Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \tag{1.1}$$

1.1. Vectors

1.2. Median

1.3. Altitude

1.4. Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2}\right)(\mathbf{B} - \mathbf{C}) = 0 \tag{1.4.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.

Solution:

(a) **BC**: given equation for the perpendicular bisector of **BC**:

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.4.1.2}$$

On substituting the values,

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix} \tag{1.4.1.3}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.4.1.4}$$

$$\tag{1.4.1.5}$$

solving using matrix multiplication

$$(\mathbf{B}-\mathbf{C})^\top \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = 0 \tag{1.4.1.6}$$

$$(\mathbf{B}-\mathbf{C})^\top = \begin{pmatrix} 3 & -2 \end{pmatrix} \tag{1.4.1.7}$$

$$(\mathbf{B}-\mathbf{C})^\top \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix} 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix} \tag{1.4.1.8}$$

$$= \frac{3}{2} \tag{1.4.1.9}$$

Therefore perpendicular bisector of $\mathbf{BC}$ is

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = \frac{3}{2} \tag{1.4.1.10}$$

(b) $\mathbf{AB}$: similarly the equation for the perpendicular bisector of $\mathbf{AB}$:

$$\left(\mathbf{x} - \frac{\mathbf{A}+\mathbf{B}}{2}\right)(\mathbf{A}-\mathbf{B}) = 0 \tag{1.4.1.11}$$

On substituting the values,

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \begin{pmatrix} \frac{1}{2} \\ \\ -1 \end{pmatrix} \tag{1.4.1.12}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -7 \\ \\ -2 \end{pmatrix} \tag{1.4.1.13}$$

$$\tag{1.4.1.14}$$

solving using matrix multiplication

$$(\mathbf{A}-\mathbf{B})^{\top}\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = 0 \tag{1.4.1.15}$$

$$(\mathbf{A}-\mathbf{B})^{\top} = \begin{pmatrix} -7 & -2 \end{pmatrix} \tag{1.4.1.16}$$

$$(\mathbf{A}-\mathbf{B})^{\top}\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix} -7 & -2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \\ -1 \end{pmatrix} \tag{1.4.1.17}$$

$$= \frac{-11}{2} \tag{1.4.1.18}$$

Therefore perpendicular bisector of **AB** is

$$\begin{pmatrix} -7 & -2 \end{pmatrix} \mathbf{x} = \frac{-11}{2} \tag{1.4.1.19}$$

(c) **CA**: similarly the equation for the perpendicular bisector of **CA**:

$$\left(\mathbf{x} - \frac{\mathbf{C}+\mathbf{A}}{2}\right)(\mathbf{C}-\mathbf{A}) = 0 \tag{1.4.1.20}$$

On substituting the values,

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix} -1 \\ \\ 2 \end{pmatrix} \tag{1.4.1.21}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix} 4 \\ \\ 4 \end{pmatrix} \tag{1.4.1.22}$$

$$\tag{1.4.1.23}$$

solving using matrix multiplication

$$(\mathbf{C}-\mathbf{A})^\top \left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = 0 \tag{1.4.1.24}$$

$$(\mathbf{C}-\mathbf{A})^\top = \begin{pmatrix} -1 & 2 \end{pmatrix} \tag{1.4.1.25}$$

$$(\mathbf{C}-\mathbf{A})^\top \left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ \\ 4 \end{pmatrix} \tag{1.4.1.26}$$

