

ASE o MLqE Story Latest

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1 Problem Description

1.1 Uncontaminated Model

Under the stochastic block model with parameters (B, ρ) , we have $X_i \stackrel{iid}{\sim} \sum_{k=1}^K \rho_k \delta_{\nu_k}$, where $\nu = [\nu_1, \dots, \nu_K]^T \in \mathbb{R}^{K \times d}$ satisfies $B = \nu^T \nu$. Define the block assignment τ such that $\tau_i = k$ if and only if $X_i = \nu_k$. Let $P = XX^T$ where $X = [X_1, \dots, X_n]^T$.

First sample τ from the multinomial distribution with parameter ρ . Then we are going to sample m conditionally i.i.d. symmetric graphs $G^{(1)}, \dots, G^{(m)}$ such that conditioning on τ , $G_{ij}^{(k)} \stackrel{ind}{\sim} \text{Exp}(P_{ij})$ for each $1 \leq k \leq m$, $1 \leq i \leq j \leq n$.

1.2 Contaminated Observations

Now we assume the observed edges are contaminated with probability ϵ .

First sample τ from the multinomial distribution with parameter ρ . Then we are going to sample m conditionally i.i.d. symmetric graphs $A^{(1)}, \dots, A^{(m)}$ such that conditioning on τ , $A_{ij}^{(t)} \stackrel{ind}{\sim} (1 - \epsilon)\text{Exp}(P_{ij}) + \epsilon\text{Exp}(C_{ij})$ for each $1 \leq t \leq m$, $1 \leq i \leq j \leq n$, where the contamination is a rank-1 matrix $C = YY^T$, $Y \in \mathbb{R}^n$.

1.3 Goal

Given the contaminated observation of adjacency matrices of m graphs, i.e. $A^{(1)}, \dots, A^{(m)}$, we want to estimate the mean of the collection of uncontaminated graphs P .

2 Candidate Estimators

After observing contaminated adjacency matrices of m graphs $A^{(1)}, \dots, A^{(m)}$, we want to propose a good estimator for the mean of the collection of graphs P .

2.1 $\hat{P}^{(1)}$ based on entry-wise MLE

Under the independent edge setting, we can simplify the problem to finding an entry-wise estimate of P . And MLE is always our first choice, which exists and happen to be \bar{A} , the entry-wise mean in this case. For consistency, we define $\hat{P}^{(1)} = \bar{A}$.

2.2 $\hat{P}^{(q)}$ based on entry-wise MLqE

Since the observations are contaminated, robust estimators are preferred. A modified MLE estimator, the maximum likelihood L- q estimator, is considered in this case. Define $\hat{P}^{(q)}$ as the entry-wise MLqE.

Remark: MLE is a special case of MLqE when $q = 1$. So we notate the entry-wise MLE to be $\hat{P}^{(1)}$ in consistent with entry-wise MLqE $\hat{P}^{(q)}$.

2.3 $\tilde{P}^{(1)}$ based on ASE of entry-wise MLE

By taking advantages of the graph structure, we expect a better performance after applying a rank-reduction procedure to the entry-wise MLE $\hat{P}^{(1)}$ under the SBM. So we first apply ASE to $\hat{P}^{(1)}$ to get the latent positions $\hat{X}^{(1)}$ in dimension $d^{(1)}$, and then define $\tilde{P}^{(1)} = \hat{X}^{(1)} \hat{X}^{(1)T}$.

2.4 $\tilde{P}^{(q)}$ based on ASE of entry-wise MLqE

Similarly, we also expect a better performance after applying a rank-reduction procedure to the entry-wise MLqE $\hat{P}^{(q)}$ under the SBM. So we first apply ASE to $\hat{P}^{(q)}$ to get the latent positions $\hat{X}^{(q)}$ in dimension $d^{(q)}$, and then define $\tilde{P}^{(q)} = \hat{X}^{(q)} \hat{X}^{(q)T}$.

3 Compare Estimators

3.1 $\hat{P}^{(q)}$ is better than $\hat{P}^{(1)}$

Lemma 3.1 *For any $0 < q, \epsilon < 1$, there exists $C_0(P_{ij}, \epsilon, q) > 0$ such that under the contaminated model with $C > C_0(P_{ij}, \epsilon, q)$,*

$$\lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}^{(q)}] - P_{ij} \right| < \lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

for $1 \leq i, j \leq n$.

Lemma 3.2 *For $1 \leq i, j \leq n$, we have*

$$\lim_{m \rightarrow \infty} \text{Var}(\hat{P}_{ij}^{(1)}) = \lim_{m \rightarrow \infty} \text{Var}(\hat{P}_{ij}^{(q)}) = 0,$$

Thus,

- By Lemma ??, when C is large enough, for every $1 \leq i, j \leq n$, $\hat{P}_{ij}^{(q)}$ has smaller asymptotic bias in absolute value than $\hat{P}_{ij}^{(1)}$ as $m \rightarrow \infty$;
- By Lemma ??, all entry-wise variances go to 0 for estimating P as $m \rightarrow \infty$;
- In terms of MSE, $\hat{P}^{(q)}$ is better than $\hat{P}^{(1)}$ when m and C are large enough.

3.2 $\tilde{P}^{(1)}$ is better than $\hat{P}^{(1)}$

Theorem 3.3 Let P and C be two n -by- n symmetric matrices satisfying element-wise conditions $0 < P_{ij} \leq C_{ij} \leq R$ for some constant $R > 0$. For $0 < \epsilon < 1$, we define m symmetric matrices as

$$A^{(t)} \stackrel{iid}{\sim} (1 - \epsilon)\text{Exp}(P) + \epsilon\text{Exp}(C),$$

for $1 \leq t \leq m$. Let $\hat{P}^{(1)}$ be the element-wise MLE based on exponential distribution with m observations. Define $H_{ij}^{(1)} = E[\hat{P}_{ij}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij}$, then for any constant $c > 0$, there exists another constant $n_0(c)$, independent of n , P , C and ϵ , such that if $n > n_0$, then for all η satisfying $n^{-c} \leq \eta \leq 1/2$,

$$P \left(\|\hat{P}^{(1)} - H^{(1)}\|_2 \leq 2R\sqrt{n \ln(n/\eta)/m} \right) \geq 1 - \eta.$$

Remark: This is the extended version of Theorem 3.1 in <http://arxiv.org/pdf/0911.0600v2.pdf>.

Define $H^{(1)} = E[\hat{P}^{(1)}] = (1 - \epsilon)P + \epsilon C$, where $P = XX^T$, $C = YY^T$. Let $Z = [\sqrt{1 - \epsilon}X, \sqrt{\epsilon}Y]$, then $H^{(1)} = ZZ^T$.

Let $d^{(1)} = \text{rank}(H^{(1)})$ be the dimension in which we are going to embed $\hat{P}^{(1)}$.

From now on, we will use \hat{P} to represent $\hat{P}^{(1)}$, use H to represent $H^{(1)}$. Assume $H = USU^T = ZZ^T$, where Z is a n -by- k matrix. Then our estimate for Z up to rotation is $\hat{Z} = \hat{U}\hat{S}^{1/2}$, where $\hat{U}\hat{S}\hat{U}^T$ is the rank- k spectral decomposition of $|\hat{P}| = (\hat{P}^T \hat{P})^{1/2}$.

Lemma 3.4 $\|S\|_2 = \Theta(n)$, $\|\hat{S}\|_2 = \Theta(n)$.

Lemma 3.5 Let $W_1 \Sigma W_2^T$ be the singular value decomposition of $U^T \hat{U}$. Then for sufficiently large n ,

$$\|U^T \hat{U} - W_1 W_2^T\|_F = O(m^{-1} n^{-1} \log n)$$

with high probability.

We will denote the orthogonal matrix $W_1 W_2^T$ by W^* .

Lemma 3.6 For sufficiently large n ,

$$\|W^* \hat{S} - S W^*\|_F = O(m^{-1/2} \log n),$$

$$\|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F = O(m^{-1/2} n^{-1/2} \log n)$$

and

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(m^{-1/2} n^{-3/2} \log n)$$

with high probability.

Lemma 3.7 There exists a rotation matrix W such that for sufficiently large n ,

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2} n^{-1/2} (\log n)^{3/2})$$

with high probability.

RT: Need proof here.

Theorem 3.8 *There exists a rotation matrix W such that for sufficiently large n ,*

$$\max_i \|\hat{Z}_i - W Z_i\|_2 = O(m^{-1/2} n^{-1/2} (\log n)^{3/2})$$

with high probability.

