ASE o MLqE Story Latest

Runze

May 29, 2016

1 Problem Description

1.1 Uncontaminated Model

Let F be a distribution on $\mathcal{X} \in \mathbb{R}^d$, satisfying $x^T y \geq 0$ for all $x, y \in \mathcal{X}$. We now generate m i.i.d. graphs under the RDPG(F) model. First sample $X_1, \cdots, X_n$ independently from distribution F, and define $X = [X_1, \cdots, X_n]^T \in \mathbb{R}^{n \times d}$, $P = XX^T \in [0, R]^{n \times n}$, where R is a constant. Then we can sample m conditionally i.i.d. symmetric and hollow graphs $G^{(1)}, \cdots, G^{(m)}$, such that conditioned on X, $G_{ij}^{(t)} \overset{ind}{\sim} \mathrm{Exp}(P_{ij})$ for each $1 \leq t \leq m$, $1 \leq i < j \leq n$.

1.2 Contaminated Observations

Now we assume the observed edges are contaminated with probability ϵ.

Let G be a distribution on $\mathcal{Y} \in \mathbb{R}^{d'}$, satisfying $x^T y \geq 0$ for all $x, y \in \mathcal{Y}$. First sample X from F and Y from G. Then we sample m conditionally i.i.d. symmetric and hollow graphs $A^{(1)}, \cdots, A^{(m)}$ such that conditioning on X and Y, $A_{ij}^{(t)} \overset{ind}{\sim} (1-\epsilon)\mathrm{Exp}(P_{ij}) + \epsilon \mathrm{Exp}(C_{ij})$ for each $1 \leq t \leq m$, $1 \leq i < j \leq n$, where the contamination is a rank-d' matrix $C = YY^T \in [0, R]^{n \times n}$, $Y \in \mathbb{R}^{n \times d'}$.

1.3 Goal

Given the contaminated observation of adjacency matrices of m graphs, i.e. $A^{(1)}, \cdots, A^{(m)}$, we want to estimate the mean of the collection of uncontaminated graphs P.

2 Candidate Estimators

After observing contaminated adjacency matrices of m graphs $A^{(1)}, \cdots, A^{(m)}$, we want to propose a good estimator for the mean of the collection of graphs P.

2.1 $\hat{P}^{(1)}$ based on entry-wise MLE

Under the independent edge setting, we can simplify the problem to finding an entry-wise estimate of P. And MLE is always our first choice, which exists and happen to be $\bar{A}$, the entry-wise mean in this case. For consistency, we define $\hat{P}^{(1)} = \bar{A}$.

2.2 $\hat{P}^{(q)}$ based on entry-wise MLqE

Since the observations are contaminated, robust estimators are preferred. A modified MLE estimator, the maximum likelihood L-q estimator, is considered in this case. Define $\hat{P}^{(q)}$ as the entry-wise MLqE.

Remark: MLE is a special case of MLqE when $q = 1$. So we notate the entry-wise MLE to be $\hat{P}^{(1)}$ in consistent with entry-wise MLqE $\hat{P}^{(q)}$.

2.3 $\widetilde{P}^{(1)}$ based on ASE of entry-wise MLE

By taking advantages of the graph structure, we expect a better performance after applying a rank-reduction procedure to the entry-wise MLE $\hat{P}^{(1)}$ under the SBM. So we first apply ASE to $\hat{P}^{(1)}$ to get the latent positions $\hat{X}^{(1)}$ in dimension $d^{(1)}$, and then define $\widetilde{P}^{(1)} = \hat{X}^{(1)}\hat{X}^{(1)T}$.

2.4 $\widetilde{P}^{(q)}$ based on ASE of entry-wise MLqE

Similarly, we also expect a better performance after applying a rank-reduction procedure to the entry-wise MLqE $\hat{P}^{(q)}$ under the SBM. So we first apply ASE to $\hat{P}^{(q)}$ to get the latent positions $\hat{X}^{(q)}$ in dimension $d^{(q)}$, and then define $\widetilde{P}^{(q)} = \hat{X}^{(q)}\hat{X}^{(q)T}$.

3 Compare Estimators

3.1 $\hat{P}^{(q)}$ is better than $\hat{P}^{(1)}$

Lemma 3.1 *For any $0 < q, \epsilon < 1$, there exists $C_0(P_{ij}, \epsilon, q) > 0$ such that under the contaminated model with $C > C_0(P_{ij}, \epsilon, q)$,*

$$\lim_{m\to\infty} \left| E[\hat{P}_{ij}^{(q)}] - P_{ij} \right| < \lim_{m\to\infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

for $1 \le i, j, \le n$ and $i \ne j$.

Lemma 3.2 *For $1 \le i, j \le n$, we have*

$$\lim_{m\to\infty} \mathrm{Var}(\hat{P}_{ij}^{(1)}) = \lim_{m\to\infty} \mathrm{Var}(\hat{P}_{ij}^{(q)}) = 0,$$

Thus,

- By Lemma 3.1, when C is large enough, for every $1 \le i, j, \le n$ and $i \ne j$, $\hat{P}_{ij}^{(q)}$ has smaller asymptotic bias in absolute value than $\hat{P}_{ij}^{(1)}$ as $m \to \infty$;
- By Lemma 3.2, all entry-wise variances go to 0 for estimating P as $m \to \infty$;
- In terms of MSE, $\hat{P}^{(q)}$ is better than $\hat{P}^{(1)}$ when m and C are large enough.

3.2 $\widetilde{P}^{(1)}$ is better than $\hat{P}^{(1)}$

Theorem 3.3 *For fixed m, $1 \le i, j \le n$,*

$$\frac{\mathrm{Var}(\widetilde{P}_{ij}^{(1)})}{\mathrm{Var}(\hat{P}_{ij}^{(1)})} = O(mn^{-1}(\log n)^3).$$

Thus

$$\mathrm{ARE}(\hat{P}_{ij}^{(1)}, \widetilde{P}_{ij}^{(1)}) = 0.$$

Then

- For each $1 \le i, j \le n$, both $\hat{P}_{ij}^{(1)}$ and $\widetilde{P}_{ij}^{(1)}$ have the same asymptotic bias as $n \to \infty$;
- Fix m, for every $1 \le i, j \le n$, $\mathrm{ARE}(\hat{P}_{ij}^{(1)}, \widetilde{P}_{ij}^{(1)}) = \lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(1)})/\mathrm{Var}(\hat{P}_{ij}^{(1)}) = 0$, which means $\widetilde{P}^{(1)}$ is better than $\hat{P}^{(1)}$;
- Actually when fixing m, for every $1 \le i, j \le n$, $\mathrm{Var}(\widetilde{P}_{ij}^{(1)})/\mathrm{Var}(\hat{P}_{ij}^{(1)})$ is of order $O(n^{-1}(\log n)^3)$ as $n \to \infty$.

3.3 $\widetilde{P}^{(q)}$ is better than $\hat{P}^{(q)}$

Define $H^{(q)} = E[\hat{P}^{(q)}]$. Let $d^{(q)} = \mathrm{rank}(H^{(q)})$ be the dimension in which we are going to embed $\hat{P}^{(q)}$. Then

- For each $1 \le i, j \le n$, both $\hat{P}_{ij}^{(q)}$ and $\widetilde{P}_{ij}^{(q)}$ have the same asymptotic bias as $n \to \infty$;
- Fix m, for every $1 \le i, j \le n$, $\mathrm{ARE}(\hat{P}_{ij}^{(q)}, \widetilde{P}_{ij}^{(q)}) = \lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(q)})/\mathrm{Var}(\hat{P}_{ij}^{(q)}) = 0$, which means $\widetilde{P}^{(q)}$ is better than $\hat{P}^{(q)}$;
- Actually, even if m is not fixed, as long as m is growing with order $o(n^{1/2}(\log n)^{-3/2})$, we still have $\mathrm{ARE}(\hat{P}_{ij}^{(q)}, \widetilde{P}_{ij}^{(q)}) = 0$,

3.4 $\widetilde{P}^{(q)}$ is better than $\widetilde{P}^{(1)}$

- When n is large enough, for every $1 \le i, j \le n$, $E[\widetilde{P}_{ij}^{(1)}]$ will be close to $E[\hat{P}_{ij}^{(1)}]$ and $E[\widetilde{P}_{ij}^{(q)}]$ will be close to $E[\hat{P}_{ij}^{(q)}]$. Combined with $\hat{P}_{ij}^{(q)}$ has smaller asymptotic bias (as $m \to \infty$) than $\hat{P}_{ij}^{(1)}$ when C is large enough, we have for sufficiently large m and n, C large enough, $\lim\limits_{m\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(1)}) > \lim\limits_{m\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(q)})$;
- Fix m, for any $1 \le i, j \le n$, when n is large enough, $\mathrm{Var}(\widetilde{P}_{ij}^{(1)})$ is less than $\mathrm{Var}(\hat{P}_{ij}^{(1)})$ times $O(n^{-1})$ and $\mathrm{Var}(\widetilde{P}_{ij}^{(q)})$ is less than $\mathrm{Var}(\hat{P}_{ij}^{(q)})$ times $O(n^{-1/2}(\log n)^{3/2})$. Thus $\lim\limits_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(1)}) = \lim\limits_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(q)}) = 0$;
- In terms of MSE, $\widetilde{P}^{(q)}$ is better than $\widetilde{P}^{(1)}$ when m, n and C are large enough.