$$= 4 \tag{1.4.1.27}$$

Therefore perpendicular bisector of **BC** is

$$\begin{pmatrix} 4 & 4 \end{pmatrix} \mathbf{x} = 4 \tag{1.4.1.28}$$

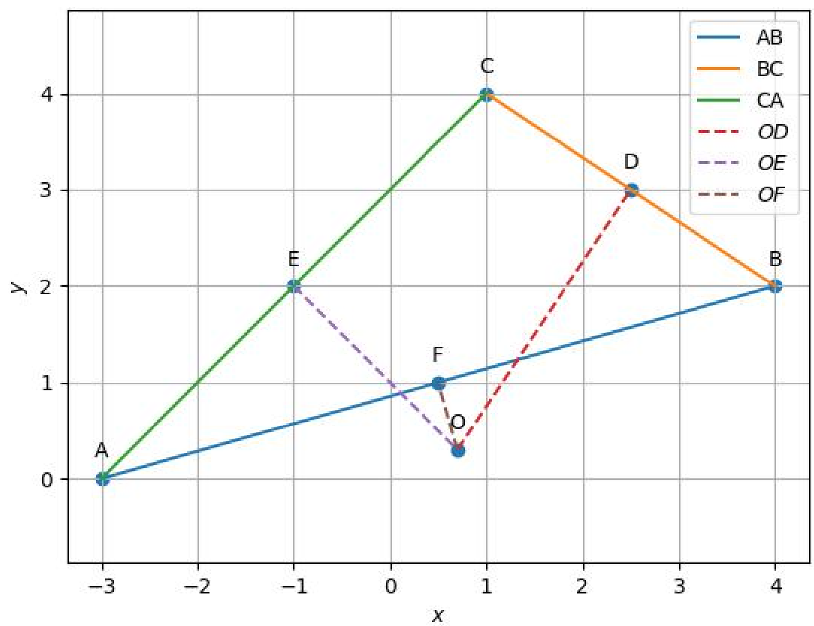

Figure 1.1: Plot of the perpendicular bisectors

1.4.2. Find the intersection **O** of the perpendicular bisectors of AB and AC.

Solution:

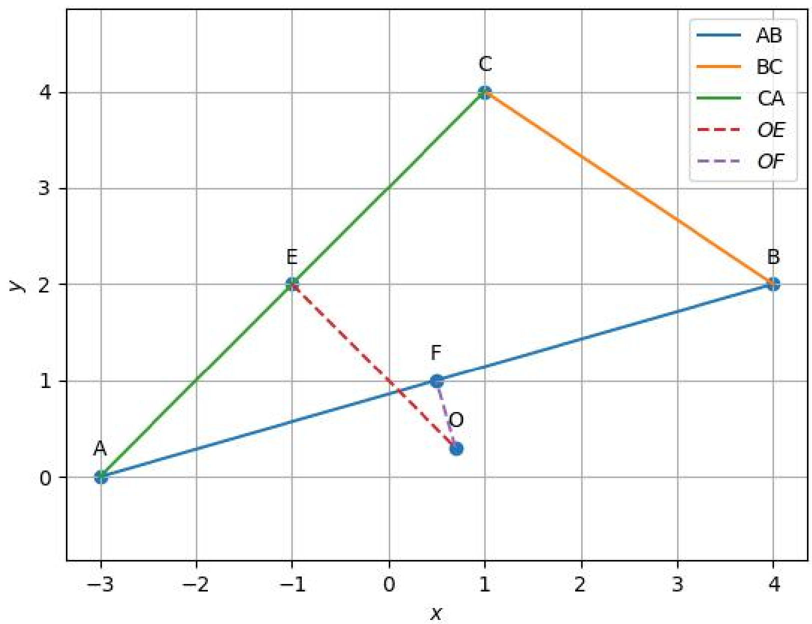

Figure 1.2: $\mathbf{O}-\mathbf{E}$ and $\mathbf{O}-\mathbf{F}$ are perpendicular bisectors of $\mathbf{A}-\mathbf{C}$ and $\mathbf{A}-\mathbf{B}$ respectively

Given,

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \tag{1.4.2.1}$$

Vector equation of perpendicular bisector of $\mathbf{A}-\mathbf{B}$ is

$$(\mathbf{A}-\mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) = 0 \tag{1.4.2.2}$$

where,

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{1.4.2.3}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.4.2.4}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{1.4.2.5}$$

$$= \begin{pmatrix} -7 \\ -2 \end{pmatrix} \tag{1.4.2.6}$$

$$\implies (\mathbf{A}-\mathbf{B})^\top = \begin{pmatrix} -7 & -2 \end{pmatrix} \tag{1.4.2.7}$$

$\therefore$ The vector equation of $\mathbf{O}-\mathbf{F}$ is

$$\begin{pmatrix} -6 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ -5 \end{pmatrix} \right) = 0 \tag{1.4.2.8}$$

$$\implies \begin{pmatrix} -6 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.4.2.9}$$

Performing matrix multiplication yields

$$\begin{pmatrix} -6 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1.4.2.10}$$