Lemma 3.9 $\left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| = O(m^{-1/2} n^{-1/2} (\log n)^{3/2})$ *with high probability.*

Corollary 3.10 *For fixed m , the estimator based on ASE of MLE has the same entry-wise asymptotic bias as MLE, i.e.*

$$\lim_{n \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(1)}] - P_{ij} = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(1)}] - P_{ij} = \lim_{n \rightarrow \infty} \text{Bias}(\hat{P}_{ij}^{(1)}).$$

Theorem 3.11 $\text{Var}(\hat{Z}_i^T \hat{Z}_j) = O(m^{-1} n^{-1} (\log n)^3)$ *with high probability.*

Corollary 3.12 *For fixed n , $1 \leq i, j \leq n$, $\text{Var}(\hat{P}_{ij}^{(1)}) = \Theta(m^{-1})$.*

Theorem 3.13 *For fixed m , $1 \leq i, j \leq n$,*

$$\frac{\text{Var}(\tilde{P}_{ij}^{(1)})}{\text{Var}(\hat{P}_{ij}^{(1)})} = O(n^{-1} (\log n)^3).$$

Thus

$$\text{ARE}(\hat{P}_{ij}^{(1)}, \tilde{P}_{ij}^{(1)}) = 0.$$

Then

- For each $1 \leq i, j \leq n$, both $\hat{P}_{ij}^{(1)}$ and $\tilde{P}_{ij}^{(1)}$ have the same asymptotic bias as $n \rightarrow \infty$;
- Fix m , for every $1 \leq i, j \leq n$, $\text{ARE}(\hat{P}_{ij}^{(1)}, \tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(1)}) / \text{Var}(\hat{P}_{ij}^{(1)}) = 0$, which means $\tilde{P}^{(1)}$ is better than $\hat{P}^{(1)}$;
- Actually when fixing m , for every $1 \leq i, j \leq n$, $\text{Var}(\tilde{P}_{ij}^{(1)}) / \text{Var}(\hat{P}_{ij}^{(1)})$ is of order $O(n^{-1})$ as $n \rightarrow \infty$.

3.3 $\tilde{P}^{(q)}$ is better than $\hat{P}^{(q)}$

Define $H^{(q)} = E[\hat{P}^{(q)}]$. Let $d^{(q)} = \text{rank}(H^{(q)})$ be the dimension in which we are going to embed $\hat{P}^{(q)}$. Then

- For each $1 \leq i, j \leq n$, both $\hat{P}_{ij}^{(q)}$ and $\tilde{P}_{ij}^{(q)}$ have the same asymptotic bias as $n \rightarrow \infty$;
- Fix m , for every $1 \leq i, j \leq n$, $\text{ARE}(\hat{P}_{ij}^{(q)}, \tilde{P}_{ij}^{(q)}) = \lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(q)}) / \text{Var}(\hat{P}_{ij}^{(q)}) = 0$, which means $\tilde{P}^{(q)}$ is better than $\hat{P}^{(q)}$;
- Actually, even if m is not fixed, as long as m is growing with order $o(n^{1/2} (\log n)^{-3/2})$, we still have $\text{ARE}(\hat{P}_{ij}^{(q)}, \tilde{P}_{ij}^{(q)}) = 0$,

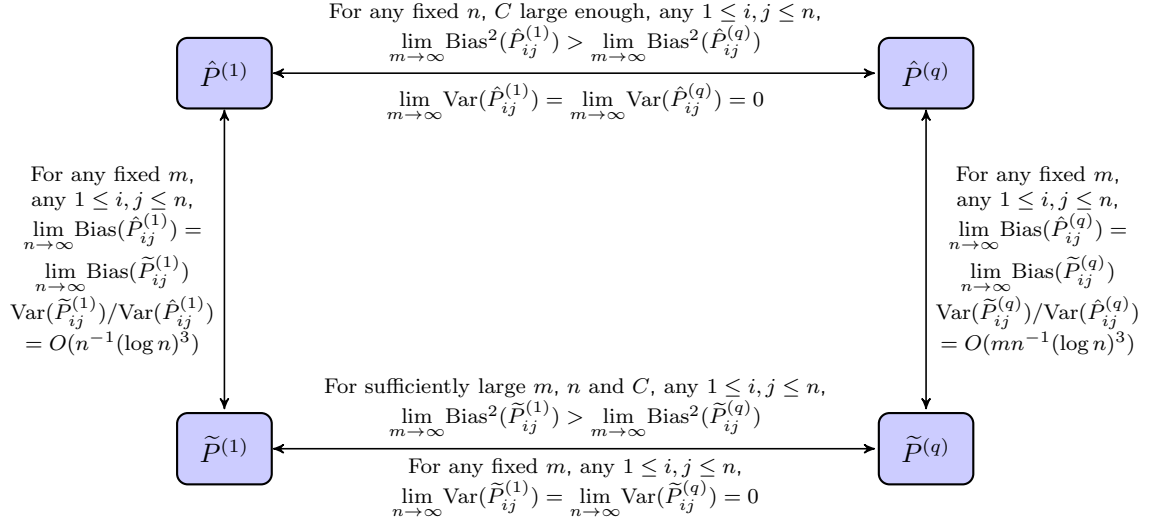


Figure 1: Relationship between four estimators.

3.4 $\tilde{P}^{(q)}$ is better than $\tilde{P}^{(1)}$

- When n is large enough, for every $1 \leq i, j \leq n$, $E[\tilde{P}_{ij}^{(1)}]$ will be close to $E[\hat{P}_{ij}^{(1)}]$ and $E[\tilde{P}_{ij}^{(q)}]$ will be close to $E[\hat{P}_{ij}^{(q)}]$. Combined with $\hat{P}_{ij}^{(q)}$ has smaller asymptotic bias (as $m \rightarrow \infty$) than $\hat{P}_{ij}^{(1)}$ when C is large enough, we have for sufficiently large m and n , C large enough, $\lim_{m \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(1)}) > \lim_{m \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(q)})$;
- Fix m , for any $1 \leq i, j \leq n$, when n is large enough, $\text{Var}(\tilde{P}_{ij}^{(1)})$ is less than $\text{Var}(\hat{P}_{ij}^{(1)})$ times $O(n^{-1})$ and $\text{Var}(\tilde{P}_{ij}^{(q)})$ is less than $\text{Var}(\hat{P}_{ij}^{(q)})$ times $O(n^{-1/2}(\log n)^{3/2})$. Thus $\lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(q)}) = 0$;
- In terms of MSE, $\tilde{P}^{(q)}$ is better than $\tilde{P}^{(1)}$ when m , n and C are large enough.

3.5 Summary

Thus, we should choose the estimator $\tilde{P}^{(q)}$.

4 Proof

4.1 $\hat{P}^{(q)}$ better than $\hat{P}^{(1)}$

Lemma 4.1 (Lemma ??) *For any $0 < q < 1$, there exists $C_0(P_{ij}, \epsilon, q) > 0$ such that under the contaminated model with $C > C_0(P_{ij}, \epsilon, q)$,*

$$\lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}^{(q)}] - P_{ij} \right| < \lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

for $1 \leq i, j \leq n$.

Proof: For the MLE $\hat{P}_{ij}^{(1)} = \bar{A}_{ij}$,

$$E[\hat{P}_{ij}^{(1)}] = E[\bar{A}_{ij}] = \frac{1}{m} \sum_{t=1}^m E[A_{ij}^{(t)}] = E[A_{ij}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij}.$$

For the MLqE $\hat{P}_{ij}^{(q)}$,

$$E[\hat{P}_{ij}^{(q)}]$$

solves a cubic equation. ■

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Mathematica

Lemma 4.2 (Lemma ??)

$$\lim_{m \rightarrow \infty} \text{Var}(\hat{P}_{ij}^{(1)}) = \lim_{m \rightarrow \infty} \text{Var}(\hat{P}_{ij}^{(q)}) = 0,$$

for $1 \leq i, j \leq n$.

Proof: Both MLE and MLqE follows a central limit theorem, which means their variances goes to 0 as $m \rightarrow \infty$. ■

4.2 $\tilde{P}^{(1)}$ better than $\hat{P}^{(1)}$

Definition 4.3 A random variable Z is called central moment bounded with real parameter $L > 0$, if for any integer $i \geq 1$ we have

$$E[|Z - E[Z]|^i] \leq i \cdot L \cdot E[|Z - E[Z]|^{i-1}].$$

Theorem 4.4 (Matrix Bernstein: Subexponential Case). Consider a finite sequence $\{X_k\}$ of independent, random, self-adjoint matrices with dimension d . Assume that

$$E[X_k] = 0 \quad \text{and} \quad E[X_k^p] \preceq \frac{p!}{2} R^{p-2} A_k^2 \quad \text{for } p = 2, 3, 4, \dots$$

Compute the variance parameter

$$\sigma^2 := \left\| \sum_k A_k^2 \right\|.$$

Then the following chain of inequalities holds for all $t \geq 0$.

$$P\left(\lambda_{\max}\left(\sum_k X_k\right) \geq t\right) \leq d \cdot \exp\left(\frac{-t^2/2}{\sigma^2 + Rt}\right).$$

Remark: See Theorem 6.2 in <http://arxiv.org/pdf/1004.4389.pdf>.

