Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \tag{1.1}$$

1.1. Angle Bisector

1.1.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$AE_3 = AF_3 = m, BD_3 = BF_3 = n, CD_3 = CE_3 = p. \tag{1.1.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Question 1.1.2.

Solution: From Question 1.1.2

$$a = \sqrt{13} \tag{1.1.1.2}$$

$$b = \sqrt{32} \tag{1.1.1.3}$$

$$c = \sqrt{53} \tag{1.1.1.4}$$

From the given information,

$$a = m + n, \tag{1.1.1.5}$$

$$b = n + p, \tag{1.1.1.6}$$

$$c = m + p \tag{1.1.1.7}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.8}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.9}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.1.1.10}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.11}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.12}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.1.1.13}$$

Therefore,

$$\begin{aligned}
p &= \frac{c+b-a}{2} = \frac{\sqrt{53}+\sqrt{32}-\sqrt{13}}{2} \\
m &= \frac{a+c-b}{2} = \frac{\sqrt{13}+\sqrt{53}-\sqrt{32}}{2} \\
n &= \frac{a+b-c}{2} = \frac{\sqrt{13}+\sqrt{32}-\sqrt{53}}{2}
\end{aligned} \tag{1.1.1.14}$$

on solving above equations we get

$$p = 4.665706432 \tag{1.1.1.15}$$

$$m = 2.614403458 \tag{1.1.1.16}$$

$$n = 0.9911478178 \tag{1.1.1.17}$$

1.1.2. Using section formula, find $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$.

Solution: Given

$$\mathbf{D}_3 = \frac{m\mathbf{C}+n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A}+p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \tag{1.1.2.1}$$

Here

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \tag{1.1.2.2}$$

$$p = 4.6657,\ m = 2.6144,\ n = 0.9911 \tag{1.1.2.3}$$

On substituting (1.1.2.2) and (1.1.2.3) in (**??**) We get

$$\mathbf{D}_3 = \frac{2.6144\begin{pmatrix}1\\4\end{pmatrix}+0.9911\begin{pmatrix}4\\2\end{pmatrix}}{2.6144+0.9911} \tag{1.1.2.4}$$

$$\mathbf{E}_3 = \frac{0.9911\begin{pmatrix}-3\\0\end{pmatrix}+4.6657\begin{pmatrix}1\\4\end{pmatrix}}{0.9911+4.6657} \tag{1.1.2.5}$$

$$\mathbf{F}_3 = \frac{4.6657\begin{pmatrix}4\\2\end{pmatrix}+2.6144\begin{pmatrix}-3\\0\end{pmatrix}}{4.6657+2.6144} \tag{1.1.2.6}$$

On solving above equations We get

$$\mathbf{D}_3 = \begin{pmatrix}1.8246\\3.4502\end{pmatrix} \tag{1.1.2.7}$$

$$\mathbf{E}_3 = \begin{pmatrix}0.2991\\3.2991\end{pmatrix} \tag{1.1.2.8}$$

$$\mathbf{F}_3 = \begin{pmatrix}1.4861\\1.2817\end{pmatrix} \tag{1.1.2.9}$$

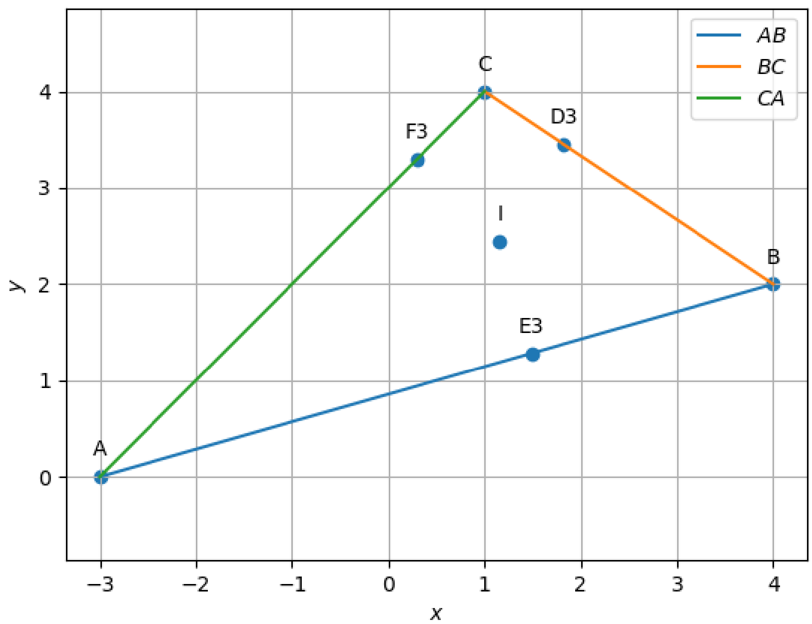

Figure 1.1: Points $D3, E3, F3$

1.1.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the incentre and inradius of $\triangle ABC$.

