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1 Problem Description

1.1 Uncontaminated Model

Let F be a distribution on $\mathcal{X} \in \mathbb{R}^d$, satisfying $x^T y \geq 0$ for all $x, y \in \mathcal{X}$. We now generate m i.i.d. graphs under the RDPG(F) model. First sample $X_1, \cdots, X_n$ independently from distribution F, and define $X = [X_1, \cdots, X_n]^T \in \mathbb{R}^{n \times d}$, $P = XX^T \in [0, R]^{n \times n}$, where R is a constant. Then we can sample m conditionally i.i.d. symmetric and hollow graphs $G^{(1)}, \cdots, G^{(m)}$, such that conditioned on X, $G_{ij}^{(t)} \stackrel{ind}{\sim} \mathrm{Exp}(P_{ij})$ for each $1 \leq t \leq m$, $1 \leq i < j \leq n$.

Note: We are now considering the SBM model as a RDPG.

1.2 Contaminated Observations

Now we assume the observed edges are contaminated with probability ϵ.

Let G be a distribution on $\mathcal{Y} \in \mathbb{R}^{d'}$, satisfying $x^T y \geq 0$ for all $x, y \in \mathcal{Y}$. First sample X from F and Y from G. Then we sample m conditionally i.i.d. symmetric and hollow graphs $A^{(1)}, \cdots, A^{(m)}$ such that conditioning on X and Y, $A_{ij}^{(t)} \stackrel{ind}{\sim} (1 - \epsilon)\mathrm{Exp}(P_{ij}) + \epsilon \mathrm{Exp}(C_{ij})$ for each $1 \leq t \leq m$, $1 \leq i < j \leq n$, where the contamination is a rank-d' matrix $C = YY^T \in [0, R]^{n \times n}$, $Y \in \mathbb{R}^{n \times d'}$.

1.3 Goal

Given the contaminated observation of adjacency matrices of m graphs, i.e. $A^{(1)}, \cdots, A^{(m)}$, we want to estimate the mean of the collection of uncontaminated graphs P.

2 Candidate Estimators

After observing contaminated adjacency matrices of m graphs $A^{(1)}, \cdots, A^{(m)}$, we want to propose a good estimator for the mean of the collection of graphs P.

2.1 $\hat{P}^{(1)}$ based on entry-wise MLE

Under the independent edge setting, we can simplify the problem to finding an entry-wise estimate of P. And MLE is always our first choice, which exists and happen to be $\bar{A}$, the entry-wise mean in this case. For consistency, we define $\hat{P}^{(1)} = \bar{A}$.

2.2 $\hat{P}^{(q)}$ based on entry-wise MLqE

Since the observations are contaminated, robust estimators are preferred. A modified MLE estimator, the maximum likelihood L-q estimator [2, 6], is considered in this case. Note that there might be multiple solution to the MLq equation, we define the MLqE to be the largest solution (which is still less than MLE when the model is exponential distribution). Denote $\hat{P}^{(q)}$ as the entry-wise MLqE.

Remark: MLE is a special case of MLqE when $q = 1$. So we notate the entry-wise MLE to be $\hat{P}^{(1)}$ in consistent with entry-wise MLqE $\hat{P}^{(q)}$.

2.3 $\widetilde{P}^{(1)}$ based on ASE of entry-wise MLE

By taking advantages of the graph structure, we expect a better performance after applying a rank-reduction procedure to the entry-wise MLE $\hat{P}^{(1)}$ under the SBM. So we first apply ASE to $\hat{P}^{(1)}$ to get the latent positions $\hat{X}^{(1)}$ in dimension $d^{(1)}$, and then define $\widetilde{P}^{(1)} = (\hat{X}^{(1)} \hat{X}^{(1)T})_{\mathrm{tr}}$, where each element is a projection of $\hat{X}_i^{(1)} \hat{X}_j^{(1)T}$ onto $[0, \max(\hat{P}_{ij}^{(1)}, R)]$.

2.4 $\widetilde{P}^{(q)}$ based on ASE of entry-wise MLqE

Similarly, we also expect a better performance after applying a rank-reduction procedure to the entry-wise MLqE $\hat{P}^{(q)}$ under the SBM. So we first apply ASE to $\hat{P}^{(q)}$ to get the latent positions $\hat{X}^{(q)}$ in dimension $d^{(q)}$, and then define $\widetilde{P}^{(q)} = (\hat{X}^{(q)} \hat{X}^{(q)T})_{\mathrm{tr}}$, where each element is a projection of $\hat{X}_i^{(q)} \hat{X}_j^{(q)T}$ onto $[0, \max(\hat{P}_{ij}^{(q)}, R)]$.

2.5 Summary

Thus, we should choose the estimator $\widetilde{P}^{(q)}$.

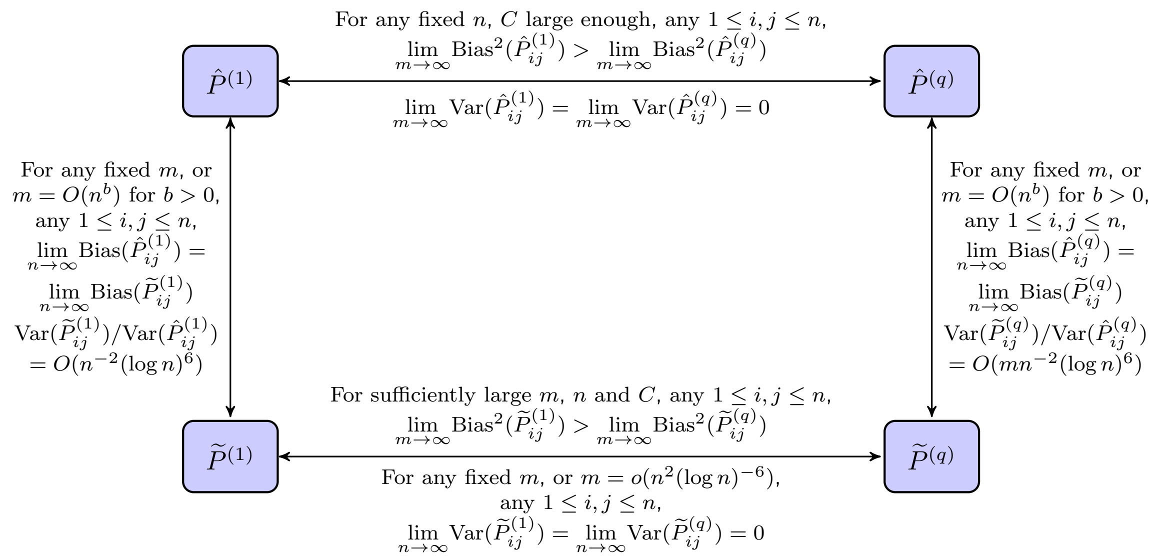

Figure 1: Relationship between four estimators.

3 Proof

3.1 $\hat{P}^{(q)}$ better than $\hat{P}^{(1)}$

Lemma 3.1 *Consider the model* $X_1, \cdots, X_m \overset{iid}{\sim} \mathrm{Exp}(P)$ *with* $m \geq 2$ *and* $E[X_1] = P$. *Given any data* $x = (x_1, \cdots, x_m)$ *such that* $x_{(1)} > 0$ *and not all* x_i*'s are the same, then no matter how the data is sampled, we have*

- *There exists at least one solution to the MLq equation;*
- *All the solutions to the MLq equation are less than the MLE.*

Thus the MLqE $\hat{P}^{(q)}$*, the root closest to the MLE, is well defined.*

Require: Exponential distribution, MLqE
Proof: The MLE is

$$\hat{P}^{(1)}(x) = \bar{x}.$$

Consider the continuous function $g(\theta, x) = \sum_{i=1}^{m} e^{-\frac{(1-q)x_i}{\theta}}(x_i - \theta)$. Then the ML$q$ equation is $g(\theta, x) = 0$.

Let $x_{(1)} \leq \cdots \leq x_{(l)} \leq \bar{x} \leq x_{(l+1)} \leq \cdots \leq x_{(m)}$. Define $s_i = \bar{x} - x_{(i)}$ for $1 \leq i \leq l$, and $t_i = x_{(l+i)} - \bar{x}$ for $1 \leq i \leq m - l$. Note that $\sum_{i=1}^{l} s_i = \sum_{i=1}^{m-l} t_i$.

Then for any $\theta \geq \bar{x}$, we have

$$
\begin{aligned}
g(\theta, x) &= \sum_{i=1}^{m} e^{-\frac{(1-q)x_{(i)}}{\theta}}(x_{(i)} - \theta) = \sum_{i=1}^{m} e^{-\frac{(1-q)x_{(i)}}{\theta}}(x_{(i)} - \bar{x} + \bar{x} - \theta) \\
&= -\sum_{i=1}^{l} e^{-\frac{(1-q)x_{(i)}}{\theta}} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i + \sum_{i=1}^{m} e^{-\frac{(1-q)x_{(i)}}{\theta}}(\bar{x} - \theta) \\
&\leq -\sum_{i=1}^{l} e^{-\frac{(1-q)x_{(i)}}{\theta}} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i \\
&\leq -e^{-\frac{(1-q)x_{(l+1)}}{\theta}} \sum_{i=1}^{l} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i \\
&\leq -e^{-\frac{(1-q)x_{(l+1)}}{\theta}} \sum_{i=1}^{m-l} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\theta}} t_i \\
&\leq -\sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&= 0,
\end{aligned}
$$

and equality holds if and only if all x_i's are the same, which is excluded by the assumption. Thus $g(\theta, x) < 0$ for any $\theta \geq \bar{x}$.

