P}^{(q)}$  as the entry-wise ML $q$ E.

**Remark:** MLE is a special case of ML $q$ E when  $q = 1$ . So we notate the entry-wise MLE to be  $\hat{P}^{(1)}$  in consistent with entry-wise ML $q$ E  $\hat{P}^{(q)}$ .

## 2.3 $\tilde{P}^{(1)}$ based on ASE of entry-wise MLE

By taking advantages of the graph structure, we expect a better performance after applying a rank-reduction procedure to the entry-wise MLE  $\hat{P}^{(1)}$  under the SBM. So we first apply ASE to  $\hat{P}^{(1)}$  to get the latent positions  $\hat{X}^{(1)}$  in dimension  $d^{(1)}$ , and then define  $\tilde{P}^{(1)} = (\hat{X}^{(1)}\hat{X}^{(1)T})_{\text{tr}}$ , where each element is a projection of  $\hat{X}_i^{(1)}\hat{X}_j^{(1)T}$  onto  $[0, \max(\hat{P}_{ij}^{(1)}, R)]$ .

## 2.4 $\tilde{P}^{(q)}$ based on ASE of entry-wise ML $q$ E

Similarly, we also expect a better performance after applying a rank-reduction procedure to the entry-wise ML $q$ E  $\hat{P}^{(q)}$  under the SBM. So we first apply ASE to  $\hat{P}^{(q)}$  to get the latent positions  $\hat{X}^{(q)}$  in dimension  $d^{(q)}$ , and then define  $\tilde{P}^{(q)} = (\hat{X}^{(q)}\hat{X}^{(q)T})_{\text{tr}}$ , where each element is a projection of  $\hat{X}_i^{(q)}\hat{X}_j^{(q)T}$  onto  $[0, \max(\hat{P}_{ij}^{(q)}, R)]$ .

## 2.5 Summary

Thus, we should choose the estimator  $\tilde{P}^{(q)}$ .

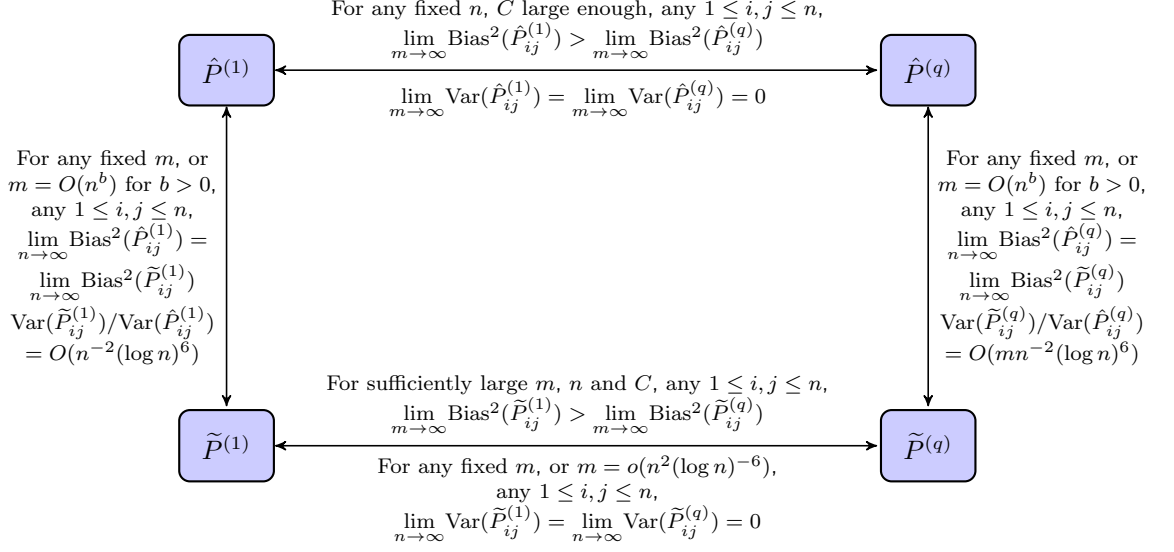


Figure 1: Relationship between four estimators.

### 3 Proof

#### 3.1 $\hat{P}^{(q)}$ better than $\hat{P}^{(1)}$

**Lemma 3.1** Consider the model  $X_1, \dots, X_m \stackrel{iid}{\sim} \text{Exp}(P)$  with  $m \geq 2$  and  $E[X_1] = P$ . Given any data  $x = (x_1, \dots, x_m)$  such that  $x_{(1)} > 0$  and not all  $x_i$ 's are the same, then no matter how the data is sampled, we have

- There exists at least one solution to the MLq equation;
- All the solutions to the MLq equation are less than the MLE.

Thus the MLqE  $\hat{P}^{(q)}$ , the root closest to the MLE, is well defined.

**Proof:** The MLE is

$$\hat{P}^{(1)}(x) = \bar{x}.$$

Consider the continuous function  $g(\theta, x) = \sum_{i=1}^m e^{-\frac{(1-q)x_i}{\theta}} (x_i - \theta)$ . Then the MLq equation is  $g(\theta, x) = 0$ .

Let  $x_{(1)} \leq \dots \leq x_{(l)} \leq \bar{x} \leq x_{(l+1)} \leq \dots \leq x_{(m)}$ . Define  $s_i = \bar{x} - x_{(i)}$  for  $1 \leq i \leq l$ , and  $t_i = x_{(l+i)} - \bar{x}$  for  $1 \leq i \leq m-l$ . Note that  $\sum_{i=1}^l s_i = \sum_{i=1}^{m-l} t_i$ .

Then for any  $\theta \geq \bar{x}$ , we have

$$\begin{aligned}
g(\theta, x) &= \sum_{i=1}^m e^{-\frac{(1-q)x_{(i)}}{\theta}} (x_{(i)} - \theta) = \sum_{i=1}^m e^{-\frac{(1-q)x_{(i)}}{\theta}} (x_{(i)} - \bar{x} + \bar{x} - \theta) \\
&= -\sum_{i=1}^l e^{-\frac{(1-q)x_{(i)}}{\theta}} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i + \sum_{i=1}^m e^{-\frac{(1-q)x_{(i)}}{\theta}} (\bar{x} - \theta) \\
&\leq -\sum_{i=1}^l e^{-\frac{(1-q)x_{(i)}}{\theta}} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i \\
&\leq -e^{-\frac{(1-q)x_{(l+1)}}{\theta}} \sum_{i=1}^l s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i \\
&\leq -e^{-\frac{(1-q)x_{(l+1)}}{\theta}} \sum_{i=1}^{m-l} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i \\
&\leq -\sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&= 0,
\end{aligned}$$

and equality holds if and only if all  $x_i$ 's are the same, which is excluded by the assumption. Thus  $g(\theta, x) < 0$  for any  $\theta \geq \bar{x}$ .

Denote any solution to the MLq equation to be  $\hat{P}^{(q)}(x)$ , then we also know:

- $g(\hat{P}^{(q)}(x), x) = 0$ ;
- $\lim_{\theta \rightarrow 0^+} g(\theta, x) = 0$ ;
- $g(\theta, x) > 0$  when  $\theta < x_{(1)}$ ;

Thus there exists at least one solution to the MLq equation. And all solutions to the MLq equation are between  $x_{(1)}$  and  $\bar{x}$ , i.e. less than the MLE. ■

**Lemma 3.2** *Consider the exponential distribution model as in Lemma 3.1 while the data is actually sampled under the contaminated model  $X, X_1, \dots, X_m \stackrel{iid}{\sim} (1-\epsilon)\text{Exp}(P) + \epsilon\text{Exp}(C)$ . Denote such contaminated distribution as  $F$ . Then there exists at least one solution  $\theta(F)$  of the population version of MLq equation, i.e.  $E_F[e^{-\frac{(1-q)X}{\theta(F)}}(X - \theta(F))] = 0$ , such that  $\theta(F) < E_F[\bar{X}] = (1-\epsilon)P + \epsilon C$ . So we can define  $\theta(F_{ij})$  to be the largest root which is less than  $E_F[\bar{X}]$ .*

**Proof:** For the MLE, i.e.  $\bar{X}$ , we have  $E[\bar{X}] = (1-\epsilon)P + \epsilon C$ . According to Equation (3.2) in [2],  $\theta(F)$  satisfies

$$\frac{\epsilon C}{(C(1-q) + \theta)^2} - \frac{\epsilon}{C(1-q) + \theta} + \frac{(1-\epsilon)P}{(P(1-q) + \theta)^2} - \frac{(1-\epsilon)}{P(1-q) + \theta} = 0,$$

i.e.

$$\frac{\epsilon(\theta - Cq)}{(C(1-q) + \theta)^2} = \frac{(1-\epsilon)(Pq - \theta)}{(P(1-q) + \theta)^2}.$$

Define  $h(\theta) = (C(1-q) + \theta)^2(1-\epsilon)(Pq - \theta) - (P(1-q) + \epsilon)^2\epsilon(\theta - Cq)$ . Then  $\lim_{\theta \rightarrow \infty} h(\theta) = -\infty$ ,  $h(0) > 0$ , and  $h(Cq) < 0$ . Consider  $q$  as the variable and

solve the equation  $h(E[\bar{X}]) = 0$ , we have three roots and one of them is  $q = 1$  obviously. The other two roots are

$$\frac{(P+C)((P-C)^2\epsilon(1-\epsilon)+2PC)}{2PC(P\epsilon+C(1-\epsilon))} \pm \frac{1}{2} \sqrt{\frac{\epsilon(1-\epsilon)(C-P)^4-4P^2C^2}{P^2C^2(P\epsilon+C(1-\epsilon))^2}}.$$

For the first part,

$$\frac{(P+C)((P-C)^2\epsilon(1-\epsilon)+2PC)}{2PC(P\epsilon+C(1-\epsilon))} > 1 + \frac{(P-C)^2\epsilon(1-\epsilon)(P+C)}{2PC(P\epsilon+C(1-\epsilon))}.$$

To prove the roots are greater or equal to 1, we just need to show

$$(P-C)^2\epsilon(1-\epsilon)(P+C)^2 \geq \epsilon(1-\epsilon)(C-P)^4 - 4P^2C^2.$$

Then it is sufficient to show that

$$(P+C)^2 \geq (C-P)^2,$$

which is true. Combined with the fact that when  $q = 0$ ,  $h(E[\bar{X}]) < 0$ , we have for any  $0 < q < 1$ ,  $h(E[\bar{X}]) < 0$ .