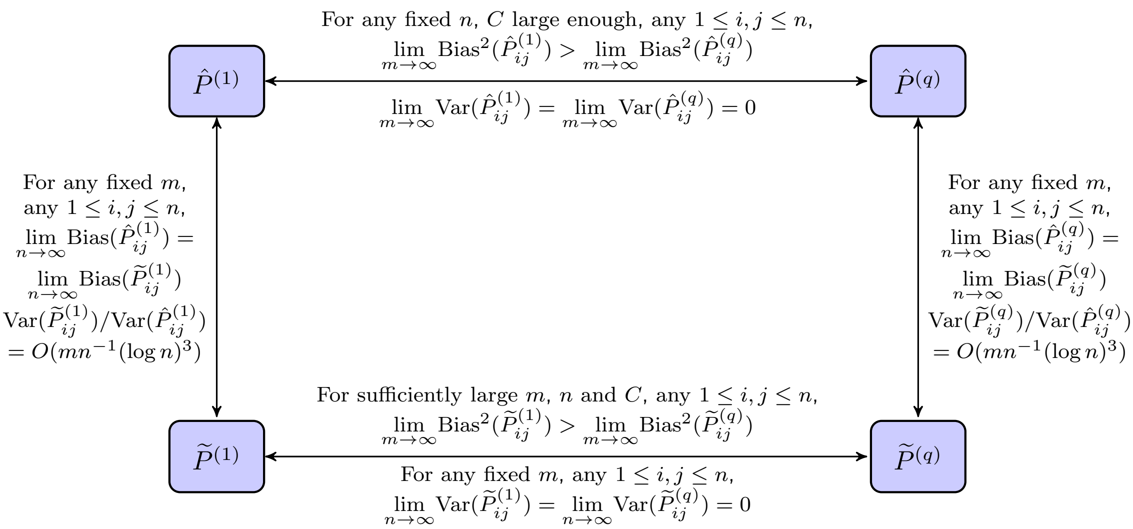

Figure 1: Relationship between four estimators.

3.5 Summary

Thus, we should choose the estimator $\widetilde{P}^{(q)}$.

RT: The figure of relationship is NOT right.

4 Proof

4.1 $\hat{P}^{(q)}$ better than $\hat{P}^{(1)}$

Lemma 4.1 (Lemma 3.1) *For any $0 < q < 1$, there exists $C_0(P_{ij}, \epsilon, q) > 0$ such that under the contaminated model with $C > C_0(P_{ij}, \epsilon, q)$,*

$$\lim_{m\to\infty} \left| E[\hat{P}_{ij}^{(q)}] - P_{ij} \right| < \lim_{m\to\infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

for $1 \le i, j, \le n$ and $i \ne j$.

Proof: For the MLE $\hat{P}_{ij}^{(1)} = \bar{A}_{ij}$,

$$E[\hat{P}_{ij}^{(1)}] = E[\bar{A}_{ij}] = \frac{1}{m}\sum_{t=1}^{m} E[A_{ij}^{(t)}] = E[A_{ij}^{(1)}] = (1-\epsilon)P_{ij} + \epsilon C_{ij}.$$

For the MLqE $\hat{P}_{ij}^{(q)}$,

$$E[\hat{P}_{ij}^{(q)}]$$

solves a cubic equation. ■

RT: Shown in Mathematica

Lemma 4.2 (Lemma 3.2)

$$\lim_{m\to\infty} \mathrm{Var}(\hat{P}_{ij}^{(1)}) = \lim_{m\to\infty} \mathrm{Var}(\hat{P}_{ij}^{(q)}) = 0,$$

for $1 \le i, j \le n$.

Proof: Both MLE and MLqE follows a central limit theorem, which means their variances goes to 0 as $m \to \infty$. ■

4.2 $\widetilde{P}^{(1)}$ better than $\hat{P}^{(1)}$

Theorem 4.3 *(Matrix Bernstein: Subexponential Case). Consider a finite sequence $\{X_k\}$ of independent, random, self-adjoint matrices with dimension d. Assume that*

$$E[X_k] = 0 \quad \text{and} \quad E[X_k^p] \preceq \frac{p!}{2} R^{p-2} A_k^2 \quad \text{for } p = 2, 3, 4, \ldots$$

Compute the variance parameter

$$\sigma^2 := \|\sum_k A_k^2\|.$$

Then the following chain of inequalities holds for all $t \geq 0$.

$$P\left(\lambda_{\max}\left(\sum_k X_k\right) \geq t\right) \leq d \cdot \exp\left(\frac{-t^2/2}{\sigma^2 + Rt}\right).$$

Remark: Theorem 6.2 in [8].

Theorem 4.4 (Theorem 3.3) *Let P and C be two n-by-n symmetric matrices satisfying element-wise conditions $0 < P_{ij} \leq C_{ij} \leq R$ for some constant $R > 0$. For $0 < \epsilon < 1$, we define m symmetric and hollow matrices as*

$$A^{(t)} \overset{iid}{\sim} (1-\epsilon)\text{Exp}(P) + \epsilon \text{Exp}(C),$$

for $1 \leq t \leq m$. Let $\hat{P}^{(1)}$ be the element-wise MLE based on exponential distribution with m observations. Define $H_{ij}^{(1)} = E[\hat{P}_{ij}^{(1)}] = (1-\epsilon)P_{ij} + \epsilon C_{ij}$, then for any constant $c > 0$, there exists another constant $n_0(c)$, independent of n, P, C and ϵ, such that if $n > n_0$, then for all η satisfying $n^{-c} \leq \eta \leq 1/2$,

$$P\left(\|\hat{P}^{(1)} - H^{(1)}\|_2 \leq 4R\sqrt{n \ln(n/\eta)}\right) \geq 1 - \eta.$$

Remark: This is the extended version of Theorem 3.1 in [4].

Proof: Let $\{e_i\}_{i=1}^n$ be the canonical basis for $\mathbb{R}^n$. For each $1 \leq i, j \leq n$, define a corresponding matrix G_{ij}:

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus $\hat{P}^{(1)} = \sum_{1 \leq i < j \leq n} \hat{P}_{ij}^{(1)} G_{ij}$ and $H^{(1)} = \sum_{1 \leq i < j \leq n} H_{ij}^{(1)} G_{ij}$. Then we have $\hat{P}^{(1)} - H^{(1)} = \sum_{1 \leq i < j \leq n} X_{ij}$, where $X_{ij} \equiv \left(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right) G_{ij}$, $1 \leq i < j \leq n$.

First consider the k-th moment of X_{ij} for $1 \leq i < j \leq n$. Since

$$\begin{aligned}
E[(A_{ij}^{(1)} - H_{ij}^{(1)})^k] \leq &(1-\epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1+k, -H_{ij}/P_{ij}) \\
&+ \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1+k, -H_{ij}/C_{ij}) \\
\leq &\left((1-\epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k\right) k! \\
\leq &\left((1-\epsilon) \cdot P_{ij}^k + \epsilon \cdot C_{ij}^k\right) k! \\
\leq &R^k k!,
\end{aligned}$$

we have

$$
\begin{aligned}
E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] =& E[(\frac{1}{m}\sum_{t=1}^{m} A_{ij}^{(t)} - H_{ij}^{(1)})^k] \\
=& E[(\frac{1}{m}\sum_{t=1}^{m}(A_{ij}^{(t)} - H_{ij}^{(1)}))^k] \\
=& \frac{1}{m^k}E[(\sum_{t=1}^{m}(A_{ij}^{(t)} - H_{ij}^{(1)}))^k] \\
\leq& R^k k!. \qquad (1)
\end{aligned}
$$

Combined with

$$
G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & \text{k is even;} \\ e_i e_j^T + e_j e_i^T, & \text{k is odd,} \end{cases}
$$

thus we have

1. When k is even,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k]G_{ij}^2 \preceq k!R^k G_{ij}^2;$$

2. When k is odd,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k]G_{ij} \preceq k!R^k G_{ij}^2.$$

So

$$E[X_{ij}^k] \preceq k!R^k G_{ij}^2.$$

Let

$$\sigma^2 := \left\| \sum_{1\leq i<j\leq n} (\sqrt{2}RG_{ij})^2 \right\|_2 = 2R^2\|(n-1)I\|_2 = 2R^2(n-1).$$