Denote any solution to the MLq equation to be $\hat{P}^{(q)}(x)$, then we also know:

- $g(\hat{P}^{(q)}(x), x) = 0$;
- $\lim_{\theta \to 0^+} g(\theta, x) = 0$;
- $g(\theta, x) > 0$ when $\theta < x_{(1)}$;

Thus there exists at least one solution to the MLq equation. And all solutions to the MLq equation are between $x_{(1)}$ and $\bar{x}$, i.e. less than the MLE. ∎

Lemma 3.2 *Consider the exponential distribution model as in Lemma 3.1 while the data is actually sampled under the contaminated model* $X, X_1, \cdots, X_m \overset{iid}{\sim} (1-\epsilon)\mathrm{Exp}(P) + \epsilon\mathrm{Exp}(C)$. *Denote such contaminated distribution as* F. *Then there exists at least one solution* $\theta(F)$ *of the population version of MLq equation, i.e.* $E_F[e^{-\frac{(1-q)X}{\theta(F)}}(X - \theta(F))] = 0$, *such that* $\theta(F) < E_F[\bar{X}] = (1-\epsilon)P + \epsilon C$. *So we can define* $\theta(F_{ij})$ *to be the largest root which is less than* $E_F[\bar{X}]$.

Require: Exponential distribution, MLqE
Proof: For the MLE, i.e. $\bar{X}$, we have $E[\bar{X}] = (1-\epsilon)P + \epsilon C$. According to Equation (3.2) in [2], $\theta(F)$ satisfies

$$
\frac{\epsilon C}{(C(1-q)+\theta)^2} - \frac{\epsilon}{C(1-q)+\theta} + \frac{(1-\epsilon)P}{(P(1-q)+\theta)^2} - \frac{(1-\epsilon)}{P(1-q)+\theta} = 0,
$$

i.e.

$$
\frac{\epsilon(\theta - Cq)}{(C(1-q)+\theta)^2} = \frac{(1-\epsilon)(Pq - \theta)}{(P(1-q)+\theta)^2}.
$$

Define $h(\theta) = (C(1-q)+\theta)^2(1-\epsilon)(Pq-\theta) - (P(1-q)+\epsilon)^2\epsilon(\theta - Cq)$. Then $\lim_{\theta\to\infty} h(\theta) = -\infty$, $h(0) > 0$, and $h(Cq) < 0$. Consider q as the variable and solve the equation $h(E[\bar{X}]) = 0$, we have three roots and one of them is $q = 1$ obviously. The other two roots are

$$\frac{(P+C)\left((P-C)^2\epsilon(1-\epsilon)+2PC\right)}{2PC(P\epsilon+C(1-\epsilon))} \pm \frac{1}{2}\sqrt{\frac{\epsilon(1-\epsilon)(C-P)^4 - 4P^2C^2}{P^2C^2(P\epsilon+C(1-\epsilon))^2}}.$$

For the first part,

$$\frac{(P+C)\left((P-C)^2\epsilon(1-\epsilon)+2PC\right)}{2PC(P\epsilon+C(1-\epsilon))} > 1 + \frac{(P-C)^2\epsilon(1-\epsilon)(P+C)}{2PC(P\epsilon+C(1-\epsilon))}.$$

To prove the roots are greater or equal to 1, we just need to show

$$(P-C)^2\epsilon(1-\epsilon)(P+C)^2 \geq \epsilon(1-\epsilon)(C-P)^4 - 4P^2C^2.$$

Then it is sufficient to show that

$$(P+C)^2 \geq (C-P)^2,$$

which is true. Combined with the fact that when $q = 0$, $h(E[\bar{X}]) < 0$, we have for any $0 < q < 1$, $h(E[\bar{X}]) < 0$.

The equation $h(\theta) = 0$ is a cubic polynomial, so it has at most three real roots. Combined with the fact that $h(0) > 0$, we have for any $0 < q < 1$, there exists at least one root of the population version of MLq equation which is less than $E[\bar{X}] = (1-\epsilon)P + \epsilon C$. ■

Lemma 3.3 *For any $a > 0$, we have*

$$\sup_{\theta\in[a,R]} \left| \frac{1}{m}\sum_{i=1}^{m} e^{-\frac{(1-q)X_i}{\theta}}(X_i - \theta) - E_F[e^{-\frac{(1-q)X}{\theta}}(X-\theta)] \right| \xrightarrow{a.s.} 0.$$

Require: Exponential distribution, MLqE. Should be easy to extend

Proof: Define $g(x,\theta) = e^{-\frac{(1-q)x}{\theta}}(x-\theta)$ and $d(x) = e^{-\frac{(1-q)x}{R}}(x+R)$. Then $E_F[d(X)] < \infty$ and $g(x,\theta) \leq d(x)$ for all $\theta \in [a,R]$. Combined with the fact that $[a,R]$ is compact and the function $g(x,\theta)$ is continuous at each θ for all $x > 0$ and measurable function of x at each θ, we have the uniform convergence by Lemma 2.4 in [4]. ■

Lemma 3.4 $\hat{P}_{ij}^{(q)} \xrightarrow{P} \theta(F_{ij})$*, where F_{ij} is the contaminated distribution $(1-\epsilon)\mathrm{Exp}(P_{ij}) + \epsilon\mathrm{Exp}(C_{ij})$. That is, $\hat{P}_{ij}^{(q)}$ is an consistent estimator of $\theta(F_{ij})$.*

Proof:

■

Lemma 3.5 $E[\hat{P}_{ij}^{(q)}] \to \theta(F_{ij})$ *as $m \to \infty$, where F_{ij} is the contaminated distribution $(1-\epsilon)\mathrm{Exp}(P_{ij}) + \epsilon\mathrm{Exp}(C_{ij})$.*

Proof: NEED PROOF HERE ■

Lemma 3.6 *For any $0 < q < 1$, there exists $C_0(P_{ij}, \epsilon, q) > 0$ such that under the contaminated model with $C > C_0(P_{ij}, \epsilon, q)$,*

$$\lim_{m\to\infty} \left| E[\hat{P}_{ij}^{(q)}] - P_{ij} \right| < \lim_{m\to\infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

for $1 \leq i, j, \leq n$ and $i \neq j$.

Require: Exponential distribution, MLqE
Proof: For the MLE $\hat{P}_{ij}^{(1)} = \bar{A}_{ij}$,

$$E[\hat{P}_{ij}^{(1)}] = E[\bar{A}_{ij}] = \frac{1}{m}\sum_{t=1}^{m} E[A_{ij}^{(t)}] = E[A_{ij}^{(1)}] = (1-\epsilon)P_{ij} + \epsilon C_{ij}.$$

As shown in Lemma 3.2, $\theta(F)$ satisfies

$$\frac{\epsilon(\theta(F) - C_{ij}q)}{(C_{ij}(1-q) + \theta(F))^2} = \frac{(1-\epsilon)(P_{ij}q - \theta(F))}{(P_{ij}(1-q) + \theta(F))^2}.$$

Thus $\theta(F) - C_{ij}q$ and $\theta(F) - P_{ij}q$ should have different signs. Combined with $C_{ij} > P_{ij}$, we have

$$qP_{ij} < \theta(F).$$

To have a smaller asymptotic bias in absolute value, combined with Lemma 3.5, we need

$$|\theta(F) - P_{ij}| < \epsilon(C_{ij} - P_{ij}).$$

Based on Lemma 3.1, we need

$$qP_{ij} > P_{ij} - \epsilon(C_{ij} - P_{ij}),$$

i.e.

$$C_{ij} > P_{ij} + \frac{(1-q)P_{ij}}{\epsilon} = C_0(P_{ij}, \epsilon, q).$$

■

Lemma 3.7

$$\lim_{m\to\infty} \mathrm{Var}(\hat{P}_{ij}^{(1)}) = \lim_{m\to\infty} \mathrm{Var}(\hat{P}_{ij}^{(q)}) = 0,$$

for $1 \leq i, j \leq n$.

Proof: MLE simply follows a central limit theorem, which means the variance goes to 0 as $m \to \infty$. For MLqE, NEED PROOF HERE ■

3.2 $\widetilde{P}^{(1)}$ better than $\hat{P}^{(1)}$

Theorem 3.8 *(Matrix Bernstein: Subexponential Case). Consider a finite sequence $\{X_k\}$ of independent, random, self-adjoint matrices with dimension d. Assume that*

$$E[X_k] = 0 \quad \textit{and} \quad E[X_k^p] \preceq \frac{p!}{2} R^{p-2} A_k^2 \quad \textit{for } p = 2, 3, 4, \ldots$$

Compute the variance parameter

$$\sigma^2 := \|\sum_k A_k^2\|.$$

Then the following chain of inequalities holds for all $t \geq 0$.

$$P\left(\lambda_{\max}\left(\sum_k X_k\right) \geq t\right) \leq d \cdot \exp\left(\frac{-t^2/2}{\sigma^2 + Rt}\right).$$

Remark: Theorem 6.2 in [9].