The equation  $h(\theta) = 0$  is a cubic polynomial, so it has at most three real roots. Combined with the fact that  $h(0) > 0$ , we have for any  $0 < q < 1$ , there exists at least one root of the population version of ML $q$  equation which is less than  $E[\bar{X}] = (1-\epsilon)P + \epsilon C$ . ■

**Lemma 3.3** *For any  $a > 0$ , we have*

$$\sup_{\theta \in [a, R]} \left| \frac{1}{m} \sum_{i=1}^m e^{-\frac{(1-q)X_i}{\theta}} (X_i - \theta) - E_F[e^{-\frac{(1-q)X}{\theta}} (X - \theta)] \right| \xrightarrow{a.s.} 0.$$

**Proof:** Define  $g(x, \theta) = e^{-\frac{(1-q)x}{\theta}}(x - \theta)$  and  $d(x) = e^{-\frac{(1-q)x}{R}}(x + R)$ . Then  $E_F[d(X)] < \infty$  and  $g(x, \theta) \leq d(x)$  for all  $\theta \in [a, R]$ . Combined with the fact that  $[a, R]$  is compact and the function  $g(x, \theta)$  is continuous at each  $\theta$  for all  $x > 0$  and measurable function of  $x$  at each  $\theta$ , we have the uniform convergence by Lemma 2.4 in [4]. ■

**Lemma 3.4**  $\hat{P}_{ij}^{(q)} \xrightarrow{P} \theta(F_{ij})$ , where  $F_{ij}$  is the contaminated distribution  $(1-\epsilon)\text{Exp}(P_{ij}) + \epsilon\text{Exp}(C_{ij})$ . That is,  $\hat{P}_{ij}^{(q)}$  is an consistent estimator of  $\theta(F_{ij})$ .

**Proof:** ■

**Lemma 3.5** *For any  $0 < q < 1$ , there exists  $C_0(P_{ij}, \epsilon, q) > 0$  such that under the contaminated model with  $C > C_0(P_{ij}, \epsilon, q)$ ,*

$$\lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}^{(q)}] - P_{ij} \right| < \lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

for  $1 \leq i, j, \leq n$  and  $i \neq j$ .

**Proof:** For the MLE  $\hat{P}_{ij}^{(1)} = \bar{A}_{ij}$ ,

$$E[\hat{P}_{ij}^{(1)}] = E[\bar{A}_{ij}] = \frac{1}{m} \sum_{t=1}^m E[A_{ij}^{(t)}] = E[A_{ij}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij}.$$

For the MLqE  $\hat{P}_{ij}^{(q)}$ , according to Equation (3.2) in [2], the expectation  $E[\hat{P}_{ij}^{(q)}]$ , denoted as  $\theta$  for simplicity, satisfies

$$\frac{\epsilon C_{ij}}{(C_{ij}(1 - q) + \theta)^2} - \frac{\epsilon}{C_{ij}(1 - q) + \theta} + \frac{(1 - \epsilon)P_{ij}}{(P_{ij}(1 - q) + \theta)^2} - \frac{(1 - \epsilon)}{P_{ij}(1 - q) + \theta} = 0,$$

i.e.

$$\frac{\epsilon(\theta - C_{ij}q)}{(C_{ij}(1 - q) + \theta)^2} = \frac{(1 - \epsilon)(P_{ij}q - \theta)}{(P_{ij}(1 - q) + \theta)^2}.$$

Thus  $\theta - C_{ij}q$  and  $\theta - P_{ij}q$  should have different signs. Combined with  $C_{ij} > P_{ij}$ , we have

$$qP_{ij} < \theta.$$

To have a smaller bias in absolute value, we need

$$|\theta - P_{ij}| < \epsilon(C_{ij} - P_{ij}).$$

Thus combined with Lemma 3.1, we need

$$qP_{ij} > P_{ij} - \epsilon(C_{ij} - P_{ij}),$$

i.e.

$$C_{ij} > P_{ij} + \frac{(1 - q)P_{ij}}{\epsilon} = C_0(P_{ij}, \epsilon, q).$$

■

### Lemma 3.6

**Proof:** According to Lemma 3.1, the expectation  $E[\hat{P}_{ij}^{(q)}]$ , denoted as  $\theta$  for simplicity, satisfies

$$\frac{\epsilon(\theta - C_{ij}q)}{(C_{ij}(1 - q) + \theta)^2} = \frac{(1 - \epsilon)(P_{ij}q - \theta)}{(P_{ij}(1 - q) + \theta)^2},$$

i.e.

$$\epsilon(\theta - C_{ij}q)(P_{ij}(1 - q) + \theta)^2 = (1 - \epsilon)(P_{ij}q - \theta)(C_{ij}(1 - q) + \theta)^2.$$

Define

$$g(\theta) = (1 - \epsilon)(P_{ij}q - \theta)(C_{ij}(1 - q) + \theta)^2 - \epsilon(\theta - C_{ij}q)(P_{ij}(1 - q) + \theta)^2,$$

then  $g(0) = (1 - \epsilon)P_{ij}q(C_{ij}(1 - q))^2 + \epsilon(C_{ij}q)(P_{ij}(1 - q))^2$ . ■

### Lemma 3.7

$$\lim_{m \rightarrow \infty} \text{Var}(\hat{P}_{ij}^{(1)}) = \lim_{m \rightarrow \infty} \text{Var}(\hat{P}_{ij}^{(q)}) = 0,$$

for  $1 \leq i, j \leq n$ .

**Proof:** MLE simply follows a central limit theorem, which means the variance goes to 0 as  $m \rightarrow \infty$ . For MLqE, ■

### 3.2 $\tilde{P}^{(1)}$ better than $\hat{P}^{(1)}$

**Theorem 3.8** (*Matrix Bernstein: Subexponential Case*). Consider a finite sequence  $\{X_k\}$  of independent, random, self-adjoint matrices with dimension  $d$ . Assume that

$$E[X_k] = 0 \quad \text{and} \quad E[X_k^p] \preceq \frac{p!}{2} R^{p-2} A_k^2 \quad \text{for } p = 2, 3, 4, \dots$$

Compute the variance parameter

$$\sigma^2 := \left\| \sum_k A_k^2 \right\|.$$

Then the following chain of inequalities holds for all  $t \geq 0$ .

$$P \left( \lambda_{\max} \left( \sum_k X_k \right) \geq t \right) \leq d \cdot \exp \left( \frac{-t^2/2}{\sigma^2 + Rt} \right).$$

**Remark:** Theorem 6.2 in [9].

**Theorem 3.9** Let  $P$  and  $C$  be two  $n$ -by- $n$  symmetric matrices satisfying element-wise conditions  $0 < P_{ij} \leq C_{ij} \leq R$  for some constant  $R > 0$ . For  $0 < \epsilon < 1$ , we define  $m$  symmetric and hollow matrices as

$$A^{(t)} \stackrel{iid}{\sim} (1 - \epsilon) \text{Exp}(P) + \epsilon \text{Exp}(C),$$

for  $1 \leq t \leq m$ . Let  $\hat{P}^{(1)}$  be the element-wise MLE based on exponential distribution with  $m$  observations. Define  $H_{ij}^{(1)} = E[\hat{P}_{ij}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij}$ , then for any constant  $c > 0$ , there exists another constant  $n_0(c)$ , independent of  $n$ ,  $P$ ,  $C$  and  $\epsilon$ , such that if  $n > n_0$ , then for all  $\eta$  satisfying  $n^{-c} \leq \eta \leq 1/2$ ,

$$P \left( \|\hat{P}^{(1)} - H^{(1)}\|_2 \leq 4R\sqrt{n \ln(n/\eta)/m} \right) \geq 1 - \eta.$$

**Remark:** This is the extended version of Theorem 3.1 in [5].

**Proof:** Let  $\{e_i\}_{i=1}^n$  be the canonical basis for  $\mathbb{R}^n$ . For each  $1 \leq i, j \leq n$ , define a corresponding matrix  $G_{ij}$ :

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus

$$\hat{P}^{(1)} = \sum_{1 \leq i < j \leq n} \hat{P}_{ij}^{(1)} G_{ij} = \frac{1}{m} \sum_{t=1}^m \sum_{1 \leq i < j \leq n} A_{ij}^{(t)} G_{ij}$$

and

$$H^{(1)} = \sum_{1 \leq i < j \leq n} H_{ij}^{(1)} G_{ij}.$$

Then we have  $\hat{P}^{(1)} - H^{(1)} = \frac{1}{m} \sum_{1 \leq t \leq m, 1 \leq i < j \leq n} X_{ij}^{(t)}$ , where  $X_{ij}^{(t)} \equiv \left( A_{ij}^{(t)} - H_{ij}^{(1)} \right) G_{ij}$  for  $1 \leq t \leq m$  and  $1 \leq i < j \leq n$ .