Notice that random matrices X_{ij} are independent, self-adjoint and have mean zero, apply Theorem 4.3 we have

$$
\begin{aligned}
P\left(\lambda_{\max}(\hat{P}^{(1)} - H^{(1)}) \geq t\right) &\leq n\exp\left(-\frac{t^2/2}{\sigma^2 + Rt}\right) \\
&\leq n\exp\left(-\frac{t^2/2}{2R^2 n + Rt}\right).
\end{aligned}
$$

Now consider $Y_{ij} \equiv \left(H^{(1)} - \hat{P}^{(1)}\right)G_{ij}$, $1 \leq i < j \leq n$. Then we have $H^{(1)} - \hat{P}^{(1)} = \sum_{1\leq i<j\leq n} Y_{ij}$. Since

$$E[(H^{(1)} - \hat{P}^{(1)})^k] = (-1)^k E[(\hat{P}^{(1)} - H^{(1)})^k],$$

1. When k is even,

$$E[Y_{ij}^k] = E[(\hat{P}^{(1)} - H^{(1)})^k]G_{ij}^2 \preceq k!R^k G_{ij}^2;$$

2. When k is odd,

$$E[Y_{ij}^k] = -E[(\hat{P}^{(1)} - H^{(1)})^k]G_{ij} \preceq k!R^k G_{ij}^2.$$

Thus by similar arguments,

$$\begin{aligned} P\left(\lambda_{\min}(\hat{P}^{(1)} - H^{(1)}) \leq -t\right) &= P\left(\lambda_{\max}(H^{(1)} - \hat{P}^{(1)}) \geq t\right) \\ &\leq n \exp\left(-\frac{t^2/2}{2R^2 n + Rt}\right). \end{aligned}$$

Therefore we have

$$P\left(\|\hat{P}^{(1)} - H^{(1)}\|_2 \geq t\right) \leq n \exp\left(-\frac{t^2/2}{2R^2 n + Rt}\right).$$

Now let $c > 0$ be given and assume $n^{-c} \leq \eta \leq 1/2$. Then there exists a $n_0(c)$ independent of n, P, C and ϵ such that whenever $n > n_0(c)$,

$$t = 4R\sqrt{n \ln(n/\eta)} \leq 6Rn.$$

Plugging this t into the equation above, we get

$$P(\|\hat{P}^{(1)} - H^{(1)}\|_2 \geq 4R\sqrt{n \ln(n/\eta)}) \leq n \exp\left(-\frac{t^2}{16R^2 n}\right) = \eta.$$

■

Define $H^{(1)} = E[\hat{P}^{(1)}] = (1-\epsilon)P + \epsilon C$, where $P = XX^T$, $X \in \mathbb{R}^{n \times d}$, $C = YY^T$, $Y \in \mathbb{R}^{n \times d'}$. Let $d^{(1)} = \text{rank}(H^{(1)})$ be the dimension in which we are going to embed $\hat{P}^{(1)}$. Then we can define $H^{(1)} = ZZ^T$ where $Z \in \mathbb{R}^{n \times d^{(1)}}$. Since $H^{(1)} = [\sqrt{1-\epsilon}X, \sqrt{\epsilon}Y][\sqrt{1-\epsilon}X, \sqrt{\epsilon}Y]^T$, we have $d^{(1)} \leq d + d'$.

For simplicity, from now on, we will use $\hat{P}$ to represent $\hat{P}^{(1)}$, use H to represent $H^{(1)}$ and use k to represent the dimension $d^{(1)}$ we are going to embed. Assume $H = USU^T = ZZ^T$, where $Z = [Z_1, \cdots, Z_n]^T$ is a n-by-k matrix. Then our estimate for Z up to rotation is $\hat{Z} = \hat{U}\hat{S}^{1/2}$, where $\hat{U}\hat{S}\hat{U}^T$ is the rank-k spectral decomposition of $|\hat{P}| = (\hat{P}^T\hat{P})^{1/2}$.

Furthermore, we assume that the second moment matrix $E[Z_1 Z_1^T]$ is rank k and has distinct eigenvalues $\lambda_i(E[Z_1 Z_1^T])$. In particular, we assume that there exists $\delta > 0$ such that

$$\delta < \min\left(\min_{i \neq j} |\lambda_i(E[Z_1 Z_1^T]) - \lambda_j(E[Z_1 Z_1^T])|, \lambda_k(E[Z_1 Z_1^T])\right)$$

Lemma 4.5 *Under the above assumptions, $\lambda_i(H) = \Theta(n)$ with high probability when $i \leq k$, i.e. the largest k eigenvalues of H is of order n. Moreover, we have $\|S\|_2 = \Theta(n)$ and $\|\hat{S}\|_2 = \Theta(n)$ with high probability.*

Remark: This is a extended version of Proposition 4.3 in [7].

Proof: Note that $\lambda_i(H) = \lambda_i(ZZ^T) = \lambda_i(Z^T Z)$ when $i \leq k$. Since each entry of $Z^T Z$ is a sum of n independent random variables each in $[0, R]$, i.e. $(Z^T Z)_{ij} = \sum_{l=1}^n Z_{li} Z_{lj}$. By Hoeffding's inequality, for each entry we have

$$P(|(Z^T Z - nE[Z_1 Z_1^T])_{ij}| \geq R\sqrt{n \log n}) \leq \frac{2}{n^2}.$$

By the union bound, we have

$$P(\|(Z^T Z - nE[Z_1 Z_1^T])_{ij}\|_F \geq kR\sqrt{n \log n}) \leq \frac{2k^2}{n^2}.$$

Then by Weyl's Theorem [3], we have

$$|\lambda_i(H) - n\lambda_i(Z_1 Z_1^T)| \leq \|Z^T Z - nE[Z_1 Z_1^T]\|_2 \leq kR\sqrt{n \log n}$$

RT: So in order to have the inequality to be small, we need the rank k increase of order $o(n)$?

with probability at least $1 - \frac{2k^2}{n^2}$. Thus $\lambda_i(H) = S_{ii} = \Theta(n)$ with probability at least $1 - \frac{2k^2}{n^2}$ when $i \leq k$.

Moreover,

RT: Here we need $k = O(n^{1/2-\epsilon})$?

$$\|H\|_2 - \|H - \hat{P}\|_2 \leq \|\hat{S}\|_2 \leq \|\hat{P} - H\|_2 + \|H\|_2.$$

Combined with Theorem 4.4, with high probability we have $\|\hat{S}\|_2 = \Theta(n)$. ■

Lemma 4.6 *Let $W_1 \Sigma W_2^T$ be the singular value decomposition of $U^T \hat{U}$. Then for sufficiently large n,*

$$\|U^T \hat{U} - W_1 W_2^T\|_F = O(kn^{-1} \log n)$$

with high probability.

Proof: Let $\sigma_1, \cdots, \sigma_d$ denote the singular values of $U^T \hat{U}$. Then $\sigma_i = \cos(\theta_i)$ where the θ_i are the principal angles between the subspaces spanned by $\hat{U}$ and U. Furthermore, by the Davis-Kahan $\sin(\Theta)$ theorem [1], combined with Theorem 4.4 and Lemma 4.5,

$$\|\hat{U}\hat{U}^T - UU^T\|_2 = \max_i |\sin(\theta_i)| \leq \frac{\|\hat{P} - H\|_2}{\lambda_k(H)} \leq \frac{C\sqrt{n \log n}}{n} = O(n^{-1/2}\sqrt{\log n}) \tag{2}$$

for sufficiently large n. Here $\lambda_k(H)$ denotes the k-th largest eigenvalue of H. We thus have

$$\begin{aligned}
\|U^T \hat{U} - W_1 W_2^T\|_F = \|\Sigma - I\|_F &= \sqrt{\sum_{i=1}^{k} (1 - \sigma_i)^2} \\
&\leq \sum_{i=1}^{k} (1 - \sigma_i) \leq \sum_{i=1}^{k} (1 - \sigma_i^2) \\
&= \sum_{i=1}^{k} \sin^2(\theta_i) \leq k\|\hat{U}\hat{U}^T - UU^T\|_2^2 \\
&= O(kn^{-1} \log n).
\end{aligned}$$

■

We will denote the orthogonal matrix $W_1 W_2^T$ by W^*.

Lemma 4.7 *For sufficiently large n,*

$$\|W^* \hat{S} - SW^*\|_F = O(k^{3/2} \log n),$$

$$\|W^*\hat{S}^{1/2} - S^{1/2}W^*\|_F = O(k^{3/2}n^{-1/2}\log n)$$

and

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(k^{3/2}n^{-3/2}\log n)$$

with high probability.