Theorem 3.9 *Let P and C be two n-by-n symmetric matrices satisfying element-wise conditions $0 < P_{ij} \leq C_{ij} \leq R$ for some constant $R > 0$. For $0 < \epsilon < 1$, we define m symmetric and hollow matrices as*

$$A^{(t)} \stackrel{iid}{\sim} (1-\epsilon)\text{Exp}(P) + \epsilon\text{Exp}(C),$$

for $1 \leq t \leq m$. Let $\hat{P}^{(1)}$ be the element-wise MLE based on exponential distribution with m observations. Define $H_{ij}^{(1)} = E[\hat{P}_{ij}^{(1)}] = (1-\epsilon)P_{ij} + \epsilon C_{ij}$, then for any constant $c > 0$, there exists another constant $n_0(c)$, independent of n, P, C and ϵ, such that if $n > n_0$, then for all η satisfying $n^{-c} \leq \eta \leq 1/2$,

$$P\left(\|\hat{P}^{(1)} - H^{(1)}\|_2 \leq 4R\sqrt{n\ln(n/\eta)/m}\right) \geq 1 - \eta.$$

Remark: This is the extended version of Theorem 3.1 in [5].
Require: Exponential distribution
Proof: Let $\{e_i\}_{i=1}^n$ be the canonical basis for $\mathbb{R}^n$. For each $1 \leq i, j \leq n$, define a corresponding matrix G_{ij}:

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus

$$\hat{P}^{(1)} = \sum_{1 \leq i < j \leq n} \hat{P}_{ij}^{(1)} G_{ij} = \frac{1}{m} \sum_{t=1}^{m} \sum_{1 \leq i < j \leq n} A_{ij}^{(t)} G_{ij}$$

and

$$H^{(1)} = \sum_{1 \leq i < j \leq n} H_{ij}^{(1)} G_{ij}.$$

Then we have $\hat{P}^{(1)} - H^{(1)} = \frac{1}{m}\sum_{1 \leq t \leq m, 1 \leq i < j \leq n} X_{ij}^{(t)}$, where $X_{ij}^{(t)} = \left(A_{ij}^{(t)} - H_{ij}^{(1)}\right) G_{ij}$ for $1 \leq t \leq m$ and $1 \leq i < j \leq n$.

First bound the k-th moment of X_{ij} for $1 \leq i < j \leq n$ as following:

$$\begin{aligned} E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k] \leq &(1-\epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1+k, -H_{ij}/P_{ij}) \\ &+ \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1+k, -H_{ij}/C_{ij}) \\ \leq &\left((1-\epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k\right) k! \\ \leq &\left((1-\epsilon) \cdot P_{ij}^k + \epsilon \cdot C_{ij}^k\right) k! \\ \leq &R^k k!, \end{aligned} \tag{1}$$

Combined with

$$G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & \text{k is even;} \\ e_i e_j^T + e_j e_i^T, & \text{k is odd,} \end{cases}$$

thus we have

1. When k is even,

$$E[(X_{ij}^{(t)})^k] = E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k]G_{ij}^2 \preceq k!R^k G_{ij}^2;$$

2. When k is odd,

$$E[(X_{ij}^{(t)})^k] = E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k]G_{ij} \preceq k!R^k G_{ij}^2.$$

So

$$E[(X_{ij}^{(t)})^k] \preceq k!R^k G_{ij}^2.$$

Let

$$\sigma^2 := \left\| \sum_{1\le t\le m, 1\le i<j\le n} (\sqrt{2}RG_{ij})^2 \right\|_2 = 2R^2 m\|(n-1)I\|_2 = 2R^2 m(n-1).$$

Notice that random matrices $X_{ij}^{(t)}$ are independent, self-adjoint and have mean zero, apply Theorem 3.8 we have

$$\begin{aligned}
P\left(\lambda_{\max}(\hat{P}^{(1)} - H^{(1)}) \ge t\right) &= P\left(\lambda_{\max}\left(\frac{1}{m}\sum_{1\le t\le m, 1\le i<j\le n} X_{ij}^{(t)}\right) \ge t\right) \\
&= P\left(\lambda_{\max}\left(\sum_{1\le t\le m, 1\le i<j\le n} X_{ij}^{(t)}\right) \ge mt\right) \\
&\le n\exp\left(-\frac{(mt)^2/2}{\sigma^2 + Rmt}\right) \\
&\le n\exp\left(-\frac{mt^2/2}{2R^2 n + Rt}\right).
\end{aligned}$$

Now consider $Y_{ij}^{(t)} \equiv \left(H_{ij}^{(1)} - A_{ij}^{(t)}\right) G_{ij}$, for $1 \le t \le m$ and $1 \le i < j \le n$. Then we have $H^{(1)} - \hat{P}^{(1)} = \frac{1}{m}\sum_{1\le t\le m, 1\le i<j\le n} Y_{ij}^{(t)}$. Since

$$E[(H^{(1)} - \hat{P}^{(1)})^k] = (-1)^k E[(\hat{P}^{(1)} - H^{(1)})^k],$$

1. When k is even,

$$E[(Y_{ij}^{(t)})^k] = E[(\hat{P}^{(1)} - H^{(1)})^k]G_{ij}^2 \preceq k!R^k G_{ij}^2;$$

2. When k is odd,

$$E[Y_{ij}^k] = -E[(\hat{P}^{(1)} - H^{(1)})^k]G_{ij} \preceq k!R^k G_{ij}^2.$$

Thus by similar arguments,

$$\begin{aligned}
P\left(\lambda_{\min}(\hat{P}^{(1)} - H^{(1)}) \le -t\right) &= P\left(\lambda_{\max}(H^{(1)} - \hat{P}^{(1)}) \ge t\right) \\
&\le n\exp\left(-\frac{mt^2/2}{2R^2 n + Rt}\right).
\end{aligned}$$

Therefore we have

$$P\left(\|\hat{P}^{(1)} - H^{(1)}\|_2 \geq t\right) \leq n \exp\left(-\frac{mt^2/2}{2R^2 n + Rt}\right).$$

Now let $c > 0$ be given and assume $n^{-c} \leq \eta \leq 1/2$. Then there exists a $n_0(c)$ independent of n, P, C and ϵ such that whenever $n > n_0(c)$,

$$t = 4R\sqrt{n \ln(n/\eta)/m} \leq 6Rn.$$

Plugging this t into the equation above, we get

$$P(\|\hat{P}^{(1)} - H^{(1)}\|_2 \geq 4R\sqrt{n \ln(n/\eta)/m}) \leq n \exp\left(-\frac{t^2}{16R^2 n}\right) = \eta.$$

■

Define $H^{(1)} = E[\hat{P}^{(1)}] = (1-\epsilon)P + \epsilon C$, where $P = XX^T$, $X \in \mathbb{R}^{n \times d}$, $C = YY^T$, $Y \in \mathbb{R}^{n \times d'}$. Let $d^{(1)} = \mathrm{rank}(H^{(1)})$ be the dimension in which we are going to embed $\hat{P}^{(1)}$. Then we can define $H^{(1)} = ZZ^T$ where $Z \in \mathbb{R}^{n \times d^{(1)}}$. Since $H^{(1)} = [\sqrt{1-\epsilon}X, \sqrt{\epsilon}Y][\sqrt{1-\epsilon}X, \sqrt{\epsilon}Y]^T$, we have $d^{(1)} \leq d + d'$.

For simplicity, from now on, we will use $\hat{P}$ to represent $\hat{P}^{(1)}$, use H to represent $H^{(1)}$ and use k to represent the dimension $d^{(1)}$ we are going to embed. Assume $H = USU^T = ZZ^T$, where $Z = [Z_1, \cdots, Z_n]^T$ is a n-by-k matrix. Then our estimate for Z up to rotation is $\hat{Z} = \hat{U}\hat{S}^{1/2}$, where $\hat{U}\hat{S}\hat{U}^T$ is the rank-k spectral decomposition of $|\hat{P}| = (\hat{P}^T\hat{P})^{1/2}$.

Furthermore, we assume that the second moment matrix $E[Z_1 Z_1^T]$ is rank k and has distinct eigenvalues $\lambda_i(E[Z_1 Z_1^T])$. In particular, we assume that there exists $\delta > 0$ such that

$$\delta < \min\left(\min_{i \neq j} |\lambda_i(E[Z_1 Z_1^T]) - \lambda_j(E[Z_1 Z_1^T])|, \lambda_k(E[Z_1 Z_1^T])\right)$$

Lemma 3.10 *Under the above assumptions, $\lambda_i(H) = \Theta(n)$ with high probability when $i \leq k$, i.e. the largest k eigenvalues of H is of order n. Moreover, we have $\|S\|_2 = \Theta(n)$ and $\|\hat{S}\|_2 = \Theta(n)$ with high probability.*

Remark: This is a extended version of Proposition 4.3 in [8].
Require: Theorem 3.9 and assumptions above
Proof: Note that $\lambda_i(H) = \lambda_i(ZZ^T) = \lambda_i(Z^T Z)$ when $i \leq k$. Since each entry of $Z^T Z$ is a sum of n independent random variables each in $[0, R]$, i.e. $(Z^T Z)_{ij} = \sum_{l=1}^n Z_{li} Z_{lj}$. By Hoeffding's inequality,

$$P(|(Z^T Z - nE[Z_1 Z_1^T])_{ij}| \geq t) \leq 2\exp(-\frac{2t^2}{nR^2}).$$

Now let $c > 0$ and assume $n^{-c} \leq \eta \leq 1/2$. Let

$$t = R\sqrt{n \ln(\sqrt{2/\eta})},$$

we have

$$P\left(|(Z^T Z - nE[Z_1 Z_1^T])_{ij}| \geq R\sqrt{n \ln(\sqrt{2/\eta})}\right) \leq \eta.$$

By the union bound, we have

$$P\left(\|Z^T Z - nE[Z_1 Z_1^T]\|_F \geq kR\sqrt{n\ln(\sqrt{2/\eta})}\right) \leq k^2\eta.$$

Then by Weyl's Theorem [3], we have

$$|\lambda_i(H) - n\lambda_i(Z_1 Z_1^T)| \leq \|Z^T Z - nE[Z_1 Z_1^T]\|_2 = O(\sqrt{n\log n})$$

with probability at least $1 - k^2\eta$. Thus $\lambda_i(H) = S_{ii} = \Theta(n)$ with probability at least $1 - \frac{2k^2}{n^2}$ when $i \leq k$.

Moreover,

$$\|H\|_2 - \|H - \hat{P}\|_2 \leq \|\hat{S}\|_2 \leq \|\hat{P} - H\|_2 + \|H\|_2.$$

Combined with Theorem 3.9, with high probability we have $\|\hat{S}\|_2 = \Theta(n)$. ■

Lemma 3.11 *Let $W_1 \Sigma W_2^T$ be the singular value decomposition of $U^T\hat{U}$. Then for sufficiently large n,*

$$\|U^T\hat{U} - W_1 W_2^T\|_F = O(m^{-1}n^{-1}\log n)$$

with high probability.