First bound the  $k$ -th moment of  $X_{ij}$  for  $1 \leq i < j \leq n$  as following:

$$\begin{aligned}
E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k] &\leq (1 - \epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1 + k, -H_{ij}/P_{ij}) \\
&\quad + \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1 + k, -H_{ij}/C_{ij}) \\
&\leq ((1 - \epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k) k! \\
&\leq ((1 - \epsilon) \cdot P_{ij}^k + \epsilon \cdot C_{ij}^k) k! \\
&\leq R^k k!,
\end{aligned} \tag{1}$$

Combined with

$$G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & k \text{ is even;} \\ e_i e_j^T + e_j e_i^T, & k \text{ is odd,} \end{cases}$$

thus we have

1. When  $k$  is even,

$$E[(X_{ij}^{(t)})^k] = E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k] G_{ij}^2 \preceq k! R^k G_{ij}^2;$$

2. When  $k$  is odd,

$$E[(X_{ij}^{(t)})^k] = E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k] G_{ij} \preceq k! R^k G_{ij}^2.$$

So

$$E[(X_{ij}^{(t)})^k] \preceq k! R^k G_{ij}^2.$$

Let

$$\sigma^2 := \left\| \sum_{1 \leq t \leq m, 1 \leq i < j \leq n} (\sqrt{2} R G_{ij})^2 \right\|_2 = 2R^2 m \|(n-1)I\|_2 = 2R^2 m(n-1).$$

Notice that random matrices  $X_{ij}^{(t)}$  are independent, self-adjoint and have mean zero, apply Theorem 3.8 we have

$$\begin{aligned}
P\left(\lambda_{\max}(\hat{P}^{(1)} - H^{(1)}) \geq t\right) &= P\left(\lambda_{\max}\left(\frac{1}{m} \sum_{1 \leq t \leq m, 1 \leq i < j \leq n} X_{ij}^{(t)}\right) \geq t\right) \\
&= P\left(\lambda_{\max}\left(\sum_{1 \leq t \leq m, 1 \leq i < j \leq n} X_{ij}^{(t)}\right) \geq mt\right) \\
&\leq n \exp\left(-\frac{(mt)^2/2}{\sigma^2 + Rmt}\right) \\
&\leq n \exp\left(-\frac{mt^2/2}{2R^2 n + Rt}\right).
\end{aligned}$$

Now consider  $Y_{ij}^{(t)} \equiv (H_{ij}^{(1)} - A_{ij}^{(t)}) G_{ij}$ , for  $1 \leq t \leq m$  and  $1 \leq i < j \leq n$ .

Then we have  $H^{(1)} - \hat{P}^{(1)} = \frac{1}{m} \sum_{1 \leq t \leq m, 1 \leq i < j \leq n} Y_{ij}^{(t)}$ . Since

$$E[(H^{(1)} - \hat{P}^{(1)})^k] = (-1)^k E[(\hat{P}^{(1)} - H^{(1)})^k],$$



1. When  $k$  is even,

$$E[(Y_{ij}^{(t)})^k] = E[(\hat{P}^{(1)} - H^{(1)})^k] G_{ij}^2 \preceq k! R^k G_{ij}^2;$$

2. When  $k$  is odd,

$$E[Y_{ij}^k] = -E[(\hat{P}^{(1)} - H^{(1)})^k] G_{ij} \preceq k! R^k G_{ij}^2.$$

Thus by similar arguments,

$$\begin{aligned} P\left(\lambda_{\min}(\hat{P}^{(1)} - H^{(1)}) \leq -t\right) &= P\left(\lambda_{\max}(H^{(1)} - \hat{P}^{(1)}) \geq t\right) \\ &\leq n \exp\left(-\frac{mt^2/2}{2R^2n + Rt}\right). \end{aligned}$$

Therefore we have

$$P\left(\|\hat{P}^{(1)} - H^{(1)}\|_2 \geq t\right) \leq n \exp\left(-\frac{mt^2/2}{2R^2n + Rt}\right).$$

Now let  $c > 0$  be given and assume  $n^{-c} \leq \eta \leq 1/2$ . Then there exists a  $n_0(c)$  independent of  $n$ ,  $P$ ,  $C$  and  $\epsilon$  such that whenever  $n > n_0(c)$ ,

$$t = 4R\sqrt{n \ln(n/\eta)/m} \leq 6Rn.$$

Plugging this  $t$  into the equation above, we get

$$P(\|\hat{P}^{(1)} - H^{(1)}\|_2 \geq 4R\sqrt{n \ln(n/\eta)/m}) \leq n \exp\left(-\frac{t^2}{16R^2n}\right) = \eta.$$

Define  $H^{(1)} = E[\hat{P}^{(1)}] = (1 - \epsilon)P + \epsilon C$ , where  $P = XX^T$ ,  $X \in \mathbb{R}^{n \times d}$ ,  $C = YY^T$ ,  $Y \in \mathbb{R}^{n \times d'}$ . Let  $d^{(1)} = \text{rank}(H^{(1)})$  be the dimension in which we are going to embed  $\hat{P}^{(1)}$ . Then we can define  $H^{(1)} = ZZ^T$  where  $Z \in \mathbb{R}^{n \times d^{(1)}}$ . Since  $H^{(1)} = [\sqrt{1 - \epsilon}X, \sqrt{\epsilon}Y][\sqrt{1 - \epsilon}X, \sqrt{\epsilon}Y]^T$ , we have  $d^{(1)} \leq d + d'$ . ■

For simplicity, from now on, we will use  $\hat{P}$  to represent  $\hat{P}^{(1)}$ , use  $H$  to represent  $H^{(1)}$  and use  $k$  to represent the dimension  $d^{(1)}$  we are going to embed. Assume  $H = USU^T = ZZ^T$ , where  $Z = [Z_1, \dots, Z_n]^T$  is a  $n$ -by- $k$  matrix. Then our estimate for  $Z$  up to rotation is  $\hat{Z} = \hat{U}\hat{S}^{1/2}$ , where  $\hat{U}\hat{S}\hat{U}^T$  is the rank- $k$  spectral decomposition of  $|\hat{P}| = (\hat{P}^T \hat{P})^{1/2}$ .

Furthermore, we assume that the second moment matrix  $E[Z_1 Z_1^T]$  is rank  $k$  and has distinct eigenvalues  $\lambda_i(E[Z_1 Z_1^T])$ . In particular, we assume that there exists  $\delta > 0$  such that

$$\delta < \min\left(\min_{i \neq j} |\lambda_i(E[Z_1 Z_1^T]) - \lambda_j(E[Z_1 Z_1^T])|, \lambda_k(E[Z_1 Z_1^T])\right)$$

**Lemma 3.10** *Under the above assumptions,  $\lambda_i(H) = \Theta(n)$  with high probability when  $i \leq k$ , i.e. the largest  $k$  eigenvalues of  $H$  is of order  $n$ . Moreover, we have  $\|S\|_2 = \Theta(n)$  and  $\|\hat{S}\|_2 = \Theta(n)$  with high probability.*

**Remark:** This is an extended version of Proposition 4.3 in [8].

**Proof:** Note that  $\lambda_i(H) = \lambda_i(ZZ^T) = \lambda_i(Z^T Z)$  when  $i \leq k$ . Since each entry of  $Z^T Z$  is a sum of  $n$  independent random variables each in  $[0, R]$ , i.e.  $(Z^T Z)_{ij} = \sum_{l=1}^n Z_{li} Z_{lj}$ . By Hoeffding's inequality,

$$P(|(Z^T Z - nE[Z_1 Z_1^T])_{ij}| \geq t) \leq 2 \exp(-\frac{2t^2}{nR^2}).$$

Now let  $c > 0$  and assume  $n^{-c} \leq \eta \leq 1/2$ . Let

$$t = R\sqrt{n \ln(\sqrt{2/\eta})},$$

we have

$$P(|(Z^T Z - nE[Z_1 Z_1^T])_{ij}| \geq R\sqrt{n \ln(\sqrt{2/\eta})}) \leq \eta.$$

By the union bound, we have

$$P(\|Z^T Z - nE[Z_1 Z_1^T]\|_F \geq kR\sqrt{n \ln(\sqrt{2/\eta})}) \leq k^2 \eta.$$

Then by Weyl's Theorem [3], we have

$$|\lambda_i(H) - n\lambda_i(Z_1 Z_1^T)| \leq \|Z^T Z - nE[Z_1 Z_1^T]\|_2 = O(\sqrt{n \log n})$$

with probability at least  $1 - k^2 \eta$ . Thus  $\lambda_i(H) = S_{ii} = \Theta(n)$  with probability at least  $1 - \frac{2k^2}{n^2}$  when  $i \leq k$ .