Proof: By Proposition 2.1 in [6] and Equation (2), we have for some orthogonal matrix W,

$$\|\hat{U} - UW\|_F^2 \leq \frac{2\|\hat{U}\hat{U}^T - UU^T\|_F^2}{\delta^2} = O(kn^{-1/2}\sqrt{\log n}).$$

Let $Q = \hat{U} - UU^T\hat{U}$. And Q is the residual after projecting $\hat{U}$ orthogonally onto the column space of U, we have

RT: Not right here, I am using the result for spectral norm as frobenius norm.

$$\|Q\|_F = \|\hat{U} - UU^T\hat{U}\|_F \leq \|\hat{U} - UT\|_F = O(kn^{-1/2}\sqrt{\log n}). \tag{3}$$

for all $k \times k$ matrices T.

Then

$$\begin{aligned}
W^*\hat{S} =&(W^* - U^T\hat{U})\hat{S} + U^T\hat{U}\hat{S} = (W^* - U^T\hat{U})\hat{S} + U^T\hat{P}\hat{U}\\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)\hat{U} + U^T H\hat{U}\\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + U^T H\hat{U}\\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + SU^T\hat{U}.
\end{aligned}$$

Combined with Theorem 4.4, Lemma 4.5, Lemma 4.6, we have

$$\begin{aligned}
&\|W^*\hat{S} - SW^*\|_F\\
=&\|(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + S(U^T\hat{U} - W^*)\|_F\\
\leq&\|W^* - U^T\hat{U}\|_F(\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F\|\hat{P} - H\|_2\|Q\|_F + \|U^T(\hat{P} - H)U\|_F\\
\leq&O(k\log n) + O(k^{3/2}\log n) + \|U^T(\hat{P} - H)U\|_F
\end{aligned}$$

with high probability. And we know $U^T(\hat{P} - H)U$ is a $k \times k$ matrix with ij-th entry to be

$$u_i^T(\hat{P} - H)u_j = \sum_{s=1}^n\sum_{t=1}^n(\hat{P}_{st} - H_{st})u_{is}u_{jt} = 2\sum_{s<t}(\hat{P}_{st} - H_{st})u_{is}u_{jt}$$

where u_i and u_j are the i-th and j-th columns of U. Thus, conditioned on H, U is fixed and $u_i^T(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables.

By Equation (1), we have

$$\begin{aligned}
&E\left[\left((\hat{P}_{st} - H_{st})u_{is}u_{jt}\right)^k\right]\\
\leq&k!R^k u_{is}^k u_{jt}^k\\
\leq&\frac{k!}{2}R^{k-2}(\sqrt{2}u_{is}u_{jt}R)^2.
\end{aligned}$$

Also we have

$$\sigma^2 := |\sum_{s<t}2R^2u_{is}^2u_{jt}^2| \leq R^2,$$

then by Theorem 4.3, we have

$$P\left(\left|2\sum_{s<t}(\hat{P}_{st}-H_{st})u_{is}u_{jt}\right|\geq t\right)\leq \exp\left(\frac{-t^2/8}{R^2+Rt}\right).$$

Let $t=\log n$, we have

$$P\left(\left|2\sum_{s<t}(\hat{P}_{st}-H_{st})u_{is}u_{jt}\right|\geq \log n\right)\leq n^{-c}$$

for some constant c. Thus each entry of $U^T(\hat{P}-H)U$ is of order $O(\log n)$ with high probability and

$$\|U^T(\hat{P}-H)U\|_F = O(k\log n) \tag{4}$$

with high probability. Hence

$$\|W^*\hat{S}-SW^*\|_F = O(k^{3/2}\log n)$$

with high probability. Also, since

$$W^*_{ij}(\lambda_j^{1/2}(\hat{P})-\lambda_i^{1/2}(H)) = W^*_{ij}\frac{\lambda_j(\hat{P})-\lambda_i(H)}{\lambda_j^{1/2}(\hat{P})+\lambda_i^{1/2}(H)}$$

and the eigenvalues $\lambda_j^{1/2}(\hat{P})$ and $\lambda_i^{1/2}(H)$ are both of order $\Theta(\sqrt{n})$, we have

$$\|W^*\hat{S}^{1/2}-S^{1/2}W^*\|_F = O(k^{3/2}n^{-1/2}\log n).$$

Similarly, since

$$W^*_{ij}(\lambda_j^{-1/2}(\hat{P})-\lambda_i^{-1/2}(H)) = W^*_{ij}\frac{\lambda_i(H)-\lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P})+\lambda_i^{-1/2}(H))\lambda_j(\hat{P})\lambda_i(H)}$$

and the eigenvalues $\lambda_j(\hat{P})$ and $\lambda_i(H)$ are both of order $\Theta(n)$, we have

$$\|W^*\hat{S}^{-1/2}-S^{-1/2}W^*\|_F = O(k^{3/2}n^{-3/2}\log n).$$

■

Lemma 4.8 *There exists a rotation matrix W such that for sufficiently large n,*

$$\|\hat{Z}-ZW\|_F = k\|(\hat{P}-H)US^{-1/2}\|_F + O(k^{5/2}n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: Let $Q_1 = UU^T\hat{U}-UW^*$, $Q_2 = W^*\hat{S}^{1/2}-S^{1/2}W^*$ and $Q_3 = \hat{U}-UW^* = \hat{U}-UU^T\hat{U}+Q_1 = Q+Q_1$. Then since $UU^TP=P$ and $\hat{U}\hat{S}^{1/2} = \hat{P}\hat{U}\hat{S}^{-1/2}$,

$$\begin{aligned}
\hat{Z}-US^{1/2}W^* =& \hat{U}\hat{S}^{1/2}-UW^*\hat{S}^{1/2}+U(W^*\hat{S}^{1/2}-S^{1/2}W^*)\\
=&(\hat{U}-UU^T\hat{U})\hat{S}^{1/2}+Q_1\hat{S}^{1/2}+UQ_2\\
=&(\hat{P}-H)\hat{U}\hat{S}^{-1/2}-UU^T(\hat{P}-H)\hat{U}\hat{S}^{-1/2}+Q_1\hat{S}^{1/2}+UQ_2\\
=&(\hat{P}-H)UW^*\hat{S}^{-1/2}-UU^T(\hat{P}-H)UW^*\hat{S}^{-1/2}\\
&+(I-UU^T)(\hat{P}-H)Q_3\hat{S}^{-1/2}+Q_1\hat{S}^{1/2}+UQ_2.
\end{aligned}$$

By Lemma 4.6,

$$\|Q_1\|_F \le \|U\|_F \|U^T \hat{U} - W^*\|_F = O(k^{3/2} n^{-1} \log n).$$

By Lemma 4.7,

$$\|Q_2\|_F = O(k^{3/2} n^{-1/2} \log n).$$

By Equation (3), if we assume $k = O(n^{1/2-\epsilon})$,

$$\|Q_3\|_F \le \|Q\|_F + \|Q_1\|_F = O(kn^{-1/2} (\log n)^{1/2}).$$

otherwise,

$$\|Q_3\|_F \le \|Q\|_F + \|Q_1\|_F = O(k^{3/2} n^{-1/2} (\log n)^{1/2}).$$

By Equation (4),

$$\|UU^T(\hat{P} - H)UW^* \hat{S}^{-1/2}\|_F \le k^{3/2} \|U^T(\hat{P} - H)U\|_F \|\hat{S}^{-1/2}\|_2 = O(k^{5/2} n^{-1/2} \log n).$$

By Lemma 4.7,

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(k^{3/2} n^{-3/2} \log n).$$

Therefore,

$$\begin{aligned}
&\|\hat{Z} - US^{1/2} W^*\|_F \\
=&\|(\hat{P} - H)UW^* \hat{S}^{-1/2}\|_F + O(k^{5/2} n^{-1/2} \log n) + \|I - UU^T\|_2 \|\hat{P} - H\|_2 O(k^{3/2} n^{-1} (\log n)^{1/2}) \\
&+ O(k^{3/2} n^{-1/2} \log n) + O(k^2 n^{-1/2} \log n) \\
=&\|(\hat{P} - H)UW^* \hat{S}^{-1/2}\|_F + O(k^{5/2} n^{-1/2} \log n) \\
\le&\|(\hat{P} - H)US^{-1/2} W^*\|_F + \|(\hat{P} - H)U(W^* \hat{S}^{-1/2} - S^{-1/2} W^*)\|_F + O(k^{5/2} n^{-1/2} \log n) \\
=&k\|(\hat{P} - H)US^{-1/2}\|_F + O(k^2 n^{-1} (\log n)^{3/2}) + O(k^{5/2} n^{-1/2} \log n) \\
=&k\|(\hat{P} - H)US^{-1/2}\|_F + O(k^{5/2} n^{-1/2} (\log n)^{3/2}).
\end{aligned}$$

Note that $Z = US^{1/2} W$ for some orthogonal matrix W. As W^* is also orthogonal, therefore $Z\tilde{W} = US^{1/2} W^*$ for some orthogonal $\tilde{W}$, which completes the proof. ■

RT: $\|I - UU^T\|_2 = O(1)$

Theorem 4.9 *There exists a rotation matrix W such that for sufficiently large n,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(k^{5/2} n^{-1/2} (\log n)^{3/2})$$

with high probability.