Require: Theorem 3.9 and Lemma 3.10
Proof: Let $\sigma_1, \cdots, \sigma_k$ denote the singular values of $U^T\hat{U}$. Then $\sigma_i = \cos(\theta_i)$ where the θ_i are the principal angles between the subspaces spanned by $\hat{U}$ and U. Furthermore, by the Davis-Kahan $\sin(\Theta)$ theorem [1], combined with Theorem 3.9 and Lemma 3.10,

$$\begin{aligned} &\|\hat{U}\hat{U}^T - UU^T\|_2 = \max_i |\sin(\theta_i)| \\ \leq &\frac{\|\hat{P} - H\|_2}{\lambda_k(H)} \leq \frac{C\sqrt{n\log n/m}}{n} \\ = &O(m^{-1/2}n^{-1/2}\sqrt{\log n}) \end{aligned} \tag{2}$$

for sufficiently large n with high probability. Here $\lambda_k(H)$ denotes the k-th largest eigenvalue of H. Thus with high probability,

$$\begin{aligned} \|U^T\hat{U} - W_1 W_2^T\|_F = \|\Sigma - I\|_F = &\sqrt{\sum_{i=1}^k (1-\sigma_i)^2} \\ &\leq \sum_{i=1}^k (1-\sigma_i) \leq \sum_{i=1}^k (1-\sigma_i^2) \\ &= \sum_{i=1}^k \sin^2(\theta_i) \leq k\|\hat{U}\hat{U}^T - UU^T\|_2^2 \\ &= O(m^{-1}n^{-1}\log n). \end{aligned}$$

■

We will denote the orthogonal matrix $W_1 W_2^T$ by W^*.

Lemma 3.12 *For sufficiently large n,*

$$\|W^*\hat{S} - SW^*\|_F = O(m^{-1/2}\log n),$$
$$\|W^*\hat{S}^{1/2} - S^{1/2}W^*\|_F = O(m^{-1/2}n^{-1/2}\log n)$$

and

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(m^{-1/2}n^{-3/2}\log n)$$

with high probability.

Require: Theorem 3.9, Lemma 3.10, Lemma 3.11, Exponential distribution
Proof: By Proposition 2.1 in [7] and Equation (2), we have for some orthogonal matrix W,

$$\begin{aligned}\|\hat{U} - UW\|_F^2 \le & \frac{2\|\hat{U}\hat{U}^T - UU^T\|_F^2}{\delta^2} \le \frac{8k^2\|\hat{U}\hat{U}^T - UU^T\|_2^2}{\delta^2} \\ =& O(m^{-1/2}n^{-1/2}\sqrt{\log n}),\end{aligned}$$

with high probability. Let $Q = \hat{U} - UU^T\hat{U}$. And Q is the residual after projecting $\hat{U}$ orthogonally onto the column space of U, we have

$$\|Q\|_F = \|\hat{U} - UU^T\hat{U}\|_F \le \|\hat{U} - UT\|_F = O(m^{-1/2}n^{-1/2}\sqrt{\log n}). \tag{3}$$

for all $k \times k$ matrices T with high probability. Then

$$\begin{aligned}W^*\hat{S} =& (W^* - U^T\hat{U})\hat{S} + U^T\hat{U}\hat{S} = (W^* - U^T\hat{U})\hat{S} + U^T\hat{P}\hat{U} \\ =& (W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)\hat{U} + U^TH\hat{U} \\ =& (W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + U^TH\hat{U} \\ =& (W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + SU^T\hat{U}.\end{aligned}$$

Combined with Theorem 3.9, Lemma 3.10, Lemma 3.11, we have

$$\begin{aligned}&\|W^*\hat{S} - SW^*\|_F \\ =& \|(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + S(U^T\hat{U} - W^*)\|_F \\ \le& \|W^* - U^T\hat{U}\|_F(\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F\|\hat{P} - H\|_2\|Q\|_F + \|U^T(\hat{P} - H)U\|_F \\ \le& O(m^{-1}\log n) + O(m^{-1/2}\log n) + \|U^T(\hat{P} - H)U\|_F\end{aligned}$$

with high probability. And we know $U^T(\hat{P} - H)U$ is a $k \times k$ matrix with ij-th entry to be

$$u_i^T(\hat{P} - H)u_j = \sum_{s=1}^n\sum_{t=1}^n(\hat{P}_{st} - H_{st})u_{is}u_{jt} = 2\sum_{s<t}(\hat{P}_{st} - H_{st})u_{is}u_{jt}$$

where u_i and u_j are the i-th and j-th columns of U. Thus, conditioned on H, U is fixed and $u_i^T(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables.

By Equation (1), we have

$$\begin{aligned}&E\left[\left((A_{st}^{(t')} - H_{st})u_{is}u_{jt}\right)^k\right] \\ \le& k!R^k u_{is}^k u_{jt}^k \\ \le& \frac{k!}{2}R^{k-2}(\sqrt{2}u_{is}u_{jt}R)^2.\end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{t',s<t} 2R^2 u_{is}^2 u_{jt}^2 \right| \le mR^2,$$

then by Theorem 3.8, we have

$$P\left(\left| 2\sum_{s<t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right| \ge t \right) \le \exp\left(\frac{-mt^2/8}{R^2 + Rt/2} \right).$$

Let $t = cRm^{-1/2} \log n$ for any $c > 0$, we have

$$P\left(\left| 2\sum_{s<t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right| \ge Cm^{-1/2} \log n \right) \le n^{-c}.$$

Thus each entry of $U^T(\hat{P} - H)U$ is of order $O(m^{-1} \log n)$ with high probability and

$$\|U^T(\hat{P} - H)U\|_F = O(m^{-1} \log n) \tag{4}$$

with high probability. Hence

$$\|W^*\hat{S} - SW^*\|_F = O(m^{-1/2} \log n)$$

with high probability. Also, since

$$W^*_{ij}(\lambda_j^{1/2}(\hat{P}) - \lambda_i^{1/2}(H)) = W^*_{ij} \frac{\lambda_j(\hat{P}) - \lambda_i(H)}{\lambda_j^{1/2}(\hat{P}) + \lambda_i^{1/2}(H)}$$

and the eigenvalues $\lambda_j^{1/2}(\hat{P})$ and $\lambda_i^{1/2}(H)$ are both of order $\Theta(\sqrt{n})$, we have

$$\|W^*\hat{S}^{1/2} - S^{1/2}W^*\|_F = O(m^{-1/2} n^{-1/2} \log n).$$

Similarly, since

$$W^*_{ij}(\lambda_j^{-1/2}(\hat{P}) - \lambda_i^{-1/2}(H)) = W^*_{ij} \frac{\lambda_i(H) - \lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P}) + \lambda_i^{-1/2}(H))\lambda_j(\hat{P})\lambda_i(H)}$$

and the eigenvalues $\lambda_j(\hat{P})$ and $\lambda_i(H)$ are both of order $\Theta(n)$, we have

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(m^{-1/2} n^{-3/2} \log n).$$

■

Lemma 3.13 *There exists a rotation matrix W such that for sufficiently large n,*

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2} n^{-1/2} (\log n)^{3/2})$$

with high probability.

Require: Lemma 3.11, Lemma 3.12

Proof: Let $Q_1 = UU^T\hat{U} - UW^*$, $Q_2 = W^*\hat{S}^{1/2} - S^{1/2}W^*$ and $Q_3 = \hat{U} - UW^* = \hat{U} - UU^T\hat{U} + Q_1 = Q + Q_1$. Then since $UU^TP = P$ and $\hat{U}\hat{S}^{1/2} = \hat{P}\hat{U}\hat{S}^{-1/2}$,

$$\begin{aligned}\hat{Z} - US^{1/2}W^* =&\hat{U}\hat{S}^{1/2} - UW^*\hat{S}^{1/2} + U(W^*\hat{S}^{1/2} - S^{1/2}W^*)\\ =&(\hat{U} - UU^T\hat{U})\hat{S}^{1/2} + Q_1\hat{S}^{1/2} + UQ_2\\ =&(\hat{P} - H)\hat{U}\hat{S}^{-1/2} - UU^T(\hat{P} - H)\hat{U}\hat{S}^{-1/2} + Q_1\hat{S}^{1/2} + UQ_2\\ =&(\hat{P} - H)UW^*\hat{S}^{-1/2} - UU^T(\hat{P} - H)UW^*\hat{S}^{-1/2}\\ &+ (I - UU^T)(\hat{P} - H)Q_3\hat{S}^{-1/2} + Q_1\hat{S}^{1/2} + UQ_2.\end{aligned}$$

By Lemma 3.11, with high probability,

$$\|Q_1\|_F \leq \|U\|_F\|U^T\hat{U} - W^*\|_F = O(m^{-1}n^{-1}\log n).$$

By Lemma 3.12, with high probability,

$$\|Q_2\|_F = O(m^{-1/2}n^{-1/2}\log n).$$

By Equation (3), with high probability,

$$\|Q_3\|_F \leq \|Q\|_F + \|Q_1\|_F = O(m^{-1/2}n^{-1/2}(\log n)^{1/2}).$$

By Equation (4), with high probability,

$$\|UU^T(\hat{P}-H)UW^*\hat{S}^{-1/2}\|_F \leq \|U^T(\hat{P}-H)U\|_F\|\hat{S}^{-1/2}\|_2 = O(m^{-1}n^{-1/2}\log n).$$

By Lemma 3.12, with high probability,

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(m^{-1/2}n^{-3/2}\log n).$$

Therefore, with high probability,

$$\begin{aligned}&\|\hat{Z} - US^{1/2}W^*\|_F\\ =&\|(\hat{P} - H)UW^*\hat{S}^{-1/2}\|_F + O(m^{-1}n^{-1/2}\log n) + \|I - UU^T\|_2\|\hat{P} - H\|_2 O(m^{-1/2}n^{-1}(\log n)^{1/2})\\ &+ O(m^{-1}n^{-1/2}\log n) + O(m^{-1/2}n^{-1/2}\log n)\\ =&\|(\hat{P} - H)UW^*\hat{S}^{-1/2}\|_F + O(m^{-1/2}n^{-1/2}\log n)\\ \leq&\|(\hat{P} - H)US^{-1/2}W^*\|_F + \|(\hat{P} - H)U(W^*\hat{S}^{-1/2} - S^{-1/2}W^*)\|_F + O(m^{-1/2}n^{-1/2}\log n)\\ =&\|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1}n^{-1}(\log n)^{3/2}) + O(m^{-1/2}n^{-1/2}(\log n)^{3/2})\\ =&\|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}).\end{aligned}$$

Note that $Z = US^{1/2}W$ for some orthogonal matrix W. As W^* is also orthogonal, therefore $Z\tilde{W} = US^{1/2}W^*$ for some orthogonal $\tilde{W}$, which completes the proof. ■

Theorem 3.14 *There exists a rotation matrix W such that for sufficiently large n,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(m^{-1/2}n^{-1/2}(\log n)^{3/2})$$

with high probability.