Moreover,

$$\|H\|_2 - \|H - \hat{P}\|_2 \leq \|\hat{S}\|_2 \leq \|\hat{P} - H\|_2 + \|H\|_2.$$

Combined with Theorem 3.9, with high probability we have  $\|\hat{S}\|_2 = \Theta(n)$ . ■

**Lemma 3.11** *Let  $W_1 \Sigma W_2^T$  be the singular value decomposition of  $U^T \hat{U}$ . Then for sufficiently large  $n$ ,*

$$\|U^T \hat{U} - W_1 W_2^T\|_F = O(m^{-1} n^{-1} \log n)$$

*with high probability.*

**Proof:** Let  $\sigma_1, \dots, \sigma_d$  denote the singular values of  $U^T \hat{U}$ . Then  $\sigma_i = \cos(\theta_i)$  where the  $\theta_i$  are the principal angles between the subspaces spanned by  $\hat{U}$  and  $U$ . Furthermore, by the Davis-Kahan  $\sin(\Theta)$  theorem [1], combined with Theorem 3.9 and Lemma 3.10,

$$\|\hat{U} \hat{U}^T - U U^T\|_2 = \max_i |\sin(\theta_i)| \leq \frac{\|\hat{P} - H\|_2}{\lambda_k(H)} \leq \frac{C\sqrt{n \log n/m}}{n} = O(m^{-1/2} n^{-1/2} \sqrt{\log n}) \quad (2)$$

for sufficiently large  $n$ . Here  $\lambda_k(H)$  denotes the  $k$ -th largest eigenvalue of  $H$ . We thus have

$$\begin{aligned}\|U^T \hat{U} - W_1 W_2^T\|_F &= \|\Sigma - I\|_F = \sqrt{\sum_{i=1}^k (1 - \sigma_i)^2} \\ &\leq \sum_{i=1}^k (1 - \sigma_i) \leq \sum_{i=1}^k (1 - \sigma_i^2) \\ &= \sum_{i=1}^k \sin^2(\theta_i) \leq k \|\hat{U} \hat{U}^T - U U^T\|_2^2 \\ &= O(m^{-1} n^{-1} \log n).\end{aligned}$$

We will denote the orthogonal matrix  $W_1 W_2^T$  by  $W^*$ . ■

**Lemma 3.12** *For sufficiently large  $n$ ,*

$$\begin{aligned}\|W^* \hat{S} - S W^*\|_F &= O(m^{-1/2} \log n), \\ \|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F &= O(m^{-1/2} n^{-1/2} \log n)\end{aligned}$$

and

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(m^{-1/2} n^{-3/2} \log n)$$

with high probability.

**Proof:** By Proposition 2.1 in [7] and Equation (2), we have for some orthogonal matrix  $W$ ,

$$\|\hat{U} - U W\|_F^2 \leq \frac{2 \|\hat{U} \hat{U}^T - U U^T\|_F^2}{\delta^2} \leq \frac{8k^2 \|\hat{U} \hat{U}^T - U U^T\|_2^2}{\delta^2} = O(m^{-1/2} n^{-1/2} \sqrt{\log n}).$$

Let  $Q = \hat{U} - U U^T \hat{U}$ . And  $Q$  is the residual after projecting  $\hat{U}$  orthogonally onto the column space of  $U$ , we have

$$\|Q\|_F = \|\hat{U} - U U^T \hat{U}\|_F \leq \|\hat{U} - U T\|_F = O(m^{-1/2} n^{-1/2} \sqrt{\log n}). \quad (3)$$

for all  $k \times k$  matrices  $T$ .

Then

$$\begin{aligned}W^* \hat{S} &= (W^* - U^T \hat{U}) \hat{S} + U^T \hat{U} \hat{S} = (W^* - U^T \hat{U}) \hat{S} + U^T \hat{P} \hat{U} \\ &= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) \hat{U} + U^T H \hat{U} \\ &= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + U^T H \hat{U} \\ &= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S U^T \hat{U}.\end{aligned}$$

Combined with Theorem 3.9, Lemma 3.10, Lemma 3.11, we have

$$\begin{aligned}\|W^* \hat{S} - S W^*\|_F &= \|(W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S(U^T \hat{U} - W^*)\|_F \\ &\leq \|W^* - U^T \hat{U}\|_F (\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F \|\hat{P} - H\|_2 \|Q\|_F + \|U^T (\hat{P} - H) U\|_F \\ &\leq O(m^{-1} \log n) + O(m^{-1/2} \log n) + \|U^T (\hat{P} - H) U\|_F\end{aligned}$$

with high probability. And we know  $U^T(\hat{P} - H)U$  is a  $k \times k$  matrix with  $ij$ -th entry to be

$$u_i^T(\hat{P} - H)u_j = \sum_{s=1}^n \sum_{t=1}^n (\hat{P}_{st} - H_{st})u_{is}u_{jt} = 2 \sum_{s < t} (\hat{P}_{st} - H_{st})u_{is}u_{jt}$$

where  $u_i$  and  $u_j$  are the  $i$ -th and  $j$ -th columns of  $U$ . Thus, conditioned on  $H$ ,  $U$  is fixed and  $u_i^T(\hat{P} - H)u_j$  is a sum of independent mean 0 random variables.

By Equation (1), we have

$$\begin{aligned} & E \left[ \left( (A_{st}^{(t')} - H_{st})u_{is}u_{jt} \right)^k \right] \\ & \leq k! R^k u_{is}^k u_{jt}^k \\ & \leq \frac{k!}{2} R^{k-2} (\sqrt{2} u_{is} u_{jt} R)^2. \end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{t', s < t} 2R^2 u_{is}^2 u_{jt}^2 \right| \leq mR^2,$$

then by Theorem 3.8, we have

$$P \left( \left| 2 \sum_{s < t} (\hat{P}_{st} - H_{st})u_{is}u_{jt} \right| \geq t \right) \leq \exp \left( \frac{-mt^2/8}{R^2 + Rt/2} \right).$$

Let  $t = cRm^{-1/2} \log n$  for any  $c > 0$ , we have

$$P \left( \left| 2 \sum_{s < t} (\hat{P}_{st} - H_{st})u_{is}u_{jt} \right| \geq Cm^{-1/2} \log n \right) \leq n^{-c}.$$

Thus each entry of  $U^T(\hat{P} - H)U$  is of order  $O(m^{-1} \log n)$  with high probability and

$$\|U^T(\hat{P} - H)U\|_F = O(m^{-1} \log n) \quad (4)$$

with high probability. Hence

$$\|W^* \hat{S} - SW^*\|_F = O(m^{-1/2} \log n)$$

with high probability. Also, since

$$W_{ij}^*(\lambda_j^{1/2}(\hat{P}) - \lambda_i^{1/2}(H)) = W_{ij}^* \frac{\lambda_j(\hat{P}) - \lambda_i(H)}{\lambda_j^{1/2}(\hat{P}) + \lambda_i^{1/2}(H)}$$

and the eigenvalues  $\lambda_j^{1/2}(\hat{P})$  and  $\lambda_i^{1/2}(H)$  are both of order  $\Theta(\sqrt{n})$ , we have

$$\|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F = O(m^{-1/2} n^{-1/2} \log n).$$

Similarly, since

$$W_{ij}^*(\lambda_j^{-1/2}(\hat{P}) - \lambda_i^{-1/2}(H)) = W_{ij}^* \frac{\lambda_i(H) - \lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P}) + \lambda_i^{-1/2}(H))\lambda_j(\hat{P})\lambda_i(H)}$$

and the eigenvalues  $\lambda_j(\hat{P})$  and  $\lambda_i(H)$  are both of order  $\Theta(n)$ , we have

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(m^{-1/2} n^{-3/2} \log n).$$

■

**Lemma 3.13** *There exists a rotation matrix  $W$  such that for sufficiently large  $n$ ,*

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2} n^{-1/2} (\log n)^{3/2})$$

*with high probability.*

**Proof:** Let  $Q_1 = UU^T \hat{U} - UW^*$ ,  $Q_2 = W^* \hat{S}^{1/2} - S^{1/2} W^*$  and  $Q_3 = \hat{U} - UW^* = \hat{U} - UU^T \hat{U} + Q_1 = Q + Q_1$ . Then since  $UU^T P = P$  and  $\hat{U} \hat{S}^{1/2} = \hat{P} \hat{U} \hat{S}^{-1/2}$ ,

$$\begin{aligned} \hat{Z} - US^{1/2} W^* &= \hat{U} \hat{S}^{1/2} - UW^* \hat{S}^{1/2} + U(W^* \hat{S}^{1/2} - S^{1/2} W^*) \\ &= (\hat{U} - UU^T \hat{U}) \hat{S}^{1/2} + Q_1 \hat{S}^{1/2} + UQ_2 \\ &= (\hat{P} - H) \hat{U} \hat{S}^{-1/2} - UU^T (\hat{P} - H) \hat{U} \hat{S}^{-1/2} + Q_1 \hat{S}^{1/2} + UQ_2 \\ &= (\hat{P} - H) UW^* \hat{S}^{-1/2} - UU^T (\hat{P} - H) UW^* \hat{S}^{-1/2} \\ &\quad + (I - UU^T) (\hat{P} - H) Q_3 \hat{S}^{-1/2} + Q_1 \hat{S}^{1/2} + UQ_2. \end{aligned}$$

By Lemma 3.11,

$$\|Q_1\|_F \leq \|U\|_F \|U^T \hat{U} - W^*\|_F = O(m^{-1} n^{-1} \log n).$$

By Lemma 3.12,

$$\|Q_2\|_F = O(m^{-1/2} n^{-1/2} \log n).$$

By Equation (3),

$$\|Q_3\|_F \leq \|Q\|_F + \|Q_1\|_F = O(m^{-1/2} n^{-1/2} (\log n)^{1/2}).$$

By Equation (4),

$$\|UU^T (\hat{P} - H) UW^* \hat{S}^{-1/2}\|_F \leq \|U^T (\hat{P} - H) U\|_F \|\hat{S}^{-1/2}\|_2 = O(m^{-1} n^{-1/2} \log n).$$

By Lemma 3.12,

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(m^{-1/2} n^{-3/2} \log n).$$