Solution: Given

$$\mathbf{D}_3 = \begin{pmatrix} 1.8246 \\ 3.4502 \end{pmatrix} \tag{1.1.3.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} 0.2991 \\ 3.2991 \end{pmatrix} \tag{1.1.3.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} 1.4861 \\ 1.2817 \end{pmatrix} \tag{1.1.3.3}$$

(a) For circumcentre

Vector equation of $\mathbf{D} - \mathbf{E}$ is

$$\left(\mathbf{D}_3 - \mathbf{E}_3\right)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{E}_3}{2}\right) = 0 \tag{1.1.3.4}$$

$$\left(\mathbf{D}_3 - \mathbf{F}_3\right)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{F}_3}{2}\right) = 0 \tag{1.1.3.5}$$

on Substituting the values of $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$ and solving We get,

$$\begin{pmatrix} 1.5255 & 0.1511 \end{pmatrix} \mathbf{x} = 2.1296 \tag{1.1.3.6}$$

$$\begin{pmatrix} 0.3385 & 2.1685 \end{pmatrix} \mathbf{x} = 5.6907 \tag{1.1.3.7}$$

Thus on solving (1.1.3.6) and (1.1.3.7) using gauss elimination We get

$$\begin{pmatrix} 1.5255 & 0.1511 & 2.1296 \\ 0.3385 & 2.1685 & 5.6907 \end{pmatrix} \tag{1.1.3.8}$$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1.1539 \\ 2.4441 \end{pmatrix} \tag{1.1.3.9}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1.1539 \\ 2.4441 \end{pmatrix} \tag{1.1.3.10}$$

(b) The circiumradius is obtained from $r = \|\mathbf{I} - \mathbf{D}_3\|$

$$\mathbf{I} = \begin{pmatrix} 1.1539 \\ \\ 2.4441 \end{pmatrix} \tag{1.1.3.11}$$

$$\mathbf{D}_3 = \begin{pmatrix} 1.8246 \\ \\ 3.4502 \end{pmatrix} \tag{1.1.3.12}$$

$$\mathbf{I} - \mathbf{D_3} = \begin{pmatrix} -0.6707 \\ \\ -1.0061 \end{pmatrix} \tag{1.1.3.13}$$

$$r = \|\mathbf{I} - \mathbf{D}_3\| = \sqrt{(\mathbf{I} - \mathbf{D}_3)^\top (\mathbf{I} - \mathbf{D}_3)} \tag{1.1.3.14}$$

$$r = 0.7423341566 \tag{1.1.3.15}$$

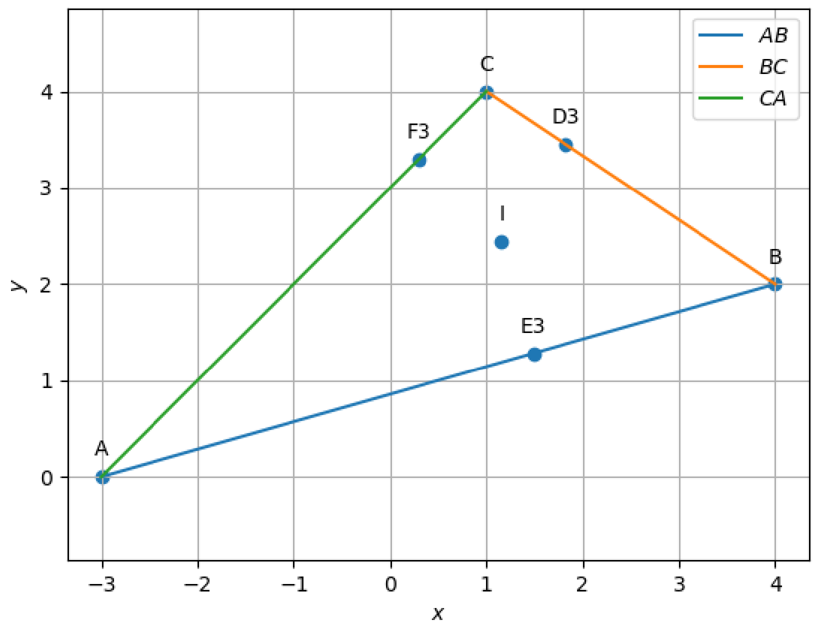

Figure 1.2: Incentre and Inradius of $\triangle ABC$

1.1.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution:

$$\mathbf{D}_3 = \begin{pmatrix} 1.8246 \\ 3.4502 \end{pmatrix} \tag{1.1.4.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} 0.2991 \\ 3.2991 \end{pmatrix} \tag{1.1.4.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} 1.4861 \\ 1.2817 \end{pmatrix} \text{Incentre} \tag{1.1.4.3}$$