Proof: By Lemma 4.8, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(k^{5/2} n^{-1/2} (\log n)^{3/2})$$

and similarly we could have the bound for each column vector

RT: Similarly

$$\begin{aligned}
\max_i \|\hat{Z}_i - WZ_i\|_2 \le& \frac{1}{\lambda_k^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(k^{5/2} n^{-1/2} (\log n)^{3/2}) \\
\le& \frac{k^{1/2}}{\lambda_k^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(k^{5/2} n^{-1/2} (\log n)^{3/2})
\end{aligned}$$

where $((\hat{P}-H)U)_i$ represents the i-th row of $(\hat{P}-H)U$ and u_j denotes the j-th column of U. Now given i and j, the i-th element of the vector $(\hat{P}-H)u_j$ is of the form

$$\sum_{s=1}^{n}(\hat{P}_{is}-H_{is})u_{js}=\sum_{s\neq i}(\hat{P}_{is}-H_{is})u_{js}.$$

Thus, conditioned on H, the i-th element of the vector $(\hat{P}-H)u_j$ is a sum of independent mean 0 random variables. By Equation (1), we have

$$\begin{aligned}&E\left[\left((\hat{P}_{is}-H_{is})u_{js}\right)^k\right]\\ \leq& k!R^k u_{js}^k\\ \leq&\frac{k!}{2}R^{k-2}(\sqrt{2}Ru_{js})^2.\end{aligned}$$

Also we have

$$\sigma^2:=|\sum_{s\neq i}2R^2u_{js}^2|\leq 2R^2,$$

then by Theorem 4.3, we have

$$P\left(\left|\sum_{s\neq i}(\hat{P}_{is}-H_{is})u_{js}\right|\geq t\right)\leq\exp\left(\frac{-t^2/2}{2R^2+Rt}\right),$$

i.e. it is of order $O(\log n)$ with high probability. Taking the union bound over all i and j, with high probability we have

$$\begin{aligned}\max_i\|\hat{Z}_i-WZ_i\|_2&\leq\frac{Ck^{1/2}}{\lambda_k^{1/2}(H)}(\log n)^{3/2}+O(k^{5/2}n^{-1/2}(\log n)^{3/2})\\&=O(k^{5/2}n^{-1/2}(\log n)^{3/2}).\end{aligned}$$

■

Lemma 4.10 $\left|\hat{Z}_i^T\hat{Z}_j-Z_i^TZ_j\right|=O(k^5n^{-1}(\log n)^3)$ *with high probability.*

Proof: Let W be the rotation matrix in Theorem 4.9, then

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j-Z_i^TZ_j\right|&=\left|\hat{Z}_i^T\hat{Z}_j-\hat{Z}_i^TWZ_j+\hat{Z}_i^TWZ_j-(WZ_i)^TWZ_j\right|\\&\leq\left|\hat{Z}_i^T(\hat{Z}_j-WZ_j)+(\hat{Z}_i^T-(WZ_i)^T)WZ_j\right|\\&\leq\|\hat{Z}_i\|_2\|\hat{Z}_j-WZ_j\|_2+\|Z_j\|_2\|\hat{Z}_i^T-(WZ_i)^T\|_2.\end{aligned}$$

Since $\|Z_i\|_2^2=Z_i^TZ_i=H_{ii}^{(1)}=E[\hat{P}_{ii}^{(1)}]=(1-\epsilon)P_{ij}+\epsilon C_{ij}\leq R$, we have $\|Z_i\|_2=O(1)$. Combined with Theorem 4.9,

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j-Z_i^TZ_j\right|=&(\|\hat{Z}_i\|_2+\|Z_j\|_2)O(k^{5/2}n^{-1/2}(\log n)^{3/2})\\ \leq&(\|\hat{Z}_i-WZ_i\|_2+\|WZ_i\|_2+\|Z_j\|_2)O(k^{5/2}n^{-1/2}(\log n)^{3/2})\\=&O(k^5n^{-1}(\log n)^3)\end{aligned}$$

with high probability. ■

Corollary 4.11 *For fixed m, the estimator based on ASE of MLE has the same entry-wise asymptotic bias as MLE, i.e.*

$$\lim_{n\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(1)}) = \lim_{n\to\infty} E[\widetilde{P}_{ij}^{(1)}] - P_{ij} = \lim_{n\to\infty} E[\hat{P}_{ij}^{(1)}] - P_{ij} = \lim_{n\to\infty} \mathrm{Bias}(\hat{P}_{ij}^{(1)}).$$

Proof: Direct result from Lemma 4.10 by noticing

$$\lim_{n\to\infty} E[\widetilde{P}_{ij}^{(1)}] = \lim_{n\to\infty} E[\hat{P}_{ij}^{(1)}].$$

■

Theorem 4.12 $\mathrm{Var}(\hat{Z}_i^T \hat{Z}_j) = O(k^{10} n^{-2} (\log n)^6)$ *with high probability.*

Proof: By Lemma 4.10,

$$\begin{aligned}
\mathrm{Var}(\hat{Z}_i^T \hat{Z}_j) =& E[(\hat{Z}_i^T \hat{Z}_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
=& E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j + Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
=& E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
& - 2E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])] \\
\leq& E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
& + 2\sqrt{E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2]} \\
\leq& 4E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] \\
=& O(k^{10} n^{-2} (\log n)^6)
\end{aligned}$$

with high probability. ■

RT: Expectation with high probability?

Corollary 4.13 *For fixed n, $1 \leq i, j \leq n$,* $\mathrm{Var}(\hat{P}_{ij}^{(1)}) = \Theta(m^{-1})$.

Proof: Direct result from central limit theorem. ■

Theorem 4.14 *For fixed m, $1 \leq i, j \leq n$ and $i \neq j$,*

$$\frac{\mathrm{Var}(\widetilde{P}_{ij}^{(1)})}{\mathrm{Var}(\hat{P}_{ij}^{(1)})} = O(mn^{-1}(\log n)^3).$$

Thus

$$\mathrm{ARE}(\hat{P}_{ij}^{(1)}, \widetilde{P}_{ij}^{(1)}) = 0.$$

Furthermore, as long as m goes to infinity of order $o(n(\log n)^{-3})$,

$$\mathrm{ARE}(\hat{P}_{ij}^{(1)}, \widetilde{P}_{ij}^{(1)}) = 0.$$

Proof: The results are direct from Theorem 4.12 and Corollary 4.13. ■

4.3 $\widetilde{P}^{(q)}$ better than $\hat{P}^{(q)}$

Lemma 4.15 *Consider the model* $X_1, \cdots, X_m \overset{iid}{\sim} \mathrm{Exp}(\theta)$ *with* $E[X_1] = \theta$. *Given any data* $x = (x_1, \cdots, x_m)$ *such that* $x_{(1)} > 0$ *and not all* x_i*'s are the same, then* $\hat{\theta}_q(x) < \hat{\theta}_1(x)$ *for* $0 < q < 1$, *i.e. MLqE [2, 5] is always less than MLE under exponential distribution no matter how the data is sampled.*

Proof: The MLE is

$$\hat{\theta}_1(x) = \bar{x}.$$

And the MLqE $\hat{\theta}_q(x)$ solves the equation

$$\sum_{i=1}^{m} e^{-\frac{(1-q)x_i}{\hat{\theta}_q(x)}} (x_i - \hat{\theta}_q(x)) = 0.$$

Consider the continuous function $g(\theta, x) = \sum_{i=1}^{m} e^{-\frac{(1-q)x_i}{\theta}} (x_i - \theta)$.

Let $x_{(1)} \leq \cdots \leq x_{(l)} \leq \bar{x} \leq x_{(l+1)} \leq \cdots \leq x_{(m)}$. Define $s_i = \bar{x} - x_{(i)}$ for $1 \leq i \leq l$, and $t_i = x_{(l+i)} - \bar{x}$ for $1 \leq i \leq m - l$. Note that $\sum_{i=1}^{l} s_i = \sum_{i=1}^{m-l} t_i$. Then we have

$$\begin{aligned}
g(\hat{\theta}_1(x), x) &= g(\bar{x}, x) \\
&= \sum_{i=1}^{m} e^{-\frac{(1-q)x_{(i)}}{\bar{x}}} (x_{(i)} - \bar{x}) \\
&= -\sum_{i=1}^{l} e^{-\frac{(1-q)x_{(i)}}{\bar{x}}} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&\leq -e^{-(1-q)} \sum_{i=1}^{l} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&\leq -e^{-(1-q)} \sum_{i=1}^{m-l} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&\leq -\sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&= 0,
\end{aligned}$$

and equality holds if and only if all x_i's are the same, which is excluded by the assumption. Thus $g(\hat{\theta}_1(x), x) < 0$.