Therefore,

$$\begin{aligned} &\|\hat{Z} - US^{1/2} W^*\|_F \\ &= \|(\hat{P} - H) UW^* \hat{S}^{-1/2}\|_F + O(m^{-1} n^{-1/2} \log n) + \|I - UU^T\|_2 \|\hat{P} - H\|_2 O(m^{-1/2} n^{-1} (\log n)^{1/2}) \\ &\quad + O(m^{-1} n^{-1/2} \log n) + O(m^{-1/2} n^{-1/2} \log n) \\ &= \|(\hat{P} - H) UW^* \hat{S}^{-1/2}\|_F + O(m^{-1/2} n^{-1/2} \log n) \\ &\leq \|(\hat{P} - H) US^{-1/2} W^*\|_F + \|(\hat{P} - H) U(W^* \hat{S}^{-1/2} - S^{-1/2} W^*)\|_F + O(m^{-1/2} n^{-1/2} \log n) \\ &= \|(\hat{P} - H) US^{-1/2}\|_F + O(m^{-1} n^{-1} (\log n)^{3/2}) + O(m^{-1/2} n^{-1/2} (\log n)^{3/2}) \\ &= \|(\hat{P} - H) US^{-1/2}\|_F + O(m^{-1/2} n^{-1/2} (\log n)^{3/2}). \end{aligned}$$

Note that  $Z = US^{1/2} W$  for some orthogonal matrix  $W$ . As  $W^*$  is also orthogonal, therefore  $Z\tilde{W} = US^{1/2} W^*$  for some orthogonal  $\tilde{W}$ , which completes the proof. ■

**Theorem 3.14** *There exists a rotation matrix  $W$  such that for sufficiently large  $n$ ,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(m^{-1/2}n^{-1/2}(\log n)^{3/2})$$

*with high probability.*

**Proof:** By Lemma 3.13, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2}n^{-1/2}(\log n)^{3/2})$$

and similarly we could have the bound for each column vector

$$\begin{aligned} \max_i \|\hat{Z}_i - WZ_i\|_2 &\leq \frac{1}{\lambda_k^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \\ &\leq \frac{k^{1/2}}{\lambda_k^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \end{aligned}$$

where  $((\hat{P} - H)U)_i$  represents the  $i$ -th row of  $(\hat{P} - H)U$  and  $u_j$  denotes the  $j$ -th column of  $U$ . Now given  $i$  and  $j$ , the  $i$ -th element of the vector  $(\hat{P} - H)u_j$  is of the form

$$\sum_{s=1}^n (\hat{P}_{is} - H_{is})u_{js} = \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}.$$

Thus, conditioned on  $H$ , the  $i$ -th element of the vector  $(\hat{P} - H)u_j$  is a sum of independent mean 0 random variables. By Equation (1), we have

$$\begin{aligned} &E \left[ \left( (A_{is}^{(t)} - H_{is})u_{js} \right)^k \right] \\ &\leq k! R^k u_{js}^k \\ &\leq \frac{k!}{2} R^{k-2} (\sqrt{2} R u_{js})^2. \end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{t, s \neq i} 2R^2 u_{js}^2 \right| \leq 2R^2 m,$$

then by Theorem 3.8, we have

$$P \left( \left| \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js} \right| \geq t \right) \leq \exp \left( \frac{-mt^2/2}{2R^2 + Rt} \right).$$

Let  $t = 3cRm^{-1/2} \log n$ , we have

$$P \left( \left| \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js} \right| \geq 3cRm^{-1/2} \log n \right) \leq n^{-c},$$

i.e. it is of order  $O(m^{-1/2} \log n)$  with high probability. Taking the union bound over all  $i$  and  $j$ , with high probability we have,

$$\begin{aligned} \max_i \|\hat{Z}_i - WZ_i\|_2 &\leq \frac{Ck^{1/2}}{\lambda_k^{1/2}(H)} m^{-1/2} (\log n)^{3/2} + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \\ &= O(m^{-1/2}n^{-1/2}(\log n)^{3/2}). \end{aligned}$$

■

**Lemma 3.15**  $\left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| = O(m^{-1/2} n^{-1} (\log n)^3)$  with high probability.

**Proof:** Let  $W$  be the rotation matrix in Theorem 3.14, then

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= \left| \hat{Z}_i^T \hat{Z}_j - \hat{Z}_i^T W Z_j + \hat{Z}_i^T W Z_j - (W Z_i)^T W Z_j \right| \\ &\leq \left| \hat{Z}_i^T (\hat{Z}_j - W Z_j) + (\hat{Z}_i^T - (W Z_i)^T) W Z_j \right| \\ &\leq \|\hat{Z}_i\|_2 \|\hat{Z}_j - W Z_j\|_2 + \|Z_j\|_2 \|\hat{Z}_i^T - (W Z_i)^T\|_2. \end{aligned}$$

Since  $\|Z_i\|_2^2 = Z_i^T Z_i = H_{ii}^{(1)} = E[\hat{P}_{ii}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij} \leq R$ , we have  $\|Z_i\|_2 = O(1)$ . Combined with Theorem 3.14,

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= (\|\hat{Z}_i\|_2 + \|Z_j\|_2) O(n^{-1/2} (\log n)^{3/2}) \\ &\leq (\|\hat{Z}_i - W Z_i\|_2 + \|W Z_i\|_2 + \|Z_j\|_2) O(m^{-1/2} n^{-1/2} (\log n)^{3/2}) \\ &= O(m^{-1/2} n^{-1} (\log n)^3) \end{aligned}$$

with high probability. ■

**Corollary 3.16** For fixed  $m$ , the estimator based on ASE of MLE has the same entry-wise asymptotic bias as MLE, i.e.

$$\lim_{n \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(1)}] - P_{ij} = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(1)}] - P_{ij} = \lim_{n \rightarrow \infty} \text{Bias}(\hat{P}_{ij}^{(1)}).$$

**Proof:** Direct result from Lemma 3.15 by noticing

$$\lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(1)}] = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(1)}].$$

Define  $(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}$ , our estimator for  $P_{ij}$ , to be a projection of  $\hat{Z}_i^T \hat{Z}_j$  onto  $[0, \max(\hat{P}_{ij}, R)]$ . ■

**Theorem 3.17** Assuming that  $m = O(n^b)$  for any  $b > 0$ , then  $\text{Var}((\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}) = O(m^{-1} n^{-2} (\log n)^6)$ .

**Proof:** By Lemma 3.15,

$$\begin{aligned} \text{Var}((\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}) &= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}]]^2 \\ &= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j + Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}]]^2 \\ &= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 + E[(Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}])]^2 \\ &\quad + 2E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j](Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}]) \\ &\leq E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 + E[(Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}])]^2 \\ &\quad + 2\sqrt{E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 E[(Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}])]^2} \\ &\leq 4E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \end{aligned}$$

Fix some  $a > 0$ , we have

$$\begin{aligned} &E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \\ &= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij} \leq a\} + E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij} > a\} \end{aligned}$$

For the first term, we have

$$\begin{aligned}
& E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij} \leq a\}] \\
& \leq E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij} \leq a\} \mathbb{I}\{\text{Lemma 4.11 holds}\}] (1 - n^{-c}) \\
& \quad + E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij} \leq a\} \mathbb{I}\{\text{Lemma 4.11 does not hold}\}] n^{-c} \\
& \leq O(m^{-1} n^{-2} (\log n)^6) (1 - n^{-c}) + 2n^{-c} E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - \hat{P}_{ij}]^2 \mathbb{I}\{\hat{P}_{ij} \leq a\}] \\
& \quad + 2n^{-c} E[(\hat{P}_{ij} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij} \leq a\}] \\
& \leq O(m^{-1} n^{-2} (\log n)^6) + 2n^{-c} E[\hat{P}_{ij}^2 \mathbb{I}\{\hat{P}_{ij} \leq a\}] + 2n^{-c} E[(\hat{P}_{ij} + R)^2 \mathbb{I}\{\hat{P}_{ij} \leq a\}] \\
& \leq O(m^{-1} n^{-2} (\log n)^6) + 2a^2 n^{-c} + 2(a + R)^2 n^{-c} \\
& \leq O(m^{-1} n^{-2} (\log n)^6) + 4n^{-c} (a + R)^2
\end{aligned}$$

Notice that

$$\begin{aligned}
& E[\hat{P}_{ij}^2 \mathbb{I}\{\hat{P}_{ij} > a\}] = E\left[\left(\frac{1}{m} \sum_{1 \leq t \leq m} A_{ij}^{(t)}\right)^2 \mathbb{I}\{\hat{P}_{ij} > a\}\right] \\
& \leq \frac{1}{m} E\left[\sum_{1 \leq t \leq m} A_{ij}^{(t)2} \mathbb{I}\{\hat{P}_{ij} > a\}\right] \leq \frac{1}{m} E\left[\sum_{1 \leq t \leq m} A_{ij}^{(t)2} \mathbb{I}\{\max_{1 \leq s \leq m} A_{ij}^{(s)} > a\}\right] \\
& \leq \frac{1}{m} E\left[\sum_{1 \leq t \leq m} A_{ij}^{(t)2} \left(\sum_{1 \leq s \leq m} \mathbb{I}\{A_{ij}^{(s)} > a\}\right)\right] = E[A_{ij}^{(1)2} \left(\sum_{1 \leq s \leq m} \mathbb{I}\{A_{ij}^{(s)} > a\}\right)] \\
& = E[A_{ij}^{(1)2} \mathbb{I}\{A_{ij}^{(1)} > a\}] + (m-1) E[A_{ij}^{(1)2} \mathbb{I}\{A_{ij}^{(2)} > a\}]] \\
& = E[A_{ij}^{(1)2} \mathbb{I}\{A_{ij}^{(1)} > a\}] + (m-1) E[A_{ij}^{(1)2}] P(A_{ij}^{(1)} > a),
\end{aligned}$$

and similarly

$$\begin{aligned}
& E[(\hat{P}_{ij} + R)^2 \mathbb{I}\{\hat{P}_{ij} > a\}] \\
& = E[\hat{P}_{ij}^2 \mathbb{I}\{\hat{P}_{ij} > a\}] + 2R \cdot E[\hat{P}_{ij} \mathbb{I}\{\hat{P}_{ij} > a\}] + R^2 P(\hat{P}_{ij} > a) \\
& \leq E[A_{ij}^{(1)2} \mathbb{I}\{A_{ij}^{(1)} > a\}] + (m-1) E[A_{ij}^{(1)2}] P(A_{ij}^{(1)} > a) \\
& \quad + 2R \left(E[A_{ij}^{(1)} \mathbb{I}\{A_{ij}^{(1)} > a\}]\right) + (m-1) E[A_{ij}^{(1)}] P(A_{ij}^{(1)} > a) \\
& \quad + R^2 \cdot m \cdot P(A_{ij}^{(1)} > a).
\end{aligned}$$