$$\mathbf{I} = \begin{pmatrix} 1.1539 \\ 2.4441 \end{pmatrix} \tag{1.1.4.4}$$

$$\text{Radius} \tag{1.1.4.5}$$

$$\mathbf{r} = 0.7423341566 \tag{1.1.4.6}$$

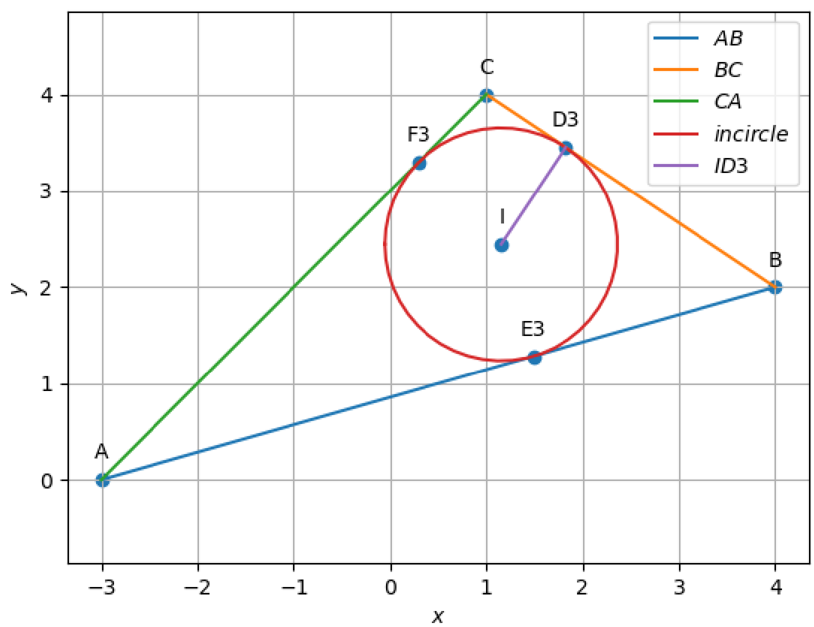

Figure 1.3: Incircle of $\triangle ABC$

1.1.5. Using (1.1.7) verify that

$$\angle BAI = \angle CAI. \tag{1.1.5.1}$$

AI is the bisector of $\angle A$.

Solution:

$$\cos \angle BAI \triangleq \frac{(\mathbf{B}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\| \, \|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.2}$$

$$\cos \angle CAI \triangleq \frac{(\mathbf{C}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{C}-\mathbf{A}\| \, \|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.3}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{I}$,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \tag{1.1.5.4}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.5.5}$$

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 4.1539 \\ 2.4441 \end{pmatrix} \tag{1.1.5.6}$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{53} \tag{1.1.5.7}$$

$$\|\mathbf{C} - \mathbf{A}\| = \sqrt{32} \tag{1.1.5.8}$$

$$\|\mathbf{I} - \mathbf{A}\| = \sqrt{23.228} \tag{1.1.5.9}$$

$$\tag{1.1.5.10}$$

(a) for $\angle BAI$:

On substtuting the values in (1.1.5.2) ,We get On solving

$$\angle BAI = 14.524611^\circ \tag{1.1.5.11}$$

(b) for $\angle CAI$:

On substtuting the values in (1.1.5.2) ,We get On solving

$$\angle CAI = 14.52559381^\circ \tag{1.1.5.12}$$

Therefore $\angle BAI = \angle CAI$. and AI is the bisector of $\angle A$.

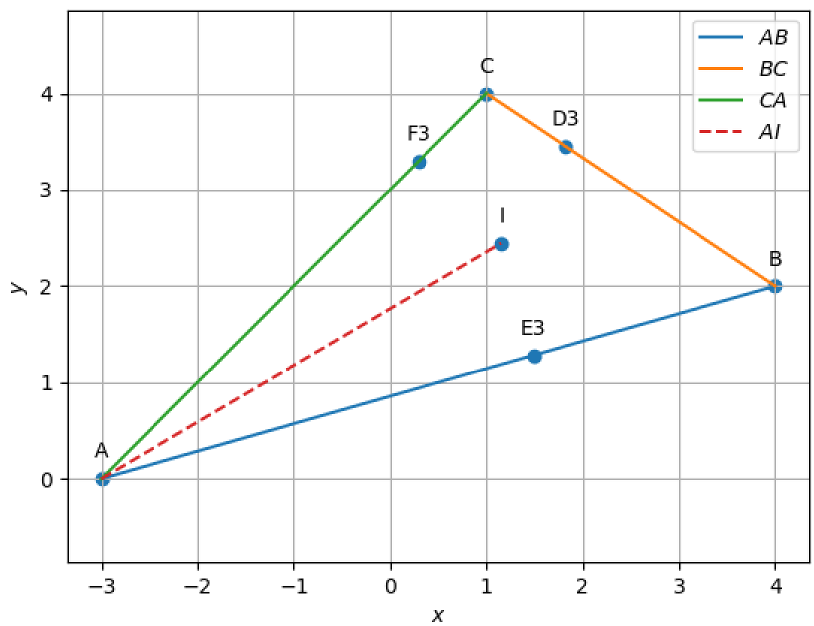

Figure 1.4: Angular bisector AI