Theorem 4.5 (Theorem 3.3) Let P and C be two n -by- n symmetric matrices satisfying element-wise conditions $0 < P_{ij} \leq C_{ij} \leq R$ for some constant $R > 0$. For $0 < \epsilon < 1$, we define m symmetric matrices as

$$A^{(t)} \stackrel{iid}{\sim} (1 - \epsilon)\text{Exp}(P) + \epsilon\text{Exp}(C),$$

for $1 \leq t \leq m$. Let $\hat{P}^{(1)}$ be the element-wise MLE based on exponential distribution with m observations. Define $H_{ij}^{(1)} = E[\hat{P}_{ij}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij}$, then for any constant $c > 0$, there exists another constant $n_0(c)$, independent of n , P , C and ϵ , such that if $n > n_0$, then for all η satisfying $n^{-c} \leq \eta \leq 1/2$,

$$P \left(\|\hat{P}^{(1)} - H^{(1)}\|_2 \leq 2R\sqrt{n \ln(n/\eta)/m} \right) \geq 1 - \eta.$$

Remark: This is the extended version of Theorem 3.1 in <http://arxiv.org/pdf/0911.0600v2.pdf>.

Proof: Let $\{e_i\}_{i=1}^n$ be the canonical basis for \mathbb{R}^n . For each $1 \leq i, j \leq n$, define a corresponding matrix G_{ij} :

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus $\hat{P}^{(1)} = \sum_{1 \leq i \leq j \leq n} \hat{P}_{ij}^{(1)} G_{ij}$ and $H^{(1)} = \sum_{1 \leq i \leq j \leq n} H_{ij}^{(1)} G_{ij}$. Then we have $\hat{P}^{(1)} - H^{(1)} = \sum_{1 \leq i \leq j \leq n} X_{ij}$, where $X_{ij} \equiv (\hat{P}_{ij}^{(1)} - H_{ij}^{(1)}) G_{ij}$, $1 \leq i \leq j \leq n$.

First consider the k -th moment of X_{ij} for $1 \leq i < j \leq n$. Since

$$\begin{aligned} E[(A_{ij}^{(1)} - H_{ij}^{(1)})^k] &\leq (1 - \epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1 + k, -H_{ij}/P_{ij}) \\ &\quad + \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1 + k, -H_{ij}/C_{ij}) \\ &\leq ((1 - \epsilon) \cdot \exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \cdot \exp(-H_{ij}/C_{ij}) C_{ij}^k) k! \\ &\leq ((1 - \epsilon) \cdot P_{ij}^k + \epsilon \cdot C_{ij}^k) k! \\ &\leq R^k k!, \end{aligned}$$

we have

$$\begin{aligned} E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] &= E\left[\left(\frac{1}{m} \sum_{t=1}^m A_{ij}^{(t)} - H_{ij}^{(1)}\right)^k\right] \\ &= E\left[\left(\frac{1}{m} \sum_{t=1}^m (A_{ij}^{(t)} - H_{ij}^{(1)})\right)^k\right] \\ &= \frac{1}{m^k} E\left[\left(\sum_{t=1}^m (A_{ij}^{(t)} - H_{ij}^{(1)})\right)^k\right] \\ &\leq \frac{1}{m^k} R^k k!. \end{aligned} \tag{1}$$

Combined with for $i \neq j$,

$$G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & k \text{ is even;} \\ e_i e_j^T + e_j e_i^T, & k \text{ is odd,} \end{cases}$$

thus we have

1. When k is even,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] G_{ij}^2 \preceq k! (R/m)^k G_{ij}^2;$$

2. When k is odd,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] G_{ij} \preceq k! (R/m)^k G_{ij}^2.$$

So

$$E[X_{ij}^k] \preceq k!(R/m)^k G_{ij}^2.$$

Let

$$\sigma^2 := \left\| \sum_{1 \leq i < j \leq n} (\sqrt{2} R G_{ij} / m)^2 \right\| = 2R^2 \|(n-1)I\|/m^2 = 2R^2(n-1)/m^2,$$

notice that random matrices X_{ij} are independent, self-adjoint and have mean zero, apply Theorem ?? we have

$$\begin{aligned} P\left(\lambda_{\max}(\hat{P}^{(1)} - H^{(1)}) \geq t\right) &\leq n \exp\left(-\frac{t^2/2}{\sigma^2 + Rt/m}\right) \\ &\leq n \exp\left(-\frac{t^2/2}{2R^2n/m^2 + Rt/m}\right). \end{aligned}$$

Now consider $Y_{ij} \equiv (H^{(1)} - \hat{P}^{(1)}) G_{ij}$, $1 \leq i < j \leq n$. Then we have $H^{(1)} - \hat{P}^{(1)} = \sum_{1 \leq i < j \leq n} Y_{ij}$. Since

$$E[(H^{(1)} - \hat{P}^{(1)})^k] = (-1)^k E[(\hat{P}^{(1)} - H^{(1)})^k],$$

1. When k is even,

$$E[Y_{ij}^k] = E[(\hat{P}^{(1)} - H^{(1)})^k] G_{ij}^2 \preceq k!(R/m)^k G_{ij}^2;$$

2. When k is odd,

$$E[Y_{ij}^k] = -E[(\hat{P}^{(1)} - H^{(1)})^k] G_{ij} \preceq k!(R/m)^k G_{ij}^2.$$

Thus

$$\begin{aligned} P\left(\lambda_{\min}(\hat{P}^{(1)} - H^{(1)}) \leq -t\right) &= P\left(\lambda_{\max}(H^{(1)} - \hat{P}^{(1)}) \geq t\right) \\ &\leq n \exp\left(-\frac{t^2/2}{2R^2n/m^2 + Rt/m}\right). \end{aligned}$$

Therefore we have

$$P\left(\|\hat{P}^{(1)} - H^{(1)}\| \geq t\right) \leq n \exp\left(-\frac{t^2/2}{2R^2n/m^2 + Rt/m}\right).$$

Now let $c > 0$ be given and assume $n^{-c} \leq \eta \leq 1/2$. Then there exists a $n_0(c)$ independent of n , P , C and ϵ such that whenever $n > n_0(c)$,

$$t = 2R\sqrt{n \ln(n/\eta)/m} \leq 2Rn.$$

Plugging this t into the equation above, we get

$$P(\|\hat{P}^{(1)} - H^{(1)}\| \geq 2R\sqrt{n \ln(n/\eta)/m}) \leq n \exp\left(-\frac{t^2}{4R^2n/m}\right) = \eta.$$

■

From now on, we will use \hat{P} to represent $\hat{P}^{(1)}$, use H to represent $H^{(1)}$. Assume $H = USU^T = ZZ^T$, where Z is a n -by- k matrix. Then our estimate for Z up to rotation is $\hat{Z} = \hat{U} \hat{S}^{1/2}$, where $\hat{U} \hat{S} \hat{U}^T$ is the rank- k spectral decomposition of $|\hat{P}| = (\hat{P}^T \hat{P})^{1/2}$.

Lemma 4.6 $\|S\|_2 = \Theta(n)$, $\|\hat{S}\|_2 = \Theta(n)$.

Lemma 4.7 Let $W_1 \Sigma W_2^T$ be the singular value decomposition of $U^T \hat{U}$. Then for sufficiently large n ,

$$\|U^T \hat{U} - W_1 W_2^T\|_F = O(m^{-1} n^{-1} \log n)$$

with high probability.

Proof: Let $\sigma_1, \dots, \sigma_k$ denote the singular values of $U^T \hat{U}$. Then $\sigma_i = \cos(\theta_i)$ where the θ_i are the principal angles between the subspaces spanned by \hat{U} and U . Furthermore, by the Davis-Kahan $\sin(\Theta)$ theorem, combined with Theorem ?? and Lemma ??,

$$\|\hat{U} \hat{U}^T - U U^T\|_2 = \max_i |\sin(\theta_i)| \leq \frac{\|\hat{P} - H\|_2}{\lambda_k(H)} \leq \frac{C \sqrt{n \log n / m}}{n} = O(m^{-1/2} n^{-1/2} \sqrt{\log n}) \quad (2)$$

for sufficient large n . Here $\lambda_k(H)$ denotes the k -th largest eigenvalue of H .

We thus have

$$\begin{aligned} \|U^T \hat{U} - W_1 W_2^T\|_F &= \|\Sigma - I\|_F = \sqrt{\sum_{i=1}^k (1 - \sigma_i)^2} \\ &\leq \sum_{i=1}^k (1 - \sigma_i) \leq \sum_{i=1}^k (1 - \sigma_i^2) \\ &= \sum_{i=1}^k \sin^2(\theta_i) \leq k \|\hat{U} \hat{U}^T - U U^T\|_2^2 \\ &= O(m^{-1} n^{-1} \log n). \end{aligned}$$

■

We will denote the orthogonal matrix $W_1 W_2^T$ by W^* .

Lemma 4.8 For sufficiently large n ,

$$\|W^* \hat{S} - S W^*\|_F = O(m^{-1/2} \log n),$$

$$\|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F = O(m^{-1/2} n^{-1/2} \log n)$$

and

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(m^{-1/2} n^{-3/2} \log n)$$

with high probability.