Thus for the second term,

$$\begin{aligned}
& E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij} > a\}] \\
& \leq 2E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - \hat{P}_{ij}]^2 \mathbb{I}\{\hat{P}_{ij} > a\}] + 2E[(\hat{P}_{ij} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij} > a\}] \\
& \leq 2E[\hat{P}_{ij}^2 \mathbb{I}\{\hat{P}_{ij} > a\}] + 2E[(\hat{P}_{ij} + R)^2 \mathbb{I}\{\hat{P}_{ij} > a\}] \\
& \leq 4E[A_{ij}^{(1)2} \mathbb{I}\{A_{ij}^{(1)} > a\}]] + 4(m-1) E[A_{ij}^{(1)2}] P(A_{ij}^{(1)} > a) \\
& \quad + 4R \cdot E[A_{ij}^{(1)} \mathbb{I}\{A_{ij}^{(1)} > a\}]] + 2R(m-1) E[A_{ij}^{(1)}] P(A_{ij}^{(1)} > a) \\
& \quad + 2R^2 \cdot m \cdot P(A_{ij}^{(1)} > a) \\
& \leq 4e^{-a/R} (a^2 + 3Ra + 3(m+1)R^2) \\
& \leq 4e^{-a/R} (a + 2m^{1/2}R)^2
\end{aligned}$$



Thus,

$$\text{Var}((\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}) \leq O(m^{-1}n^{-2}(\log n)^6) + 16(a+R)^2n^{-c} + 16(a+2m^{1/2}R)^2e^{-a/R}.$$

Let  $a = m^{-1/2}n^b$  for any  $b > 0$ , and  $c = 2b + 3$ , combined with the assumption  $m = O(n^b)$ , we have

$$\begin{aligned} \text{Var}((\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}) &= O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) + O(m^{-1}n^{2b}) \cdot O(e^{-m^{-1/2}n^b}) \\ &= O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) + O(m^{-1}n^{2b}) \cdot O(e^{-n^{b/2}}) \\ &= O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) + O(m^{-1}n^{2b}) \cdot O(n^{-2b-3}) \\ &= O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) \\ &= O(m^{-1}n^{-2}(\log n)^6). \end{aligned}$$

■

**Corollary 3.18** For fixed  $n$ ,  $1 \leq i, j \leq n$ ,  $\text{Var}(\hat{P}_{ij}^{(1)}) = \Theta(m^{-1})$ .

**Proof:** Direct result from central limit theorem. ■

**Theorem 3.19** For fixed  $m$ ,  $1 \leq i, j \leq n$  and  $i \neq j$ ,

$$\frac{\text{Var}(\tilde{P}_{ij}^{(1)})}{\text{Var}(\hat{P}_{ij}^{(1)})} = O(n^{-2}(\log n)^6).$$

Thus

$$\text{ARE}(\hat{P}_{ij}^{(1)}, \tilde{P}_{ij}^{(1)}) = 0.$$

Furthermore, as long as  $m$  goes to infinity of order  $O(n^b)$  for any  $b > 0$ ,

$$\text{ARE}(\hat{P}_{ij}^{(1)}, \tilde{P}_{ij}^{(1)}) = 0.$$

**Proof:** The results are direct from Theorem 3.17 and Corollary 3.18. ■

### 3.3 $\tilde{P}^{(q)}$ better than $\hat{P}^{(q)}$

**Theorem 3.20** Let  $P$  and  $C$  be two  $n$ -by- $n$  symmetric and hollow matrices satisfying element-wise conditions  $0 < P_{ij} \leq C_{ij} \leq R$  for some constant  $R > 0$ . For  $0 < \epsilon < 1$ , we define  $m$  symmetric and hollow matrices as

$$A^{(t)} \stackrel{iid}{\sim} (1 - \epsilon)\text{Exp}(P) + \epsilon\text{Exp}(C)$$

for  $1 \leq t \leq m$ . Let  $\hat{P}^{(q)}$  be the entry-wise MLqE based on exponential distribution with  $m$  observations. Define  $H^{(q)} = E[\hat{P}^{(q)}]$ , then for any constant  $c > 0$  there exists another constant  $n_0(c)$ , independent of  $n$ ,  $P$ ,  $C$  and  $\epsilon$ , such that if  $n > n_0$ , then for all  $\eta$  satisfying  $n^{-c} \leq \eta \leq 1/2$ ,

$$P\left(\|\hat{P}^{(q)} - H^{(q)}\|_2 \leq 8R\sqrt{2n \ln(n/\eta)}\right) \geq 1 - \eta.$$

**Remark:** This is the extended version of Theorem 3.1 in [5].

**Proof:** Let  $\{e_i\}_{i=1}^n$  be the canonical basis for  $\mathbb{R}^n$ . For each  $1 \leq i, j \leq n$ , define a corresponding matrix  $G_{ij}$ :

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus  $\hat{P}^{(q)} = \sum_{1 \leq i < j \leq n} \hat{P}_{ij}^{(q)} G_{ij}$  and  $H^{(q)} = \sum_{1 \leq i < j \leq n} H_{ij}^{(q)} G_{ij}$ . Then we have  $\hat{P}^{(q)} - H^{(q)} = \sum_{1 \leq i < j \leq n} X_{ij}$ , where  $X_{ij} \equiv (\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}) G_{ij}$ ,  $1 \leq i < j \leq n$ .

First consider the  $k$ -th moment of  $X_{ij}$  for  $1 \leq i < j \leq n$ . By Lemma 3.1 we have

$$\begin{aligned} |\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}| &= |\hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)} + \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} + H_{ij}^{(1)} - H_{ij}^{(q)}| \\ &\leq |\hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)}| + |\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}| + |H_{ij}^{(1)} - H_{ij}^{(q)}| \\ &\leq \hat{P}_{ij}^{(1)} + |\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}| + H_{ij}^{(1)} \\ &\leq 2 \left( |\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}| + H_{ij}^{(1)} \right). \end{aligned}$$

Since

$$\begin{aligned} E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] &\leq (1 - \epsilon) \exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1 + k, -H_{ij}/P_{ij}) \\ &\quad + \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1 + k, -H_{ij}/C_{ij}) \\ &\leq ((1 - \epsilon) \exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k) k! \\ &\leq ((1 - \epsilon) P_{ij}^k + \epsilon C_{ij}^k) k! \\ &\leq C_{ij}^k k!, \end{aligned}$$

Then

$$\begin{aligned} E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] &\leq E \left[ \left| \hat{P}_{ij}^{(q)} - H_{ij}^{(q)} \right|^k \right] \\ &\leq 2^k E \left[ \left( |\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}| + H_{ij}^{(1)} \right)^k \right] \\ &\leq 2^k \sum_{s=0}^k \binom{k}{s} E \left[ \left| \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} \right|^s \right] \left( H_{ij}^{(1)} \right)^{k-s} \\ &\leq 2^k \sum_{s=0}^k \binom{k}{s} C_{ij}^s s! \left( H_{ij}^{(1)} \right)^{k-s} \\ &\leq 2^k k! \sum_{s=0}^k \binom{k}{s} C_{ij}^s \left( H_{ij}^{(1)} \right)^{k-s} \\ &= 2^k k! \left( C_{ij} + H_{ij}^{(1)} \right)^k. \end{aligned} \tag{5}$$

Combined with for  $i \neq j$ ,

$$G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & k \text{ is even}; \\ e_i e_j^T + e_j e_i^T, & k \text{ is odd}, \end{cases}$$

thus we have

1. When  $k$  is even,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij}^2 \preceq 2^{2k} k! R^k G_{ij}^2;$$

2. When  $k$  is odd,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij} \preceq 2^{2k} k! R^k G_{ij}^2.$$

So

$$E[X_{ij}^k] \preceq 2^{2k} k! R^k G_{ij}^2.$$

Let

$$\sigma^2 := \left\| \sum_{1 \leq i < j \leq n} (4\sqrt{2} R G_{ij})^2 \right\| = 32R^2 \|(n-1)I\| = 32R^2(n-1),$$

notice that random matrices  $X_{ij}$  are independent, self-adjoint and have mean zero, apply Theorem 3.8 we have

$$\begin{aligned} P\left(\lambda_{\max}(\hat{P}^{(q)} - H^{(q)}) \geq t\right) &\leq n \exp\left(-\frac{t^2/2}{\sigma^2 + 4Rt}\right) \\ &\leq n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right). \end{aligned}$$