Require: Lemma 3.13, Exponential distribution
Proof: By Lemma 3.13, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2}n^{-1/2}(\log n)^{3/2})$$

with high probability and similarly we could have the bound for each column vector with high probability that

$$\begin{aligned}\max_i \|\hat{Z}_i - WZ_i\|_2 \leq& \frac{1}{\lambda_k^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \\ \leq& \frac{k^{1/2}}{\lambda_k^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(m^{-1/2}n^{-1/2}(\log n)^{3/2})\end{aligned}$$

where $((\hat{P} - H)U)_i$ represents the i-th row of $(\hat{P} - H)U$ and u_j denotes the j-th column of U. Now given i and j, the i-th element of the vector $(\hat{P} - H)u_j$ is of the form

$$\sum_{s=1}^n (\hat{P}_{is} - H_{is})u_{js} = \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}.$$

Thus, conditioned on H, the i-th element of the vector $(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables. By Equation (1), we have

$$\begin{aligned}&E\left[\left((A_{is}^{(t)} - H_{is})u_{js}\right)^k\right] \\ \leq& k! R^k u_{js}^k \\ \leq& \frac{k!}{2} R^{k-2} (\sqrt{2}Ru_{js})^2.\end{aligned}$$

Also we have

$$\sigma^2 := |\sum_{t, s \neq i} 2R^2 u_{js}^2| \leq 2R^2 m,$$

then by Theorem 3.8, we have

$$P\left(\left|\sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}\right| \geq t\right) \leq \exp\left(\frac{-mt^2/2}{2R^2 + Rt}\right).$$

Let $t = 3cRm^{-1/2}\log n$, we have

$$P\left(\left|\sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}\right| \geq 3cRm^{-1/2}\log n\right) \leq n^{-c},$$

i.e. it is of order $O(m^{-1/2}\log n)$ with high probability. Taking the union bound over all i and j, with high probability we have,

$$\begin{aligned}\max_i \|\hat{Z}_i - WZ_i\|_2 \leq& \frac{Ck^{1/2}}{\lambda_k^{1/2}(H)} m^{-1/2}(\log n)^{3/2} + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \\ =& O(m^{-1/2}n^{-1/2}(\log n)^{3/2}).\end{aligned}$$

■

Lemma 3.15 $\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| = O(m^{-1/2}n^{-1}(\log n)^3)$ *with high probability.*

Require: Theorem 3.14
Proof: Let W be the rotation matrix in Theorem 3.14, then

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| =& \left|\hat{Z}_i^T\hat{Z}_j - \hat{Z}_i^T W Z_j + \hat{Z}_i^T W Z_j - (WZ_i)^T W Z_j\right| \\ \leq& \left|\hat{Z}_i^T(\hat{Z}_j - WZ_j) + (\hat{Z}_i^T - (WZ_i)^T)WZ_j\right| \\ \leq& \|\hat{Z}_i\|_2\|\hat{Z}_j - WZ_j\|_2 + \|Z_j\|_2\|\hat{Z}_i^T - (WZ_i)^T\|_2.\end{aligned}$$

Since $\|Z_i\|_2^2 = Z_i^T Z_i = H_{ii}^{(1)} = E[\hat{P}_{ii}^{(1)}] = (1-\epsilon)P_{ij} + \epsilon C_{ij} \leq R$, we have $\|Z_i\|_2 = O(1)$. Combined with Theorem 3.14,

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| =& (\|\hat{Z}_i\|_2 + \|Z_j\|_2)O(n^{-1/2}(\log n)^{3/2}) \\ \leq& (\|\hat{Z}_i - WZ_i\|_2 + \|WZ_i\|_2 + \|Z_j\|_2)O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \\ =& O(m^{-1/2}n^{-1}(\log n)^3)\end{aligned}$$

with high probability. ■

Definition 3.16 *Define* $\widetilde{P}_{ij}^{(1)} = (\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}$*, our estimator for* P_{ij}*, to be a projection of* $\hat{Z}_i^T\hat{Z}_j$ *onto* $[0, \min(\hat{P}_{ij}^{(1)}, R)]$.

Corollary 3.17 *For fixed* m*, the estimator based on ASE of MLE has the same entry-wise asymptotic bias as MLE, i.e.*

$$\lim_{n\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(1)}) = \lim_{n\to\infty} E[\widetilde{P}_{ij}^{(1)}] - P_{ij} = \lim_{n\to\infty} E[\hat{P}_{ij}^{(1)}] - P_{ij} = \lim_{n\to\infty} \mathrm{Bias}(\hat{P}_{ij}^{(1)}).$$

Proof: From Lemma 3.15, we have

$$\lim_{n\to\infty} E[\hat{Z}_i^T\hat{Z}_j] = \lim_{n\to\infty} E[Z_i^T Z_j] = \lim_{n\to\infty} E[\hat{P}_{ij}^{(1)}].$$

And

$$\lim_{n\to\infty} E[\widetilde{P}_{ij}^{(1)}] = \lim_{n\to\infty} E[\hat{P}_{ij}^{(1)}].$$

■

Theorem 3.18 *Assuming that* $m = O(n^b)$ *for any* $b > 0$*, then* $\mathrm{Var}((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}) = O(m^{-1}n^{-2}(\log n)^6)$.

Proof: By Lemma 3.15,

$$\begin{aligned}\mathrm{Var}((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}) =& E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ =& E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j + Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ =& E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ &+ 2E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])] \\ \leq& E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ &+ 2\sqrt{E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2]E[(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2]} \\ \leq& 4E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2]\end{aligned}$$

Fix some $a > 0$, we have

$$
\begin{aligned}
&E[((\hat{Z}_i^T\hat{Z}_j)_{\text{tr}} - Z_i^T Z_j)^2] \\
=&E[((\hat{Z}_i^T\hat{Z}_j)_{\text{tr}} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij} \le a\}] + E[((\hat{Z}_i^T\hat{Z}_j)_{\text{tr}} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij} > a\}]
\end{aligned}
$$

For the first term, we have

$$
\begin{aligned}
&E[((\hat{Z}_i^T\hat{Z}_j)_{\text{tr}} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij} \le a\}] \\
\le&E[((\hat{Z}_i^T\hat{Z}_j)_{\text{tr}} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij} \le a\}\mathbb{I}\{\text{Lemma4.11holds}\}](1-n^{-c}) \\
&+ E[((\hat{Z}_i^T\hat{Z}_j)_{\text{tr}} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij} \le a\}\mathbb{I}\{\text{Lemma4.11doesnothold}\}]n^{-c} \\
\le&O(m^{-1}n^{-2}(\log n)^6)(1-n^{-c}) + 2n^{-c}E[((\hat{Z}_i^T\hat{Z}_j)_{\text{tr}} - \hat{P}_{ij})^2\mathbb{I}\{\hat{P}_{ij} \le a\}] \\
&+ 2n^{-c}E[(\hat{P}_{ij} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij} \le a\}] \\
\le&O(m^{-1}n^{-2}(\log n)^6) + 2n^{-c}E[\hat{P}_{ij}^2\mathbb{I}\{\hat{P}_{ij} \le a\}] + 2n^{-c}E[(\hat{P}_{ij} + R)^2\mathbb{I}\{\hat{P}_{ij} \le a\}] \\
\le&O(m^{-1}n^{-2}(\log n)^6) + 2a^2n^{-c} + 2(a+R)^2n^{-c} \\
\le&O(m^{-1}n^{-2}(\log n)^6) + 4n^{-c}(a+R)^2
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&E[\hat{P}_{ij}^2\mathbb{I}\{\hat{P}_{ij} > a\}] = E[(\frac{1}{m}\sum_{1\le t\le m} A_{ij}^{(t)})^2\mathbb{I}\{\hat{P}_{ij} > a\}] \\
\le&\frac{1}{m}E[\sum_{1\le t\le m} A_{ij}^{(t)2}\mathbb{I}\{\hat{P}_{ij} > a\}] \le \frac{1}{m}E[\sum_{1\le t\le m} A_{ij}^{(t)2}\mathbb{I}\{\max_{1\le s\le m} A_{ij}^{(s)} > a\}] \\
\le&\frac{1}{m}E[\sum_{1\le t\le m} A_{ij}^{(t)2}(\sum_{1\le s\le m}\mathbb{I}\{A_{ij}^{(s)} > a\})] = E[A_{ij}^{(1)2}(\sum_{1\le s\le m}\mathbb{I}\{A_{ij}^{(s)} > a\})] \\
=&E[A_{ij}^{(1)2}\mathbb{I}\{A_{ij}^{(1)} > a\})] + (m-1)E[A_{ij}^{(1)2}\mathbb{I}\{A_{ij}^{(2)} > a\})] \\
=&E[A_{ij}^{(1)2}\mathbb{I}\{A_{ij}^{(1)} > a\})] + (m-1)E[A_{ij}^{(1)2}]P(A_{ij}^{(1)} > a),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&E[(\hat{P}_{ij} + R)^2\mathbb{I}\{\hat{P}_{ij} > a\}] \\
=&E[\hat{P}_{ij}^2\mathbb{I}\{\hat{P}_{ij} > a\}] + 2R\cdot E[\hat{P}_{ij}\mathbb{I}\{\hat{P}_{ij} > a\}] + R^2P(\hat{P}_{ij} > a) \\
\le&E[A_{ij}^{(1)2}\mathbb{I}\{A_{ij}^{(1)} > a\})] + (m-1)E[A_{ij}^{(1)2}]P(A_{ij}^{(1)} > a) \\
&+ 2R\left(E[A_{ij}^{(1)}\mathbb{I}\{A_{ij}^{(1)} > a\})] + (m-1)E[A_{ij}^{(1)}]P(A_{ij}^{(1)} > a)\right) \\
&+ R^2\cdot m\cdot P(A_{ij}^{(1)} > a).
\end{aligned}
$$