Proof: Let $Q = \hat{U} - U U^T \hat{U}$. And Q is the residual after projecting \hat{U} orthogonally onto the column space of U , we have

$$\|\hat{U} - U U^T \hat{U}\|_F \leq \|\hat{U} - U T\|_F$$

for all $k \times k$ matrices T . Also, by Equation (??) we have

$$\|Q\|_F = \|\hat{U} - U U^T \hat{U}\|_F = \|\hat{U} \hat{U}^T - U U^T\|_F = O(m^{-1/2} n^{-1/2} \sqrt{\log n}). \quad (3)$$

Then

$$\begin{aligned}
W^* \hat{S} &= (W^* - U^T \hat{U}) \hat{S} + U^T \hat{U} \hat{S} = (W^* - U^T \hat{U}) \hat{S} + U^T \hat{P} \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) \hat{U} + U^T H \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + U^T H \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S U^T \hat{U}.
\end{aligned}$$

Combined with Theorem ??, Lemma ??, Lemma ??, we have

$$\begin{aligned}
&\|W^* \hat{S} - S W^*\|_F \\
&= \|(W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S(U^T \hat{U} - W^*)\|_F \\
&\leq \|W^* - U^T \hat{U}\|_F (\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F \|\hat{P} - H\|_2 \|Q\|_F + \|U^T (\hat{P} - H) U\|_F \\
&\leq O(m^{-1} \log n) + O(m^{-1/2} \log n) + \|U^T (\hat{P} - H) U\|_F
\end{aligned}$$

with high probability. And we know $U^T (\hat{P} - H) U$ is a $k \times k$ matrix with ij -th entry to be

$$u_i^T (\hat{P} - H) u_j = \sum_{s=1}^n \sum_{t=1}^n (\hat{P}_{st} - H_{st}) u_{is} u_{jt} = 2 \sum_{s < t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt} - \sum_{s=1}^n H_{ss} u_{is} u_{js}$$

where u_i and u_j are the i -th and j -th columns of U . Thus, conditioned on H , $u_i^T (\hat{P} - H) u_j$ is a sum of independent mean 0 random variables and a term of order $O(1)$.

By Equation ??, we have

$$\begin{aligned}
&E \left[\left((\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right)^k \right] \\
&\leq k! (R/m)^k u_{is}^k u_{jt}^k \\
&\leq \frac{k!}{2} (R/m)^{k-2} (\sqrt{2} u_{is} u_{jt} R/m)^2.
\end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{s < t} 2R^2 u_{is}^2 u_{jt}^2 / m^2 \right| \leq 2R^2 / m^2,$$

then by Theorem ??, we have

$$P \left(\left| 2 \sum_{s < t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right| \geq t \right) \leq \exp \left(\frac{-t^2/2}{2R^2/m^2 + Rt/m} \right),$$

thus each entry of $U^T (\hat{P} - H) U$ is of order $O(m^{-1})$ with high probability and thus

$$\|U^T (\hat{P} - H) U\|_F = O(m^{-1}) \quad (4)$$

with high probability. Hence

$$\|W^* \hat{S} - S W^*\|_F = O(m^{-1/2} \log n)$$

with high probability. Also, since

$$W_{ij}^* (\lambda_j^{1/2}(\hat{P}) - \lambda_i^{1/2}(H)) = W_{ij}^* \frac{\lambda_j(\hat{P}) - \lambda_i(H)}{\lambda_j^{1/2}(\hat{P}) + \lambda_i^{1/2}(H)}$$

and the eigenvalues $\lambda_j^{1/2}(\hat{P})$ and $\lambda_i^{1/2}(H)$ are both of order $\Theta(\sqrt{n})$, we have

$$\|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F = O(m^{-1/2} n^{-1/2} \log n).$$

Similarly, since

$$W_{ij}^* (\lambda_j^{-1/2}(\hat{P}) - \lambda_i^{-1/2}(H)) = W_{ij}^* \frac{\lambda_i(H) - \lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P}) + \lambda_i^{-1/2}(H)) \lambda_j(\hat{P}) \lambda_i(H)}$$

and the eigenvalues $\lambda_j(\hat{P})$ and $\lambda_i(H)$ are both of order $\Theta(n)$, we have

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(m^{-1/2} n^{-3/2} \log n).$$

■

Lemma 4.9 *There exists a rotation matrix W such that for sufficiently large n ,*

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2} n^{-1/2} (\log n)^{3/2})$$

with high probability.

Proof: Let $Q_1 = UU^T \hat{U} - UW^*$, $Q_2 = W^* \hat{S}^{1/2} - S^{1/2} W^*$ and $Q_3 = \hat{U} - UW^* = \hat{U} - UU^T \hat{U} + Q_1 = Q + Q_1$. Then since $UU^T P = P$ and $\hat{U} \hat{S}^{1/2} = \hat{P} \hat{U} \hat{S}^{-1/2}$,

$$\begin{aligned} \hat{Z} - US^{1/2} W^* &= \hat{U} \hat{S}^{1/2} - UW^* \hat{S}^{1/2} + U(W^* \hat{S}^{1/2} - S^{1/2} W^*) \\ &= (\hat{U} - UU^T \hat{U}) \hat{S}^{1/2} + Q_1 \hat{S}^{1/2} + UQ_2 \\ &= (\hat{P} - H) \hat{U} \hat{S}^{-1/2} - UU^T (\hat{P} - H) \hat{U} \hat{S}^{-1/2} + Q_1 \hat{S}^{1/2} + UQ_2 \\ &= (\hat{P} - H) UW^* \hat{S}^{-1/2} - UU^T (\hat{P} - H) UW^* \hat{S}^{-1/2} \\ &\quad + (I - UU^T) (\hat{P} - H) Q_3 \hat{S}^{-1/2} + Q_1 \hat{S}^{1/2} + UQ_2. \end{aligned}$$

By Lemma ??,

$$\|Q_1\|_F \leq \|U\|_F \|U^T \hat{U} - W^*\|_F = O(m^{-1} n^{-1} \log n).$$

By Lemma ??,

$$\|Q_2\|_F = O(m^{-1/2} n^{-1/2} \log n).$$

By Equation ??,

$$\|Q_3\|_F \leq \|Q\|_F + \|Q_1\|_F = O(m^{-1/2} n^{-1/2} (\log n)^{1/2}).$$

By Equation ??,

$$\|UU^T (\hat{P} - H) UW^* \hat{S}^{-1/2}\|_F \leq \|U^T (\hat{P} - H) U\|_F \|\hat{S}^{-1/2}\|_2 = O(m^{-1} n^{-1/2}).$$

By Lemma ??,

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(m^{-1/2} n^{-3/2} \log n).$$

Therefore,

$$\begin{aligned}
& \|\hat{Z} - US^{1/2}W^*\|_F \\
&= \|(\hat{P} - H)UW^*\hat{S}^{-1/2}\|_F + O(m^{-1}n^{-1/2}) + \|I - UU^T\|_2 \|\hat{P} - H\|_2 O(m^{-1/2}n^{-1}(\log n)^{1/2}) \\
&\quad + O(m^{-1}n^{-1/2} \log n) + O(m^{-1/2}n^{-1/2} \log n) \\
&= \|(\hat{P} - H)UW^*\hat{S}^{-1/2}\|_F + O(m^{-1/2}n^{-1/2} \log n) \\
&\leq \|(\hat{P} - H)US^{-1/2}W^*\|_F + \|(\hat{P} - H)U(W^*\hat{S}^{-1/2} - S^{-1/2}W^*)\|_F + O(m^{-1/2}n^{-1/2} \log n) \\
&= \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1}n^{-1}(\log n)^{3/2}) + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \\
&= \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2}n^{-1/2}(\log n)^{3/2})
\end{aligned}$$

Note that $Z = US^{1/2}W$ for some orthogonal matrix W . As W^* is also orthogonal, therefore $Z\tilde{W} = US^{1/2}W^*$ for some orthogonal \tilde{W} , which completes the proof. \blacksquare

Theorem 4.10 *There exists a rotation matrix W such that for sufficiently large n ,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(m^{-1/2}n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: By Lemma ??, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(m^{-1/2}n^{-1/2}(\log n)^{3/2})$$

and similarly we could have the bound for each row vector

$$\begin{aligned}
\max_i \|\hat{Z}_i - WZ_i\|_2 &\leq \frac{1}{\lambda_k^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(m^{-1/2}n^{-1/2}(\log n)^{3/2}) \\
&\leq \frac{k^{1/2}}{\lambda_k^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(m^{-1/2}n^{-1/2}(\log n)^{3/2})
\end{aligned}$$

RT: k should be d here?

where u_j denotes the j -th column of U . Now given i and j , the i -th element of the vector $(\hat{P} - H)u_j$ is of the form

$$\sum_{s=1}^n (\hat{P}_{is} - H_{is})u_{js} = \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js} - H_{ii}u_{ji}.$$

Thus, conditioned on H , the i -th element of the vector $(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables and a term of order $O(1)$. By Equation ??, we have

RT: We don't have $O(m^{-1})$ here!

$$\begin{aligned}
& E \left[\left((\hat{P}_{is} - H_{is})u_{js} \right)^k \right] \\
&\leq k! (R/m)^k u_{js}^k \\
&\leq \frac{k!}{2} (R/m)^{k-2} (\sqrt{2}Ru_{js}/m)^2.
\end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{s \neq i} 2R^2 u_{js}^2 / m^2 \right| \leq 2R^2 / m^2,$$

then by Theorem ??, we have

$$P \left(\left| \sum_{s \neq i} (\hat{P}_{is} - H_{is}) u_{js} \right| \geq t \right) \leq \exp \left(\frac{-t^2/2}{2R^2/m^2 + Rt/m} \right),$$

i.e. it is of order $O(m^{-1})$ with high probability. Taking the union bound over all i and j , we have

$$\begin{aligned} \max_i \|\hat{Z}_i - W Z_i\|_2 &\leq \frac{Ck^{1/2}}{\lambda_k^{1/2}(H)} m^{-1/2} \log n + O(m^{-1/2} n^{-1/2} (\log n)^{3/2}) \\ &= O(m^{-1/2} n^{-1/2} (\log n)^{3/2}). \end{aligned}$$

■

RT: $\log n$

Lemma 4.11 $\left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| = O(m^{-1/2} n^{-1/2} (\log n)^{3/2})$ with high probability.