Now consider  $Y_{ij} \equiv (H^{(q)} - \hat{P}^{(q)}) G_{ij}$ ,  $1 \leq i < j \leq n$ . Then we have  $H^{(q)} - \hat{P}^{(q)} = \sum_{1 \leq i < j \leq n} Y_{ij}$ . Since

$$E[(H^{(q)} - \hat{P}^{(q)})^k] = (-1)^k E[(\hat{P}^{(q)} - H^{(q)})^k],$$

1. When  $k$  is even,

$$E[Y_{ij}^k] = E[(\hat{P}^{(q)} - H^{(q)})^k] G_{ij}^2 \preceq 2^{2k} k! R^k G_{ij}^2;$$

2. When  $k$  is odd,

$$E[Y_{ij}^k] = -E[(\hat{P}^{(q)} - H^{(q)})^k] G_{ij} \preceq 2^{2k} k! R^k G_{ij}^2.$$

Thus

$$\begin{aligned} P\left(\lambda_{\min}(\hat{P}^{(q)} - H^{(q)}) \leq -t\right) &= P\left(\lambda_{\max}(H^{(q)} - \hat{P}^{(q)}) \geq t\right) \\ &\leq n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right). \end{aligned}$$

Therefore we have

$$P\left(\|\hat{P}^{(q)} - H^{(q)}\| \geq t\right) \leq n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right).$$

Now let  $c > 0$  be given and assume  $n^{-c} \leq \eta \leq 1/2$ . Then there exists a  $n_0(c)$  independent of  $n$ ,  $P$ ,  $C$  and  $\epsilon$  such that whenever  $n > n_0(c)$ ,

$$t = 8R\sqrt{2n \ln(n/\eta)} \leq 32Rn.$$

Plugging this  $t$  into the equation above, we get

$$P(\|\hat{P}^{(q)} - H^{(q)}\| \geq 8R\sqrt{2n\ln(n/\eta)}) \leq n \exp\left(-\frac{t^2}{64R^2n}\right) = \eta.$$

■

As we define  $H^{(q)} = E[\hat{P}^{(q)}]$ , let  $d^{(q)} = \text{rank}(H^{(q)})$  be the dimension in which we are going to embed  $\hat{P}^{(q)}$ . Notice that it is less than or equal to  $K$  since the SBM assumption. Then we can define  $H^{(q)} = ZZ^T$  where  $Z \in \mathbb{R}^{n \times d^{(q)}}$ .

For simplicity, from now on, we will use  $\hat{P}$  to represent  $\hat{P}^{(q)}$ , use  $H$  to represent  $H^{(q)}$  and use  $k$  to represent the dimension  $d^{(q)}$  we are going to embed. Assume  $H = USU^T = ZZ^T$ , where  $Z$  is a  $n$ -by- $k$  matrix. Then our estimate for  $Z$  up to rotation is  $\hat{Z} = \hat{U}\hat{S}^{1/2}$ , where  $\hat{U}\hat{S}\hat{U}^T$  is the rank- $d$  spectral decomposition of  $|\hat{P}| = (\hat{P}^T \hat{P})^{1/2}$ .

**Lemma 3.21** *Under the above assumptions,  $\lambda_i(H) = \Theta(n)$  with high probability when  $i \leq k$ , i.e. the largest  $k$  eigenvalues of  $H$  is of order  $n$ . Moreover, we have  $\|S\|_2 = \Theta(n)$  and  $\|\hat{S}\|_2 = \Theta(n)$  with high probability.*

**Remark:** This is an extended version of Proposition 4.3 in [8].

**Proof:** Exactly the same as proof for Lemma 3.10. ■

**Lemma 3.22** *Let  $W_1\Sigma W_2^T$  be the singular value decomposition of  $U^T\hat{U}$ . Then for sufficiently large  $n$ ,*

$$\|U^T\hat{U} - W_1W_2^T\|_F = O(n^{-1}\log n)$$

*with high probability.*

**Proof:** Exactly the same as proof for Lemma 3.11. ■

We will denote the orthogonal matrix  $W_1W_2^T$  by  $W^*$ .

**Lemma 3.23** *For sufficiently large  $n$ ,*

$$\|W^*\hat{S} - SW^*\|_F = O(\log n),$$

$$\|W^*\hat{S}^{1/2} - S^{1/2}W^*\|_F = O(n^{-1/2}\log n)$$

*and*

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(n^{-3/2}\log n)$$

*with high probability.*

**Proof:**

By Proposition 2.1 in [7] and Equation (2), we have for some orthogonal matrix  $W$ ,

$$\|\hat{U} - UW\|_F^2 \leq \frac{2\|\hat{U}\hat{U}^T - UU^T\|_F^2}{\delta^2} \leq \frac{8k^2\|\hat{U}\hat{U}^T - UU^T\|_2^2}{\delta^2} = O(n^{-1/2}\sqrt{\log n}).$$

Let  $Q = \hat{U} - UU^T\hat{U}$ . And  $Q$  is the residual after projecting  $\hat{U}$  orthogonally onto the column space of  $U$ , we have

$$\|Q\|_F = \|\hat{U} - UU^T\hat{U}\|_F \leq \|\hat{U} - U\|_F = O(n^{-1/2}\sqrt{\log n}). \quad (6)$$

for all  $k \times k$  matrices  $T$ . Then

$$\begin{aligned}
W^* \hat{S} &= (W^* - U^T \hat{U}) \hat{S} + U^T \hat{U} \hat{S} = (W^* - U^T \hat{U}) \hat{S} + U^T \hat{P} \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) \hat{U} + U^T H \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + U^T H \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S U^T \hat{U}.
\end{aligned}$$

Combined with Theorem 3.20, Lemma 3.21, Lemma 3.22, we have

$$\begin{aligned}
&\|W^* \hat{S} - S W^*\|_F \\
&= \|(W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S(U^T \hat{U} - W^*)\|_F \\
&\leq \|W^* - U^T \hat{U}\|_F (\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F \|\hat{P} - H\|_2 \|Q\|_F + \|U^T (\hat{P} - H) U\|_F \\
&\leq O(\log n) + O(\log n) + \|U^T (\hat{P} - H) U\|_F
\end{aligned}$$

with high probability. And we know  $U^T (\hat{P} - H) U$  is a  $k \times k$  matrix with  $ij$ -th entry to be

$$u_i^T (\hat{P} - H) u_j = \sum_{s=1}^n \sum_{t=1}^n (\hat{P}_{st} - H_{st}) u_{is} u_{jt} = 2 \sum_{s < t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt}$$

where  $u_i$  and  $u_j$  are the  $i$ -th and  $j$ -th columns of  $U$ . Thus, conditioned on  $H$ ,  $u_i^T (\hat{P} - H) u_j$  is a sum of independent mean 0 random variables.

By Equation (5), we have

$$\begin{aligned}
&E \left[ \left( (\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right)^k \right] \\
&\leq 2^k k! (C_{st} + H_{st}^{(1)})^k u_{is}^k u_{jt}^k \\
&\leq \frac{k!}{2} (4R)^{k-2} (4\sqrt{2} R u_{is} u_{jt})^2.
\end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{s < t} 32R^2 u_{is}^2 u_{jt}^2 \right| \leq 32R^2,$$

then by Theorem 3.8, we have

$$P \left( \left| 2 \sum_{s < t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right| \geq t \right) \leq \exp \left( \frac{-t^2/8}{32R^2 + 2Rt} \right),$$

thus each entry of  $U^T (\hat{P} - H) U$  is of order  $O(\log n)$  with high probability and thus

$$\|U^T (\hat{P} - H) U\|_F = O(\log n) \quad (7)$$

with high probability. Hence

$$\|W^* \hat{S} - S W^*\|_F = O(\log n)$$

with high probability. Also, since

$$W_{ij}^* (\lambda_j^{1/2}(\hat{P}) - \lambda_i^{1/2}(H)) = W_{ij}^* \frac{\lambda_j(\hat{P}) - \lambda_i(H)}{\lambda_j^{1/2}(\hat{P}) + \lambda_i^{1/2}(H)}$$

and the eigenvalues  $\lambda_j^{1/2}(\hat{P})$  and  $\lambda_i^{1/2}(H)$  are both of order  $\Theta(\sqrt{n})$ , we have

$$\|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F = O(n^{-1/2} \log n).$$

Similarly, since

$$W_{ij}^* (\lambda_j^{-1/2}(\hat{P}) - \lambda_i^{-1/2}(H)) = W_{ij}^* \frac{\lambda_i(H) - \lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P}) + \lambda_i^{-1/2}(H)) \lambda_j(\hat{P}) \lambda_i(H)}$$

and the eigenvalues  $\lambda_j(\hat{P})$  and  $\lambda_i(H)$  are both of order  $\Theta(n)$ , we have

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(n^{-3/2} \log n).$$

■

**Lemma 3.24** *There exists a rotation matrix  $W$  such that for sufficiently large  $n$ ,*

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2})$$

*with high probability.*

**Proof:** Exactly the same as proof for Lemma 3.13. ■

**Theorem 3.25** *There exists a rotation matrix  $W$  such that for sufficiently large  $n$ ,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(n^{-1/2}(\log n)^{3/2})$$

*with high probability.*

**Proof:** By Lemma 3.24, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2})$$

and similarly we could have the bound for each row vector

$$\begin{aligned} \max_i \|\hat{Z}_i - WZ_i\|_2 &\leq \frac{1}{\lambda_k^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(n^{-1/2}(\log n)^{3/2}) \\ &\leq \frac{k^{1/2}}{\lambda_k^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(n^{-1/2}(\log n)^{3/2}) \end{aligned}$$

where  $u_j$  denotes the  $j$ -th column of  $U$ . Now given  $i$  and  $j$ , the  $i$ -th element of the vector  $(\hat{P} - H)u_j$  is of the form

$$\sum_{s=1}^n (\hat{P}_{is} - H_{is})u_{js} = \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}.$$