Also we know:

- $g(\hat{\theta}_q(x), x) = 0$;
- $\lim_{\theta \to 0^+} g(\theta, x) = 0$;
- $g(\theta, x) > 0$ when $\theta < x_{(1)}$;

Combined with $g(\hat{\theta}_1(x), x) < 0$, we have $\hat{\theta}_q(x) < \hat{\theta}_1(x)$ for $0 < q < 1$. ∎

Theorem 4.16 *Let P and C be two n-by-n symmetric and hollow matrices satisfying element-wise conditions $0 < P_{ij} \leq C_{ij} \leq R$ for some constant $R > 0$. For $0 < \epsilon < 1$, we define m symmetric and hollow matrices as*

$$A^{(t)} \stackrel{iid}{\sim} (1-\epsilon)\mathrm{Exp}(P) + \epsilon \mathrm{Exp}(C)$$

for $1 \leq t \leq m$. Let $\hat{P}^{(q)}$ be the entry-wise MLqE based on exponential distribution with m observations. Define $H^{(q)} = E[\hat{P}^{(q)}]$, then for any constant $c > 0$ there exists another constant $n_0(c)$, independent of n, P, C and ϵ, such that if $n > n_0$, then for all η satisfying $n^{-c} \leq \eta \leq 1/2$,

$$P\left(\|\hat{P}^{(q)} - H^{(q)}\|_2 \leq 8R\sqrt{2n\ln(n/\eta)}\right) \geq 1-\eta.$$

Remark: This is the extended version of Theorem 3.1 in [4].
Proof: Let $\{e_i\}_{i=1}^n$ be the canonical basis for $\mathbb{R}^n$. For each $1 \leq i, j \leq n$, define a corresponding matrix G_{ij}:

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus $\hat{P}^{(q)} = \sum_{1\leq i<j\leq n} \hat{P}_{ij}^{(q)} G_{ij}$ and $H^{(q)} = \sum_{1\leq i<j\leq n} H_{ij}^{(q)} G_{ij}$. Then we have $\hat{P}^{(q)} - H^{(q)} = \sum_{1\leq i<j\leq n} X_{ij}$, where $X_{ij} \equiv \left(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}\right) G_{ij}$, $1 \leq i < j \leq n$.

First consider the k-th moment of X_{ij} for $1 \leq i < j \leq n$. By Lemma 4.15 we have

$$\begin{aligned}
\left|\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}\right| &= \left|\hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)} + \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} + H_{ij}^{(1)} - H_{ij}^{(q)}\right| \\
&\leq \left|\hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)}\right| + \left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + \left|H_{ij}^{(1)} - H_{ij}^{(q)}\right| \\
&\leq \hat{P}_{ij}^{(1)} + \left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + H_{ij}^{(1)} \\
&\leq 2\left(\left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + H_{ij}^{(1)}\right).
\end{aligned}$$

Since

$$\begin{aligned}
E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] \leq& (1-\epsilon)\exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1+k, -H_{ij}/P_{ij}) \\
&+ \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1+k, -H_{ij}/C_{ij}) \\
\leq& \left((1-\epsilon)\exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k\right) k! \\
\leq& \left((1-\epsilon) P_{ij}^k + \epsilon C_{ij}^k\right) k! \\
\leq& C_{ij}^k k!,
\end{aligned}$$

Then

$$
\begin{aligned}
E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] \leq & E\left[\left|\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}\right|^k\right] \\
\leq & 2^k E\left[\left(\left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + H_{ij}^{(1)}\right)^k\right] \\
\leq & 2^k \sum_{s=0}^{k} \binom{k}{s} E\left[\left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right|^s\right] \left(H_{ij}^{(1)}\right)^{k-s} \\
\leq & 2^k \sum_{s=0}^{k} \binom{k}{s} C_{ij}^s s! \left(H_{ij}^{(1)}\right)^{k-s} \\
\leq & 2^k k! \sum_{s=0}^{k} \binom{k}{s} C_{ij}^s \left(H_{ij}^{(1)}\right)^{k-s} \\
= & 2^k k! \left(C_{ij} + H_{ij}^{(1)}\right)^k .
\end{aligned} \tag{5}
$$

Combined with for $i \neq j$,

$$
G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & \text{k is even;} \\ e_i e_j^T + e_j e_i^T, & \text{k is odd,} \end{cases}
$$

thus we have

1. When k is even,

$$
E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij}^2 \preceq 2^{2k} k! R^k G_{ij}^2;
$$

2. When k is odd,

$$
E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij} \preceq 2^{2k} k! R^k G_{ij}^2.
$$

So

$$
E[X_{ij}^k] \preceq 2^{2k} k! R^k G_{ij}^2.
$$

Let

$$
\sigma^2 := \left\| \sum_{1 \leq i < j \leq n} (4\sqrt{2} R G_{ij})^2 \right\| = 32 R^2 \|(n-1) I\| = 32 R^2 (n-1),
$$

notice that random matrices X_{ij} are independent, self-adjoint and have mean zero, apply Theorem 4.3 we have

$$
\begin{aligned}
P\left(\lambda_{\max}(\hat{P}^{(q)} - H^{(q)}) \geq t\right) \leq & \, n \exp\left(-\frac{t^2/2}{\sigma^2 + 4Rt}\right) \\
\leq & \, n \exp\left(-\frac{t^2/2}{32 R^2 n + Rt}\right).
\end{aligned}
$$

Now consider $Y_{ij} \equiv \left(H^{(q)} - \hat{P}^{(q)}\right) G_{ij}$, $1 \leq i < j \leq n$. Then we have $H^{(q)} - \hat{P}^{(q)} = \sum_{1 \leq i < j \leq n} Y_{ij}$. Since

$$
E[(H^{(q)} - \hat{P}^{(q)})^k] = (-1)^k E[(\hat{P}^{(q)} - H^{(q)})^k],
$$

1. When k is even,
$$E[Y_{ij}^k] = E[(\hat{P}^{(q)} - H^{(q)})^k]G_{ij}^2 \preceq 2^{2k}k!R^kG_{ij}^2;$$
2. When k is odd,
$$E[Y_{ij}^k] = -E[(\hat{P}^{(q)} - H^{(q)})^k]G_{ij} \preceq 2^{2k}k!R^kG_{ij}^2.$$

Thus
$$\begin{aligned} P\left(\lambda_{\min}(\hat{P}^{(q)} - H^{(q)}) \le -t\right) &= P\left(\lambda_{\max}(H^{(q)} - \hat{P}^{(q)}) \ge t\right) \\ &\le n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right). \end{aligned}$$

Therefore we have
$$P\left(\|\hat{P}^{(q)} - H^{(q)}\| \ge t\right) \le n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right).$$

Now let $c > 0$ be given and assume $n^{-c} \le \eta \le 1/2$. Then there exists a $n_0(c)$ independent of n, P, C and ϵ such that whenever $n > n_0(c)$,
$$t = 8R\sqrt{2n \ln(n/\eta)} \le 32Rn.$$

Plugging this t into the equation above, we get
$$P(\|\hat{P}^{(q)} - H^{(q)}\| \ge 8R\sqrt{2n \ln(n/\eta)}) \le n \exp\left(-\frac{t^2}{64R^2n}\right) = \eta.$$

■

As we define $H^{(q)} = E[\hat{P}^{(q)}]$, let $d^{(q)} = \text{rank}(H^{(q)})$ be the dimension in which we are going to embed $\hat{P}^{(q)}$. Then we can define $H^{(q)} = ZZ^T$ where $Z \in \mathbb{R}^{n \times d^{(q)}}$.

For simplicity, from now on, we will use $\hat{P}$ to represent $\hat{P}^{(q)}$, use H to represent $H^{(q)}$ and use k to represent the dimension $d^{(q)}$ we are going to embed. Assume $H = USU^T = ZZ^T$, where Z is a n-by-k matrix. Then our estimate for Z up to rotation is $\hat{Z} = \hat{U}\hat{S}^{1/2}$, where $\hat{U}\hat{S}\hat{U}^T$ is the rank-d spectral decomposition of $|\hat{P}| = (\hat{P}^T\hat{P})^{1/2}$.

RT: We don't have any bound on $d^{(q)}$?

Lemma 4.17 *Under the above assumptions, $\lambda_i(H) = \Theta(n)$ with high probability when $i \le k$, i.e. the largest k eigenvalues of H is of order n. Moreover, we have $\|S\|_2 = \Theta(n)$ and $\|\hat{S}\|_2 = \Theta(n)$ with high probability.*

Remark: This is a extended version of Proposition 4.3 in [7].
Proof: Exactly the same as proof for Lemma 4.5. ■

Lemma 4.18 *Let $W_1\Sigma W_2^T$ be the singular value decomposition of $U^T\hat{U}$. Then for sufficiently large n,*
$$\|U^T\hat{U} - W_1W_2^T\|_F = O(kn^{-1}\log n)$$
with high probability.