Proof: Let W be the rotation matrix in Theorem ??, then

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= \left| \hat{Z}_i^T \hat{Z}_j - \hat{Z}_i^T W Z_j + \hat{Z}_i^T W Z_j - (W Z_i)^T W Z_j \right| \\ &\leq \left| \hat{Z}_i^T (\hat{Z}_j - W Z_j) + (\hat{Z}_i^T - (W Z_i)^T) W Z_j \right| \\ &\leq \|\hat{Z}_i\|_2 \|\hat{Z}_j - W Z_j\|_2 + \|Z_j\|_2 \|\hat{Z}_i^T - (W Z_i)^T\|_2. \end{aligned}$$

Since $\|Z_i\|_2^2 = Z_i^T Z_i = H_{ii}^q = E[\hat{P}_{ii}^{(q)}] \leq E[\hat{P}_{ii}^{(1)}] = (1 - \epsilon) P_{ij} + \epsilon C_{ij} \leq R$, we have $\|Z_i\|_2 = O(1)$. Combined with Theorem ??,

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= (\|\hat{Z}_i\|_2 + \|Z_j\|_2) O(m^{-1/2} n^{-1/2} (\log n)^{3/2}) \\ &\leq (\|\hat{Z}_i - W Z_i\|_2 + \|W Z_i\|_2 + \|Z_j\|_2) O(m^{-1/2} n^{-1/2} (\log n)^{3/2}) \\ &= O(m^{-1/2} n^{-1/2} (\log n)^{3/2}) \end{aligned}$$

with high probability. ■

Corollary 4.12 For fixed m , the estimator based on ASE of MLE has the same entry-wise asymptotic bias as MLE, i.e.

$$\lim_{n \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(1)}] - P_{ij} = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(1)}] - P_{ij} = \lim_{n \rightarrow \infty} \text{Bias}(\hat{P}_{ij}^{(1)}).$$

Proof: Direct result from Lemma ?? by noticing

$$\lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(1)}] = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(1)}].$$

■

Theorem 4.13 $\text{Var}(\hat{Z}_i^T \hat{Z}_j) = O(m^{-1} n^{-1} (\log n)^3)$ with high probability.

Proof: By Lemma ??,

$$\begin{aligned}
\text{Var}(\hat{Z}_i^T \hat{Z}_j) &= E[(\hat{Z}_i^T \hat{Z}_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&= E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j + Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&= E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&\quad - 2E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])] \\
&\leq E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&\quad + 2\sqrt{E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2]E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2]} \\
&\leq 4E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] \\
&= O(m^{-1}n^{-1}(\log n)^3)
\end{aligned}$$

with high probability. ■

Corollary 4.14 For fixed n , $1 \leq i, j \leq n$, $\text{Var}(\hat{P}_{ij}^{(1)}) = \Theta(m^{-1})$.

Proof: Direct result from central limit theorem. ■

Theorem 4.15 For fixed m , $1 \leq i, j \leq n$,

$$\frac{\text{Var}(\tilde{P}_{ij}^{(1)})}{\text{Var}(\hat{P}_{ij}^{(1)})} = O(n^{-1}(\log n)^3).$$

Thus

$$\text{ARE}(\hat{P}_{ij}^{(1)}, \tilde{P}_{ij}^{(1)}) = 0.$$

Proof: The results are direct from Theorem ?? and Corollary ??. ■

4.3 $\tilde{P}^{(q)}$ better than $\hat{P}^{(q)}$

Lemma 4.16 Consider the model $X_1, \dots, X_m \stackrel{iid}{\sim} \text{Exp}(\theta)$ with $E[X_1] = \theta$. Given any data $x = (x_1, \dots, x_m)$ such that $x_{(1)} > 0$ and not all x_i 's are the same, then $\hat{\theta}_q(x) < \hat{\theta}_1(x)$ for $0 < q < 1$, i.e. MLqE is always less than MLE under exponential distribution no matter how the data is sampled.

Proof: The MLE is

$$\hat{\theta}_1(x) = \bar{x}.$$

And the MLqE $\hat{\theta}_q(x)$ solves the equation

$$\sum_{i=1}^m e^{-\frac{(1-q)x_i}{\hat{\theta}_q(x)}} (x_i - \hat{\theta}_q(x)) = 0.$$

Consider the continuous function $g(\theta, x) = \sum_{i=1}^m e^{-\frac{(1-q)x_i}{\theta}} (x_i - \theta)$.

Let $x_{(1)} \leq \dots \leq x_{(l)} \leq \bar{x} \leq x_{(l+1)} \leq \dots \leq x_{(m)}$. Define $s_i = \bar{x} - x_{(i)}$ for $1 \leq i \leq l$, and $t_i = x_{(l+i)} - \bar{x}$ for $1 \leq i \leq m-l$. Note that $\sum_{i=1}^l s_i = \sum_{i=1}^{m-l} t_i$.

Then we have

$$\begin{aligned}
g(\hat{\theta}_1(x), x) &= g(\bar{x}, x) \\
&= \sum_{i=1}^m e^{-\frac{(1-q)x_{(i)}}{\bar{x}}} (x_{(i)} - \bar{x}) \\
&= -\sum_{i=1}^l e^{-\frac{(1-q)x_{(i)}}{\bar{x}}} s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&\leq -e^{-(1-q)} \sum_{i=1}^l s_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&\leq -e^{-(1-q)} \sum_{i=1}^{m-l} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&\leq -\sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i + \sum_{i=1}^{m-l} e^{-\frac{(1-q)x_{(i+l)}}{\bar{x}}} t_i \\
&= 0,
\end{aligned}$$

and equality holds if and only if all x_i 's are the same, which is excluded by the assumption. Thus $g(\hat{\theta}_1(x), x) < 0$.

Also we know:

- $g(\hat{\theta}_q(x), x) = 0$;
- $\lim_{\theta \rightarrow 0^+} g(\theta, x) = 0$;
- $g(\theta, x) > 0$ when $\theta < x_{(1)}$;

Combined with $g(\hat{\theta}_1(x), x) < 0$, we have $\hat{\theta}_q(x) < \hat{\theta}_1(x)$ for $0 < q < 1$. ■

Theorem 4.17 Let $P, C : [n]^2 \rightarrow (0, \infty)$ be symmetric and hollow, and

$$A^{(t)} \stackrel{iid}{\sim} (1 - \epsilon)\text{Exp}(P) + \epsilon\text{Exp}(C)$$

also be symmetric and hollow. Further assume $C_{ij} \geq P_{ij}$ for all i, j . Let $\hat{P}^{(q)}$ be the elementwise MLqE based on exponential distribution. Define $H^{(q)} = E[\hat{P}^{(q)}]$ and $R = \max C_{ij}$, then for any constant $c > 0$ there exists another constant $n_0(c)$, independent of n, P, C and ϵ , such that if $n > n_0$, then for all η satisfying $n^{-c} \leq \eta \leq 1/2$,

$$P \left(\|\hat{P}^{(q)} - H^{(q)}\| \leq 8R\sqrt{2n \ln(n/\eta)} \right) \geq 1 - \eta.$$

Remark: This is the extended version of Theorem 3.1 in <http://arxiv.org/pdf/0911.0600v2.pdf>.

Proof: Let $\{e_i\}_{i=1}^n$ be the canonical basis for \mathbb{R}^n . For each $1 \leq i, j \leq n$, define a corresponding matrix G_{ij} :

$$G_{ij} \equiv \begin{cases} e_i e_j^T + e_j e_i^T, & i \neq j; \\ e_i e_i^T, & i = j. \end{cases}$$

Thus $\hat{P}^{(q)} = \sum_{1 \leq i < j \leq n} \hat{P}_{ij}^{(q)} G_{ij}$ and $H^{(q)} = \sum_{1 \leq i < j \leq n} H_{ij}^{(q)} G_{ij}$. Then we have $\hat{P}^{(q)} - H^{(q)} = \sum_{1 \leq i < j \leq n} X_{ij}$, where $X_{ij} \equiv \left(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)} \right) G_{ij}$, $1 \leq i < j \leq n$.