Thus, conditioned on  $H$ , the  $i$ -th element of the vector  $(\hat{P} - H)u_j$  is a sum of independent mean 0 random variables. By Equation (5), we have

$$\begin{aligned} &E \left[ \left( (\hat{P}_{is} - H_{is})u_{js} \right)^k \right] \\ &\leq 2^k k! (C_{is} + H_{is}^{(1)})^k u_{js}^k \\ &\leq \frac{k!}{2} (4R)^{k-2} (4\sqrt{2}R u_{js})^2. \end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{s \neq i} 32R^2 u_{js}^2 \right| \leq 32R^2,$$

then by Theorem 3.8, we have with high probability,

$$P \left( \left| \sum_{s \neq i} (\hat{P}_{is} - H_{is}) u_{js} \right| \geq t \right) \leq \exp \left( \frac{-t^2/2}{32R^2 + Rt} \right).$$

Let  $t = 2cR \log n$ , we have

$$P \left( \left| \sum_{s \neq i} (\hat{P}_{is} - H_{is}) u_{js} \right| \geq 2cR \log n \right) \leq n^{-c},$$

i.e. it can be bounded by  $O(\log n)$  with high probability. Taking the union bound over all  $i$  and  $j$ , with high probability, we have

$$\max_i \|\hat{Z}_i - WZ_i\|_2 \leq \frac{Cd^{1/2}}{\lambda_d^{1/2}(H)} (\log n)^{3/2} + O(n^{-1/2}(\log n)^{3/2}) = O(n^{-1/2}(\log n)^{3/2}).$$

■

**Lemma 3.26**  $\left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| = O(n^{-1}(\log n)^3)$  with high probability.

**Proof:** Let  $W$  be the rotation matrix in Theorem 3.25, then

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= \left| \hat{Z}_i^T \hat{Z}_j - \hat{Z}_i^T WZ_j + \hat{Z}_i^T WZ_j - (WZ_i)^T WZ_j \right| \\ &\leq \left| \hat{Z}_i^T (\hat{Z}_j - WZ_j) + (\hat{Z}_i^T - (WZ_i)^T) WZ_j \right| \\ &\leq \|\hat{Z}_i\|_2 \|\hat{Z}_j - WZ_j\|_2 + \|Z_j\|_2 \|\hat{Z}_i^T - (WZ_i)^T\|_2. \end{aligned}$$

Since  $\|Z_i\|_2^2 = Z_i^T Z_i = H_{ii}^q = E[\hat{P}_{ii}^{(q)}] \leq E[\hat{P}_{ii}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij} \leq R$ , we have  $\|Z_i\|_2 = O(1)$ . Combined with Theorem 3.25,

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= (\|\hat{Z}_i\|_2 + \|Z_j\|_2) O(n^{-1/2}(\log n)^{3/2}) \\ &\leq (\|\hat{Z}_i - WZ_i\|_2 + \|WZ_i\|_2 + \|Z_j\|_2) O(n^{-1/2}(\log n)^{3/2}) \\ &= O(n^{-1}(\log n)^3) \end{aligned}$$

with high probability. ■

**Corollary 3.27** For fixed  $m$ , the estimator based on ASE of MLqE has the same entry-wise asymptotic bias as MLqE, i.e.

$$\lim_{n \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(q)}) = \lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(q)}] - P_{ij} = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(q)}] - P_{ij} = \lim_{n \rightarrow \infty} \text{Bias}(\hat{P}_{ij}^{(q)}).$$

**Proof:** Direct result from Lemma 3.26 by noticing

$$\lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(q)}] = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(q)}].$$

■

**Theorem 3.28** Assuming that  $m = O(n^b)$  for any  $b > 0$ , then  $\text{Var}((\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}) = O(n^{-2}(\log n)^6)$ .

**Proof:** By Lemma 3.26,

$$\begin{aligned}
\text{Var}((\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}) &= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}]]^2 \\
&= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j + Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}]]^2 \\
&= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 + E[(Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}])]^2 \\
&\quad + 2E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j](Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}]) \\
&\leq E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 + E[(Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}])]^2 \\
&\quad + 2\sqrt{E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 E[(Z_i^T Z_j - E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}])]^2} \\
&\leq 4E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2
\end{aligned}$$

Fix some  $a > 0$ , we have

$$\begin{aligned}
&E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \\
&= E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\} + E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}
\end{aligned}$$

Note that we are thresholding according to  $\hat{P}^{(1)}$  instead of  $\hat{P}^{(q)}$ . By Lemma 3.1, we know  $\hat{P}^{(q)} < \hat{P}^{(1)}$  given any data. For the first term, we have

$$\begin{aligned}
&E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\} \\
&\leq E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\} \mathbb{I}\{\text{Lemma 3.22 holds}\} (1 - n^{-c}) \\
&\quad + E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\} \mathbb{I}\{\text{Lemma 3.22 does not hold}\} n^{-c} \\
&\leq O(n^{-2}(\log n)^6)(1 - n^{-c}) + 2n^{-c} E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - \hat{P}_{ij}^{(q)}]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\} \\
&\quad + 2n^{-c} E[(\hat{P}_{ij}^{(q)} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\}] \\
&\leq O(n^{-2}(\log n)^6) + 2n^{-c} E[\hat{P}_{ij}^{(q)2} \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\}] + 2n^{-c} E[(\hat{P}_{ij}^{(q)} + R)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\}] \\
&\leq O(n^{-2}(\log n)^6) + 2n^{-c} E[\hat{P}_{ij}^{(1)2} \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\}] + 2n^{-c} E[(\hat{P}_{ij}^{(1)} + R)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \leq a\}] \\
&\leq O(n^{-2}(\log n)^6) + 2a^2 n^{-c} + 2(a + R)^2 n^{-c} \\
&\leq O(n^{-2}(\log n)^6) + 4n^{-c}(a + R)^2
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
&E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - Z_i^T Z_j]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\} \\
&\leq 2E[(\hat{Z}_i^T \hat{Z}_j)_{\text{tr}} - \hat{P}_{ij}^{(q)}]^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\} + 2E[(\hat{P}_{ij}^{(q)} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] \\
&\leq 2E[\hat{P}_{ij}^{(q)2} \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] + 2E[(\hat{P}_{ij}^{(q)} + R)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] \\
&\leq 2E[\hat{P}_{ij}^{(1)2} \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] + 2E[(\hat{P}_{ij}^{(1)} + R)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] \\
&\leq 4e^{-a/R}(a + 2m^{1/2}R)^2
\end{aligned}$$

Similarly, assuming  $m = O(n^b)$  for any  $b > 0$ , we have

$$\text{Var}((\hat{Z}_i^T \hat{Z}_j)_{\text{tr}}) = O(n^{-2}(\log n)^6).$$

■



**Theorem 3.29** Let  $u_q(\theta) = E_\theta[\hat{\theta}_{q,n}]$ ,  $\phi_q(x; \theta) = \frac{\partial}{\partial \theta} L_q(f(x; \theta))$ , and  $\phi'_q(x; \theta) = \frac{\partial^2}{\partial \theta^2} L_q(f(x; \theta))$ . Then the asymptotic distribution of  $\hat{\theta}_{q,n}$  is  $\sqrt{n}(\hat{\theta}_{q,n} - u_q(\theta)) \sim \mathcal{N}(0, V_q(\theta))$ , where  $V_q(\theta) = E[\phi_q(X; \theta)^2]/E[\phi'_q(X; \theta)]^2$ .

**Remark:** See Theorem 1 in <http://arxiv.org/pdf/1310.7278.pdf>.

**Corollary 3.30**  $\text{Var}(\hat{P}_{ij}^{(q)}) = \Theta(m^{-1})$ .

**Proof:** Direct result from Theorem 3.29. ■

**Theorem 3.31** For fixed  $m$ ,  $1 \leq i, j \leq n$ ,

$$\frac{\text{Var}(\tilde{P}_{ij}^{(q)})}{\text{Var}(\hat{P}_{ij}^{(q)})} = O(mn^{-2}(\log n)^6).$$

Thus

$$\text{ARE}(\hat{P}_{ij}^{(q)}, \tilde{P}_{ij}^{(q)}) = 0.$$

Furthermore, as long as  $m$  goes to infinity of order  $o(n^2(\log n)^{-6})$ ,

$$\text{ARE}(\hat{P}_{ij}^{(q)}, \tilde{P}_{ij}^{(q)}) = 0.$$

**Proof:** The results are direct from Theorem 3.28 and Corollary 3.30. ■

### 3.4 $\tilde{P}^{(q)}$ better than $\tilde{P}^{(1)}$

**Theorem 3.32** For sufficiently large  $n$  and  $C$ , any  $1 \leq i, j \leq n$ ,

$$\lim_{m \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(1)}) > \lim_{m \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(q)})$$

**Proof:** Direct result from Lemma 3.1, Corollary 3.16 and Corollary 3.27. ■

**Theorem 3.33** For any fixed  $m$ , any  $1 \leq i, j \leq n$ ,

$$\lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(q)}) = 0.$$

Furthermore, as long as  $m$  goes to infinity of order  $o(n^2(\log n)^{-6})$ , any  $1 \leq i, j \leq n$ ,

$$\lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(q)}) = 0$$

**Proof:** Direct result from Theorem 3.17 and Theorem 3.28. ■

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