Proof: Exactly the same as proof for Lemma 4.6. ■

We will denote the orthogonal matrix $W_1 W_2^T$ by W^*.

Lemma 4.19 *For sufficiently large n,*

$$\|W^*\hat{S} - SW^*\|_F = O(k^{3/2}\log n),$$

$$\|W^*\hat{S}^{1/2} - S^{1/2}W^*\|_F = O(k^{3/2}n^{-1/2}\log n)$$

and

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(k^{3/2}n^{-3/2}\log n)$$

with high probability.

Proof:

By Proposition 2.1 in [6] and Equation (2), we have for some orthogonal matrix W,

$$\|\hat{U} - UW\|_F^2 \leq \frac{2\|\hat{U}\hat{U}^T - UU^T\|_F^2}{\delta^2} = O(kn^{-1/2}\sqrt{\log n}).$$

Let $Q = \hat{U} - UU^T\hat{U}$. And Q is the residual after projecting $\hat{U}$ orthogonally onto the column space of U, we have

$$\|Q\|_F = \|\hat{U} - UU^T\hat{U}\|_F \leq \|\hat{U} - UT\|_F = O(n^{-1/2}\sqrt{\log n}). \quad (6)$$

RT: Not right here, I am using the result for spectral norm as frobenius norm.

for all $k \times k$ matrices T. Then

$$\begin{aligned}
W^*\hat{S} =&(W^* - U^T\hat{U})\hat{S} + U^T\hat{U}\hat{S} = (W^* - U^T\hat{U})\hat{S} + U^T\hat{P}\hat{U} \\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)\hat{U} + U^T H\hat{U} \\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + U^T H\hat{U} \\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + SU^T\hat{U}.
\end{aligned}$$

Combined with Theorem 4.16, Lemma 4.17, Lemma 4.18, we have

$$\begin{aligned}
&\|W^*\hat{S} - SW^*\|_F \\
=&\|(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + S(U^T\hat{U} - W^*)\|_F \\
\leq&\|W^* - U^T\hat{U}\|_F(\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F\|\hat{P} - H\|_2\|Q\|_F + \|U^T(\hat{P} - H)U\|_F \\
\leq&O(k\log n) + O(k^{3/2}\log n) + \|U^T(\hat{P} - H)U\|_F
\end{aligned}$$

with high probability. And we know $U^T(\hat{P} - H)U$ is a $k \times k$ matrix with ij-th entry to be

$$u_i^T(\hat{P} - H)u_j = \sum_{s=1}^{n}\sum_{t=1}^{n}(\hat{P}_{st} - H_{st})u_{is}u_{jt} = 2\sum_{s<t}(\hat{P}_{st} - H_{st})u_{is}u_{jt}$$

where u_i and u_j are the i-th and j-th columns of U. Thus, conditioned on H, $u_i^T(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables.

By Equation (5), we have

$$\begin{aligned}
&E\left[\left((\hat{P}_{st}-H_{st})u_{is}u_{jt}\right)^k\right]\\
\le&2^k k!(C_{st}+H_{st}^{(1)})^k u_{is}^k u_{jt}^k\\
\le&\frac{k!}{2}(4R)^{k-2}(4\sqrt{2}Ru_{is}u_{jt})^2.
\end{aligned}$$

Also we have

$$\sigma^2 := |\sum_{s<t} 32R^2u_{is}^2u_{jt}^2| \le 32R^2,$$

then by Theorem 4.3, we have

$$P\left(\left|2\sum_{s<t}(\hat{P}_{st}-H_{st})u_{is}u_{jt}\right| \ge t\right) \le \exp\left(\frac{-t^2/8}{32R^2+2Rt}\right),$$

thus each entry of $U^T(\hat{P}-H)U$ is of order $O(\log n)$ with high probability and thus

$$\|U^T(\hat{P}-H)U\|_F = O(k\log n) \tag{7}$$

with high probability. Hence

$$\|W^*\hat{S}-SW^*\|_F = O(k^{3/2}\log n)$$

with high probability. Also, since

$$W_{ij}^*(\lambda_j^{1/2}(\hat{P})-\lambda_i^{1/2}(H)) = W_{ij}^*\frac{\lambda_j(\hat{P})-\lambda_i(H)}{\lambda_j^{1/2}(\hat{P})+\lambda_i^{1/2}(H)}$$

and the eigenvalues $\lambda_j^{1/2}(\hat{P})$ and $\lambda_i^{1/2}(H)$ are both of order $\Theta(\sqrt{n})$, we have

$$\|W^*\hat{S}^{1/2}-S^{1/2}W^*\|_F = O(k^{3/2}n^{-1/2}\log n).$$

Similarly, since

$$W_{ij}^*(\lambda_j^{-1/2}(\hat{P})-\lambda_i^{-1/2}(H)) = W_{ij}^*\frac{\lambda_i(H)-\lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P})+\lambda_i^{-1/2}(H))\lambda_j(\hat{P})\lambda_i(H)}$$

and the eigenvalues $\lambda_j(\hat{P})$ and $\lambda_i(H)$ are both of order $\Theta(n)$, we have

$$\|W^*\hat{S}^{-1/2}-S^{-1/2}W^*\|_F = O(k^{3/2}n^{-3/2}\log n).$$

■

Lemma 4.20 *There exists a rotation matrix W such that for sufficiently large n,*

$$\|\hat{Z}-ZW\|_F = k\|(\hat{P}-H)US^{-1/2}\|_F + O(k^{5/2}n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: Exactly the same as proof for Lemma 4.8. ■

Theorem 4.21 *There exists a rotation matrix W such that for sufficiently large n,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(k^{5/2}n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: By Lemma 4.20, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(k^{5/2}n^{-1/2}(\log n)^{3/2})$$

and similarly we could have the bound for each row vector

$$\begin{aligned}\max_i \|\hat{Z}_i - WZ_i\|_2 \leq & \frac{1}{\lambda_d^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(k^{5/2}n^{-1/2}(\log n)^{3/2}) \\ \leq & \frac{d^{1/2}}{\lambda_d^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(k^{5/2}n^{-1/2}(\log n)^{3/2})\end{aligned}$$

where u_j denotes the j-th column of U. Now given i and j, the i-th element of the vector $(\hat{P} - H)u_j$ is of the form

$$\sum_{s=1}^{n}(\hat{P}_{is} - H_{is})u_{js} = \sum_{s\neq i}(\hat{P}_{is} - H_{is})u_{js}.$$

Thus, conditioned on H, the i-th element of the vector $(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables. By Equation (5), we have

$$\begin{aligned}&E\left[\left((\hat{P}_{is} - H_{is})u_{js}\right)^k\right] \\ \leq & 2^k k!(C_{is} + H_{is}^{(1)})^k u_{js}^k \\ \leq & \frac{k!}{2}(4R)^{k-2}(4\sqrt{2}Ru_{js})^2.\end{aligned}$$

Also we have

$$\sigma^2 := |\sum_{s\neq i} 32R^2u_{js}^2| \leq 32R^2,$$

then by Theorem 4.3, we have

$$P\left(\left|\sum_{s\neq i}(\hat{P}_{is} - H_{is})u_{js}\right| \geq t\right) \leq \exp\left(\frac{-t^2/2}{32R^2 + Rt}\right),$$

i.e. it can be bounded by a constant with high probability. Taking the union bound over all i and j, we have

$$\max_i \|\hat{Z}_i - WZ_i\|_2 \leq \frac{Ck^{1/2}}{\lambda_d^{1/2}(H)}(\log n)^{3/2} + O(k^{5/2}n^{-1/2}(\log n)^{3/2}) = O(k^{5/2}n^{-1/2}(\log n)^{3/2}).$$

■

Lemma 4.22 $\left|\hat{Z}_i^T\hat{Z}_j - Z_i^TZ_j\right| = O(k^5n^{-1}(\log n)^3)$ *with high probability.*

Proof: Let W be the rotation matrix in Theorem 4.21, then

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| &= \left|\hat{Z}_i^T\hat{Z}_j - \hat{Z}_i^T W Z_j + \hat{Z}_i^T W Z_j - (WZ_i)^T W Z_j\right| \\ &\leq \left|\hat{Z}_i^T(\hat{Z}_j - WZ_j) + (\hat{Z}_i^T - (WZ_i)^T)WZ_j\right| \\ &\leq \|\hat{Z}_i\|_2\|\hat{Z}_j - WZ_j\|_2 + \|Z_j\|_2\|\hat{Z}_i^T - (WZ_i)^T\|_2.\end{aligned}$$

Since $\|Z_i\|_2^2 = Z_i^T Z_i = H_{ii}^q = E[\hat{P}_{ii}^{(q)}] \leq E[\hat{P}_{ii}^{(1)}] = (1-\epsilon)P_{ij} + \epsilon C_{ij} \leq R$, we have $\|Z_i\|_2 = O(1)$. Combined with Theorem 4.21,