First consider the k -th moment of X_{ij} for $1 \leq i < j \leq n$. By Lemma ?? we have

$$\begin{aligned}
\left| \hat{P}_{ij}^{(q)} - H_{ij}^{(q)} \right| &= \left| \hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)} + \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} + H_{ij}^{(1)} - H_{ij}^{(q)} \right| \\
&\leq \left| \hat{P}_{ij}^{(q)} - \hat{P}_{ij}^{(1)} \right| + \left| \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} \right| + \left| H_{ij}^{(1)} - H_{ij}^{(q)} \right| \\
&\leq \hat{P}_{ij}^{(1)} + \left| \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} \right| + H_{ij}^{(1)} \\
&\leq 2 \left(\left| \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} \right| + H_{ij}^{(1)} \right).
\end{aligned}$$

Since

$$\begin{aligned}
E[(\hat{P}_{ij}^{(1)} - H_{ij}^{(1)})^k] &\leq (1 - \epsilon) \exp(-H_{ij}/P_{ij}) P_{ij}^k \Gamma(1 + k, -H_{ij}/P_{ij}) \\
&\quad + \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k \Gamma(1 + k, -H_{ij}/C_{ij}) \\
&\leq ((1 - \epsilon) \exp(-H_{ij}/P_{ij}) P_{ij}^k + \epsilon \exp(-H_{ij}/C_{ij}) C_{ij}^k) k! \\
&\leq ((1 - \epsilon) P_{ij}^k + \epsilon C_{ij}^k) k! \\
&\leq C_{ij}^k k!,
\end{aligned}$$

Then

$$\begin{aligned}
E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] &\leq E \left[\left| \hat{P}_{ij}^{(q)} - H_{ij}^{(q)} \right|^k \right] \\
&\leq 2^k E \left[\left(\left| \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} \right| + H_{ij}^{(1)} \right)^k \right] \\
&\leq 2^k \sum_{s=0}^k \binom{k}{s} E \left[\left| \hat{P}_{ij}^{(1)} - H_{ij}^{(1)} \right|^s \right] \left(H_{ij}^{(1)} \right)^{k-s} \\
&\leq 2^k \sum_{s=0}^k \binom{k}{s} C_{ij}^s s! \left(H_{ij}^{(1)} \right)^{k-s} \\
&\leq 2^k k! \sum_{s=0}^k \binom{k}{s} C_{ij}^s \left(H_{ij}^{(1)} \right)^{k-s} \\
&= 2^k k! \left(C_{ij} + H_{ij}^{(1)} \right)^k. \tag{5}
\end{aligned}$$

Combined with for $i \neq j$,

$$G_{ij}^k \equiv \begin{cases} e_i e_i^T + e_j e_j^T, & k \text{ is even;} \\ e_i e_j^T + e_j e_i^T, & k \text{ is odd,} \end{cases}$$

thus we have

1. When k is even,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij}^2 \preceq 2^{2k} k! R^k G_{ij}^2;$$

2. When k is odd,

$$E[X_{ij}^k] = E[(\hat{P}_{ij}^{(q)} - H_{ij}^{(q)})^k] G_{ij} \preceq 2^{2k} k! R^k G_{ij}^2.$$

So

$$E[X_{ij}^k] \preceq 2^{2k} k! R^k G_{ij}^2.$$

Let

$$\sigma^2 := \left\| \sum_{1 \leq i < j \leq n} (4\sqrt{2} R G_{ij})^2 \right\| = 32R^2 \|(n-1)I\| = 32R^2(n-1),$$

notice that random matrices X_{ij} are independent, self-adjoint and have mean zero, apply Theorem ?? we have

$$\begin{aligned} P\left(\lambda_{\max}(\hat{P}^{(q)} - H^{(q)}) \geq t\right) &\leq n \exp\left(-\frac{t^2/2}{\sigma^2 + 4Rt}\right) \\ &\leq n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right). \end{aligned}$$

Now consider $Y_{ij} \equiv (H^{(q)} - \hat{P}^{(q)}) G_{ij}$, $1 \leq i < j \leq n$. Then we have $H^{(q)} - \hat{P}^{(q)} = \sum_{1 \leq i < j \leq n} Y_{ij}$. Since

$$E[(H^{(q)} - \hat{P}^{(q)})^k] = (-1)^k E[(\hat{P}^{(q)} - H^{(q)})^k],$$

1. When k is even,

$$E[Y_{ij}^k] = E[(\hat{P}^{(q)} - H^{(q)})^k] G_{ij}^2 \preceq 2^{2k} k! R^k G_{ij}^2;$$

2. When k is odd,

$$E[Y_{ij}^k] = -E[(\hat{P}^{(q)} - H^{(q)})^k] G_{ij}^2 \preceq 2^{2k} k! R^k G_{ij}^2.$$

Thus

$$\begin{aligned} P\left(\lambda_{\min}(\hat{P}^{(q)} - H^{(q)}) \leq -t\right) &= P\left(\lambda_{\max}(H^{(q)} - \hat{P}^{(q)}) \geq t\right) \\ &\leq n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right). \end{aligned}$$

Therefore we have

$$P\left(\|\hat{P}^{(q)} - H^{(q)}\| \geq t\right) \leq n \exp\left(-\frac{t^2/2}{32R^2n + Rt}\right).$$

Now let $c > 0$ be given and assume $n^{-c} \leq \eta \leq 1/2$. Then there exists a $n_0(c)$ independent of n , P , C and ϵ such that whenever $n > n_0(c)$,

$$t = 8R\sqrt{2n \ln(n/\eta)} \leq 32Rn.$$

Plugging this t into the equation above, we get

$$P(\|\hat{P}^{(q)} - H^{(q)}\| \geq 8R\sqrt{2n \ln(n/\eta)}) \leq n \exp\left(-\frac{t^2}{64R^2n}\right) = \eta.$$

■

From now on, we will use \hat{P} to represent $\hat{P}^{(q)}$, use H to represent $H^{(q)}$. Assume $H = USU^T = ZZ^T$, where Z is a n -by- k matrix. Then our estimate for Z up to rotation is $\hat{Z} = \hat{U} \hat{S}^{1/2}$, where $\hat{U} \hat{S} \hat{U}^T$ is the rank- k spectral decomposition of $|\hat{P}| = (\hat{P}^T \hat{P})^{1/2}$.

Lemma 4.18 $\|S\|_2 = \Theta(n)$, $\|\hat{S}\|_2 = \Theta(n)$.

Lemma 4.19 Let $W_1 \Sigma W_2^T$ be the singular value decomposition of $U^T \hat{U}$. Then for sufficiently large n ,

$$\|U^T \hat{U} - W_1 W_2^T\|_F = O(n^{-1} \log n)$$

with high probability.

Proof: Let $\sigma_1, \dots, \sigma_k$ denote the singular values of $U^T \hat{U}$. Then $\sigma_i = \cos(\theta_i)$ where the θ_i are the principal angles between the subspaces spanned by \hat{U} and U . Furthermore, by the Davis-Kahan $\sin(\Theta)$ theorem, combined with Theorem ?? and Lemma ??,

$$\|\hat{U} \hat{U}^T - U U^T\|_2 = \max_i |\sin(\theta_i)| \leq \frac{\|\hat{P} - H\|_2}{\lambda_k(H)} \leq \frac{C \sqrt{n \log n}}{n} = O(n^{-1/2} \sqrt{\log n}) \quad (6)$$

for sufficient large n . Here $\lambda_k(H)$ denotes the k -th largest eigenvalue of H .

We thus have

$$\begin{aligned} \|U^T \hat{U} - W_1 W_2^T\|_F &= \|\Sigma - I\|_F = \sqrt{\sum_{i=1}^k (1 - \sigma_i)^2} \\ &\leq \sum_{i=1}^k (1 - \sigma_i) \leq \sum_{i=1}^k (1 - \sigma_i^2) \\ &= \sum_{i=1}^k \sin^2(\theta_i) \leq k \|\hat{U} \hat{U}^T - U U^T\|_2 \\ &= O(n^{-1} \log n). \end{aligned}$$

■

We will denote the orthogonal matrix $W_1 W_2^T$ by W^* .

Lemma 4.20 For sufficiently large n ,

$$\|W^* \hat{S} - S W^*\|_F = O(\log n),$$

$$\|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F = O(n^{-1/2} \log n)$$

and

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(n^{-3/2} \log n)$$

with high probability.