Vector equation of perpendicular bisector of $\mathbf{A}-\mathbf{C}$ is

$$(\mathbf{A}-\mathbf{C})^\top \left(\mathbf{x}-\frac{\mathbf{A}+\mathbf{C}}{2}\right)=0 \tag{1.4.2.11}$$

where,

$$\mathbf{A}+\mathbf{C}=\begin{pmatrix}-3\\-5\end{pmatrix}+\begin{pmatrix}-4\\-3\end{pmatrix} \tag{1.4.2.12}$$

$$=\begin{pmatrix}-7\\-8\end{pmatrix} \tag{1.4.2.13}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-3\\-5\end{pmatrix}-\begin{pmatrix}-4\\-3\end{pmatrix} \tag{1.4.2.14}$$

$$=\begin{pmatrix}1\\-2\end{pmatrix} \tag{1.4.2.15}$$

$$\implies (\mathbf{A}-\mathbf{C})^\top=\begin{pmatrix}1 & -2\end{pmatrix} \tag{1.4.2.16}$$

$\therefore$ The vector equation of $\mathbf{O}-\mathbf{E}$ is

$$\begin{pmatrix}1 & -2\end{pmatrix}\left(\mathbf{x}-\frac{1}{2}\begin{pmatrix}-7\\-8\end{pmatrix}\right)=0 \tag{1.4.2.17}$$

$$\implies \begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x}=\frac{1}{2}\begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}-7\\-8\end{pmatrix} \tag{1.4.2.18}$$

Performing matrix multiplication yields

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = \frac{9}{2} \tag{1.4.2.19}$$

Thus,

$$\begin{pmatrix} -6 & 0 & 0 \\ 1 & -2 & \frac{9}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{-1}{6} R_1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & \frac{9}{2} \end{pmatrix} \tag{1.4.2.20}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & \frac{9}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & \frac{-9}{2} \end{pmatrix} \tag{1.4.2.21}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & \frac{-9}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{-1}{2} R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-9}{4} \end{pmatrix} \tag{1.4.2.22}$$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} \tag{1.4.2.23}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix} \tag{1.4.2.24}$$

Therefore, the point of intersection of perpendicular bisectors of $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}-\mathbf{C}$ is $\mathbf{O} = \begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix}$

1.4.3. Verify that $\mathbf{O}$ satisfies (1.4.1.1). $\mathbf{O}$ is known as the circumcentre.

Solution: From the previous question we get,

$$\mathbf{O} = \begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix} \tag{1.4.3.1}$$

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.4.3.2}$$

when substituted in the above equation,

$$= \left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right).(\mathbf{B}-\mathbf{C}) \tag{1.4.3.3}$$

$$= \left(\begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix}\right)^{\top} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.4.3.4}$$

$$= \begin{pmatrix} \frac{-18}{10} & \frac{-27}{10} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.4.3.5}$$

$$= 0 \tag{1.4.3.6}$$

It is hence proved that **O** satisfies the equation (1.4.1.1)

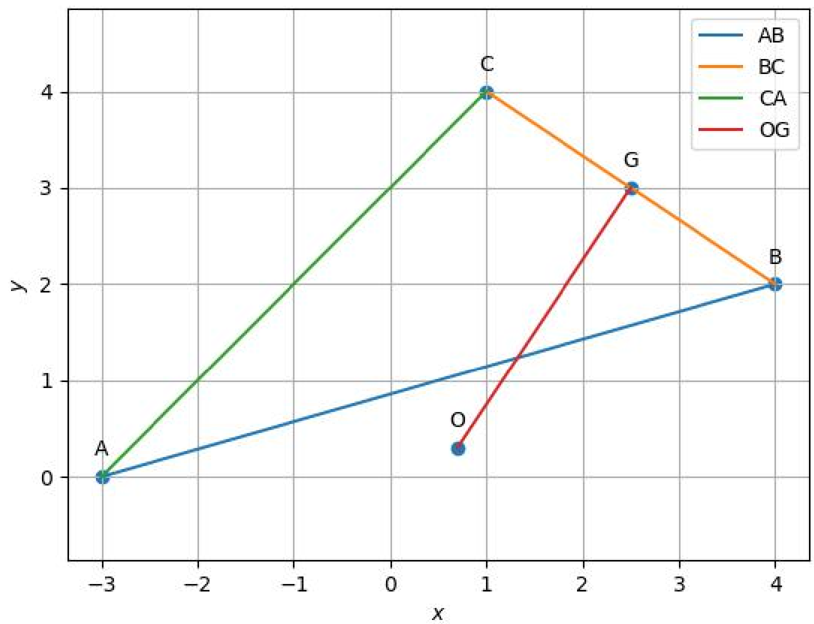

Figure 1.3: Circumcenter plotted using python

1.4.4. Verify that

$$OA = OB = OC \tag{1.4.4.1}$$

Solution: Given

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.4.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{1.4.4.3}$$