Thus for the second term,

$$
\begin{aligned}
&E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^TZ_j)^2\mathbb{I}\{\hat{P}_{ij} > a\}] \\
\leq &2E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - \hat{P}_{ij})^2\mathbb{I}\{\hat{P}_{ij} > a\}] + 2E[(\hat{P}_{ij} - Z_i^TZ_j)^2\mathbb{I}\{\hat{P}_{ij} > a\}] \\
\leq &2E[\hat{P}_{ij}^2\mathbb{I}\{\hat{P}_{ij} > a\}] + 2E[(\hat{P}_{ij} + R)^2\mathbb{I}\{\hat{P}_{ij} > a\}] \\
\leq &4E[A_{ij}^{(1)2}\mathbb{I}\{A_{ij}^{(1)} > a\})] + 4(m-1)E[A_{ij}^{(1)2}]P(A_{ij}^{(1)} > a) \\
&+ 4R \cdot E[A_{ij}^{(1)}\mathbb{I}\{A_{ij}^{(1)} > a\})] + 2R(m-1)E[A_{ij}^{(1)}]P(A_{ij}^{(1)} > a) \\
&+ 2R^2 \cdot m \cdot P(A_{ij}^{(1)} > a) \\
\leq &4e^{-a/R}\left(a^2 + 3Ra + 3(m+1)R^2\right) \\
\leq &4e^{-a/R}(a + 2m^{1/2}R)^2
\end{aligned}
$$

Thus,

$$\mathrm{Var}((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}) \leq O(m^{-1}n^{-2}(\log n)^6) + 16(a+R)^2n^{-c} + 16(a+2m^{1/2}R)^2e^{-a/R}.$$

Let $a = m^{-1/2}n^b$ for any $b > 0$, and $c = 2b+3$, combined with the assumption $m = O(n^b)$, we have

$$
\begin{aligned}
\mathrm{Var}((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}) =&O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) + O(m^{-1}n^{2b}) \cdot O(e^{-m^{-1/2}n^b}) \\
=&O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) + O(m^{-1}n^{2b}) \cdot O(e^{-n^{b/2}}) \\
=&O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) + O(m^{-1}n^{2b}) \cdot O(n^{-2b-3}) \\
=&O(m^{-1}n^{-2}(\log n)^6) + O(m^{-1}n^{-3}) \\
=&O(m^{-1}n^{-2}(\log n)^6).
\end{aligned}
$$

■

Corollary 3.19 *For fixed n, $1 \leq i, j \leq n$, $\mathrm{Var}(\hat{P}_{ij}^{(1)}) = \Theta(m^{-1})$.*

Proof: Direct result from central limit theorem. ■

Theorem 3.20 *For fixed m, $1 \leq i, j \leq n$ and $i \neq j$,*

$$\frac{\mathrm{Var}(\widetilde{P}_{ij}^{(1)})}{\mathrm{Var}(\hat{P}_{ij}^{(1)})} = O(n^{-2}(\log n)^6).$$

Thus

$$\mathrm{ARE}(\hat{P}_{ij}^{(1)}, \widetilde{P}_{ij}^{(1)}) = 0.$$

Furthermore, as long as m goes to infinity of order $O(n^b)$ for any $b > 0$,

$$\mathrm{ARE}(\hat{P}_{ij}^{(1)}, \widetilde{P}_{ij}^{(1)}) = 0.$$

Proof: The results are direct from Theorem 3.18 and Corollary 3.19. ■

3.3 $\widetilde{P}^{(q)}$ better than $\hat{P}^{(q)}$

Theorem 3.21 *Let P and C be two n-by-n symmetric and hollow matrices satisfying element-wise conditions $0 < P_{ij} \leq C_{ij} \leq R$ for some constant $R > 0$. For $0 < \epsilon < 1$, we define m symmetric and hollow matrices as*

$$A^{(t)} \stackrel{iid}{\sim} (1-\epsilon)\mathrm{Exp}(P) + \epsilon \mathrm{Exp}(C)$$

for $1 \leq t \leq m$. Let $\hat{P}^{(q)}$ be the entry-wise MLqE based on exponential distribution with m observations. Define $H^{(q)} = E[\hat{P}^{(q)}]$, then for any constant $c > 0$ there exists another constant $n_0(c)$, independent of n, P, C and ϵ, such that if $n > n_0$, then for all η satisfying $n^{-c} \leq \eta \leq 1/2$,

$$P\left(\|\hat{P}^{(q)} - H^{(q)}\|_2 \leq 8R\sqrt{2n\ln(n/\eta)}\right) \geq 1 - \eta.$$

Remark: This is the extended version of Theorem 3.1 in [5].
Proof: Let $\{e_i\}_{i=1}^n$ be the canonical basis for $\mathbb{R}^n$. For each $1 \leq i, j \leq n$, define a corresponding matrix G_{ij}:

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus $\hat{P}^{(q)} = \sum_{1 \leq i < j \leq n} \hat{P}_{ij}^{(q)} G_{ij}$ and $H^{(q)} = \sum_{1 \leq i < j \leq n} H_{ij}^{(q)} G_{ij}$. Then we have $\hat{P}^{(q)} - H^{(q)} = \sum_{1 \leq i < j \leq n} X_{ij}$, where $X_{ij} \equiv \left(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}\right) G_{ij}$, $1 \leq i < j \leq n$.

First consider the k-th moment of X_{ij} for $1 \leq i < j \leq n$. By Lemma 3.1 we have

$$\begin{aligned}
\left|\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}\right| &= \left|\hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)} + \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} + H_{ij}^{(1)} - H_{ij}^{(q)}\right| \\
&\leq \left|\hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)}\right| + \left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + \left|H_{ij}^{(1)} - H_{ij}^{(q)}\right| \\
&\leq \hat{P}_{ij}^{(1)} + \left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + H_{ij}^{(1)} \\
&\leq 2\left(\left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + H_{ij}^{(1)}\right).
\end{aligned}$$

Since

$$\begin{aligned}
E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] \leq &(1-\epsilon)\exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1+k, -H_{ij}/P_{ij}) \\
&+ \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1+k, -H_{ij}/C_{ij}) \\
\leq &\left((1-\epsilon)\exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k\right) k! \\
\leq &\left((1-\epsilon) P_{ij}^k + \epsilon C_{ij}^k\right) k! \\
\leq &C_{ij}^k k!,
\end{aligned}$$

Then

$$\begin{aligned}
E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] \leq & E\left[\left|\hat{P}_{ij}^{(q)} - H_{ij}^{(q)}\right|^k\right] \\
\leq & 2^k E\left[\left(\left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right| + H_{ij}^{(1)}\right)^k\right] \\
\leq & 2^k \sum_{s=0}^{k} \binom{k}{s} E\left[\left|\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}\right|^s\right] \left(H_{ij}^{(1)}\right)^{k-s} \\
\leq & 2^k \sum_{s=0}^{k} \binom{k}{s} C_{ij}^s s! \left(H_{ij}^{(1)}\right)^{k-s} \\
\leq & 2^k k! \sum_{s=0}^{k} \binom{k}{s} C_{ij}^s \left(H_{ij}^{(1)}\right)^{k-s} \\
= & 2^k k! \left(C_{ij} + H_{ij}^{(1)}\right)^k . \qquad (5)
\end{aligned}$$

Combined with for $i \neq j$,

$$G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & \text{k is even;} \\ e_i e_j^T + e_j e_i^T, & \text{k is odd,} \end{cases}$$

thus we have

1. When k is even,
$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij}^2 \preceq 2^{2k} k! R^k G_{ij}^2;$$
2. When k is odd,
$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij} \preceq 2^{2k} k! R^k G_{ij}^2.$$

So
$$E[X_{ij}^k] \preceq 2^{2k} k! R^k G_{ij}^2.$$
Let

$$\sigma^2 := \left\| \sum_{1 \leq i < j \leq n} (4\sqrt{2} R G_{ij})^2 \right\| = 32R^2 \|(n-1)I\| = 32R^2(n-1),$$

notice that random matrices X_{ij} are independent, self-adjoint and have mean zero, apply Theorem 3.8 we have

$$\begin{aligned}
P\left(\lambda_{\max}(\hat{P}^{(q)} - H^{(q)}) \geq t\right) &\leq n \exp\left(-\frac{t^2/2}{\sigma^2 + 4Rt}\right) \\
&\leq n \exp\left(-\frac{t^2/2}{32R^2 n + Rt}\right).
\end{aligned}$$