Proof: Let $Q = \hat{U} - U U^T \hat{U}$. And Q is the residual after projecting \hat{U} orthogonally onto the column space of U , we have

$$\|\hat{U} - U U^T \hat{U}\|_F \leq \|\hat{U} - U T\|_F$$

for all $k \times k$ matrices T . Also, by Equation (??) we have

$$\|Q\|_F = \|\hat{U} - U U^T \hat{U}\|_F = \|\hat{U} \hat{U}^T - U U^T\|_F = O(n^{-1/2} \sqrt{\log n}). \quad (7)$$

Then

$$\begin{aligned}
W^* \hat{S} &= (W^* - U^T \hat{U}) \hat{S} + U^T \hat{U} \hat{S} = (W^* - U^T \hat{U}) \hat{S} + U^T \hat{P} \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) \hat{U} + U^T H \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + U^T H \hat{U} \\
&= (W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S U^T \hat{U}.
\end{aligned}$$

Combined with Theorem ??, Lemma ??, Lemma ??, we have

$$\begin{aligned}
&\|W^* \hat{S} - S W^*\|_F \\
&= \|(W^* - U^T \hat{U}) \hat{S} + U^T (\hat{P} - H) Q + U^T (\hat{P} - H) U U^T \hat{U} + S(U^T \hat{U} - W^*)\|_F \\
&\leq \|W^* - U^T \hat{U}\|_F (\|\hat{S}\|_2 + \|S\|_2) + \|U^T\|_F \|\hat{P} - H\|_2 \|Q\|_F + \|U^T (\hat{P} - H) U\|_F \\
&\leq O(\log n) + O(\log n) + \|U^T (\hat{P} - H) U\|_F
\end{aligned}$$

with high probability. And we know $U^T (\hat{P} - H) U$ is a $k \times k$ matrix with ij -th entry to be

$$u_i^T (\hat{P} - H) u_j = \sum_{s=1}^n \sum_{t=1}^n (\hat{P}_{st} - H_{st}) u_{is} u_{jt} = 2 \sum_{s < t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt} - \sum_{s=1}^n H_{ss} u_{is} u_{js}$$

where u_i and u_j are the i -th and j -th columns of U . Thus, conditioned on H , $u_i^T (\hat{P} - H) u_j$ is a sum of independent mean 0 random variables and a term of order $O(1)$.

By Equation ??, we have

$$\begin{aligned}
&E \left[\left((\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right)^k \right] \\
&\leq 2^k k! (C_{st} + H_{st}^{(1)})^k u_{is}^k u_{jt}^k \\
&\leq \frac{k!}{2} (4R)^{k-2} (4\sqrt{2}R u_{is} u_{jt})^2.
\end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{s < t} 32R^2 u_{is}^2 u_{jt}^2 \right| \leq 32R^2,$$

then by Theorem ??, we have

$$P \left(\left| 2 \sum_{s < t} (\hat{P}_{st} - H_{st}) u_{is} u_{jt} \right| \geq t \right) \leq \exp \left(\frac{-t^2/8}{32R^2 + 2Rt} \right),$$

thus each entry of $U^T (\hat{P} - H) U$ is of order $O(1)$ with high probability and thus

$$\|U^T (\hat{P} - H) U\|_F = O(1) \quad (8)$$

with high probability. Hence

$$\|W^* \hat{S} - S W^*\|_F = O(\log n)$$

with high probability. Also, since

$$W_{ij}^* (\lambda_j^{1/2}(\hat{P}) - \lambda_i^{1/2}(H)) = W_{ij}^* \frac{\lambda_j(\hat{P}) - \lambda_i(H)}{\lambda_j^{1/2}(\hat{P}) + \lambda_i^{1/2}(H)}$$

RT: Does it make sense here to be $O(1)$?

and the eigenvalues $\lambda_j^{1/2}(\hat{P})$ and $\lambda_i^{1/2}(H)$ are both of order $\Theta(\sqrt{n})$, we have

$$\|W^* \hat{S}^{1/2} - S^{1/2} W^*\|_F = O(n^{-1/2} \log n).$$

Similarly, since

$$W_{ij}^* (\lambda_j^{-1/2}(\hat{P}) - \lambda_i^{-1/2}(H)) = W_{ij}^* \frac{\lambda_i(H) - \lambda_j(\hat{P})}{(\lambda_j^{-1/2}(\hat{P}) + \lambda_i^{-1/2}(H)) \lambda_j(\hat{P}) \lambda_i(H)}$$

and the eigenvalues $\lambda_j(\hat{P})$ and $\lambda_i(H)$ are both of order $\Theta(n)$, we have

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(n^{-3/2} \log n).$$

■

Lemma 4.21 *There exists a rotation matrix W such that for sufficiently large n ,*

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: Let $Q_1 = UU^T \hat{U} - UW^*$, $Q_2 = W^* \hat{S}^{1/2} - S^{1/2} W^*$ and $Q_3 = \hat{U} - UW^* = \hat{U} - UU^T \hat{U} + Q_1 = Q + Q_1$. Then since $UU^T P = P$ and $\hat{U} \hat{S}^{1/2} = \hat{P} \hat{U} \hat{S}^{-1/2}$,

$$\begin{aligned} \hat{Z} - US^{1/2} W^* &= \hat{U} \hat{S}^{1/2} - UW^* \hat{S}^{1/2} + U(W^* \hat{S}^{1/2} - S^{1/2} W^*) \\ &= (\hat{U} - UU^T \hat{U}) \hat{S}^{1/2} + Q_1 \hat{S}^{1/2} + UQ_2 \\ &= (\hat{P} - H) \hat{U} \hat{S}^{-1/2} - UU^T (\hat{P} - H) \hat{U} \hat{S}^{-1/2} + Q_1 \hat{S}^{1/2} + UQ_2 \\ &= (\hat{P} - H) UW^* \hat{S}^{-1/2} - UU^T (\hat{P} - H) UW^* \hat{S}^{-1/2} \\ &\quad + (I - UU^T) (\hat{P} - H) Q_3 \hat{S}^{-1/2} + Q_1 \hat{S}^{1/2} + UQ_2. \end{aligned}$$

By Lemma ??,

$$\|Q_1\|_F \leq \|U\|_F \|U^T \hat{U} - W^*\|_F = O(n^{-1} \log n).$$

By Lemma ??,

$$\|Q_2\|_F = O(n^{-1/2} \log n).$$

By Equation ??,

$$\|Q_3\|_F \leq \|Q\|_F + \|Q_1\|_F = O(n^{-1/2} \log n).$$

By Equation ??,

$$\|UU^T (\hat{P} - H) UW^* \hat{S}^{-1/2}\|_F \leq \|U^T (\hat{P} - H) U\|_F \|\hat{S}^{-1/2}\|_2 = O(n^{-1/2}).$$

By Lemma ??,

$$\|W^* \hat{S}^{-1/2} - S^{-1/2} W^*\|_F = O(n^{-3/2} \log n).$$

Therefore,

$$\begin{aligned}
& \|\hat{Z} - US^{1/2}W^*\|_F \\
&= \|(\hat{P} - H)UW^*\hat{S}^{-1/2}\|_F + O(n^{-1/2}) + \|I - UU^T\|_2 \|\hat{P} - H\|_2 O(n^{-1} \log n) \\
&\quad + O(n^{-1/2} \log n) + O(n^{-1/2} \log n) \\
&= \|(\hat{P} - H)UW^*\hat{S}^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2}) \\
&\leq \|(\hat{P} - H)US^{-1/2}W^*\|_F + \|(\hat{P} - H)U(W^*\hat{S}^{-1/2} - S^{-1/2}W^*)\|_F + O(n^{-1/2}(\log n)^{3/2}) \\
&= \|(\hat{P} - H)US^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2}).
\end{aligned}$$

Note that $Z = US^{1/2}W$ for some orthogonal matrix W . As W^* is also orthogonal, therefore $Z\tilde{W} = US^{1/2}W^*$ for some orthogonal \tilde{W} , which completes the proof. \blacksquare

Theorem 4.22 *There exists a rotation matrix W such that for sufficiently large n ,*

$$\max_i \|\hat{Z}_i - WZ_i\|_2 = O(n^{-1/2}(\log n)^{3/2})$$

with high probability.

Proof: By Lemma ??, we have

$$\|\hat{Z} - ZW\|_F = \|(\hat{P} - H)US^{-1/2}\|_F + O(n^{-1/2}(\log n)^{3/2})$$

and similarly we could have the bound for each row vector

$$\begin{aligned}
\max_i \|\hat{Z}_i - WZ_i\|_2 &\leq \frac{1}{\lambda_d^{1/2}(H)} \max_i \|((\hat{P} - H)U)_i\|_2 + O(n^{-1/2}(\log n)^{3/2}) \\
&\leq \frac{d^{1/2}}{\lambda_d^{1/2}(H)} \max_j \|(\hat{P} - H)u_j\|_\infty + O(n^{-1/2}(\log n)^{3/2})
\end{aligned}$$

where u_j denotes the j -th column of U . Now given i and j , the i -th element of the vector $(\hat{P} - H)u_j$ is of the form

$$\sum_{s=1}^n (\hat{P}_{is} - H_{is})u_{js} = \sum_{s \neq i} (\hat{P}_{is} - H_{is})u_{js} - H_{ii}u_{ji}.$$

Thus, conditioned on H , the i -th element of the vector $(\hat{P} - H)u_j$ is a sum of independent mean 0 random variables and a term of order $O(1)$. By Equation ??, we have

$$\begin{aligned}
& E \left[\left((\hat{P}_{is} - H_{is})u_{js} \right)^k \right] \\
&\leq 2^k k! (C_{is} + H_{is}^{(1)})^k u_{js}^k \\
&\leq \frac{k!}{2} (4R)^{k-2} (4\sqrt{2}Ru_{js})^2.
\end{aligned}$$

Also we have

$$\sigma^2 := \left| \sum_{s \neq i} 32R^2 u_{js}^2 \right| \leq 32R^2,$$

then by Theorem ??, we have

$$P \left(\left| \sum_{s \neq i} (\hat{P}_{is} - H_{is}) u_{js} \right| \geq t \right) \leq \exp \left(\frac{-t^2/2}{32R^2 + Rt} \right),$$

i.e. it can be bounded by a constant with high probability. Taking the union bound over all i and j , we have

$$\max_i \|\hat{Z}_i - W Z_i\|_2 \leq \frac{C d^{1/2}}{\lambda_d^{1/2}(H)} \log n + O(n^{-1/2}(\log n)^{3/2}) = O(n^{-1/2}(\log n)^{3/2}).$$

■

RT: $\log n$ in the first term?