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| &= (\|\hat{Z}_i\|_2 + \|Z_j\|_2)O(n^{-1/2}(\log n)^{3/2}) \\ &\leq (\|\hat{Z}_i - WZ_i\|_2 + \|WZ_i\|_2 + \|Z_j\|_2)O(k^{5/2}n^{-1/2}(\log n)^{3/2}) \\ &= O(k^5 n^{-1}(\log n)^3)\end{aligned}$$

with high probability. ■

Corollary 4.23 *For fixed m, the estimator based on ASE of MLqE has the same entry-wise asymptotic bias as MLqE, i.e.*

$$\lim_{n\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(q)}) = \lim_{n\to\infty} E[\widetilde{P}_{ij}^{(q)}] - P_{ij} = \lim_{n\to\infty} E[\hat{P}_{ij}^{(q)}] - P_{ij} = \lim_{n\to\infty} \mathrm{Bias}(\hat{P}_{ij}^{(q)}).$$

Proof: Direct result from Lemma 4.22 by noticing

$$\lim_{n\to\infty} E[\widetilde{P}_{ij}^{(q)}] = \lim_{n\to\infty} E[\hat{P}_{ij}^{(q)}].$$

■

Theorem 4.24 $\mathrm{Var}(\hat{Z}_i^T\hat{Z}_j) = O(n^{-1}(\log n)^3)$ *with high probability.*

Proof:

$$\begin{aligned}\mathrm{Var}(\hat{Z}_i^T\hat{Z}_j) &= E[(\hat{Z}_i^T\hat{Z}_j - E[\hat{Z}_i^T\hat{Z}_j])^2] \\ &= E[(\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j + Z_i^T Z_j - E[\hat{Z}_i^T\hat{Z}_j])^2] \\ &= E[(\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T\hat{Z}_j])^2] \\ &\quad - 2E[(\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j)(Z_i^T Z_j - E[\hat{Z}_i^T\hat{Z}_j])] \\ &\leq E[(\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T\hat{Z}_j])^2] \\ &\quad + 2\sqrt{E[(\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j)^2]E[(Z_i^T Z_j - E[\hat{Z}_i^T\hat{Z}_j])^2]} \\ &\leq 4E[(\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j)^2] \\ &= O(k^{10}n^{-2}(\log n)^6)\end{aligned}$$

with high probability. ■

Theorem 4.25 *Let $u_q(\theta) = E_\theta[\hat{\theta}_{q,n}]$, $\phi_q(x;\theta) = \frac{\partial}{\partial\theta}L_q(f(x;\theta))$, and $\phi_q'(x;\theta) = \frac{\partial^2}{\partial\theta^2}L_q(f(x;\theta))$. Then the asymptotic distribution of $\hat{\theta}_{q,n}$ is $\sqrt{n}(\hat{\theta}_{q,n} - u_q(\theta)) \sim \mathcal{N}(0, V_q(\theta))$, where $V_q(\theta) = E[\phi_q(X;\theta)^2]/E[\phi_q'(X;\theta)]^2$.*

Remark: See Theorem 1 in http://arxiv.org/pdf/1310.7278.pdf.

Corollary 4.26 $\mathrm{Var}(\hat{P}_{ij}^{(q)}) = \Theta(m^{-1})$.

Proof: Direct result from Theorem 4.25. ■

Theorem 4.27 *For fixed* m*,* $1 \le i, j \le n$*,*

$$\frac{\mathrm{Var}(\widetilde{P}_{ij}^{(q)})}{\mathrm{Var}(\hat{P}_{ij}^{(q)})} = O(k^10mn^{-2}(\log n)^6).$$

Thus

$$\mathrm{ARE}(\hat{P}_{ij}^{(q)}, \widetilde{P}_{ij}^{(q)}) = 0.$$

Furthermore, as long as m *goes to infinity of order* $o(n(\log n)^{-3})$*,*

$$\mathrm{ARE}(\hat{P}_{ij}^{(q)}, \widetilde{P}_{ij}^{(q)}) = 0.$$

Proof: The results are direct from Theorem 4.24 and Corollary 4.26. ■

4.4 $\widetilde{P}^{(q)}$ better than $\widetilde{P}^{(1)}$

Theorem 4.28 *For sufficiently large* n *and* C*, any* $1 \le i, j \le n$*,*

$$\lim_{m\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(1)}) > \lim_{m\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(q)})$$

Proof: Direct result from Lemma 3.1, Corollary 4.11 and Corollary 4.23. ■

Theorem 4.29 *For any fixed* m*, any* $1 \le i, j \le n$*,*

$$\lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(1)}) = \lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(q)}) = 0$$

Proof: Direct result from Theorem 4.12 and Theorem 4.24. ■

5 Generalization

We can generalize the exponential distribution to F and generalize MLqE to $\hat{P}$ with the following assumptions:

- There exists $C_0(P_{ij}, \epsilon) > 0$ such that under the contaminated model with $C > C_0(P_{ij}, \epsilon)$,

$$\lim_{m \to \infty} \left| E[\hat{P}_{ij}] - P_{ij} \right| < \lim_{m \to \infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

  for $1 \leq i, j \leq n$.

- Let $A^{(t)} \stackrel{iid}{\sim} (1-\epsilon)F(P) + \epsilon F(C)$ and $H_{ij}^{(1)} = E[\hat{P}_{ij}^{(1)}] = (1-\epsilon)E_F(P_{ij}) + \epsilon E_F(C_{ij})$, then $E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k] \leq \text{const} \cdot k!$.

- $\hat{P}_{ij} \leq \text{const} \cdot \hat{P}_{ij}^{(1)}$. This might be generalized to with high probability later.

- $\lim_{m \to \infty} \text{Var}(\hat{P}_{ij}) = 0$

6 Analysis of MLqE under Exponential Model

Let $f(x,\theta) = \sum_{i=1}^{m} e^{-\frac{(1-q)x_i}{\theta}}(x-\theta)$, then $\frac{\partial f}{\partial x_i}(x,\theta) = e^{-\frac{(1-q)x_i}{\theta}}\left(-\frac{(1-q)x_i}{\theta} + 2 - q\right)$. Define $c_q = 1 + \frac{1}{1-q}$, then

$$\frac{\partial f}{\partial x_i}(x,\theta) = 0 \Leftrightarrow x_i \to \infty \text{ or } x_i = (1 + \frac{1}{1-q})\theta$$

Define $x = (x_1, \cdots, x_m)$ and $x^- = (x_1, \cdots, x_m)\backslash x_i$, then

1. $\theta_{\max}$, $x_i = c_q\theta \le c_q\bar{x}$
   $\frac{1}{m}(c_q\bar{x} + (m-1)\bar{x}^-) = \bar{x}$
   Thus $\theta_{\max} \le \frac{m-1}{m-C_q}\bar{x}^-$

2. $\theta_{\min}$
   2 candidates: $x_i = 0$ or $x_i = \infty$.
   When $x_i = 0$, It's like $m-1$ points x^-;
   When $x_i = \infty$, $f(x,\theta) = f(x^-,\theta) - \theta$.
   Thus $\theta_{\min}$ solves $f(x,\theta)|_{x_i=0} = 0$.

7 Appendix

Definition 7.1 *A random variable Z is called central moment bounded with real parameter $L > 0$, if for any integer $i \ge 1$ we have*

$$E\left[|Z - E[Z]|^i\right] \le i \cdot L \cdot E\left[|Z - E[Z]|^{i-1}\right].$$

References

[1] Chandler Davis and William Morton Kahan. The rotation of eigenvectors by a perturbation. iii. *SIAM Journal on Numerical Analysis*, 7(1):1–46, 1970.

[2] Davide Ferrari and Yuhong Yang. Maximum lq-likelihood estimation. *Ann. Statist.*, 38(2):753–783, 04 2010.

[3] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

[4] Roberto Imbuzeiro Oliveira. Concentration of the adjacency matrix and of the laplacian in random graphs with independent edges. *arXiv preprint arXiv:0911.0600*, 2009.

[5] Yichen Qin and Carey E Priebe. Maximum l q-likelihood estimation via the expectation-maximization algorithm: A robust estimation of mixture models. *Journal of the American Statistical Association*, 108(503):914–928, 2013.

[6] Karl Rohe, Sourav Chatterjee, and Bin Yu. Spectral clustering and the high-dimensional stochastic blockmodel. *The Annals of Statistics*, pages 1878–1915, 2011.

[7] Daniel L Sussman, Minh Tang, and Carey E Priebe. Consistent latent position estimation and vertex classification for random dot product graphs. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 36(1):48–57, 2014.

[8] Joel A Tropp. User-friendly tail bounds for sums of random matrices. *Foundations of computational mathematics*, 12(4):389–434, 2012.