Now consider $Y_{ij} \equiv \left(H^{(q)} - \hat{P}^{(q)}\right) G_{ij}$, $1 \leq i < j \leq n$. Then we have $H^{(q)} - \hat{P}^{(q)} = \sum_{1 \leq i < j \leq n} Y_{ij}$. Since

$$E[(H^{(q)} - \hat{P}^{(q)})^k] = (-1)^k E[(\hat{P}^{(q)} - H^{(q)})^k],$$

1. When k is even,

$$E[Y_{ij}^k] = E[(\hat{P}^{(q)} - H^{(q)})^k]G_{ij}^2 \preceq 2^{2k}k!R^kG_{ij}^2;$$

2. When k is odd,

$$E[Y_{ij}^k] = -E[(\hat{P}^{(q)} - H^{(q)})^k]G_{ij} \preceq 2^{2k}k!R^kG_{ij}^2.$$

Thus

$$\begin{aligned} P\left(\lambda_{\min}(\hat{P}^{(q)} - H^{(q)}) \le -t\right) &= P\left(\lambda_{\max}(H^{(q)} - \hat{P}^{(q)}) \ge t\right) \\ &\le n\exp\left(-\frac{t^2/2}{32R^2n + Rt}\right). \end{aligned}$$

Therefore we have

$$P\left(\|\hat{P}^{(q)} - H^{(q)}\| \ge t\right) \le n\exp\left(-\frac{t^2/2}{32R^2n + Rt}\right).$$

Now let $c > 0$ be given and assume $n^{-c} \le \eta \le 1/2$. Then there exists a $n_0(c)$ independent of n, P, C and ϵ such that whenever $n > n_0(c)$,

$$t = 8R\sqrt{2n\ln(n/\eta)} \le 32Rn.$$

Plugging this t into the equation above, we get

$$P(\|\hat{P}^{(q)} - H^{(q)}\| \ge 8R\sqrt{2n\ln(n/\eta)}) \le n\exp\left(-\frac{t^2}{64R^2n}\right) = \eta.$$

■

As we define $H^{(q)} = E[\hat{P}^{(q)}]$, let $d^{(q)} = \operatorname{rank}(H^{(q)})$ be the dimension in which we are going to embed $\hat{P}^{(q)}$. Notice that it is less than or equal to K since the SBM assumption. Then we can define $H^{(q)} = ZZ^T$ where $Z \in \mathbb{R}^{n\times d^{(q)}}$.

For simplicity, from now on, we will use $\hat{P}$ to represent $\hat{P}^{(q)}$, use H to represent $H^{(q)}$ and use k to represent the dimension $d^{(q)}$ we are going to embed. Assume $H = USU^T = ZZ^T$, where Z is a n-by-k matrix. Then our estimate for Z up to rotation is $\hat{Z} = \hat{U}\hat{S}^{1/2}$, where $\hat{U}\hat{S}\hat{U}^T$ is the rank-d spectral decomposition of $|\hat{P}| = (\hat{P}^T\hat{P})^{1/2}$.

Lemma 3.22 *Under the above assumptions, $\lambda_i(H) = \Theta(n)$ with high probability when $i \le k$, i.e. the largest k eigenvalues of H is of order n. Moreover, we have $\|S\|_2 = \Theta(n)$ and $\|\hat{S}\|_2 = \Theta(n)$ with high probability.*

Remark: This is a extended version of Proposition 4.3 in [8].
Proof: Exactly the same as proof for Lemma 3.10. ■

Lemma 3.23 *Let $W_1\Sigma W_2^T$ be the singular value decomposition of $U^T\hat{U}$. Then for sufficiently large n,*

$$\|U^T\hat{U} - W_1W_2^T\|_F = O(n^{-1}\log n)$$

with high probability.

Proof: Exactly the same as proof for Lemma 3.11. ∎

We will denote the orthogonal matrix $W_1 W_2^T$ by W^*.

Lemma 3.24 *For sufficiently large n,*

$$\|W^*\hat{S} - SW^*\|_F = O(\log n),$$

$$\|W^*\hat{S}^{1/2} - S^{1/2}W^*\|_F = O(n^{-1/2}\log n)$$

and

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(n^{-3/2}\log n)$$

with high probability.

Proof:

By Proposition 2.1 in [7] and Equation (2), we have for some orthogonal matrix W,

$$\|\hat{U} - UW\|_F^2 \leq \frac{2\|\hat{U}\hat{U}^T - UU^T\|_F^2}{\delta^2} \leq \frac{8k^2\|\hat{U}\hat{U}^T - UU^T\|_2^2}{\delta^2} = O(n^{-1/2}\sqrt{\log n}).$$

Let $Q = \hat{U} - UU^T\hat{U}$. And Q is the residual after projecting $\hat{U}$ orthogonally onto the column space of U, we have

$$\|Q\|_F = \|\hat{U} - UU^T\hat{U}\|_F \leq \|\hat{U} - UT\|_F = O(n^{-1/2}\sqrt{\log n}). \tag{6}$$

for all $k \times k$ matrices T. Then

$$\begin{aligned}
W^*\hat{S} =&(W^* - U^T\hat{U})\hat{S} + U^T\hat{U}\hat{S} = (W^* - U^T\hat{U})\hat{S} + U^T\hat{P}\hat{U}\\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)\hat{U} + U^T H\hat{U}\\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + U^T H\hat{U}\\
=&(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + SU^T\hat{U}.
\end{aligned}$$

Combined with Theorem 3.21, Lemma 3.22, Lemma 3.23, we have

$$\begin{aligned}
&\|W^*\hat{S} - SW^*\|_F\\
=&\|(W^* - U^T\hat{U})\hat{S} + U^T(\hat{P} - H)Q + U^T(\hat{P} - H)UU^T\hat{U} + S(U^T\hat{U} - W^*)\|_F\\
\leq&\|W^* - U^T\hat{U}\|_F(\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F\|\hat{P} - H\|_2\|Q\|_F + \|U^T(\hat{P} - H)U\|_F\\
\leq&O(\log n) + O(\log n) + \|U^T(\hat{P} - H)U\|_F
\end{aligned}$$

with high probability. And we know $U^T(\hat{P} - H)U$ is a $k \times k$ matrix with ij-th entry to be

$$u_i^T(\hat{P} - H)u_j = \sum_{s=1}^{n}\sum_{t=1}^{n}(\hat{P}_{st} - H_{st})u_{is}u_{jt} = 2\sum_{s<t}(\hat{P}_{st} - H_{st})u_{is}u_{jt}$$

where u_i and u_j are the i-th and j-th columns of U. Thus, conditioned on H, $u_i^T(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables.

By Equation (5), we have

$$
\begin{aligned}
&E\left[\left((\hat{P}_{st}-H_{st})u_{is}u_{jt}\right)^k\right]\\
\le&2^k k!(C_{st}+H_{st}^{(1)})^k u_{is}^k u_{jt}^k\\
\le&\frac{k!}{2}(4R)^{k-2}(4\sqrt{2}Ru_{is}u_{jt})^2.
\end{aligned}
$$

Also we have

$$\sigma^2 := |\sum_{s<t} 32R^2 u_{is}^2 u_{jt}^2| \le 32R^2,$$

then by Theorem 3.8, we have

$$P\left(\left|2\sum_{s<t}(\hat{P}_{st}-H_{st})u_{is}u_{jt}\right| \ge t\right) \le \exp\left(\frac{-t^2/8}{32R^2+2Rt}\right),$$

thus each entry of $U^T(\hat{P}-H)U$ is of order $O(\log n)$ with high probability and thus

$$\|U^T(\hat{P}-H)U\|_F = O(\log n) \tag{7}$$

with high probability. Hence

$$\|W^*\hat{S} - SW^*\|_F = O(\log n)$$

with high probability. Also, since

$$W_{ij}^*(\lambda_j^{1/2}(\hat{P}) - \lambda_i^{1/2}(H)) = W_{ij}^* \frac{\lambda_j(\hat{P}) - \lambda_i(H)}{\lambda_j^{1/2}(\hat{P}) + \lambda_i^{1/2}(H)}$$

and the eigenvalues $\lambda_j^{1/2}(\hat{P})$ and $\lambda_i^{1/2}(H)$ are both of order $\Theta(\sqrt{n})$, we have

$$\|W^*\hat{S}^{1/2} - S^{1/2}W^*\|_F = O(n^{-1/2}\log n).$$

Similarly, since

$$W_{ij}^*(\lambda_j^{-1/2}(\hat{P}) - \lambda_i^{-1/2}(H)) = W_{ij}^* \frac{\lambda_i(H) - \lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P}) + \lambda_i^{-1/2}(H))\lambda_j(\hat{P})\lambda_i(H)}$$

and the eigenvalues $\lambda_j(\hat{P})$ and $\lambda_i(H)$ are both of order $\Theta(n)$, we have

$$\|W^*\hat{S}^{-1/2} - S^{-1/2}W^*\|_F = O(n^{-3/2}\log n).$$

■

Lemma 3.25 *There exists a rotation matrix W such that for sufficiently large n,*

$$\|\hat{Z} - ZW\|_F = \|(\hat{P}-H)US^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: Exactly the same as proof for Lemma 3.13. ■

Theorem 3.26 *There exists a rotation matrix W such that for sufficiently large n,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: By Lemma 3.25, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2})$$

and similarly we could have the bound for each row vector

$$\begin{aligned}\max_i \|\hat{Z}_i - WZ_i\|_2 \leq & \frac{1}{\lambda_k^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(n^{-1/2}(\log n)^{3/2}) \\ \leq & \frac{k^{1/2}}{\lambda_k^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(n^{-1/2}(\log n)^{3/2})\end{aligned}$$

where u_j denotes the j-th column of U. Now given i and j, the i-th element of the vector $(\hat{P} - H)u_j$ is of the form

$$\sum_{s=1}^{n} (\hat{P}_{is} - H_{is})u_{js} = \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}.$$