Lemma 4.23 $\left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| = O(n^{-1/2}(\log n)^{3/2})$ with high probability.

Proof: Let W be the rotation matrix in Theorem ??, then

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= \left| \hat{Z}_i^T \hat{Z}_j - \hat{Z}_i^T W Z_j + \hat{Z}_i^T W Z_j - (W Z_i)^T W Z_j \right| \\ &\leq \left| \hat{Z}_i^T (\hat{Z}_j - W Z_j) + (\hat{Z}_i^T - (W Z_i)^T) W Z_j \right| \\ &\leq \|\hat{Z}_i\|_2 \|\hat{Z}_j - W Z_j\|_2 + \|Z_j\|_2 \|\hat{Z}_i^T - (W Z_i)^T\|_2. \end{aligned}$$

Since $\|Z_i\|_2^2 = Z_i^T Z_i = H_{ii}^q = E[\hat{P}_{ii}^{(q)}] \leq E[\hat{P}_{ii}^{(1)}] = (1 - \epsilon)P_{ij} + \epsilon C_{ij} \leq R$, we have $\|Z_i\|_2 = O(1)$. Combined with Theorem ??,

$$\begin{aligned} \left| \hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j \right| &= (\|\hat{Z}_i\|_2 + \|Z_j\|_2) O(n^{-1/2}(\log n)^{3/2}) \\ &\leq (\|\hat{Z}_i - W Z_i\|_2 + \|W Z_i\|_2 + \|Z_j\|_2) O(n^{-1/2}(\log n)^{3/2}) \\ &= O(n^{-1/2}(\log n)^{3/2}) \end{aligned}$$

with high probability. ■

Corollary 4.24 For fixed m , the estimator based on ASE of MLqE has the same entry-wise asymptotic bias as MLqE, i.e.

$$\lim_{n \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(q)}) = \lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(q)}] - P_{ij} = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(q)}] - P_{ij} = \lim_{n \rightarrow \infty} \text{Bias}(\hat{P}_{ij}^{(q)}).$$

Proof: Direct result from Lemma ?? by noticing

$$\lim_{n \rightarrow \infty} E[\tilde{P}_{ij}^{(q)}] = \lim_{n \rightarrow \infty} E[\hat{P}_{ij}^{(q)}].$$

■

Theorem 4.25 $\text{Var}(\hat{Z}_i^T \hat{Z}_j) = O(n^{-1}(\log n)^3)$ with high probability.

Proof:

$$\begin{aligned}
\text{Var}(\hat{Z}_i^T \hat{Z}_j) &= E[(\hat{Z}_i^T \hat{Z}_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&= E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j + Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&= E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&\quad - 2E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])] \\
&\leq E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] + E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2] \\
&\quad + 2\sqrt{E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2]E[(Z_i^T Z_j - E[\hat{Z}_i^T \hat{Z}_j])^2]} \\
&\leq 4E[(\hat{Z}_i^T \hat{Z}_j - Z_i^T Z_j)^2] \\
&= O(n^{-1}(\log n)^3)
\end{aligned}$$

with high probability. ■

Theorem 4.26 Let $u_q(\theta) = E_\theta[\hat{\theta}_{q,n}]$, $\phi_q(x; \theta) = \frac{\partial}{\partial \theta} L_q(f(x; \theta))$, and $\phi'_q(x; \theta) = \frac{\partial^2}{\partial \theta^2} L_q(f(x; \theta))$. Then the asymptotic distribution of $\hat{\theta}_{q,n}$ is $\sqrt{n}(\hat{\theta}_{q,n} - u_q(\theta)) \sim \mathcal{N}(0, V_q(\theta))$, where $V_q(\theta) = E[\phi_q(X; \theta)^2]/E[\phi'_q(X; \theta)]^2$.

Remark: See Theorem 1 in <http://arxiv.org/pdf/1310.7278.pdf>.

Corollary 4.27 $\text{Var}(\hat{P}_{ij}^{(q)}) = \Theta(m^{-1})$.

Proof: Direct result from Theorem ?? ■

Theorem 4.28 For fixed m , $1 \leq i, j \leq n$,

$$\frac{\text{Var}(\tilde{P}_{ij}^{(q)})}{\text{Var}(\hat{P}_{ij}^{(q)})} = O(mn^{-1}(\log n)^3).$$

Thus

$$\text{ARE}(\hat{P}_{ij}^{(q)}, \tilde{P}_{ij}^{(q)}) = 0.$$

Furthermore, as long as m is of order $o(n(\log n)^{-3})$,

$$\text{ARE}(\hat{P}_{ij}^{(q)}, \tilde{P}_{ij}^{(q)}) = 0.$$

Proof: The results are direct from Theorem ?? and Corollary ?? ■

4.4 $\tilde{P}^{(q)}$ better than $\tilde{P}^{(1)}$

Theorem 4.29 For sufficiently large n and C , any $1 \leq i, j \leq n$,

$$\lim_{m \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(1)}) > \lim_{m \rightarrow \infty} \text{Bias}(\tilde{P}_{ij}^{(q)})$$

Proof: Direct result from Lemma ??, Corollary ?? and Corollary ?? ■

Theorem 4.30 For any fixed m , any $1 \leq i, j \leq n$,

$$\lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(1)}) = \lim_{n \rightarrow \infty} \text{Var}(\tilde{P}_{ij}^{(q)}) = 0$$

Proof: Direct result from Theorem ?? and Theorem ?? ■

5 Generalization

We can generalize the exponential distribution to F and generalize MLqE to \hat{P} with the following assumptions:

- There exists $C_0(P_{ij}, \epsilon) > 0$ such that under the contaminated model with $C > C_0(P_{ij}, \epsilon)$,

$$\lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}] - P_{ij} \right| < \lim_{m \rightarrow \infty} \left| E[\hat{P}_{ij}^{(1)}] - P_{ij} \right|,$$

for $1 \leq i, j \leq n$.

- Let $A^{(t)} \stackrel{iid}{\sim} (1 - \epsilon)F(P) + \epsilon F(C)$ and $H_{ij}^{(1)} = E[\hat{P}_{ij}^{(1)}] = (1 - \epsilon)E_F(P_{ij}) + \epsilon E_F(C_{ij})$, then $E[(A_{ij}^{(t)} - H_{ij}^{(1)})^k] \leq \text{const} \cdot k!$.
- $\hat{P}_{ij} \leq \text{const} \cdot \hat{P}_{ij}^{(1)}$. This might be generalized to with high probability later.
- $\lim_{m \rightarrow \infty} \text{Var}(\hat{P}_{ij}) = 0$

6 Analysis of ML q E under Exponential Model

Let $f(x, \theta) = \sum_{i=1}^m e^{-\frac{(1-q)x_i}{\theta}} (x - \theta)$, then $\frac{\partial f}{\partial x_i}(x, \theta) = e^{-\frac{(1-q)x_i}{\theta}} \left(-\frac{(1-q)x_i}{\theta} + 2 - q \right)$.

Define $c_q = 1 + \frac{1}{1-q}$, then

$$\frac{\partial f}{\partial x_i}(x, \theta) = 0 \Leftrightarrow x_i \rightarrow \infty \text{ or } x_i = \left(1 + \frac{1}{1-q}\right)\theta$$

Define $x = (x_1, \dots, x_m)$ and $x^- = (x_1, \dots, x_m) \setminus x_i$, then

1. $\theta_{\max}, x_i = c_q \theta \leq c_q \bar{x}$
 $\frac{1}{m} (c_q \bar{x} + (m-1)\bar{x}^-) = \bar{x}$
Thus $\theta_{\max} \leq \frac{m-1}{m-C_q} \bar{x}^-$
2. θ_{\min}
2 candidates: $x_i = 0$ or $x_i = \infty$.
When $x_i = 0$, It's like $m-1$ points x^- ;
When $x_i = \infty$, $f(x, \theta) = f(x^-, \theta) - \theta$.
Thus θ_{\min} solves $f(x, \theta)|_{x_i=0} = 0$.