$$\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.4.4.4}$$

From problem-1.4.2 :

$$O = \begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix} \tag{1.4.4.5}$$

$$= \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \tag{1.4.4.6}$$

(a)

$$OA = \sqrt{(\mathbf{O}-\mathbf{A})^\top(\mathbf{O}-\mathbf{A})} \tag{1.4.4.7}$$

$$= \sqrt{\begin{pmatrix} 3.7 & -0.3 \end{pmatrix}\begin{pmatrix} 3.7 \\ -0.3 \end{pmatrix}} \tag{1.4.4.8}$$

$$= \sqrt{13.78} \tag{1.4.4.9}$$

$$\tag{1.4.4.10}$$

(b)

$$OB = \sqrt{(\mathbf{O}-\mathbf{B})^\top(\mathbf{O}-\mathbf{B})} \tag{1.4.4.11}$$

$$= \sqrt{\begin{pmatrix} -3.3 & -1.7 \end{pmatrix} \begin{pmatrix} -3.3 \\ -1.7 \end{pmatrix}} \tag{1.4.4.12}$$

$$= \sqrt{13.78} \tag{1.4.4.13}$$

$$\tag{1.4.4.14}$$

(c)

$$OC = \sqrt{(\mathbf{O}-\mathbf{C})^\top(\mathbf{O}-\mathbf{C})} \tag{1.4.4.15}$$

$$= \sqrt{\begin{pmatrix} -0.3 & -3.7 \end{pmatrix} \begin{pmatrix} -0.3 \\ -3.7 \end{pmatrix}} \tag{1.4.4.16}$$

$$= \sqrt{13.78} \tag{1.4.4.17}$$

$$\tag{1.4.4.18}$$

From above,

$$OA = OB = OC \tag{1.4.4.19}$$

Hence verified.

1.4.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \tag{1.4.5.1}$$

This is known as the circumradius.

Solution: Given

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.4.5.2}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{1.4.5.3}$$

$$\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.4.5.4}$$

From Q1.4.2, the circumcentre is

$$\mathbf{O} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \tag{1.4.5.5}$$

Now we will calculate the radius,

$$R = OA \tag{1.4.5.6}$$

$$= \|\mathbf{A} - \mathbf{O}\| \tag{1.4.5.7}$$

$$= \left\| \begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \right\| \tag{1.4.5.8}$$

$$= 3.712 \tag{1.4.5.9}$$

see Fig. 1.4

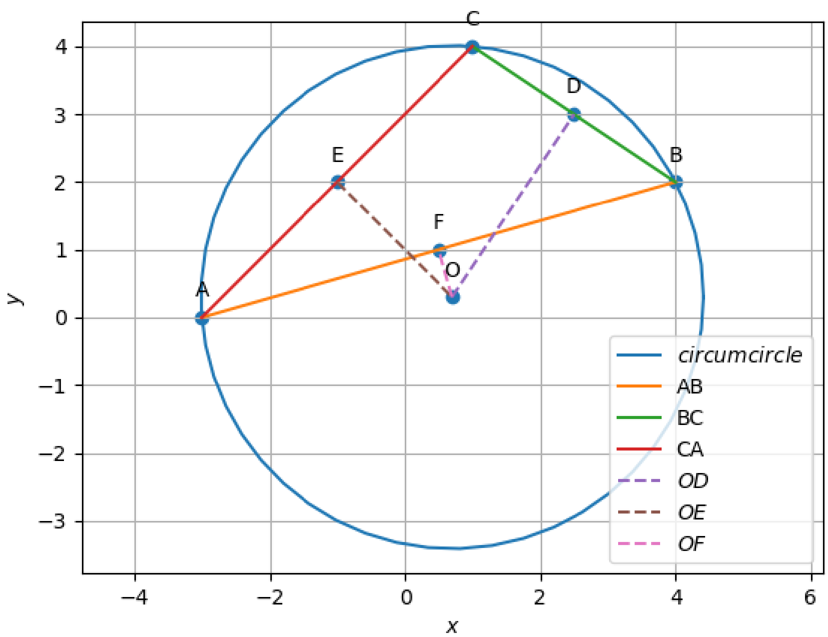

Figure 1.4: circumcircle of Triangle ABC with centre O

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \tag{1.4.6.1}$$

Solution:

(a) To find the value of $\angle BOC$:

$$\mathbf{B} - \mathbf{O} = \begin{pmatrix} \frac{33}{10} \\ \frac{17}{10} \end{pmatrix} \tag{1.4.6.2}$$

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{3}{10} \\ \frac{37}{10} \end{pmatrix} \tag{1.4.6.3}$$

$$\implies (\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O}) = \frac{-159}{16} \tag{1.4.6.4}$$

$$\implies \|\mathbf{B} - \mathbf{O}\| = \frac{\sqrt{265}}{4} \tag{1.4.6.5}$$

$$\|\mathbf{C} - \mathbf{O}\| = \frac{\sqrt{265}}{4} \tag{1.4.6.6}$$

Thus,

$$\cos BOC = \frac{(\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O})}{\|\mathbf{B} - \mathbf{O}\| \, \|\mathbf{C} - \mathbf{O}\|} = \frac{-159}{265} \tag{1.4.6.7}$$

$$\implies \angle BOC = \cos^{-1}\left(\frac{-159}{265}\right) \tag{1.4.6.8}$$

$$= 116.5650^\circ \tag{1.4.6.9}$$

Taking the reflex of above angle we get

$$\angle BOC = 360^\circ - 116.5650^\circ \tag{1.4.6.10}$$

$$= 233.130^\circ \tag{1.4.6.11}$$

(b) To find the value of $\angle BAC$:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \tag{1.4.6.12}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.4.6.13}$$

$$\implies (\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = -6 \tag{1.4.6.14}$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{36} = 6 \, \|\mathbf{C} - \mathbf{A}\| = \sqrt{5} \tag{1.4.6.15}$$

Thus,

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\|} = \frac{-1}{\sqrt{5}} \tag{1.4.6.16}$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-1}{\sqrt{5}}\right) \tag{1.4.6.17}$$

$$= 116.565^\circ \tag{1.4.6.18}$$

$$2 \times \angle BAC = 233.130 \tag{1.4.6.19}$$

From (1.4.6.18) and (**??**),

$$2 \times \angle BAC = \angle BOC \tag{1.4.6.20}$$

Hence Verified

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.4.7.1}$$

Find θ if

$$\mathbf{C} - \mathbf{O} = \mathbf{P}\left(\mathbf{A} - \mathbf{O}\right) \tag{1.4.7.2}$$

Solution:

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{3}{10} \\ \frac{37}{10} \end{pmatrix} \tag{1.4.7.3}$$

$$\mathbf{A} - \mathbf{O} = \begin{pmatrix} -3.7 \\ 0.3 \end{pmatrix} \tag{1.4.7.4}$$

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.4.7.5}$$

$$\mathbf{C} - \mathbf{O} = \mathbf{P}\left(\mathbf{A} - \mathbf{O}\right) \tag{1.4.7.6}$$

Now from (1.4.7.6)

$$\begin{pmatrix} -4 \\ \frac{-3}{4} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -3 \\ \frac{-11}{4} \end{pmatrix} \tag{1.4.7.7}$$

solving using matrix multiplication,we get

$$\begin{pmatrix} -4 \\ \frac{-3}{4} \end{pmatrix} = \begin{pmatrix} -3\cos\theta + \frac{11}{4}\sin\theta \\ -3\sin\theta + \frac{-11}{4}\cos\theta \end{pmatrix} \tag{1.4.7.8}$$

Comparing on Both sides ,we get

$$-3\cos\theta + \frac{11}{4}\sin\theta = -4 \tag{1.4.7.9}$$

$$-3\sin\theta + \frac{-11}{4}\cos\theta = \frac{-3}{4} \tag{1.4.7.10}$$

On solving equations (1.4.7.9) and (1.4.7.10)

$$\cos\theta = \frac{42}{51} \tag{1.4.7.11}$$

$$\sin\theta = \frac{-28}{51} \tag{1.4.7.12}$$

$$\theta = \cos^{-1}\frac{42}{51} \tag{1.4.7.13}$$

$$= 34.5608 \tag{1.4.7.14}$$

$$\therefore \theta = 34.5608 \tag{1.4.7.15}$$