Thus, conditioned on H, the i-th element of the vector $(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables. By Equation (5), we have

$$\begin{aligned} & E\left[\left((\hat{P}_{is} - H_{is})u_{js}\right)^k\right] \\ \leq & 2^k k!(C_{is} + H_{is}^{(1)})^k u_{js}^k \\ \leq & \frac{k!}{2}(4R)^{k-2}(4\sqrt{2}Ru_{js})^2.\end{aligned}$$

Also we have

$$\sigma^2 := |\sum_{s \neq i} 32R^2 u_{js}^2| \leq 32R^2,$$

then by Theorem 3.8, we have with high probability,

$$P\left(\left|\sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}\right| \geq t\right) \leq \exp\left(\frac{-t^2/2}{32R^2 + Rt}\right).$$

Let $t = 2cR\log n$, we have

$$P\left(\left|\sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js}\right| \geq 2cR\log n\right) \leq n^{-c},$$

i.e. it can be bounded by $O(\log n)$ with high probability. Taking the union bound over all i and j, with high probability, we have

$$\max_i \|\hat{Z}_i - WZ_i\|_2 \leq \frac{Cd^{1/2}}{\lambda_d^{1/2}(H)}(\log n)^{3/2} + O(n^{-1/2}(\log n)^{3/2}) = O(n^{-1/2}(\log n)^{3/2}).$$

■

Lemma 3.27 $\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| = O(n^{-1}(\log n)^3)$ *with high probability.*

Proof: Let W be the rotation matrix in Theorem 3.26, then

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| &= \left|\hat{Z}_i^T\hat{Z}_j - \hat{Z}_i^T W Z_j + \hat{Z}_i^T W Z_j - (WZ_i)^T W Z_j\right| \\ &\le \left|\hat{Z}_i^T(\hat{Z}_j - WZ_j) + (\hat{Z}_i^T - (WZ_i)^T)WZ_j\right| \\ &\le \|\hat{Z}_i\|_2\|\hat{Z}_j - WZ_j\|_2 + \|Z_j\|_2\|\hat{Z}_i^T - (WZ_i)^T\|_2.\end{aligned}$$

Since $\|Z_i\|_2^2 = Z_i^T Z_i = H_{ii}^q = E[\hat{P}_{ii}^{(q)}] \le E[\hat{P}_{ii}^{(1)}] = (1-\epsilon)P_{ij} + \epsilon C_{ij} \le R$, we have $\|Z_i\|_2 = O(1)$. Combined with Theorem 3.26,

$$\begin{aligned}\left|\hat{Z}_i^T\hat{Z}_j - Z_i^T Z_j\right| &= (\|\hat{Z}_i\|_2 + \|Z_j\|_2)O(n^{-1/2}(\log n)^{3/2}) \\ &\le (\|\hat{Z}_i - WZ_i\|_2 + \|WZ_i\|_2 + \|Z_j\|_2)O(n^{-1/2}(\log n)^{3/2}) \\ &= O(n^{-1}(\log n)^3)\end{aligned}$$

with high probability. ■

Corollary 3.28 *For fixed m, the estimator based on ASE of MLqE has the same entry-wise asymptotic bias as MLqE, i.e.*

$$\lim_{n\to\infty}\mathrm{Bias}(\widetilde{P}_{ij}^{(q)}) = \lim_{n\to\infty} E[\widetilde{P}_{ij}^{(q)}] - P_{ij} = \lim_{n\to\infty} E[\hat{P}_{ij}^{(q)}] - P_{ij} = \lim_{n\to\infty}\mathrm{Bias}(\hat{P}_{ij}^{(q)}).$$

Proof: Direct result from Lemma 3.27 by noticing

$$\lim_{n\to\infty} E[\widetilde{P}_{ij}^{(q)}] = \lim_{n\to\infty} E[\hat{P}_{ij}^{(q)}].$$

■

Theorem 3.29 *Assuming that $m = O(n^b)$ for any $b > 0$, then* $\mathrm{Var}((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}) = O(n^{-2}(\log n)^6)$.

Proof: By Lemma 3.27,

$$\begin{aligned}\mathrm{Var}((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}) &= E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ &= E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j + Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ &= E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ &\quad + 2E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])] \\ &\le E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2] \\ &\quad + 2\sqrt{E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2]E[(Z_i^T Z_j - E[(\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}])^2]} \\ &\le 4E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2]\end{aligned}$$

Fix some $a > 0$, we have

$$\begin{aligned}&E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2] \\ =& E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] + E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2\mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}]\end{aligned}$$

Note that we are thresholding according to $\hat{P}^{(1)}$ instead of $\hat{P}^{(q)}$. By Lemma 3.1, we know $\hat{P}^{(q)} < \hat{P}^{(1)}$ given any data. For the first term, we have

$$
\begin{aligned}
&E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] \\
\le& E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}\mathbb{I}\{\mathrm{Lemma 3.22 holds}\}](1-n^{-c}) \\
&+ E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}\mathbb{I}\{\mathrm{Lemma 3.22 does not hold}\}]n^{-c} \\
\le& O(n^{-2}(\log n)^6)(1-n^{-c}) + 2n^{-c}E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - \hat{P}_{ij}^{(q)})^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] \\
&+ 2n^{-c}E[(\hat{P}_{ij}^{(q)} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] \\
\le& O(n^{-2}(\log n)^6) + 2n^{-c}E[\hat{P}_{ij}^{(q)2}\mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] + 2n^{-c}E[(\hat{P}_{ij}^{(q)} + R)^2\mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] \\
\le& O(n^{-2}(\log n)^6) + 2n^{-c}E[\hat{P}_{ij}^{(1)2}\mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] + 2n^{-c}E[(\hat{P}_{ij}^{(1)} + R)^2\mathbb{I}\{\hat{P}_{ij}^{(1)} \le a\}] \\
\le& O(n^{-2}(\log n)^6) + 2a^2 n^{-c} + 2(a+R)^2 n^{-c} \\
\le& O(n^{-2}(\log n)^6) + 4n^{-c}(a+R)^2
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
&E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] \\
\le& 2E[((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}} - \hat{P}_{ij}^{(q)})^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] + 2E[(\hat{P}_{ij}^{(q)} - Z_i^T Z_j)^2 \mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] \\
\le& 2E[\hat{P}_{ij}^{(q)2}\mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] + 2E[(\hat{P}_{ij}^{(q)} + R)^2\mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] \\
\le& 2E[\hat{P}_{ij}^{(1)2}\mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] + 2E[(\hat{P}_{ij}^{(1)} + R)^2\mathbb{I}\{\hat{P}_{ij}^{(1)} > a\}] \\
\le& 4e^{-a/R}(a + 2m^{1/2}R)^2
\end{aligned}
$$

Similarly, assuming $m = O(n^b)$ for any $b > 0$, we have

$$\mathrm{Var}((\hat{Z}_i^T\hat{Z}_j)_{\mathrm{tr}}) = O(n^{-2}(\log n)^6).$$

■

Theorem 3.30 *Let $u_q(\theta) = E_\theta[\hat{\theta}_{q,n}]$, $\phi_q(x;\theta) = \frac{\partial}{\partial\theta}L_q(f(x;\theta))$, and $\phi'_q(x;\theta) = \frac{\partial^2}{\partial\theta^2}L_q(f(x;\theta))$. Then the asymptotic distribution of $\hat{\theta}_{q,n}$ is $\sqrt{n}(\hat{\theta}_{q,n} - u_q(\theta)) \sim \mathcal{N}(0, V_q(\theta))$, where $V_q(\theta) = E[\phi_q(X;\theta)^2]/E[\phi'_q(X;\theta)]^2$.*

Remark: See Theorem 1 in http://arxiv.org/pdf/1310.7278.pdf.

Corollary 3.31 $\mathrm{Var}(\hat{P}_{ij}^{(q)}) = \Theta(m^{-1})$.

Proof: Direct result from Theorem 3.30. ■

Theorem 3.32 *For fixed m, $1 \le i, j \le n$,*

$$\frac{\mathrm{Var}(\widetilde{P}_{ij}^{(q)})}{\mathrm{Var}(\hat{P}_{ij}^{(q)})} = O(mn^{-2}(\log n)^6).$$

Thus

$$\mathrm{ARE}(\hat{P}_{ij}^{(q)}, \widetilde{P}_{ij}^{(q)}) = 0.$$

Furthermore, as long as m goes to infinity of order $o(n^2(\log n)^{-6})$,

$$\mathrm{ARE}(\hat{P}_{ij}^{(q)}, \widetilde{P}_{ij}^{(q)}) = 0.$$

Proof: The results are direct from Theorem 3.29 and Corollary 3.31. ■

3.4 $\widetilde{P}^{(q)}$ better than $\widetilde{P}^{(1)}$

Theorem 3.33 *For sufficiently large n and C, any $1 \le i, j \le n$,*

$$\lim_{m\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(1)}) > \lim_{m\to\infty} \mathrm{Bias}(\widetilde{P}_{ij}^{(q)})$$

Proof: Direct result from Lemma 3.6, Corollary 3.17 and Corollary 3.28. ■

Theorem 3.34 *For any fixed m, any $1 \le i, j \le n$,*

$$\lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(1)}) = \lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(q)}) = 0.$$

Furthermore, as long as m goes to infinity of order $o(n^2(\log n)^{-6})$, any $1 \le i, j \le n$,

$$\lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(1)}) = \lim_{n\to\infty} \mathrm{Var}(\widetilde{P}_{ij}^{(q)}) = 0$$

Proof: Direct result from Theorem 3.18 and Theorem 3.29. ■

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